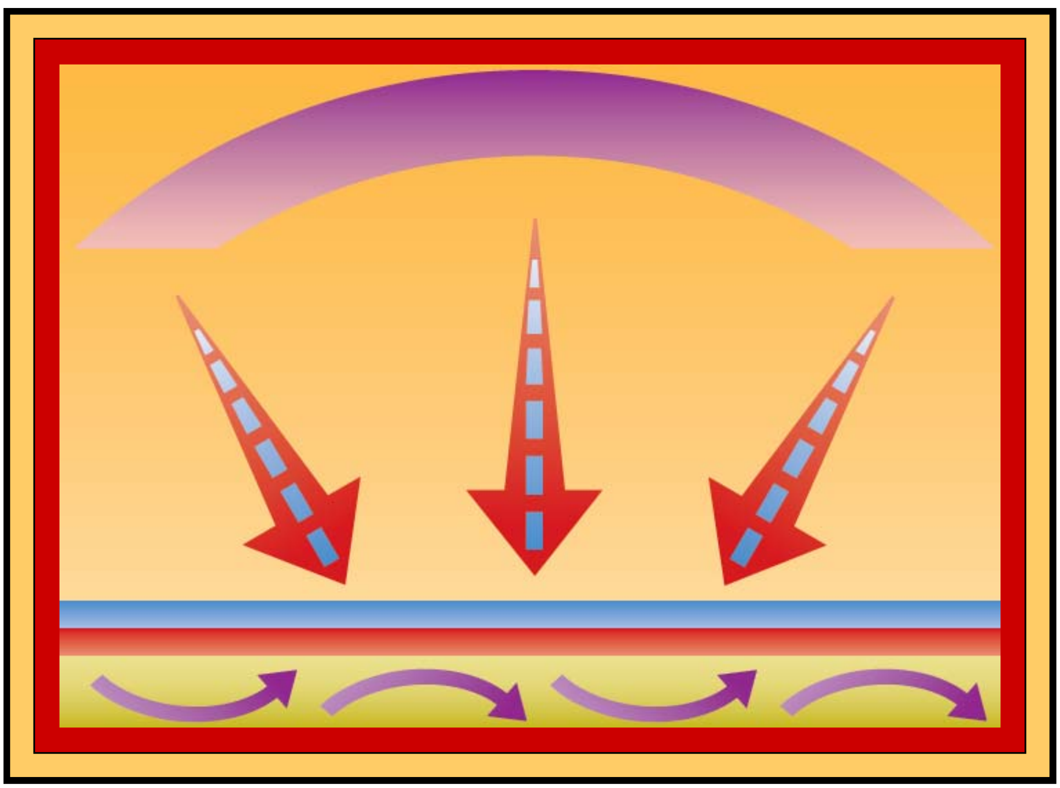
SEVENTH
EDITION

PRINCIPLES OF

HEAT and MASS TRANSFER

SUPPLEMENTAL MATERIAL



INCROPERA / DEWITT / BERGMAN / LAVINE

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4S.1 The Graphical Method

The graphical method may be employed for two-dimensional problems involving adiabatic and isothermal boundaries. Although the approach has been superseded by computer solutions based on numerical procedures, it may be used to obtain a first estimate of the temperature distribution and to develop a physical appreciation for the nature of the temperature field and heat flow.

4S.1.1 Methodology of Constructing a Flux Plot

The rationale for the graphical method comes from the fact that lines of constant temperature must be perpendicular to lines that indicate the direction of heat flow (see Figure 4.1). The objective of the graphical method is to systematically construct such a network of isotherms and heat flow lines. This network, commonly termed a *flux plot,* is used to infer the temperature distribution and the rate of heat flow through the system.

Consider a square, two-dimensional channel whose inner and outer surfaces are maintained at T_1 and T_2, respectively. A cross section of the channel is shown in Figure 4S.1*a*. A procedure for constructing the flux plot, a portion of which is shown in Figure 4S.1*b*, is as follows.

1. The first step is to *identify all relevant lines of symmetry.* Such lines are determined by thermal, as well as geometrical, conditions. For the square channel of Figure 4S.1*a*, such lines include the designated vertical, horizontal, and diagonal lines. For this system it is therefore possible to consider only one-eighth of the configuration, as shown in Figure 4S.1*b*.
2. *Lines of symmetry are adiabatic* in the sense that there can be no heat transfer in a direction perpendicular to the lines. They are therefore heat flow lines and should be treated as such. Since there is no heat flow in a direction perpendicular to a heat flow line, it can be termed an *adiabat.*
3. After all known lines of constant temperature are identified, an attempt should be made to sketch lines of constant temperature within the system. Note that *isotherms should always be perpendicular to adiabats.*
4. Heat flow lines should then be drawn with an eye toward creating a network of *curvilinear squares.* This is done by having the *heat flow lines and isotherms intersect at right angles* and by requiring that *all sides of each square be of approximately the same length.* It is often impossible to satisfy this second requirement exactly, and it is more realistic to strive for equivalence between the sums of the opposite sides of each square, as shown in Figure 4S.1*c*. Assigning the *x*-coordinate to the direction of heat

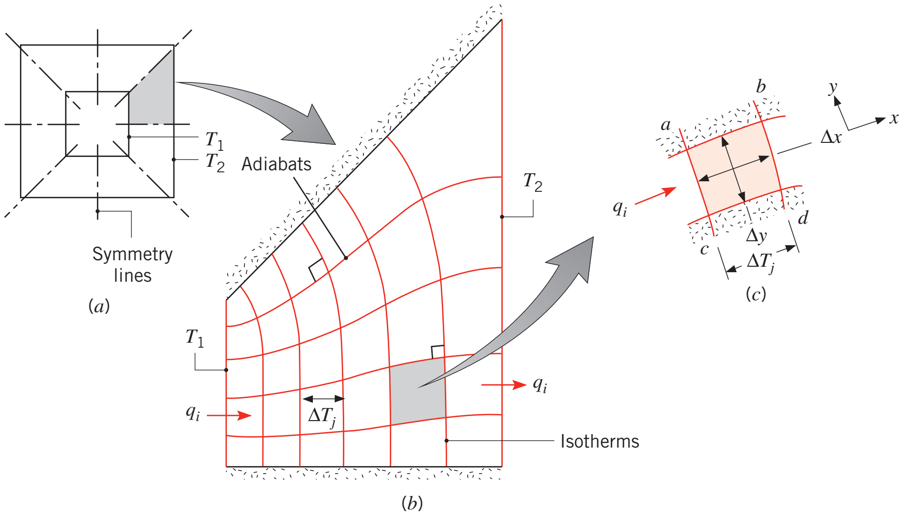


FIGURE 4S.1 Two-dimensional conduction in a square channel of length *l*. (*a*) Symmetry planes. (*b*) Flux plot. (*c*) Typical curvilinear square.

flow and the y-coordinate to the direction normal to this flow, the requirement may be expressed as

$$\Delta x \equiv \frac{ab + cd}{2} \approx \Delta y \equiv \frac{ac + bd}{2} \tag{4S.1}$$

It is difficult to create a satisfactory network of curvilinear squares in the first attempt, and several iterations must often be made. This trial-and-error process involves adjusting the isotherms and adiabats until satisfactory curvilinear squares are obtained for most of the network.[1] Once the flux plot has been obtained, it may be used to infer the temperature distribution in the medium. From a simple analysis, the heat transfer rate may then be obtained.

4S.1.2 Determination of the Heat Transfer Rate

The rate at which energy is conducted through a *lane,* which is the region between adjoining adiabats, is designated as q_i. If the flux plot is properly constructed, the value of q_i will be approximately the same for all lanes and the total heat transfer rate may be expressed as

$$q \approx \sum_{i=1}^{M} q_i = M q_i \tag{4S.2}$$

where M is the *number of lanes* associated with the plot. From the curvilinear square of Figure 4S.1*c* and the application of Fourier's law, q_i may be expressed as

$$q_i \approx k A_i \frac{\Delta T_j}{\Delta x} \approx k(\Delta y \cdot l) \frac{\Delta T_j}{\Delta x} \tag{4S.3}$$

[1]In certain regions, such as corners, it may be impossible to approach the curvilinear square requirements. However, such difficulties generally have a small effect on the overall accuracy of the results obtained from the flux plot.

where ΔT_j is the temperature difference between successive isotherms, A_i is the conduction heat transfer area for the lane, and l is the length of the channel normal to the page. However, since the temperature increment is approximately the same for all adjoining isotherms, the overall temperature difference between boundaries, ΔT_{1-2}, may be expressed as

$$\Delta T_{1-2} = \sum_{j=1}^{N} \Delta T_j \approx N \, \Delta T_j \tag{4S.4}$$

where N is the total number of temperature increments. Combining Equations 4S.2 through 4S.4 and recognizing that $\Delta x \approx \Delta y$ for curvilinear squares, we obtain

$$q \approx \frac{Ml}{N} k \, \Delta T_{1-2} \tag{4S.5}$$

The manner in which a flux plot may be used to obtain the heat transfer rate for a two-dimensional system is evident from Equation 4S.5. The ratio of the number of heat flow lanes to the number of temperature increments (the value of M/N) may be obtained from the plot. Recall that specification of N is based on step 3 of the foregoing procedure, and the value, which is an integer, may be made large or small depending on the desired accuracy. The value of M is then a consequence of following step 4. Note that M is not necessarily an integer, since a fractional lane may be needed to arrive at a satisfactory network of curvilinear squares. For the network of Figure 4S.1*b*, $N = 6$ and $M = 5$. Of course, as the network, or *mesh,* of curvilinear squares is made finer, N and M increase and the estimate of M/N becomes more accurate.

4S.1.3 The Conduction Shape Factor

Equation 4S.5 may be used to define the *shape factor, S*, of a two-dimensional system. That is, the heat transfer rate may be expressed as

$$q = Sk\Delta T_{1-2} \tag{4S.6}$$

where, for a flux plot,

$$S \equiv \frac{Ml}{N} \tag{4S.7}$$

From Equation 4S.6, it also follows that a *two-dimensional conduction resistance* may be expressed as

$$R_{t,\text{cond(2D)}} = \frac{1}{Sk} \tag{4S.8}$$

Shape factors have been obtained for numerous two-dimensional systems, and results are summarized in Table 4.1 for some common configurations. In cases 1 through 9 and case 11, conduction is presumed to occur between boundaries that are maintained at uniform temperatures, with $\Delta T_{1-2} \equiv T_1 - T_2$. In case 10 conduction is between an isothermal surface (T_1) and a semi-infinite medium of uniform temperature (T_2) at locations well removed from the surface. Shape factors may also be defined for one-dimensional geometries, and from the results of Table 3.3, it follows that for plane, cylindrical, and spherical walls, respectively, the shape factors are A/L, $2\pi L/\ln(r_2/r_1)$, and $4\pi r_1 r_2/(r_2 - r_1)$. Results are available for many other configurations [1–4].

EXAMPLE 4S.1

A hole of diameter $D = 0.25$ m is drilled through the center of a solid block of square cross section with $w = 1$ m on a side. The hole is drilled along the length, $l = 2$ m, of the block, which has a thermal conductivity of $k = 150$ W/m·K. A warm fluid passing through the hole maintains an inner surface temperature of $T_1 = 75°C$, while the outer surface of the block is kept at $T_2 = 25°C$.

1. Using the flux plot method, determine the shape factor for the system.
2. What is the rate of heat transfer through the block?

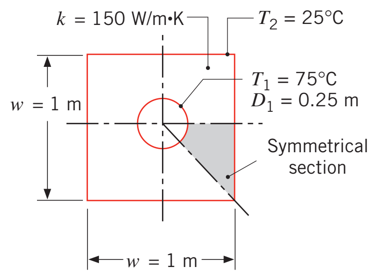
SOLUTION

Known: Dimensions and thermal conductivity of a block with a circular hole drilled along its length.

Find:

1. Shape factor.
2. Heat transfer rate for prescribed surface temperatures.

Schematic:

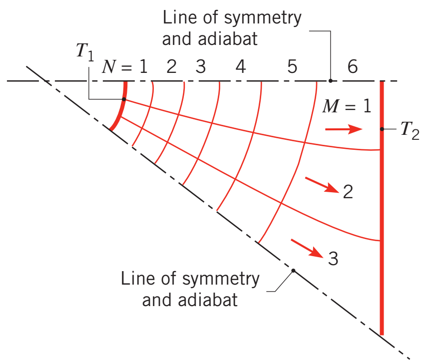


Assumptions:

1. Steady-state conditions.
2. Two-dimensional conduction.
3. Constant properties.
4. Ends of block are well insulated.

Analysis:

1. The flux plot may be simplified by identifying lines of symmetry and reducing the system to the one-eighth section shown in the schematic. Using a fairly coarse grid involving $N = 6$ temperature increments, the flux plot was generated. The resulting network of curvilinear squares is as follows.



With the number of heat flow lanes for the section corresponding to $M = 3$, it follows from Equation 4S.7 that the shape factor for the entire block is

$$S = 8\frac{Ml}{N} = 8\frac{3 \times 2 \text{ m}}{6} = 8 \text{ m} \quad \triangleleft$$

where the factor of 8 results from the number of symmetrical sections. The accuracy of this result may be determined by referring to Table 4.1, where, for the prescribed system, case 6, it follows that

$$S = \frac{2\pi L}{\ln(1.08\, w/D)} = \frac{2\pi \times 2 \text{ m}}{\ln(1.08 \times 1 \text{ m}/0.25 \text{ m})} = 8.59 \text{ m}$$

Hence the result of the flux plot underpredicts the shape factor by approximately 7%. Note that, although the requirement $l \gg w$ is not satisfied for this problem, the shape factor result from Table 4.1 remains valid if there is negligible axial conduction in the block. This condition is satisfied if the ends are insulated.

2. Using $S = 8.59$ m with Equation 4S.6, the heat rate is

$$q = Sk(T_1 - T_2)$$

$$q = 8.59 \text{ m} \times 150 \text{ W/m} \cdot \text{K}\,(75 - 25)^\circ\text{C} = 64.4 \text{ kW} \quad \triangleleft$$

Comments: The accuracy of the flux plot may be improved by using a finer grid (increasing the value of N). How would the symmetry and heat flow lines change if the vertical sides were insulated? If one vertical and one horizontal side were insulated? If both vertical and one horizontal side were insulated?

4S.2 *The Gauss-Seidel Method: Example of Usage*

The Gauss-Seidel method, described in Appendix D, is utilized in the following example.

