

CHAPTER 8

FREQUENCY RESPONSE: MODAL FORM

8.1 Introduction

Now that the theory behind the modal analysis method has been covered, we will solve our tdof problem for its frequency response.

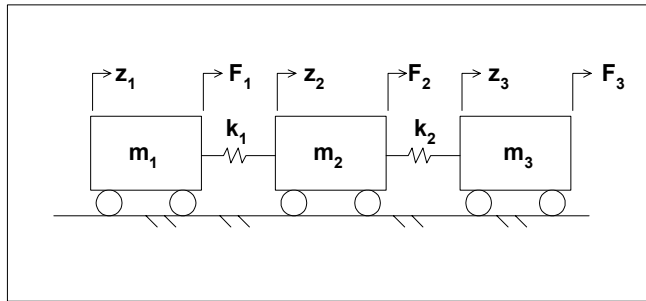


Figure 8.1: tdof undamped model for modal analysis.

We will use eigenvalue/eigenvector results from Chapter 7 to define the equations of motion in principal coordinates and to transform forces to principal coordinates. We will then use Laplace transforms to solve for the transfer functions in principal coordinates and back-transform to physical coordinates, where the individual mode contributions will be evident. We will discuss the relationship between the partial fraction expansion transfer function form and the modal form derived here. We discussed in Section 5.13 how to excite only a single mode of vibration by judicious choice of initial conditions. Here we will describe the forcing function combination required to excite only a single mode.

We will spend considerable time in this chapter on developing a greater understanding of how individual modes of vibration combine to give the overall frequency response. MATLAB code is supplied for the tdof problem to illustrate the point. ANSYS is also used to solve the tdof problem and the ANSYS results are described and compared with the MATLAB results.

8.2 Review from Previous Results

Since the problem we are solving is frequency response, or finding the steady state motion of each mass as a function of frequency and of applied forces, initial conditions are not required.

From previous analyses, (7.85) to (7.88), we know the eigenvalues and eigenvectors normalized with respect to mass, ω_i , $\mathbf{z}_n$:

$$\omega_1 = 0 \quad \omega_2 = \pm\sqrt{\frac{k}{m}} \quad \omega_3 = \pm\sqrt{\frac{3k}{m}} \quad (8.1)$$

$$\mathbf{z}_n = \frac{1}{\sqrt{m}} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \quad (8.2)$$

Knowing that in principal coordinates the mass matrix is the identity matrix and the stiffness matrix is a diagonal matrix with the squares of the respective eigenvalues as terms, we can write the matrices by inspection:

$$\mathbf{m}_p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{k}_p = \left(\frac{k}{m}\right) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad (8.3)$$

The force vector in principal coordinates is:

$$\mathbf{F}_p = \mathbf{z}_n^T \mathbf{F} = \begin{bmatrix} z_{n11} & z_{n21} & z_{n31} \\ z_{n12} & z_{n22} & z_{n32} \\ z_{n13} & z_{n23} & z_{n33} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \quad (8.4)$$

Expanding:

$$F_{p1} = z_{n11}F_1 + z_{n21}F_2 + z_{n31}F_3$$

$$F_{p2} = z_{n12}F_1 + z_{n22}F_2 + z_{n32}F_3 \quad (8.5a,b,c)$$

$$F_{p3} = z_{n13}F_1 + z_{n23}F_2 + z_{n33}F_3$$

Performing the actual problem multiplication:

$$\mathbf{F}_p = \mathbf{z}_n^T \mathbf{F} = \frac{1}{\sqrt{m}} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \frac{1}{\sqrt{m}} \begin{bmatrix} \frac{F_1}{\sqrt{3}} + \frac{F_2}{\sqrt{3}} + \frac{F_3}{\sqrt{3}} \\ \frac{F_1}{\sqrt{2}} + 0 - \frac{F_3}{\sqrt{2}} \\ \frac{F_1}{\sqrt{6}} - \frac{2F_2}{\sqrt{6}} + \frac{F_3}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} F_{p1} \\ F_{p2} \\ F_{p3} \end{bmatrix} \quad (8.6)$$

Writing the resulting equations of motion in principal coordinates in matrix form:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{z}_{p1} \\ \ddot{z}_{p2} \\ \ddot{z}_{p3} \end{bmatrix} + \left(\frac{k}{m} \right) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} z_{p1} \\ z_{p2} \\ z_{p3} \end{bmatrix} = \frac{1}{\sqrt{m}} \begin{bmatrix} \frac{F_1}{\sqrt{3}} + \frac{F_2}{\sqrt{3}} + \frac{F_3}{\sqrt{3}} \\ \frac{F_1}{\sqrt{2}} + 0 - \frac{F_3}{\sqrt{2}} \\ \frac{F_1}{\sqrt{6}} - \frac{2F_2}{\sqrt{6}} + \frac{F_3}{\sqrt{6}} \end{bmatrix} \quad (8.7)$$

Writing out the equations in expanded form:

$$\ddot{z}_{p1} = (F_1 + F_2 + F_3) \frac{1}{\sqrt{3m}} = F_{p1} \quad (8.8)$$

$$\ddot{z}_{p2} + \frac{k}{m} z_{p2} = (F_1 - F_3) \frac{1}{\sqrt{2m}} = F_{p2} \quad \frac{k}{m} = \omega_2^2 \quad (8.9)$$

$$\ddot{z}_{p3} + \frac{3k}{m} z_{p3} = (F_1 - 2F_2 + F_3) \frac{1}{\sqrt{6m}} = F_{p3} \quad \frac{3k}{m} = \omega_3^2 \quad (8.10)$$

8.3 Transfer Functions – Laplace Transforms in Principal Coordinates

We now solve for the transfer functions. Taking the Laplace transform of each equation, ignoring initial conditions and collecting the displacement terms, where $z_{pi}(s)$ is the Laplace transform of z_{pi} (Appendix 2):

$$\begin{aligned} s^2 z_{p1}(s) &= (F_1(s) + F_2(s) + F_3(s)) \frac{1}{\sqrt{3m}} \\ z_{p2}(s)(s^2 + \omega_2^2) &= (F_1(s) - F_3(s)) \frac{1}{\sqrt{2m}} \\ z_{p3}(s)(s^2 + \omega_3^2) &= (F_1(s) - 2F_2(s) + F_3(s)) \frac{1}{\sqrt{6m}} \end{aligned} \quad (8.11a,b,c)$$

Solving for the three principal displacements and eliminating the “(s)” for simplicity:

$$\begin{aligned} z_{p1} &= (F_1 + F_2 + F_3) \frac{1}{s^2 \sqrt{3m}} \\ z_{p2} &= (F_1 - F_3) \frac{1}{(s^2 + \omega_2^2) \sqrt{2m}} \\ z_{p3} &= (F_1 - 2F_2 + F_3) \frac{1}{(s^2 + \omega_3^2) \sqrt{6m}} \end{aligned} \quad (8.12a,b,c)$$

Taking the forces one at a time, the elements of a transfer function matrix can be defined.

$$\begin{aligned} \frac{z_{p1}}{F_1} &= \frac{1}{s^2 \sqrt{3m}} \\ \frac{z_{p1}}{F_2} &= \frac{1}{s^2 \sqrt{3m}} \\ \frac{z_{p1}}{F_3} &= \frac{1}{s^2 \sqrt{3m}} \end{aligned} \quad (8.13a,b,c)$$

$$\begin{aligned}\frac{z_{p2}}{F_1} &= \frac{1}{(s^2 + \omega_2^2)\sqrt{2m}} \\ \frac{z_{p2}}{F_2} &= 0 \\ \frac{z_{p2}}{F_3} &= \frac{-1}{(s^2 + \omega_2^2)\sqrt{2m}}\end{aligned}\tag{8.14a,b,c}$$

$$\begin{aligned}\frac{z_{p3}}{F_1} &= \frac{1}{(s^2 + \omega_3^2)\sqrt{6m}} \\ \frac{z_{p3}}{F_2} &= \frac{-2}{(s^2 + \omega_3^2)\sqrt{6m}} \\ \frac{z_{p3}}{F_3} &= \frac{1}{(s^2 + \omega_3^2)\sqrt{6m}}\end{aligned}\tag{8.15a,b,c}$$

Writing out the principal coordinate transfer functions for each external force, F_1 , F_2 , and F_3 :

$$\frac{z_p}{F_1} = \begin{bmatrix} \frac{z_{p1}}{F_1} \\ \frac{z_{p2}}{F_1} \\ \frac{z_{p3}}{F_1} \end{bmatrix} = \begin{bmatrix} \frac{1}{s^2\sqrt{3m}} \\ \frac{1}{(s^2 + \omega_2^2)\sqrt{2m}} \\ \frac{1}{(s^2 + \omega_3^2)\sqrt{6m}} \end{bmatrix} = \begin{bmatrix} z_{p11} \\ z_{p21} \\ z_{p31} \end{bmatrix}\tag{8.16}$$

$$\frac{z_p}{F_2} = \begin{bmatrix} \frac{z_{p1}}{F_2} \\ \frac{z_{p2}}{F_2} \\ \frac{z_{p3}}{F_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{s^2\sqrt{3m}} \\ 0 \\ \frac{-2}{(s^2 + \omega_3^2)\sqrt{6m}} \end{bmatrix} = \begin{bmatrix} z_{p12} \\ z_{p22} \\ z_{p32} \end{bmatrix}\tag{8.17}$$

$$\frac{\mathbf{z}_p}{F_3} = \begin{bmatrix} \frac{z_{p1}}{F_3} \\ \frac{z_{p2}}{F_3} \\ \frac{z_{p3}}{F_3} \end{bmatrix} = \begin{bmatrix} \frac{1}{s^2 \sqrt{3m}} \\ \frac{-1}{(s^2 + \omega_2^2) \sqrt{2m}} \\ \frac{1}{(s^2 + \omega_3^2) \sqrt{6m}} \end{bmatrix} = \begin{bmatrix} z_{p13} \\ z_{p23} \\ z_{p33} \end{bmatrix} \quad (8.18)$$

