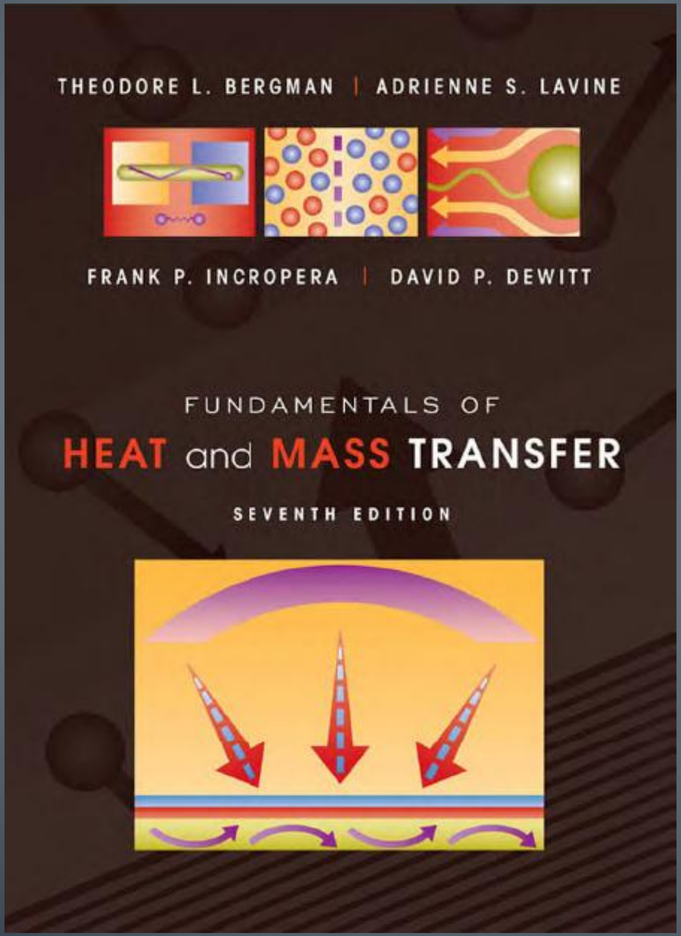
Heat Transfer I

Farbod Fakhrabadi
Islamic Azad University of Qazvin

Textbooks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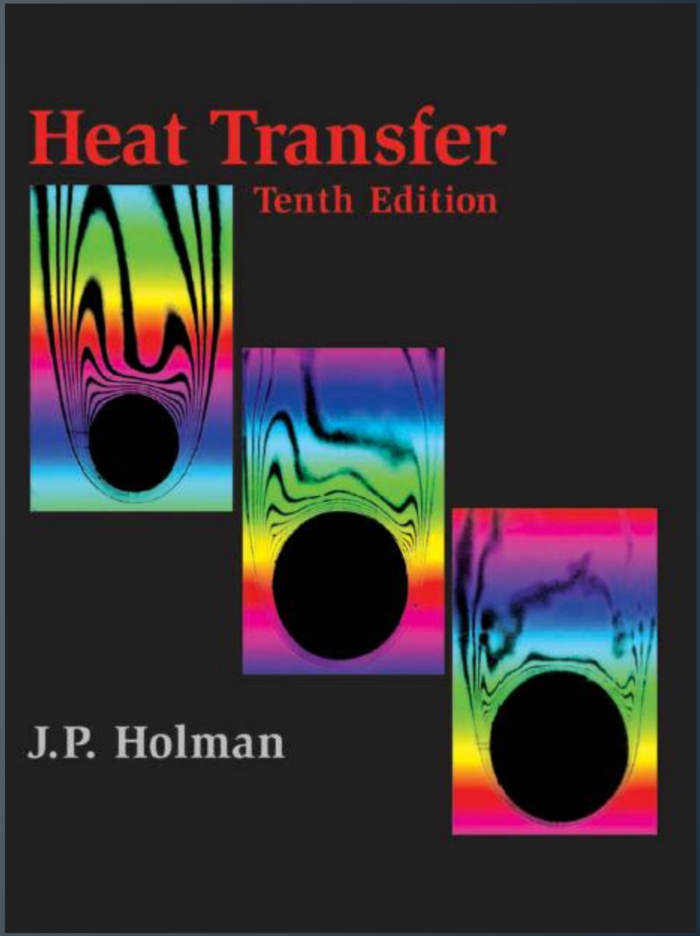
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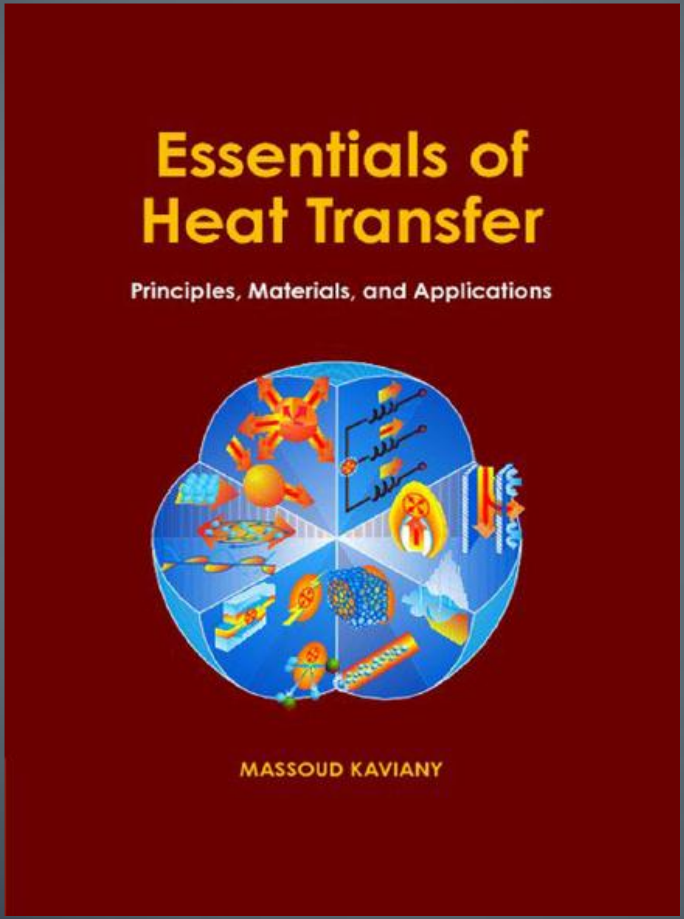
Essentials of
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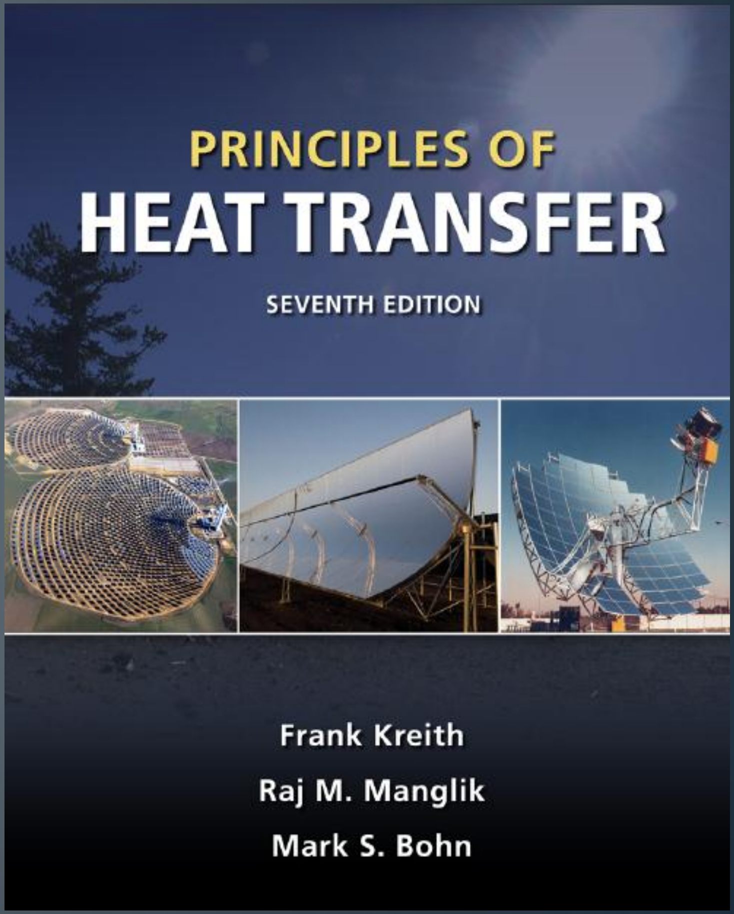
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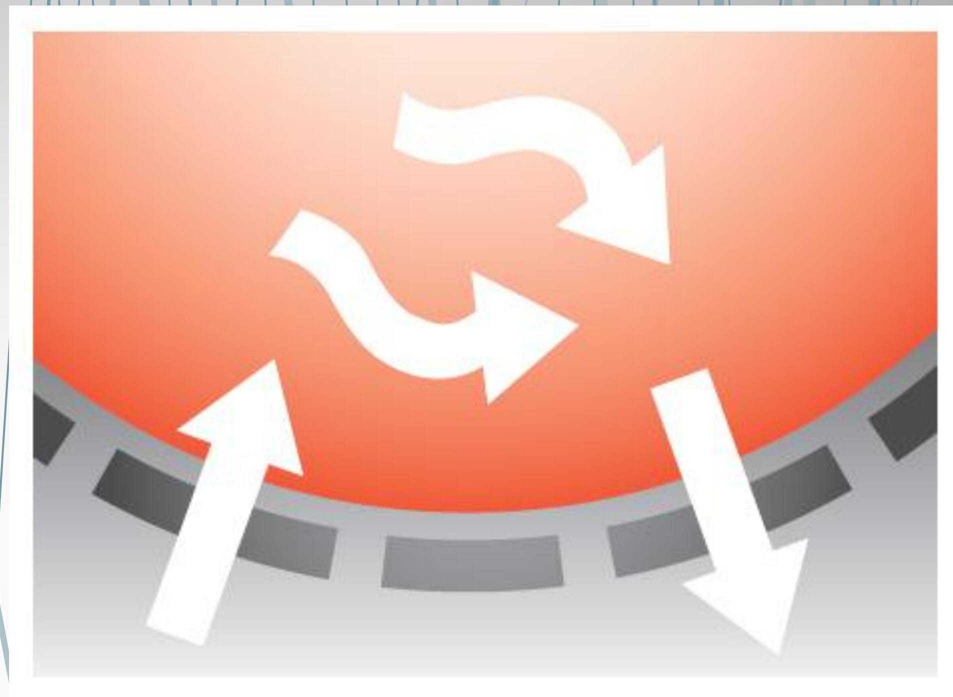


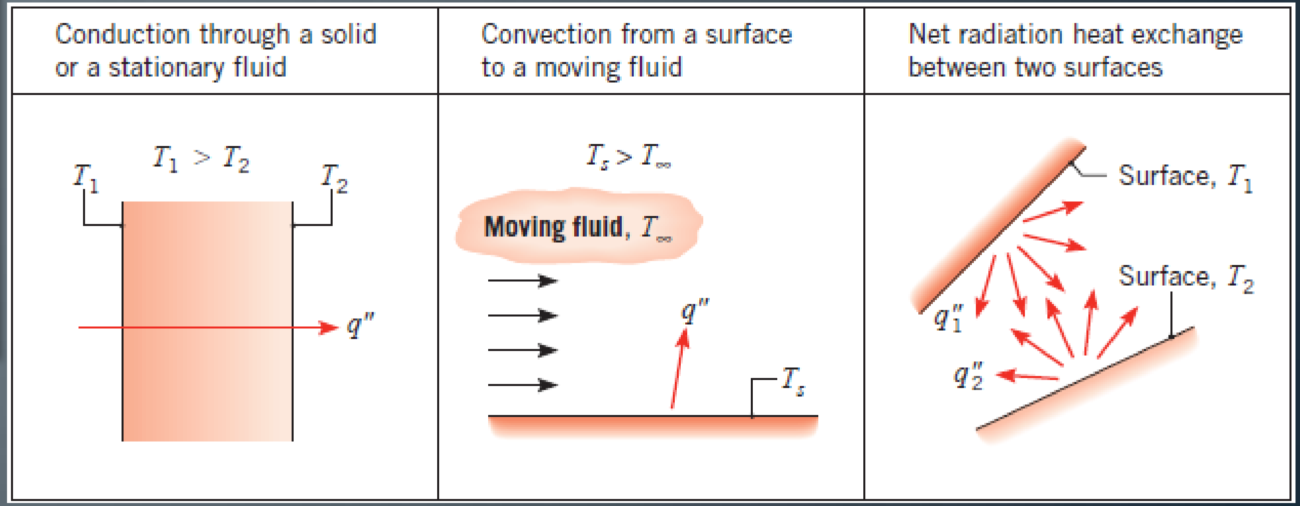
Introduction

Chapter 1

What and How?

- What is heat transfer?

- *Heat transfer (or heat) is thermal energy in transit due to a spatial temperature difference.*



What and How?

- When a temperature gradient exists in a stationary medium, which may be a solid or a fluid, we use the term *conduction* to refer to the heat transfer that will occur across the medium.

- In contrast, the term *convection* refers to heat transfer that will occur between a surface and a moving fluid when they are at different temperatures.

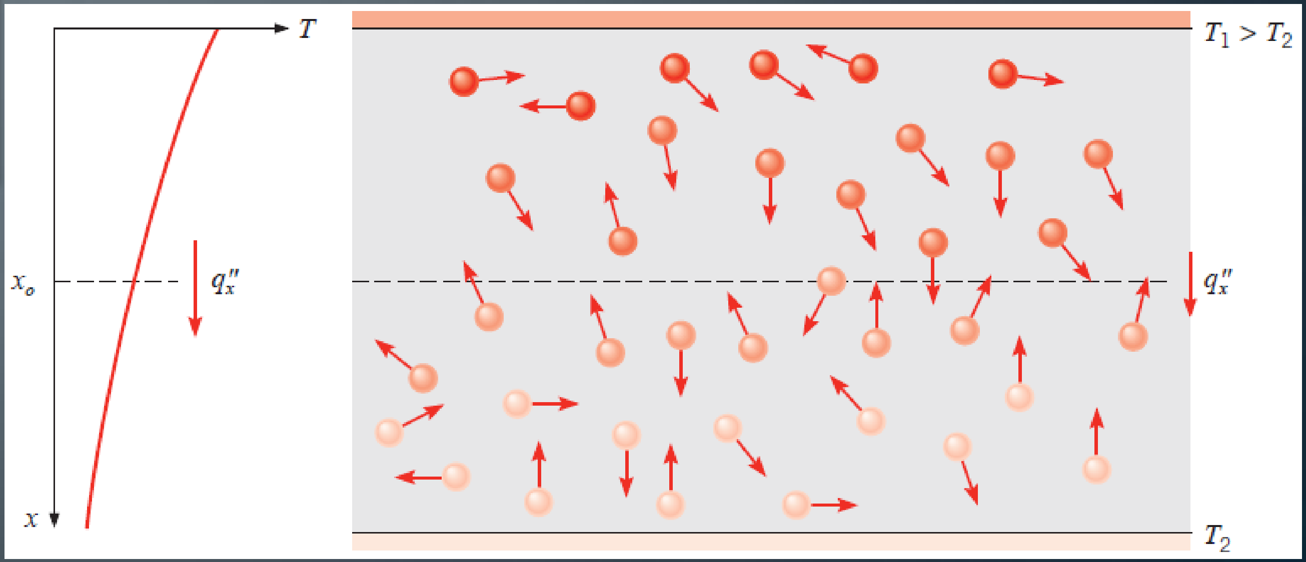
- The third mode of heat transfer is termed *thermal radiation*. All surfaces of finite temperature emit energy in the form of electromagnetic waves. Hence, in the absence of an intervening medium, there is net heat transfer by radiation between two surfaces at different temperatures.

Conduction

- Conduction may be viewed as the transfer of energy from the more energetic to the less energetic particles of a substance due to interactions between the particles.

- We associate the temperature at any point with the energy of gas molecules in proximity to the point.

- This energy is related to the random translational motion, as well as to the internal rotational and vibrational motions, of the molecules.

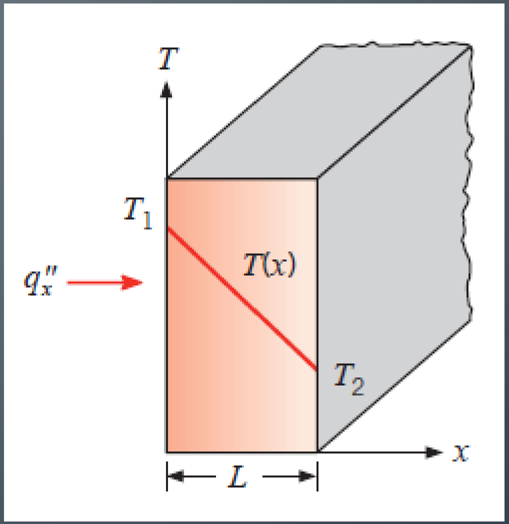


Conduction

- We may speak of the net transfer of energy by random molecular motion as a *diffusion* of energy.
- The situation is much the same in liquids, although the molecules are more closely spaced and the molecular interactions are stronger and more frequent.
- Similarly, in a solid, conduction may be attributed to atomic activity in the form of lattice vibrations. In an electrical nonconductor, the energy transfer is exclusively via these lattice waves; in a conductor, it is also due to the translational motion of the free electrons.

Conduction

- Heat transfer processes can be quantified in terms of appropriate *rate equations*.
- These equations may be used to compute the amount of energy being transferred per unit time.
- For heat conduction, the rate equation is known as *Fourier's law*.
- For the one-dimensional plane wall having a temperature distribution $T(x)$, the rate equation is expressed as



$$q''_x = -k\frac{dT}{dx}$$

Conduction

- The *heat flu* $(\mathrm{W/m^2})$ is the heat transfer rate in the *x*-direction *per* unit area *perpendicular* to the direction of transfer, and it is proportional to the *temperature gradient*, dT/dx, in this direction. The parameter *k* is a *transport* property known as the *thermal conductivity* $(\mathrm{W/m \cdot K})$ and is a characteristic of the wall material.

- The *minus sign* is a consequence of the fact that heat is transferred in the direction of decreasing temperature.

- Under the *steady-state conditions*, where the temperature distribution is *linear*, the temperature gradient may be expressed as

$$\frac{dT}{dx} = \frac{T_2 - T_1}{L}$$

Conduction

- and the heat flux is then

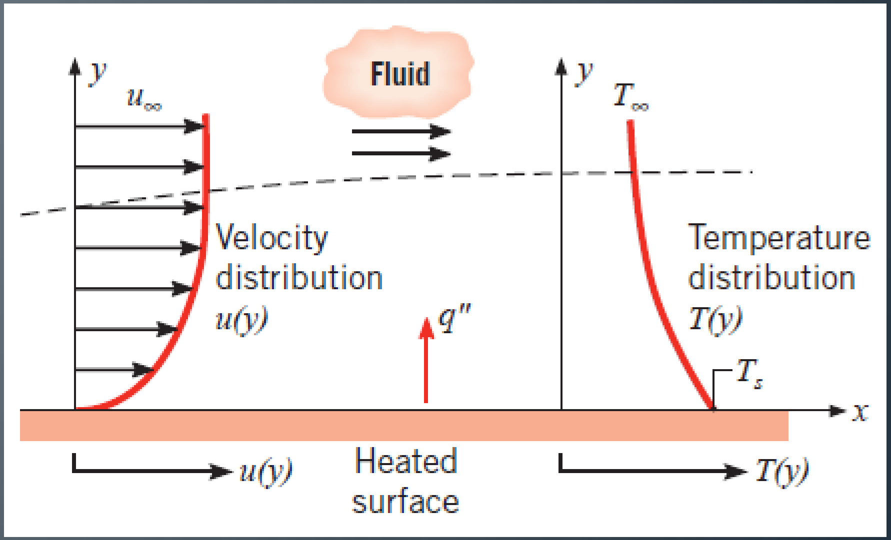
$$q''_x = k\frac{T_1 - T_2}{L} = k\frac{\Delta T}{L}$$

- The *heat rate* by conduction, q_x (W), through a plane wall of area *A* is then the product of the flux and the area,

$$q_x = q''_x \cdot A$$

Convection

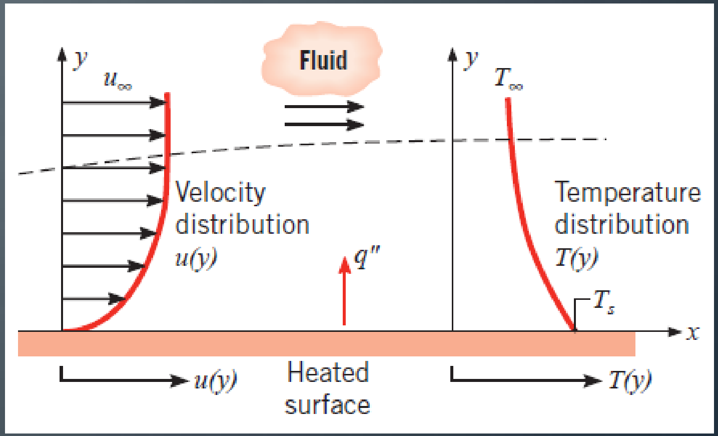
- The convection heat transfer *mode* is comprised of *two mechanisms*. In addition to energy transfer due to *random molecular motion (diffusion)*, energy is also transferred by the *bulk*, or *macroscopic*, *motion* of the fluid (advection).



Convection

- Consider fluid flow over the heated surface . A consequence of the fluid–surface interaction is the development of a region in the fluid through which the velocity varies from zero at the surface to a finite value u_∞ associated with the flow.

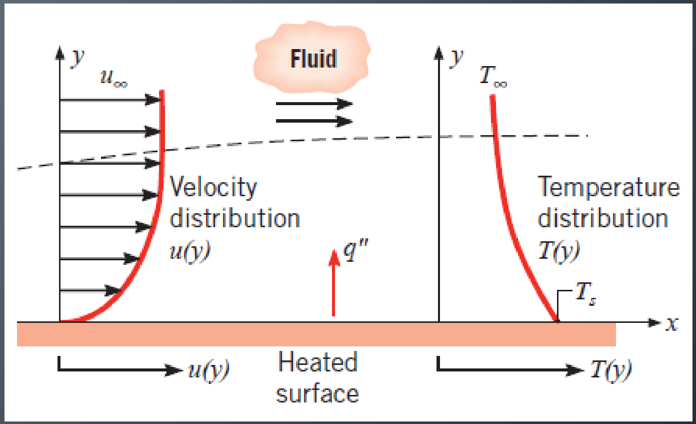
- This region of the fluid is known as the *hydrodynamic*, or *velocity boundary layer*. Moreover, if the surface and flow temperatures differ, there will be a region of the fluid through which the temperature varies from at to in the outer flow. This region, called the *thermal boundary layer*, may be smaller, larger, or the same size as that through which the velocity varies.



Convection

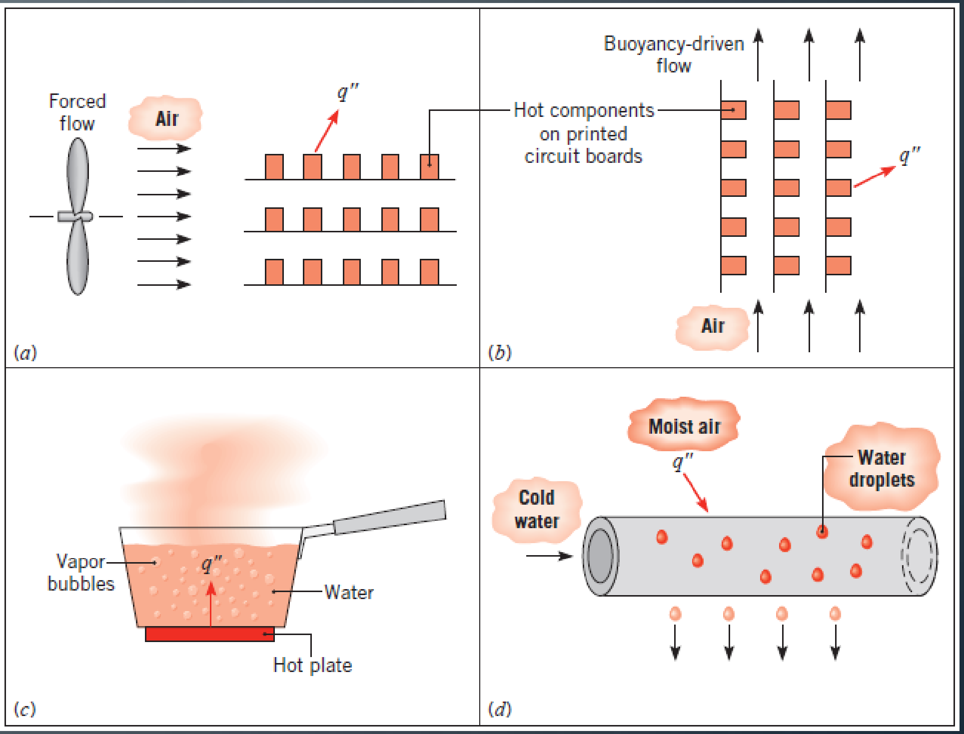
- The contribution due to random molecular motion (diffusion) dominates near the surface where the fluid velocity is low. In fact, at the interface between the surface and the fluid the fluid velocity is zero, and heat is transferred by this mechanism only.

- The contribution due to bulk fluid motion originates from the fact that the boundary layer grows as the flow progresses in the *x*-direction. In effect, the heat that is conducted into this layer is swept downstream and is eventually transferred to the fluid outside the boundary layer.



Convection

- Convection heat transfer may be classified according to the nature of the flow.
- We speak of *forced convection* when the flow is caused by external means, such as by a fan, a pump, or atmospheric winds.
- In contrast, for *free* (or *natural*) *convection*, the flow is induced by buoyancy forces, which are due to density differences caused by temperature variations in the fluid.



Convection

- Regardless of the nature of the convection heat transfer process, the appropriate rate equation is of the form

$$q'' = h(T_s - T_\infty)$$

- where , the convective *heat flux* ($\mathrm{W/m^2}$), is proportional to the difference between the surface and fluid temperatures, T_s and T , respectively. This expression is known as *Newton's law of cooling*, and the parameter h $(\mathrm{W/m^2 \cdot K})$ is termed the *convection heat transfer coefficient*. This coefficient depends on conditions in the boundary layer, which are influenced by surface geometry, the nature of the fluid motion, and an assortment of fluid thermodynamic and transport properties.

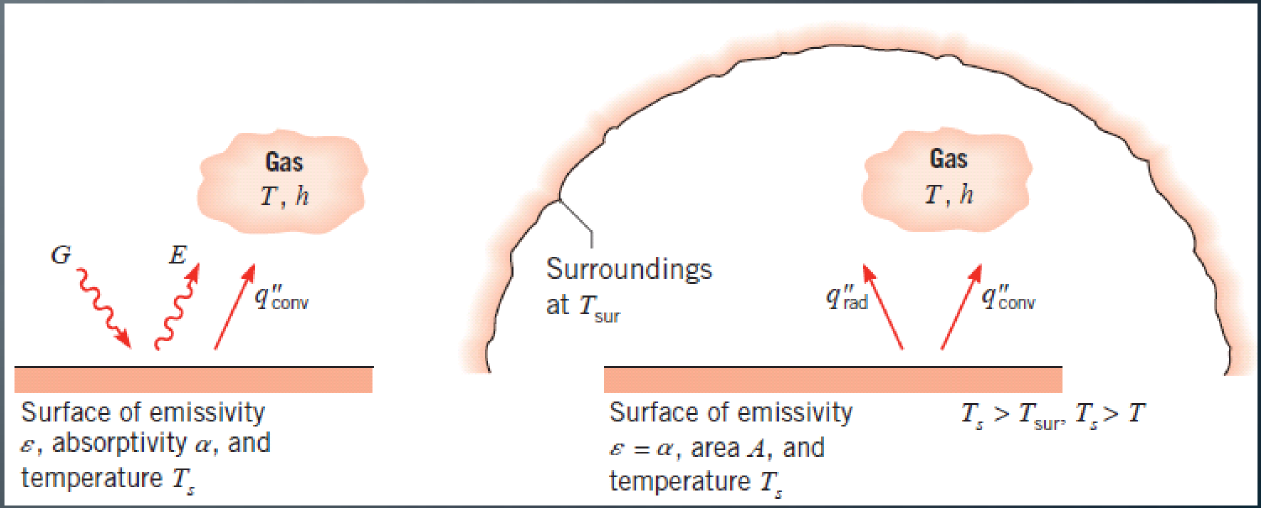
Convection

Typical values of the convection heat transfer coefficient

Process	***h*** **($W/m^2 \cdot K$)**
Free convection	
Gases	2–25
Liquids	50–1000
Forced convection	
Gases	25–250
Liquids	100–20,000
Convection with phase change	
Boiling or condensation	2500–100,000

Radiation

- Thermal radiation is energy *emitted* by matter that is at a nonzero temperature.
- The energy of the radiation field is transported by electromagnetic waves (or alternatively, photons).
- While the transfer of energy by conduction or convection requires the presence of a material medium, radiation does not. In fact, radiation transfer occurs most efficiently in a vacuum.



Radiation

- Consider radiation transfer processes. Radiation that is *emitted* by the surface originates from the thermal energy of matter bounded by the surface, and the rate at which energy is released per unit area (W/m^2) is termed the surface *emissive power*, E.

- There is an upper limit to the emissive power, which is prescribed by the *Stefan-Boltzmann law*

$$E_b = \sigma T_s^4$$

- where T_s is the *absolute temperature* (K) of the surface and σ is the Stefan-Boltzmann constant.

$$\sigma = 5.67 \times 10^{-8}\ \mathrm{W/m^2 \cdot K^4}$$

- Such a surface is called an ideal radiator or *blackbody*.

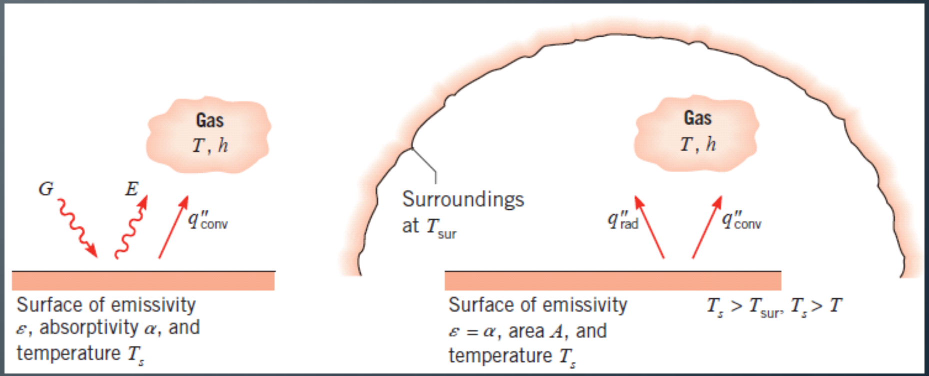
Radiation

- The heat flux emitted by a real surface is less than that of a blackbody at the same temperature and is given by

$$E = \varepsilon \sigma T_s^4$$

- where ε is a radiative property of the surface termed the *emissivity*. With values in the range $0 \leq \varepsilon \leq 1$, this property provides a measure of how efficiently a surface emits energy relative to a blackbody.

- Radiation may also be *incident* on a surface from its surroundings. We designate the rate at which all such radiation is incident on a unit area of the surface as the *irradiation G*.



Radiation

- A portion, or all, of the irradiation may be *absorbed* by the surface, thereby increasing the thermal energy of the material. The rate at which radiant energy is absorbed per unit surface area may be evaluated from knowledge of a surface radiative property termed the *absorptivity* α.

$$G_{abs} = \alpha G$$

- Where $0 \leq \alpha \leq 1$. If $\alpha \leq 1$ and the surface is *opaque*, portions of the irradiation are *reflected* If the surface is semitransparent, portions of the irradiation may also be *transmitted*.

- Note that the value of α depends on the *nature of the irradiation*, as well as on the *surface* itself.

Radiation

- A special case that occurs frequently involves radiation exchange between a small surface at T_s and a much larger, isothermal surface that completely surrounds the smaller one.

- We will show that, for such a condition, the irradiation may be approximated by emission from a blackbody at T_{sur}, in which case $G = \sigma T_{sur}^4$. If the surface is assumed to be one for which $\alpha = \varepsilon$ (a *gray surface*), the *net* rate of radiation heat transfer *from* the surface, expressed per unit area of the surface, is

$$q''_{\text{rad}} = \frac{q}{A} = \varepsilon E_b(T_s) - \alpha G = \varepsilon\sigma(T_s^4 - T_{\text{sur}}^4)$$

$$q_{\text{rad}} = h_r A(T_s - T_{\text{sur}})$$

- The *radiation heat transfer coefficient*

$$h_r \equiv \varepsilon\sigma(T_s + T_{\text{sur}})(T_s^2 + T_{\text{sur}}^2)$$

Conservation of Energy

- ***First Law of Thermodynamics over a Time Interval (Δt)***

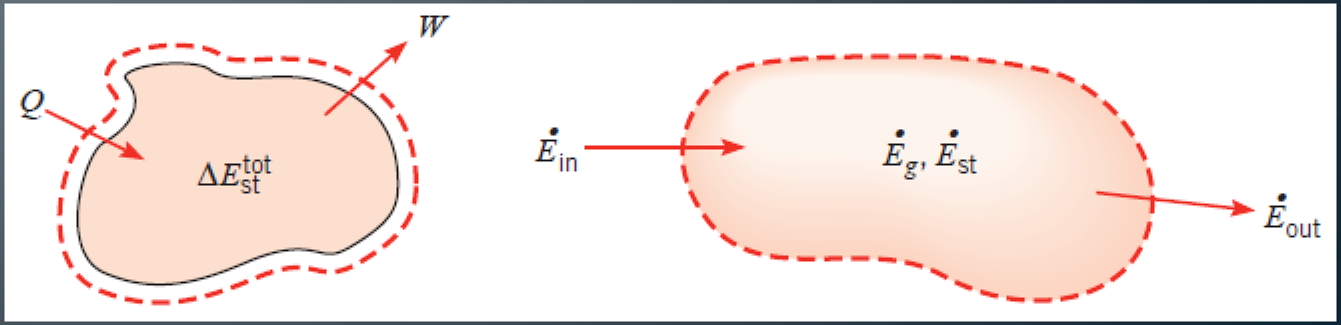
$$\Delta E_{st}^{tot} = Q - W$$

- ***Thermal and Mechanical Energy Equation over a Time Interval (Δt)***

$$\Delta E_{st} = E_{in} - E_{out} + E_g$$

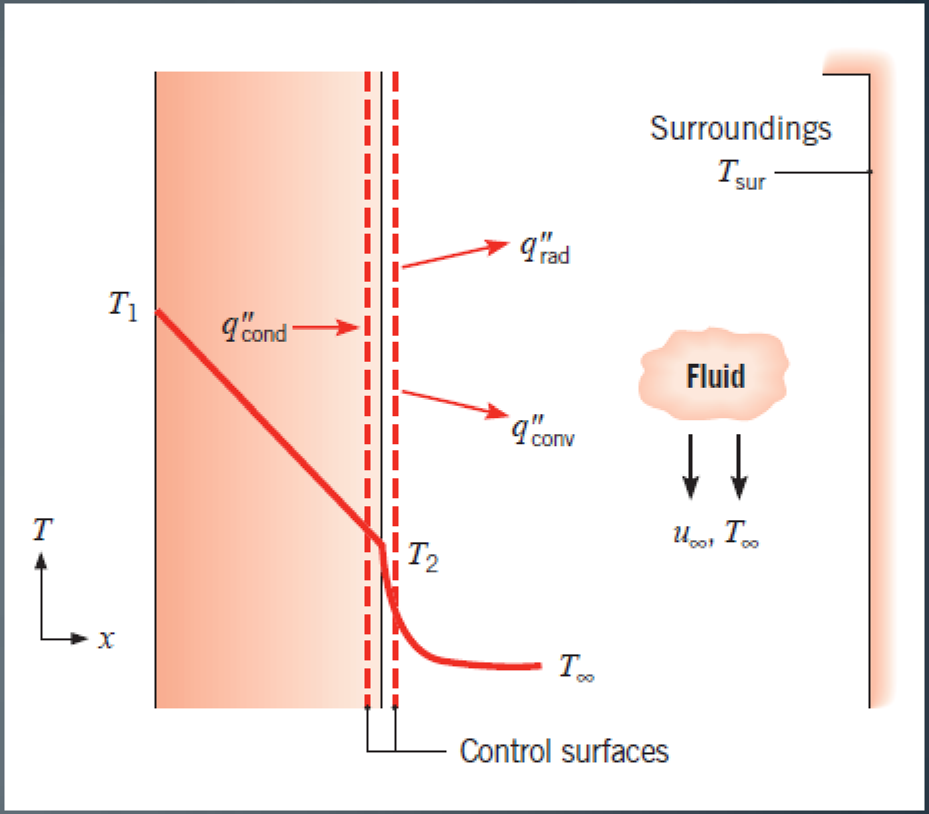
- ***Thermal and Mechanical Energy Equation at an Instant (t)***

$$\dot{E}_{st} \equiv \frac{dE_{st}}{dt} = \dot{E}_{in} - \dot{E}_{out} + \dot{E}_g$$



Conservation of Energy

- *The Surface Energy Balance*



$$\dot{E}_{in} - \dot{E}_{out} = 0$$

$$q''_{cond} - q''_{conv} - q''_{rad} = 0$$

Conservation of Energy

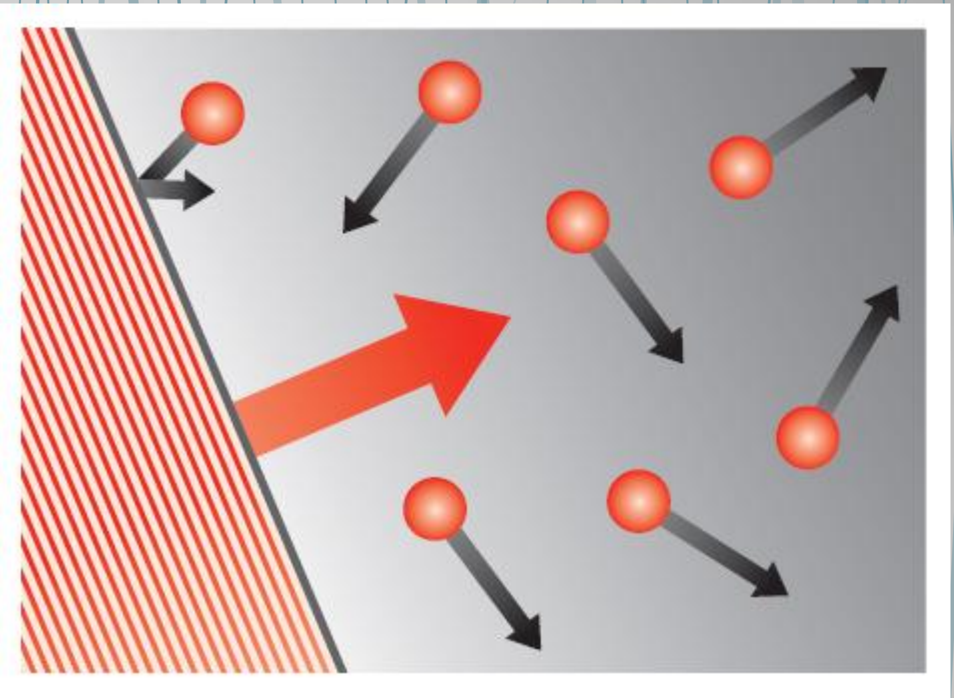
- ***Application of the Conservation Laws: Methodology***

- **1.** The appropriate control volume must be defined, with the control surfaces represented by a dashed line or lines.

- **2.** The appropriate time basis must be identified.

- **3.** The relevant energy processes must be identified, and each process should be shown on the control volume by an appropriately labeled arrow.

- **4.** The conservation equation must then be written, and appropriate rate expressions must be substituted for the relevant terms in the equation.

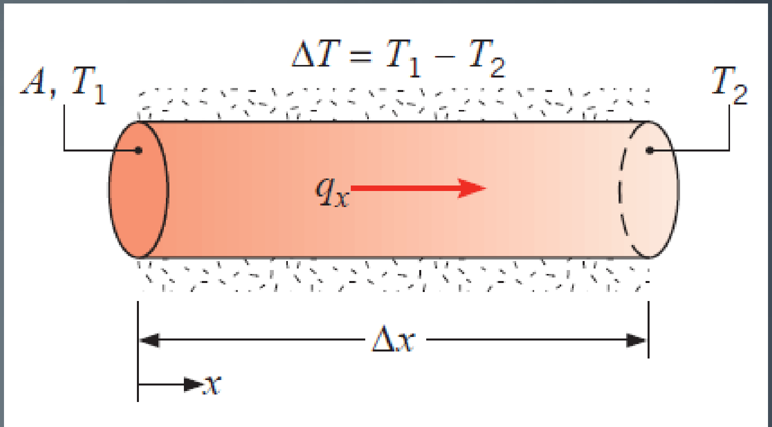


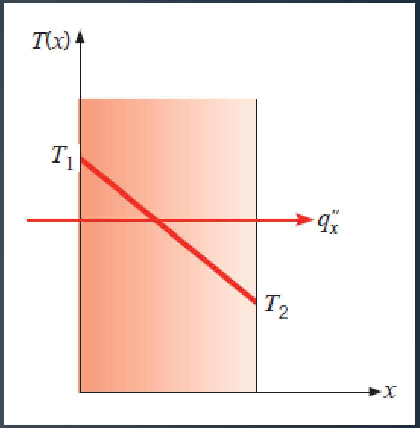
Introduction to Conduction

Chapter 2

The Conduction Rate Equation

- Fourier's law is *phenomenological*; that is, it is developed from observed phenomena rather than being derived from first principles.





$$q_x \propto A \frac{\Delta T}{\Delta x}$$

$$q_x = kA \frac{\Delta T}{\Delta x}$$

$$q_x = -kA \frac{dT}{dx}$$

$$q''_x = \frac{q_x}{A} = -k \frac{dT}{dx}$$

The Conduction Rate Equation

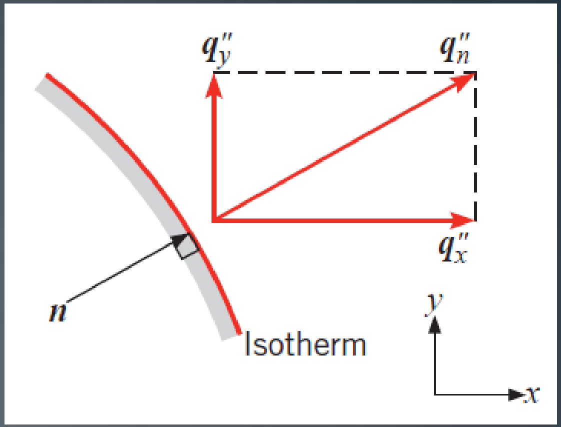
- Fourier's law implies that the heat flux is a *directional* quantity.