Example 4S.2

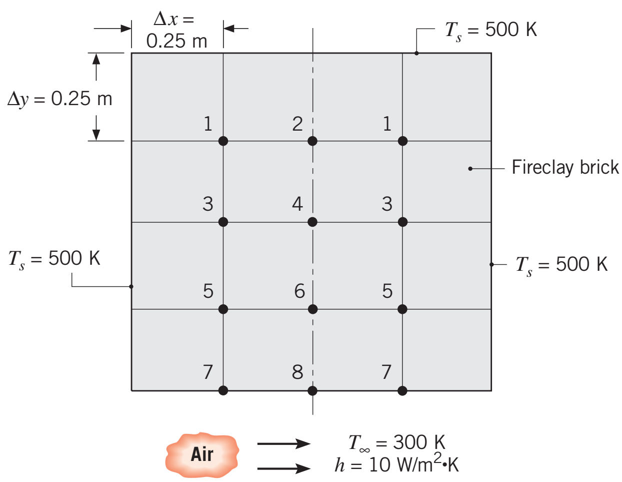
A large industrial furnace is supported on a long column of fireclay brick, which is 1 m × 1 m on a side. During steady-state operation, installation is such that three surfaces of the column are maintained at 500 K while the remaining surface is exposed to an airstream for which $T_\infty = 300$ K and $h = 10\ \text{W/m}^2 \cdot \text{K}$. Using a grid of $\Delta x = \Delta y = 0.25$ m, determine the two-dimensional temperature distribution in the column and the heat rate to the airstream per unit length of column.

Solution

Known: Dimensions and surface conditions of a support column.

Find: Temperature distribution and heat rate per unit length.

Schematic:



Assumptions:

1. Steady-state conditions.
2. Two-dimensional conduction.
3. Constant properties.
4. No internal heat generation.

Properties: Table A.3, fireclay brick ($T \approx 478$ K): $k = 1\ \text{W/m} \cdot \text{K}$.

Analysis: The prescribed grid consists of 12 nodal points at which the temperature is unknown. However, the number of unknowns is reduced to 8 through symmetry, in which case the temperature of nodal points to the left of the symmetry line must equal the temperature of those to the right.

Nodes 1, 3, and 5 are interior points for which the finite-difference equations may be inferred from Equation 4.29. Hence

Node 1: $T_2 + T_3 + 1000 - 4T_1 = 0$

Node 3: $T_1 + T_4 + T_5 + 500 - 4T_3 = 0$

Node 5: $T_3 + T_6 + T_7 + 500 - 4T_5 = 0$

Equations for points 2, 4, and 6 may be obtained in a like manner or, since they lie on a symmetry adiabat, by using Equation 4.42 with $h = 0$. Hence

Node 2: $2T_1 + T_4 + 500 - 4T_2 = 0$

Node 4: $T_2 + 2T_3 + T_6 - 4T_4 = 0$

Node 6: $T_4 + 2T_5 + T_8 - 4T_6 = 0$

From Equation 4.42 and the fact that $h\,\Delta x/k = 2.5$, it also follows that

Node 7: $2T_5 + T_8 + 2000 - 9T_7 = 0$

Node 8: $2T_6 + 2T_7 + 1500 - 9T_8 = 0$

Having the required finite-difference equations, the temperature distribution will be determined by using the Gauss–Seidel iteration method. Referring to the arrangement of finite-difference equations, it is evident that the order is already characterized by diagonal dominance. This behavior is typical of finite-difference solutions to conduction problems. We therefore begin with step 2 and express the equations in explicit form

$$T_1^{(k)} = 0.25T_2^{(k-1)} + 0.25T_3^{(k-1)} + 250$$
$$T_2^{(k)} = 0.50T_1^{(k)} + 0.25T_4^{(k-1)} + 125$$
$$T_3^{(k)} = 0.25T_1^{(k)} + 0.25T_4^{(k-1)} + 0.25T_5^{(k-1)} + 125$$
$$T_4^{(k)} = 0.25T_2^{(k)} + 0.50T_3^{(k)} + 0.25T_6^{(k-1)}$$
$$T_5^{(k)} = 0.25T_3^{(k)} + 0.25T_6^{(k-1)} + 0.25T_7^{(k-1)} + 125$$
$$T_6^{(k)} = 0.25T_4^{(k)} + 0.50T_5^{(k)} + 0.25T_8^{(k-1)}$$
$$T_7^{(k)} = 0.2222T_5^{(k)} + 0.1111T_8^{(k-1)} + 222.22$$
$$T_8^{(k)} = 0.2222T_6^{(k)} + 0.2222T_7^{(k)} + 166.67$$

Having the finite-difference equations in the required form, the iteration procedure may be implemented by using a table that has one column for the iteration (step) number and a column for each of the nodes labeled as T_i. The calculations proceed as follows:

1. For each node, the initial temperature estimate is entered on the row for $k = 0$. Values are selected rationally to reduce the number of required iterations.
2. Using the N finite-difference equations and values of T_i from the first and second rows, the new values of T_i are calculated for the first iteration ($k = 1$). These new values are entered on the second row.
3. This procedure is repeated to calculate $T_i^{(k)}$ from the previous values of $T_i^{(k-1)}$ and the current values of $T_i^{(k)}$, until the temperature difference between iterations meets the prescribed criterion, $\varepsilon \leq 0.2$ K, at every nodal point.

k	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
0	480	470	440	430	400	390	370	350
1	477.5	471.3	451.9	441.3	428.0	411.8	356.2	337.3
2	480.8	475.7	462.5	453.1	432.6	413.9	355.8	337.7
3	484.6	480.6	467.6	457.4	434.3	415.9	356.2	338.3
4	487.0	482.9	469.7	459.6	435.5	417.2	356.6	338.6
5	488.1	484.0	470.8	460.7	436.1	417.9	356.7	338.8
6	488.7	484.5	471.4	461.3	436.5	418.3	356.9	338.9
7	489.0	484.8	471.7	461.6	436.7	418.5	356.9	339.0
8	489.1	485.0	471.9	461.8	436.8	418.6	356.9	339.0

The results given in row 8 are in excellent agreement with those that would be obtained by an exact solution of the matrix equation, although better agreement could be obtained by reducing the value of ε. However, given the approximate nature of the finite-difference equations, the results still represent approximations to the actual temperatures. The accuracy of the approximation may be improved by using a finer grid (increasing the number of nodes).

The heat rate from the column to the airstream may be computed from the expression

$$\left(\frac{q}{L}\right) = 2h\left[\left(\frac{\Delta x}{2}\right)(T_s - T_\infty) + \Delta x\,(T_7 - T_\infty) + \left(\frac{\Delta x}{2}\right)(T_8 - T_\infty)\right]$$

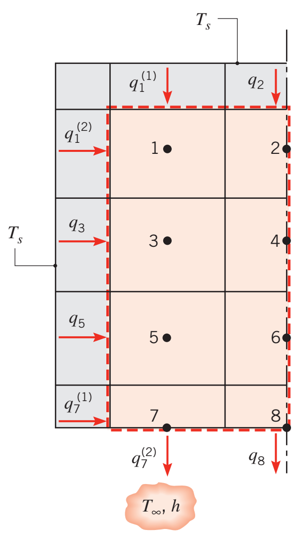
where the factor of 2 outside the brackets originates from the symmetry condition. Hence

$$\left(\frac{q}{L}\right) = 2 \times 10\ \text{W/m}^2 \cdot \text{K}[0.125\ \text{m}\ (200\ \text{K})$$

$$+ 0.25\ \text{m}\ (56.9\ \text{K}) + 0.125\ \text{m}\ (39.0\ \text{K})] = 882\ \text{W/m} \qquad \triangleleft$$

Comments:

1. To ensure that no errors have been made in formulating the finite-difference equations or in effecting their solution, a check should be made to verify that the results satisfy conservation of energy for the nodal network. For steady-state conditions, the requirement dictates that the rate of energy inflow be balanced by the rate of outflow *for a control surface surrounding all nodal regions whose temperatures have been evaluated.*



For the one-half symmetrical section shown schematically above, it follows that conduction into the nodal regions must be balanced by convection from the regions. Hence

$$q_1^{(1)} + q_1^{(2)} + q_2 + q_3 + q_5 + q_7^{(1)} = q_7^{(2)} + q_8$$

The cumulative conduction rate is then

$$\frac{q_{\text{cond}}}{L} = k\left[\Delta x \frac{(T_s - T_1)}{\Delta y} + \Delta y \frac{(T_s - T_1)}{\Delta x} + \frac{\Delta x}{2}\frac{(T_s - T_2)}{\Delta y}\right.$$
$$\left. + \Delta y \frac{(T_s - T_3)}{\Delta x} + \Delta y \frac{(T_s - T_5)}{\Delta x} + \frac{\Delta y}{2}\frac{(T_s - T_7)}{\Delta x}\right]$$
$$= 192.1 \text{ W/m}$$

and the convection rate is

$$\frac{q_{\text{conv}}}{L} = h\left[\Delta x(T_7 - T_\infty) + \frac{\Delta x}{2}(T_8 - T_\infty)\right] = 191.0 \text{ W/m}$$

Agreement between the conduction and convection rates is excellent, confirming that mistakes have not been made in formulating and solving the finite-difference equations. Note that convection transfer from the entire bottom surface (882 W/m) is obtained by adding transfer from the edge node at 500 K (250 W/m) to that from the interior nodes (191.0 W/m) and multiplying by 2 from symmetry.

2. Although the computed temperatures satisfy the finite-difference equations, they do not provide us with the exact temperature field. Remember that the equations are approximations whose accuracy may be improved by reducing the grid size (increasing the number of nodal points).

3. A second software package accompanying this text, *Finite-Element Heat Transfer (FEHT)*, may also be used to solve one- and two-dimensional forms of the heat equation. This example is provided as a solved model in *FEHT* and may be accessed through *Examples* on the *Toolbar*.

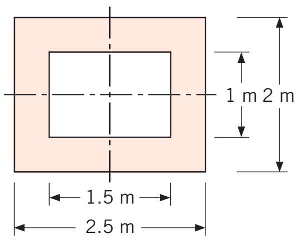
References

1. Sunderland, J. E., and K. R. Johnson, *Trans. ASHRAE*, **10**, 237–241, 1964.
2. Kutateladze, S. S., *Fundamentals of Heat Transfer*, Academic Press, New York, 1963.
3. General Electric Co. (Corporate Research and Development), *Heat Transfer Data Book*, Section 502, General Electric Company, Schenectady, NY, 1973.
4. Hahne, E., and U. Grigull, *Int. J. Heat Mass Transfer*, **18**, 751–767, 1975.

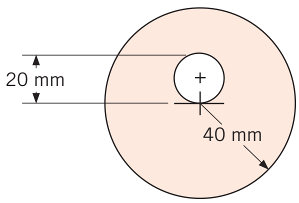
Problems

Flux Plotting

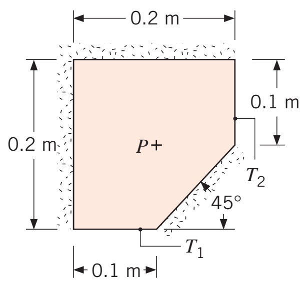
4S.1 A long furnace, constructed from refractory brick with a thermal conductivity of 1.2 W/m · K, has the cross section shown with inner and outer surface temperatures of 600 and 60°C, respectively. Determine the shape factor and the heat transfer rate per unit length using the flux plot method.



4S.2 A hot pipe is embedded eccentrically as shown in a material of thermal conductivity 0.5 W/m·K. Using the flux plot method, determine the shape factor and the heat transfer per unit length when the pipe and outer surface temperatures are 150 and 35°C, respectively.



4S.3 A supporting strut fabricated from a material with a thermal conductivity of 75 W/m·K has the cross section shown. The end faces are at different temperatures T_1 = 100°C and T_2 = 0°C, while the remaining sides are insulated.



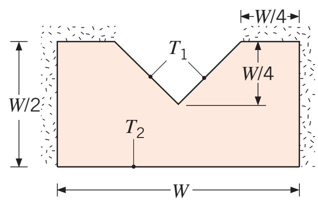
(a) Estimate the temperature at the location *P*.

(b) Using the flux plot method, estimate the shape factor and the heat transfer rate through the strut per unit length.

(c) Sketch the 25, 50, and 75°C isotherms.

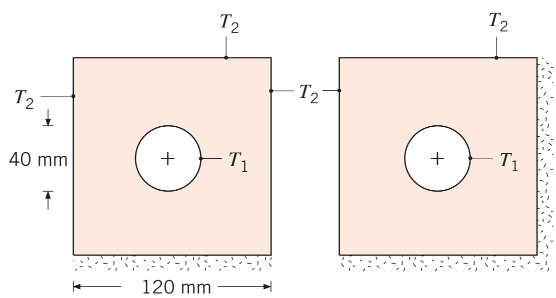
(d) Consider the same geometry, but now with the 0.1-m-wide surfaces insulated, the 45° surface maintained at T_1 = 100°C, and the 0.2-m-wide surfaces maintained at T_2 = 0°C. Using the flux plot method, estimate the corresponding shape factor and the heat rate per unit length. Sketch the 25, 50, and 75°C isotherms.

4S.4 A hot liquid flows along a V-groove in a solid whose top and side surfaces are well insulated and whose bottom surface is in contact with a coolant.

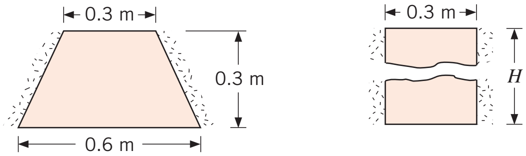


Accordingly, the V-groove surface is at a temperature T_1, which exceeds that of the bottom surface, T_2. Construct an appropriate flux plot and determine the shape factor of the system.

4S.5 A very long conduit of inner circular cross section and a thermal conductivity of 1 W/m·K passes a hot fluid, which maintains the inner surface at T_1 = 50°C. The outer surfaces of square cross section are insulated or maintained at a uniform temperature of T_2 = 20°C, depending on the application. Find the shape factor and the heat rate for each case.



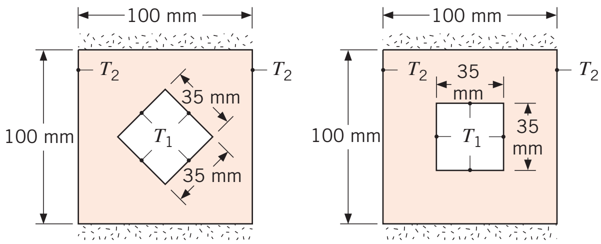
4S.6 A long support column of trapezoidal cross section is well insulated on its sides, and temperatures of 100 and 0°C are maintained at its top and bottom surfaces, respectively. The column is fabricated from AISI 1010 steel, and its widths at the top and bottom surfaces are 0.3 and 0.6 m, respectively.