8.4 Back-Transforming Mode Contributions to Transfer Functions in Physical Coordinates

Now the transfer functions in principal coordinates can be back-transformed to physical coordinates. This allows one to see the contributions of each mode, where z_{ij} is the physical displacement at dof “i” due to a force at dof “j.”

$$\mathbf{z} = \mathbf{z}_n \mathbf{z}_p = \begin{bmatrix} z_{n11} & z_{n12} & z_{n13} \\ z_{n21} & z_{n22} & z_{n23} \\ z_{n31} & z_{n32} & z_{n33} \end{bmatrix} \begin{bmatrix} z_{p11} & z_{p12} & z_{p13} \\ z_{p21} & z_{p22} & z_{p23} \\ z_{p31} & z_{p32} & z_{p33} \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix} \quad (8.19)$$

The equations below show how the results from each of the principal equations (modes) combine to give the overall response. The overall transfer function is seen to be a combination of the three modes of vibration and is referred to as the “modal form.”

$$\frac{z_1}{F_1} = \underbrace{z_{n11} z_{p11}}_{1\text{st mode}} + \underbrace{z_{n12} z_{p21}}_{2\text{nd mode}} + \underbrace{z_{n13} z_{p31}}_{3\text{rd mode}} \quad \text{contributions to total } \frac{z_1}{F_1} \text{ transfer function.}$$

$$\frac{z_2}{F_1} = \underbrace{z_{n21} z_{p11}}_{1\text{st mode}} + \underbrace{z_{n22} z_{p21}}_{2\text{nd mode}} + \underbrace{z_{n23} z_{p31}}_{3\text{rd mode}} \quad \text{contributions to total } \frac{z_2}{F_1} \text{ transfer function.}$$

$$\frac{z_3}{F_1} = \underbrace{z_{n31} z_{p11}}_{1\text{st mode}} + \underbrace{z_{n32} z_{p21}}_{2\text{nd mode}} + \underbrace{z_{n33} z_{p31}}_{3\text{rd mode}} \quad \text{contributions to total } \frac{z_3}{F_1} \text{ transfer function.}$$

$$\frac{z_1}{F_2} = \underbrace{z_{n11} z_{p12}}_{1\text{st mode}} + \underbrace{z_{n12} z_{p22}}_{2\text{nd mode}} + \underbrace{z_{n13} z_{p32}}_{3\text{rd mode}} \quad \text{contributions to total } \frac{z_1}{F_2} \text{ transfer function.}$$

$$\frac{Z_2}{F_2} = \underbrace{Z_{n21}Z_{p12}}_{1\text{st mode}} + \underbrace{Z_{n22}Z_{p22}}_{2\text{nd mode}} + \underbrace{Z_{n23}Z_{p32}}_{3\text{rd mode}} \quad \text{contributions to total } \frac{Z_2}{F_2} \text{ transfer function.}$$

$$\frac{Z_3}{F_2} = \underbrace{Z_{n31}Z_{p12}}_{1\text{st mode}} + \underbrace{Z_{n32}Z_{p22}}_{2\text{nd mode}} + \underbrace{Z_{n33}Z_{p32}}_{3\text{rd mode}} \quad \text{contributions to total } \frac{Z_3}{F_2} \text{ transfer function.}$$

$$\frac{Z_1}{F_3} = \underbrace{Z_{n11}Z_{p13}}_{1\text{st mode}} + \underbrace{Z_{n12}Z_{p23}}_{2\text{nd mode}} + \underbrace{Z_{n13}Z_{p33}}_{3\text{rd mode}} \quad \text{contributions to total } \frac{Z_1}{F_3} \text{ transfer function.}$$

$$\frac{Z_2}{F_3} = \underbrace{Z_{n21}Z_{p13}}_{1\text{st mode}} + \underbrace{Z_{n22}Z_{p23}}_{2\text{nd mode}} + \underbrace{Z_{n23}Z_{p33}}_{3\text{rd mode}} \quad \text{contributions to total } \frac{Z_2}{F_3} \text{ transfer function.}$$

$$\frac{Z_3}{F_3} = \underbrace{Z_{n31}Z_{p13}}_{1\text{st mode}} + \underbrace{Z_{n32}Z_{p23}}_{2\text{nd mode}} + \underbrace{Z_{n33}Z_{p33}}_{3\text{rd mode}} \quad \text{contributions to total } \frac{Z_3}{F_3} \text{ transfer function.}$$

We saw earlier that because of symmetry there are only four distinctly different transfer functions of the total of nine:

$$\frac{Z_1}{F_1}, \frac{Z_2}{F_1}, \frac{Z_3}{F_1} \text{ and } \frac{Z_2}{F_2}$$

Expanding the four transfer functions:

$$\begin{aligned} \frac{Z_1}{F_1} &= \frac{1}{\sqrt{m}} \left[\frac{Z_{p11}}{\sqrt{3}} + \frac{Z_{p21}}{\sqrt{2}} + \frac{Z_{p31}}{\sqrt{6}} \right] \\ &= \frac{1}{\sqrt{m}} \left[\frac{1}{s^2 \sqrt{3m} \sqrt{3}} + \frac{1}{(s^2 + \omega_2^2) \sqrt{2m} \sqrt{2}} + \frac{1}{(s^2 + \omega_3^2) \sqrt{6m} \sqrt{6}} \right] \\ &= \frac{1}{s^2 (3m)} + \frac{1}{(s^2 + \omega_2^2) 2m} + \frac{1}{(s^2 + \omega_3^2) 6m} \end{aligned}$$

$$= \frac{\left(\frac{1}{3m}\right)}{s^2} + \frac{\left(\frac{1}{2m}\right)}{s^2 + \omega_2^2} + \frac{\left(\frac{1}{6m}\right)}{s^2 + \omega_3^2} \quad (8.20)$$

$$\begin{aligned} \frac{z_2}{F_1} &= \frac{1}{\sqrt{m}} \left[\frac{1}{s^2 \sqrt{3m} \sqrt{3}} + 0 - \frac{2}{(s^2 + \omega_3^2) \sqrt{6m} \sqrt{6}} \right] \\ &= \frac{\left(\frac{1}{3m}\right)}{s^2} + 0 - \frac{\left(\frac{2}{6m}\right)}{s^2 + \omega_3^2} \end{aligned} \quad (8.21)$$

$$\begin{aligned} \frac{z_3}{F_1} &= \frac{1}{\sqrt{m}} \left[\frac{1}{s^2 \sqrt{3m} \sqrt{3}} - \frac{1}{(s^2 + \omega_2^2) \sqrt{2m} \sqrt{2}} + \frac{1}{(s^2 + \omega_3^2) \sqrt{6m} \sqrt{6}} \right] \\ &= \frac{\left(\frac{1}{3m}\right)}{s^2} - \frac{\left(\frac{1}{2m}\right)}{s^2 + \omega_2^2} + \frac{\left(\frac{1}{6m}\right)}{s^2 + \omega_3^2} \end{aligned} \quad (8.22)$$

$$\begin{aligned} \frac{z_2}{F_2} &= \frac{1}{\sqrt{m}} \left[\frac{1}{s^2 \sqrt{3m} \sqrt{3}} + 0 + \frac{4}{(s^2 + \omega_3^2) \sqrt{6m} \sqrt{6}} \right] \\ &= \frac{\left(\frac{1}{3m}\right)}{s^2} + 0 + \frac{\left(\frac{4}{6m}\right)}{s^2 + \omega_3^2} \end{aligned} \quad (8.23)$$