- the direction of heat flow will always be normal to a surface of constant temperature, called an *isothermal* surface.

- Recognizing that the heat flux is a vector quantity, we can write a more general statement of the conduction rate equation (*Fourier's law*) as follows:

$$\mathbf{q}'' = -k\nabla T = -k\left(\mathbf{i}\frac{\partial T}{\partial x} + \mathbf{j}\frac{\partial T}{\partial y} + \mathbf{k}\frac{\partial T}{\partial z}\right)$$

The Conduction Rate Equation



- An alternative form of Fourier's law is therefore

$$\boldsymbol{q}'' = q''_n \boldsymbol{n} = -k \frac{\partial T}{\partial n} \boldsymbol{n}$$

$$\boldsymbol{q}'' = \boldsymbol{i} q''_x + \boldsymbol{j} q''_y + \boldsymbol{k} q''_z$$

$$q''_x = -k \frac{\partial T}{\partial x} \qquad q''_y = -k \frac{\partial T}{\partial y} \qquad q''_z = -k \frac{\partial T}{\partial z}$$

The Heat Diffusion Equation

- A major objective in a conduction analysis is to determine the *temperature field* in a medium resulting from conditions imposed on its *boundaries*.

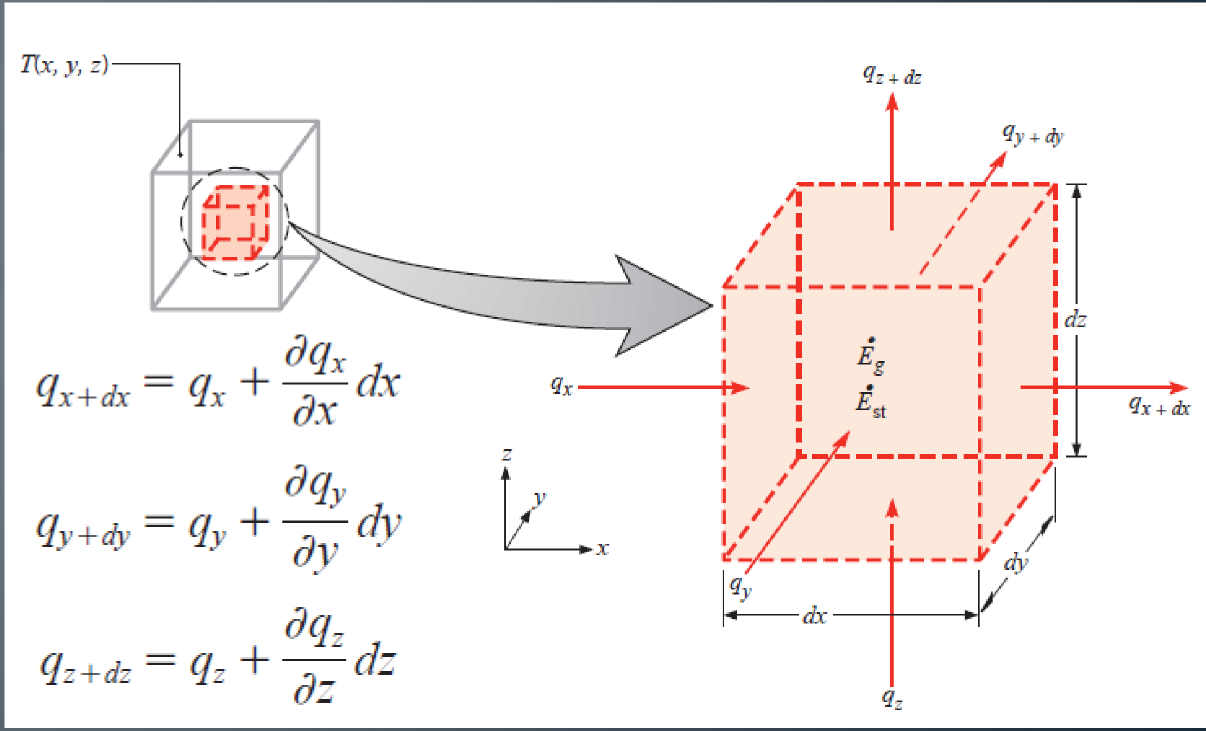
- Once this distribution is known, the conduction heat flux at any point in the medium or on its surface may be computed from Fourier's law.

- For a solid, knowledge of the temperature distribution could be used to ascertain structural integrity through determination of thermal stresses, expansions, and deflections.

- The temperature distribution could also be used to optimize the thickness of an insulating material or to determine the compatibility of special coatings or adhesives used with the material.

The Heat Diffusion Equation

- Consider a homogeneous medium within which there is no bulk motion (advection) and the temperature distribution $T(x, y, z)$ is expressed in Cartesian coordinates.



$$q_{x+dx} = q_x + \frac{\partial q_x}{\partial x} dx$$

$$q_{y+dy} = q_y + \frac{\partial q_y}{\partial y} dy$$

$$q_{z+dz} = q_z + \frac{\partial q_z}{\partial z} dz$$

The Heat Diffusion Equation

- Within the medium there may also be an *energy source* term associated with the rate of thermal energy generation. This term is represented as

$$\dot{E}_g = \dot{q}\, dx\, dy\, dz$$

- In addition, changes may occur in the amount of the *internal thermal energy* stored by the material in the control volume. If the material is not experiencing a change in phase, latent energy effects are not pertinent, and the *energy storage* term may be expressed as

$$\dot{E}_{\mathrm{st}} = \rho c_p \frac{\partial T}{\partial t}\, dx\, dy\, dz$$

The Heat Diffusion Equation

- On a *rate* basis, the general form of the conservation of energy requirement is

$$\dot{E}_{\text{in}} + \dot{E}_{\text{g}} - \dot{E}_{\text{out}} = \dot{E}_{\text{st}}$$

$$q_x + q_y + q_z + \dot{q}\, dx\, dy\, dz - q_{x+dx} - q_{y+dy} - q_{z+dz} = \rho c_p \frac{\partial T}{\partial t} dx\, dy\, dz$$

$$-\frac{\partial q_x}{\partial x} dx - \frac{\partial q_y}{\partial y} dy - \frac{\partial q_z}{\partial z} dz + \dot{q}\, dx\, dy\, dz = \rho c_p \frac{\partial T}{\partial t} dx\, dy\, dz$$

$$q_x = -k\, dy\, dz \frac{\partial T}{\partial x}$$

$$q_y = -k\, dx\, dz \frac{\partial T}{\partial y}$$

$$q_z = -k\, dx\, dy \frac{\partial T}{\partial z}$$

$$\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

The Heat Diffusion Equation

- If the thermal conductivity is constant, the heat equation is

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

- where $\alpha = k/\rho c_p$ is the *thermal diffusivity*.

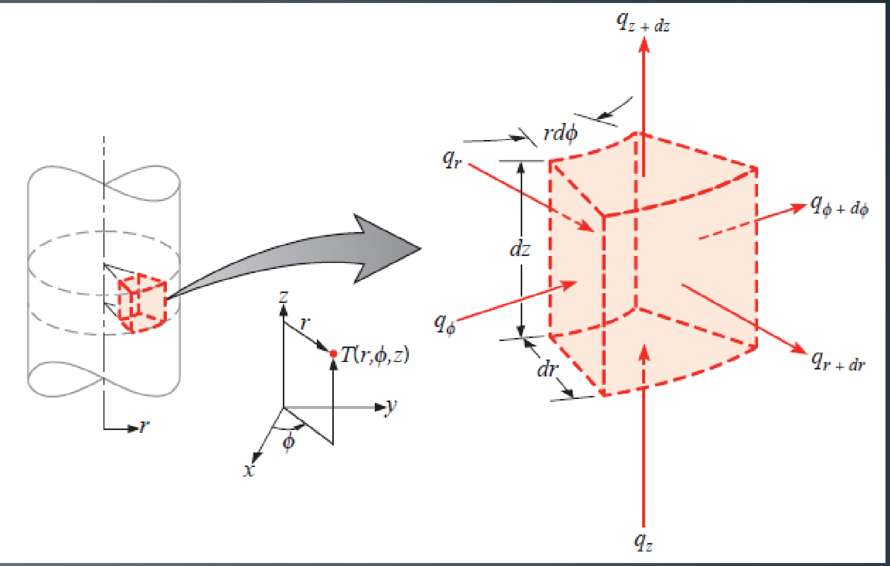
- It measures the ability of a material to conduct thermal energy relative to its ability to store thermal energy. Materials of large will respond quickly to changes in their thermal environment, whereas materials of small will respond more sluggishly, taking longer to reacha new equilibrium condition.

- Under *steady-state* conditions, there can be no change in the amount of energy storage; hence

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=0$$

- *Cylindrical Coordinates*

The Heat Diffusion Equation



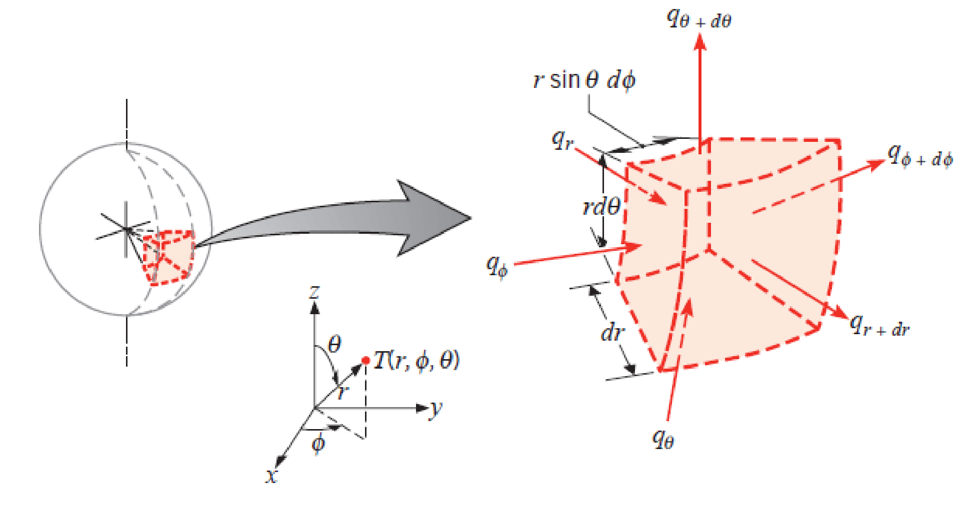
$$q'' = -k\nabla T = -k\left(\boldsymbol{i}\frac{\partial T}{\partial r} + \boldsymbol{j}\frac{1}{r}\frac{\partial T}{\partial \phi} + \boldsymbol{k}\frac{\partial T}{\partial z}\right)$$

$$q''_r = -k\frac{\partial T}{\partial r} \qquad q''_\phi = -\frac{k}{r}\frac{\partial T}{\partial \phi} \qquad q''_z = -k\frac{\partial T}{\partial z}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho c_p\frac{\partial T}{\partial t}$$

The Heat Diffusion Equation

- *Spherical Coordinates*



$$q'' = -k\nabla T = -k\left(\boldsymbol{i}\frac{\partial T}{\partial r} + \boldsymbol{j}\frac{1}{r}\frac{\partial T}{\partial \theta} + \boldsymbol{k}\frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\right)$$

$$q''_r = -k\frac{\partial T}{\partial r} \qquad q''_\theta = -\frac{k}{r}\frac{\partial T}{\partial \theta} \qquad q''_\phi = -\frac{k}{r\sin\theta}\frac{\partial T}{\partial \phi}$$

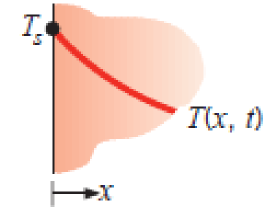
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right) + \dot{q} = \rho c_p\frac{\partial T}{\partial t}$$

Boundary and Initial Conditions

Boundary conditions for the heat diffusion equation at the surface ($x = 0$)

1. Constant surface temperature

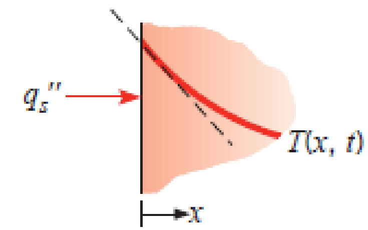
$$T(0, t) = T_s \qquad (2.31)$$



2. Constant surface heat flux

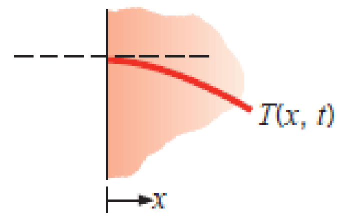
(a) Finite heat flux

$$-k\frac{\partial T}{\partial x}\bigg|_{x=0} = q_s'' \qquad (2.32)$$



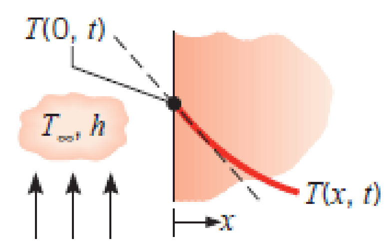
(b) Adiabatic or insulated surface

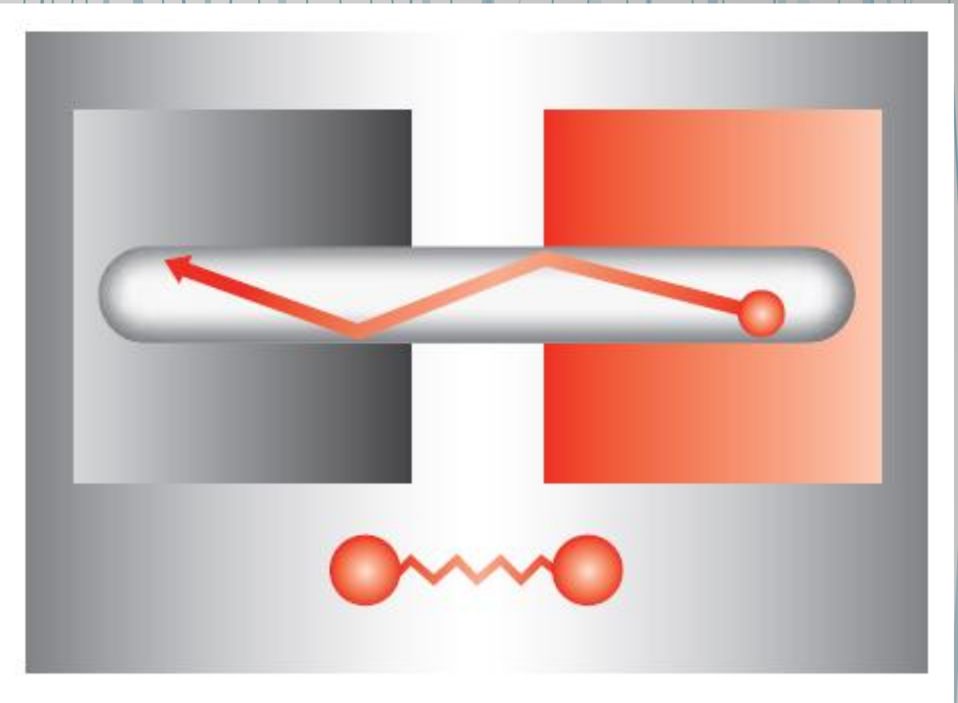
$$\frac{\partial T}{\partial x}\bigg|_{x=0} = 0 \qquad (2.33)$$



3. Convection surface condition

$$-k\frac{\partial T}{\partial x}\bigg|_{x=0} = h[T_\infty - T(0, t)] \qquad (2.34)$$





One-Dimensional, Steady-State Conduction

Chapter 3

The Plane Wall

- **Temperature Distribution**

- For steady-state conditions with no distributed source or sink of energy within the wall, the appropriate form of the heat equation is

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)=0$$

- *For one-dimensional, steady-state conduction in a plane wall with no heat generation, the heat flux is a constant, independent of x.*

- If the thermal conductivity of the wall material is assumed to be constant, the equation may be integrated twice to obtain the *general solution*

$$T(x)=C_1x+C_2$$

The Plane Wall

- **Temperature Distribution**

- To obtain the constants of integration, C_1 and C_2, boundary conditions must be introduced.

$$T(0) = T_{s,1} \quad \text{and} \quad T(L) = T_{s,2}$$

- Applying the conditions, the temperature distribution is then

$$T(x) = (T_{s,2} - T_{s,1})\frac{x}{L} + T_{s,1}$$

- *For one-dimensional, steady-state conduction in a plane wall with no heat generation and constant thermal conductivity, the temperature varies linearly with x.*

The Plane Wall

- **Temperature Distribution**

- Now that we have the temperature distribution, we may use Fourier's law to determine the conduction heat transfer rate. That is,

$$q_x = -kA\frac{dT}{dx} = \frac{kA}{L}(T_{s,1} - T_{s,2})$$

- Note that A is the area of the wall *normal* to the direction of heat transfer and, for the plane wall, it is a constant independent of x. The heat flux is then

$$q''_x = \frac{q_x}{A} = \frac{k}{L}(T_{s,1} - T_{s,2})$$

The Plane Wall

- **Thermal Resistance**

- In particular, an analogy exists between the diffusion of heat and electrical charge.

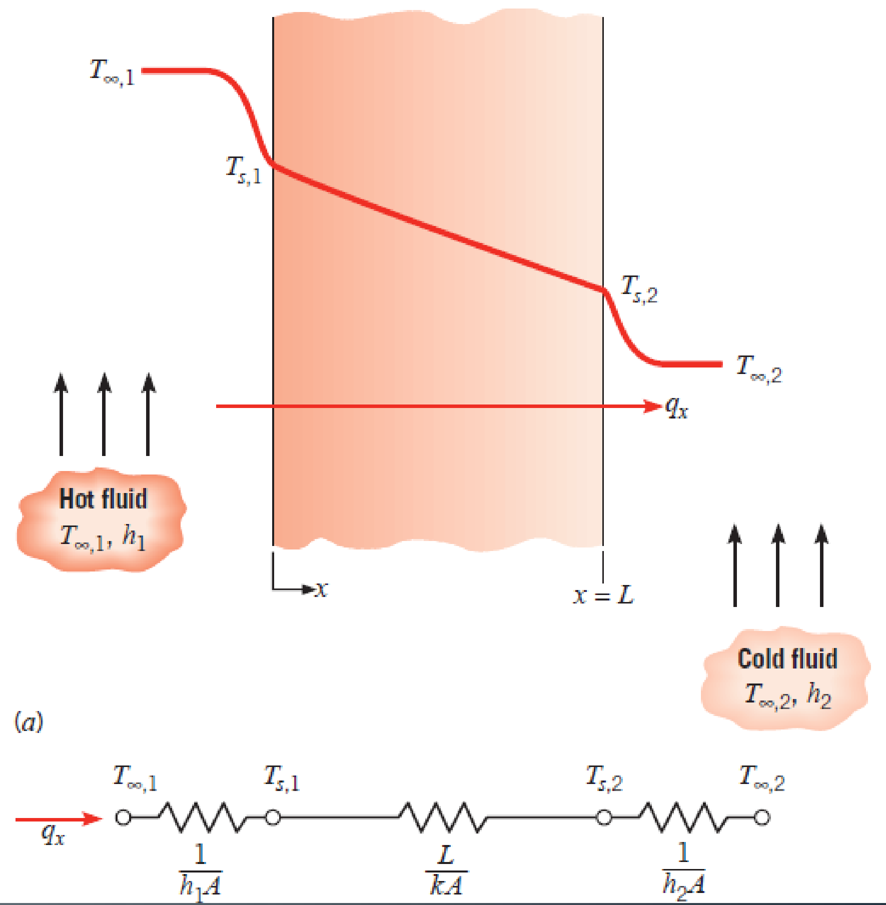
- Just as an electrical resistance is associated with the conduction of electricity, a thermal resistance may be associated with the conduction of heat.

- Defining resistance as the ratio of a driving potential to the corresponding transfer rate, it follows that the *thermal resistance for conduction in a plane wall* is

$$R_{t,\text{cond}} \equiv \frac{T_{s,1} - T_{s,2}}{q_x} = \frac{L}{kA}$$

The Plane Wall

- **Thermal Resistance**



- A thermal resistance may also be associated with heat transfer by convection at a surface. The *thermal resistance for convection* is then

$$R_{t,\text{conv}} \equiv \frac{T_s - T_\infty}{q} = \frac{1}{hA}$$

The Plane Wall

- **Thermal Resistance**
- Since q_x is constant throughout the network, it follows that

$$q_x = \frac{T_{\infty,1} - T_{s,1}}{1/h_1 A} = \frac{T_{s,1} - T_{s,2}}{L/kA} = \frac{T_{s,2} - T_{\infty,2}}{1/h_2 A}$$

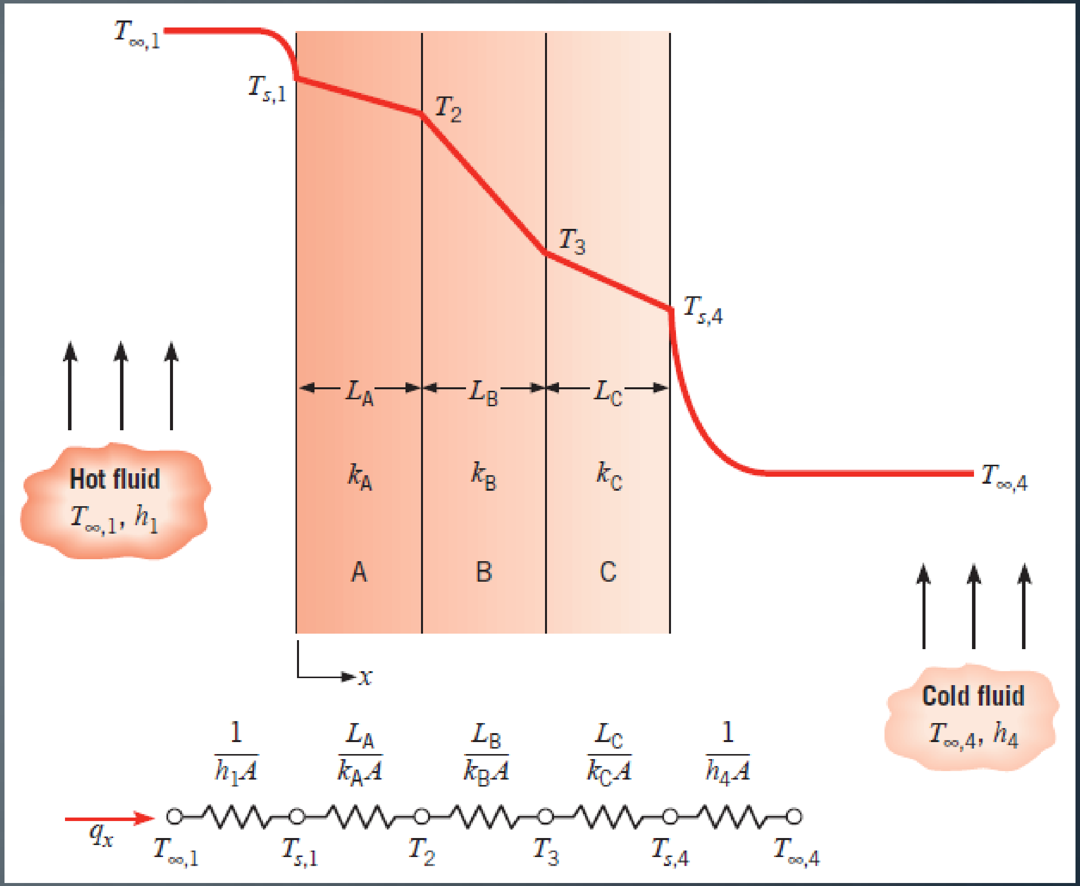
- In terms of the *overall temperature difference*, $T_{\infty,1} - T_{\infty,2}$, and the total thermal resistance, R_{tot}, the heat transfer rate may also be expressed as

$$q_x = \frac{T_{\infty,1} - T_{\infty,2}}{R_{\text{tot}}}$$

$$R_{\text{tot}} = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A}$$

The Plane Wall

- **The Composite Wall**
- Equivalent thermal circuits may also be used for more complex systems, such as *composite walls*.



$$q_x = \frac{T_{\infty,1} - T_{\infty,4}}{\Sigma R_t}$$

The Plane Wall

- **The Composite Wall**

- Equivalent thermal circuits may also be used for more complex systems, such as *composite walls*.

$$q_x = \frac{T_{\infty,1} - T_{\infty,4}}{[(1/h_1A) + (L_A/k_AA) + (L_B/k_BA) + (L_C/k_CA) + (1/h_4A)]}$$

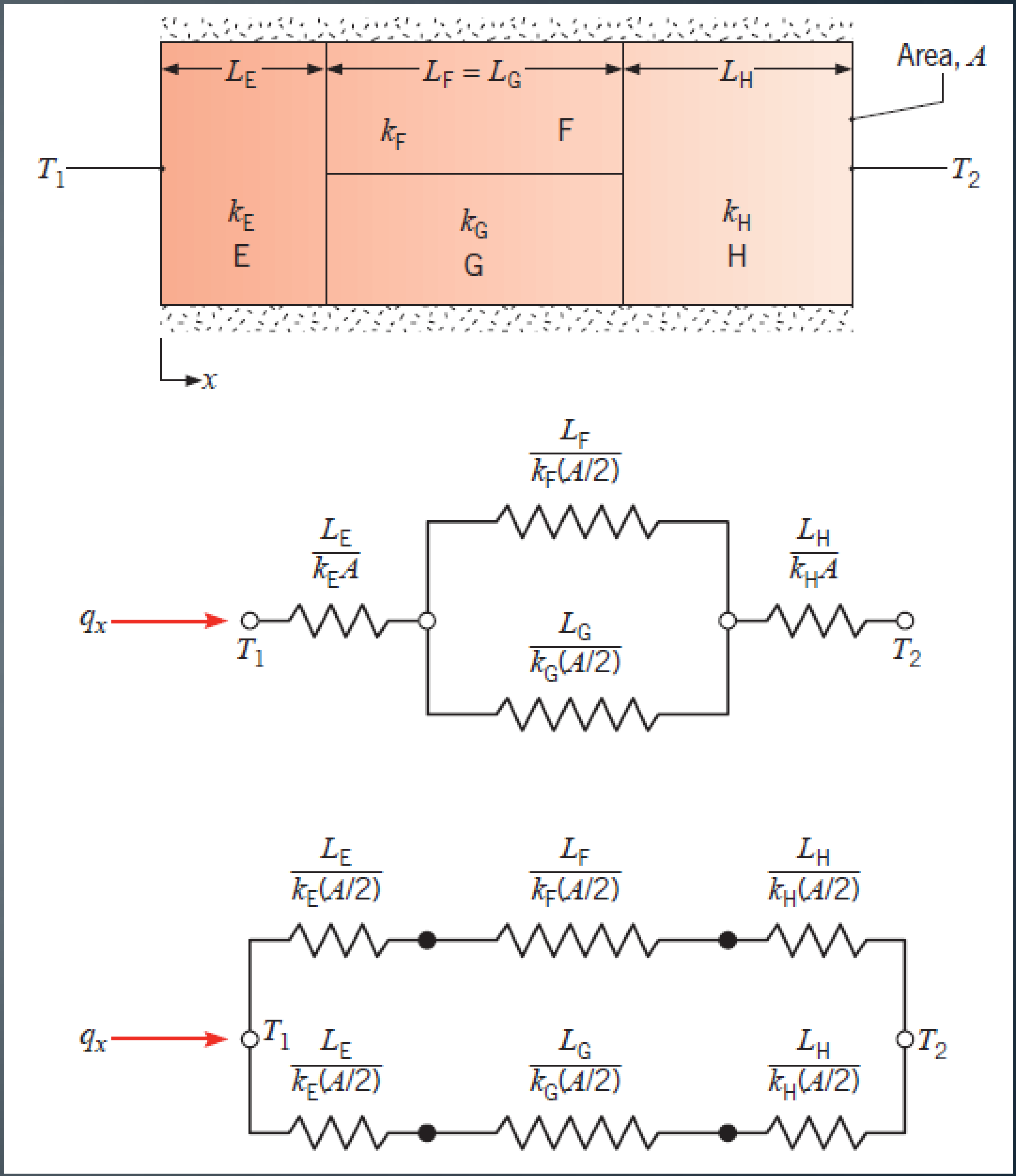
$$q_x = \frac{T_{\infty,1} - T_{s,1}}{(1/h_1A)} = \frac{T_{s,1} - T_2}{(L_A/k_AA)} = \frac{T_2 - T_3}{(L_B/k_BA)} = \cdots$$

- With composite systems, it is often convenient to work with an *overall heat transfer coefficient U*, which is defined by an expression analogous to Newton's law of cooling.

$$q_x \equiv UA\,\Delta T$$

The Plane Wall

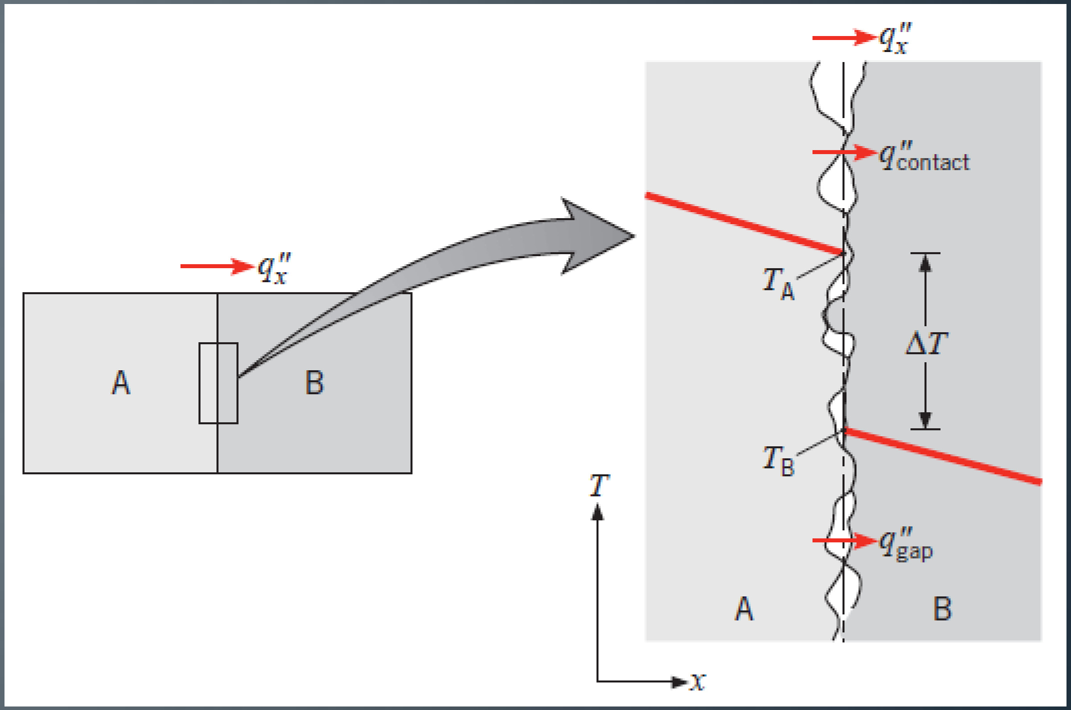
- **The Composite Wall**



The Plane Wall

- **Contact Resistance**

- In composite systems, the temperature drop across the interface between materials may be appreciable. This temperature change is attributed to what is known as the *thermal contact resistance*, $R''_{t,c}$.



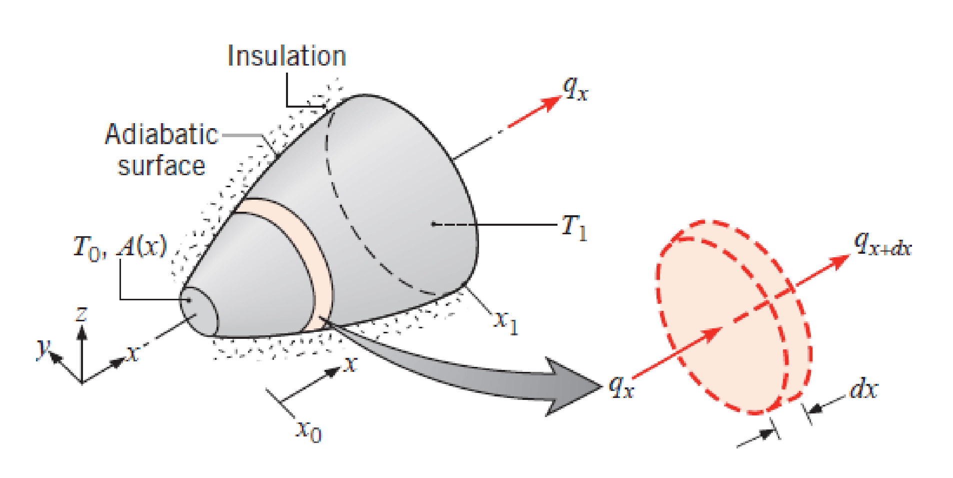
$$R''_{t,c} = \frac{T_A - T_B}{q''_x}$$

EXAMPLE 3.1

EXAMPLE 3.3

An Alternative Conduction Analysis

- Considering conduction in the system below, we recognize that, for *steady-state conditions* with *no heat generation* and *no heat loss from the sides*, the heat transfer rate q_x must be a constant independent of x.



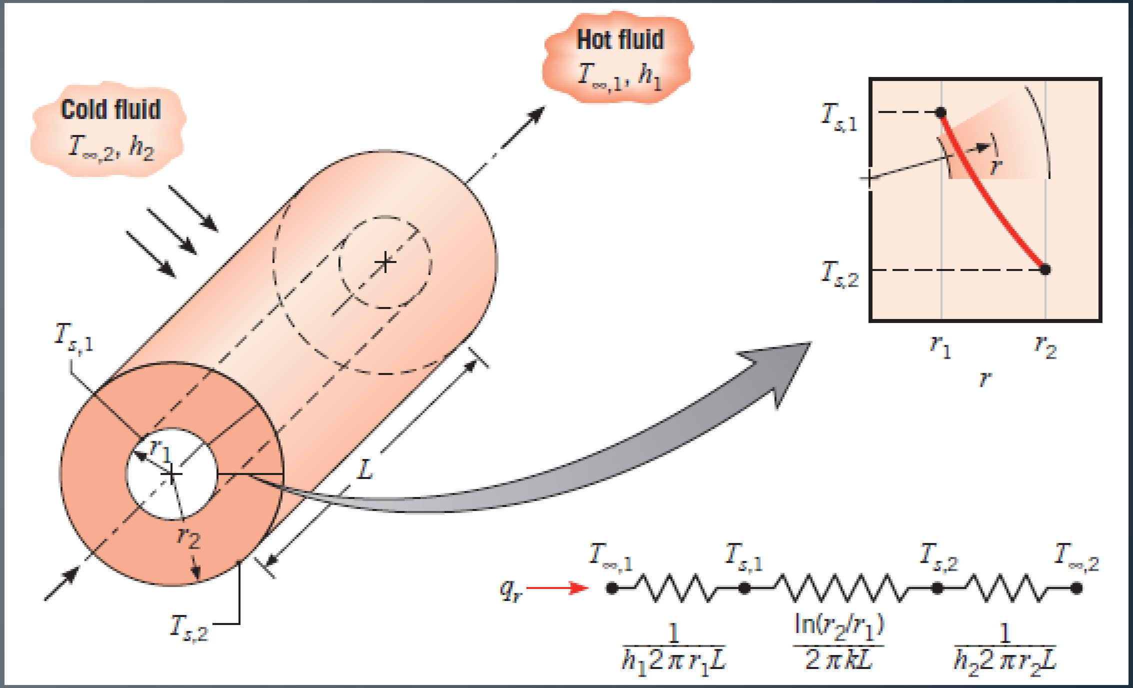
- In particular, since the conduction rate is a *constant*, the rate equation may be *integrated*, even though neither the rate nor the temperature distribution is known.

$$q_x \int_{x_0}^{x} \frac{dx}{A(x)} = -\int_{T_0}^{T} k(T)\, dT$$

EXAMPLE 3.5

Radial Systems

- **The Cylinder**
- A common example is the hollow cylinder whose inner and outer surfaces are exposed to fluids at different temperatures.



- For steady-state conditions with no heat generation, the appropriate form of the heat equation is

$$\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right) = 0$$

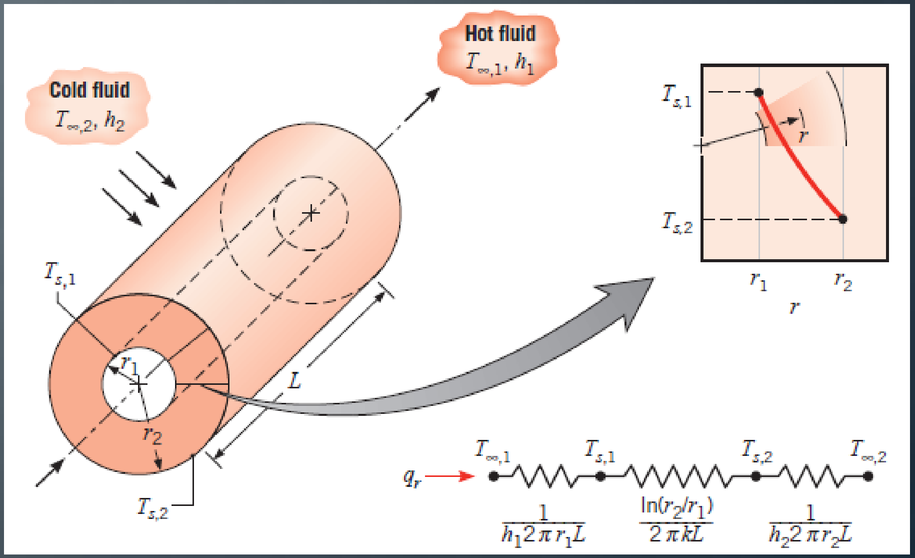
Radial Systems

- **The Cylinder**

- The rate at which energy is conducted across any cylindrical surface in the solid may be expressed as

$$q_r = -kA\frac{dT}{dr} = -k(2\pi rL)\frac{dT}{dr}$$

- Since the heat equation dictates that the quantity $kr(dT/dr)$ is independent of r, it follows that the conduction heat transfer rate q_r (not the heat flux q_r'') is a constant in the radial direction.



Radial Systems

- **The Cylinder**

- We may determine the temperature distribution in the cylinder by solving the heat equation and applying appropriate boundary conditions.

$$T(r) = C_1 \ln r + C_2$$

$$T(r_1) = T_{s,1} \quad \text{and} \quad T(r_2) = T_{s,2}$$

$$T(r) = \frac{T_{s,1} - T_{s,2}}{\ln (r_1/r_2)} \ln\left(\frac{r}{r_2}\right) + T_{s,2}$$

- Note that the temperature distribution associated with radial conduction through a cylindrical wall is *logarithmic*, not linear, as it is for the plane wall under the same conditions.