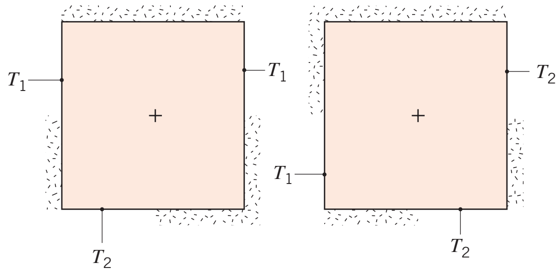
(a) Using the flux plot method, determine the heat transfer rate per unit length of the column.

(b) If the trapezoidal column is replaced by a bar of rectangular cross section 0.3 m wide and the same material, what height H must the bar be to provide an equivalent thermal resistance?

4S.7 Hollow prismatic bars fabricated from plain carbon steel are 1 m long with top and bottom surfaces, as well as both ends, well insulated. For each bar, find the shape factor and the heat rate per unit length of the bar when $T_1 = 500$ K and $T_2 = 300$ K.



4S.8 The two-dimensional, square shapes, 1 m to a side, are maintained at uniform temperatures, $T_1 = 100°C$ and $T_2 = 0°C$, on portions of their boundaries and are well insulated elsewhere.



(*a*) (*b*)

Use the flux plot method to estimate the heat rate per unit length normal to the page if the thermal conductivity is 50 W/m·K.

5S.1 *Graphical Representation of One-Dimensional, Transient Conduction in the Plane Wall, Long Cylinder, and Sphere*

In Sections 5.5 and 5.6, one-term approximations have been developed for transient, one-dimensional conduction in a plane wall (with symmetrical convection conditions) and radial systems (long cylinder and sphere). The results apply for $Fo > 0.2$ and can conveniently be represented in graphical forms that illustrate the functional dependence of the transient temperature distribution on the Biot and Fourier numbers.

Results for the plane wall (Figure 5.6*a*) are presented in Figures 5S.1 through 5S.3. Figure 5S.1 may be used to obtain the *midplane* temperature of the wall, $T(0, t) \equiv T_o(t)$, at any time during the transient process. If T_o is known for particular values of Fo and Bi, Figure 5S.2 may be used to determine the corresponding temperature at any location *off the midplane*. Hence Figure 5S.2 must be used in conjunction with Figure 5S.1. For example, if one wishes to determine the surface temperature ($x^* = \pm 1$) at some time t, Figure 5S.1 would first be used to determine T_o at t. Figure 5S.2 would then be used to determine the surface temperature from knowledge of T_o. The procedure would be inverted if the problem were one of determining the time required for the surface to reach a prescribed temperature.

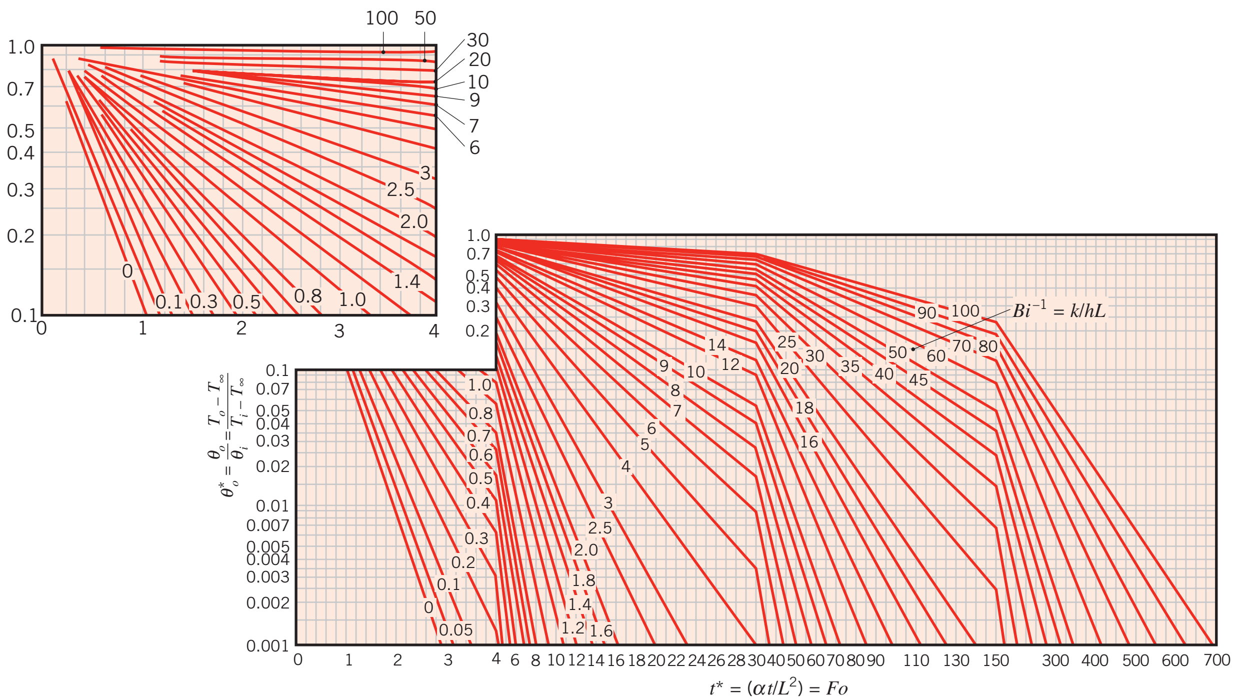


FIGURE 5S.1 Midplane temperature as a function of time for a plane wall of thickness $2L$ [1]. (Used with permission.)

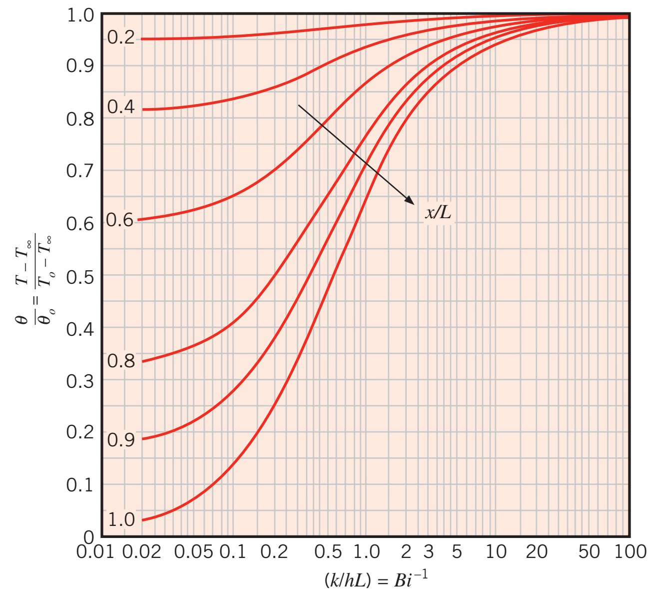


FIGURE 5S.2 Temperature distribution in a plane wall of thickness $2L$ [1]. (Used with permission.)

Graphical results for the energy transferred from a plane wall over the time interval t are presented in Figure 5S.3. These results were generated from Equation 5.49. The dimensionless energy transfer Q/Q_o is expressed exclusively in terms of Fo and Bi.

Results for the infinite cylinder are presented in Figures 5S.4 through 5S.6, and those for the sphere are presented in Figures 5S.7 through 5S.9, where the Biot number is defined in terms of the radius r_o.

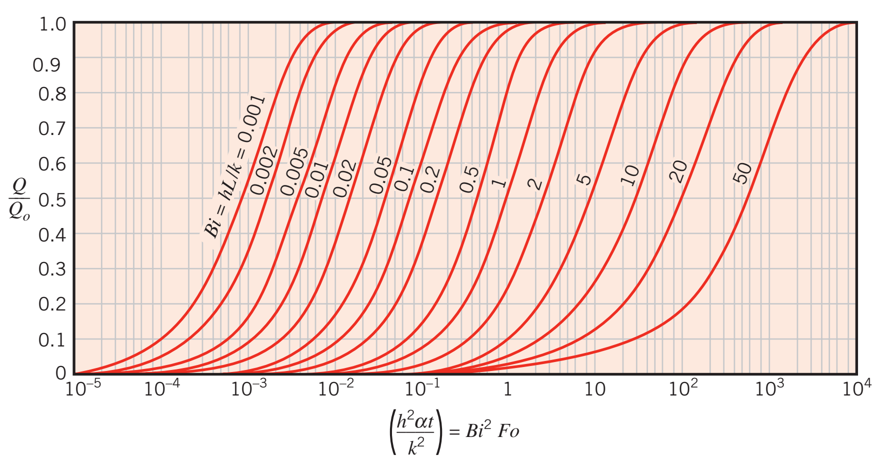


FIGURE 5S.3 Internal energy change as a function of time for a plane wall of thickness $2L$ [2]. (Adapted with permission.)

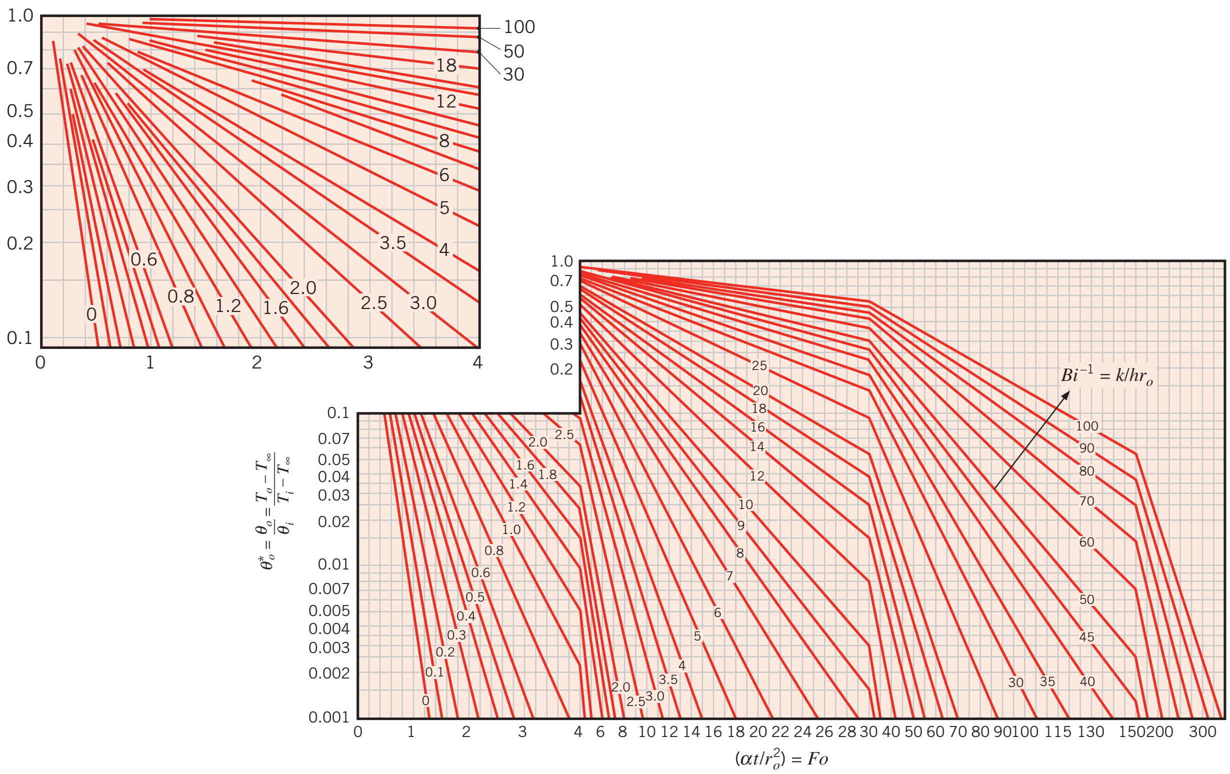


FIGURE 5S.4 Centerline temperature as a function of time for an infinite cylinder of radius r_o [1]. (Used with permission.)

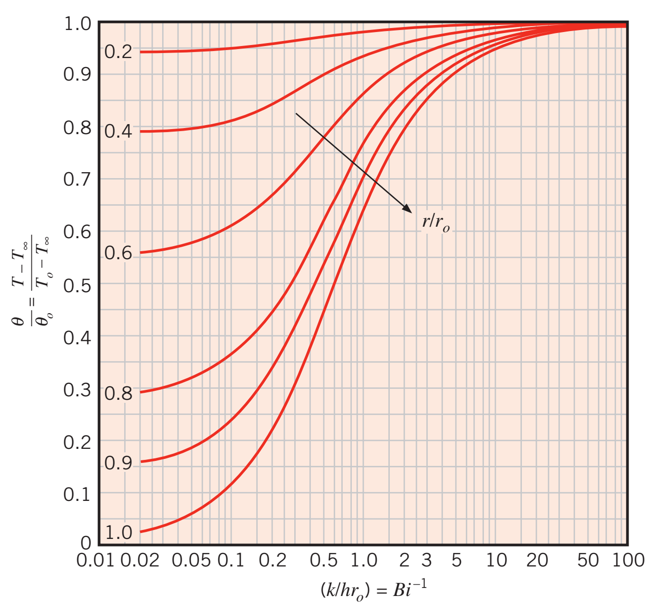


FIGURE 5S.5 Temperature distribution in an infinite cylinder of radius r_o [1]. (Used with permission.)

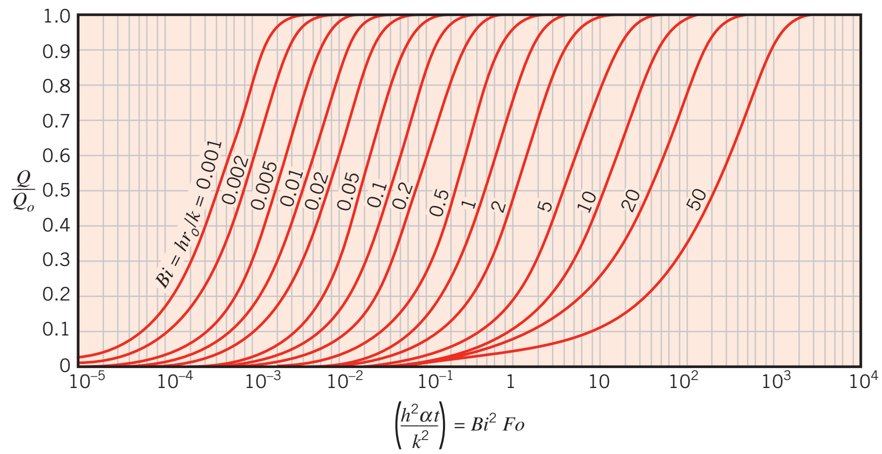


FIGURE 5S.6 Internal energy change as a function of time for an infinite cylinder of radius r_o [2]. (Adapted with permission.)