Taking $m = k = 1$ yields: $\omega_1^2 = 0$, $\omega_2^2 = \frac{k}{m} = 1$, $\omega_3^2 = \frac{3k}{m} = 3$, and substituting above:

$$\frac{z_1}{F_1} = \frac{\frac{1}{3}}{s^2} + \frac{\frac{1}{2}}{s^2 + 1} + \frac{\frac{1}{6}}{s^2 + 3} \quad (8.24)$$

$$\frac{z_2}{F_1} = \frac{\frac{1}{3}}{s^2} - \frac{\frac{1}{3}}{s^2 + 3} \quad (8.25)$$

$$\frac{z_3}{F_1} = \frac{1}{s^2} - \frac{1}{s^2 + 1} + \frac{1}{s^2 + 3} \quad (8.26)$$

$$\frac{z_2}{F_2} = \frac{1}{s^2} + \frac{2}{s^2 + 3} \quad (8.27)$$

8.5 Partial Fraction Expansion and the Modal Form

Another way of finding the modal form (not as insightful, but does not require solving the eigenvalue problem) is to take the original transfer functions derived in the Chapter 2 and perform a partial fraction expansion. Partial fraction expansion gives the same results as the modal form in Section 8.4. The four unique transfer functions from (2.62) to (2.65) are repeated below:

$$\frac{z_1}{F_1} = \frac{m^2 s^4 + 3mks^2 + k^2}{s^2 (m^3 s^4 + 4m^2 ks^2 + 3mk^2)} \quad (8.28)$$

$$\frac{z_2}{F_1} = \frac{k}{s^2 (m^2 s^2 + 3km)} \quad (8.29)$$

$$\frac{z_3}{F_1} = \frac{k^2}{s^2 (m^3 s^4 + 4m^2 ks^2 + 3mk^2)} \quad (8.30)$$

$$\frac{z_2}{F_2} = \frac{m^2 s^4 + 2mks^2 + k^2}{s^2 (m^3 s^4 + 4m^2 ks^2 + 3mk^2)} \quad (8.31)$$

In order to perform a partial fraction expansion, we need the roots of the characteristic equation, found earlier to be:

$$\omega_1^2 = 0 \quad \omega_2^2 = \frac{k}{m} \quad \omega_3^2 = \frac{3k}{m} \quad (8.32)$$

Taking the z_1 / F_1 transfer function and expanding in partial fraction form, setting $m = k = 1$;

$$\frac{z_1}{F_1} = \frac{m^2 s^4 + 3mks^2 + k^2}{s^2 (m^3 s^4 + 4m^2 ks^2 + 3mk^2)} = \frac{s^4 + 3s^2 + 1}{s^2 (s^4 + 4s^2 + 3)} = \frac{s^4 + 3s^2 + 1}{s^2 (s^2 + \omega_2^2)(s^2 + \omega_3^2)}$$

$$\frac{s^4 + 3s^2 + 1}{s^2 (s^2 + \omega_2^2)(s^2 + \omega_3^2)} = \frac{A}{s^2 + \omega_1^2} + \frac{B}{s^2 + \omega_2^2} + \frac{C}{s^2 + \omega_3^2}$$

(8.33a,b)

The terms A, B and C, known as “residues,” are evaluated using the “cover-up” method, where each coefficient is evaluated by “covering up” its term in the transfer function and evaluating the remaining expression at $s^2 = -\omega_1^2$.

Evaluating A:

$$A = \frac{s^4 + 3s^2 + 1}{(s^2 + \omega_2^2)(s^2 + \omega_3^2)} \text{ evaluated at } s^2 = -\omega_1^2 = 0$$

$$= \frac{1}{\omega_2^2 \omega_3^2} = \frac{1}{(1)(3)} = \frac{1}{3}$$

(8.34)

Evaluating B:

$$B = \frac{s^4 + 3s^2 + 1}{s^2 (s^2 + \omega_3^2)} \text{ evaluated at } s^2 = -\omega_2^2 = -1$$

$$= \frac{\omega_2^4 - 3\omega_2^2 + 1}{-\omega_2^2 (-\omega_2^2 + \omega_3^2)} = \frac{1 - 3 + 1}{-1(-1 + 3)} = \frac{-1}{-2} = \frac{1}{2}$$

(8.35)

Evaluating C:

$$\begin{aligned}
 C &= \frac{s^4 + 3s^2 + 1}{s^2(s^2 + \omega_2^2)(s^2 + \omega_3^2)} \quad \text{evaluated at } s^2 = -\omega_3^2 = -3 \\
 &= \frac{\omega_3^4 - 3\omega_3^2 + 1}{-\omega_3^2(-\omega_3^2 + \omega_2^2)} = \frac{3^2 - 3(-3) + 1}{-3(-3 + 1)} = \frac{1}{6}
 \end{aligned} \tag{8.36}$$

Combining terms:

$$\frac{z_1}{F_1} = \frac{s^4 + 3s^2 + 1}{s^2(s^2 + \omega_2^2)(s^2 + \omega_3^2)} = \frac{\left[\frac{1}{3}\right]}{s^2 + \omega_1^2} + \frac{\left[\frac{1}{2}\right]}{s^2 + \omega_2^2} + \frac{\left[\frac{1}{6}\right]}{s^2 + \omega_3^2} \tag{8.37}$$

This expression is the same as the term for z_1/F_1 in (8.20). Converting the other three transfer functions to partial fraction form also reveals their modal form.

8.6 Forcing Function Combinations to Excite Single Mode

It is instructive at this point to see what types of forcing function combinations will excite each of the three modes separately. From the definition of normal modes, we know that if the system is started from initial displacement conditions that match one of the normal modes, the system will respond at only that mode. An analogous situation exists for combinations of forcing functions. Repeating the transformed equations of motion in principal coordinates from (8.12a,b,c) with $m = 1$.

$$\begin{aligned}
 z_{p1} &= (F_1 + F_2 + F_3) \frac{1}{s^2 \sqrt{3}} = F_{p1} \\
 z_{p2} &= (F_1 - F_3) \frac{1}{(s^2 + \omega_2^2) \sqrt{2}} = F_{p2} \\
 z_{p3} &= (F_1 - 2F_2 + F_3) \frac{1}{(s^2 + \omega_3^2) \sqrt{6}} = F_{p3}
 \end{aligned} \tag{8.38a,b,c}$$

To excite only the first mode, we can start with initial displacements being any multiple of the first eigenvector, which has equal displacements for all masses.

Now let us see if applying equal forces to all three masses with zero initial conditions excites only the first mode. Set $F_1 = F_2 = F_3 \neq 0$, which should excite only the first, rigid body mode:

$$\begin{aligned} z_{p1} &= (F_1 + F_2 + F_3) \frac{1}{s^2 \sqrt{3}} = 3F_1 \frac{1}{s^2 \sqrt{3}} \\ z_{p2} &= (F_1 - F_3) \frac{1}{(s^2 + \omega_2^2) \sqrt{2}} = 0 \\ z_{p3} &= (F_1 - 2F_2 + F_3) \frac{1}{(s^2 + \omega_3^2) \sqrt{6}} = 0 \end{aligned} \quad (8.39a,b,c)$$

We can see above that the motion for the second and third modes is zero. It is the information contained in the eigenvector, which, when multiplied by the force vector in physical coordinates, determines the force vector in principal coordinates.

To excite the second mode only, applying zero force at mass 1 and equal and opposite sign forces at masses 1 and 2 should work: $F_1 = -F_3$, $F_2 = 0$:

$$\begin{aligned} z_{p1} &= (F_1 + F_2 + F_3) \frac{1}{s^2 \sqrt{3}} = 0 \\ z_{p2} &= (F_1 - F_3) \frac{1}{(s^2 + \omega_2^2) \sqrt{2}} = 2F_1 \frac{1}{(s^2 + \omega_2^2) \sqrt{2}} \\ z_{p3} &= (F_1 - 2F_2 + F_3) \frac{1}{(s^2 + \omega_3^2) \sqrt{6}} = 0 \end{aligned} \quad (8.40a,b,c)$$

In this case the combination of the eigenvectors and forcing function signs cancel out the first and third modes, leaving only the second mode.

To excite the third mode only, applying the same force to masses 1 and 3 and twice the force with opposite sign to mass 2 should work: $F_1 = F_3$, $F_2 = -2F_1$:

$$\begin{aligned}
z_{p1} &= (F_1 + F_2 + F_3) \frac{1}{s^2 \sqrt{3}} = 0 \\
z_{p2} &= (F_1 - F_3) \frac{1}{(s^2 + \omega_2^2) \sqrt{2}} = 0 \\
z_{p3} &= (F_1 - 2F_2 + F_3) \frac{1}{(s^2 + \omega_3^2) \sqrt{6}} = 6F_1 \frac{1}{(s^2 + \omega_3^2) \sqrt{6}}
\end{aligned} \tag{8.41a,b,c}$$

In this case the combination of the eigenvectors and forcing function signs cancel out the first and second modes, leaving only the third mode.