Radial Systems

- **The Cylinder**

- If the temperature distribution, is now used with Fourier's law, we obtain the following expression for the heat transfer rate:

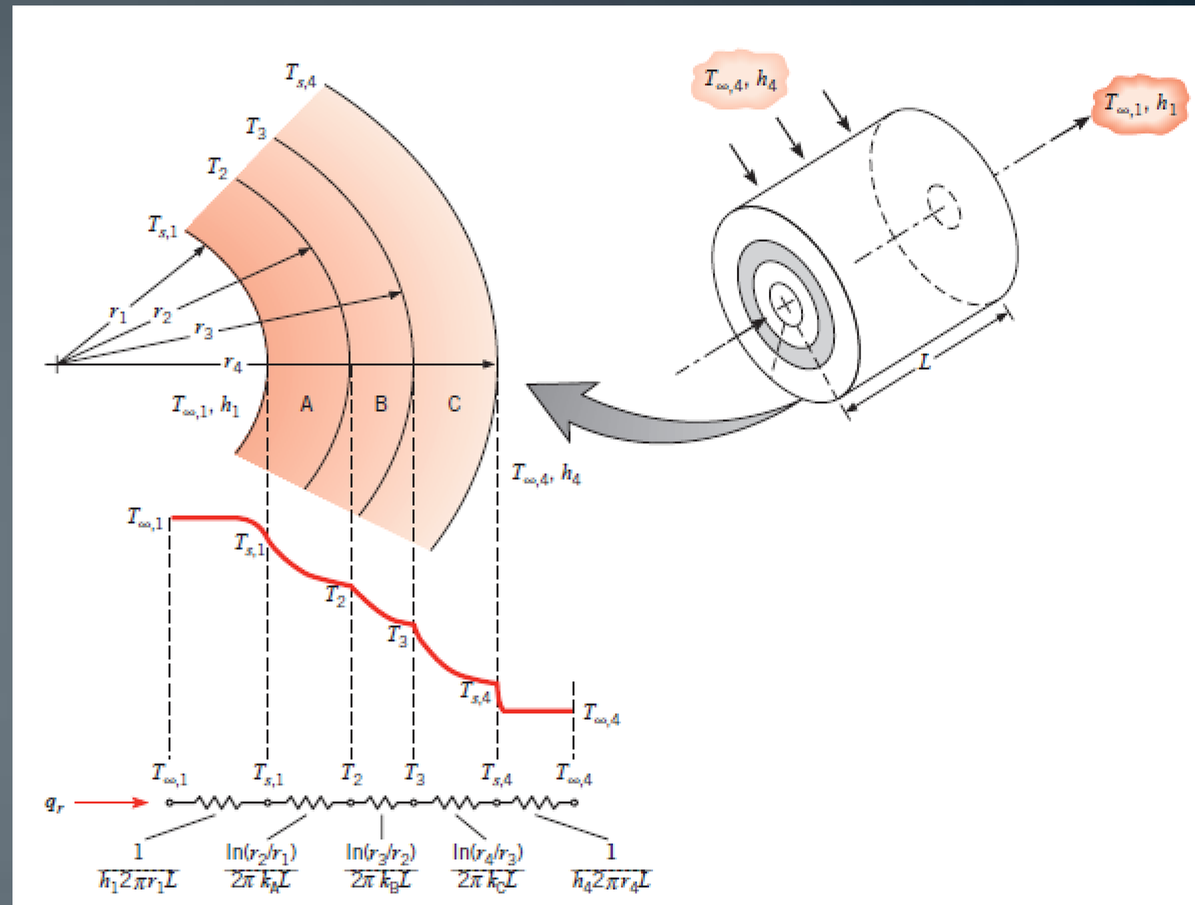
$$q_r = \frac{2\pi Lk(T_{s,1} - T_{s,2})}{\ln(r_2/r_1)}$$

- From this result it is evident that, for radial conduction in a cylindrical wall, the thermal resistance is of the form

$$R_{t,\text{cond}} = \frac{\ln(r_2/r_1)}{2\pi Lk}$$

Radial Systems

- **The Cylinder**



$$q_r = \frac{T_{\infty,1} - T_{\infty,4}}{\dfrac{1}{2\pi r_1 L h_1} + \dfrac{\ln(r_2/r_1)}{2\pi k_A L} + \dfrac{\ln(r_3/r_2)}{2\pi k_B L} + \dfrac{\ln(r_4/r_3)}{2\pi k_C L} + \dfrac{1}{2\pi r_4 L h_4}}$$

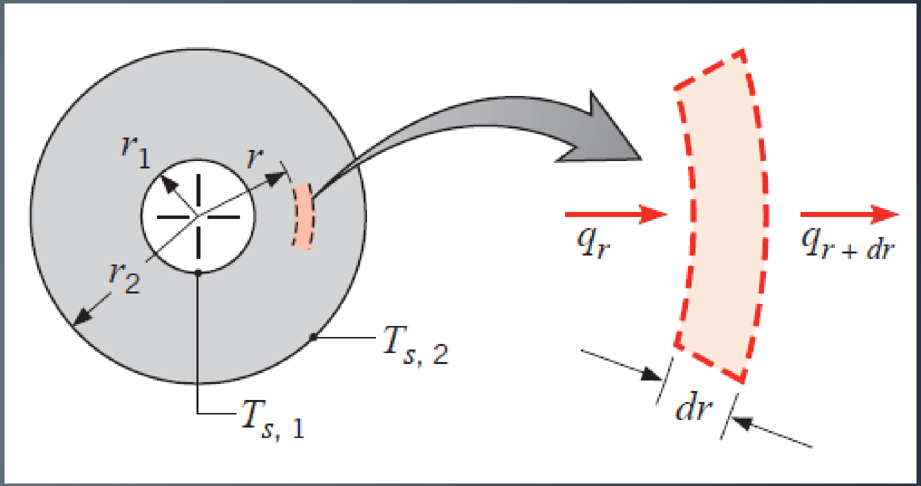
$$q_r = \frac{T_{\infty,1} - T_{\infty,4}}{R_{tot}} = UA(T_{\infty,1} - T_{\infty,4})$$

EXAMPLE 3.6

Radial Systems

- **The Sphere**

- Now consider applying the *alternative method* to analyzing conduction in the hollow sphere.



- For the differential control volume of the figure, energy conservation requires that $q_r = q_{r+dr}$ for steady-state, one-dimensional conditions with no heat generation.

$$q_r = -kA\frac{dT}{dr} = -k(4\pi r^2)\frac{dT}{dr}$$

Radial Systems

- **The Sphere**
- Acknowledging that q_r is a constant, independent of r, the heat transfer rate equation may be expressed in the integral form

$$\frac{q_r}{4\pi}\int_{r_1}^{r_2}\frac{dr}{r^2} = -\int_{T_{s,1}}^{T_{s,2}} k(T)\,dT$$

- Assuming constant k, we then obtain

$$q_r = \frac{4\pi k(T_{s,1} - T_{s,2})}{(1/r_1) - (1/r_2)}$$

- Remembering that the thermal resistance is defined as the temperature difference divided by the heat transfer rate, we obtain

$$R_{t,\text{cond}} = \frac{1}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Summary of One-Dimensional Conduction Results

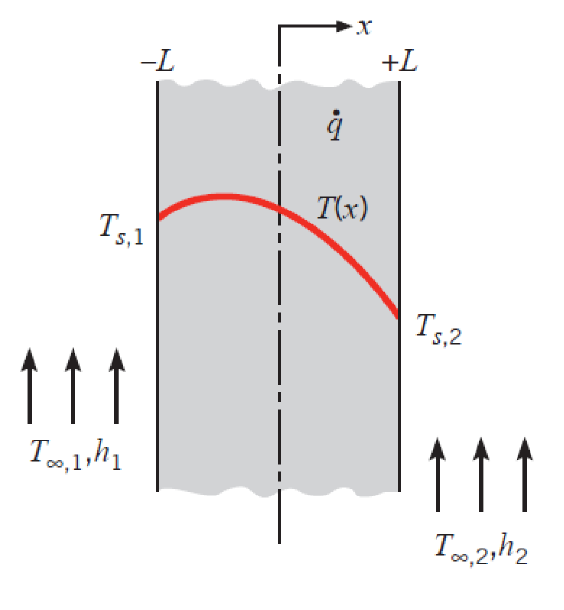
One-dimensional, steady-state solutions to the heat equation with no generation

	Plane Wall	**Cylindrical Wall**[a]	**Spherical Wall**[a]
Heat equation	$\frac{d^2T}{dx^2} = 0$	$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$	$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0$
Temperature distribution	$T_{s,1} - \Delta T\frac{x}{L}$	$T_{s,2} + \Delta T\frac{\ln(r/r_2)}{\ln(r_1/r_2)}$	$T_{s,1} - \Delta T\left[\frac{1-(r_1/r)}{1-(r_1/r_2)}\right]$
Heat flux (q'')	$k\frac{\Delta T}{L}$	$\frac{k\,\Delta T}{r\ln(r_2/r_1)}$	$\frac{k\,\Delta T}{r^2[(1/r_1)-(1/r_2)]}$
Heat rate (q)	$kA\frac{\Delta T}{L}$	$\frac{2\pi Lk\,\Delta T}{\ln(r_2/r_1)}$	$\frac{4\pi k\,\Delta T}{(1/r_1)-(1/r_2)}$
Thermal resistance ($R_{t,\text{cond}}$)	$\frac{L}{kA}$	$\frac{\ln(r_2/r_1)}{2\pi Lk}$	$\frac{(1/r_1)-(1/r_2)}{4\pi k}$

[a]The critical radius of insulation is $r_{cr} = k/h$ for the cylinder and $r_{cr} = 2k/h$ for the sphere.

Conduction with Thermal Energy Generation

- **The Plane Wall**
- Consider the plane wall in which there is *uniform* energy generation per unit volume ($\dot{q}$ is constant) and the surfaces are maintained at $T_{s,1}$ and $T_{s,2}$.



- For constant thermal conductivity k, the appropriate form of the heat equation is

$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0$$

Conduction with Thermal Energy Generation

- **The Plane Wall**

- We may determine the temperature distribution in the plane wall by solving the heat equation and applying appropriate boundary conditions.

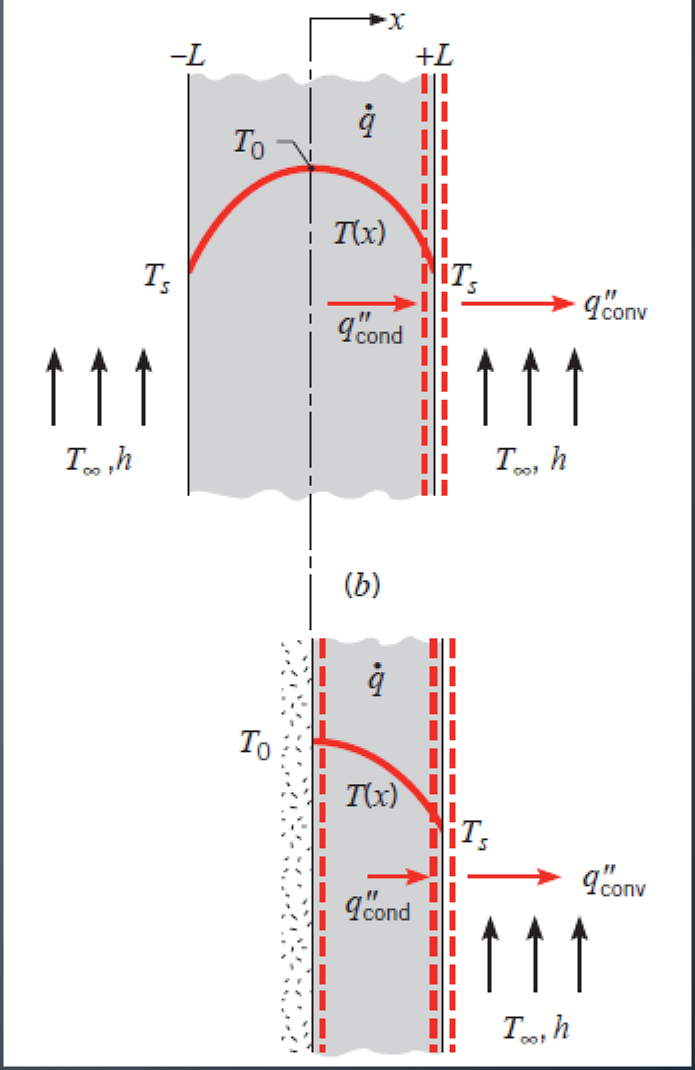
$$T = -\frac{\dot{q}}{2k}x^2 + C_1 x + C_2$$

$$T(-L) = T_{s,1} \qquad \text{and} \qquad T(L) = T_{s,2}$$

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + \frac{T_{s,2} - T_{s,1}}{2}\frac{x}{L} + \frac{T_{s,1} + T_{s,2}}{2}$$

- Note, however, that *with generation the heat fluxis no longer independent of* x.

- **The Plane Wall**
- The preceding result simplifies when both surfaces are maintained at a common temperature, $T_{s,1} = T_{s,2} = T_s$.



Conduction with Thermal Energy Generation

Conduction with Thermal Energy Generation

- **The Plane Wall**

- The temperature distribution is then *symmetrical* about the midplane and is given by

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_s$$

- The maximum temperature exists at the midplane

$$T(0) \equiv T_0 = \frac{\dot{q}L^2}{2k} + T_s$$

- In which case the temperature distribution may be expressed as

$$\frac{T(x) - T_0}{T_s - T_0} = \left(\frac{x}{L}\right)^2$$

Conduction with Thermal Energy Generation

- **The Plane Wall**

- It is important to note that at the plane of symmetry, the temperature gradient is zero, $(dT/dx)_{x=0} = 0$. Accordingly, there is no heat transfer across this plane, and it may be represented by the *adiabatic* surface.

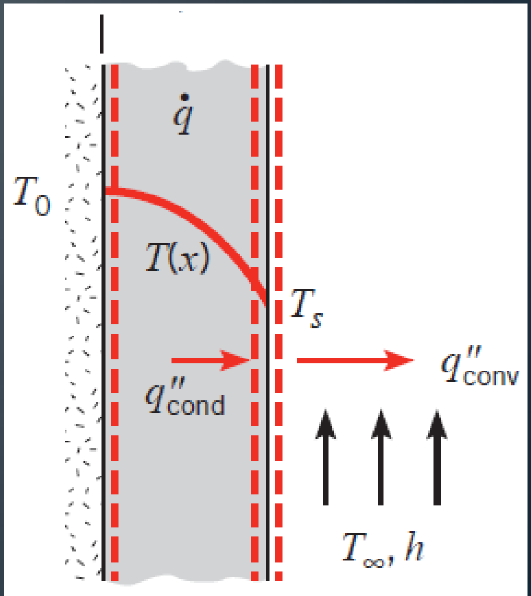
- A common situation is one for which it is the temperature of an adjoining fluid, T_∞, and not T_s, which is known.

$$-k\left.\frac{dT}{dx}\right|_{x=L} = h(T_s - T_\infty)$$

$$T_s = T_\infty + \frac{\dot{q}L}{h}$$

$$\dot{E}_g = \dot{E}_{out}$$

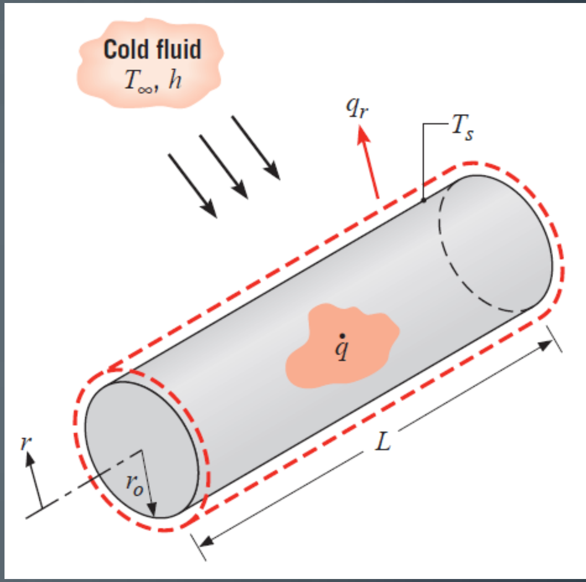
$$\dot{q}L = h(T_s - T_\infty)$$



EXAMPLE 3.7

Conduction with Thermal Energy Generation

- **Radial Systems**
- Heat generation may occur in a variety of radial geometries. Consider the long, solid cylinder which could represent a current-carrying wire or a fuel element in a nuclear reactor.



- To determine the temperature distribution in the cylinder, we begin with the appropriate form of the heat equation.

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{q}}{k}=0$$

Conduction with Thermal Energy Generation

- **Radial Systems**

- We may determine the temperature distribution in the cylinder by solving the heat equation and applying appropriate boundary conditions.

$$T(r) = -\frac{\dot{q}}{4k} r^2 + C_1 \ln r + C_2$$

$$\left.\frac{dT}{dr}\right|_{r=0} = 0 \quad \text{and} \quad T(r_0) = T_s$$

$$T(r) = \frac{\dot{q} r_o^2}{4k}\left(1 - \frac{r^2}{r_o^2}\right) + T_s$$

Conduction with Thermal Energy Generation

- **Radial Systems**

- Evaluating the previous equation at the centerline and dividing the result into it, we obtain the temperature distribution in nondimensional form,

$$\frac{T(r) - T_s}{T_o - T_s} = 1 - \left(\frac{r}{r_o}\right)^2$$

- To relate the surface temperature, T_s , to the temperature of the cold fluid T_∞, either a *surface energy balance* or an *overall energy balance* may be used. Choosing the second approach, we obtain

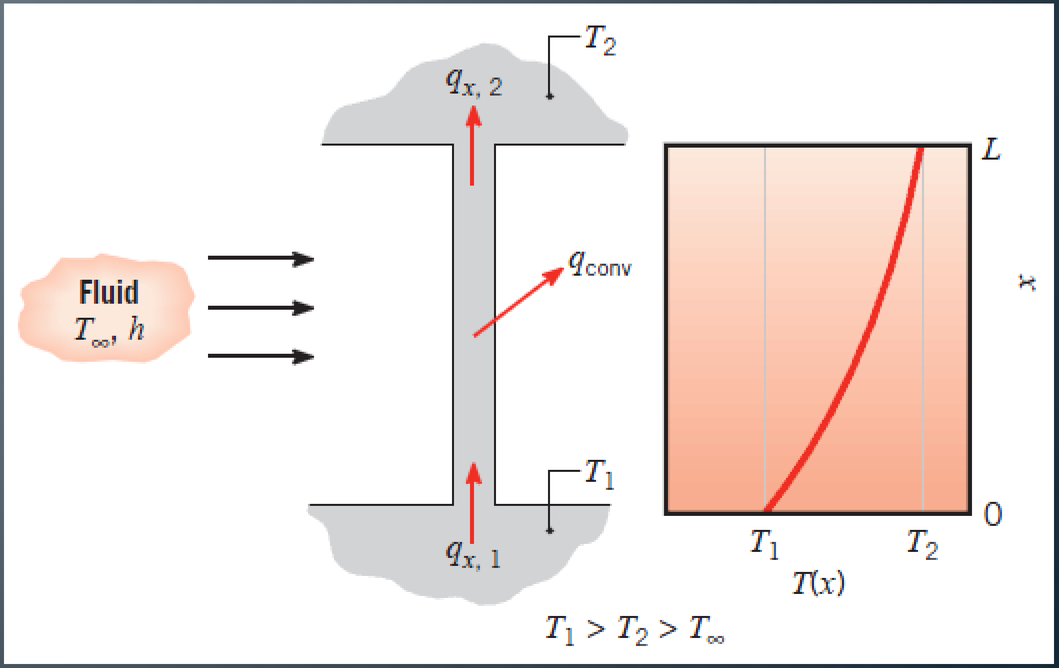
$$\dot{q}(\pi r_o^2 L) = h(2\pi r_o L)(T_s - T_\infty)$$

$$T_s = T_\infty + \frac{\dot{q} r_o}{2h}$$

EXAMPLE 3.8

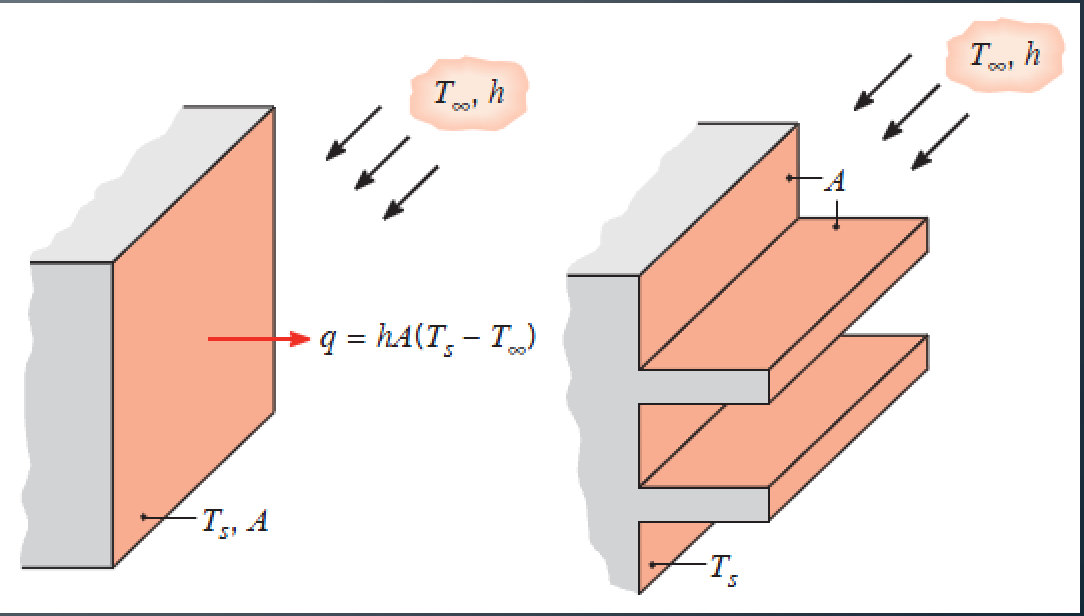
Heat Transfer from Extended Surfaces

- The term *extended surface* is commonly used to depict an important special case involving heat transfer by conduction within a solid and heat transfer by convection (and/or radiation) from the boundaries of the solid.
- Until now, we have considered heat transfer from the boundaries of a solid to be in the same direction as heat transfer by conduction in the solid. In contrast, for an extended surface, the direction of heat transfer from the boundaries is *perpendicular* to the principal direction of heat transfer in the solid.



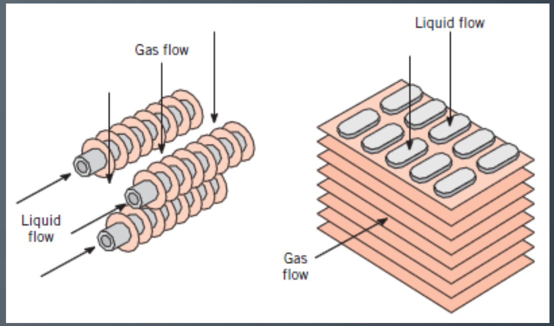
Heat Transfer from Extended Surfaces

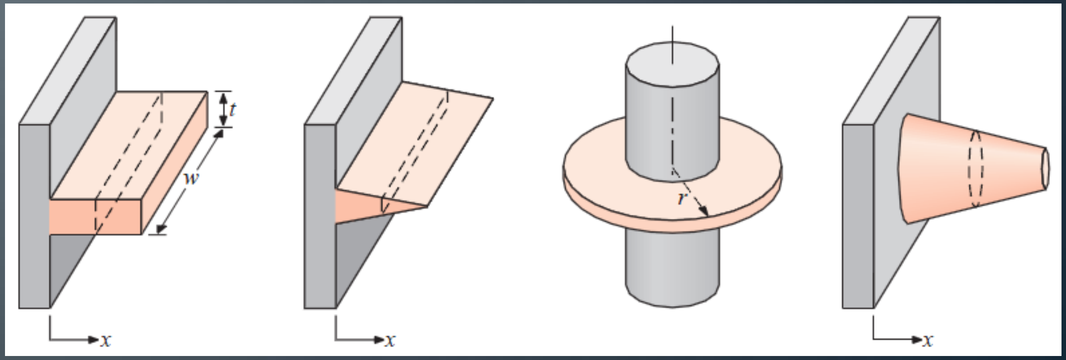
- The most frequent application is one in which an extended surface is used specifically to *enhance* heat transfer between a solid and an adjoining fluid. Such an extended surface is termed a *fin*.
- The heat transfer rate may be increased by increasing the surface area across which the convection occurs. This may be done by employing *fin* that *extend* from the wall into the surrounding fluid.
- Ideally, the fin material should have a large thermal conductivity to *minimize temperature variations* from its base to its tip.



Heat Transfer from Extended Surfaces

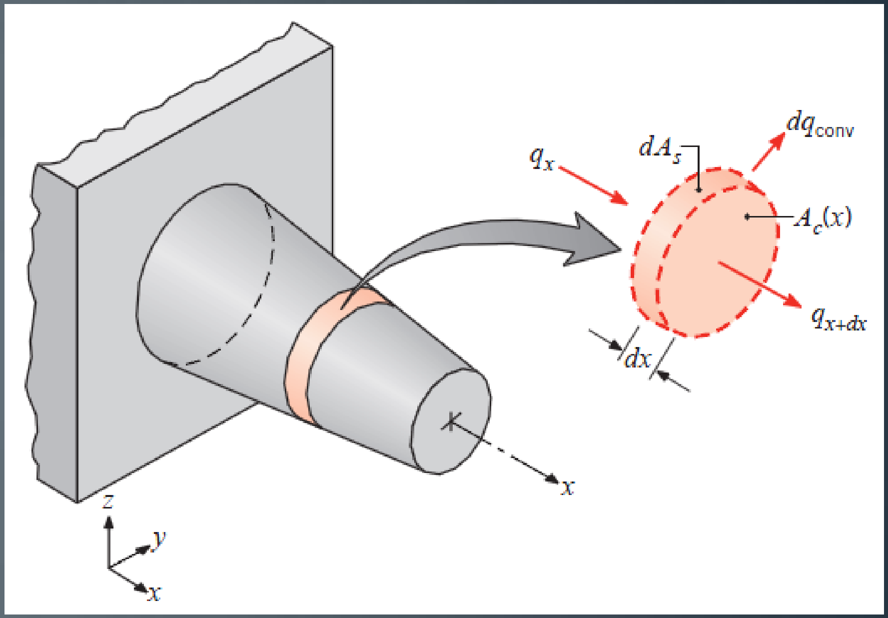
- A *straight fin* is any extended surface that is attached to a plane wall. It may be of uniform cross-sectional area, or its cross-sectional area may vary with the distance from the wall.
- An *annular fin* is one that is circumferentially attached to a cylinder, and its cross section varies with radius from the wall of the cylinder.
- A *pin fin* or *spine*, is an extended surface of circular cross section. Pin fins may also be of uniform or nonuniform cross section.





A General Conduction Analysis

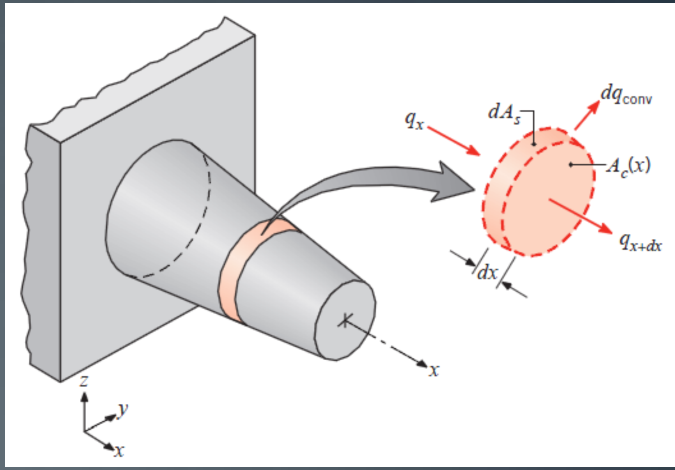
- To determine the heat transfer rate associated with a fin, we must first obtain the *temperature distribution* along the fin.
- We begin by performing an *energy balance* on an appropriate differential element.



- We may assume that the temperature is *uniform across the fin thickness*, that is, it is only a function of x. We will consider *steady-state conditions* and also assume that the *thermal conductivity is constant*, that *radiation from the surface is negligible*, that *heat generation effects are absent*, and that the *convection heat transfer coefficient h is uniform* over the surface.

A General Conduction Analysis

- Applying the conservation of energy requirement, to the differential element, we obtain



$$q_x = q_{x+dx} + dq_{conv}$$

$$q_x = -kA_c \frac{dT}{dx}$$

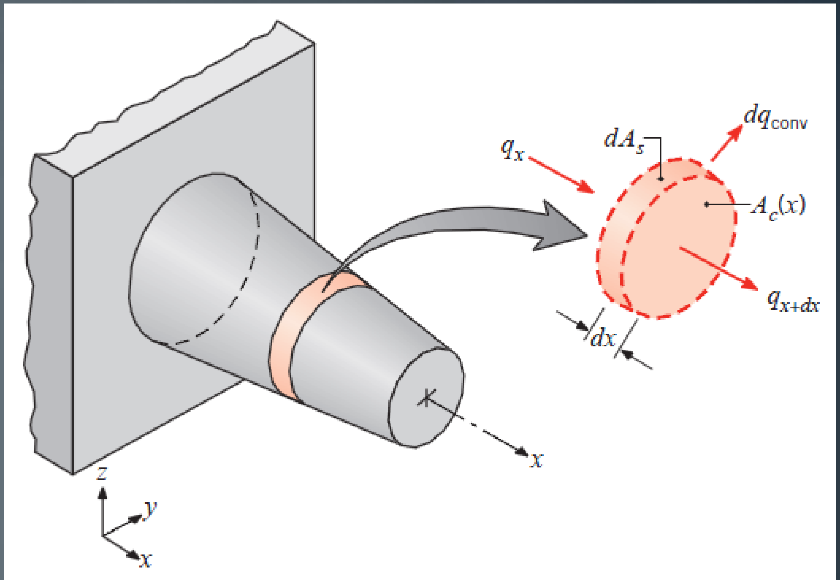
$$q_{x+dx} = q_x + \frac{dq_x}{dx} dx$$

$$q_{x+dx} = -kA_c \frac{dT}{dx} - k \frac{d}{dx}\left(A_c \frac{dT}{dx}\right) dx$$

$$dq_{conv} = h dA_s (T - T_\infty)$$

A General Conduction Analysis

- Substituting the foregoing rate equations into the energy balance, we obtain



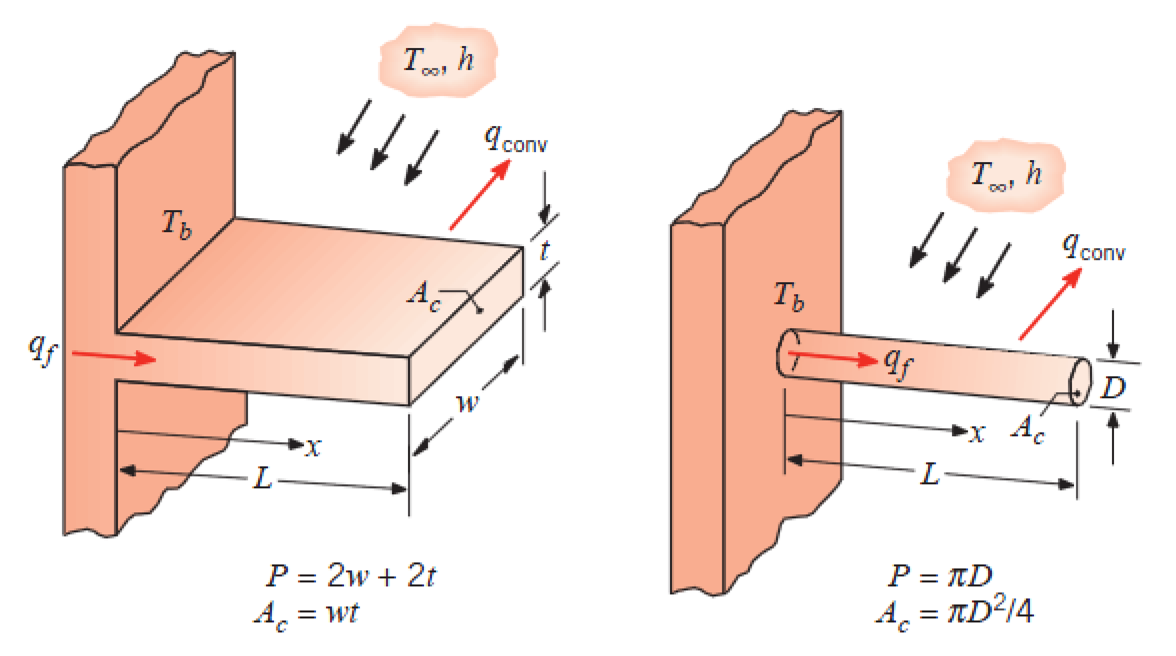
$$\frac{d}{dx}\left(A_c \frac{dT}{dx}\right) - \frac{h}{k}\frac{dA_s}{dx}(T - T_\infty) = 0$$

$$\frac{d^2T}{dx^2} + \left(\frac{1}{A_c}\frac{dA_c}{dx}\right)\frac{dT}{dx} - \left(\frac{1}{A_c}\frac{h}{k}\frac{dA_s}{dx}\right)(T - T_\infty) = 0$$

Fins of Uniform Cross-Sectional Area

- We begin with the simplest case of straight rectangular and pin fins of uniform cross section.

- Each fin is attached to a base surface of temperature $T(0) = T_b$ and extends into a fluid of temperature T_∞.



Fins of Uniform Cross-Sectional Area

- For the prescribed fins, A_c is a constant and $A_s = Px$, where A_s is the surface area measured from the base to x and P is the fin perimeter. Accordingly, with $dA_c/dx = 0$ and $dA_s/dx = P$, the above equation reduces to

$$\frac{d^2T}{dx^2} - \frac{hP}{kA_c}(T - T_\infty) = 0$$

- To simplify the form of this equation, we transform the dependent variable by defining an *excess temperature* as

$$\theta(x) \equiv T(x) - T_\infty$$

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0$$

$$m^2 \equiv \frac{hP}{kA_c}$$

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx}$$

Fins of Uniform Cross-Sectional Area

- To evaluate the constants C_1 and C_2 of the above equation, it is necessary to specify appropriate boundary conditions.
- One such condition may be specified in terms of the temperature at the *base* of the fin $(x = 0)$

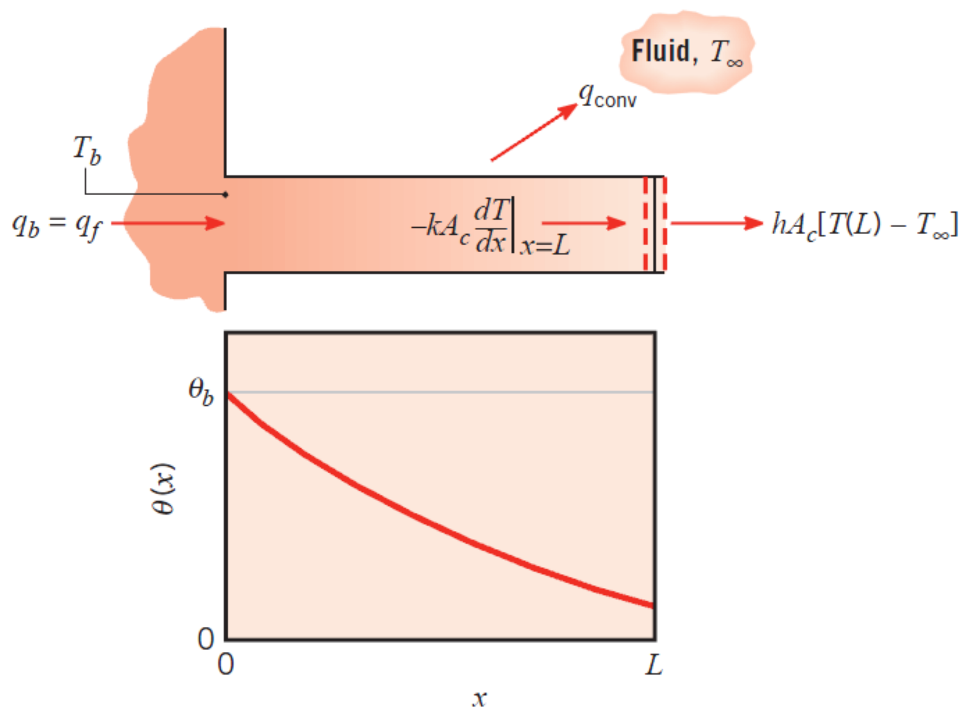
$$\theta(0) = T_b - T_\infty \equiv \theta_b$$

- The second condition, specified at the fin tip $(x = L)$, may correspond to one of four different physical situations.
- The first condition, *Case A*, considers convection heat transfer from the fin tip.

$$hA_c[T(L) - T_\infty] = -kA_c \left.\frac{dT}{dx}\right|_{x=L}$$

$$h\theta(L) = -k \left.\frac{d\theta}{dx}\right|_{x=L}$$

Fins of Uniform Cross-Sectional Area



- Solving for C_1 and C_2, it may be shown, after some manipulation, that

$$\frac{\theta}{\theta_b} = \frac{\cosh m(L - x) + (h/mk) \sinh m(L - x)}{\cosh mL + (h/mk) \sinh mL}$$

- Note that the magnitude of the temperature gradient decreases with increasing x.

Fins of Uniform Cross-Sectional Area

- We are particularly interested in the amount of heat transferred from the entire fin.
- The fin heat transfer rate qf may be evaluated in two alternative ways, both of which involve use of the *temperature distribution*.
- The simpler procedure, and the one that we will use, involves applying Fourier's law at the fin base. That is,

$$q_f = q_b = -kA_c \frac{dT}{dx}\bigg|_{x=0} = -kA_c \frac{d\theta}{dx}\bigg|_{x=0}$$

- However, conservation of energy dictates that the rate at which heat is transferred by convection from the fin must equal the rate at which it is conducted through the base of the fin.

$$q_f = \int_{A_f} h[T(x) - T_\infty]\, dA_s$$

$$q_f = \int_{A_f} h\theta(x)\, dA_s$$

Fins of Uniform Cross-Sectional Area

- Hence, knowing the temperature distribution, $\theta(x)$, q_f may be evaluated, giving

$$q_f = \sqrt{hPkA_c}\,\theta_b \frac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$$

- The second tip condition, Case B, corresponds to the assumption that the convective heat loss from the fin tip is negligible, in which case the tip may be treated as adiabatic and

$$\left.\frac{d\theta}{dx}\right|_{x=L} = 0$$

$$\frac{\theta}{\theta_b} = \frac{\cosh m(L-x)}{\cosh mL}$$

$$q_f = \sqrt{hPkA_c}\,\theta_b \tanh mL$$

Fins of Uniform Cross-Sectional Area

- In the same manner, we can obtain the fin temperature distribution and heat transfer rate for Case C, where the temperature is prescribed at the fin tip. That is,

$$\theta(L) = \theta_L$$

$$\frac{\theta}{\theta_b} = \frac{(\theta_L/\theta_b)\sinh mx + \sinh m(L-x)}{\sinh mL}$$

$$q_f = \sqrt{hPkA_c}\,\theta_b \frac{\cosh mL - \theta_L/\theta_b}{\sinh mL}$$

- The *very long fin Case D*, is an interesting extension of these results. In particular, as $L \to \infty$, $\theta_L \to 0$ and it is easily verified that

$$\frac{\theta}{\theta_b} = e^{-mx}$$

$$q_f = \sqrt{hPkA_c}\,\theta_b$$

Temperature distribution and heat loss for fins of uniform cross section

Case	Tip Condition $(x = L)$	Temperature Distribution θ/θ_b	Fin Heat Transfer Rate q_f
A	Convection heat transfer: $h\theta(L) = -k\,d\theta/dx\rvert_{x=L}$	$\dfrac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$ (3.75)	$M\dfrac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$ (3.77)
B	Adiabatic: $d\theta/dx\rvert_{x=L} = 0$	$\dfrac{\cosh m(L-x)}{\cosh mL}$ (3.80)	$M \tanh mL$ (3.81)
C	Prescribed temperature: $\theta(L) = \theta_L$	$\dfrac{(\theta_L/\theta_b)\sinh mx + \sinh m(L-x)}{\sinh mL}$ (3.82)	$M\dfrac{(\cosh mL - \theta_L/\theta_b)}{\sinh mL}$ (3.83)
D	Infinite fin $(L \rightarrow \infty)$: $\theta(L) = 0$	e^{-mx} (3.84)	M (3.85)

$\theta \equiv T - T_\infty$ $\qquad m^2 \equiv hP/kA_c$

$\theta_b = \theta(0) = T_b - T_\infty$ $\qquad M \equiv \sqrt{hPkA_c}\,\theta_b$

EXAMPLE 3.9

Fin Performance

- The *fin effectiveness* ε_f is defined as the *ratio of the fin heat transfer rate to the heat transfer rate that would exist without the fin*, Therefore

$$\varepsilon_f = \frac{q_f}{hA_{c,b}\theta_b}$$

- In any rational design the value of ε_f should be as large as possible, and in general, the use of fins may rarely be justified unless

$$\varepsilon_f \gtrsim 2$$

- Assuming the convection coefficient of the finned surface to be equivalent to that of the unfinned base, it follows that, for the infinite fin approximation (Case D), the result is

$$\varepsilon_f = \left(\frac{kP}{hA_c}\right)^{1/2}$$

Fin Performance

$$\varepsilon_f = \left(\frac{kP}{hA_c}\right)^{1/2}$$

- Fin effectiveness is enhanced by the choice of a material of high thermal conductivity.
- Fin effectiveness is also enhanced by increasing the ratio of the perimeter to the cross-sectional area. For this reason, the use of *thin*, but closely spaced fins, is preferred, with the proviso that the fin gap not be reduced to a value for which flow between the fins is severely impeded, thereby reducing the convection coefficient.
- The use of fins can be better justified under conditions for which the convection coefficient h is *small*. Hence it is evident that the need for fins is stronger when the fluid is a gas rather than a liquid and when the surface heat transfer is by *free convection*.
- As seen in Example 3.8, 99% of the maximum possible fin heat transfer rate is achieved for $mL = 2.65$. Hence, it would make no sense to extend the fins beyond $L = 2.65/m$.

Fin Performance

- Fin performance may also be quantified in terms of a thermal resistance.
- Treating the difference between the base and fluid temperatures as the driving potential, a *fin resistance* may be defined as

$$R_{t,f} = \frac{\theta_b}{q_f}$$

$$R_{t,b} = \frac{1}{hA_{c,b}}$$

$$\varepsilon_f = \frac{R_{t,b}}{R_{t,f}}$$

- Hence the fin effectiveness may be interpreted as a ratio of thermal resistances, and to increase ε_f it is necessary to reduce the conduction/convection resistance of the fin.

Fin Performance

- Another measure of fin thermal performance is provided by the *fin efficiency* η_f.
- The maximum rate at which a fin could dissipate energy is the rate that would exist if the entire fin surface were at the base temperature. A logical definition of fin efficiency is therefore

$$\eta_f \equiv \frac{q_f}{q_{\max}} = \frac{q_f}{hA_f\theta_b}$$

- For a straight fin of uniform cross section and an adiabatic tip,

$$\eta_f = \frac{M \tanh mL}{hPL\theta_b} = \frac{\tanh mL}{mL}$$

- This result tells us that η_f approaches its maximum and minimum values of 1 and 0, respectively, as L approaches 0 and ∞.