The foregoing charts may also be used to determine the transient response of a plane wall, an infinite cylinder, or sphere subjected to a *sudden change in surface temperature*. For such a condition it is only necessary to replace T_∞ by the prescribed surface temperature T_s and to set Bi^{-1} equal to zero. In so doing, the convection coefficient is tacitly assumed to be infinite, in which case $T_\infty = T_s$.

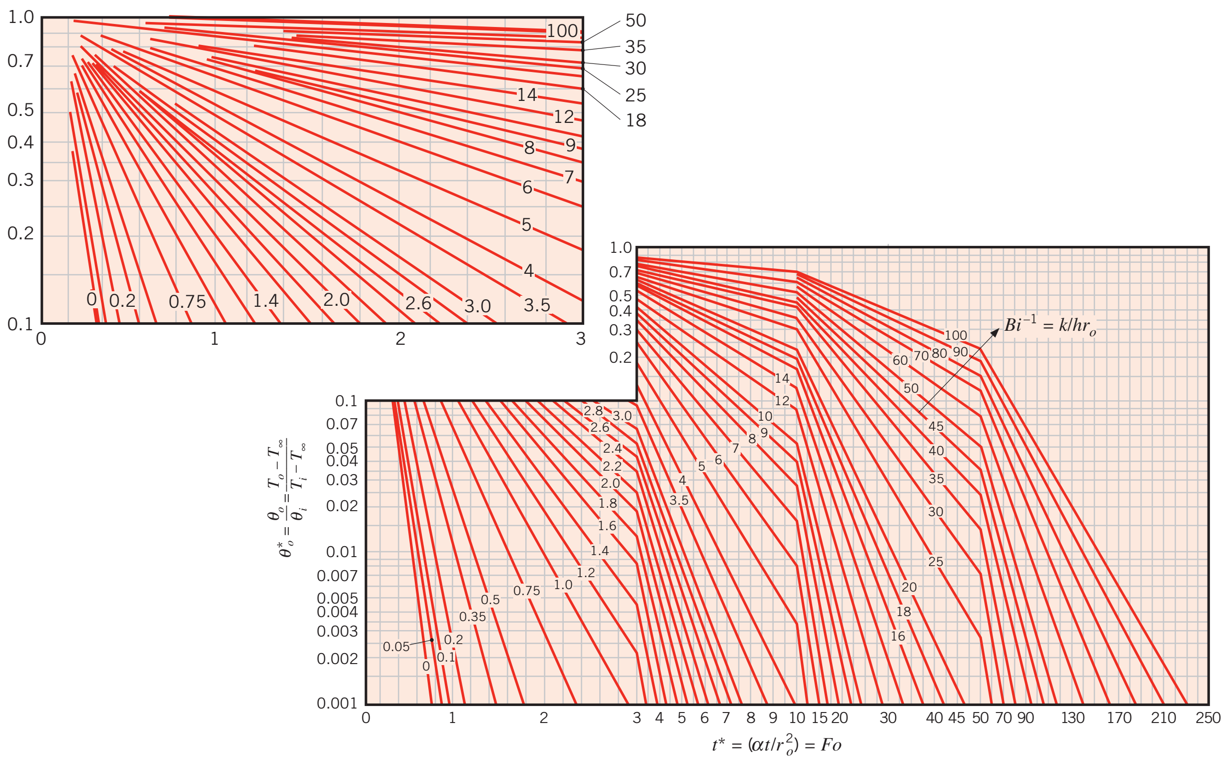


FIGURE 5S.7 Center temperature as a function of time in a sphere of radius r_o [1]. (Used with permission.)

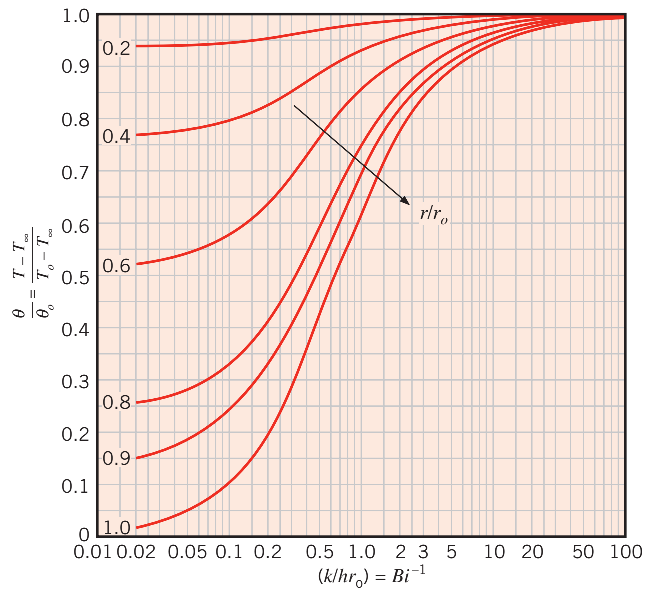


FIGURE 5S.8 Temperature distribution in a sphere of radius r_o [1]. (Used with permission.)

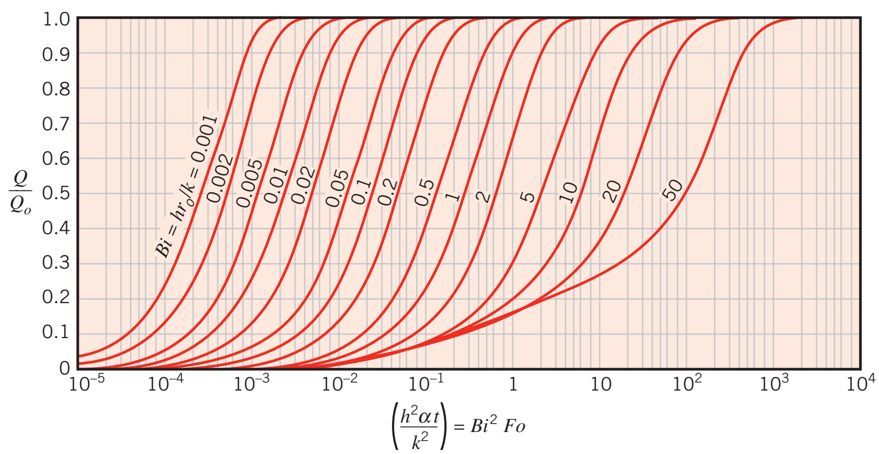


FIGURE 5S.9 Internal energy change as a function of time for a sphere of radius r_o [2]. (Adapted with permission.)

5S.2 *Analytical Solution of Multidimensional Effects*

Transient problems are frequently encountered for which two- and even three-dimensional effects are significant. Solution to a class of such problems can be obtained from the one-dimensional analytical results of Sections 5.5 through 5.7.

Consider immersing the *short* cylinder of Figure 5S.10, which is initially at a uniform temperature T_i, in a fluid of temperature $T_\infty \neq T_i$. Because the length and diameter are comparable, the subsequent transfer of energy by conduction will be significant for both the r- and x-coordinate directions. The temperature within the cylinder will therefore depend on r, x, and t.

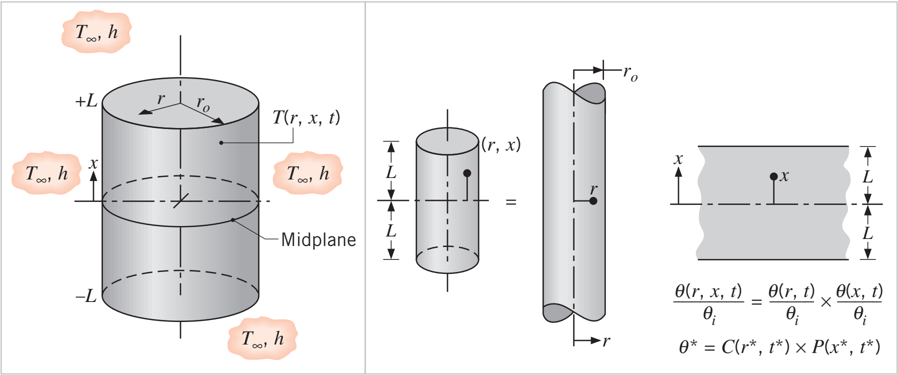


FIGURE 5S.10 Two-dimensional, transient conduction in a short cylinder. (*a*) Geometry. (*b*) Form of the product solution.

Assuming constant properties and no generation, the appropriate form of the heat equation is, from Equation 2.26,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

where x has been used in place of z to designate the axial coordinate. A closed-form solution to this equation may be obtained by the separation of variables method. Although we will not consider the details of this solution, it is important to note that the end result may be expressed in the following form:

$$\frac{T(r, x, t) - T_\infty}{T_i - T_\infty} = \left.\frac{T(x, t) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Plane}\\ \text{wall}}} \cdot \left.\frac{T(r, t) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Infinite}\\ \text{cylinder}}}$$

That is, the two-dimensional solution may be expressed as a *product* of one-dimensional solutions that correspond to those for a plane wall of thickness $2L$ and an infinite cylinder of radius r_o. For $Fo > 0.2$, these solutions are provided by the one-term approximations of Equations 5.43 and 5.52, as well as by Figures 5S.1 and 5S.2 for the plane wall and Figures 5S.4 and 5S.5 for the infinite cylinder.

Results for other multidimensional geometries are summarized in Figure 5S.11. In each case the multidimensional solution is prescribed in terms of a product involving one or more of the following one-dimensional solutions:

$$S(x, t) \equiv \left.\frac{T(x, t) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Semi-infinite}\\ \text{solid}}} \tag{5S.1}$$

$$P(x, t) \equiv \left.\frac{T(x, t) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Plane}\\ \text{wall}}} \tag{5S.2}$$

$$C(r, t) \equiv \left.\frac{T(r, t) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Infinite}\\ \text{cylinder}}} \tag{5S.3}$$

The x-coordinate for the semi-infinite solid is measured from the surface, whereas for the plane wall it is measured from the midplane. In using Figure 5S.11 the coordinate origins

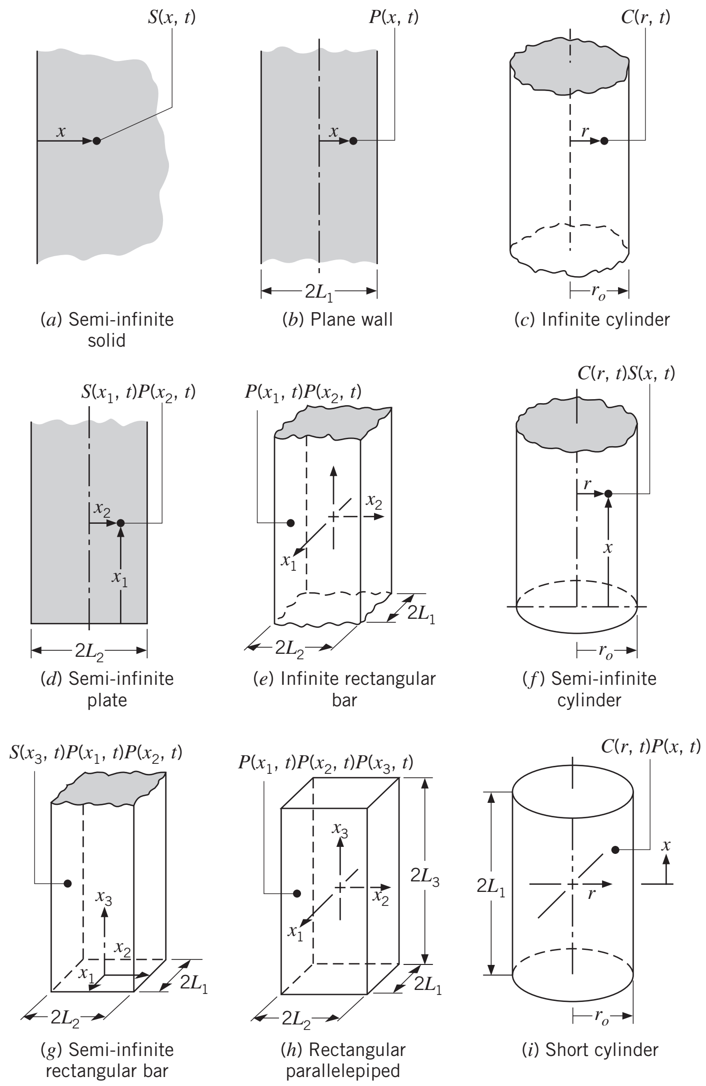


FIGURE 5S.11 Solutions for multidimensional systems expressed as products of one-dimensional results.

should carefully be noted. The transient, three-dimensional temperature distribution in a rectangular parallelepiped, Figure 5S.11*h*, is then, for example, the product of three one-dimensional solutions for plane walls of thicknesses $2L_1$, $2L_2$, and $2L_3$. That is,

$$\frac{T(x_1, x_2, x_3, t) - T_\infty}{T_i - T_\infty} = P(x_1, t) \cdot P(x_2, t) \cdot P(x_3, t)$$

The distances x_1, x_2, and x_3 are all measured with respect to a rectangular coordinate system whose origin is at the center of the parallelepiped.

The amount of energy Q transferred to or from a solid during a multidimensional transient conduction process may also be determined by combining one-dimensional results, as shown by Langston [3].

EXAMPLE 5S.1

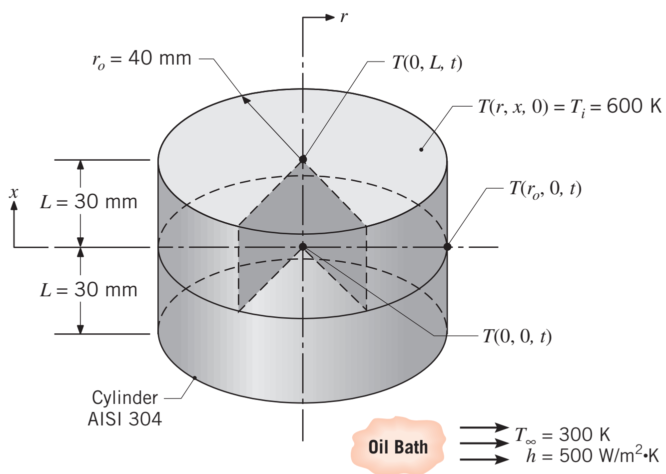
In a manufacturing process stainless steel cylinders (AISI 304) initially at 600 K are quenched by submersion in an oil bath maintained at 300 K with $h = 500\ \text{W/m}^2 \cdot \text{K}$. Each cylinder is of length $2L = 60$ mm and diameter $D = 80$ mm. Consider a time 3 min into the cooling process and determine temperatures at the center of the cylinder, at the center of a circular face, and at the midheight of the side. Note that Problem 5.147 requires a numerical solution of the same problem using *FEHT*.