8.7 How Modes Combine to Create Transfer Functions

We have shown that both the normal mode method and partial fraction expansion of transfer functions yield additive combinations of sdof systems for the overall response. The purpose of this section is to develop a general equation for any transfer function, again showing that the system frequency response is an additive combination of sdof systems. **Each sdof system has a gain determined by the appropriate eigenvector entries and a resonant frequency given by the appropriate eigenvalue.**

The three equations of motion in principal coordinates are:

$$\begin{aligned}
\ddot{z}_{p1} + \omega_1^2 z_{p1} &= F_{p1} \\
\ddot{z}_{p2} + \omega_2^2 z_{p2} &= F_{p2} \\
\ddot{z}_{p3} + \omega_3^2 z_{p3} &= F_{p3}
\end{aligned} \tag{8.42a,b,c}$$

$$\omega_1^2 = 0 \quad \omega_2^2 = \frac{k}{m} \quad \omega_3^2 = \frac{3k}{m} \tag{8.43a,b,c}$$

Where the forces in principal coordinates are given by:

$$\begin{aligned}
\mathbf{F}_p &= \mathbf{z}_n^T \mathbf{F} = \begin{bmatrix} z_{n11} & z_{n21} & z_{n31} \\ z_{n12} & z_{n22} & z_{n32} \\ z_{n13} & z_{n23} & z_{n33} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \\
&= \begin{bmatrix} z_{n11} F_1 + z_{n21} F_2 + z_{n31} F_3 \\ z_{n12} F_1 + z_{n22} F_2 + z_{n32} F_3 \\ z_{n13} F_1 + z_{n23} F_2 + z_{n33} F_3 \end{bmatrix}
\end{aligned} \tag{8.44}$$

Taking the Laplace transform of the differential equations (8.42a,b,c) and dividing by the coefficients of each principal displacement:

$$\mathbf{z}_p = \begin{bmatrix} z_{p1} \\ z_{p2} \\ z_{p3} \end{bmatrix} = \begin{bmatrix} \frac{F_{p1}}{s^2 + \omega_1^2} \\ \frac{F_{p2}}{s^2 + \omega_2^2} \\ \frac{F_{p3}}{s^2 + \omega_3^2} \end{bmatrix} = \begin{bmatrix} \frac{z_{n11}F_1 + z_{n21}F_2 + z_{n31}F_3}{s^2 + \omega_1^2} \\ \frac{z_{n12}F_1 + z_{n22}F_2 + z_{n32}F_3}{s^2 + \omega_2^2} \\ \frac{z_{n13}F_1 + z_{n23}F_2 + z_{n33}F_3}{s^2 + \omega_3^2} \end{bmatrix} \quad (8.45)$$

The equation above shows how the individual eigenvector matrix elements contribute to the displacements in principal coordinates.

Since we are only interested in SISO transfer functions that arise from a force applied to a single dof, we will look at the F_1 , F_2 , F_3 cases individually.

For force F_1 :

$$\mathbf{z}_p = \begin{bmatrix} z_{p1} \\ z_{p2} \\ z_{p3} \end{bmatrix} = \begin{bmatrix} \frac{F_{p1}}{s^2 + \omega_1^2} \\ \frac{F_{p2}}{s^2 + \omega_2^2} \\ \frac{F_{p3}}{s^2 + \omega_3^2} \end{bmatrix} = \begin{bmatrix} \frac{z_{n11}F_1}{s^2 + \omega_1^2} \\ \frac{z_{n12}F_1}{s^2 + \omega_2^2} \\ \frac{z_{n13}F_1}{s^2 + \omega_3^2} \end{bmatrix} \quad (8.46)$$

For force F_2 :

$$\mathbf{z}_p = \begin{bmatrix} z_{p1} \\ z_{p2} \\ z_{p3} \end{bmatrix} = \begin{bmatrix} \frac{F_{p1}}{s^2 + \omega_1^2} \\ \frac{F_{p2}}{s^2 + \omega_2^2} \\ \frac{F_{p3}}{s^2 + \omega_3^2} \end{bmatrix} = \begin{bmatrix} \frac{z_{n21}F_2}{s^2 + \omega_1^2} \\ \frac{z_{n22}F_2}{s^2 + \omega_2^2} \\ \frac{z_{n23}F_2}{s^2 + \omega_3^2} \end{bmatrix} \quad (8.47)$$

For force F_3

$$\mathbf{z}_p = \begin{bmatrix} z_{p1} \\ z_{p2} \\ z_{p3} \end{bmatrix} = \begin{bmatrix} \frac{F_{p1}}{s^2 + \omega_1^2} \\ \frac{F_{p2}}{s^2 + \omega_2^2} \\ \frac{F_{p3}}{s^2 + \omega_3^2} \end{bmatrix} = \begin{bmatrix} \frac{z_{n31}F_3}{s^2 + \omega_1^2} \\ \frac{z_{32}F_3}{s^2 + \omega_2^2} \\ \frac{z_{n33}F_3}{s^2 + \omega_3^2} \end{bmatrix} \quad (8.48)$$

The equations for displacements in physical coordinates are found by premultiplying the above three equations by $\mathbf{z}_n$ (7.107).

For F_1 :

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} \frac{z_{n11}z_{n11}F_1}{s^2 + \omega_1^2} + \frac{z_{n12}z_{n12}F_1}{s^2 + \omega_2^2} + \frac{z_{n13}z_{n13}F_1}{s^2 + \omega_3^2} \\ \frac{z_{n21}z_{n11}F_1}{s^2 + \omega_1^2} + \frac{z_{n22}z_{n12}F_1}{s^2 + \omega_2^2} + \frac{z_{n23}z_{n13}F_1}{s^2 + \omega_3^2} \\ \frac{z_{n31}z_{n11}F_1}{s^2 + \omega_1^2} + \frac{z_{n32}z_{n12}F_1}{s^2 + \omega_2^2} + \frac{z_{n33}z_{n13}F_1}{s^2 + \omega_3^2} \end{bmatrix} \quad (8.49)$$

Dividing by F_1 :

$$\frac{\mathbf{z}}{F_1} = \begin{bmatrix} \frac{z_1}{F_1} \\ \frac{z_2}{F_1} \\ \frac{z_3}{F_1} \end{bmatrix} = \begin{bmatrix} \frac{z_{n11}z_{n11}}{s^2 + \omega_1^2} + \frac{z_{n12}z_{n12}}{s^2 + \omega_2^2} + \frac{z_{n13}z_{n13}}{s^2 + \omega_3^2} \\ \frac{z_{n21}z_{n11}}{s^2 + \omega_1^2} + \frac{z_{n22}z_{n12}}{s^2 + \omega_2^2} + \frac{z_{n23}z_{n13}}{s^2 + \omega_3^2} \\ \frac{z_{n31}z_{n11}}{s^2 + \omega_1^2} + \frac{z_{n32}z_{n12}}{s^2 + \omega_2^2} + \frac{z_{n33}z_{n13}}{s^2 + \omega_3^2} \end{bmatrix} \quad (8.50)$$

Similarly for F_1 and F_2 :

$$\frac{\mathbf{z}}{F_2} = \begin{bmatrix} \frac{z_1}{F_2} \\ \frac{z_2}{F_2} \\ \frac{z_3}{F_2} \end{bmatrix} = \begin{bmatrix} \frac{Z_{n11}Z_{n21}}{s^2 + \omega_1^2} + \frac{Z_{n12}Z_{n22}}{s^2 + \omega_2^2} + \frac{Z_{n13}Z_{n23}}{s^2 + \omega_3^2} \\ \frac{Z_{n21}Z_{n21}}{s^2 + \omega_1^2} + \frac{Z_{n22}Z_{n22}}{s^2 + \omega_2^2} + \frac{Z_{n23}Z_{n23}}{s^2 + \omega_3^2} \\ \frac{Z_{n31}Z_{n21}}{s^2 + \omega_1^2} + \frac{Z_{n32}Z_{n22}}{s^2 + \omega_2^2} + \frac{Z_{n33}Z_{n23}}{s^2 + \omega_3^2} \end{bmatrix} \quad (8.51)$$

$$\frac{\mathbf{z}}{F_3} = \begin{bmatrix} \frac{z_1}{F_3} \\ \frac{z_2}{F_3} \\ \frac{z_3}{F_3} \end{bmatrix} = \begin{bmatrix} \frac{Z_{n11}Z_{n31}}{s^2 + \omega_1^2} + \frac{Z_{n12}Z_{n32}}{s^2 + \omega_2^2} + \frac{Z_{n13}Z_{n33}}{s^2 + \omega_3^2} \\ \frac{Z_{n21}Z_{n31}}{s^2 + \omega_1^2} + \frac{Z_{n22}Z_{n32}}{s^2 + \omega_2^2} + \frac{Z_{n23}Z_{n33}}{s^2 + \omega_3^2} \\ \frac{Z_{n31}Z_{n31}}{s^2 + \omega_1^2} + \frac{Z_{n32}Z_{n32}}{s^2 + \omega_2^2} + \frac{Z_{n33}Z_{n33}}{s^2 + \omega_3^2} \end{bmatrix} \quad (8.52)$$

The nine transfer functions above may be generalized by the following equation. For the transfer function with the force applied at dof “k,” the displacement taken at dof “j” and for mode “i”:

$$\frac{Z_j}{F_k} = \frac{Z_{nj1}Z_{nk1}}{s^2 + \omega_1^2} + \frac{Z_{nj2}Z_{nk2}}{s^2 + \omega_2^2} + \frac{Z_{nj3}Z_{nk3}}{s^2 + \omega_3^2} \quad (8.53)$$

Rewriting in summation form, and generalizing from our tdof system to a general system where “m” is the total number of modes for the system for an undamped (8.54a) and damped (8.54b) system:

$$\frac{Z_j}{F_k} = \sum_{i=1}^m \frac{Z_{nji}Z_{nki}}{s^2 + \omega_i^2} \quad (8.54a,b)$$

$$\frac{Z_j}{F_k} = \sum_{i=1}^m \frac{Z_{nji}Z_{nki}}{s^2 + 2\zeta_i\omega_i s + \omega_i^2}$$

Equations (8.54a,b) shows that in general every transfer function is made up of additive combinations of single degree of freedom systems, with each system having its dc gain (transfer function evaluated with $s = j0$) determined by the appropriate eigenvector entry product divided by the

square of the eigenvalue, $z_{nji}z_{nki} / \omega_i^2$, and with resonant frequency defined by the appropriate eigenvalue, ω_i .