Efficiency of common fin shapes

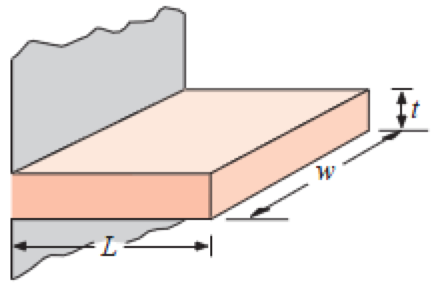
Straight Fins

Rectangular[a]

$A_f = 2wL_c$

$L_c = L + (t/2)$

$A_p = tL$

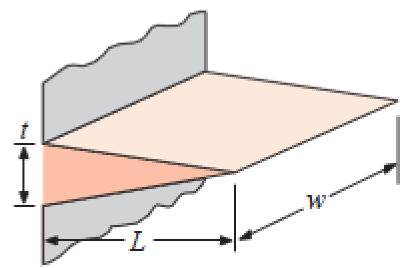


$$\eta_f = \frac{\tanh mL_c}{mL_c}$$

Triangular[a]

$A_f = 2w[L^2 + (t/2)^2]^{1/2}$

$A_p = (t/2)L$



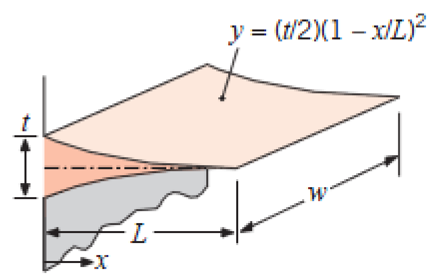
$$\eta_f = \frac{1}{mL}\frac{I_1(2mL)}{I_0(2mL)}$$

Parabolic[a]

$A_f = w[C_1L +$
$(L^2/t)\ln(t/L + C_1)]$

$C_1 = [1 + (t/L)^2]^{1/2}$

$A_p = (t/3)L$



$$\eta_f = \frac{2}{[4(mL)^2 + 1]^{1/2} + 1}$$

Continued

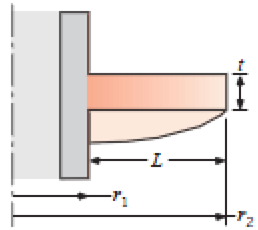
Circular Fin

Rectangular[a]

$A_f = 2\pi\,(r_{2c}^2 - r_1^2)$

$r_{2c} = r_2 + (t/2)$

$V = \pi\,(r_2^2 - r_1^2)t$



$$\eta_f = C_2\,\frac{K_1(mr_1)I_1(mr_{2c}) - I_1(mr_1)K_1(mr_{2c})}{I_0(mr_1)K_1(mr_{2c}) + K_0(mr_1)I_1(mr_{2c})}$$

$$C_2 = \frac{(2r_1/m)}{(r_{2c}^2 - r_1^2)}$$

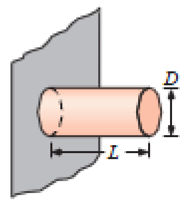
Pin Fins

Rectangular[b]

$A_f = \pi D L_c$

$L_c = L + (D/4)$

$V = (\pi D^2/4)L$

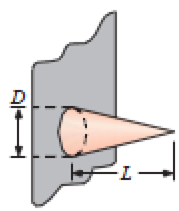


$$\eta_f = \frac{\tanh mL_c}{mL_c}$$

Triangular[b]

$A_f = \frac{\pi D}{2}\,[L^2 + (D/2)^2]^{1/2}$

$V = (\pi/12)D^2 L$



$$\eta_f = \frac{2}{mL}\,\frac{I_2(2mL)}{I_1(2mL)}$$

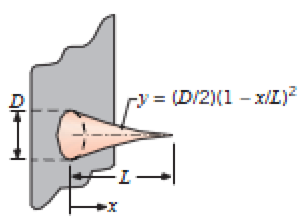
Parabolic[b]

$A_f = \frac{\pi L^3}{8D}\,\{C_3 C_4 - \frac{L}{2D}\ln\,[(2DC_4/L) + C_3]\}$

$C_3 = 1 + 2(D/L)^2$

$C_4 = [1 + (D/L)^2]^{1/2}$

$V = (\pi/20)D^2\,L$



$$\eta_f = \frac{2}{[4/9(mL)^2 + 1]^{1/2} + 1}$$

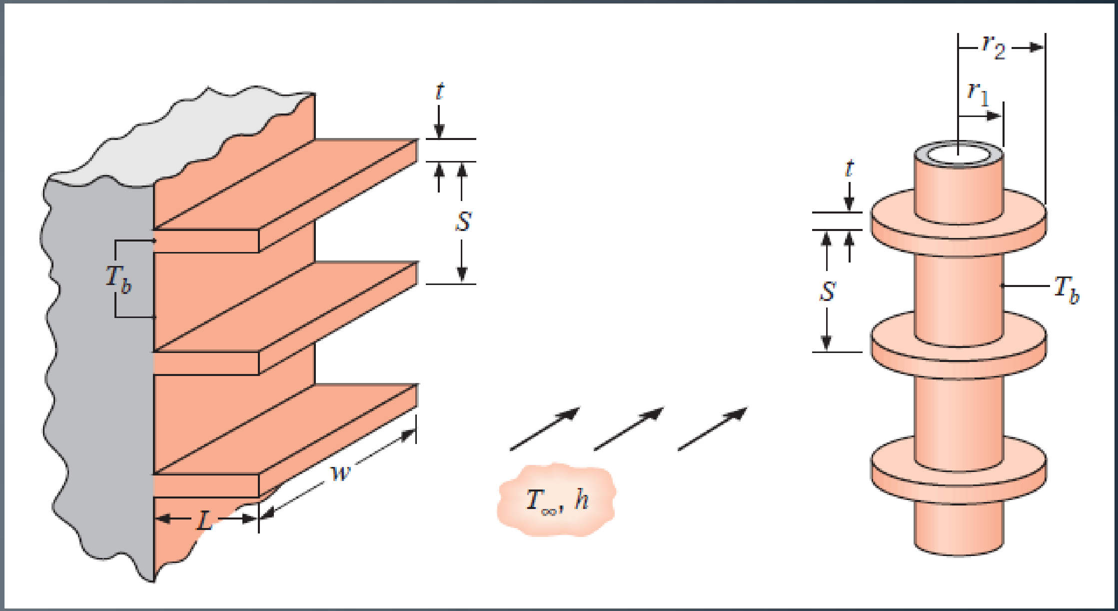
---

[a] $m = (2h/kt)^{1/2}$.

[b] $m = (4h/kD)^{1/2}$.

Overall Surface Efficiency

- In contrast to the fin efficiency η_f, which characterizes the performance of a single fin, the *overall surface efficiency* η_o characterizes an *array* of fins and the base surface to which they are attached.



$$\eta_o = \frac{q_t}{q_{\max}} = \frac{q_t}{hA_t\theta_b}$$

$$A_t = NA_f + A_b$$

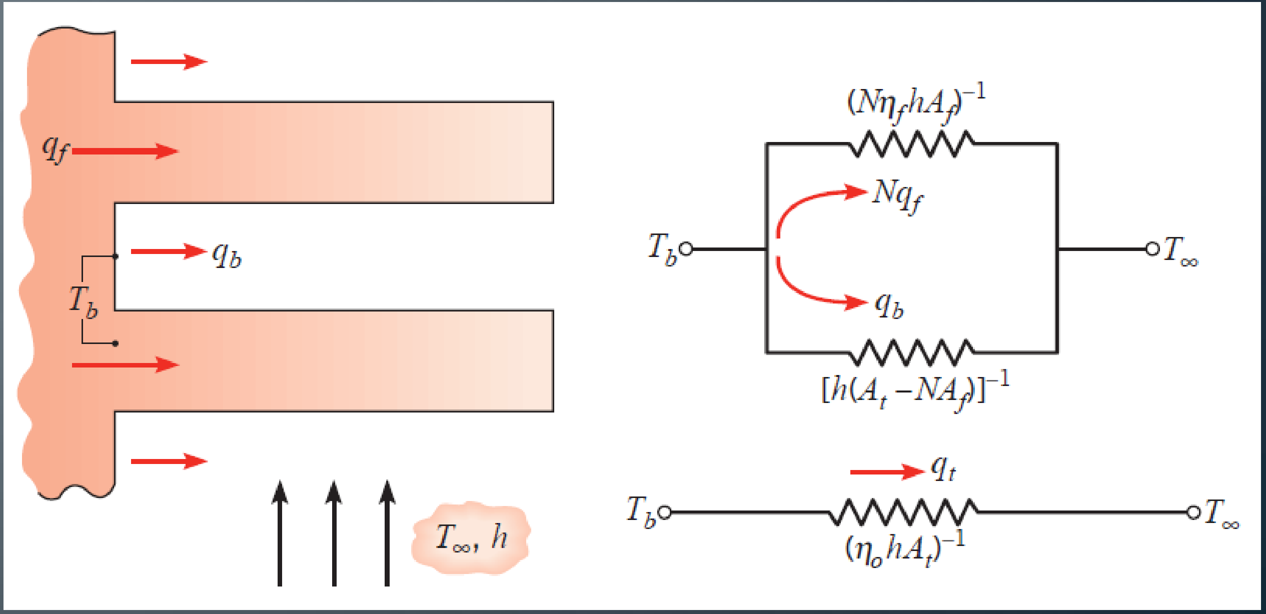
$$q_t = N\eta_f hA_f\theta_b + hA_b\theta_b$$

Overall Surface Efficiency

$$\eta_o = 1 - \frac{NA_f}{A_t}(1 - \eta_f)$$

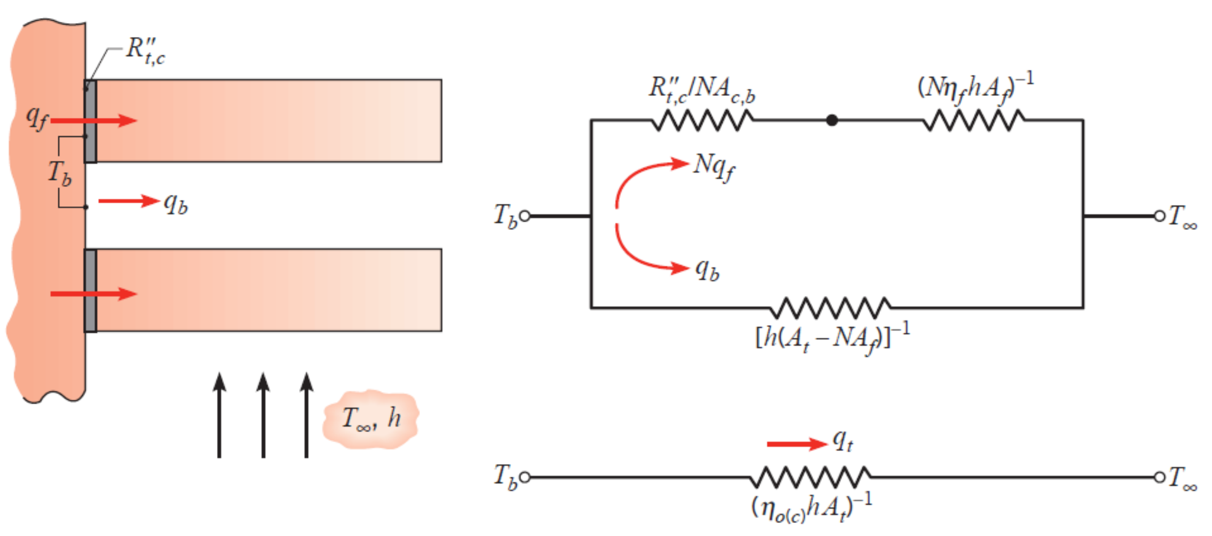
- Recalling the definition of the fin thermal resistance,

$$R_{t,o} = \frac{\theta_b}{q_t} = \frac{1}{\eta_o h A_t}$$



Overall Surface Efficiency

- If fins are machined as an integral part of the wall from which they extend, there is no contact resistance at their base. However, more commonly, fins are manufactured separately and are attached to the wall by a metallurgical or adhesive joint.



$$R_{t,o(c)} = \frac{\theta_b}{q_t} = \frac{1}{\eta_{o(c)} h A_t}$$

$$\eta_{o(c)} = 1 - \frac{NA_f}{A_t}\left(1 - \frac{\eta_f}{C_1}\right)$$

$$C_1 = 1 + \eta_f h A_f (R''_{t,c}/A_{c,b})$$

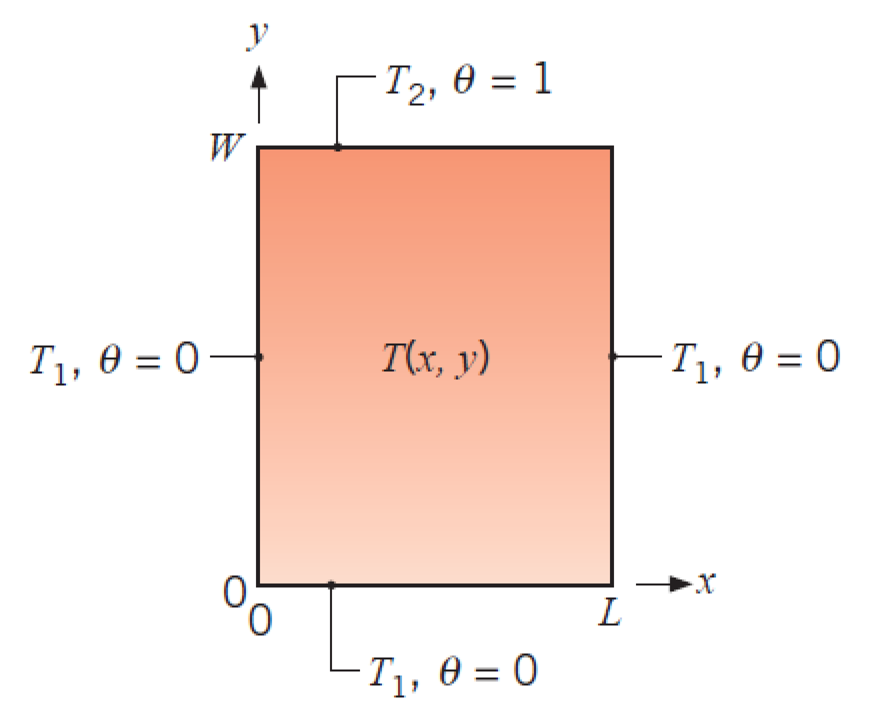
EXAMPLE 3.10

Two-Dimensional, Steady-State Conduction

Chapter 4

The Method of Separation of Variables

- Three sides of a thin rectangular plate or a long rectangular rod are maintained at a constant temperature T_1, while the fourth side is maintained at a constant temperature $T_2 \neq T_1$.



$$\theta \equiv \frac{T - T_1}{T_2 - T_1}$$

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$$

The Method of Separation of Variables

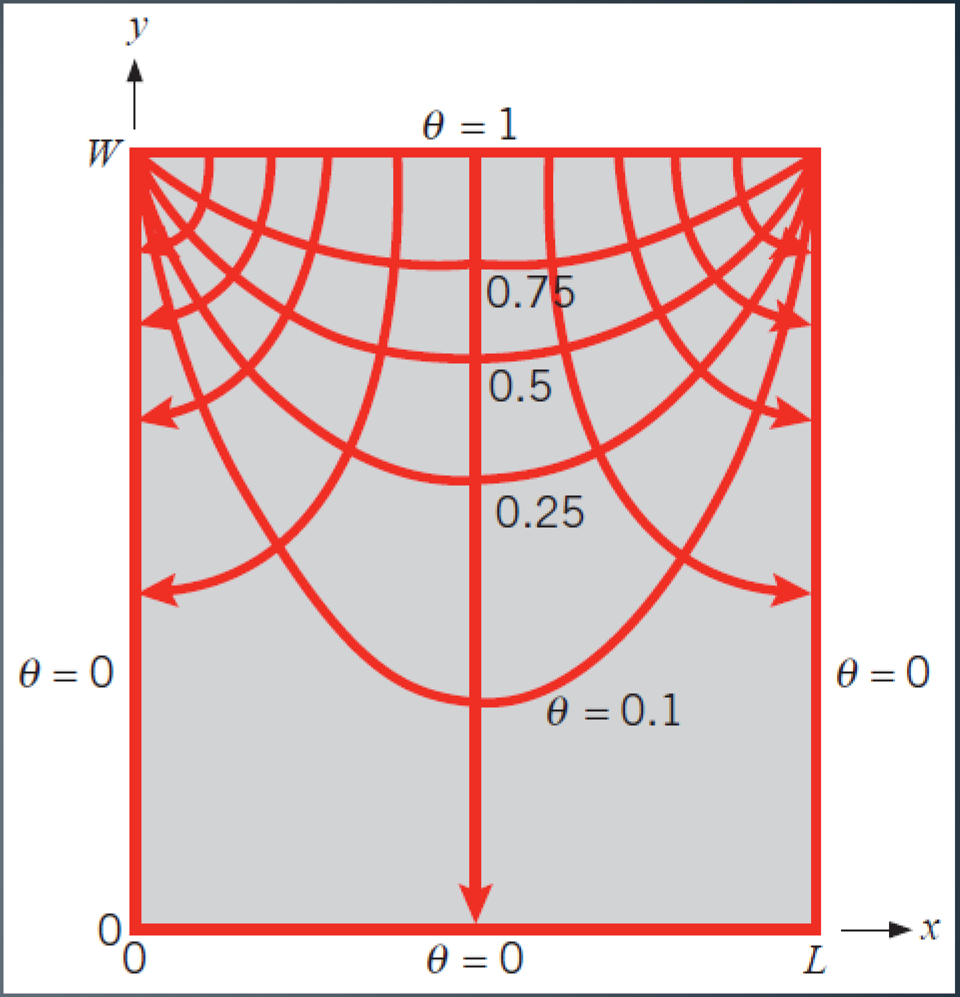
- Since the equation is second order in both x and y, two boundary conditions are needed for each of the coordinates. They are

$$\theta(0, y) = 0 \quad \text{and} \quad \theta(x, 0) = 0$$
$$\theta(L, y) = 0 \quad \text{and} \quad \theta(x, W) = 1$$

- We now apply the separation of variables technique by assuming that the desired solution can be expressed as the product of two functions, one of which depends only on x while the other depends only on y.

$$\theta(x, y) = X(x) \cdot Y(y)$$

The Method of Separation of Variables



$$\theta(x, y) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} + 1}{n} \sin \frac{n\pi x}{L} \frac{\sinh (n\pi y/L)}{\sinh (n\pi W/L)}$$

The Conduction Shape Factor

- In many instances, two- or three-dimensional conduction problems may be rapidly solved by utilizing *existing* solutions to the heat diffusion equation.

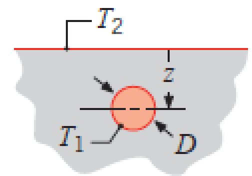
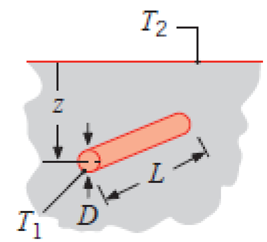
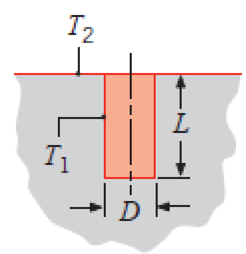
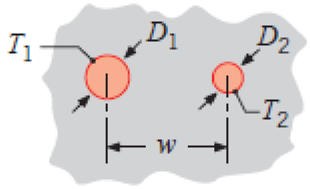
- These solutions are reported in terms of a *shape factor S*. The shape factor is defined such that

$$q = Sk\Delta T_{1-2}$$

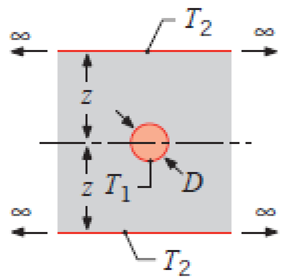
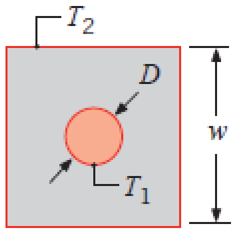
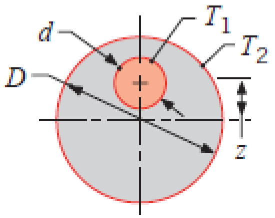
- It also follows that a two-dimensional conduction resistance may be expressed as

$$R_{t,\text{cond(2D)}} = \frac{1}{Sk}$$

Conduction shape factors for selected systems.

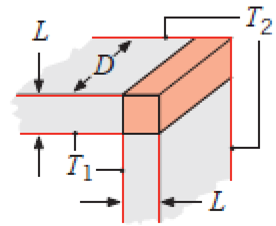
System	Schematic	Restrictions	Shape Factor
Case 1 Isothermal sphere buried in a semi-infinite medium		$z > D/2$	$\dfrac{2\pi D}{1 - D/4z}$
Case 2 Horizontal isothermal cylinder of length L buried in a semi-infinite medium		$L \gg D$	$\dfrac{2\pi L}{\cosh^{-1}(2z/D)}$
		$L \gg D$ $z > 3D/2$	$\dfrac{2\pi L}{\ln(4z/D)}$
Case 3 Vertical cylinder in a semi-infinite medium		$L \gg D$	$\dfrac{2\pi L}{\ln(4L/D)}$
Case 4 Conduction between two cylinders of length L in infinite medium		$L \gg D_1, D_2$ $L \gg w$	$\dfrac{2\pi L}{\cosh^{-1}\left(\dfrac{4w^2 - D_1^2 - D_2^2}{2D_1D_2}\right)}$

Continued

System	Schematic	Restrictions	Shape Factor
Case 5 Horizontal circular cylinder of length L midway between parallel planes of equal length and infinite width		$z \gg D/2$ $L \gg z$	$\dfrac{2\pi L}{\ln(8z/\pi D)}$
Case 6 Circular cylinder of length L centered in a square solid of equal length		$w > D$ $L \gg w$	$\dfrac{2\pi L}{\ln(1.08\, w/D)}$
Case 7 Eccentric circular cylinder of length L in a cylinder of equal length		$D > d$ $L \gg D$	$\dfrac{2\pi L}{\cosh^{-1}\left(\dfrac{D^2 + d^2 - 4z^2}{2Dd}\right)}$

Case 8

Conduction through the edge of adjoining walls

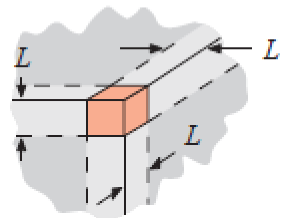


$D > 5L$

$0.54D$

Case 9

Conduction through corner of three walls with a temperature difference ΔT_{1-2} across the walls

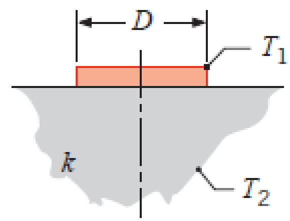


$L \ll$ length and width of wall

$0.15L$

Case 10

Disk of diameter D and temperature T_1 on a semi-infinite medium of thermal conductivity k and temperature T_2

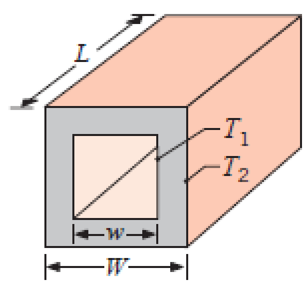


None

$2D$

Case 11

Square channel of length L



$\frac{W}{w} < 1.4$: $\dfrac{2\pi L}{0.785 \ln (W/w)}$

$\frac{W}{w} > 1.4$: $\dfrac{2\pi L}{0.930 \ln (W/w) - 0.050}$

$L \gg W$

EXAMPLE 4.1

Finite-Difference Equations

- As discussed in previous section, analytical methods may be used, in certain cases, to effect exact mathematical solutions to steady, two-dimensional conduction problems.

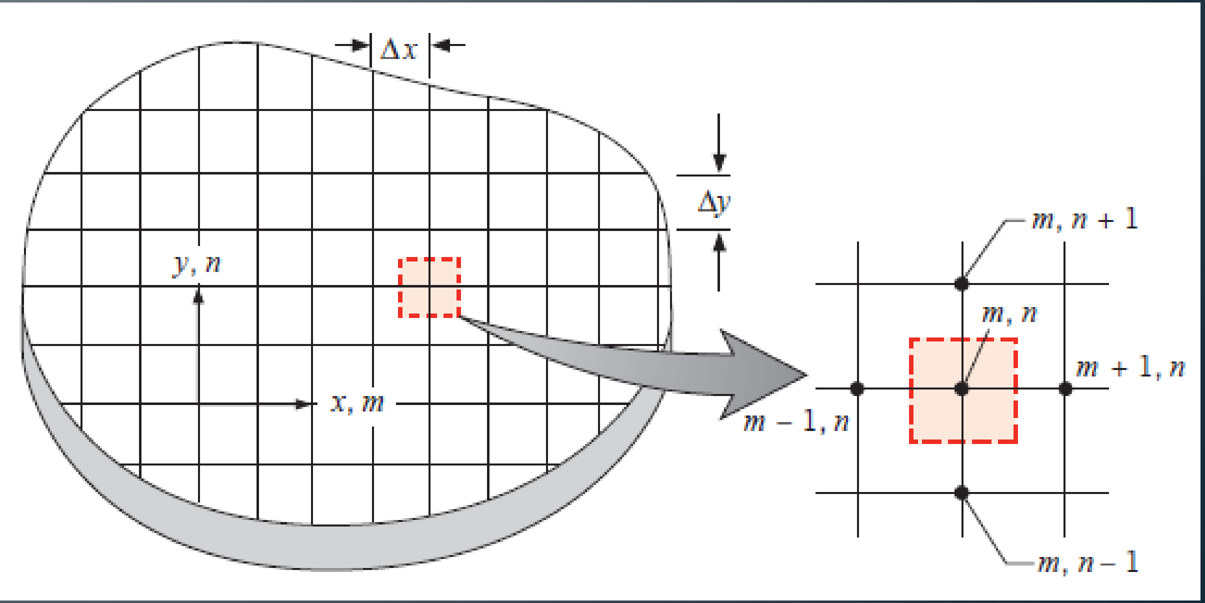
- These solutions have been generated for an assortment of simple geometries and boundary conditions and are well documented in the literature. However, more often than not, two-dimensional problems involve geometries and/or boundary conditions that preclude such solutions.

- In these cases, the best alternative is often one that uses a *numerical technique* such as the *finite-difference*, *finite-element* or *boundary-element method*. Another strength of numerical methods is that they can be readily extended to three-dimensional problems.

The Nodal Network

- In contrast to an analytical solution, which allows for temperature determination at *any* point of interest in a medium, a numerical solution enables determination of the temperature at only *discrete* points.

- The first step in any numerical analysis must therefore be to select these points. Referring to the below figure, this may be done by subdividing the medium of interest into a number of *small regions* and assigning to each a *reference point* that is at its *center*.



The Nodal Network

- The reference point is frequently termed a *nodal point* (or simply a *node*), and the aggregate of points is termed a *nodal network*, *grid*, or *mesh*.

- The x and y locations are designated by the m and n indices, respectively.

- Each node represents a certain region, and its temperature is a measure of the *average* temperature of the region.

- The selection of nodal points is rarely arbitrary, depending often on matters such as geometric convenience and the desired accuracy.

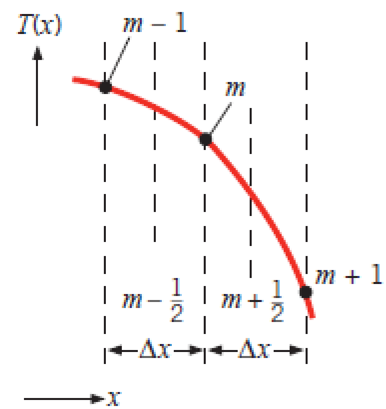
- The numerical accuracy of the calculations depends strongly on the number of designated nodal points. If this number is large (a *fine mesh*), accurate solutions can be obtained.

Finite-Difference Form of the Heat Equation

- Determination of the temperature distribution numerically dictates that an appropriate conservation equation be written for *each* of the nodal points of unknown temperature.
- The resulting set of equations may then be solved simultaneously for the temperature at each node.
- For *any interior* node of a two-dimensional system with *no generation* and *uniform thermal conductivity*, the *exact* form of the energy conservation requirement is given by the *heat equation*.
- However, if the system is characterized in terms of a nodal network, it is necessary to work with an *approximate*, or *finite-difference* form of this equation.

$$\left.\frac{\partial T}{\partial x}\right|_{m-1/2,n} = \frac{T_{m,n} - T_{m-1,n}}{\Delta x}$$

$$\left.\frac{\partial T}{\partial x}\right|_{m+1/2,n} = \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$



Finite-Difference Form of the Heat Equation

- Finite-difference equation that is suitable for the interior nodes of a two-dimensional system may be inferred directly from the heat equation.

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} \approx \frac{\partial T/\partial x|_{m+1/2,n} - \partial T/\partial x|_{m-1/2,n}}{\Delta x}$$

$$\left.\frac{\partial T}{\partial x}\right|_{m+1/2,n} \approx \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$\left.\frac{\partial T}{\partial x}\right|_{m-1/2,n} \approx \frac{T_{m,n} - T_{m-1,n}}{\Delta x}$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} \approx \frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2}$$

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{m,n} \approx \frac{\partial T/\partial y|_{m,n+1/2} - \partial T/\partial y|_{m,n-1/2}}{\Delta y} \approx \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2}$$

Finite-Difference Form of the Heat Equation

- Using a network for which $\Delta x = \Delta y$ and substituting the above equations into the heat equation, we obtain

$$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} - 4T_{m,n} = 0$$

- Hence for the (m, n) node, the heat equation, which is an *exact differential equation*, is reduced to an *approximate algebraic equation*. This *approximate, finite-difference form of the heat equation* may be applied to any interior node that is equidistant from its four neighboring nodes.

- It requires simply that the temperature of an interior node be equal to the average of the temperatures of the four neighboring nodes.

The Energy Balance Method

- In many cases, it is desirable to develop the finite-difference equations by an alternative method called the *energy balance method*.

- As will become evident, this approach enables one to analyze many different phenomena such as problems involving *multiple materials*, *embedded heat sources*, or *exposed surfaces that do not align with an axis of the coordinate system*.

- In the energy balance method, the finite-difference equation for a node is obtained by applying conservation of energy to a control volume about the nodal region.

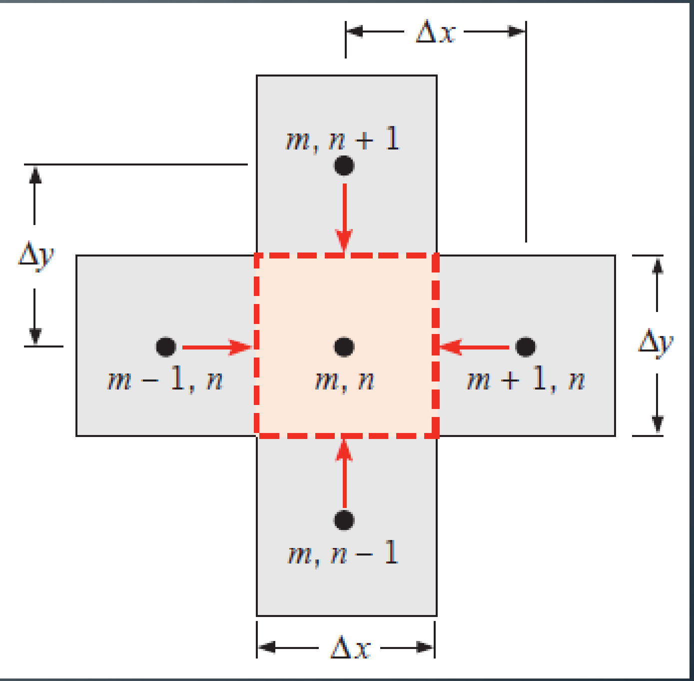
- Since the actual direction of heat flow (into or out of the node) is often unknown, it is convenient to formulate the energy balance by *assuming* that *all* the heat flow is *into the node*.

The Energy Balance Method

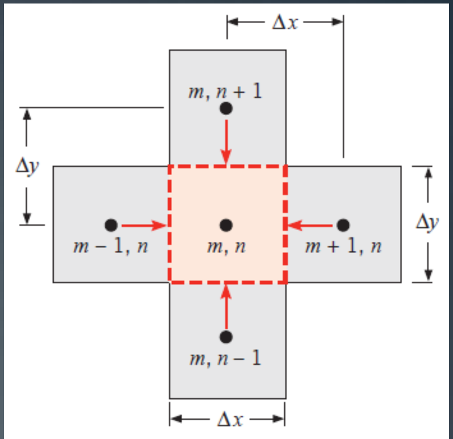
- For steady-state conditions with generation, the appropriate form of the heat equation is then

$$\dot{E}_{in} + \dot{E}_g = 0$$

$$\sum_{i=1}^{4} q_{(i) \to (m,n)} + \dot{q}(\Delta x \cdot \Delta y \cdot 1) = 0$$



The Energy Balance Method



$$q_{(m-1,n)\to(m,n)} = k(\Delta y \cdot 1)\frac{T_{m-1,n} - T_{m,n}}{\Delta x}$$

$$q_{(m+1,n)\to(m,n)} = k(\Delta y \cdot 1)\frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

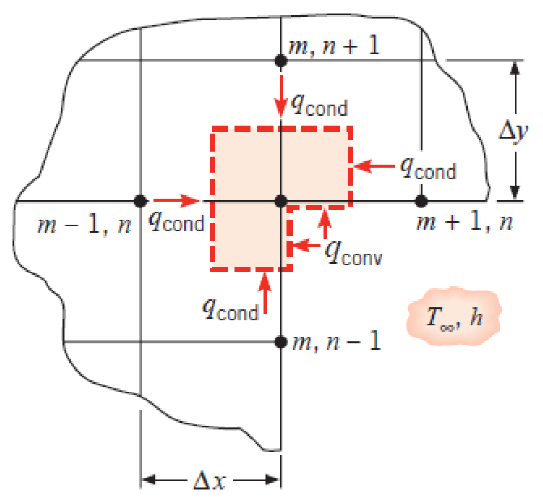
$$q_{(m,n+1)\to(m,n)} = k(\Delta x \cdot 1)\frac{T_{m,n+1} - T_{m,n}}{\Delta y}$$

$$q_{(m,n-1)\to(m,n)} = k(\Delta x \cdot 1)\frac{T_{m,n-1} - T_{m,n}}{\Delta y}$$

- Substituting the above equations into the energy balance and remembering that $\Delta x = \Delta y$, it follows that the finite-difference equation for an interior node with generation is

$$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} + \frac{\dot{q}(\Delta x)^2}{k} - 4T_{m,n} = 0$$

The Energy Balance Method



$$q_{(m-1,n)\to(m,n)} = k(\Delta y \cdot 1)\frac{T_{m-1,n} - T_{m,n}}{\Delta x}$$

$$q_{(m,n+1)\to(m,n)} = k(\Delta x \cdot 1)\frac{T_{m,n+1} - T_{m,n}}{\Delta y}$$

$$q_{(m+1,n)\to(m,n)} = k\left(\frac{\Delta y}{2}\cdot 1\right)\frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

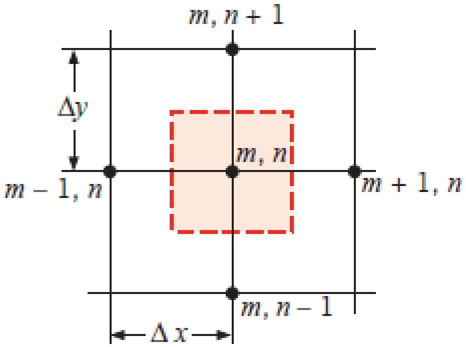
$$q_{(m,n-1)\to(m,n)} = k\left(\frac{\Delta x}{2}\cdot 1\right)\frac{T_{m,n-1} - T_{m,n}}{\Delta y}$$

$$q_{(\infty)\to(m,n)} = h\left(\frac{\Delta x}{2}\cdot 1\right)(T_\infty - T_{m,n}) + h\left(\frac{\Delta y}{2}\cdot 1\right)(T_\infty - T_{m,n})$$

$$T_{m-1,n} + T_{m,n+1} + \frac{1}{2}(T_{m+1,n} + T_{m,n-1}) + \frac{h\Delta x}{k}T_\infty - \left(3 + \frac{h\Delta x}{k}\right)T_{m,n} = 0$$

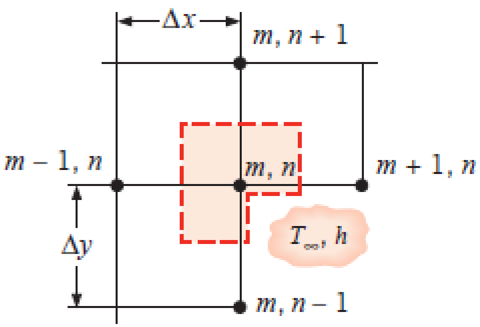
Summary of nodal finite-difference equations

Configuratio | **Finite-Difference Equation for $\Delta x = \Delta y$**



$$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} - 4T_{m,n} = 0$$

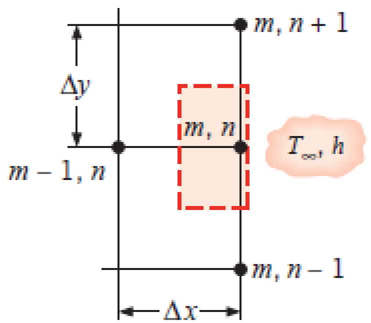
Case 1. Interior node



$$2(T_{m-1,n} + T_{m,n+1}) + (T_{m+1,n} + T_{m,n-1})$$

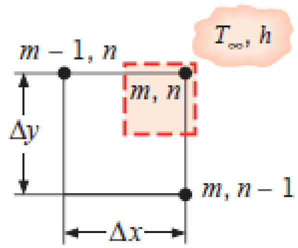
$$+ 2\frac{h\,\Delta x}{k}T_\infty - 2\left(3 + \frac{h\,\Delta x}{k}\right)T_{m,n} = 0$$

Case 2. Node at an internal corner with convection



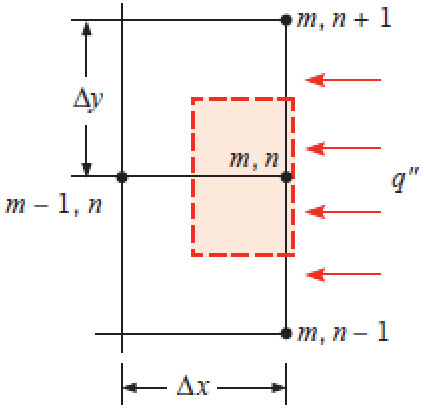
$$(2T_{m-1,n} + T_{m,n+1} + T_{m,n-1}) + \frac{2h\,\Delta x}{k}T_\infty - 2\left(\frac{h\,\Delta x}{k} + 2\right)T_{m,n} = 0$$

Case 3. Node at a plane surface with convection



$$(T_{m,n-1} + T_{m-1,n}) + 2\frac{h\,\Delta x}{k}T_\infty - 2\left(\frac{h\,\Delta x}{k} + 1\right)T_{m,n} = 0$$

Case 4. Node at an external corner with convection



$$(2T_{m-1,n} + T_{m,n+1} + T_{m,n-1}) + \frac{2q''\,\Delta x}{k} - 4T_{m,n} = 0$$

Case 5. Node at a plane surface with uniform heat flux

[a,b] To obtain the finite-difference equation for an adiabatic surface (or surface of symmetry), simply set h or q'' equal to zero.

EXAMPLE 4.2

Solving the Finite-Difference Equations

- **Formulation as a Matrix Equation**

- Once the nodal network has been established and an appropriate finite-difference equation has been written for each node, the temperature distribution may be determined.

- The problem reduces to one of solving a system of linear, algebraic equations.

$$
\begin{aligned}
a_{11}T_1 + a_{12}T_2 + a_{13}T_3 + \cdots + a_{1N}T_N &= C_1 \\
a_{21}T_1 + a_{22}T_2 + a_{23}T_3 + \cdots + a_{2N}T_N &= C_2 \\
\vdots \qquad \vdots \qquad \vdots \qquad & \vdots \\
a_{N1}T_1 + a_{N2}T_2 + a_{N3}T_3 + \cdots + a_{NN}T_N &= C_N
\end{aligned}
$$

$$[A][T] = [C]$$

$$
A \equiv \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}, \qquad T \equiv \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_N \end{bmatrix}, \qquad C \equiv \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix}
$$

Solving the Finite-Difference Equations

- **Verifying the Accuracy of the Solution**

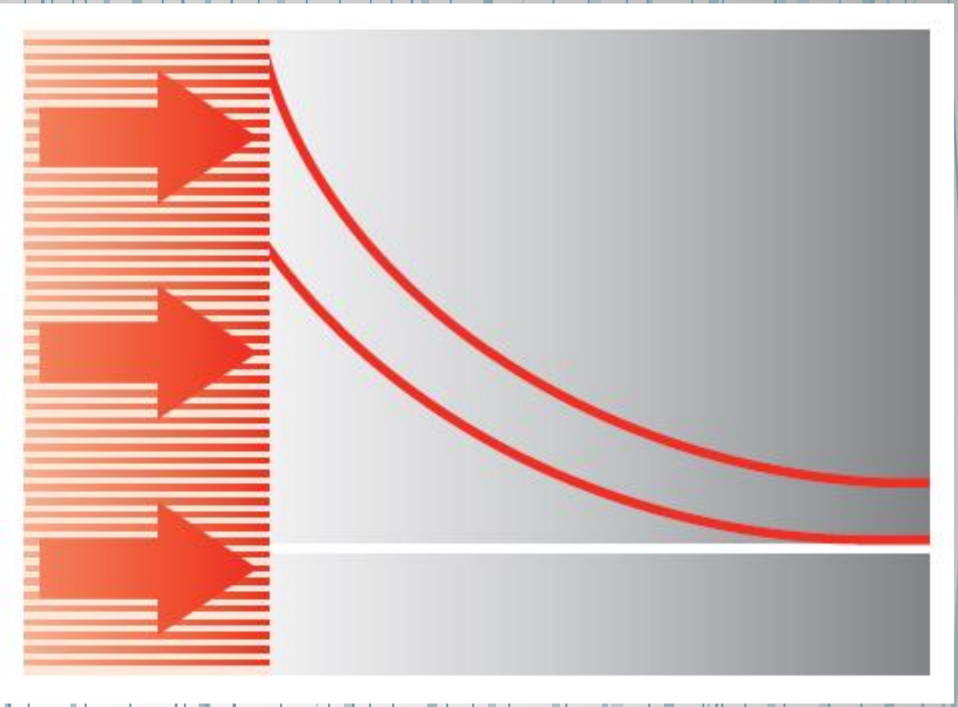
- It is good practice to verify that a numerical solution has been correctly formulated by performing an energy balance on a control surface surrounding all nodal regions whose temperatures have been evaluated.

- Even when the finite-difference equations have been properly formulated and solved, the results may still represent a coarse approximation to the actual temperature field. Hence, if accurate results are desired, grid studies should be performed. Such *grid-independent* results would provide an accurate solution to the physical problem.

- Another option for validating a numerical solution involves comparing results with those obtained from an exact solution.