SOLUTION

Known: Initial temperature and dimensions of cylinder and temperature and convection conditions of an oil bath.

Find: Temperatures $T(r, x, t)$ after 3 min at the cylinder center, $T(0, 0, 3\ \text{min})$, at the center of a circular face, $T(0, L, 3\ \text{min})$, and at the midheight of the side, $T(r_o, 0, 3\ \text{min})$.

Schematic:



Assumptions:

1. Two-dimensional conduction in r and x.
2. Constant properties.

Properties: Table A.1, stainless steel, AISI 304 [$T = (600 + 300)/2 = 450$ K]: $\rho = 7900\ \text{kg/m}^3$, $c = 526\ \text{J/kg} \cdot \text{K}$, $k = 17.4\ \text{W/m} \cdot \text{K}$, $\alpha = k/\rho c = 4.19 \times 10^{-6}\ \text{m}^2\text{/s}$.

Analysis: The solid steel cylinder corresponds to case (i) of Figure 5S.11, and the temperature at any point in the cylinder may be expressed as the following product of one-dimensional solutions.

$$\frac{T(r, x, t) - T_\infty}{T_i - T_\infty} = P(x, t)C(r, t)$$

where $P(x, t)$ and $C(r, t)$ are defined by Equations 5S.2 and 5S.3, respectively. Accordingly, for the center of the cylinder,

$$\frac{T(0, 0, 3\text{ min}) - T_\infty}{T_i - T_\infty} = \left.\frac{T(0, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Plane}\\ \text{wall}}} \cdot \left.\frac{T(0, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Infinite}\\ \text{cylinder}}}$$

Hence, for the plane wall, with

$$Bi^{-1} = \frac{k}{hL} = \frac{17.4\text{ W/m}\cdot\text{K}}{500\text{ W/m}^2\cdot\text{K} \times 0.03\text{ m}} = 1.16$$

$$Fo = \frac{\alpha t}{L^2} = \frac{4.19 \times 10^{-6}\text{ m}^2\text{/s} \times 180\text{ s}}{(0.03\text{ m})^2} = 0.84$$

it follows from Equation 5.44 that

$$\theta_o^* = \frac{\theta_o}{\theta_i} = C_1 \exp(-\zeta_1^2 Fo)$$

where, with $Bi = 0.862$, $C_1 = 1.109$ and $\zeta_1 = 0.814$ rad from Table 5.1. With $Fo = 0.84$,

$$\frac{\theta_o}{\theta_i} = \left.\frac{T(0, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Plane}\\ \text{wall}}} = 1.109\ \exp[-(0.814\text{ rad})^2 \times 0.84] = 0.636$$

Similarly, for the infinite cylinder, with

$$Bi^{-1} = \frac{k}{hr_o} = \frac{17.4\text{ W/m}\cdot\text{K}}{500\text{ W/m}^2\cdot\text{K} \times 0.04\text{ m}} = 0.87$$

$$Fo = \frac{\alpha t}{r_o^2} = \frac{4.19 \times 10^{-6}\text{ m}^2\text{/s} \times 180\text{ s}}{(0.04\text{ m})^2} = 0.47$$

it follows from Equation 5.52c that

$$\theta_o^* = \frac{\theta_o}{\theta_i} = C_1 \exp(-\zeta_1^2 Fo)$$

where, with $Bi = 1.15$, $C_1 = 1.227$ and $\zeta_1 = 1.307$ rad from Table 5.1. With $Fo = 0.47$,

$$\left.\frac{\theta_o}{\theta_i}\right|_{\substack{\text{Infinite}\\ \text{cylinder}}} = 1.109 \exp[-(1.307\text{ rad})^2 \times 0.47] = 0.550$$

Hence, for the center of the cylinder,

$$\frac{T(0, 0, 3\text{ min}) - T_\infty}{T_i - T_\infty} = 0.636 \times 0.550 = 0.350$$

$$T(0, 0, 3\text{ min}) = 300\text{ K} + 0.350(600 - 300)\text{ K} = 405\text{ K} \quad \triangleleft$$

The temperature at the center of a circular face may be obtained from the requirement that

$$\frac{T(0, L, 3\text{ min}) - T_\infty}{T_i - T_\infty} = \left.\frac{T(L, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Plane}\\ \text{wall}}} \cdot \left.\frac{T(0, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Infinite}\\ \text{cylinder}}}$$

where, from Equation 5.43b,

$$\frac{\theta^*}{\theta_o^*} = \frac{\theta}{\theta_o} = \cos(\zeta_1 x^*)$$

Hence, with $x^* = 1$, we have

$$\frac{\theta(L)}{\theta_o} = \left.\frac{T(L, 3\text{ min}) - T_\infty}{T(0, 3\text{ min}) - T_\infty}\right|_{\substack{\text{Plane}\\\text{wall}}} = \cos(0.814\text{ rad} \times 1) = 0.687$$

Hence

$$\left.\frac{T(L, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Plane}\\\text{wall}}} = \left.\frac{T(L, 3\text{ min}) - T_\infty}{T(0, 3\text{ min}) - T_\infty}\right|_{\substack{\text{Plane}\\\text{wall}}} \cdot \left.\frac{T(0, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Plane}\\\text{wall}}}$$

$$\left.\frac{T(L, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Plane}\\\text{wall}}} = 0.687 \times 0.636 = 0.437$$

Hence

$$\frac{T(0, L, 3\text{ min}) - T_\infty}{T_i - T_\infty} = 0.437 \times 0.550 = 0.240$$

$$T(0, L, 3\text{ min}) = 300\text{ K} + 0.24(600 - 300)\text{ K} = 372\text{ K} \qquad \triangleleft$$

The temperature at the midheight of the side may be obtained from the requirement that

$$\frac{T(r_o, 0, 3\text{ min}) - T_\infty}{T_i - T_\infty} = \left.\frac{T(0, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Plane}\\\text{wall}}} \cdot \left.\frac{T(r_o, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Infinite}\\\text{cylinder}}}$$

where, from Equation 5.52b,

$$\frac{\theta^*}{\theta_o^*} = \frac{\theta}{\theta_o} = J_0(\zeta_1 r^*)$$

With $r^* = 1$ and the value of the Bessel function determined from Table B.4,

$$\frac{\theta(r_o)}{\theta_o} = \left.\frac{T(r_o, 3\text{ min}) - T_\infty}{T(0, 3\text{ min}) - T_\infty}\right|_{\substack{\text{Infinite}\\\text{cylinder}}} = J_0(1.307\text{ rad} \times 1) = 0.616$$

Hence

$$\left.\frac{T(r_o, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Infinite}\\\text{cylinder}}} = \left.\frac{T(r_o, 3\text{ min}) - T_\infty}{T(0, 3\text{ min}) - T_\infty}\right|_{\substack{\text{Infinite}\\\text{cylinder}}} \cdot \left.\frac{T(0, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Infinite}\\\text{cylinder}}}$$

$$\left.\frac{T(r_o, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Infinite}\\\text{cylinder}}} = 0.616 \times 0.550 = 0.339$$

Hence

$$\frac{T(r_o, 0, 3\text{ min}) - T_\infty}{T_i - T_\infty} = 0.636 \times 0.339 = 0.216$$

$$T(r_o, 0, 3\text{ min}) = 300\text{ K} + 0.216(600 - 300)\text{ K} = 365\text{ K} \qquad \triangleleft$$

Comments:

1. Verify that the temperature at the edge of the cylinder is $T(r_o, L, 3\text{ min}) = 344$ K.
2. The Heisler charts of Section 5S.1 could also be used to obtain the desired results. Accessing these charts, one would obtain $\theta_o/\theta_i|_{\text{Plane wall}} \approx 0.64$, $\theta_o/\theta_i|_{\text{Infinite cylinder}} \approx 0.55$, $\theta(L)/\theta_o|_{\text{Plane wall}} \approx 0.68$, and $\theta(r_o)/\theta_o|_{\text{Infinite cylinder}} \approx 0.61$, which are in good agreement with results obtained from the one-term approximations.
3. The *IHT Models*, *Transient Conduction* option for the *Plane Wall* and *Infinite Cylinder* may be used to calculate temperature ratios required for the foregoing product solution.

References

1. Heisler, M. P., *Trans. ASME,* **69,** 227–236, 1947.
2. Gröber, H., S. Erk, and U. Grigull, *Fundamentals of Heat Transfer,* McGraw-Hill, New York, 1961.
3. Langston, L. S., *Int. J. Heat Mass Transfer*, **25**, 149–150, 1982.

Problems

One-Dimensional Conduction: The Plane Wall

5S.1 Consider the thermal energy storage unit of Problem 5.16, but with a masonry material of $\rho = 1900$ kg/m^3, $c = 800$ J/kg·K, and $k = 0.70$ W/m·K used in place of the aluminum. How long will it take to achieve 75% of the maximum possible energy storage? What are the maximum and minimum temperatures of the masonry at this time?

5S.2 An ice layer forms on a 5-mm-thick windshield of a car while parked during a cold night for which the ambient temperature is -20°C. Upon start-up, using a new defrost system, the interior surface is suddenly exposed to an airstream at 30°C. Assuming that the ice behaves as an insulating layer on the exterior surface, what interior convection coefficient would allow the exterior surface to reach 0°C in 60 s? The windshield thermophysical properties are $\rho = 2200$ kg/m^3, $c_p = 830$ J/kg·K, and $k = 1.2$ W/m·K.

One-Dimensional Conduction: The Long Cylinder

5S.3 Cylindrical steel rods (AISI 1010), 50 mm in diameter, are heat treated by drawing them through an oven 5 m long in which air is maintained at 750°C. The rods enter at 50°C and achieve a centerline temperature of 600°C before leaving. For a convection coefficient of 125 W/m^2·K, estimate the speed at which the rods must be drawn through the oven.

5S.4 Estimate the time required to cook a hot dog in boiling water. Assume that the hot dog is initially at 6°C, that the convection heat transfer coefficient is 100 W/m^2·K, and that the final temperature is 80°C at the centerline. Treat the hot dog as a long cylinder of 20-mm diameter having the properties: $\rho = 880$ kg/m^3, $c = 3350$ J/kg·K, and $k = 0.52$ W/m·K.

5S.5 A long bar of 70-mm diameter and initially at 90°C is cooled by immersing it in a water bath that is at 40°C and provides a convection coefficient of 20 W/m^2·K.

The thermophysical properties of the bar are $\rho = 2600\ \text{kg/m}^3$, $c = 1030\ \text{J/kg}\cdot\text{K}$, and $k = 3.50\ \text{W/m}\cdot\text{K}$.

(a) How long should the bar remain in the bath in order that, when it is removed and allowed to equilibrate while isolated from any surroundings, it achieves a uniform temperature of 55°C?

(b) What is the surface temperature of the bar when it is removed from the bath?

One-Dimensional Conduction: The Sphere

5S.6 A sphere of 80-mm diameter ($k = 50\ \text{W/m}\cdot\text{K}$ and $\alpha = 1.5 \times 10^{-6}\ \text{m}^2/\text{s}$) is initially at a uniform, elevated temperature and is quenched in an oil bath maintained at 50°C. The convection coefficient for the cooling process is $1000\ \text{W/m}^2\cdot\text{K}$. At a certain time, the surface temperature of the sphere is measured to be 150°C. What is the corresponding center temperature of the sphere?

5S.7 A spherical hailstone that is 5 mm in diameter is formed in a high-altitude cloud at −30°C. If the stone begins to fall through warmer air at 5°C, how long will it take before the outer surface begins to melt? What is the temperature of the stone's center at this point in time, and how much energy (J) has been transferred to the stone? A convection heat transfer coefficient of $250\ \text{W/m}^2\cdot\text{K}$ may be assumed, and the properties of the hailstone may be taken to be those of ice.

5S.8 In a process to manufacture glass beads ($k = 1.4\ \text{W/m}\cdot\text{K}$, $\rho = 2200\ \text{kg/m}^3$, $c_p = 800\ \text{J/kg}\cdot\text{K}$) of 3-mm diameter, the beads are suspended in an upwardly directed airstream that is at $T_\infty = 15°\text{C}$ and maintains a convection coefficient of $h = 400\ \text{W/m}^2\cdot\text{K}$.

(a) If the beads are at an initial temperature of $T_i = 477°\text{C}$, how long must they be suspended to achieve a center temperature of 80°C? What is the corresponding surface temperature?

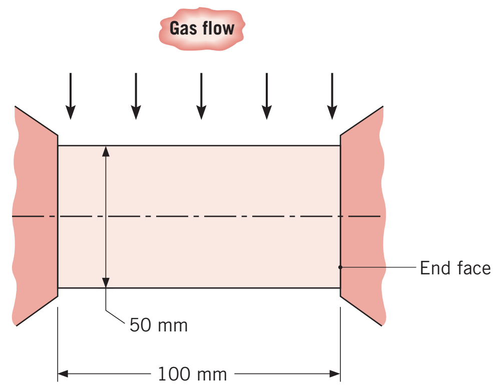
(b) Compute and plot the center and surface temperatures as a function of time for $0 \le t \le 20$ s and $h = 100$, 400, and $1000\ \text{W/m}^2\cdot\text{K}$.

Multidimensional Conduction

5S.9 A long steel (plain carbon) billet of square cross section 0.3 m × 0.3 m, initially at a uniform temperature of 30°C, is placed in a soaking oven having a temperature of 750°C. If the convection heat transfer coefficient for the heating process is $100\ \text{W/m}^2\cdot\text{K}$, how long must the billet remain in the oven before its center temperature reaches 600°C?

5S.10 Fireclay brick of dimensions 0.06 m × 0.09 m × 0.20 m is removed from a kiln at 1600 K and cooled in air at 40°C with $h = 50\ \text{W/m}^2\cdot\text{K}$. What is the temperature at the center and at the corners of the brick after 50 min of cooling?

5S.11 A cylindrical copper pin 100 mm long and 50 mm in diameter is initially at a uniform temperature of 20°C. The end faces are suddenly subjected to an intense heating rate that raises them to a temperature of 500°C. At the same time, the cylindrical surface is subjected to heating by gas flow with a temperature of 500°C and a heat transfer coefficient of $100\ \text{W/m}^2\cdot\text{K}$.



(a) Determine the temperature at the center point of the cylinder 8 s after sudden application of the heat.

(b) Considering the parameters governing the temperature distribution in transient heat diffusion problems, can any simplifying assumptions be justified in analyzing this particular problem? Explain briefly.

5S.12 Recalling that your mother once said that meat should be cooked until every portion has attained a temperature of 80°C, how long will it take to cook a 2.25-kg roast? Assume that the meat is initially at 6°C and that the oven temperature is 175°C with a convection heat transfer coefficient of $15\ \text{W/m}^2\cdot\text{K}$. Treat the roast as a cylinder with properties of liquid water, having a diameter equal to its length.