For our tdof system, substituting for the ω_i values:

$$\frac{Z_j}{F_k} = \frac{z_{nj1}z_{nk1}}{s^2} + \frac{z_{nj2}z_{nk2}}{s^2 + k/m} + \frac{z_{nj3}z_{nk3}}{s^2 + 3k/m} \quad (8.55)$$

This equation makes the graphical combining of modal contributions to the final transfer function more clear. The contribution of each mode is a simple harmonic oscillator at frequency ω_i with dc gain $z_{nji}z_{nki} / \omega_i^2$, where “i” is the mode number.

8.8 Plotting Individual Mode Contributions

Taking z_1 / F_1 for example, the separate contributions of each mode to the total response can be plotted as follows. First we calculate the DC response of the non-rigid body mode:

Rigid body response: at $\omega = 0.1$ rad/sec $z_{111} = \frac{1}{0.03} = 33.33 = 30.457$ db,
slope = -2

Now we calculate the dc gain of the non-rigid body modes:

Second mode response: at DC, $z_{112} = \frac{1}{2} = 0.5 = -6$ db, slope = 0
resonance at $\omega_2^2 = 1$, slope at $\infty = -2$

Third mode response: at DC, $z_{113} = \frac{1}{18} = 0.0555 = -25.1$ db, slope = 0
resonance at $\omega_3^2 = 3$, slope at $\infty = -2$, where the
“ijk” notation in z_{ijk} indicates: dof “i,” due to force
“j,” for mode k.

Thus, the total response is defined by the additive combination of three single degree of freedom responses, each with its own spring-dominated low

frequency section, damping dominated resonant section and mass dominated high frequency section.

The MATLAB code **tdof_modal_xfer.m** is used to calculate and plot the individual mode contributions to the overall frequency response of all four unique transfer functions for the tdof model. The program plots the frequency responses using several different magnitude scalings. We will discuss below the results for the z_1 / F_1 frequency response, using plots from the MATLAB code to illustrate. The notation “z113” below signifies the transfer function z_1 / F_1 for mode 3, and so forth.

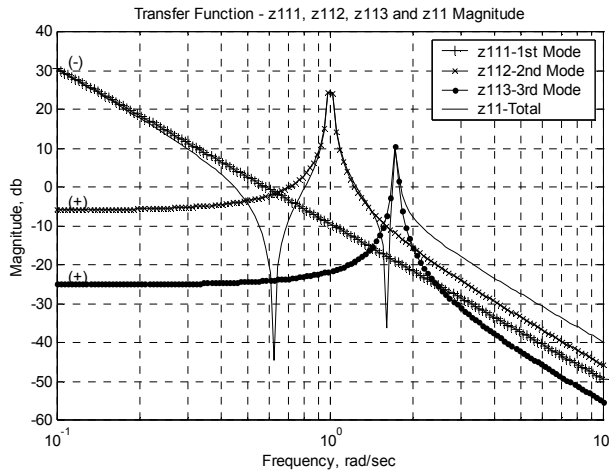


Figure 8.2: z11 transfer function frequency response plot with individual mode contributions overlaid.

Figure 8.2 shows the overall z_1 / F_1 (z11) transfer function and the individual modal contributions which add to create it. Because the magnitude scale in Figure 8.2 is in log or “db” units, the individual mode contributions cannot be added graphically. To add graphically requires a linear magnitude axis. We cannot use the log magnitude or db scale for adding directly because adding with log or db coordinates is equivalent to the multiplication of responses, not addition.

There is zero damping in this model, so the amplitudes at the two poles in Figure 8.2 should go to infinity. The peak amplitudes do not go to infinity because they are limited by the resolution of the frequency scale chosen for the plot. The two zeros should go to zero, but once again they do not because of the frequency resolution chosen.

Figures 8.3 and 8.4 show the same frequency responses plotted on a linear magnitude scale.

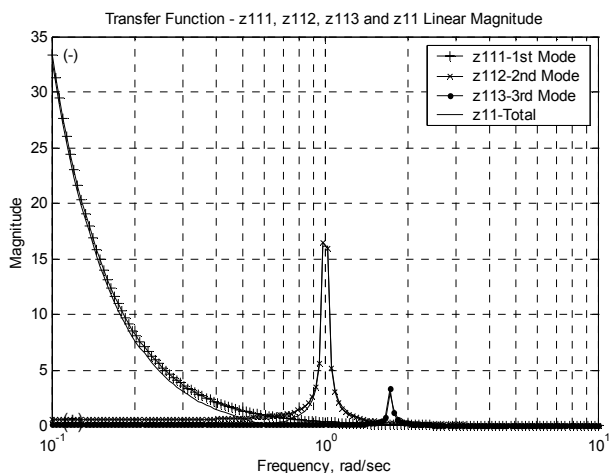


Figure 8.3: z11 frequency response and modal contributions plotted with linear magnitude scale.

Figure 8.4 uses an expanded magnitude axis to more clearly show how the three individual mode sdof responses combine graphically to create the overall frequency response. It also contains notation that shows how the signs change through the resonance. In Chapter 3 we learned how to sketch frequency response plots by hand, knowing the high and low frequency asymptotes and the locations of the poles and zeros. Similarly, we can combine modes by hand if we know the signs (phases) of the individual modes that are being combined. In our tdof example, it just so happens that the signs of the low frequency portions of the second and third modes (1.0 and 1.7 rps) were both positive. In general, it is not the case that all low frequency signs will be positive (see the z31 example below). The discussion below will show how to define the sign (phase) of the low frequency portion of each mode by knowing the signs of the eigenvector entries for the input and output degrees of freedom.

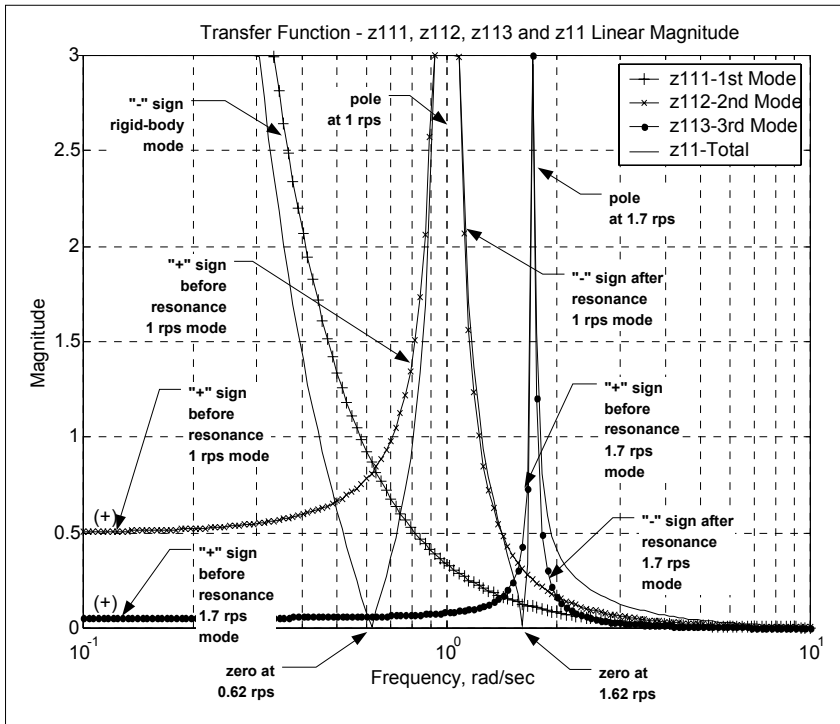


Figure 8.4: z11 frequency response with expanded magnitude scale to see contributors to the zeros.

Since the phase at frequencies much lower than the resonant frequency is zero for a spring mass sdof system (2.19b), and since each mode in principal coordinates is a sdof system, **the phase for each mode contribution to the overall response at low frequency is given by the sign of the eigenvector for the dof whose displacement is desired times the sign of the dof where the force is applied.** For the three modes and the transfer function z11, where we are interested in measuring the displacement of mass 1 and in the force being applied to mass 1, the signs for the three modes at low frequencies are found as follows. The normalized modal matrix is repeated to see the signs of the entries:

	Mode 1	Mode 2	Mode 3
dof 1	-0.5774	-0.7071	0.4082
dof2	-0.5774	0.0000	-0.8165
dof3	-0.5774	0.7071	0.4082

Table 8.1: Normalized modal matrix.

Sign of mode 1 low frequency asymptote for z11 frequency response:

dof 1, mode 1: -0.5774 (-)

dof 1, mode 1: -0.5774 4 (-)

Low frequency sign (phase) = (-) times (-) = (+), but since resonance is rigid body at zero rad/sec, all frequencies of interest to us are “after resonance” and thus the sign is (-) because the phase is -180° .