EXAMPLE 4.3

Problems



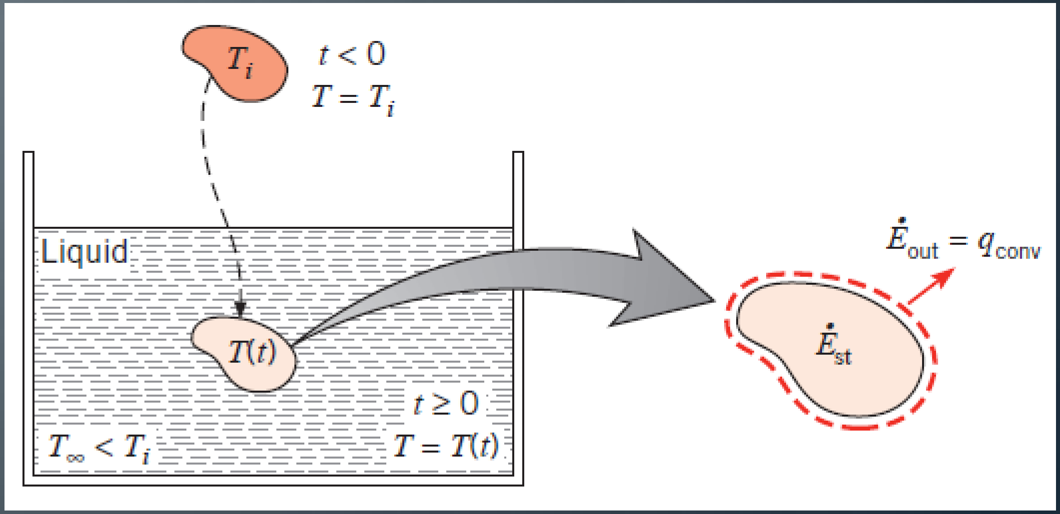
Transient Conduction

Chapter 5

The Lumped Capacitance Method

- A simple, yet common, transient conduction problem is one for which a solid experiences a sudden change in its thermal environment.

- Consider a hot metal forging that is initially at a uniform temperature T_i and is quenched by immersing it in a liquid of lower temperature $T_\infty < T_i$. If the quenching is said to begin at time $t = 0$, the temperature of the solid will decrease for time $t > 0$, until it eventually reaches T_∞.

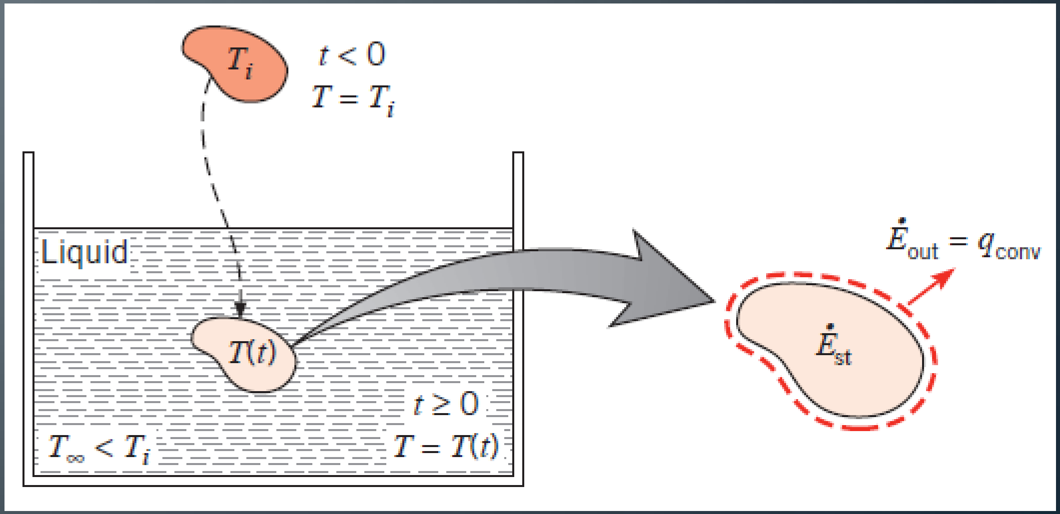


The Lumped Capacitance Method

- The essence of the lumped capacitance method is the assumption that the temperature of the solid is *spatially uniform* at any instant during the transient process.

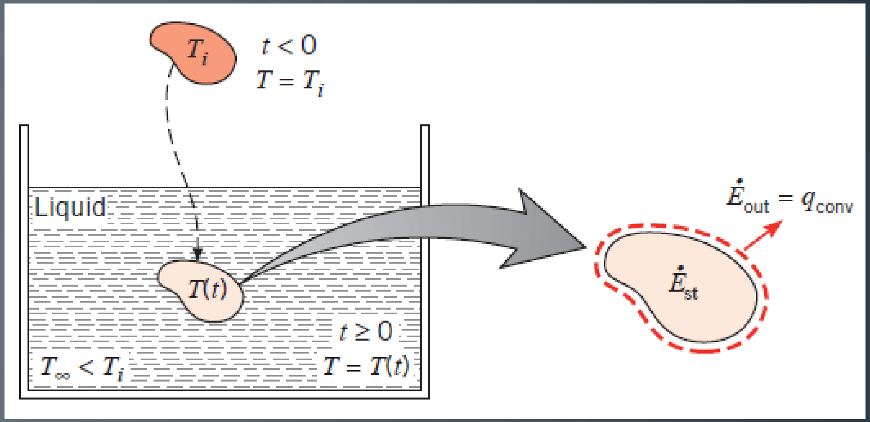
- From Fourier's law, heat conduction in the absence of a temperature gradient implies the existence of *infinite thermal conductivity*.

- Such a condition is clearly impossible. However, the condition is closely approximated if the resistance to conduction within the solid is small compared with the resistance to heat transfer between the solid and its surroundings.



The Lumped Capacitance Method

- The transient temperature response is determined by formulating an overall energy balance on the entire solid.



$$-\dot{E}_{\text{out}} = \dot{E}_{\text{st}}$$

$$-hA_s(T - T_\infty) = \rho V c \frac{dT}{dt}$$

The Lumped Capacitance Method

- Introducing the temperature difference

$$\theta \equiv T - T_\infty$$

$$\frac{\rho V c}{h A_s} \frac{d\theta}{dt} = -\theta$$

$$\frac{\rho V c}{h A_s} \int_{\theta_i}^{\theta} \frac{d\theta}{\theta} = -\int_0^t dt$$

- Where

$$\theta_i \equiv T_i - T_\infty$$

The Lumped Capacitance Method

- Evaluating the integrals, it follows that

$$\frac{\rho V c}{h A_s} \ln \frac{\theta_i}{\theta} = t$$

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp\left[- \left(\frac{h A_s}{\rho V c} \right) t \right]$$

- The quantity $(\rho V c / h A_s)$ may be interpreted as a *thermal time constant* expressed as

$$\tau_t = \left(\frac{1}{h A_s} \right) (\rho V c) = R_t C_t$$

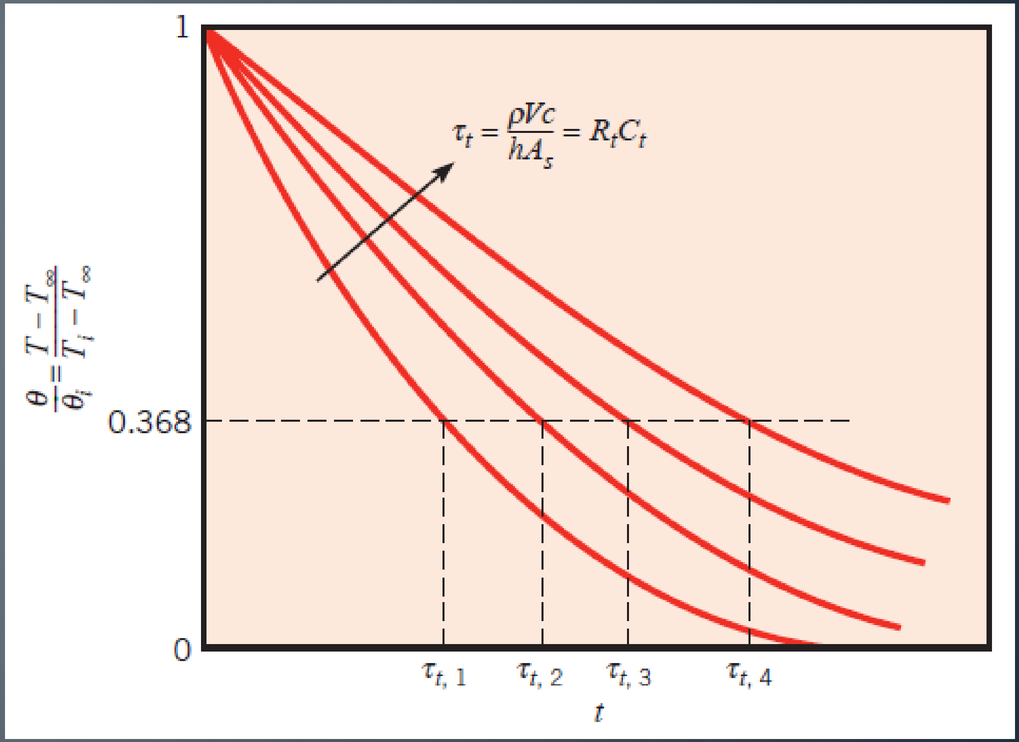
- where R_t is the *resistance to convection heat transfer* and C_t is the *lumped thermal capacitance* of the solid.

- Any increase in R_t or C_t will cause a solid to respond more *slowly* to changes in its thermal environment.

The Lumped Capacitance Method

- To determine the total energy transfer Q occurring up to some time t, we simply write

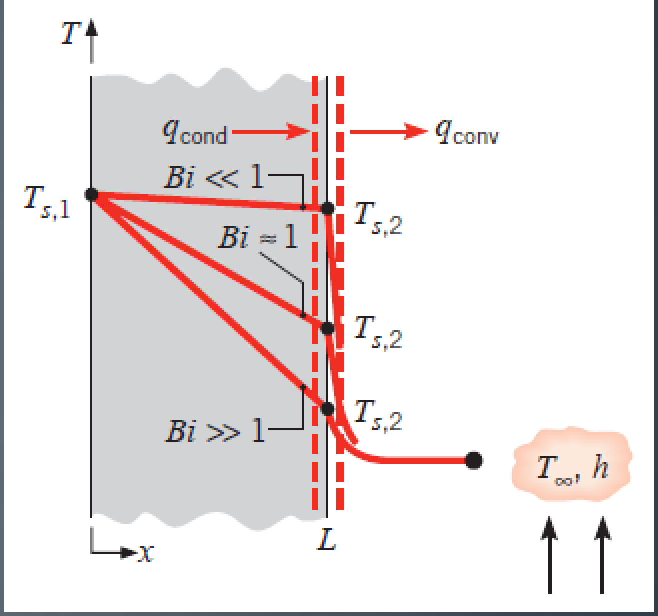
$$Q = \int_0^t q\,dt = hA_s \int_0^t \theta\,dt$$



$$Q = (\rho V c)\theta_i \left[1 - \exp\left(-\frac{t}{\tau_t}\right)\right]$$

Validity of the Lumped Capacitance Method

- It is important to determine under what conditions the lumped capacitance method may be used with reasonable accuracy.
- To develop a suitable criterion consider steady-state conduction through the plane wall of area *A*.



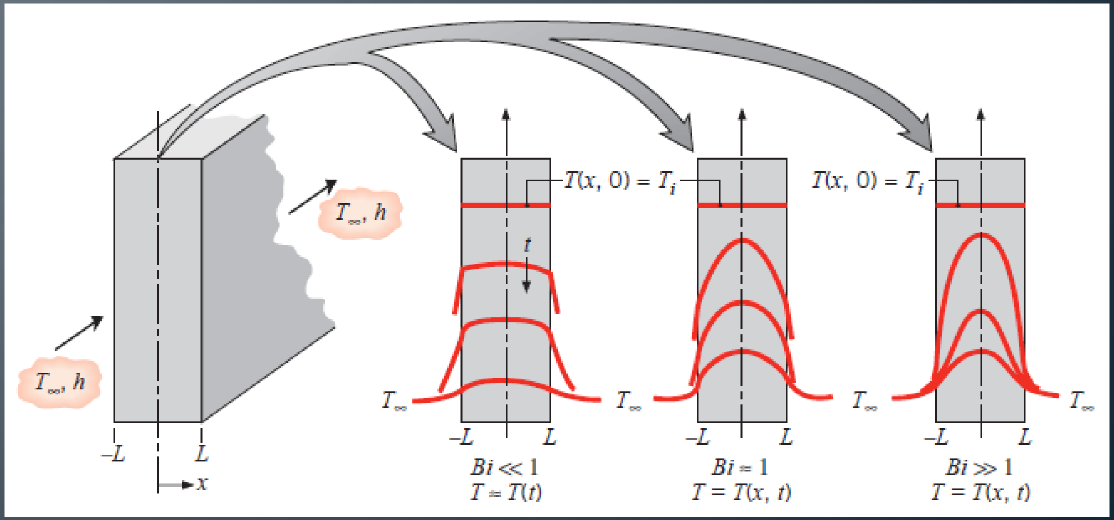
$$\frac{kA}{L}(T_{s,1} - T_{s,2}) = hA(T_{s,2} - T_\infty)$$

$$\frac{T_{s,1} - T_{s,2}}{T_{s,2} - T_\infty} = \frac{(L/kA)}{(1/hA)} = \frac{R_{t,cond}}{R_{t,conv}} = \frac{hL}{k} \equiv Bi$$

Validity of the Lumped Capacitance Method

- The quantity (hL_c/k) is a dimensionless parameter. It is termed the *Biot number*.
- The Biot number provides a measure of the *temperature drop in the solid* relative to the *temperature difference between the solid's surface and the fluid*.
- It is also evident that the Biot number may be interpreted as a *ratio of thermal resistances*.
- If the following condition is satisfied the error associated with using the lumped capacitance method is small.

$$Bi = \frac{hL_c}{k} < 0.1$$



Validity of the Lumped Capacitance Method

- For convenience, it is customary to define the *characteristic length* as the ratio of the solid's volume to surface area

$$L_c \equiv V/A_s$$

- Finally, we note that, with $L_c \equiv V/A_s$, the exponent of Equation 5.6 may be expressed as

$$\frac{hA_s t}{\rho V c} = \frac{ht}{\rho c L_c} = \frac{hL_c}{k}\frac{k}{\rho c}\frac{t}{L_c^2} = \frac{hL_c}{k}\frac{\alpha t}{L_c^2}$$

$$\frac{hA_s t}{\rho V c} = Bi \cdot Fo$$

- where

$$Fo \equiv \frac{\alpha t}{L_c^2}$$

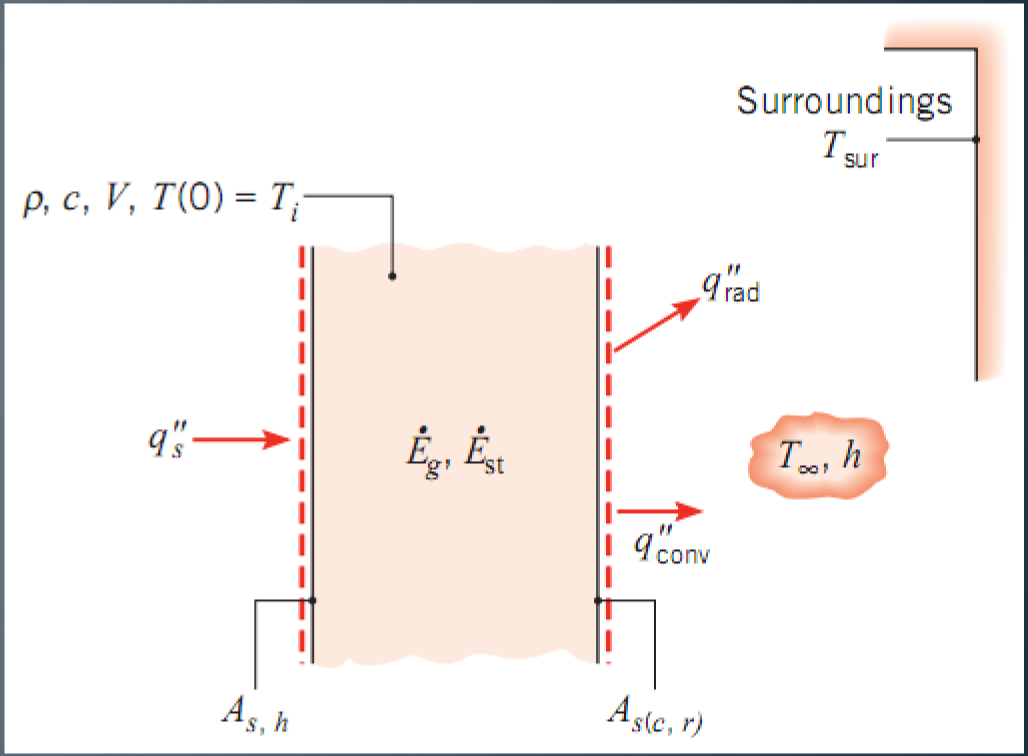
- is a dimensionless time termed the *Fourier number*.

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp(-Bi \cdot Fo)$$

EXAMPLE 5.1

General Lumped Capacitance Analysis

- The below figure depicts the general situation for which thermal conditions within a solid may be influenced simultaneously by *convection*, *radiation*, an *applied surface heat flux*, and *internal energy generation*.

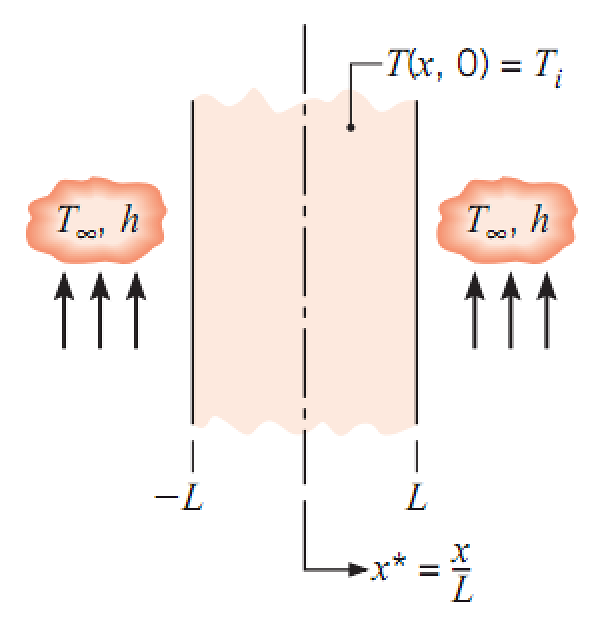


$$q''_s A_{s,h} + \dot{E}_g - [h(T - T_\infty) + \varepsilon\sigma(T^4 - T_{sur}^4)]A_{s(c,r)} = \rho V c \frac{dT}{dt}$$

EXAMPLE 5.2

Spatial Effects

- Situations frequently arise for which the Biot number is *not small*, and we must cope with the fact that temperature gradients within the medium are no longer negligible.



$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Spatial Effects

- To solve the above equation for the temperature distribution $T(x,t)$, it is necessary to specify *an initial condition* and *two boundary conditions*.

$$T(x, 0) = T_i$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h[T(L, t) - T_\infty]$$

- It is evident that, in addition to depending on x and t, temperatures in the wall also depend on a number of physical parameters. In particular

$$T = T(x, t, T_i, T_\infty, L, k, \alpha, h)$$

Spatial Effects

- It is important to note the advantages that may be obtained by *nondimensionalizing* the governing equations. This may be done by arranging the relevant variables into suitable *groups*.

$$\theta^* \equiv \frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty}$$

$$x^* \equiv \frac{x}{L}$$

$$t^* \equiv \frac{\alpha t}{L^2} \equiv Fo$$

- Substituting the definitions, the heat equation and the initial and boundary conditions become

Spatial Effects

$$\frac{\partial^2\theta^*}{\partial x^{*2}} = \frac{\partial\theta^*}{\partial Fo}$$

$$\theta^*(x^*, 0) = 1$$

$$\left.\frac{\partial\theta^*}{\partial x^*}\right|_{x^*=0} = 0$$

$$\left.\frac{\partial\theta^*}{\partial x^*}\right|_{x^*=1} = -Bi\,\theta^*(1, t^*)$$

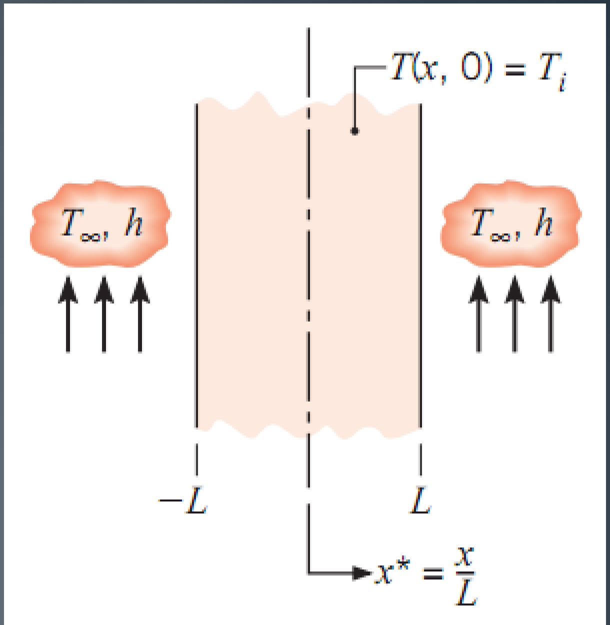
- In dimensionless form the functional dependence may now be expressed as

$$\theta^* = f(x^*, Fo, Bi)$$

- The above equation implies that for a *prescribed geometry, the transient temperature distribution is a universal function of x^*, Fo, and Bi*.

The Plane Wall with Convection

- **Exact Solution**



$$\theta^* = \sum_{n=1}^{\infty} C_n \exp\left(-\zeta_n^2 Fo\right) \cos\left(\zeta_n x^*\right)$$

$$C_n = \frac{4 \sin \zeta_n}{2\zeta_n + \sin(2\zeta_n)}$$

$$\zeta_n \tan \zeta_n = Bi$$

The Plane Wall with Convection

- **Approximate Solution**

- It can be shown that for values of $Fo > 0.2$, the infinite series solution can be approximated by the first term of the series, $n = 1$.

$$\theta^* = C_1 \exp\left(-\zeta_1^2 Fo\right) \cos\left(\zeta_1 x^*\right)$$

$$\theta^* = \theta_o^* \cos\left(\zeta_1 x^*\right)$$

$$\theta_o^* \equiv (T_o - T_\infty)/(T_i - T_\infty)$$

$$\theta_o^* = C_1 \exp\left(-\zeta_1^2 Fo\right)$$

- An important implication of the above equation is that *the time dependence of the temperature at any location within the wall is the same as that of the midplane temperature*.

The Plane Wall with Convection

- **Total Energy Transfer**

- In many situations it is useful to know the total energy that has left (or entered) the wall up to any time *t* in the transient process.

$$E_{in} - E_{out} = \Delta E_{st}$$

$$Q = -[E(t) - E(0)]$$

$$Q = -\int \rho c[T(x, t) - T_i]dV$$

- It is convenient to nondimensionalize this result by introducing the quantity

$$Q_o = \rho c V(T_i - T_\infty)$$

The Plane Wall with Convection

- **Total Energy Transfer**

- which may be interpreted as the initial internal energy of the wall relative to the fluid temperature. It is also the *maximum* amount of energy transfer that could occur if the process were continued to time $t = \infty$.

$$\frac{Q}{Q_o} = \int \frac{-[T(x,t) - T_i]}{T_i - T_\infty} \frac{dV}{V} = \frac{1}{V} \int (1 - \theta^*) dV$$

- Employing the approximate form of the temperature distribution for the plane wall the integration can be performed to obtain

$$\frac{Q}{Q_o} = 1 - \frac{\sin \zeta_1}{\zeta_1} \theta_o^*$$

The Plane Wall with Convection

- **Additional Considerations**

- Because the mathematical problem is precisely the same, the foregoing results may also be applied to a plane wall of thickness L that is insulated on one side $(x^* = 0)$ and experiences convective transport on the other side $(x^* = 1)$.

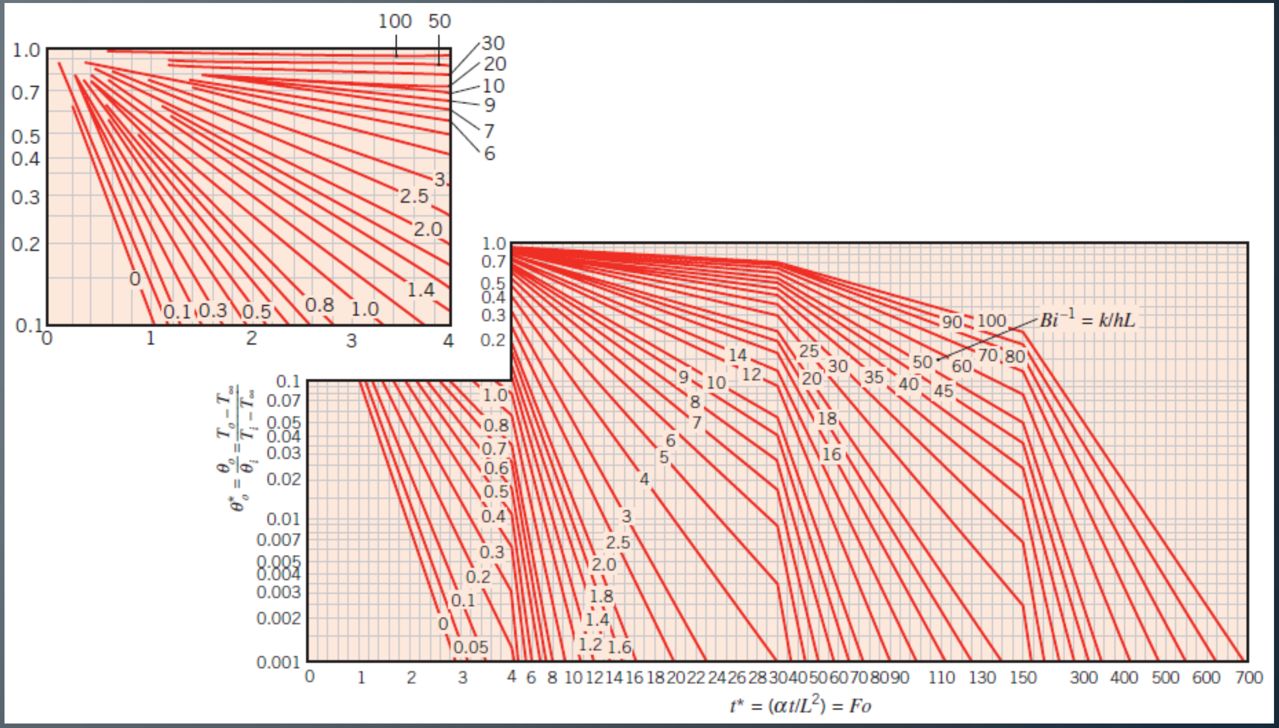
- Also note that the foregoing results may be used to determine the transient response of a plane wall to a sudden change in *surface* temperature.

- The process is equivalent to having an infinite convection coefficient, in which case the Biot number is infinite $(Bi = \infty)$ and the fluid temperature T_∞ is replaced by the prescribed surface temperature T_s.

The Plane Wall with Convection

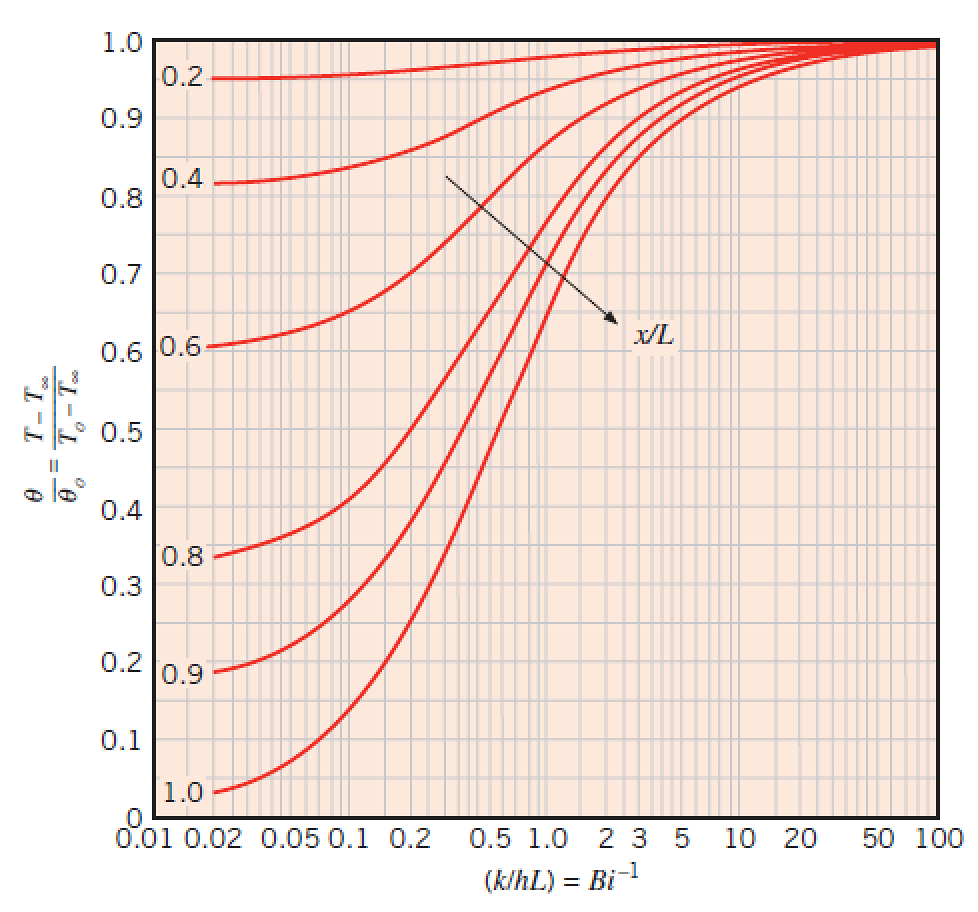
- **Heisler Charts**

- The results apply for $Fo > 0.2$ and can conveniently be represented in graphical forms that illustrate the functional dependence of the transient temperature distribution on the Biot and Fourier numbers.



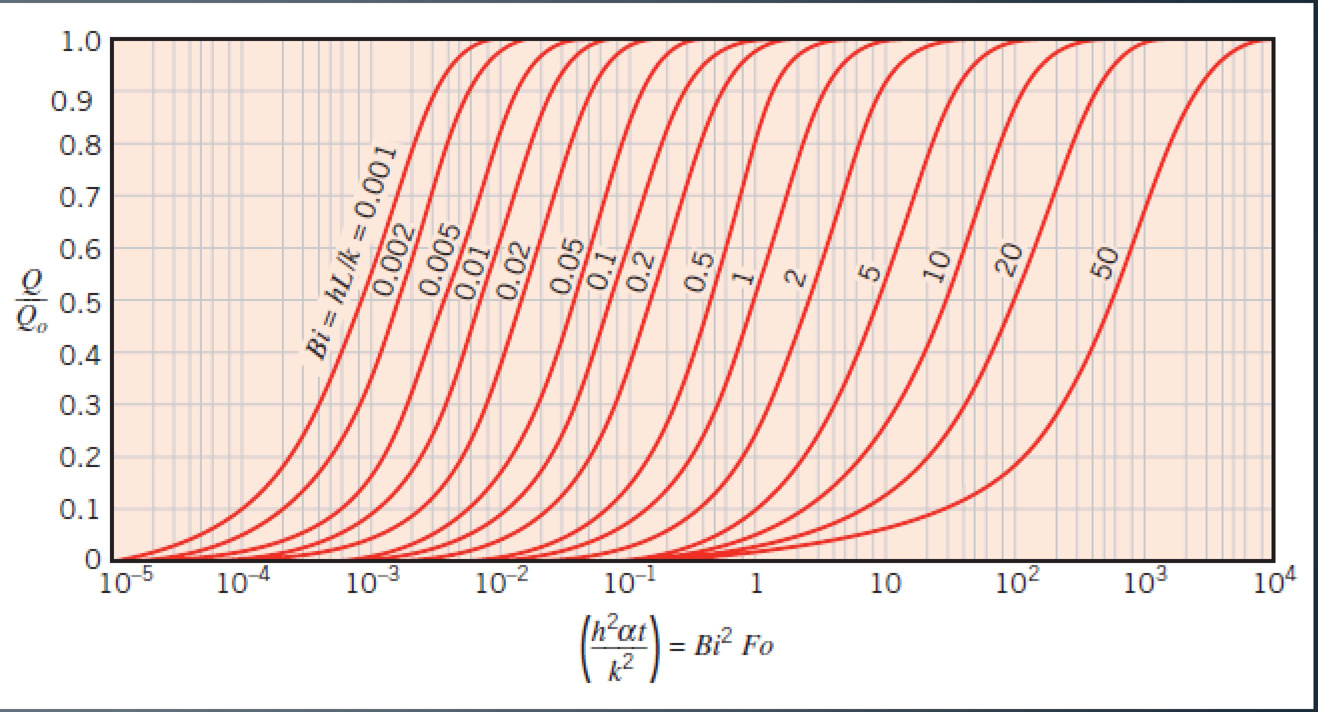
- **Heisler Charts**

The Plane Wall with Convection



- **Heisler Charts**

The Plane Wall with Convection



Radial Systems with Convection

- **Exact Solutions**
- *Infinite cylinder*

$$\theta^* = \sum_{n=1}^{\infty} C_n \exp(-\zeta_n^2 Fo) J_0(\zeta_n r^*)$$

$$C_n = \frac{2}{\zeta_n} \frac{J_1(\zeta_n)}{J_0^2(\zeta_n) + J_1^2(\zeta_n)}$$

$$\zeta_n \frac{J_1(\zeta_n)}{J_0(\zeta_n)} = Bi$$

- *Sphere*

$$\theta^* = \sum_{n=1}^{\infty} C_n \exp(-\zeta_n^2 Fo) \frac{1}{\zeta_n r^*} \sin(\zeta_n r^*)$$

$$C_n = \frac{4[\sin(\zeta_n) - \zeta_n \cos(\zeta_n)]}{2\zeta_n - \sin(2\zeta_n)}$$

$$1 - \zeta_n \cot \zeta_n = Bi$$

Radial Systems with Convection

- **Approximate Solutions**

- *Infinite cylinder*

$$\theta^* = C_1 \exp(-\zeta_1^2 Fo) J_0(\zeta_1 r^*)$$

$$\theta^* = \theta_o^* J_0(\zeta_1 r^*)$$

$$\theta_o^* = C_1 \exp(-\zeta_1^2 Fo)$$

- *Sphere*

$$\theta^* = C_1 \exp(-\zeta_1^2 Fo) \frac{1}{\zeta_1 r^*} \sin(\zeta_1 r^*)$$

$$\theta^* = \theta_o^* \frac{1}{\zeta_1 r^*} \sin(\zeta_1 r^*)$$

$$\theta_o^* = C_1 \exp(-\zeta_1^2 Fo)$$

Radial Systems with Convection

- **Total Energy Transfer**

- *Infinite cylinder*

$$\frac{Q}{Q_o} = 1 - \frac{2\theta_o^*}{\zeta_1} J_1(\zeta_1)$$

- *Sphere*

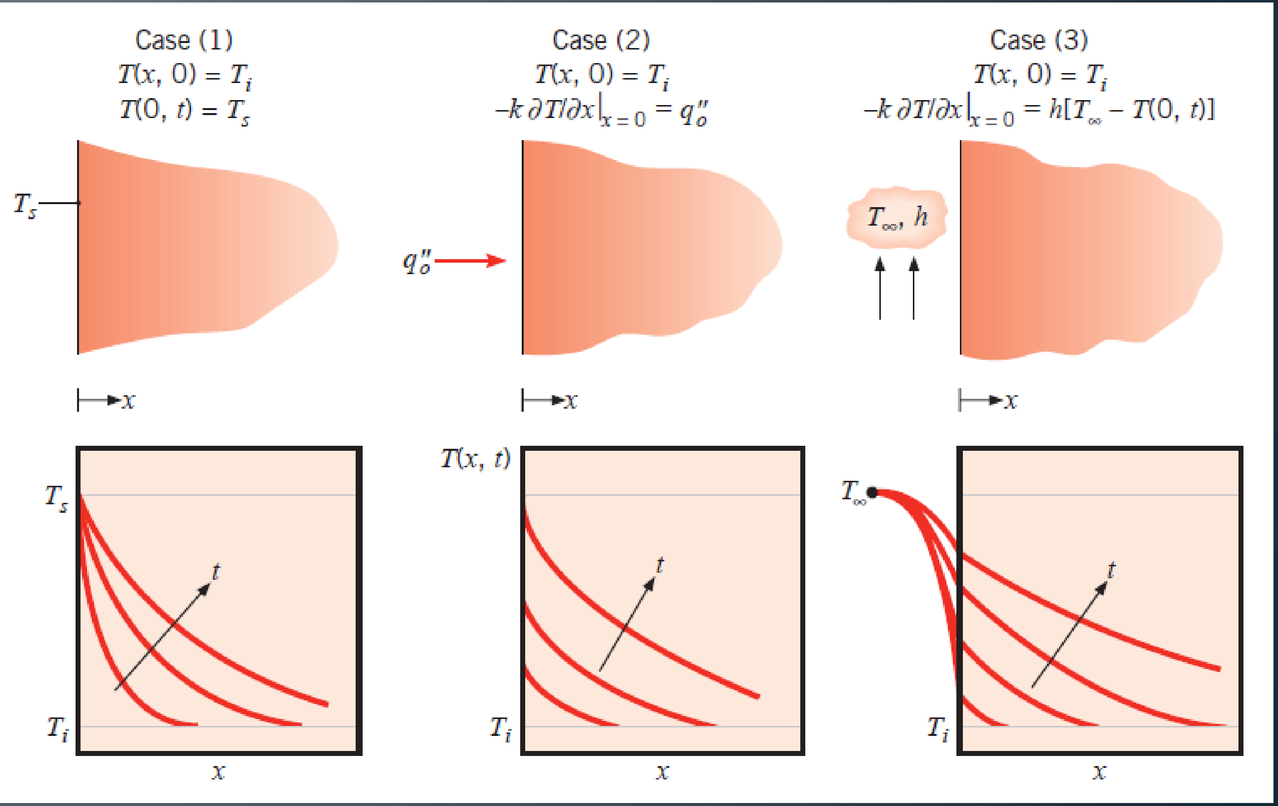
$$\frac{Q}{Q_o} = 1 - \frac{3\theta_o^*}{\zeta_1^3}\left[\sin(\zeta_1) - \zeta_1 \cos(\zeta_1)\right]$$

EXAMPLE 5.6

The Semi-Infinite Solid

- An important simple geometry for which analytical solutions may be obtained is the *semi-infinite solid*.

- If a sudden change of conditions is imposed at this surface, transient, one-dimensional conduction will occur within the solid.



The Semi-Infinite Solid

- The semi-infinite solid provides a *useful idealization* for many practical problems.

- Although the exact solutions of the preceding sections could be used to determine the temperature distributions, many terms might be required to evaluate the infinite series expressions.

- The following semi-infinite solid solutions often eliminate the need to evaluate the cumbersome infinite series exact solutions at small Fo.

- It can be shown that a plane wall of thickness $2L$ can be accurately approximated as a semi-infinite solid for $Fo = \alpha t/L^2 \leq 0.2$.

The Semi-Infinite Solid

- Closed-form solutions have been obtained for three important surface conditions, instantaneously applied at $t = 0$.

Case 1 Constant Surface Temperature: $T(0, t) = T_s$

$$\frac{T(x, t) - T_s}{T_i - T_s} = \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

$$q''_s(t) = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}}$$

Case 2 Constant Surface Heat Flux: $q''_s = q''_o$

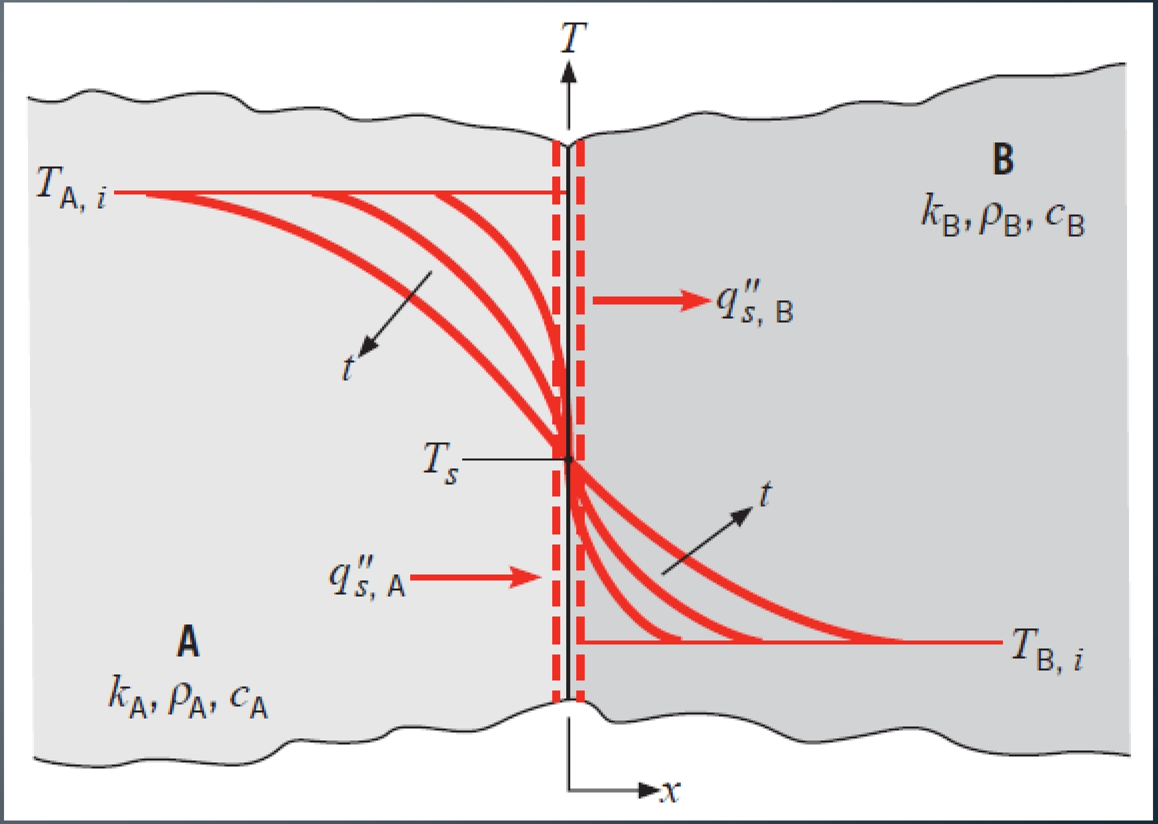
$$T(x, t) - T_i = \frac{2q''_o(\alpha t/\pi)^{1/2}}{k} \exp\left(\frac{-x^2}{4\alpha t}\right) - \frac{q''_o x}{k} \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

Case 3 Surface Convection: $-k \left.\frac{\partial T}{\partial x}\right|_{x=0} = h[T_\infty - T(0, t)]$

$$\frac{T(x, t) - T_i}{T_\infty - T_i} = \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - \left[\exp\left(\frac{hx}{k} + \frac{h^2 \alpha t}{k^2}\right)\right]\left[\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)\right]$$

The Semi-Infinite Solid

- An interesting permutation of case 1 occurs when two semi-infinite solids, initially at different uniform temperatures are placed in contact at their free surfaces.