5S.13 A long rod 20 mm in diameter is fabricated from alumina (polycrystalline aluminum oxide) and is initially at a uniform temperature of 850 K. The rod is suddenly exposed to fluid at 350 K with $h = 500\ \text{W/m}^2\cdot\text{K}$. Estimate the centerline temperature of the rod after 30 s at an exposed end and at an axial distance of 6 mm from the end.

5S.14 Consider the stainless steel cylinder of Example 5S.1, which is initially at 600 K and suddenly quenched in an oil bath at 300 K with $h = 500\ \text{W/m}^2 \cdot \text{K}$. Use the *Transient Conduction*, *Plane Wall and Cylinder* models of *IHT* to obtain the following solutions.

(a) Calculate the temperatures, $T(r, x, t)$, after 3 min at the cylinder center, $T(0, 0, 3\ \text{min})$, at the center of a circular face, $T(0, L, 3\ \text{min})$, and at the midheight of the side, $T(r_o, 0, 3\ \text{min})$. Compare your results with those in the example.

(b) Use the *Explore* and *Graph* options of *IHT* to calculate and plot temperature histories at the cylinder center, $T(0, 0, t)$, and the midheight of the side, $T(r_o, 0, t)$, for $0 \leq t \leq 10$ min. Comment on the gradients occurring at these locations and what effect they might have on phase transformations and thermal stresses. *Hint*: In your sweep over the time variable, start at 1 s rather than zero.

(c) For $0 \leq t \leq 10$ min, calculate and plot temperature histories at the cylinder center, $T(0, 0, t)$, for convection coefficients of $500\ \text{W/m}^2 \cdot \text{K}$ and $1000\ \text{W/m}^2 \cdot \text{K}$.

6S.1 *Derivation of the Convection Transfer Equations*

In Chapter 2 we considered a stationary substance in which heat is transferred by conduction and developed means for determining the temperature distribution within the substance. We did so by applying *conservation of energy* to a differential control volume (Figure 2.11) and deriving a differential equation that was termed the *heat equation*. For a prescribed geometry and boundary conditions, the equation may be solved to determine the corresponding temperature distribution.

If the substance is not stationary, conditions become more complex. For example, if conservation of energy is applied to a differential control volume in a moving fluid, the effects of fluid motion (*advection*) on energy transfer across the surfaces of the control volume must be considered, along with those of conduction. The resulting differential equation, which provides the basis for predicting the temperature distribution, now requires knowledge of the velocity field. This field must, in turn, be determined by solving additional differential equations derived by applying *conservation of mass* and *Newton's second law of motion* to a differential control volume.

In this supplemental material we consider conditions involving flow of a *viscous fluid* in which there is concurrent *heat* and *mass transfer*. Our objective is to develop differential equations that may be used to predict velocity, temperature, and species concentration fields within the fluid, and we do so by applying Newton's second law of motion and conservation of mass, energy, and species to a differential control volume. To simplify this development, we restrict our attention to *steady, two-dimensional flow* in the x- and y-directions of a Cartesian coordinate system. A unit depth may therefore be assigned to the z-direction, thereby providing a differential control volume of extent $(dx \cdot dy \cdot 1)$.

6S.1.1 Conservation of Mass

One conservation law that is pertinent to the flow of a viscous fluid is that matter may neither be created nor destroyed. Stated in the context of the differential control volume of Figure 6S.1, this law requires that, for steady flow, *the net rate at which mass enters the control volume* (inflow − outflow) *must equal zero*. Mass enters and leaves the control volume exclusively through gross fluid motion. Transport due to such motion is often referred to as *advection*. If one corner of the control volume is located at (x, y), the rate at which mass enters the control volume through the surface perpendicular to x may be expressed as $(\rho u)\, dy$, where ρ is the total mass density $(\rho = \rho_A + \rho_B)$ and u is the x-component of the *mass average velocity*. The control volume is of unit depth in the z-direction. Since ρ and u may vary with x, the rate at which mass leaves the surface at $x + dx$ may be expressed by a Taylor series expansion of the form

$$\left[(\rho u) + \frac{\partial(\rho u)}{\partial x}\, dx\right] dy$$

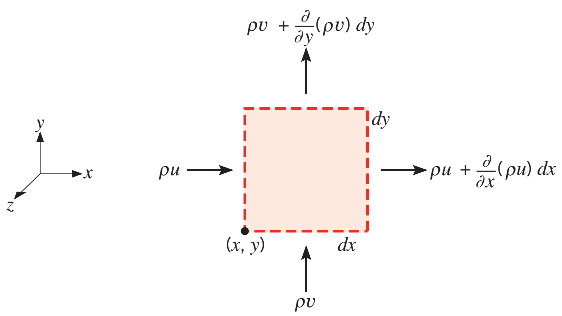


FIGURE 6S.1 Differential control volume $(dx \cdot dy \cdot 1)$ for mass conservation in two-dimensional flow of a viscous fluid.

Using a similar result for the y-direction, the conservation of mass requirement becomes

$$(\rho u)\, dy + (\rho v)\, dx - \left[\rho u + \frac{\partial(\rho u)}{\partial x}\, dx\right] dy - \left[\rho v + \frac{\partial(\rho v)}{\partial y}\, dy\right] dx = 0$$

Canceling terms and dividing by $dx\, dy$, we obtain

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \qquad (6S.1)$$

Equation 6S.1, the *continuity equation*, is a general expression of the *overall mass* conservation requirement, and it must be satisfied at every point in the fluid. The equation applies for a single species fluid, as well as for mixtures in which species diffusion and chemical reactions may be occurring. If the fluid is *incompressible*, the density ρ is a constant, and the continuity equation reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (6S.2)$$

6S.1.2 Newton's Second Law of Motion

The second fundamental law that is pertinent to the flow of a viscous fluid is *Newton's second law of motion*. For a differential control volume in the fluid, this requirement states that the sum of all forces acting on the control volume must equal the net rate at which momentum leaves the control volume (outflow − inflow).

Two kinds of forces may act on the fluid: *body forces*, which are proportional to the volume, and *surface forces*, which are proportional to area. Gravitational, centrifugal, magnetic, and/or electric fields may contribute to the total body force, and we designate the x- and y-components of this force per unit volume of fluid as X and Y, respectively. The surface forces F_s are due to the fluid static pressure as well as to *viscous stresses*. At any point in the fluid, the viscous stress (a force per unit area) may be resolved into two perpendicular components, which include a *normal stress* σ_{ii} and a *shear stress* τ_{ij} (Figure 6S.2). A double subscript notation is used to specify the stress components. The first subscript indicates the surface orientation by providing the direction of its outward normal, and the second subscript indicates the direction of the force component. Accordingly, for the x surface of Figure 6S.2, the normal stress σ_{xx} corresponds to a force component normal to the surface, and the shear stress τ_{xy} corresponds to a force in the y-direction

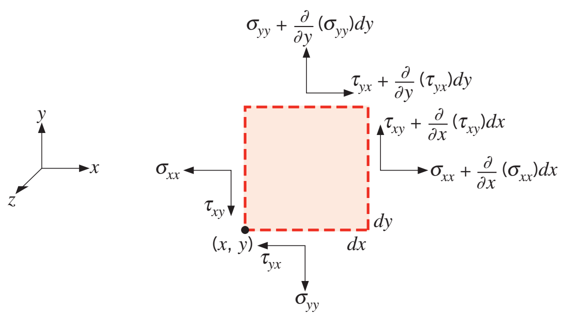


FIGURE 6S.2 Normal and shear viscous stresses for a differential control volume ($dx \cdot dy \cdot 1$) in two-dimensional flow of a viscous fluid.

along the surface. All the stress components shown are positive in the sense that *both* the surface normal and the force component are in the same direction. That is, they are both in either the positive coordinate direction or the negative coordinate direction. By this convention the normal viscous stresses are *tensile stresses*. In contrast the static pressure originates from an external force acting on the fluid in the control volume and is therefore a *compressive* stress.

Several features of the viscous stress should be noted. The associated force is between adjoining fluid elements and is a natural consequence of the fluid motion and viscosity. The surface forces of Figure 6S.2 are therefore presumed to act on the fluid within the control volume and are attributed to its interaction with the surrounding fluid. These stresses would vanish if the fluid velocity, or the velocity gradient, went to zero. In this respect the normal viscous stresses (σ_{xx} and σ_{yy}) must not be confused with the static pressure, which does not vanish for zero velocity.

Each of the stresses may change continuously in each of the coordinate directions. Using a Taylor series expansion for the stresses, the *net* surface force for each of the two directions may be expressed as

$$F_{s,x} = \left(\frac{\partial \sigma_{xx}}{\partial x} - \frac{\partial p}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}\right) dx\, dy \tag{6S.3}$$

$$F_{s,y} = \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} - \frac{\partial p}{\partial y}\right) dx\, dy \tag{6S.4}$$

To use Newton's second law, the fluid momentum fluxes for the control volume must also be evaluated. If we focus on the x-direction, the relevant fluxes are as shown in Figure 6S.3. A contribution to the total x-momentum flux is made by the mass flow in each of the two directions. For example, the mass flux through the x surface (in the y-z plane) is (ρu), the corresponding x-momentum flux is $(\rho u)u$. Similarly, the x-momentum flux due to mass flow through the y surface (in the x-z plane) is $(\rho v)u$. These fluxes may change in each of the coordinate directions, and the *net* rate at which x-momentum leaves the control volume is

$$\frac{\partial[(\rho u)u]}{\partial x} dx\,(dy) + \frac{\partial[(\rho v)u]}{\partial y} dy\,(dx)$$

Equating the rate of change in the x-momentum of the fluid to the sum of the forces in the x-direction, we then obtain

$$\frac{\partial[(\rho u)u]}{\partial x} + \frac{\partial[(\rho v)u]}{\partial y} = \frac{\partial \sigma_{xx}}{\partial x} - \frac{\partial p}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + X \tag{6S.5}$$

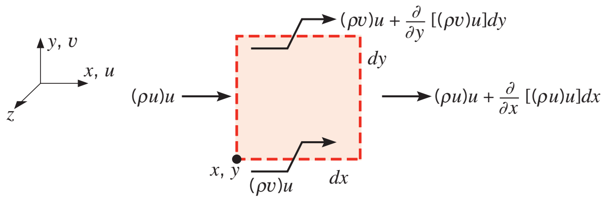


FIGURE 6S.3 Momentum fluxes for a differential control volume ($dx \cdot dy \cdot 1$) in two-dimensional flow of a viscous fluid.

This expression may be put in a more convenient form by expanding the derivatives on the left-hand side and substituting from the continuity equation, Equation 6S.1, giving

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(\sigma_{xx} - p) + \frac{\partial \tau_{yx}}{\partial y} + X \tag{6S.6}$$

A similar expression may be obtained for the y-direction and is of the form

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial}{\partial y}(\sigma_{yy} - p) + Y \tag{6S.7}$$

We should not lose sight of the physics represented by Equations 6S.6 and 6S.7. The two terms on the left-hand side of each equation represent the *net* rate of momentum flow from the control volume. The terms on the right-hand side account for the net viscous and pressure forces, as well as the body force. These equations must be satisfied at each point in the fluid, and with Equation 6S.1 they may be solved for the velocity field.

Before a solution to the foregoing equations can be obtained, it is necessary to relate the viscous stresses to other flow variables. These stresses are associated with the deformation of the fluid and are a function of the fluid viscosity and velocity gradients. From Figure 6S.4 it is evident that a *normal stress* must produce a *linear deformation* of the fluid, whereas a *shear stress* produces an *angular deformation*. Moreover, the magnitude of a stress is proportional to the *rate* at which the deformation occurs. The deformation rate is, in turn, related to the fluid viscosity and to the velocity gradients in the flow. For a *Newtonian fluid*[1] the stresses are proportional to the velocity gradients, where the proportionality constant is the fluid viscosity. Because of its complexity, however, development of the specific relations is left to the literature [1], and we limit ourselves to a presentation of the results. In particular, it has been shown that

$$\sigma_{xx} = 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{6S.8}$$

$$\sigma_{yy} = 2\mu\frac{\partial v}{\partial y} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{6S.9}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{6S.10}$$

Substituting Equations 6S.8 through 6S.10 into Equations 6S.6 and 6S.7, the x- and y-momentum equations become

[1]A Newtonian fluid is one for which the shear stress is linearly proportional to the rate of angular deformation. All fluids of interest in the text are Newtonian.

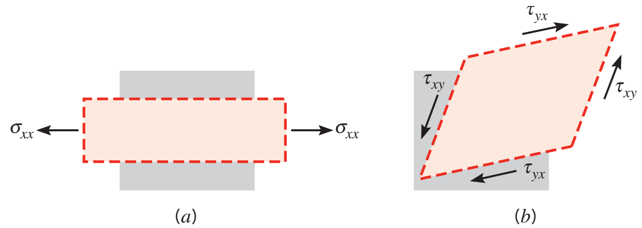


FIGURE 6S.4 Deformations of a fluid element due to viscous stresses. (*a*) Linear deformation due to a normal stress. (*b*) Angular deformation due to shear stresses.

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left\{\mu\left[2\frac{\partial u}{\partial x}-\frac{2}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right]\right\}+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+X \qquad (6S.11)$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial y}\left\{\mu\left[2\frac{\partial v}{\partial y}-\frac{2}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right]\right\}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+Y \qquad (6S.12)$$

Equations 6S.1, 6S.11, and 6S.12 provide a complete representation of conditions in a two-dimensional viscous flow, and the corresponding velocity field may be determined by solving the equations. Once the velocity field is known, it is a simple matter to obtain the wall shear stress τ_s from Equation 6.2.