Sign of mode 2 low frequency asymptote for z11 frequency response:

dof 1, mode 2: -0.7071 (-)

dof 1, mode 2: -0.7071 (-)

Low frequency sign (phase) = (-) times (-) = (+), 0°

Sign of mode 3 low frequency asymptote for z11 frequency response:

dof 1, mode 3: +0.4082 (-)

dof 1, mode 3: +0.4082 (-)

Low frequency sign (phase) = (+) times (+) = (+), 0°

As mentioned earlier, the signs of the low frequency portions of the second and third modes were both (+). The signs of the eigenvector entries above show why this was the case.

The “sign” of the rigid body mode is always “–” because the phase is always -180° . The signs of the 1 rad/sec (rps) and 1.732 rps modes are both “+” at low frequencies because their phases are 0° . After the resonance, their signs change to “–” as phase goes to -180° . Exactly at resonance the phase of each is -90° , however, away from resonance the phases are either 0° or -180° because the problem has no damping.

Thus, if the low frequency asymptote sign (phase) is known for each mode, the **SISO frequency response zeros can be identified as frequencies where the appropriate modes add to zero algebraically**, as can be seen graphically on [Figure 8.4](#).

The z11 zeros at 0.62 and 1.62 rps arise when the contributions of the three modes combine algebraically to zero.

For other transfer functions, for example z31, the low frequency signs would be different, as can be seen below:

Sign of mode 1 low frequency asymptote for z31 frequency response:

dof 3, mode 1: -0.5774 (–)

dof 1, mode 1: -0.5774 (–)

Low frequency sign (phase) = (–) times (–) = (+), but is after resonance so sign is (–)

Sign of mode 2 low frequency asymptote for z31 frequency response:

dof 3, mode 2: $+0.7071$ (–)

dof 1, mode 2: -0.7071 (–)

Low frequency sign (phase) = (+) times (–) = (–), -180°

Sign of mode 3 low frequency asymptote for z31 frequency response:

dof 3, mode 3: $+0.4082$ (–)

dof 1, mode 3: $+0.4082$ (–)

Low frequency sign (phase) = (+) times (+) = (+), 0°

Now that the low frequency phases of the individual modes are defined, we can follow the combining of modes to get the overall response, indicated by the “+” signs.

Because we are dealing with a linear magnitude axis above, we can graphically add or subtract the contribution of each to get the overall response.

To get the overall response we combine the amplitudes of each mode depending on its sign. For example, at 0.4 rad/sec frequency, we combine the amplitude of the rigid body mode with a negative sign with the two oscillatory modes, each of which has a positive sign:

Rigid body response: at $\omega = 0.4$ rad/sec, $\omega_1 = 0$:

$$z_{111} = \frac{1}{3s^2} = \frac{1}{3(j\omega)^2} = \frac{-1}{3\omega^2} = \frac{-1}{3(0.4)^2} = -2.083 \quad (8.56)$$

Second mode response: at $\omega = 0.4$ rad/sec, $\omega_2 = 1$:

$$z_{112} = \frac{1}{2(s^2 + \omega_2^2)} = \frac{1}{2[(j\omega)^2 + \omega_2^2]} = \frac{1}{2[-\omega^2 + \omega_2^2]} = \frac{1}{2[-(0.4)^2 + 1]} = 0.595 \quad (8.57)$$

Third mode response: at $\omega = 0.4$ rad/sec, $\omega_3 = 1.732$:

$$z_{113} = \frac{1}{6(s^2 + \omega_3^2)} = \frac{1}{6[(j\omega)^2 + \omega_3^2]} = \frac{1}{6[-\omega^2 + \omega_3^2]} = \frac{1}{6[-(0.4)^2 + 3]} = 0.0586 \quad (8.58)$$

Adding (with proper signs) the three contributions at 0.4 rad/sec gives the amplitude and phase of the overall response at 0.4 rad/sec:

$$-2.083 + 0.595 + 0.0586 = -1.4294 \quad (8.59)$$

The amplitude is 1.4294 and the phase is -180° , as indicated by the negative sign. Because the model has no damping, each mode has 0° phase before

resonance and immediately after resonance switches phase to -180° . Exactly at resonance the amplitudes are theoretically infinite.

Let us now track what happens at the frequency of the first zero, which we showed in (2.85) to be 0.618 rad/sec. We will carry out the same calculations as above for a frequency of 0.618 rad/sec:

Rigid body response: at $\omega = 0.618$ rad/sec, $\omega_1 = 0$:

$$z_{111} = \frac{1}{3s^2} = \frac{1}{3(j\omega)^2} = \frac{-1}{3\omega^2} = \frac{-1}{3(0.618)^2} = -0.8727 \quad (8.60)$$

Second mode response: at $\omega = 0.618$ rad/sec, $\omega_2 = 1$:

$$z_{112} = \frac{1}{2(s^2 + \omega_2^2)} = \frac{1}{2[(j\omega)^2 + \omega_2^2]} = \frac{1}{2[-(0.618)^2 + 1]} = 0.8089 \quad (8.61)$$

Third mode response: at $\omega = 0.618$ rad/sec, $\omega_3 = 1.732$:

$$z_{113} = \frac{1}{6(s^2 + \omega_3^2)} = \frac{1}{6[(j\omega)^2 + \omega_3^2]} = \frac{1}{6[-(0.618)^2 + 3]} = 0.0636 \quad (8.62)$$

Adding (with proper signs) the three contributions gives the amplitude and phase of the overall response at 0.618 rad/sec:

$$-0.8727 + 0.8089 + 0.0636 = -0.0002 \cong 0 \quad (8.63)$$

The amplitude is -0.0002 . With greater accuracy in the values used for the eigenvalues and the frequency of the zero, the solution would have been exactly zero.

In Chapter 2, we showed that the zeros for SISO transfer functions arose from the roots of the numerator. The modal analysis method shows another explanation of how zeros of transfer functions arise: **when modes combine with appropriate signs (phases) it is possible at some frequencies to have no motion.**

We will calculate the response at one more frequency to show how the phase changes for a mode when the frequency is higher than the resonant frequency.

We will choose a frequency of 1.3 rad/sec, which is higher than the second mode but lower than the third mode. We should see that the sign of the contribution for the second mode changes sign from positive to negative. Signs for the first and third mode should remain unchanged.

Rigid body response: at $\omega = 1.3$ rad/sec, $\omega_1 = 0$:

$$z_{111} = \frac{1}{3s^2} = \frac{1}{3(j\omega)^2} = \frac{-1}{3\omega^2} = \frac{-1}{3(1.3)^2} = -0.1972 \quad (8.64)$$

Second mode response: at $\omega = 1.3$ rad/sec, $\omega_2 = 1$:

$$z_{112} = \frac{1}{2(s^2 + \omega_2^2)} = \frac{1}{2[(j\omega)^2 + \omega_2^2]} = \frac{1}{2[-(1.3)^2 + 1]} = -0.7246 \quad (8.65)$$

Third mode response: at $\omega = 1.3$ rad/sec, $\omega_3 = 1.732$:

$$z_{113} = \frac{1}{6(s^2 + \omega_3^2)} = \frac{1}{6[(j\omega)^2 + \omega_3^2]} = \frac{1}{6[-(1.3)^2 + 3]} = 0.1272 \quad (8.66)$$

Adding (with proper signs) the three contributions at 1.3 rad/sec gives the amplitude and phase of the overall response at 1.3 rad/sec:

$$-0.1972 - 0.7246 + 0.1272 = -0.7946 \quad (8.67)$$

The amplitude is 0.7946 and the phase is -180° . Note that the sign of the second mode contribution changed from positive to negative when the resonant frequency was passed.

The same calculations can be repeated for any desired frequency. Also, knowing the high and low frequency asymptotes, their signs and resonant frequencies, one can plot the overall frequency response roughly by hand, similar to what was done in Section 3.3. Here, unlike the previous hand plotting, we have not calculated any zeros; they occur by additive combinations of individual modes.

8.9 MATLAB Code tdof_modal_xfer.m – Plotting Frequency Responses, Modal Contributions

8.9.1 Code Overview

Figures 8.2 to 8.4 were plotted using this code. The code uses (8.24 to 8.27) to evaluate the four transfer functions z_{11} , z_{21} , z_{31} and z_{22} . Each of the transfer functions also has its modal contributions calculated and plotted as overlays. The frequency response plots are all plotted with log and db magnitude scales as well as a linear scale which is expanded in the fourth plot of the series. Because of the amount of code used for the plotting, only the code for the z_{11} transfer function will be listed. All the other transfer functions are calculated in a similar fashion.