The Semi-Infinite Solid

- The equilibrium surface temperature may be determined from a surface energy balance, which requires that

$$q''_{s,\mathrm{A}} = q''_{s,\mathrm{B}}$$

$$\frac{-k_{\mathrm{A}}(T_s - T_{\mathrm{A},i})}{(\pi\alpha_{\mathrm{A}}t)^{1/2}} = \frac{k_{\mathrm{B}}(T_s - T_{\mathrm{B},i})}{(\pi\alpha_{\mathrm{B}}t)^{1/2}}$$

$$T_s = \frac{(k\rho c)_{\mathrm{A}}^{1/2} T_{\mathrm{A},i} + (k\rho c)_{\mathrm{B}}^{1/2} T_{\mathrm{B},i}}{(k\rho c)_{\mathrm{A}}^{1/2} + (k\rho c)_{\mathrm{B}}^{1/2}}$$

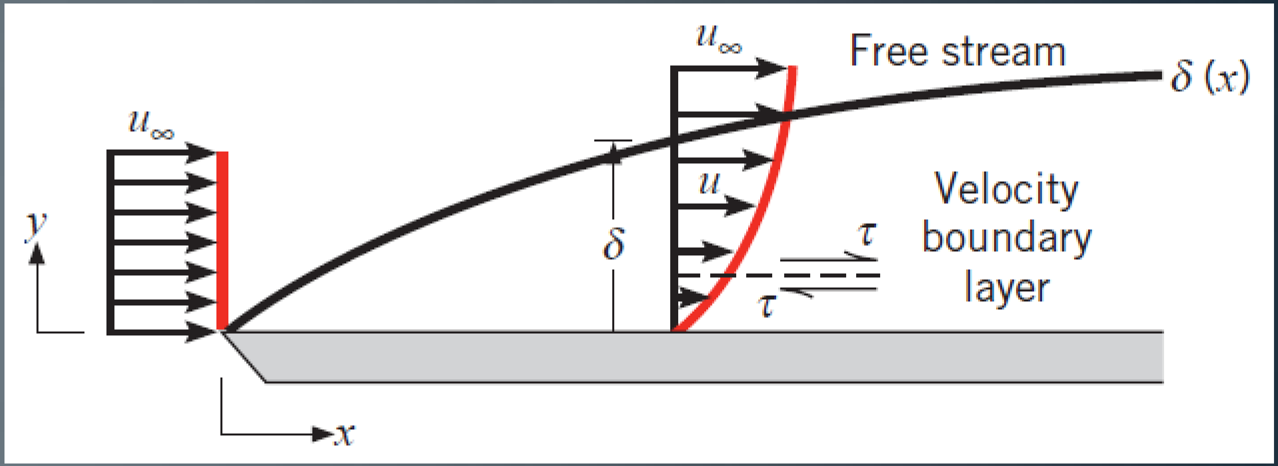
Introduction to Convection

Chapter 6

The Convection Boundary Layers

- **The Velocity Boundary Layer**

- The quantity δ is termed the *boundary layer thickness*, and it is typically defined as the value of y for which $u = 0.99u_\infty$.

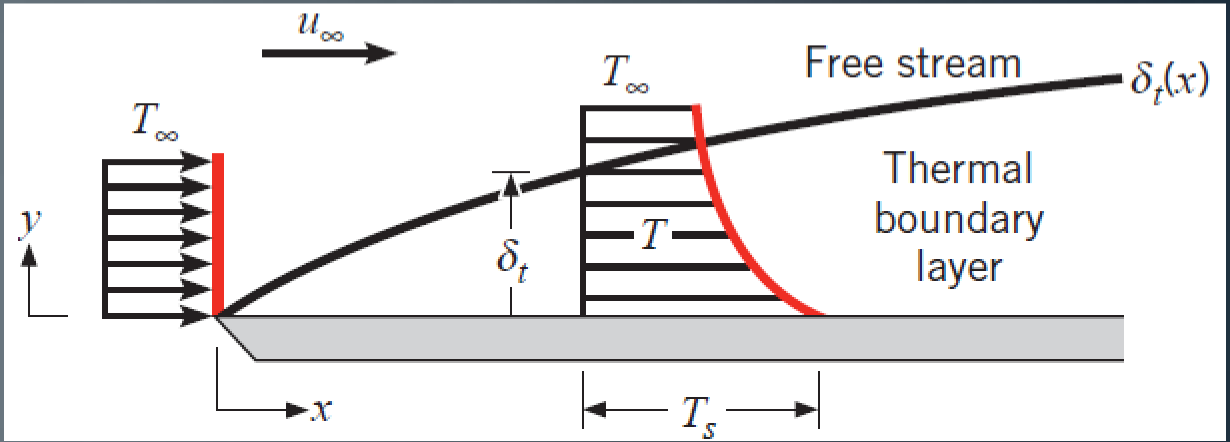


$$C_f \equiv \frac{\tau_s}{\rho u_\infty^2/2}$$

$$\tau_s = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$$

The Convection Boundary Layers

- **The Thermal Boundary Layer**
- The region of the fluid in which these temperature gradients exist is the *thermal boundary layer*, and its thickness δ_t is typically defined as the value of y for which the ratio $[(T_s - T)/(T_s - T_\infty)] = 0.99$.



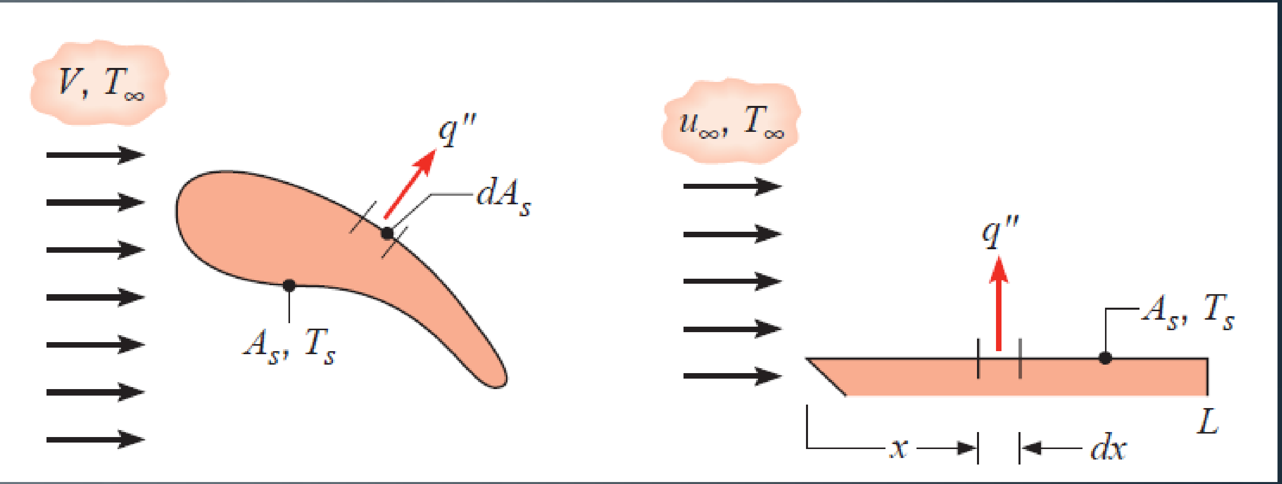
$$q_s'' = -k_f \frac{\partial T}{\partial y}\bigg|_{y=0}$$

$$q_s'' = h(T_s - T_\infty)$$

$$h = \frac{-k_f \, \partial T / \partial y|_{y=0}}{T_s - T_\infty}$$

Local and Average Convection Coefficients

- **Heat Transfer**
- The *total heat transfer rate q* may be obtained by integrating the local flux over the entire surface.



$$q = \int_{A_s} q'' dA_s$$

$$q = (T_s - T_\infty) \int_{A_s} h dA_s$$

- Defining an *average convection coefficient $\bar{h}$* for the entire surface, the total heat transfer rate may also be expressed as

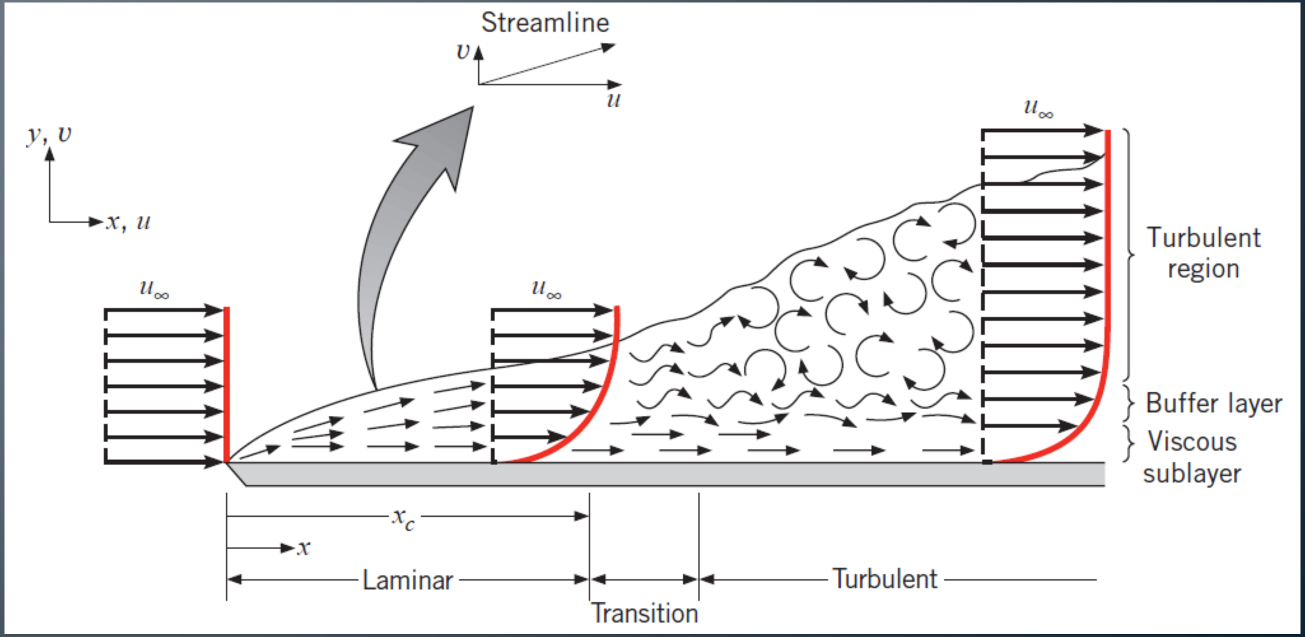
$$q = \bar{h} A_s (T_s - T_\infty)$$

$$\bar{h} = \frac{1}{A_s} \int_{A_s} h dA_s$$

EXAMPLE 6.1

Laminar and Turbulent Flow

- **Laminar and Turbulent Velocity Boundary Layers**
- In the *laminar boundary layer*, the fluid flow is highly ordered and it is possible to identify streamlines along which fluid particles move.
- The highly ordered behavior continues until a *transition* zone is reached, across which a conversion from laminar to turbulent conditions occurs.
- Flow in the fully *turbulent boundary layer* is, in general, highly irregular and is characterized by random, three-dimensional motion of relatively large parcels of fluid.



Laminar and Turbulent Flow

- **Laminar and Turbulent Velocity Boundary Layers**

- Three different regions may be delineated within the turbulent boundary layer as a function of distance from the surface.

- We may speak of a *viscous sublayer* in which transport is dominated by diffusion and the velocity profile is nearly linear.

- There is an adjoining *buffer layer* in which diffusion and turbulent mixing are comparable.

- And there is a *turbulent zone* in which transport is dominated by turbulent mixing.

Laminar and Turbulent Flow

- **Laminar and Turbulent Velocity Boundary Layers**

- The onset of turbulence depends on a dimensionless grouping of parameters called the *Reynolds number*,

$$Re_x = \frac{\rho u_\infty x}{\mu}$$

- It will be shown later that the Reynolds number represents the *ratio of the inertia to viscous forces*.

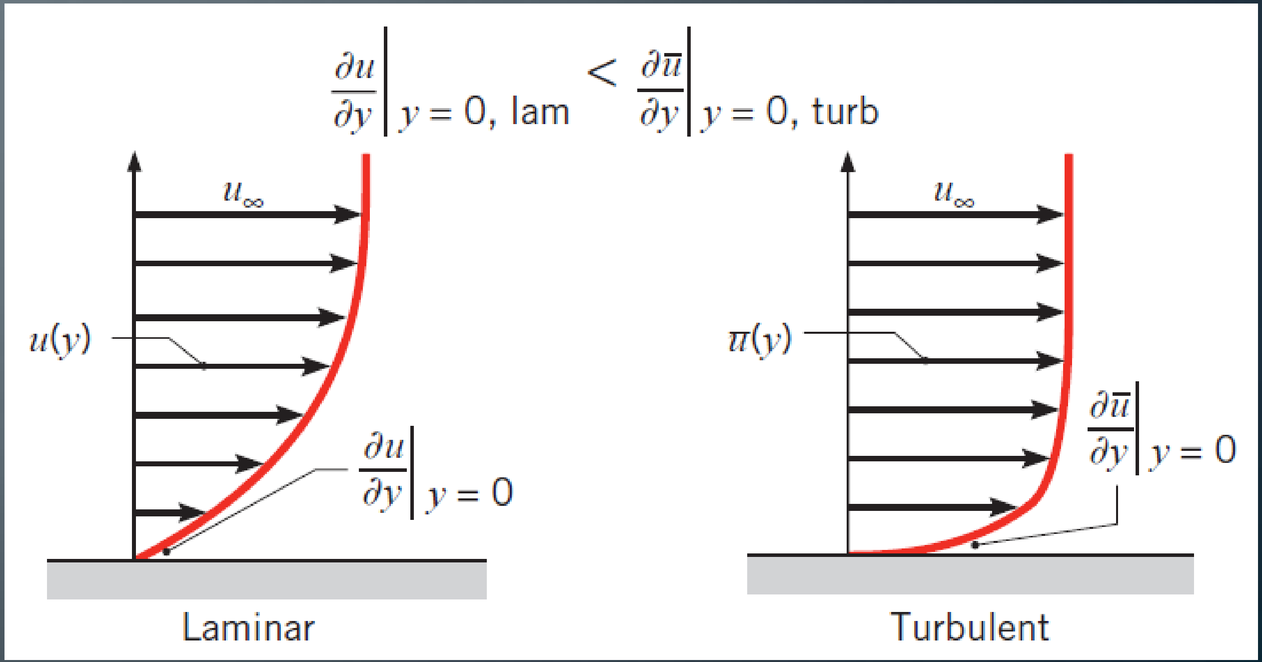
- In determining whether the boundary layer is laminar or turbulent, it is frequently reasonable to assume that transition begins at some location x_c. This location is determined by the *critical Reynolds number, $Re_{x,c}$*. For flow over a flat plate,

$$Re_{x,c} \equiv \frac{\rho u_\infty x_c}{\mu} = 5 \times 10^5$$

Laminar and Turbulent Flow

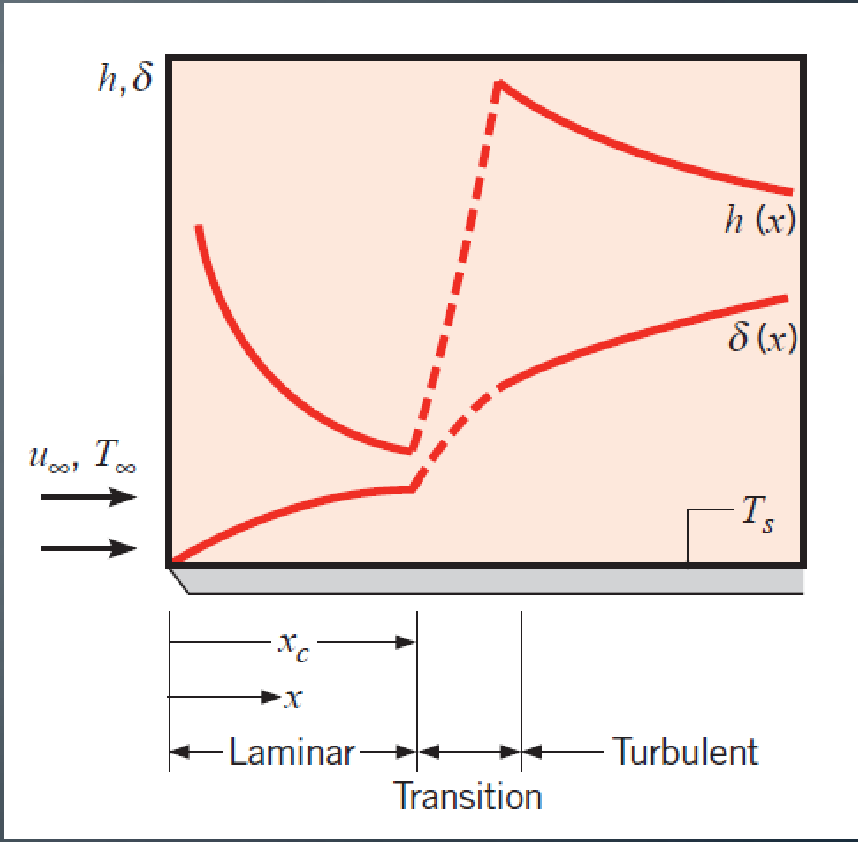
- **Laminar and Turbulent Velocity Boundary Layers**

- A comparison of the laminar and turbulent boundary layer profiles for the x-component of the velocity shows that the turbulent velocity profile is relatively *flat* due to the mixing that occurs within the buffer layer and turbulent region, giving rise to large velocity gradients within the viscous sublayer.



Laminar and Turbulent Flow

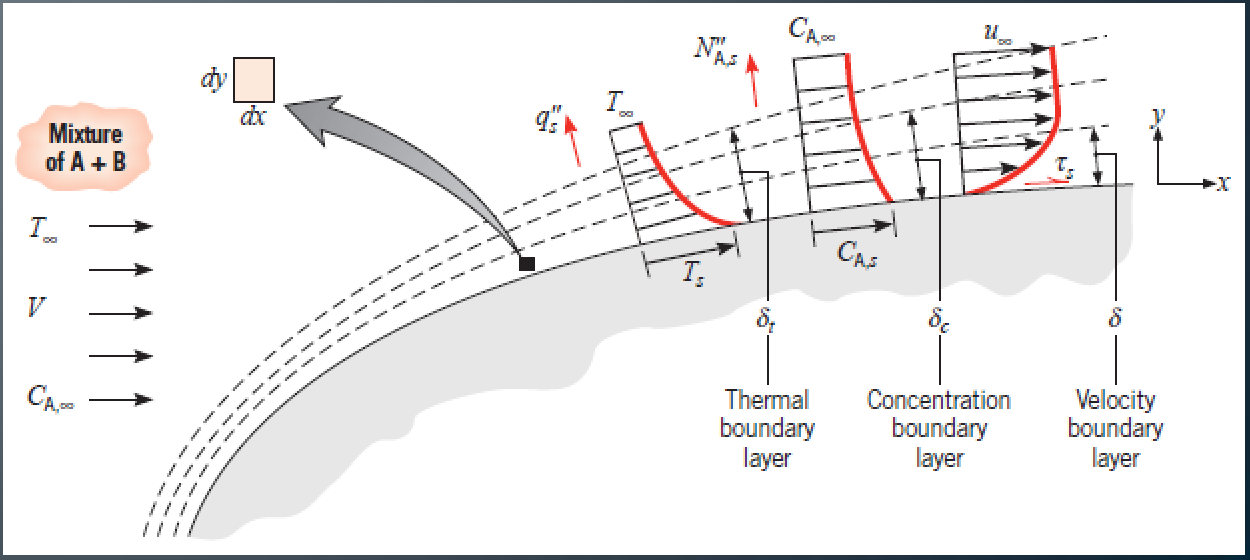
- **Laminar and Turbulent Thermal Boundary Layer**
- Because turbulence induces mixing, which in turn reduces the importance of conduction and diffusion in determining the thermal and species boundary layer thicknesses, *differences* in the thicknesses of the velocity, thermal, and species boundary layers tend to be much smaller in turbulent flow than in laminar flow.



EXAMPLE 6.4

The Boundary Layer Equations

- **The Convection Transfer Equations**
- Motion of a fluid in which there are coexisting *velocity and temperature gradients* must comply with several fundamental laws of nature.
- In particular, at each point in the fluid, *conservation of mass and energy*, as well as Newton's second law of motion, must be satisfied.
- Equations representing these requirements are derived by applying the laws to a differential control volume situated in the flow.
- The resulting equations, in Cartesian coordinates, for the *steady*, *two-dimensional flow* of an *incompressible fluid* with *constant properties* are as follow.



- **The Convection Transfer Equations**

The Boundary Layer Equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + X$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + Y$$

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \mu\Phi + \dot{q}$$

$$\mu\Phi = \mu\left\{\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]\right\}$$

The Boundary Layer Equations

- **Boundary Layer Equations for Laminar Flow**
- We begin by restricting attention to applications for which *body forces are negligible* and there is *no thermal energy generation* in the fluid.
- The *boundary layer thicknesses* are typically *very small* relative to the size of the object upon which they form.
- Therefore, *gradients normal to the object's surface* are *much larger* than those along the surface. As a result, we can neglect terms that represent x-direction diffusion of momentum and thermal energy, relative to their y-direction counterparts.

$$v \ll u \qquad \frac{\partial^2 u}{\partial x^2} \ll \frac{\partial^2 u}{\partial y^2} \qquad \frac{\partial^2 T}{\partial x^2} \ll \frac{\partial^2 T}{\partial y^2}$$

- Furthermore, because the boundary layer is so thin, the x-direction pressure gradient within the boundary layer can be approximated as the free stream pressure gradient

$$\frac{\partial p}{\partial x} \approx \frac{dp_\infty}{dx}$$

The Boundary Layer Equations

- **Boundary Layer Equations for Laminar Flow**

- With the foregoing simplifications and approximations, the overall continuity equation is unchanged

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

- The two terms represent the *net outflow of mass in the x- and y-directions*, the sum of which must be zero for steady flow.

- The x-momentum equation reduces to

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

- The left-hand side represents the *net rate at which x-momentum leaves the control volume* due to fluid motion across its boundaries. The first term on the right-hand side represents the *net pressure force*, and the second term represents the *net force due to viscous shear stresses*.

The Boundary Layer Equations

- **Boundary Layer Equations for Laminar Flow**
- The y-momentum equation reduces to

$$\frac{\partial p}{\partial y} = 0$$

- That is, the *pressure does not vary in the direction normal to the surface*.
- The energy equation reduces to

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2$$

- Terms on the left-hand side account for the *net rate at which thermal energy leaves the control volume due to bulk fluid motion (advection)*. The first term on the right-hand side accounts for the *net inflow of thermal energy due to y-direction conduction*. The last term on the right-hand side is what remains of the *viscous dissipation*.

EXAMPLE 6S.1

Boundary Layer Similarity: The Normalized Boundary Layer Equations

- If we examine the boundary layer equations, we note a strong *similarity*.
- In fact, if the *pressure gradient* appearing in the x-momentum equation and the *viscous dissipation* term of the energy equation are negligible, the three equations are of the same form.
- Each equation is characterized by *advection terms* on the left-hand side and a *diffusion term* on the right-hand side.
- This situation describes *low-speed, forced convection flows* which are found in many engineering applications.
- Implications of this similarity may be developed in a rational manner by first *nondimensionalizing* the governing equations.

Boundary Layer Similarity: The Normalized Boundary Layer Equations

- **Boundary Layer Similarity Parameters**
- The boundary layer equations are *normalized* by first defining dimensionless independent variables of the forms

$$x^* \equiv \frac{x}{L} \qquad y^* \equiv \frac{y}{L} \qquad T^* \equiv \frac{T - T_s}{T_\infty - T_s}$$

- By normalizing the boundary layer equations, two very important dimensionless *similarity parameters* evolve. They are the *Reynolds number*, Re_L; *and Prandtl number*, Pr.
- Such similarity parameters are important because they allow us to apply results obtained for a surface experiencing one set of convective conditions to *geometrically similar* surfaces experiencing entirely different conditions.
- As long as the *similarity parameters* and *dimensionless boundary conditions* are the same for two sets of conditions, the solutions of the differential equations for the nondimensional velocity and temperature will be *identical*.

Boundary Layer Similarity: The Normalized Boundary Layer Equations

- **Boundary Layer Similarity Parameters**

Boundary Layer	Conservation Equation
Velocity	$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{dp^*}{dx^*} + \frac{1}{Re_L} \frac{\partial^2 u^*}{\partial y^{*2}}$ (6.35)
Thermal	$u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{1}{Re_L\, Pr} \frac{\partial^2 T^*}{\partial y^{*2}}$ (6.36)

Boundary Conditions: Wall	Boundary Conditions: Free Stream		Similarity Parameter(s)
$u^*(x^*, 0) = 0$	$u^*(x^*, \infty) = \frac{u_\infty(x^*)}{V}$	(6.38)	$Re_L = \frac{VL}{\nu}$
$T^*(x^*, 0) = 0$	$T^*(x^*, \infty) = 1$	(6.39)	$Re_L,\ Pr = \frac{\nu}{\alpha}$

Boundary Layer Similarity: The Normalized Boundary Layer Equations

- **Functional Form of the Solutions**
- The solution to the x-momentum equation may be expressed in the form

$$u^* = f\left(x^*, y^*, Re_L, \frac{dp^*}{dx^*}\right)$$

$$\tau_s = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \left(\frac{\mu V}{L}\right) \frac{\partial u^*}{\partial y^*}\bigg|_{y^*=0}$$

$$C_f = \frac{\tau_s}{\rho V^2/2} = \frac{2}{Re_L} \frac{\partial u^*}{\partial y^*}\bigg|_{y^*=0}$$

$$\frac{\partial u^*}{\partial y^*}\bigg|_{y^*=0} = f\left(x^*, Re_L, \frac{dp^*}{dx^*}\right)$$

- Hence, *for a prescribed geometry*

$$C_f = \frac{2}{Re_L} f(x^*, Re_L)$$

Boundary Layer Similarity: The Normalized Boundary Layer Equations

- **Functional Form of the Solutions**
- The solution to the energy equation may be expressed in the form

$$T^* = f\left(x^*, y^*, Re_L, Pr, \frac{dp^*}{dx^*}\right)$$

$$h = -\frac{k_f}{L}\frac{(T_\infty - T_s)}{(T_s - T_\infty)}\frac{\partial T^*}{\partial y^*}\bigg|_{y^*=0} = +\frac{k_f}{L}\frac{\partial T^*}{\partial y^*}\bigg|_{y^*=0}$$

- This expression suggests defining an important dependent dimensionless parameter termed the *Nusselt number*.

$$Nu \equiv \frac{hL}{k_f} = +\frac{\partial T^*}{\partial y^*}\bigg|_{y^*=0}$$

- It follows that, *for a prescribed geometry*

$$Nu = f(x^*, Re_L, Pr)$$

$$\overline{Nu} = \frac{\bar{h}L}{k_f} = f(Re_L, Pr)$$

EXAMPLE 6.5

EXAMPLE 6.6

Physical Interpretation of the Dimensionless Parameters

- The *Reynolds number* Re_L may be interpreted as the *ratio of inertia to viscous forces* in a region of characteristic dimension L.

$$\frac{F_I}{F_s} \approx \frac{\rho V^2/L}{\mu V/L^2} = \frac{\rho V L}{\mu} = Re_L$$

- The *Prandtl number* is defined as the ratio of the kinematic viscosity, also referred to as the *momentum diffusivity*, ϑ, to the *thermal diffusivity* α.

- The Prandtl number of gases is near unity, in which case energy and momentum transfer by diffusion are comparable. In a liquid metal, $Pr \ll 1$ and the energy diffusion rate greatly exceeds the momentum diffusion rate. The opposite is true for oils, for which $Pr \gg 1$.

Physical Interpretation of the Dimensionless Parameters

- In fact for *laminar boundary layers* (in which transport by *diffusion* is *not* overshadowed by *turbulent mixing*), it is reasonable to expect that

$$\frac{\delta}{\delta_t} \approx Pr^n$$

- Hence for a *gas* $\delta_t \approx \delta$; *for a liquid metal* $\delta_t \approx \delta$; *for an oil* $\delta_t \approx \delta$.

- Note also that, although similar in form, the Nusselt and Biot numbers differ in both definition and interpretation. Whereas the *Nusselt number* is defined in terms of the *thermal conductivity of the fluid*, the *Biot number* is based on the *solid thermal conductivity*.

Biot number (Bi)	$\frac{hL}{k_s}$

Nusselt number (Nu_L)	$\frac{hL}{k_f}$

Boundary Layer Analogies

- **The Reynolds Analogy**
- If two or more processes are governed by dimensionless equations of the same form, the processes are said to be *analogous*.

Functional relations pertinent to the boundary layer analogies

Fluid Flow	Heat Transfer
$u^* = f\left(x^*, y^*, Re_L, \frac{dp^*}{dx^*}\right)$	$T^* = f\left(x^*, y^*, Re_L, Pr, \frac{dp^*}{dx^*}\right)$
$C_f = \frac{2}{Re_L} \frac{\partial u^*}{\partial y^*}\Big\vert_{y^*=0}$	$Nu = \frac{hL}{k} = +\frac{\partial T^*}{\partial y^*}\Big\vert_{y^*=0}$
$C_f = \frac{2}{Re_L} f(x^*, Re_L)$	$Nu = f(x^*, Re_L, Pr)$
	$\overline{Nu} = f(Re_L, Pr)$

- For $dp^*/dx^* = 0$ and $Pr = 1$, the boundary layer equations, are of precisely the same form.
- For a flat plate parallel to the incoming flow, we have $dp^*/dx^* = 0$ and there is no variation in the free stream velocity outside the boundary layer.

Boundary Layer Analogies

- **The Reynolds Analogy**
- From the boundary layer equations it follows that

$$C_f \frac{Re_L}{2} = Nu$$

- Replacing *Nu by the Stanton number (St)*

$$St \equiv \frac{h}{\rho V c_p} = \frac{Nu}{Re\,Pr}$$

- It may also be expressed in the form

$$\frac{C_f}{2} = St$$

- The above equation is known as the *Reynolds analogy*, and it relates the key engineering parameters of the velocity and thermal boundary layers.
- If the velocity parameter is known, the analogy may be used to obtain the other parameters, and vice versa.

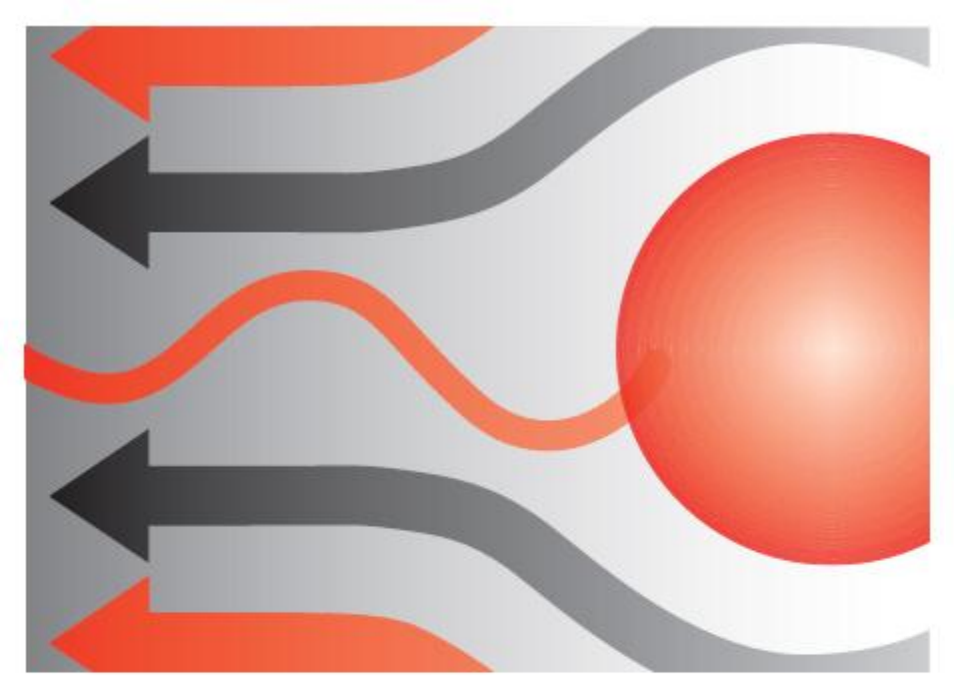
Boundary Layer Analogies

- **The Reynolds Analogy**

- However, there are restrictions associated with using this result. In addition to relying on the *validity of the boundary layer approximations*, the accuracy of the Reynolds analogy depends on having $Pr = 1$ *and* $dp^*/dx^* = 0$.
- However, it has been shown that the analogy may be applied over a wide range of Pr, if certain corrections are added. In particular the *modified Reynolds*, or *Chilton-Colburn*, *analogies* have the form

$$\frac{C_f}{2} = St\, Pr^{2/3} \equiv j_H \qquad 0.6 < Pr < 60$$

- For *laminar flow* the above equations is only appropriate when $dp^*/dx^* = 0$, but in *turbulent flow*, conditions are *less sensitive to the effect of pressure* gradients and these equations remain approximately valid.

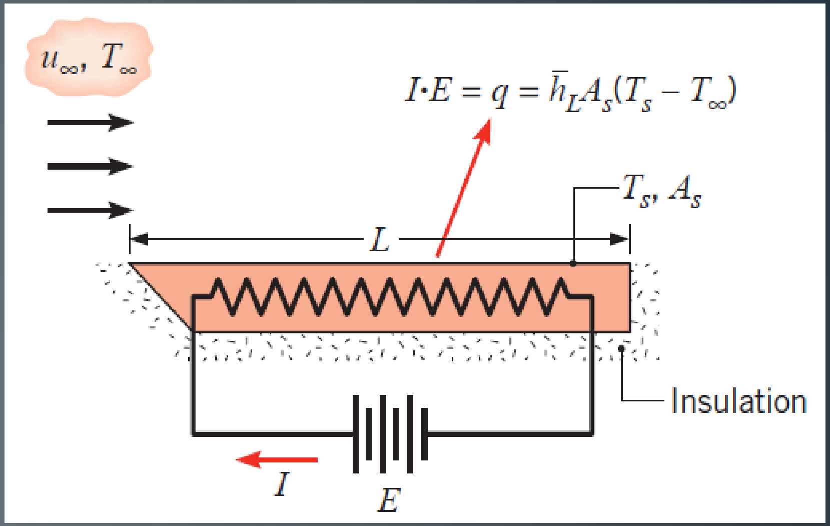


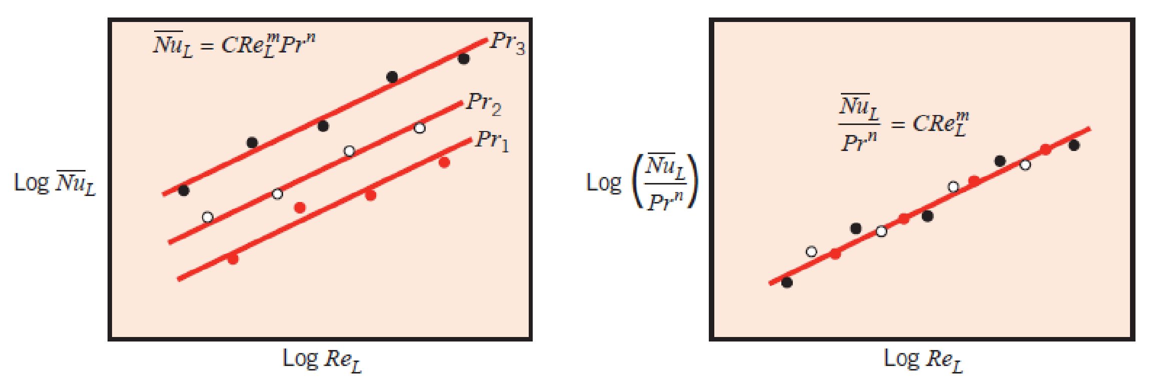
External Flow

Chapter 7

The Empirical Method

- The manner in which a convection heat transfer correlation may be obtained experimentally is illustrated in the below figure.