Equations 6S.11 and 6S.12 may be simplified for an *incompressible fluid* of *constant viscosity*. Rearranging the right-hand side of each expression and substituting from Equation 6S.2, the x- and y-momentum equations become

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+X \qquad (6S.13)$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+Y \qquad (6S.14)$$

6S.1.3 Conservation of Energy

To apply the energy conservation requirement (Equation 1.12c) to a differential control volume in a viscous fluid with heat transfer (Figure 6S.5), it is necessary to first delineate the relevant physical processes. If potential energy effects are treated as work done by the body forces, the energy per unit mass of the fluid includes the thermal internal energy e and the kinetic energy $V^2/2$, where $V^2 \equiv u^2 + v^2$. Accordingly, thermal and kinetic energy are *advected* with the *bulk fluid* motion across the control surfaces, and for the x-direction, the *net* rate at which this energy *enters* the control volume is

$$\dot{E}_{\text{adv},x} - \dot{E}_{\text{adv},x+dx} \equiv \rho u\left(e + \frac{V^2}{2}\right)dy - \left\{\rho u\left(e + \frac{V^2}{2}\right) + \frac{\partial}{\partial x}\left[\rho u\left(e + \frac{V^2}{2}\right)\right]dx\right\}dy$$

$$= -\frac{\partial}{\partial x}\left[\rho u\left(e + \frac{V^2}{2}\right)\right]dx\,dy \qquad (6S.15)$$

Energy is also transferred across the control surface by *molecular processes*. There may be two contributions: that due to *conduction* and energy transfer due to the *diffusion of species* A and B. However, it is only in chemically reacting flows that species diffusion strongly influences thermal conditions. Hence the effect is neglected in this development. For the conduction process, the *net* transfer of energy into the control volume is

$$\dot{E}_{\text{cond},x} - \dot{E}_{\text{cond},x+dx} = -\left(k\frac{\partial T}{\partial x}\right)dy - \left[-k\frac{\partial T}{\partial x} - \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx\right]dy$$

$$= \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx\,dy \qquad (6S.16)$$

Energy may also be transferred to and from the fluid in the control volume by *work* interactions involving the *body* and *surface forces*. The *net* rate at which work is done on the fluid by forces in the x-direction may be expressed as

$$\dot{W}_{\text{net},x} = (Xu)\,dx\,dy + \frac{\partial}{\partial x}[(\sigma_{xx} - p)\,u]\,dx\,dy + \frac{\partial}{\partial y}(\tau_{yx}u)\,dx\,dy \qquad (6S.17)$$

The first term on the right-hand side of Equation 6S.17 represents the work done by the body force, and the remaining terms account for the *net* work done by the pressure and viscous forces.

Using Equations 6S.15 through 6S.17, as well as analogous equations for the y-direction, the energy conservation requirement (Equation 1.12c) may be expressed as

$$-\frac{\partial}{\partial x}\left[\rho u\left(e + \frac{V^2}{2}\right)\right] - \frac{\partial}{\partial y}\left[\rho v\left(e + \frac{V^2}{2}\right)\right]$$
$$+ \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + (Xu + Yv) - \frac{\partial}{\partial x}(pu) - \frac{\partial}{\partial y}(pv)$$
$$+ \frac{\partial}{\partial x}(\sigma_{xx}u + \tau_{xy}v) + \frac{\partial}{\partial y}(\tau_{yx}u + \sigma_{yy}v) + \dot{q} = 0 \qquad (6S.18)$$

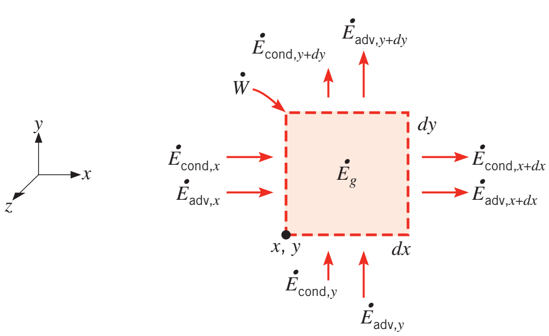


FIGURE 6S.5 Differential control volume $(dx \cdot dy \cdot 1)$ for energy conservation in two-dimensional flow of a viscous fluid with heat transfer.

where $\dot{q}$ is the rate at which thermal energy is generated per unit volume. This expression provides a general form of the energy conservation requirement for flow of a viscous fluid with heat transfer.

Because Equation 6S.18 represents conservation of *kinetic* and *thermal internal* energy, it is rarely used in solving heat transfer problems. Instead, a more convenient form, which is termed the *thermal energy equation*, is obtained by multiplying Equations 6S.6 and 6S.7 by u and v, respectively, and subtracting the results from Equation 6S.18. After considerable manipulation, it follows that [2]

$$\rho u \frac{\partial e}{\partial x} + \rho v \frac{\partial e}{\partial y} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) - p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \mu\Phi + \dot{q} \qquad (6S.19)$$

where the term $p(\partial u/\partial x + \partial v/\partial y)$ represents a reversible conversion between mechanical work and thermal energy, and $\mu\Phi$, the *viscous dissipation*, is defined as

$$\mu\Phi \equiv \mu\left\{\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2\right\} \qquad (6S.20)$$

The first term on the right-hand side of Equation 6S.20 originates from the viscous shear stresses, and the remaining terms arise from the viscous normal stresses. Collectively, the terms account for the rate at which *mechanical work is irreversibly converted to thermal energy due to viscous effects in the fluid*.

If the fluid is incompressible, Equations 6S.19 and 6S.20 may be simplified by substituting Equation 6S.2. Moreover, with $de = c_v dT$ and $c_v = c_p$ for an incompressible fluid, the thermal energy equation may then be expressed as

$$\rho c_p\left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \mu\Phi + \dot{q} \qquad (6S.21)$$

where

$$\mu\Phi = \mu\left\{\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]\right\} \qquad (6S.22)$$

The thermal energy equation may also be cast in terms of the fluid enthalpy i, instead of its internal energy e. Introducing the definition of the enthalpy,

$$i = e + \frac{p}{\rho} \qquad (6S.23)$$

and using Equation 6S.1 to replace the third term on the right-hand side of Equation 6S.19 by spatial derivatives of p and (p/ρ), the energy equation may be expressed as [2]

$$\rho u \frac{\partial i}{\partial x} + \rho v \frac{\partial i}{\partial y} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \left(u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y}\right) + \mu\Phi + \dot{q} \qquad (6S.24)$$

If the fluid may be approximated as a *perfect gas*, $di = c_p dT$, Equation 6S.24 becomes

$$\rho c_p\left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \left(u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y}\right) + \mu\Phi + \dot{q} \qquad (6S.25)$$

6S.1.4 Conservation of Species

If the viscous fluid consists of a binary mixture in which there are species concentration gradients (Figure 6.9), there will be *relative* transport of the species, and *species conservation* must be satisfied at each point in the fluid. The pertinent form of the conservation equation may be obtained by identifying the processes that affect the *transport* and *generation* of species A for a differential control volume in the fluid.

Consider the control volume of Figure 6S.6. Species A may be transported by *advection* (with the mean velocity of the mixture) and by *diffusion* (relative to the mean motion) in each of the coordinate directions. The concentration may also be affected by chemical reactions, and we designate the rate at which the mass of species A is generated per unit volume due to such reactions as $\dot{n}_A$.

The *net* rate at which species A *enters* the control volume due to *advection* in the x-direction is

$$\dot{M}_{A,\text{adv},x} - \dot{M}_{A,\text{adv},x+dx} = (\rho_A u)\, dy - \left[(\rho_A u) + \frac{\partial(\rho_A u)}{\partial x} dx\right] dy$$

$$= -\frac{\partial(\rho_A u)}{\partial x} dx\, dy \tag{6S.26}$$

Similarly, multiplying both sides of Fick's law (Equation 6.6) by the molecular weight $\mathcal{M}_A$ (kg/kmol) of species A to evaluate the diffusion flux, the *net* rate at which species A *enters* the control volume due to *diffusion* in the x-direction is

$$\dot{M}_{A,\text{dif},x} - \dot{M}_{A,\text{dif},x+dx} = \left(-D_{AB}\frac{\partial \rho_A}{\partial x}\right) dy - \left[\left(-D_{AB}\frac{\partial \rho_A}{\partial x}\right)\right.$$

$$\left. + \frac{\partial}{\partial x}\left(-D_{AB}\frac{\partial \rho_A}{\partial x}\right) dx\right] dy = \frac{\partial}{\partial x}\left(D_{AB}\frac{\partial \rho_A}{\partial x}\right) dx\, dy \tag{6S.27}$$

Expressions similar to Equations 6S.26 and 6S.27 may be formulated for the y-direction.

Referring to Figure 6S.6, the species conservation requirement is

$$\dot{M}_{A,\text{adv},x} - \dot{M}_{A,\text{adv},x+dx} + \dot{M}_{A,\text{adv},y} - \dot{M}_{A,\text{adv},y+dy}$$

$$+ \dot{M}_{A,\text{dif},x} - \dot{M}_{A,\text{dif},x+dx} + \dot{M}_{A,\text{dif},y} - \dot{M}_{A,\text{dif},y+dy} + \dot{M}_{A,g} = 0 \tag{6S.28}$$

Substituting from Equations 6S.26 and 6S.27, as well as from similar forms for the y-direction, it follows that

$$\frac{\partial(\rho_A u)}{\partial x} + \frac{\partial(\rho_A v)}{\partial y} = \frac{\partial}{\partial x}\left(D_{AB}\frac{\partial \rho_A}{\partial x}\right) + \frac{\partial}{\partial y}\left(D_{AB}\frac{\partial \rho_A}{\partial y}\right) + \dot{n}_A \tag{6S.29}$$

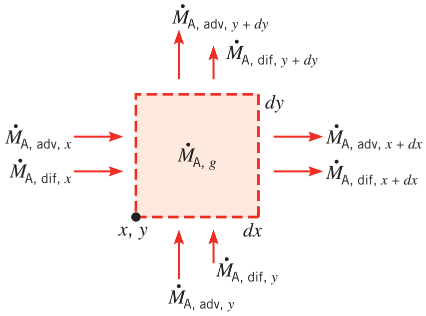


FIGURE 6S.6 Differential control volume ($dx \cdot dy \cdot 1$) for species conservation in two-dimensional flow of a viscous fluid with mass transfer.

A more useful form of this equation may be obtained by expanding the terms on the left-hand side and substituting from the overall continuity equation for an incompressible fluid. Equation 6S.29 then reduces to

$$u\frac{\partial \rho_A}{\partial x} + v\frac{\partial \rho_A}{\partial y} = \frac{\partial}{\partial x}\left(D_{AB}\frac{\partial \rho_A}{\partial x}\right) + \frac{\partial}{\partial y}\left(D_{AB}\frac{\partial \rho_A}{\partial y}\right) + \dot{n}_A \tag{6S.30}$$

or in molar form

$$u\frac{\partial C_A}{\partial x} + v\frac{\partial C_A}{\partial y} = \frac{\partial}{\partial x}\left(D_{AB}\frac{\partial C_A}{\partial x}\right) + \frac{\partial}{\partial y}\left(D_{AB}\frac{\partial C_A}{\partial y}\right) + \dot{N}_A \tag{6S.31}$$

Example 6S.1

One of the few situations for which *exact* solutions to the convection transfer equations may be obtained involves what is termed *parallel flow*. In this case fluid motion is only in one direction. Consider a special case of parallel flow involving stationary and moving plates of infinite extent separated by a distance L, with the intervening space filled by an incompressible fluid. This situation is referred to as Couette flow and occurs, for example, in a journal bearing.

1. What is the appropriate form of the continuity equation (Equation E.1)?
2. Beginning with the momentum equation (Equation E.2), determine the velocity distribution between the plates.
3. Beginning with the energy equation (Equation E.4), determine the temperature distribution between the plates.
4. Consider conditions for which the fluid is engine oil with $L = 3$ mm. The speed of the moving plate is $U = 10$ m/s, and the temperatures of the stationary and moving plates are $T_0 = 10°C$ and $T_L = 30°C$, respectively. Calculate the heat flux to each of the plates and determine the maximum temperature in the oil.

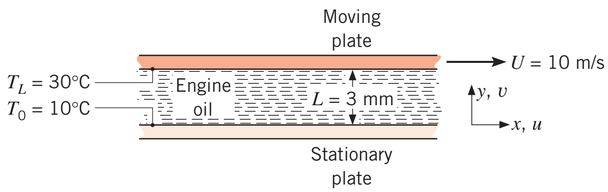
Solution

Known: Couette flow with heat transfer.

Find:

1. Form of the continuity equation.
2. Velocity distribution.
3. Temperature distribution.
4. Surface heat fluxes and maximum temperature for prescribed conditions.

Schematic:



Assumptions:

1. Steady-state conditions.
2. Two-dimensional flow (no variations in z).
3. Incompressible fluid with constant properties.
4. No body forces.
5. No internal energy generation.

Properties: Table A.8, engine oil (20°C): $\rho = 888.2\ \text{kg/m}^3$, $k = 0.145\ \text{W/m}\cdot\text{K}$, $\nu = 900 \times 10^{-6}\ \text{m}^2/\text{s}$, $\mu = \nu\rho = 0.799\ \text{N}\cdot\text{s/m}^2$.

Analysis:

1. For an incompressible fluid (constant ρ) and parallel flow ($v = 0$), Equation E.1 reduces to

$$\frac{\partial u}{\partial x} = 0 \qquad \triangleleft$$

The important implication of this result is that, although depending on y, the x velocity component u is independent of x. It may then be said that the velocity field is *fully developed.*

2. For two-dimensional, steady-state conditions with $v = 0$, $(\partial u/\partial x) = 0$, and $X = 0$, Equation E.2 reduces to

$$0 = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2}\right)$$

However, in Couette flow, motion of the fluid is not sustained by the pressure gradient, $\partial p/\partial x$, but by an external force that provides for motion of the top plate relative to the bottom plate. Hence $(\partial p/\partial x) = 0$. Accordingly, the x-momentum equation reduces to

$$\frac{\partial^2 u}{\partial y^2} = 0$$

The desired velocity distribution may be obtained by solving this equation. Integrating twice, we obtain

$$u(y) = C_1 y + C_2$$

where C_1 and C_2 are the constants of integration. Applying the boundary conditions

$$u(0) = 0 \qquad u(L) = U$$

it follows that $C_2 = 0$ and $C_1 = U/L$. The velocity distribution is then

$$u(y) = \frac{y}{L}\,U \qquad \triangleleft$$

3. The energy equation (E.4) may be simplified for the prescribed conditions. In particular, with $v = 0$, $(\partial u/\partial x) = 0$, and $\dot{q} = 0$, it follows that

$$\rho c_p u \frac{\partial T}{\partial x} = k\frac{\partial^2 T}{\partial x^2} + k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2$$

However, because the top and bottom plates are at uniform temperatures, the temperature field must also be fully developed, in which case $(\partial T/\partial x) = 0$. The appropriate form of the energy equation is then

$$0 = k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2$$

The desired temperature distribution may be obtained by solving this equation. Rearranging and substituting for the velocity distribution,

$$k\frac{d^2T}{dy^2} = -\mu\left(\frac{du}{dy}\right)^2 = -\mu\left(\frac{U}{L}\right)^2$$