8.9.2 Code Listing, Partial

```
%      tdof_modal_xfer.m   plotting modal transfer functions of three dof model

clf;

legend off;

subplot(1,1,1);

clear all;

%      Define a vector of frequencies to use, radians/sec. The logspace command uses
%      the log10 value as limits, i.e. -1 is  $10^{-1} = 0.1$  rad/sec, and 1 is
%       $10^1 = 10$  rad/sec. The 200 defines 200 frequency points.

w = logspace(-1,1,150);

%      calculate the rigid-body motions for low and high frequency portions
%      of all the transfer functions

%      z11, output 1 due to force 1 transfer functions

z111num = 1/3;

z111den = [1 0 0];

z112num = 1/2;

z112den = [1 0 1];

z113num = 1/6;

z113den = [1 0 3];

[z111mag,z111phs] = bode(z111num,z111den,w);
```

```

[z112mag,z112phs] = bode(z112num,z112den,w);

[z113mag,z113phs] = bode(z113num,z113den,w);

if abs(z111phs(1)) >= 10

    z111text = '(-)';

else

    z111text = '(+)';

end

if abs(z112phs(1)) >= 10

    z112text = '(-)';

else

    z112text = '(+)';

end

if abs(z113phs(1)) >= 10

    z113text = '(-)';

else

    z113text = '(+)';

end

z111magdb = 20*log10(z111mag);

z112magdb = 20*log10(z112mag);

z113magdb = 20*log10(z113mag);

% calculate the complete transfer function

z11 = ((1/3)./(j*w).^2) + ((1/2)./(j*w).^2 + 1)) + ((1/6)./(j*w).^2 + 3));

z11mag = abs(z11);

z11magdb = 20*log10(z11mag);

z11phs = 180*angle(z11)/pi ;

% truncate peaks for microsoft word plotting of expanded linear scale

z11plotmag = z11mag;

z111plotmag = z111mag;

```

```

z112plotmag = z112mag;

z113plotmag = z113mag;

for cnt = 1:length(z11mag)

    if z11plotmag(cnt) >= 3.0

        z11plotmag(cnt) = 3.0;

    end

    if z111plotmag(cnt) >= 3.0

        z111plotmag(cnt) = 3.0;

    end

    if z112plotmag(cnt) >= 3.0

        z112plotmag(cnt) = 3.0;

    end

    if z113plotmag(cnt) >= 3.0

        z113plotmag(cnt) = 3.0;

    end

end

% plot the three modal contribution transfer functions and the total using
% log magnitude versus frequency

loglog(w,z111mag,'k+-',w,z112mag,'kx-',w,z113mag,'k.-',w,z11mag,'k-')
title('Transfer Functions - z111, z112, z113 and z11 magnitude')
legend('z111-1st Mode','z112-2nd Mode','z113-3rd Mode','z11-Total')
text(.11,1.2*z111mag(1),z111text)
text(.11,1.2*z112mag(1),z112text)
text(.11,1.2*z113mag(1),z113text)
xlabel('Frequency, rad/sec')
ylabel('Magnitude')
grid

disp('execution paused to display figure, "enter" to continue'); pause

% plot the four transfer functions using db

semilogx(w,z111magdb,'k+-',w,z112magdb,'kx-',w,z113magdb,'k.-',w,z11magdb,'k-')
title('Transfer Function - z111, z112, z113 and z11 Magnitude')
legend('z111-1st Mode','z112-2nd Mode','z113-3rd Mode','z11-Total')

```

```

text(.11,2+z111magdb(1),z111text)
text(.11,2+z112magdb(1),z112text)
text(.11,2+z113magdb(1),z113text)
xlabel('Frequency, rad/sec')
ylabel('Magnitude, db')
grid

disp('execution paused to display figure, "enter" to continue'); pause

%
%
plot the four transfer functions using a linear magnitude scale so that
the amplitudes can be added directly

semilogx(w,z111mag,'k+-',w,z112mag,'kx-',w,z113mag,'k-',w,z11mag,'k-')
title('Transfer Function - z111, z112, z113 and z11 Linear Magnitude')
legend('z111-1st Mode','z112-2nd Mode','z113-3rd Mode','z11-Total')
text(.11,1.0*z111mag(1),z111text)
text(.11,1.1*z112mag(1),z112text)
text(.11,1.1*z113mag(1),z113text)
xlabel('Frequency, rad/sec')
ylabel('Magnitude')
grid

disp('execution paused to display figure, "enter" to continue'); pause

semilogx(w,z111plotmag,'k+-',w,z112plotmag,'kx-',w,z113plotmag,'k-',w,z11plotmag,'k-')
title('Transfer Function - z111, z112, z113 and z11 Linear Magnitude')
legend('z111-1st Mode','z112-2nd Mode','z113-3rd Mode','z11-Total')
text(.11,1.0*z111mag(1),z111text)
text(.11,1.1*z112mag(1),z112text)
text(.11,1.1*z113mag(1),z113text)
xlabel('Frequency, rad/sec')
ylabel('Magnitude')
axis([.1 10 0 3]);
grid

disp('execution paused to display figure, "enter" to continue'); pause

%
plot phase

semilogx(w,z111phs,'k+-',w,z112phs,'kx-',w,z113phs,'k-',w,z11phs,'k-')
title('Transfer Function - z111, z112, z113 and z11 Phase')
legend('z111-1st Mode','z112-2nd Mode','z113-3rd Mode','z11-Total')
xlabel('Frequency, rad/sec')
ylabel('Phase, Deg')
grid

disp('execution paused to display figure, "enter" to continue'); pause

```

8.10 tdof Eigenvalue Problem Using ANSYS

An ANSYS solution to the tdof problem is now shown in order to start becoming familiar with how ANSYS presents its eigenvalue/eigenvector results.

8.10.1 ANSYS Code threedof.inp Description

The ANSYS code **threedof.inp** below is used to build the model, calculate eigenvalues and eigenvectors, output the frequency listing and eigenvectors, plot the mode shapes, calculate and plot all three transfer functions for a forcing function applied to mass 1: z_1/F_1 , z_2/F_1 , and z_3/F_1 . The hand calculated values for masses and stiffnesses are used, $m_1 = m_2 = m_3 = 1.0$, $k_1 = k_2 = 1.0$.

To run the code, from the “begin” level in ANSYS, type “/input,threedof.inp,” and the program will run unattended. The various outputs are available as follows:

threedof.frq	frequency list, ascii file
threedof.eig	eigenvector list, ascii file
threedof.grp2	mode shape plots
threedof.grp1	frequency response plots

Use the ANSYS Display program to load and display the two plot files.

8.10.2 ANSYS Code Listing

```
/title, threedof.inp, three dof vibration class model, Ansys Version 5.5

/prep7                                ! enter model preparation section

! element type definitions

et,1,21                               ! element type for mass
et,2,14                               ! element type for spring

! real value definitions

r,1,1,1,1                             ! mass of 1kg
r,2,1                                 ! spring stiffness of 1mn/mm, or 1n/m

! define plotting characteristics

/view,1,-1,0,0                       ! z-y plane
/angle,1,0                           ! not iso
```

```

/pnum,real,1      ! color by real
/num,1            ! numbers off
/type,1,0         ! hidden plot
/pbc,all,1        ! show all boundary conditions

csys,0            ! define global coordinate system

! nodes

n,1,0,0,-1        ! left hand mass at x = -1.0 mm
n,2,0,0,0         ! middle mass at x = 0 mm
n,3,0,0,1         ! right hand mass at x = +1.0 mm

! define masses

type,1
real,1
e,1
e,2
e,3

! define springs

type,2
real,2
e,1,2
e,2,3

! define constraints, ux and uy zero, leaving only uz motion

nsel,s,node,,1,3
d,all,ux,0
d,all,uy,0

allsel
eplo

! ***** eigenvalue run *****

fini              ! fini just in case not in begin

/solu             ! enters the solution processor, needs to be here to do editing below

allsel           ! default selects all items of specified entity type, typically nodes, elements

! define masters for frequency response (transfer function) run

m,1,uz
m,2,uz
m,3,uz

antype,modal,new
modopt,reduc,3    ! method – Block Lanczost
expass,off        ! key = off, no expansion pass, key = on, do expansion
mxpand,3,,no      ! number of modes to expand

```

```

total,3,1                ! total masters, all translational dof
allsel

solve                    ! starts the solution of one load step of a solution sequence, modal
fini

! ***** output frequencies *****

/post1

/output,threedof,frq      ! write out frequency list to ascii file .frq

set,list

/output,term              ! returns output to terminal

! ***** output eigenvectors *****

! define nodes for output
allsel

/output,threedof,eig      ! write out eigenvectors to ascii file .eig

*do,i,1,3
    set,,i
    prdisp
*enddo

/output,term
***** plot modes *****

/show,threedof,grp2,0     ! raster plot, 1 is vector plot, write out to graph file .grp2
allsel

*do,i,1,3
    set,1,i
    pldi,1
*enddo

/show,term

! ***** calculate and plot transfer functions *****

fini

/assign,rst,junk,rst      ! reassigns a file name to an ANSYS identifier

/solu

dmpmat,0                  ! sets a constant damping ratio for all modes, zeta = 0