The Flat Plate in Parallel Flow

- **Laminar Flow over an Isothermal Plate: A Similarity Solution**

- ***Hydrodynamic Solution***

$$\delta = \frac{5.0}{\sqrt{u_\infty/\nu x}} = \frac{5x}{\sqrt{Re_x}}$$

$$C_{f,x} \equiv \frac{\tau_{s,x}}{\rho u_\infty^2/2} = 0.664\, Re_x^{-1/2}$$

- ***Heat Transfer Solution***

$$Nu_x \equiv \frac{h_x x}{k} = 0.332\, Re_x^{1/2}\, Pr^{1/3}$$

$$\frac{\delta}{\delta_t} \approx Pr^{1/3} \qquad Pr \gtrsim 0.6$$

The Flat Plate in Parallel Flow

- **Laminar Flow over an Isothermal Plate: A Similarity Solution**
- *Average Boundary Layer Parameters*

$$\overline{C}_{f,x} \equiv \frac{\overline{\tau}_{s,x}}{\rho u_\infty^2/2}$$

$$\overline{\tau}_{s,x} \equiv \frac{1}{x}\int_0^x \tau_{s,x}\,dx$$

$$\overline{C}_{f,x} = 1.328\,Re_x^{-1/2}$$

$$\overline{h}_x = \frac{1}{x}\int_0^x h_x dx = 0.332\left(\frac{k}{x}\right)Pr^{1/3}\left(\frac{u_\infty}{\nu}\right)^{1/2}\int_0^x \frac{dx}{x^{1/2}}$$

$$\overline{h}_x = 2h_x$$

$$\overline{Nu}_x \equiv \frac{\overline{h}_x x}{k} = 0.664\,Re_x^{1/2}\,Pr^{1/3} \qquad Pr \gtrsim 0.6$$

The Flat Plate in Parallel Flow

- **Turbulent Flow over an Isothermal Plate**
- It is *not possible* to obtain *exact analytical solutions* for *turbulent boundary layers*, which are inherently unsteady.

$$C_{f,x} = 0.0592\, Re_x^{-1/5} \qquad Re_{x,c} \lesssim Re_x \lesssim 10^8$$

$$\delta = 0.37x\, Re_x^{-1/5}$$

- *Turbulent boundary layer growth* is much *more rapid* (δ varies as $x^{4/5}$ in contrast to $x^{1/2}$ for laminar flow) and that the *decay in the friction coefficient* is *more gradual* ($x^{-1/5}$ versus $x^{-1/2}$).
- For *turbulent flow*, boundary layer development is influenced strongly by random fluctuations in the fluid and not by molecular diffusion. Hence relative boundary layer growth *does not depend on* the value of Pr. That is, for turbulent flow, $\delta_t \approx \delta$.
- Using the modified Reynolds, or Chilton–Colburn, analogy, the *local Nusselt number for turbulent flow is*

$$Nu_x = St\, Re_x\, Pr = 0.0296\, Re_x^{4/5}\, Pr^{1/3} \qquad 0.6 \lesssim Pr \lesssim 60$$

The Flat Plate in Parallel Flow

- **Mixed Boundary Layer Conditions**

- Integrating over the laminar region $(0 \le x \le x_c)$ and then over the turbulent region $(x_c < x \le L)$, the average convection heat transfer coefficient for the entire plate may be expressed as

$$\bar{h}_L = \frac{1}{L}\left(\int_0^{x_c} h_{\text{lam}}\, dx + \int_{x_c}^{L} h_{\text{turb}}\, dx\right)$$

$$\bar{h}_L = \left(\frac{k}{L}\right)\left[0.332\left(\frac{u_\infty}{\nu}\right)^{1/2}\int_0^{x_c}\frac{dx}{x^{1/2}} + 0.0296\left(\frac{u_\infty}{\nu}\right)^{4/5}\int_{x_c}^{L}\frac{dx}{x^{1/5}}\right] Pr^{1/3}$$

$$\overline{Nu}_L = (0.037\, Re_L^{4/5} - A)\, Pr^{1/3}$$

$$\left[\begin{array}{l} 0.6 \lesssim Pr \lesssim 60 \\ Re_{x,c} \lesssim Re_L \lesssim 10^8 \end{array}\right]$$

$$A = 0.037\, Re_{x,c}^{4/5} - 0.664\, Re_{x,c}^{1/2}$$

The Flat Plate in Parallel Flow

- **Mixed Boundary Layer Conditions**

- Similarly, the average friction coefficient may be found using the expression

$$\overline{C}_{f,L} = \frac{1}{L}\left(\int_0^{x_c} C_{f,x,\text{lam}}\,dx + \int_{x_c}^{L} C_{f,x,\text{turb}}\,dx\right)$$

$$\overline{C}_{f,L} = 0.074\,Re_L^{-1/5} - \frac{2A}{Re_L}$$

$$[Re_{x,c} \lesssim Re_L \lesssim 10^8]$$

The Flat Plate in Parallel Flow

- **Flat Plates with Constant Heat Flux Conditions**

- It is also possible to have a uniform surface heat flux, rather than a uniform temperature, imposed at the plate. For laminar flow, it may be shown that

$$Nu_x = 0.453\, Re_x^{1/2}\, Pr^{1/3}$$

$$Nu_x = 0.0308\, Re_x^{4/5}\, Pr^{1/3}$$

- If the heat flux is known, the convection coefficient may be used to determine the local surface temperature

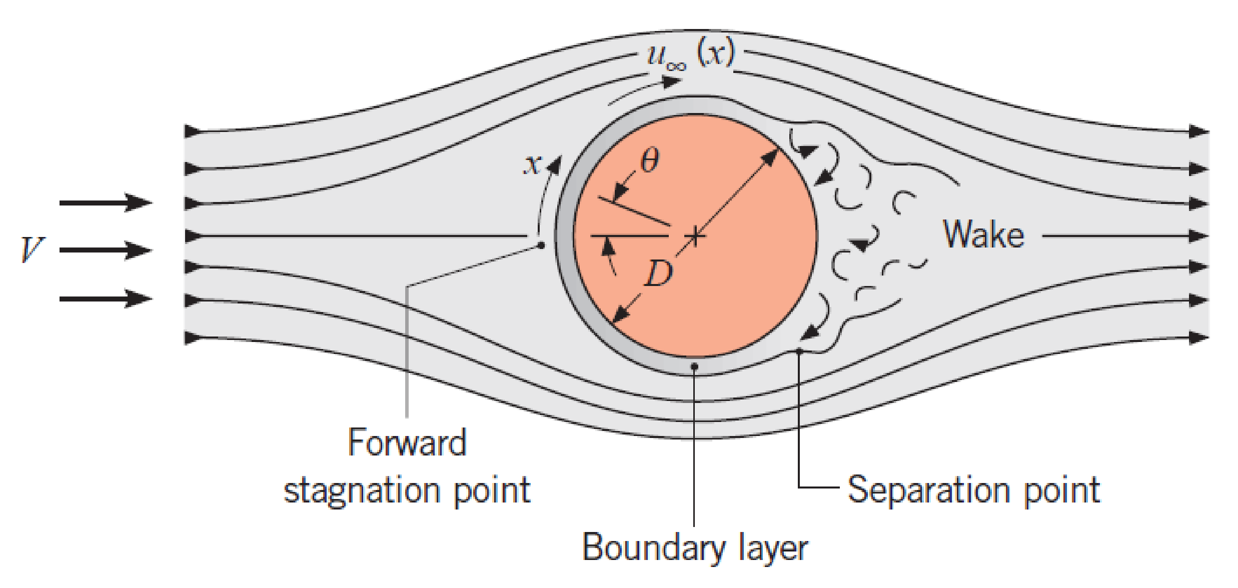
$$T_s(x) = T_\infty + \frac{q_s''}{h_x}$$

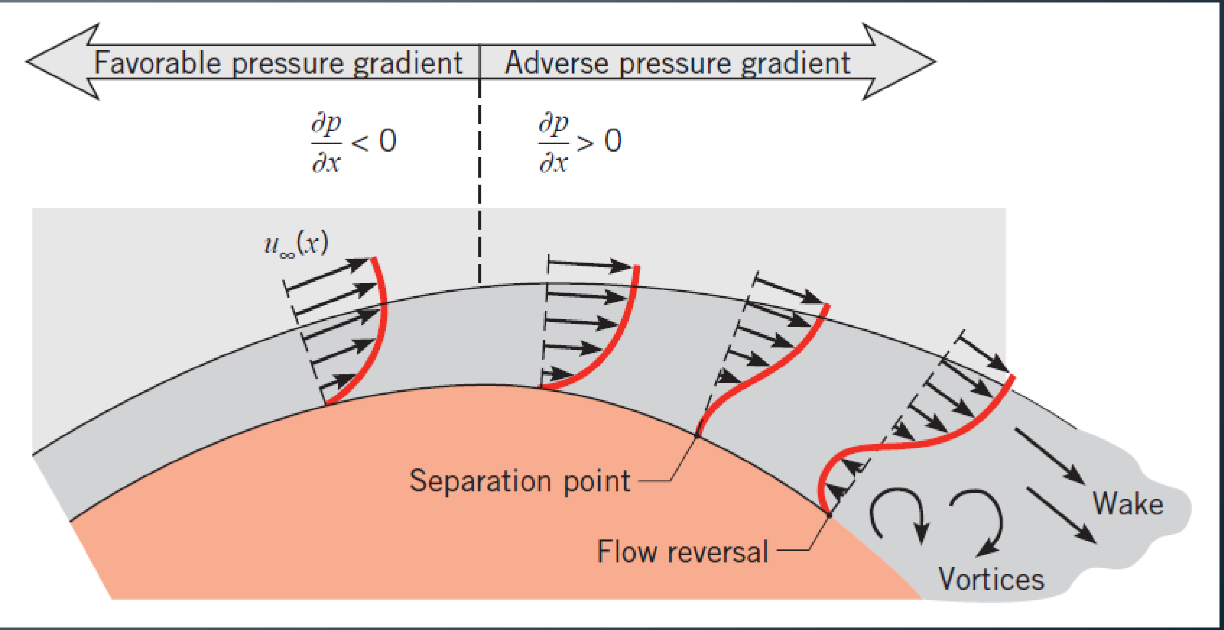
EXAMPLE 7.1

EXAMPLE 7.2

- **Flow Considerations**

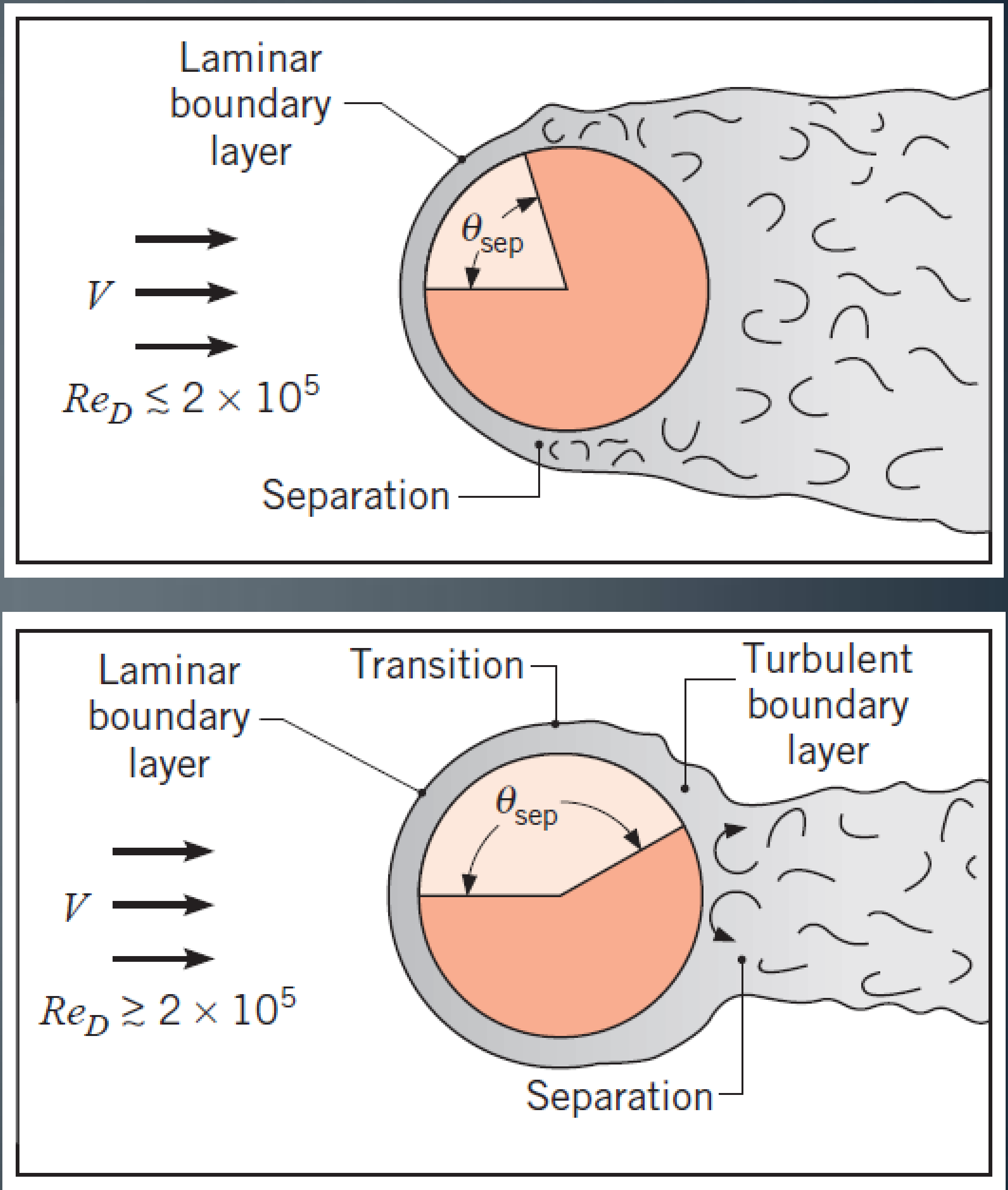
The Cylinder in Cross Flow





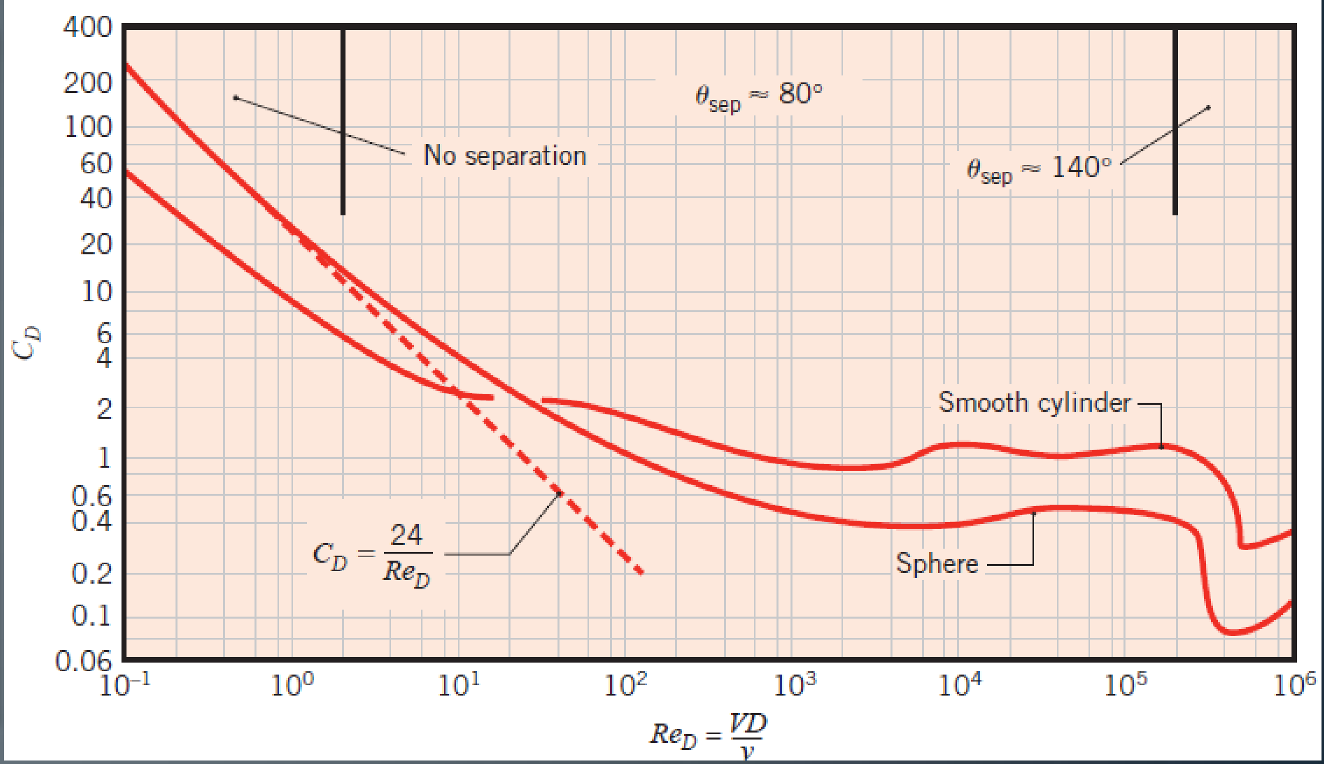
The Cylinder in Cross Flow

- **Flow Considerations**



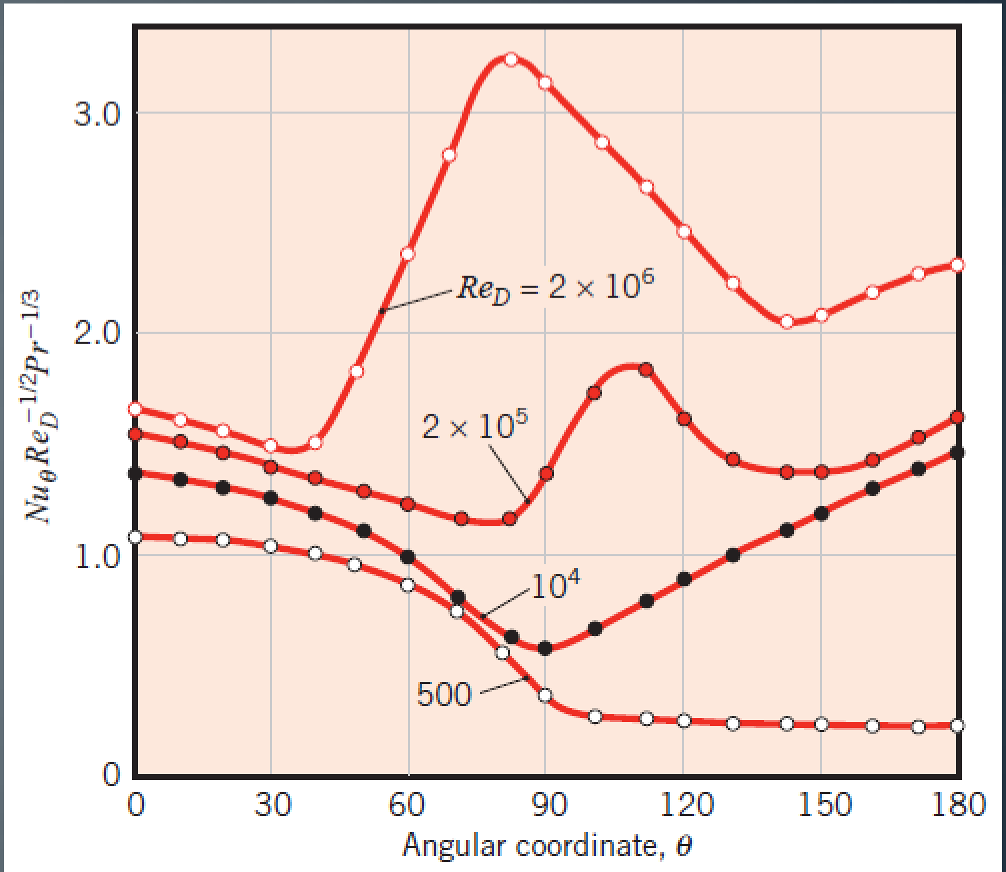
• **Flow Considerations**

The Cylinder in Cross Flow



- Convection Heat and Mass Transfer

The Cylinder in Cross Flow



- **Convection Heat and Mass Transfer**

The Cylinder in Cross Flow

$$\overline{Nu}_D \equiv \frac{\overline{h}D}{k} = C\,Re_D^m\,Pr^{1/3}$$

Re_D	C	m
0.4–4	0.989	0.330
4–40	0.911	0.385
40–4000	0.683	0.466
4000–40,000	0.193	0.618
40,000–400,000	0.027	0.805

- In working with the Hilpert equation all properties are evaluated at the film temperature.

$$T_f \equiv \frac{T_s + T_\infty}{2}$$

EXAMPLE 7.4

The Sphere

- Numerous heat transfer correlations have been proposed, and Whitaker recommends an expression of the form

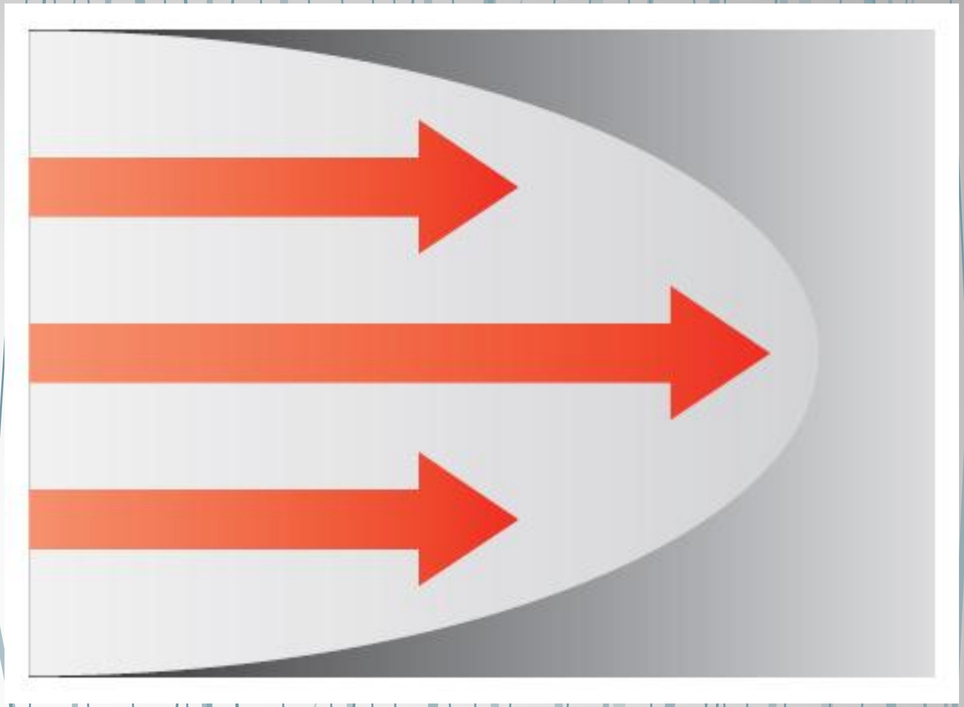
$$\overline{Nu}_D = 2 + (0.4\, Re_D^{1/2} + 0.06\, Re_D^{2/3}) Pr^{0.4} \left(\frac{\mu}{\mu_s} \right)^{1/4}$$

$$\left[\begin{array}{l} 0.71 \lesssim Pr \lesssim 380 \\ 3.5 \lesssim Re_D \lesssim 7.6 \times 10^4 \\ 1.0 \lesssim (\mu/\mu_s) \lesssim 3.2 \end{array} \right]$$

- All properties except μ_s are evaluated at T_∞.

- In the limit $Re_D \rightarrow 0$, the above equation reduce to $\overline{Nu}_D = 2$, which corresponds to heat transfer by conduction from a spherical surface to a stationary, infinite medium around the surface.

EXAMPLE 7.6

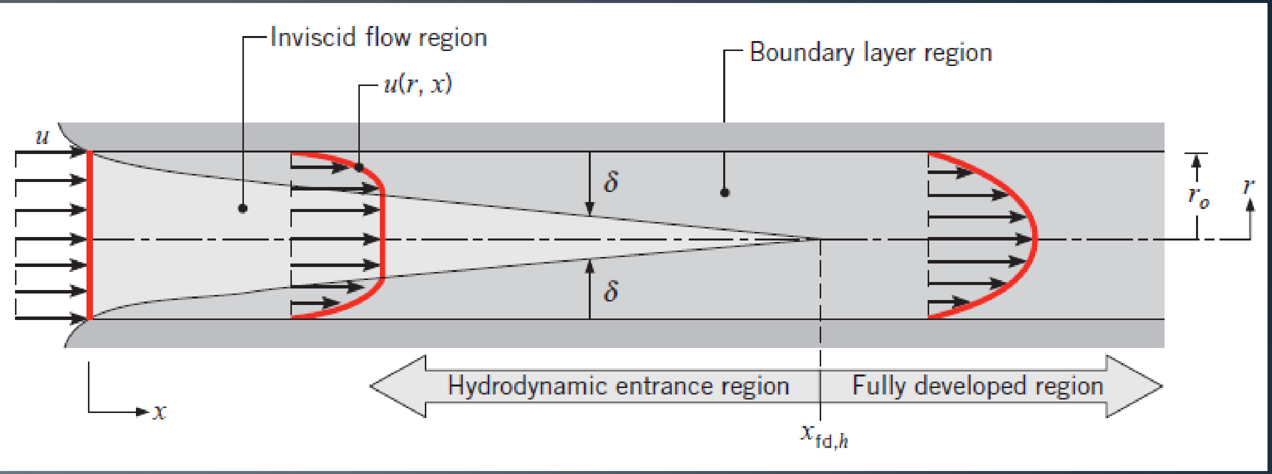


External Flow

Chapter 8

Hydrodynamic Considerations

- **Flow Conditions**



$$Re_D \equiv \frac{\rho u_m D}{\mu} = \frac{u_m D}{\nu}$$

$$Re_{D,c} \approx 2300$$

$$\left(\frac{x_{fd,h}}{D}\right)_{lam} \approx 0.05\, Re_D$$

$$10 \lesssim \left(\frac{x_{fd,h}}{D}\right)_{turb} \lesssim 60$$

Hydrodynamic Considerations

- **The Mean Velocity**

$$\dot{m} = \rho u_m A_c$$

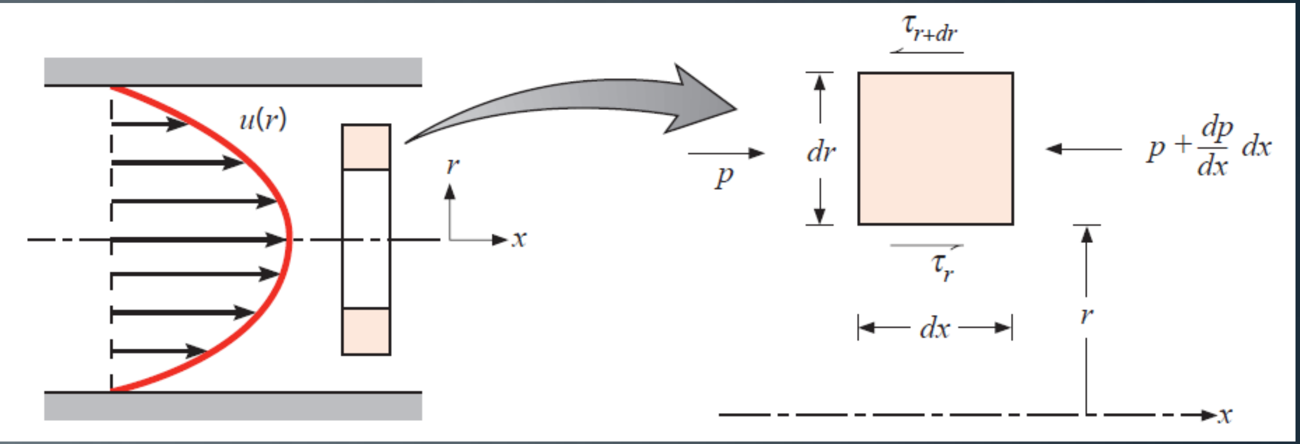
$$Re_D = \frac{4\dot{m}}{\pi D \mu}$$

$$\dot{m} = \int_{A_c} \rho u(r, x)\, dA_c$$

$$u_m = \frac{\int_{A_c} \rho u(r, x)\, dA_c}{\rho A_c} = \frac{2\pi\rho}{\rho\pi r_o^2}\int_0^{r_o} u(r, x) r\, dr = \frac{2}{r_o^2}\int_0^{r_o} u(r, x) r\, dr$$

- **Velocity Profile in the Fully Developed Region**

Hydrodynamic Considerations



$$u(r) = -\frac{1}{4\mu}\left(\frac{dp}{dx}\right) r_o^2\left[1-\left(\frac{r}{r_o}\right)^2\right]$$

$$u_m = -\frac{r_o^2}{8\mu}\frac{dp}{dx}$$

$$\frac{u(r)}{u_m} = 2\left[1-\left(\frac{r}{r_o}\right)^2\right]$$

Hydrodynamic Considerations

- **Pressure Gradient and Friction Factor in Fully Developed Flow**
- To determine the pressure drop, it is convenient to work with the *Moody* (or *Darcy*) *friction factor*, which is a dimensionless parameter defined as

$$f \equiv \frac{-(dp/dx)D}{\rho u_m^2/2}$$

$$C_f \equiv \frac{\tau_s}{\rho u_m^2/2} \qquad C_f = \frac{f}{4}$$

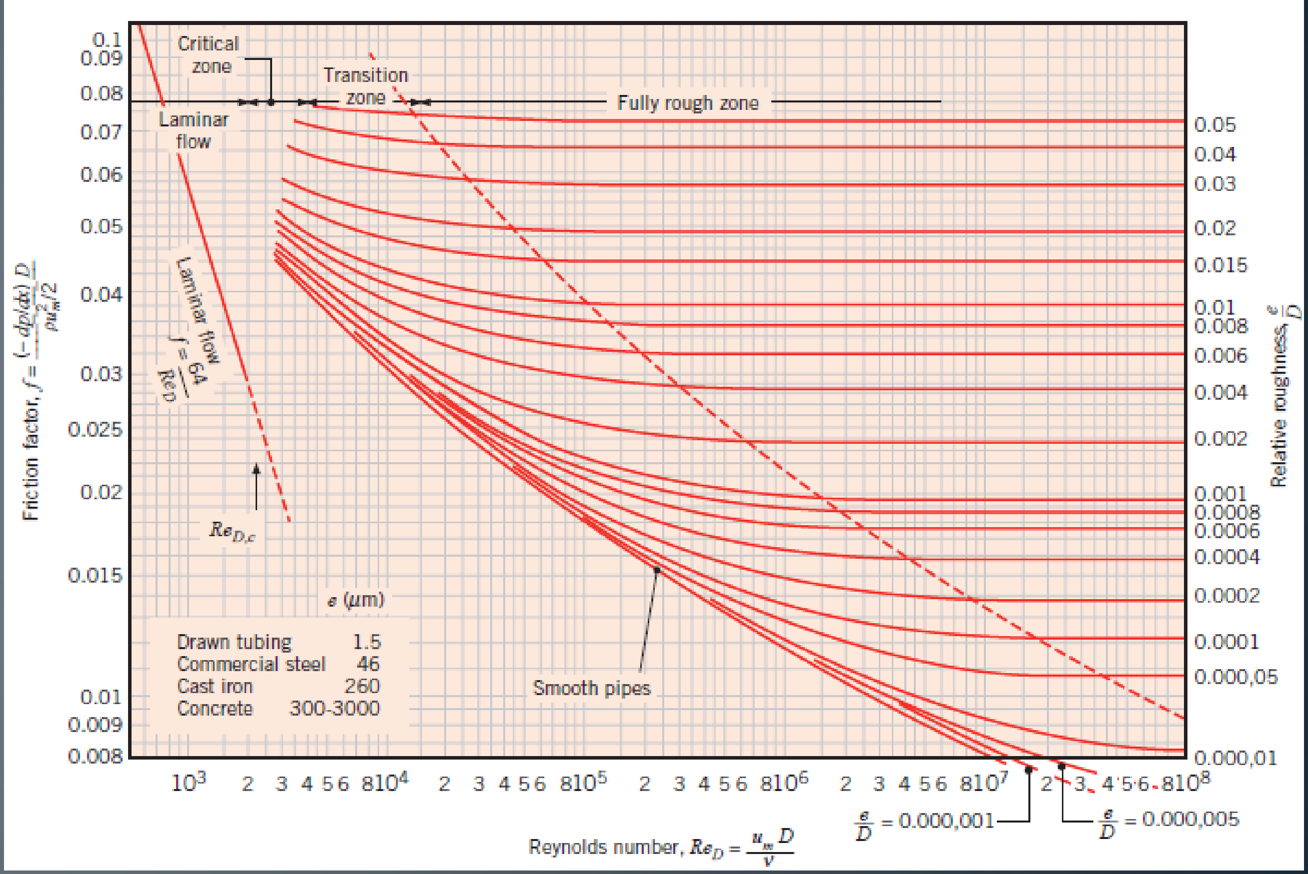
- For fully developed laminar flow

$$f = \frac{64}{Re_D}$$

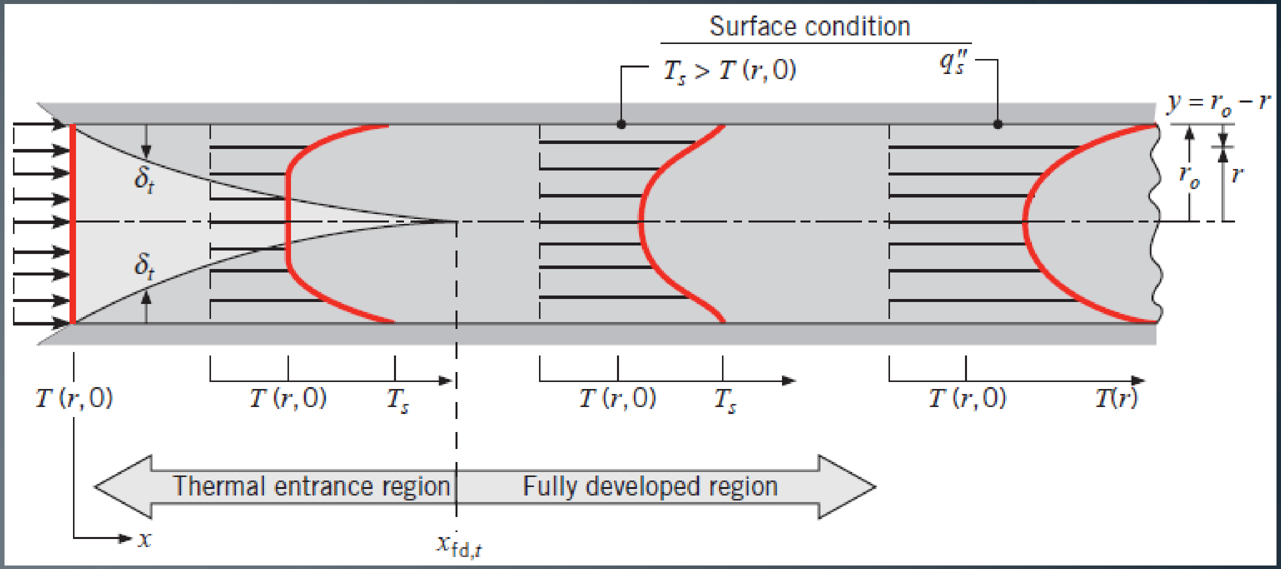
- For fully developed turbulent flow

$$\frac{1}{\sqrt{f}} = -2.0 \log\left[\frac{e/D}{3.7} + \frac{2.51}{Re_D\sqrt{f}}\right]$$

Hydrodynamic Considerations



Thermal Considerations



$$\left(\frac{x_{fd,t}}{D}\right)_{lam} \approx 0.05\, Re_D\, Pr$$

$$(x_{fd,t}/D) = 10$$

Thermal Considerations

- **The Mean Temperature**

$$q = \dot{m}\, c_p (T_{\text{out}} - T_{\text{in}})$$

$$\dot{m} c_p T_m = \int_{A_c} \rho u c_p T \, dA_c$$

$$T_m = \frac{\int_{A_c} \rho u c_p T \, dA_c}{\dot{m} c_p}$$

$$T_m = \frac{2}{u_m r_o^2} \int_0^{r_o} u T r \, dr$$

Thermal Considerations

- **Fully Developed Conditions**

$$\frac{\partial}{\partial x}\left[\frac{T_s(x) - T(r,x)}{T_s(x) - T_m(x)}\right]_{\mathrm{fd},t} = 0$$

$$\frac{\partial}{\partial r}\left(\frac{T_s - T}{T_s - T_m}\right)\Bigg|_{r=r_o} = \frac{-\partial T/\partial r|_{r=r_o}}{T_s - T_m} \neq f(x)$$

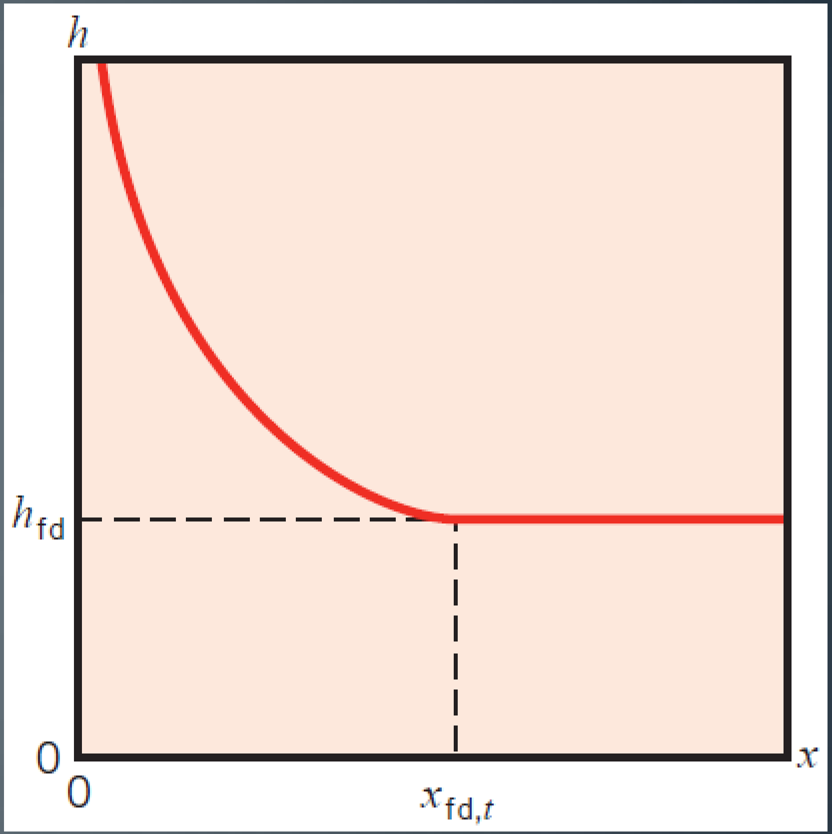
$$q_s'' = -k\frac{\partial T}{\partial y}\bigg|_{y=0} = k\frac{\partial T}{\partial r}\bigg|_{r=r_o}$$

$$\frac{h}{k} \neq f(x)$$

- Hence in the thermally *fully developed flow* of a fluid with *constant properties*, the *local convection coefficient is a constant, independent of x*.

- **Fully Developed Conditions**

Thermal Considerations



- Hence in the thermally *fully developed flow* of a fluid with *constant properties*, the *local convection coefficient is a constant, independent of x*.

Thermal Considerations

- **Fully Developed Conditions**

$$\left.\frac{dT_s}{dx}\right|_{\text{fd},t} = \left.\frac{dT_m}{dx}\right|_{\text{fd},t} \qquad q_s'' = \text{constant}$$

$$\left.\frac{\partial T}{\partial x}\right|_{\text{fd},t} = \left.\frac{dT_s}{dx}\right|_{\text{fd},t} - \left.\frac{(T_s - T)}{(T_s - T_m)}\frac{dT_s}{dx}\right|_{\text{fd},t} + \left.\frac{(T_s - T)}{(T_s - T_m)}\frac{dT_m}{dx}\right|_{\text{fd},t}$$

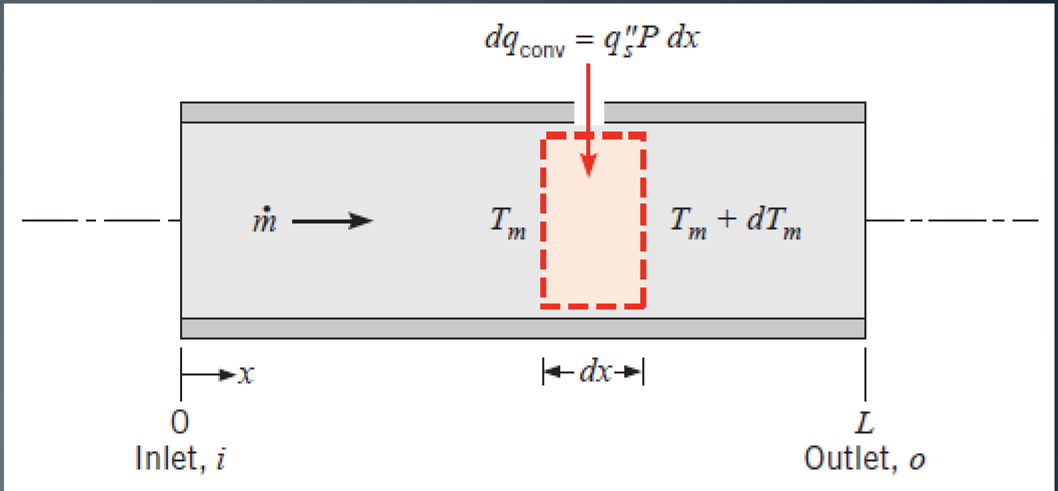
$$\left.\frac{\partial T}{\partial x}\right|_{\text{fd},t} = \left.\frac{dT_m}{dx}\right|_{\text{fd},t} \qquad q_s'' = \text{constant}$$

$$\left.\frac{\partial T}{\partial x}\right|_{\text{fd},t} = \left.\frac{(T_s - T)}{(T_s - T_m)}\frac{dT_m}{dx}\right|_{\text{fd},t} \qquad T_s = \text{constant}$$

EXAMPLE 8.1

- **General Considerations**

The Energy Balance



$$q_{conv} = \dot{m} c_p (T_{m,o} - T_{m,i})$$

$$dq_{conv} = \dot{m} c_p [(T_m + dT_m) - T_m]$$

$$dq_{conv} = \dot{m} c_p dT_m$$

$$\frac{dT_m}{dx} = \frac{q_s'' P}{\dot{m} c_p} = \frac{P}{\dot{m} c_p} h (T_s - T_m)$$

- **Constant Surface Heat Flux**

The Energy Balance

$$\frac{dT_m}{dx} = \frac{q_s''P}{\dot{m}c_p} = \frac{P}{\dot{m}c_p}h(T_s - T_m)$$

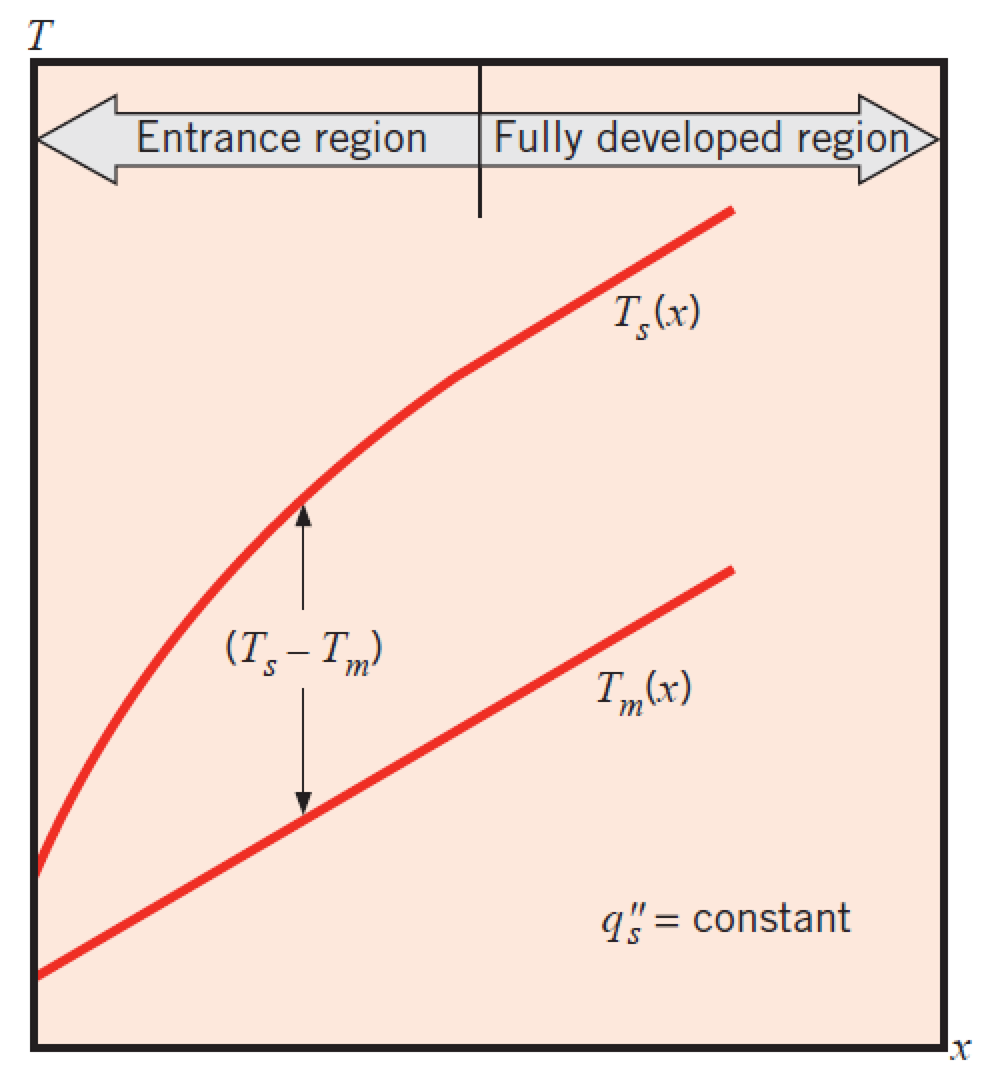
$$q_{\text{conv}} = q_s''(P \cdot L)$$

$$\frac{dT_m}{dx} = \frac{q_s''P}{\dot{m}c_p} \neq f(x)$$

$$T_m(x) = T_{m,i} + \frac{q_s''P}{\dot{m}c_p}x \qquad q_s'' = \text{constant}$$

- **Constant Surface Heat Flux**

The Energy Balance



EXAMPLE 8.2

The Energy Balance

- **Constant Surface Temperature**

$$\frac{dT_m}{dx} = -\frac{d(\Delta T)}{dx} = \frac{P}{\dot{m}c_p} h\,\Delta T$$

$$\int_{\Delta T_i}^{\Delta T_o} \frac{d(\Delta T)}{\Delta T} = -\frac{P}{\dot{m}c_p}\int_0^L h\,dx$$

$$\ln\frac{\Delta T_o}{\Delta T_i} = -\frac{PL}{\dot{m}c_p}\left(\frac{1}{L}\int_0^L h\,dx\right)$$

$$\ln\frac{\Delta T_o}{\Delta T_i} = -\frac{PL}{\dot{m}c_p}\bar{h}_L \qquad T_s = \text{constant}$$

The Energy Balance

- **Constant Surface Temperature**

$$\frac{\Delta T_o}{\Delta T_i} = \frac{T_s - T_{m,o}}{T_s - T_{m,i}} = \exp\left(-\frac{PL}{\dot{m}c_p}\bar{h}\right) \qquad T_s = \text{constant}$$

$$\frac{T_s - T_m(x)}{T_s - T_{m,i}} = \exp\left(-\frac{Px}{\dot{m}c_p}\bar{h}\right) \qquad T_s = \text{constant}$$

- This result tells us that the temperature difference $(T_s - T_m)$ *decays exponentially* with distance along the tube axis.

$$q_{\text{conv}} = \dot{m}c_p[(T_s - T_{m,i}) - (T_s - T_{m,o})] = \dot{m}c_p(\Delta T_i - \Delta T_o)$$

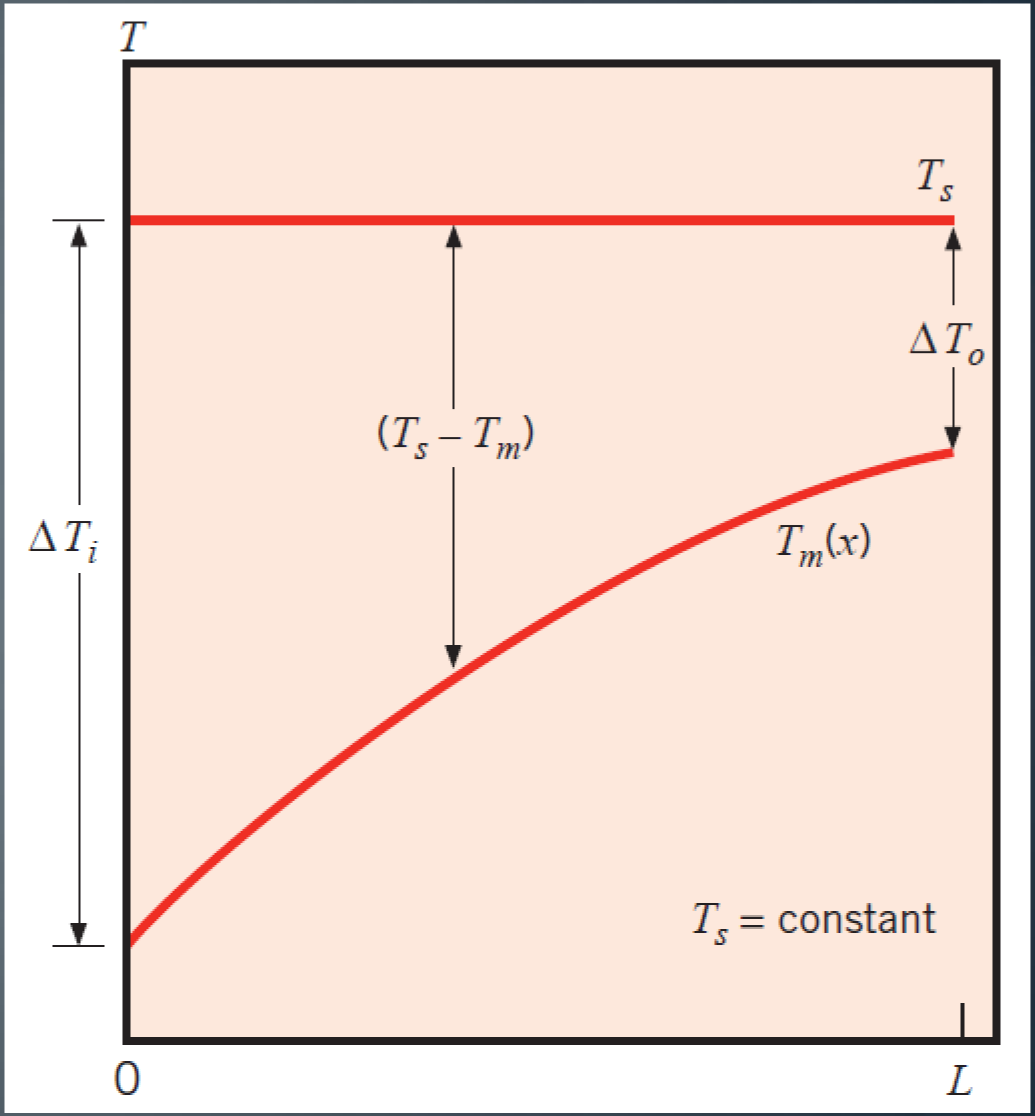
$$q_{\text{conv}} = \bar{h}A_s\Delta T_{\text{lm}} \qquad T_s = \text{constant}$$

- T_{lm} is the *log mean temperature difference*,

$$\Delta T_{\text{lm}} \equiv \frac{\Delta T_o - \Delta T_i}{\ln(\Delta T_o/\Delta T_i)}$$

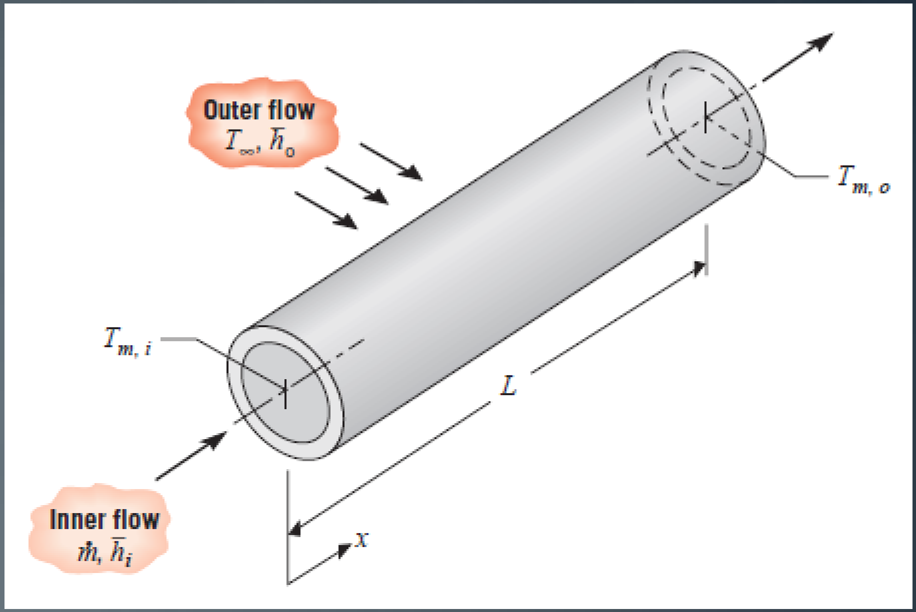
- **Constant Surface Temperature**

The Energy Balance



The Energy Balance

- **Constant Surface Temperature**



$$\frac{\Delta T_o}{\Delta T_i} = \frac{T_\infty - T_{m,o}}{T_\infty - T_{m,i}} = \exp\left(-\frac{\overline{U}A_s}{\dot{m}c_p}\right)$$

$$q = \overline{U}A_s \Delta T_{\text{lm}}$$

$$\frac{\Delta T_o}{\Delta T_i} = \frac{T_\infty - T_{m,o}}{T_\infty - T_{m,i}} = \exp\left(-\frac{1}{\dot{m}c_p R_{\text{tot}}}\right)$$

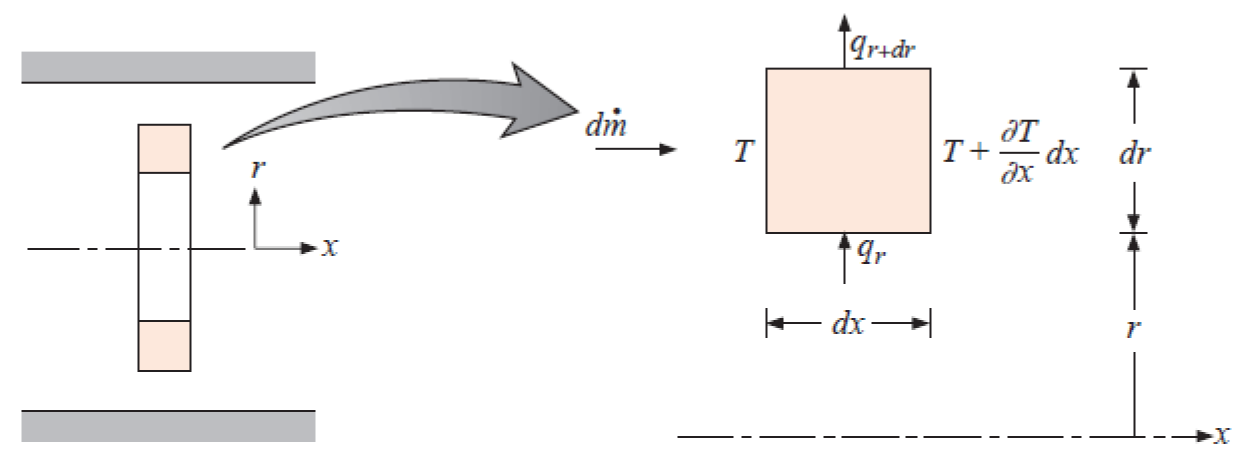
$$q = \frac{\Delta T_{\text{lm}}}{R_{\text{tot}}}$$

EXAMPLE 8.3

Laminar Flow in Circular Tubes: Thermal Analysis and Convection Correlations

- **The Fully Developed Region**

- The problem of heat transfer in *laminar flow* of an *incompressible*, *constant property fluid* in the *fully developed region* of a *circular tube* is treated theoretically.



$$q_r - q_{r+dr} = (d\dot{m})c_p\left[\left(T + \frac{\partial T}{\partial x}dx\right) - T\right]$$

$$(d\dot{m})c_p \frac{\partial T}{\partial x} dx = q_r - \left(q_r + \frac{\partial q_r}{\partial r} dr\right) = -\frac{\partial q_r}{\partial r} dr$$

- **The Fully Developed Region**

Laminar Flow in Circular Tubes: Thermal Analysis and Convection Correlations

$$(d\dot{m})c_p \frac{\partial T}{\partial x} dx = q_r - \left(q_r + \frac{\partial q_r}{\partial r} dr \right) = -\frac{\partial q_r}{\partial r} dr$$

$$d\dot{m} = \rho u 2\pi r dr$$

$$q_r = -k(\partial T/\partial r) 2\pi r dx$$

$$u \frac{\partial T}{\partial x} = \frac{\alpha}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right)$$

$$\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right) = \frac{2u_m}{\alpha}\left(\frac{dT_m}{dx} \right)\left[1 - \left(\frac{r}{r_o} \right)^2 \right]$$

$$q_s'' = \text{constant}$$

Laminar Flow in Circular Tubes: Thermal Analysis and Convection Correlations

- **The Fully Developed Region**

$$T(r,x) = \frac{2u_m}{\alpha}\left(\frac{dT_m}{dx}\right)\left[\frac{r^2}{4} - \frac{r^4}{16r_o^2}\right] + C_1 \ln r + C_2$$

$$C_2 = T_s(x) - \frac{2u_m}{\alpha}\left(\frac{dT_m}{dx}\right)\left(\frac{3r_o^2}{16}\right)$$

$$T(r,x) = T_s(x) - \frac{2u_m r_o^2}{\alpha}\left(\frac{dT_m}{dx}\right)\left[\frac{3}{16} + \frac{1}{16}\left(\frac{r}{r_o}\right)^4 - \frac{1}{4}\left(\frac{r}{r_o}\right)^2\right]$$

$$T_m(x) = T_s(x) - \frac{11}{48}\left(\frac{u_m r_o^2}{\alpha}\right)\left(\frac{dT_m}{dx}\right)$$

$$\frac{dT_m}{dx} = \frac{q_s'' P}{\dot{m} c_p}$$

$$\dot{m} = \rho u_m (\pi D^2/4)$$

$$P = \pi D$$

Laminar Flow in Circular Tubes: Thermal Analysis and Convection Correlations

- **The Fully Developed Region**

$$T_m(x) - T_s(x) = -\frac{11}{48}\frac{q_s''D}{k}$$

$$h = \frac{48}{11}\left(\frac{k}{D}\right)$$

$$Nu_D \equiv \frac{hD}{k} = 4.36 \qquad q_s'' = \text{constant}$$

- Hence in a *circular tube* characterized by uniform *surface heat flux* and *laminar*, *fully developed conditions*, the *Nusselt number is a constant*, independent of Re_D, Pr, and axial location.

Laminar Flow in Circular Tubes: Thermal Analysis and Convection Correlations

- **The Fully Developed Region**

- For *laminar, fully developed conditions* with a *constant surface temperature*, the assumption of negligible axial conduction is often reasonable.
- Substituting for the velocity profile and for the axial temperature gradient, the energy equation becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{2u_m}{\alpha}\left(\frac{dT_m}{dx}\right)\left[1-\left(\frac{r}{r_o}\right)^2\right]\frac{T_s - T}{T_s - T_m}$$

$$T_s = \text{constant}$$

- A solution to this equation may be obtained by an iterative procedure, which involves making successive approximations to the temperature profile. The resulting profile is not described by a simple algebraic expression, but the resulting Nusselt number may be shown to be

$$Nu_D = 3.66 \qquad T_s = \text{constant}$$

EXAMPLE 8.4

Convection Correlations: Turbulent Flow in Circular Tubes

- **The Fully Developed Region**

- Since the analysis of turbulent flow conditions is a good deal more involved, greater emphasis is placed on determining empirical correlations.
- For *fully developed (hydrodynamically and thermally) turbulent flow* in a *smooth circular tube*, *the local* Nusselt number may be obtained from the *Dittus-Boelter equation*

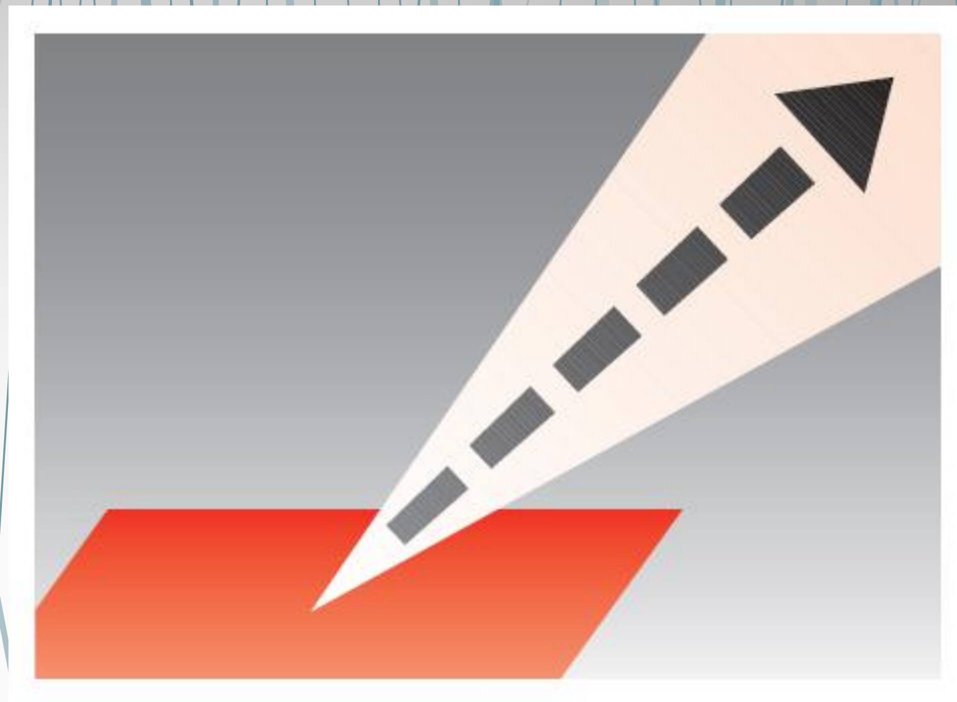
$$Nu_D = 0.023\, Re_D^{4/5}\, Pr^n$$

- where $n = 0.4$ for heating $(T_s > T_m)$ and 0.3 for cooling $(T_s < T_m)$.

$$\begin{bmatrix} 0.6 \lesssim Pr \lesssim 160 \\ Re_D \gtrsim 10{,}000 \\ \dfrac{L}{D} \gtrsim 10 \end{bmatrix}$$

EXAMPLE 8.5

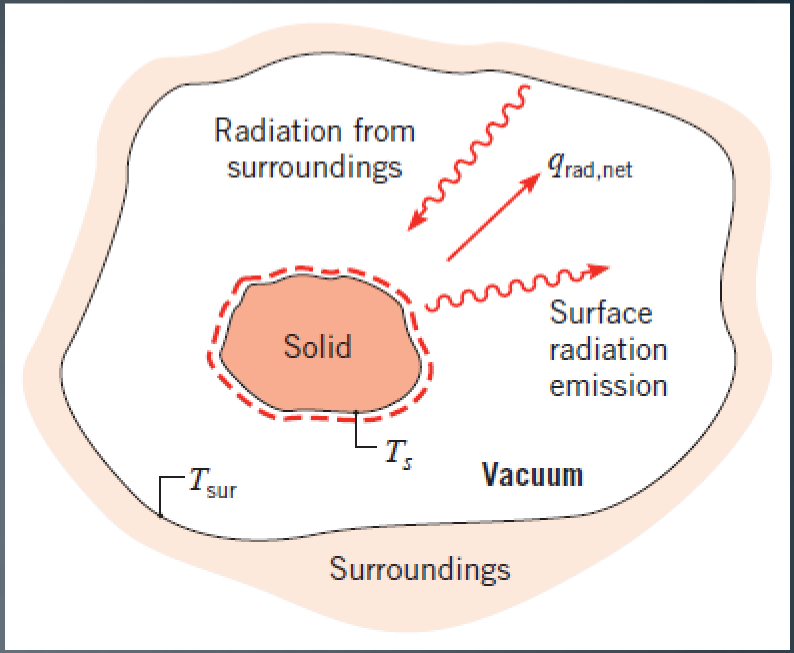
EXAMPLE 8.6

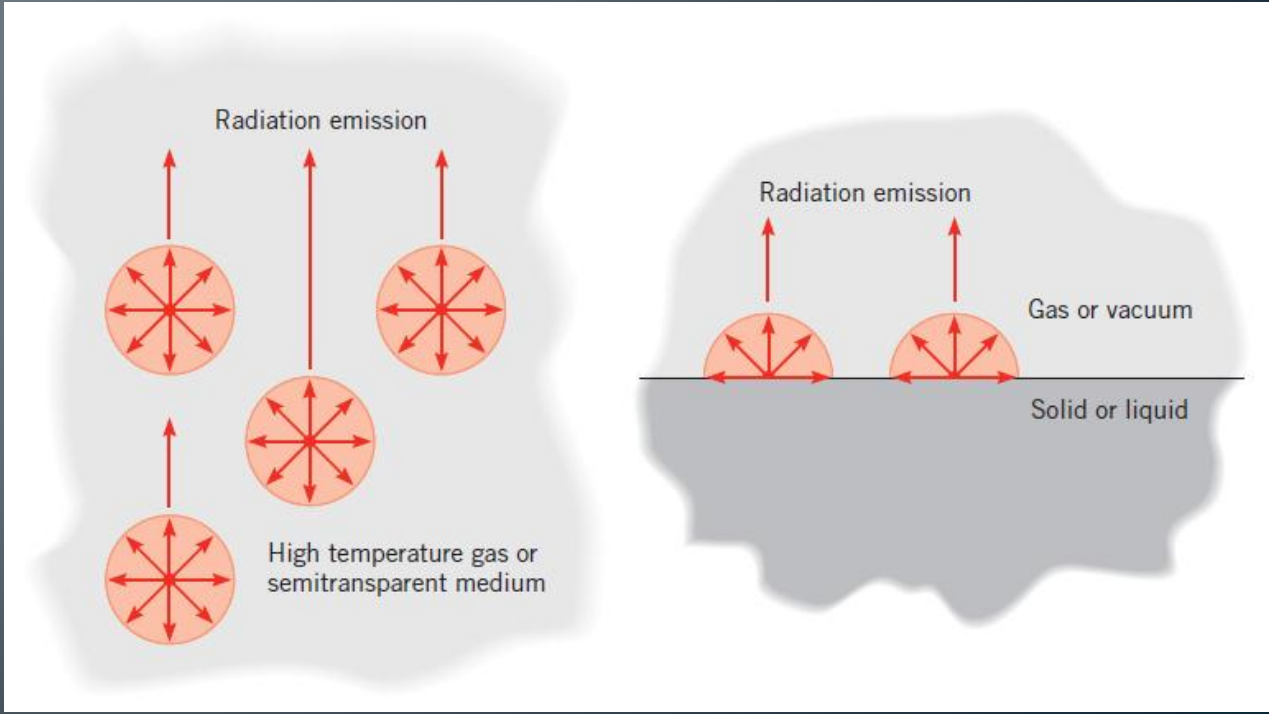


Radiation: Processes and Properties

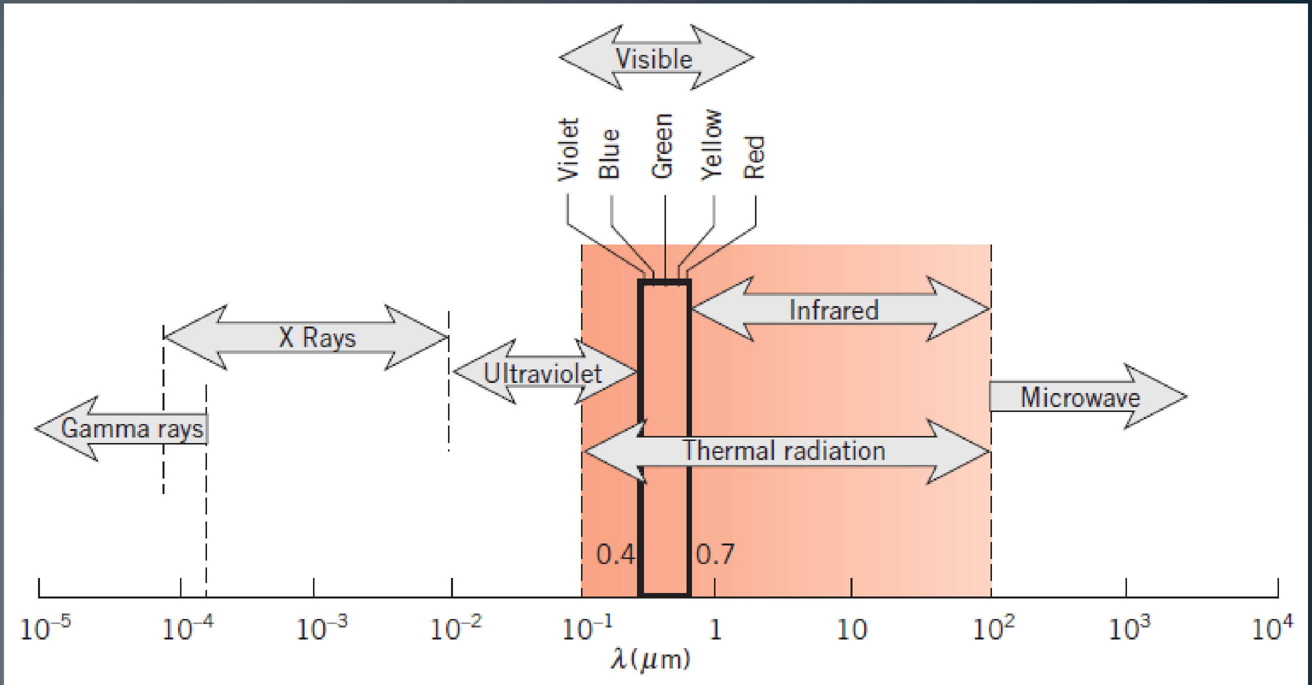
Chapter 12

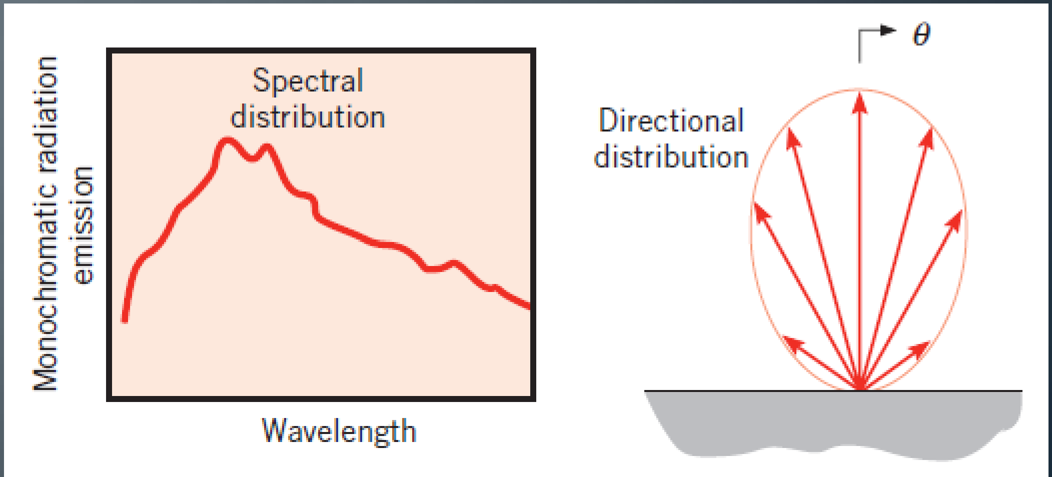
Fundamental Concepts





Fundamental Concepts

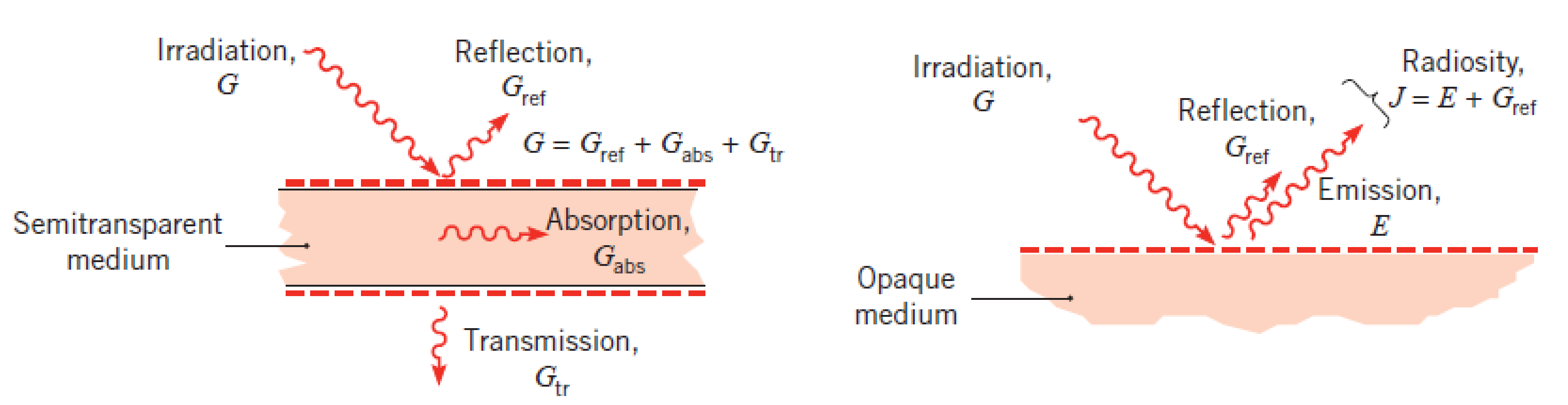




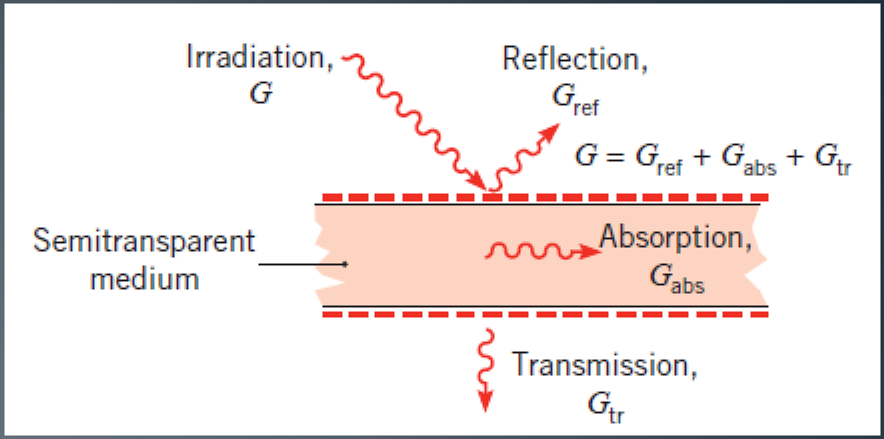
Radiation Heat Fluxes

Radiative fluxes (over all wavelengths and in all directions)

Flux (W/m²)	Description	Comment
Emissive power, E	Rate at which radiation is emitted from a surface per unit area	$E = \varepsilon\sigma T_s^4$
Irradiation, G	Rate at which radiation is incident upon a surface per unit area	Irradiation can be reflected, absorbed, or transmitted
Radiosity, J	Rate at which radiation leaves a surface per unit area	For an opaque surface $J = E + \rho G$
Net radiative flux, $q''_{rad} = J - G$	Net rate of radiation leaving a surface per unit area	For an opaque surface $q''_{rad} = \varepsilon\sigma T_s^4 - \alpha G$



Radiation Heat Fluxes



$$\rho + \alpha + \tau = 1$$

- A medium that experiences no transmission $(\tau = 0)$ is *opaque*, in which case

$$\rho + \alpha = 1$$

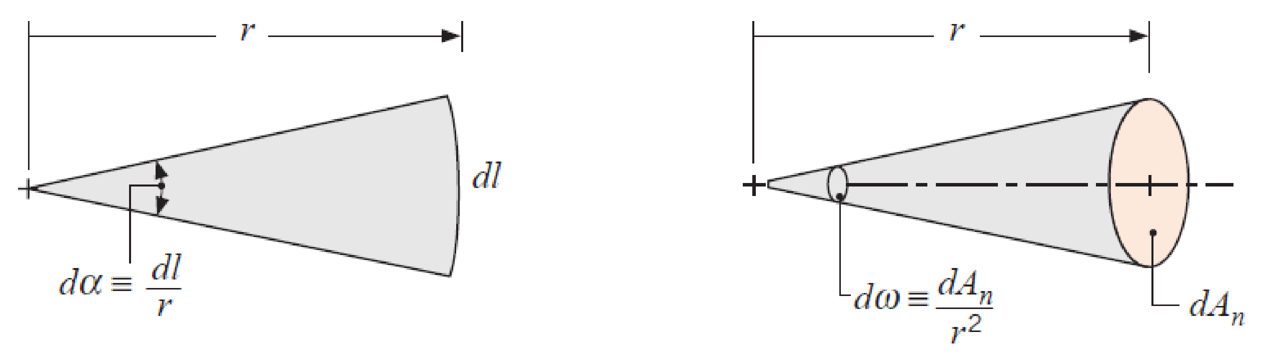
$$J = E + G_{\text{ref}} = E + \rho G$$

$$q''_{\text{rad}} = J - G$$

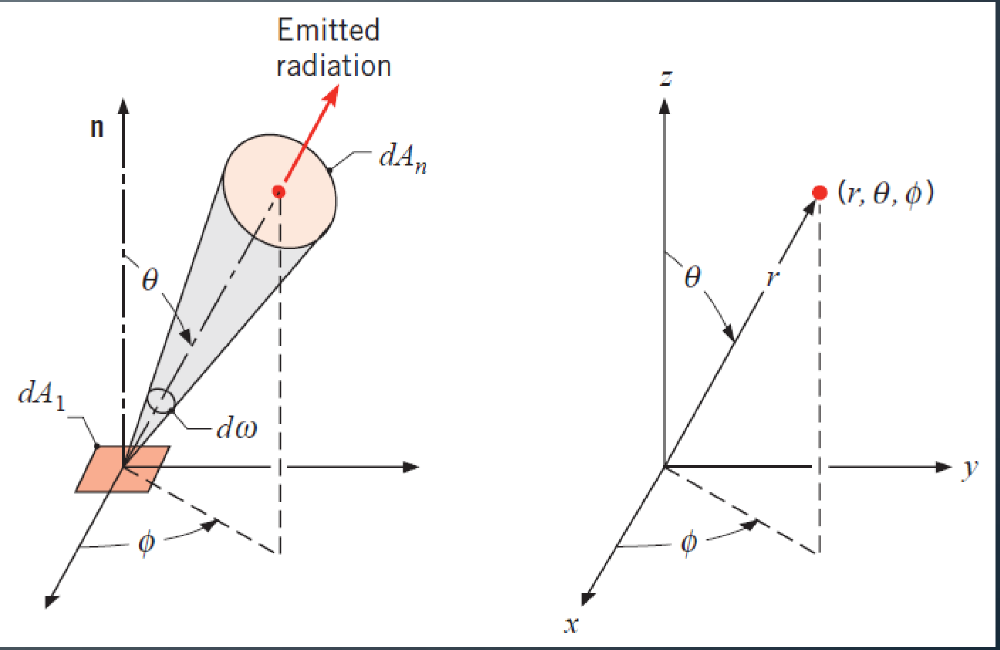
$$q''_{\text{rad}} = E + \rho G - G = \varepsilon \sigma T_s^4 - \alpha G$$

- **Mathematical Definitions**

Radiation Intensity

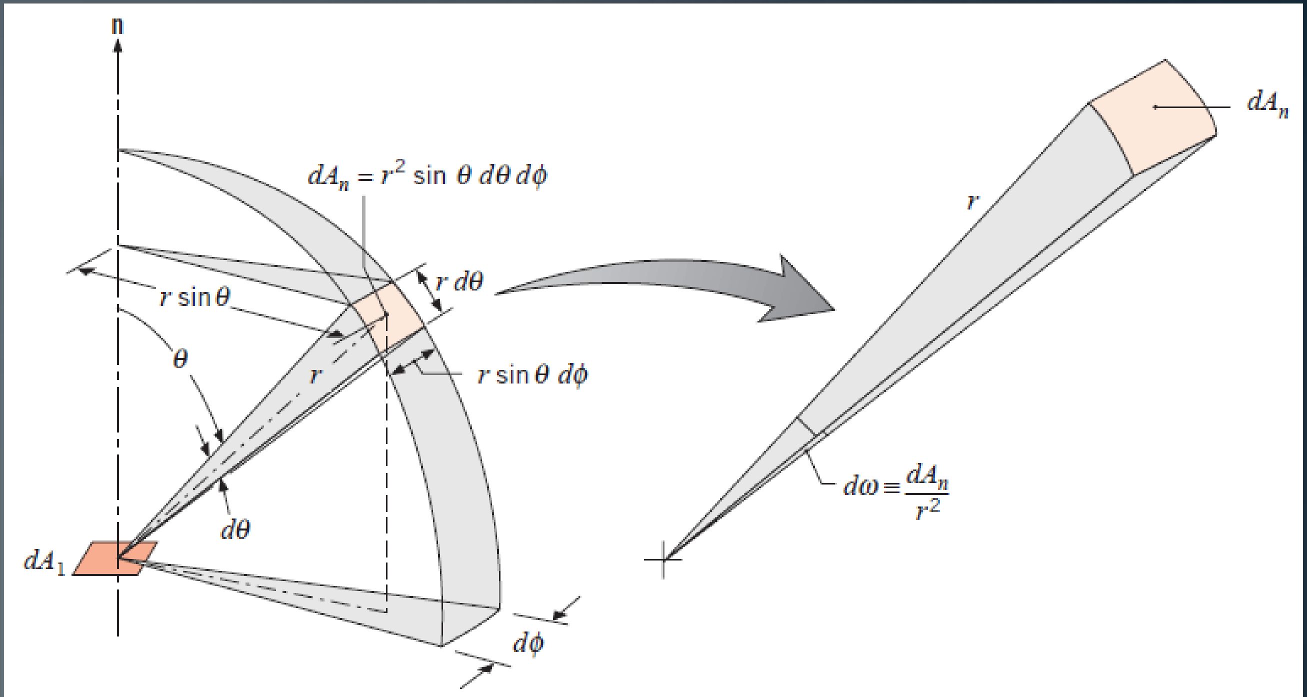


$$d\omega \equiv \frac{dA_n}{r^2}$$



- **Mathematical Definitions**

Radiation Intensity



$$dA_n = r^2 \sin\theta \, d\theta \, d\phi$$

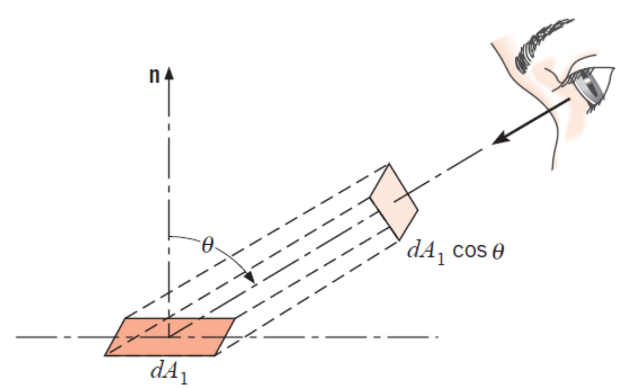
$$d\omega = \sin\theta \, d\theta \, d\phi$$

$$\int_h d\omega = \int_0^{2\pi}\int_0^{\pi/2} \sin\theta \, d\theta \, d\phi = 2\pi \int_0^{\pi/2} \sin\theta \, d\theta = 2\pi \text{ sr}$$

Radiation Intensity

- **Radiation Intensity and Its Relation to Emission**
- We formally define *spectral intensity* $I_{\lambda,e}$ as the *rate at which radiant energy is emitted at the wavelength* λ *in the* (θ,φ) *direction, per unit area of the emitting surface normal to this direction, per unit solid angle about this direction, and per unit wavelength interval* $d\lambda$ *about* λ.

$$I_{\lambda,e}(\lambda,\theta,\phi) \equiv \frac{dq}{dA_1\cos\theta\cdot d\omega\cdot d\lambda}$$

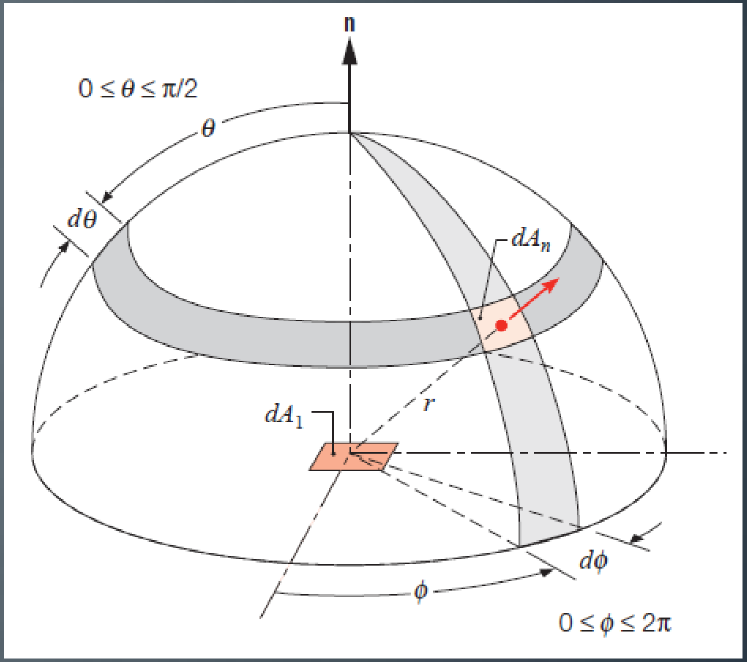


$$dq_\lambda = I_{\lambda,e}(\lambda,\theta,\phi)dA_1\cos\theta\,d\omega$$

$$dq''_\lambda = I_{\lambda,e}(\lambda,\theta,\phi)\cos\theta\sin\theta\,d\theta\,d\phi$$

Radiation Intensity

- **Radiation Intensity and Its Relation to Emission**
- We define the *spectral, hemispherical emissive power* $E_\lambda\ (W/m^2 \cdot \mu m)$ as the rate at which radiation of wavelength λ is emitted in *all directions* from a surface per unit wavelength interval $d\lambda$ about λ and per unit surface area.



$$E_\lambda(\lambda) = q''_\lambda(\lambda) = \int_0^{2\pi} \int_0^{\pi/2} I_{\lambda,e}(\lambda, \theta, \phi) \cos\theta \sin\theta \, d\theta \, d\phi$$

Radiation Intensity

- **Radiation Intensity and Its Relation to Emission**
- The *total, hemispherical emissive power,* $E\ (\mathrm{W/m^2})$,is the rate at which radiation is emitted per unit area at all possible wavelengths and in all possible directions.

$$E = \int_0^\infty E_\lambda(\lambda)\, d\lambda$$

$$E = \int_0^\infty \int_0^{2\pi} \int_0^{\pi/2} I_{\lambda,e}(\lambda, \theta, \phi) \cos\theta \sin\theta \, d\theta \, d\phi \, d\lambda$$

- We speak of a *diffuse emitter* as a surface for which the intensity of the emitted radiation is independent of direction, in which case

$$I_{\lambda,e}(\lambda, \theta, \phi) = I_{\lambda,e}(\lambda)$$

$$E_\lambda(\lambda) = \pi I_{\lambda,e}(\lambda)$$

$$E = \pi I_e$$

EXAMPLE 12.1

Radiation Intensity

- **Radiation Intensity and Its Relation to Emission**
- The *total, hemispherical emissive power,* $E\ (\mathrm{W/m^2})$,is the rate at which radiation is emitted per unit area at all possible wavelengths and in all possible directions.

$$E = \int_0^\infty E_\lambda(\lambda)\, d\lambda$$

$$E = \int_0^\infty \int_0^{2\pi} \int_0^{\pi/2} I_{\lambda,e}(\lambda, \theta, \phi) \cos\theta \sin\theta\, d\theta\, d\phi\, d\lambda$$

- We speak of a *diffuse emitter* as a surface for which the intensity of the emitted radiation is independent of direction, in which case

$$I_{\lambda,e}(\lambda,\ \theta,\ \phi) = I_{\lambda,e}(\lambda)$$

$$E_\lambda(\lambda) = \pi I_{\lambda,e}(\lambda)$$

$$E = \pi I_e$$

Radiation Intensity

- **Radiation Intensity and Its Relation to Emission**
- The *spectral irradiation* $G_\lambda\ (\mathrm{W/m^2 \cdot \mu m})$ is defined as the rate at which radiation of wavelength λ is incident on a surface, per unit area of the surface and per unit wavelength interval $d\lambda$ about λ.

$$G_\lambda(\lambda) = \int_0^{2\pi}\int_0^{\pi/2} I_{\lambda,i}(\lambda,\theta,\phi)\cos\theta\sin\theta\,d\theta\,d\phi$$