```

```

allsel
eplo                      ! show forces applied

f,1,fz,1                  ! 1 mn force applied to node 1, left-hand mass

/title, threedof.inp, tdof, force at mass 1

antype,harmic             ! harmonic (frequency response) analysis

hropt,msup,3              ! mode superposition method, nummodes modes used

harfrq,0.0159,1.59        ! frequency range, hz, for solution, -1 to 10 rad/sec

hrout,off,off             ! amplitude/phase, cluster off

kbc,1

nsubst,200                ! 200 frequency points

outres,nsol,all,          ! controls solution set written to database, nodal dof solution, all
                          ! frequencies, component name for selected set of nodes

solve

fini

/post26

file,,rfrq                ! frequency response results

xvar,0                     ! display versus frequency

lines,10000                ! specifies the length of a printed page for frequency response listing

nsol,2,1,u,z,z1           ! specifies nodal data to be stored in results file
                          ! u - displacement, z direction
                          ! note that nsol,1 is frequency vector

nsol,3,2,u,z,z2

nsol,4,3,u,z,z3

! plot magnitude

plcplx,0
/grid,1
/axlab,x,frequency, hz
/axlab,y,amplitude, mm
/gropt,logx,1              ! log plot for frequency
/gropt,logy,1              ! log plot for amplitude

/show,threedof.grp1        ! file name for storing

plvar,2,3,4
/show,term

```

```

! plot phase

plcplx,1
/grid,1
/axlab,x,freq
/axlab,y,phase, deg           ! label for y axis
/gropt,logx,1                ! log plot for frequency
/gropt,logy,0                ! linear plot for phase

/show,threedof,grp1
plvar,2,3,4
/show,term

! save ascii data to file

prcplx,1                      ! stores phase angle in ascii file .dat

/output,threedof,dar
prvar,2,3,4
/output,term

fini

```

8.10.3 ANSYS Results

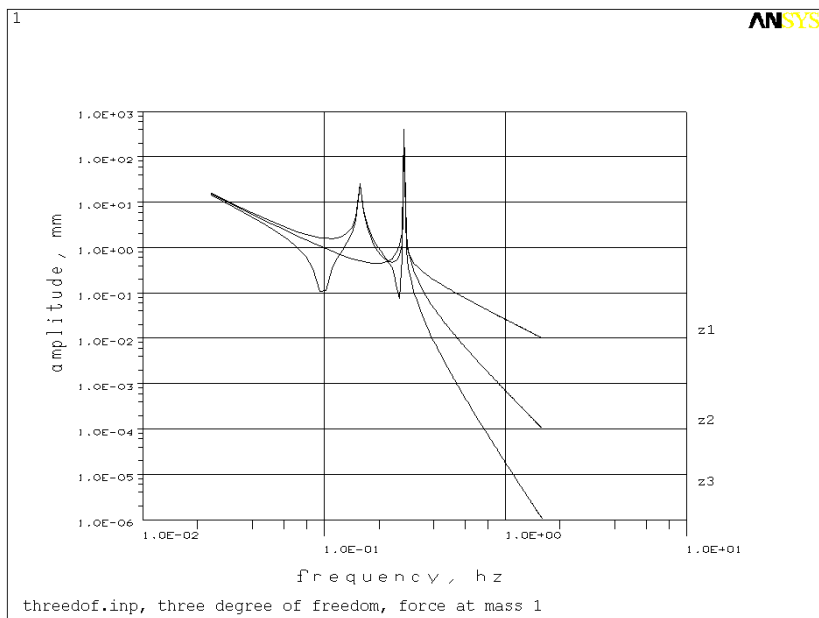


Figure 8.5: ANSYS frequency responses for force at mass 1.

The resulting ANSYS transfer function plot is shown in [Figure 8.5](#), with the frequency axis in Hz, not rad/sec.

The ANSYS frequency listing from **threedof.frq** is shown below, in hz units:

***** INDEX OF DATA SETS ON RESULTS FILE *****				
SET	TIME/FREQ	LOAD STEP	SUBSTEP	CUMULATIVE
1	0.47280E-06	1	1	1
2	0.15915	1	2	2
3	0.27566	1	3	3

Note that the rigid body mode is calculated to be 0.4726e-6, close to 0 hz. The second and third modes are calculated to be 0.15915 and 0.27566 hz, or 0.999969 and 1.732 rad/sec, respectively. This is the same as our hand-calculated results.

The ANSYS eigenvector listing from threedof.eig is below:

*DO LOOP ON PARAMETER=I FROM 1.0000 TO 3.0000 BY 1.0000						
USE LOAD STEP 1 SUBSTEP 1 FOR LOAD CASE 0						
SET COMMAND GOT LOAD STEP= 1 SUBSTEP= 1 CUMULATIVE ITERATION= 1						
TIME/FREQUENCY= 0.47280E-06						
TITLE= threedof.inp, three dof vibration class model, Ansys Version 5.5						
PRINT DOF NODAL SOLUTION PER NODE						
***** POST1 NODAL DEGREE OF FREEDOM LISTING *****						
LOAD STEP= 1 SUBSTEP= 1						
FREQ= 0.47280E-06 LOAD CASE= 0						
THE FOLLOWING DEGREE OF FREEDOM RESULTS ARE IN GLOBAL COORDINATES						
NODE	UX	UY	UZ	ROTX	ROTY	ROTZ

1	0.0000	0.0000	0.57735			
2	0.0000	0.0000	0.57735			
3	0.0000	0.0000	0.57735			
MAXIMUM ABSOLUTE VALUES						
NODE	0	0	1	0	0	0
VALUE	0.0000	0.0000	0.57735	0.0000	0.0000	0.0000
*ENDDO INDEX= I						
***** POST1 NODAL DEGREE OF FREEDOM LISTING *****						
LOAD STEP= 1 SUBSTEP= 2						
FREQ= 0.15915 LOAD CASE= 0						
THE FOLLOWING DEGREE OF FREEDOM RESULTS ARE IN GLOBAL COORDINATES						
NODE	UX	UY	UZ	ROTX	ROTY	ROTZ
1	0.0000	0.0000	-0.70711			
2	0.0000	0.0000	0.75552E-14			
3	0.0000	0.0000	0.70711			
MAXIMUM ABSOLUTE VALUES						
NODE	0	0	3	0	0	0
VALUE	0.0000	0.0000	0.70711	0.0000	0.0000	0.0000
***** POST1 NODAL DEGREE OF FREEDOM LISTING *****						
LOAD STEP= 1 SUBSTEP= 3						
FREQ= 0.27566 LOAD CASE= 0						
THE FOLLOWING DEGREE OF FREEDOM RESULTS ARE IN GLOBAL COORDINATES						
NODE	UX	UY	UZ	ROTX	ROTY	ROTZ
1	0.0000	0.0000	-0.40825			
2	0.0000	0.0000	0.81650			
3	0.0000	0.0000	-0.40825			
MAXIMUM ABSOLUTE VALUES						
NODE	0	0	2	0	0	0
VALUE	0.0000	0.0000	0.81650	0.0000	0.0000	0.0000

The ANSYS calculated eigenvectors, the three “UZ” listings highlighted in bold type above, arranged in the modal matrix:

$$\text{ANSYS } z_n = \begin{bmatrix} 0.57735 & -0.707 & -0.40825 \\ 0.57735 & 0 & 0.81649 \\ 0.57735 & 0.707 & -0.40825 \end{bmatrix} \tag{8.68}$$

The hand-calculated modal matrix is below, only differing from the ANSYS calculated values in the arbitrary “-1” multiplier for the second and third modes:

$$z_n = \frac{1}{\sqrt{m}} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} 0.57735 & 0.707 & 0.40825 \\ 0.57735 & 0 & -0.81649 \\ 0.57735 & -0.707 & 0.40825 \end{bmatrix} \quad (8.69)$$

Problems

Note: All the problems refer to the two dof system shown in [Figure P2.2](#).

P8.1 Using the eigenvalues and eigenvectors normalized with respect to mass from P7.2 and forces F_1 and F_2 applied to mass 1 and mass 2, respectively, write the equations in motion and physical and principal coordinates in matrix form. Identify the components of the forcing function vector in principal coordinates – which eigenvector components and which force, F_1 or F_2 , are involved.

P8.2 Solve for the four transfer functions for the system of P8.1 and write them in transfer function matrix form. Separate each transfer function in principal coordinates to show z_p/F_1 and z_p/F_2 as in (8.16).

P8.3 Back transform the transfer functions in principal coordinates to physical coordinates. Identify the contributions to the transfer function from mode 1 and from mode 2 for all transfer functions.

P8.4 Take the transfer function results of P2.2 with $m_1 = m_2 = m = 1$, $k_1 = k_2 = k = 1$ and zero damping and perform a partial fraction expansion on each transfer function. Show that the results are identical to P8.3, the modal form.

P8.5 What is the relationship between F_1 and F_2 in order to excite mode 1 only? To excite only mode 2?

P8.6 Plot by hand the individual mode contributions to the z_2/F_1 frequency response for zero damping. Note the sign of the dc gain portion of each contribution and add the two contributions appropriately to obtain the overall frequency response. Extra Credit: Plot all three unique frequency responses, showing the individual mode contributions.

P8.7 (MATLAB) Modify **tdof_modal_xfer.m** for the undamped two dof system with $m_1 = m_2 = m = 1$, $k_1 = k_2 = k = 1$ and plot the overlaid frequency responses.

P8.8 (ANSYS) Modify the **threedof.inp** code for the two dof system. Run the code and plot the frequency responses for both masses for $F_1 = 1, F_2 = 0$. Print out the eigenvalue and eigenvector results. Pick out the appropriate entries of the eigenvector output and write out the modal matrix that ANSYS calculates. Compare it with the modal matrix from P7.1 and identify any differences.