Integrating twice, we obtain

$$T(y) = -\frac{\mu}{2k}\left(\frac{U}{L}\right)^2 y^2 + C_3 y + C_4$$

The constants of integration may be obtained from the boundary conditions

$$T(0) = T_0 \qquad T(L) = T_L$$

in which case

$$C_4 = T_0 \qquad \text{and} \qquad C_3 = \frac{T_L - T_0}{L} + \frac{\mu}{2k}\frac{U^2}{L}$$

and

$$T(y) = T_0 + \frac{\mu}{2k}U^2\left[\frac{y}{L} - \left(\frac{y}{L}\right)^2\right] + (T_L - T_0)\frac{y}{L} \quad \triangleleft$$

4. Knowing the temperature distribution, the surface heat fluxes may be obtained by applying Fourier's law. Hence

$$q''_y = -k\frac{dT}{dy} = -k\left[\frac{\mu}{2k}U^2\left(\frac{1}{L} - \frac{2y}{L^2}\right) + \frac{T_L - T_0}{L}\right]$$

At the bottom and top surfaces, respectively, it follows that

$$q''_0 = -\frac{\mu U^2}{2L} - \frac{k}{L}(T_L - T_0) \qquad \text{and} \qquad q''_L = +\frac{\mu U^2}{2L} - \frac{k}{L}(T_L - T_0)$$

Hence, for the prescribed numerical values,

$$q''_0 = -\frac{0.799\ \text{N}\cdot\text{s/m}^2 \times 100\ \text{m}^2/\text{s}^2}{2 \times 3 \times 10^{-3}\ \text{m}} - \frac{0.145\ \text{W/m}\cdot\text{K}}{3 \times 10^{-3}\ \text{m}}(30 - 10)^\circ\text{C}$$

$$q''_0 = -13{,}315\ \text{W/m}^2 - 967\ \text{W/m}^2 = -14.3\ \text{kW/m}^2 \quad \triangleleft$$

$$q''_L = +13{,}315\ \text{W/m}^2 - 967\ \text{W/m}^2 = 12.3\ \text{kW/m}^2 \quad \triangleleft$$

The location of the maximum temperature in the oil may be found from the requirement that

$$\frac{dT}{dy} = \frac{\mu}{2k}U^2\left(\frac{1}{L} - \frac{2y}{L^2}\right) + \frac{T_L - T_0}{L} = 0$$

Solving for y, it follows that

$$y_{\max} = \left[\frac{k}{\mu U^2}(T_L - T_0) + \frac{1}{2}\right]L$$

or for the prescribed conditions

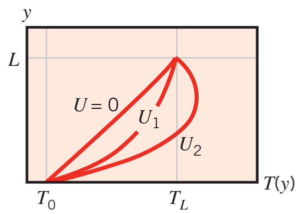
$$y_{\max} = \left[\frac{0.145\ \text{W/m}\cdot\text{K}}{0.799\ \text{N}\cdot\text{s/m}^2 \times 100\ \text{m}^2/\text{s}^2}(30 - 10)^\circ\text{C} + \frac{1}{2}\right]L = 0.536L$$

Substituting the value of $y_{\max}$ into the expression for $T(y)$,

$$T_{\max} = 89.2^\circ\text{C} \quad \triangleleft$$

Comments:

1. Given the strong effect of viscous dissipation for the prescribed conditions, the maximum temperature occurs in the oil and there is heat transfer to the hot, as well as to the cold, plate. The temperature distribution is a function of the velocity of the moving plate, and the effect is shown schematically below.



For velocities less than U_1 the maximum temperature corresponds to that of the hot plate. For $U = 0$ there is no viscous dissipation, and the temperature distribution is linear.

2. Recognize that the properties were evaluated at $\overline{T} = (T_L + T_0)/2 = 20°C$, which is *not* a good measure of the average oil temperature. For more precise calculations, the properties should be evaluated at a more appropriate value of the average temperature (e.g., $\overline{T} \approx 55°C$), and the calculations should be repeated.

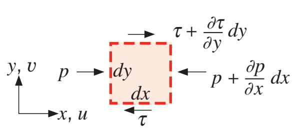
References

1. Schlichting, H., *Boundary Layer Theory*, 7th ed., McGraw-Hill, New York, 1979.
2. Bird, R. B., W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, Wiley, Hoboken, NJ, 1966.

Problems

Conservation Equations and Solutions

6S.1 Consider the control volume shown for the special case of steady-state conditions with $v = 0$, $T = T(y)$, and $\rho = \text{const}$.



(a) Prove that $u = u(y)$ if $v = 0$ everywhere.

(b) Derive the x-momentum equation and simplify it as much as possible.

(c) Derive the energy equation and simplify it as much as possible.

6S.2 Consider a lightly loaded journal bearing using oil having the constant properties $\mu = 10^{-2}$ kg/s · m and $k = 0.15$ W/m · K. If the journal and the bearing are each maintained at a temperature of 40°C, what is the maximum temperature in the oil when the journal is rotating at 10 m/s?

6S.3 Consider a lightly loaded journal bearing using oil having the constant properties $\rho = 800$ kg/m^3, $\nu = 10^{-5}$ m^2/s, and $k = 0.13$ W/m · K. The journal diameter is 75 mm; the clearance is 0.25 mm; and the bearing operates at 3600 rpm.

(a) Determine the temperature distribution in the oil film assuming that there is no heat transfer into the journal and that the bearing surface is maintained at 75°C.

(b) What is the rate of heat transfer from the bearing, and how much power is needed to rotate the journal?

6S.4 Consider two large (infinite) parallel plates, 5 mm apart. One plate is stationary, while the other plate is moving at a speed of 200 m/s. Both plates are maintained at 27°C. Consider two cases, one for which the plates are separated by water and the other for which the plates are separated by air.

(a) For each of the two fluids, what is the force per unit surface area required to maintain the above condition? What is the corresponding power requirement?

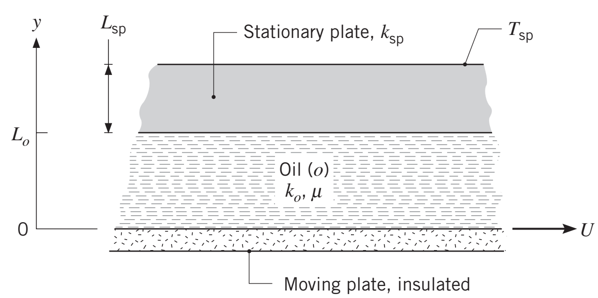
(b) What is the viscous dissipation associated with each of the two fluids?

(c) What is the maximum temperature in each of the two fluids?

6S.5 A judgment concerning the influence of viscous dissipation in forced convection heat transfer may be made by calculating the quantity $Pr \cdot Ec$, where the Prandtl number $Pr = c_p\mu/k$ and the Eckert number $Ec = U^2/c_p\,\Delta T$ are *dimensionless groups.* The characteristic velocity and temperature difference of the problem are designated as U and ΔT, respectively. If $Pr \cdot Ec \ll 1$, dissipation effects may be neglected. Consider Couette flow for which one plate moves at 10 m/s and a temperature difference of 25°C is maintained between the plates. Evaluating properties at 27°C, determine the value of $Pr \cdot Ec$ for air, water, and engine oil. What is the value of $Pr \cdot Ec$ for air if the plate is moving at the sonic velocity?

6S.6 Consider Couette flow for which the moving plate is maintained at a uniform temperature and the stationary plate is insulated. Determine the temperature of the insulated plate, expressing your result in terms of fluid properties and the temperature and speed of the moving plate. Obtain an expression for the heat flux at the moving plate.

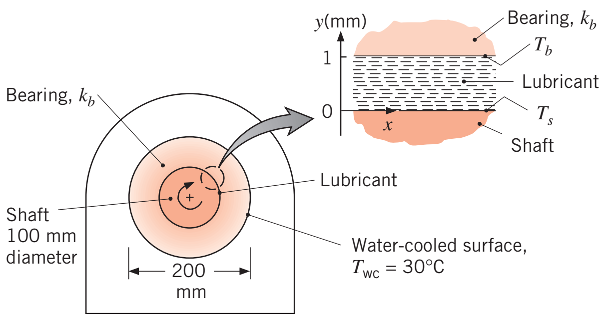
6S.7 Consider Couette flow with heat transfer for which the lower plate (mp) moves with a speed of $U = 5$ m/s and is perfectly insulated. The upper plate (sp) is stationary and is made of a material with thermal conductivity $k_{sp} = 1.5$ W/m · K and thickness $L_{sp} = 3$ mm. Its outer surface is maintained at $T_{sp} = 40$°C. The plates are separated by a distance $L_o = 5$ mm, which is filled with an engine oil of viscosity $\mu = 0.799$ N · s/m^2 and thermal conductivity $k_o = 0.145$ W/m · K.



(a) On $T(y)$–y coordinates, sketch the temperature distribution in the oil film and the moving plate.

(b) Obtain an expression for the temperature at the lower surface of the oil film, $T(0) = T_o$, in terms of the plate speed U, the stationary plate parameters (T_{sp}, k_{sp}, L_{sp}) and the oil parameters (μ, k_o, L_o). Calculate this temperature for the prescribed conditions.

6S.8 A shaft with a diameter of 100 mm rotates at 9000 rpm in a journal bearing that is 70 mm long. A uniform lubricant gap of 1 mm separates the shaft and the bearing. The lubricant properties are $\mu = 0.03$ N · s/m^2 and $k = 0.15$ W/m · K, while the bearing material has a thermal conductivity of $k_b = 45$ W/m · K.



(a) Determine the viscous dissipation, $\mu\Phi$ (W/m^3), in the lubricant.

(b) Determine the rate of heat transfer (W) from the lubricant, assuming that no heat is lost through the shaft.

(c) If the bearing housing is water-cooled, such that the outer surface of the bearing is maintained at 30°C, determine the temperatures of the bearing and shaft, T_b and T_s.

6S.9 Consider Couette flow with heat transfer as described in Example 6S.1.

(a) Rearrange the temperature distribution to obtain the dimensionless form

$$\theta(\eta) = \eta[1 + \tfrac{1}{2} Pr\, Ec(1 - \eta)]$$

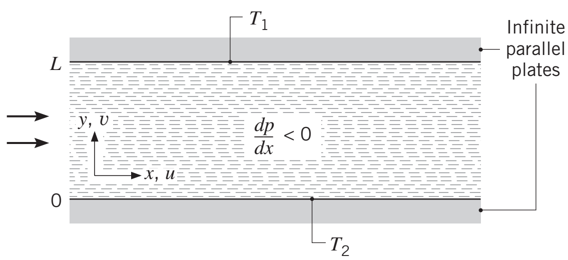
where $\theta \equiv [T(y) - T_0]/[T_L - T_0]$ and $\eta = y/L$. The *dimensionless groups* are the Prandtl number $Pr = \mu c_p/k$ and the Eckert number $Ec = U^2/c_p(T_L - T_0)$.

(b) Derive an expression that prescribes the conditions under which there will be no heat transfer to the upper plate.

(c) Derive an expression for the heat transfer rate to the lower plate for the conditions identified in part (b).

(d) Generate a plot of θ versus η for $0 \le \eta \le 1$ and values of $Pr\, Ec = 0, 1, 2, 4, 10$. Explain key features of the temperature distributions.

6S.10 Consider the problem of steady, incompressible laminar flow between two stationary, infinite parallel plates maintained at different temperatures.



Referred to as *Poiseuille flow* with heat transfer, this special case of parallel flow is one for which the x velocity component is finite, but the y- and z-components (v and w) are zero.

(a) What is the form of the continuity equation for this case? In what way is the flow *fully developed*?

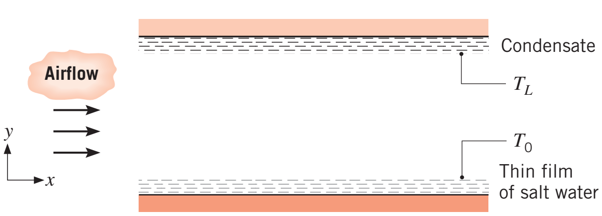
(b) What forms do the x- and y-momentum equations take? What is the form of the velocity profile? Note that, unlike Couette flow, fluid motion between the plates is now sustained by a finite pressure gradient. How is this pressure gradient related to the maximum fluid velocity?

(c) Assuming viscous dissipation to be significant and recognizing that conditions must be thermally fully developed, what is the appropriate form of the energy equation? Solve this equation for the temperature distribution. What is the heat flux at the upper ($y = L$) surface?

Species Conservation Equation and Solution

6S.11 Consider Problem 6S.10, when the fluid is a binary mixture with different molar concentrations $C_{A,1}$ and $C_{A,2}$ at the top and bottom surfaces, respectively. For the region between the plates, what is the appropriate form of the species A continuity equation? Obtain expressions for the species concentration distribution and the species flux at the upper surface.

6S.12 A simple scheme for desalination involves maintaining a thin film of saltwater on the lower surface of two large (infinite) parallel plates that are slightly inclined and separated by a distance L.



A slow, incompressible, laminar airflow exists between the plates, such that the x velocity component is finite while the y- and z-components are zero. Evaporation occurs from the liquid film on the lower surface, which is maintained at an elevated temperature T_0, while condensation occurs at the upper surface, which is maintained at a reduced temperature T_L. The corresponding molar concentrations of water vapor at the lower and upper surfaces are designated as $C_{A,0}$ and $C_{A,L}$, respectively. The species concentration and temperature may be assumed to be independent of x and z.

(a) Obtain an expression for the distribution of the water vapor molar concentration $C_A(y)$ in the air. What is the mass rate of pure water production per unit surface area? Express your results in terms of $C_{A,0}$, $C_{A,L}$, L, and the vapor–air diffusion coefficient D_{AB}.

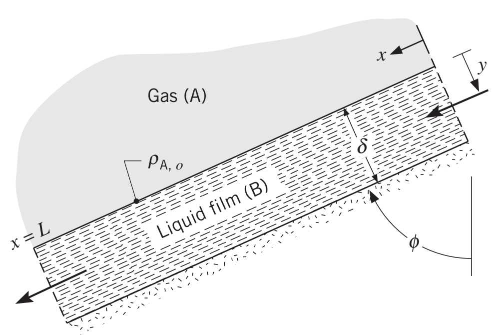
(b) Obtain an expression for the rate at which heat must be supplied per unit area to maintain the lower surface at T_0. Express your result in terms of $C_{A,0}$, $C_{A,L}$, T_0, T_L, L, D_{AB}, h_{fg} (the latent heat of vaporization of water), and the thermal conductivity k.

6S.13 Consider the conservation equations (6S.24) and (6S.31).

(a) Describe the physical significance of each term.

(b) Identify the approximations and special conditions needed to reduce these expressions to the boundary layer equations (6.29 and 6.30). Comparing these equations, identify the conditions under which they have the same form. Comment on the existence of a heat and mass transfer analogy.

6S.14 The *falling film* is widely used in chemical processing for the removal of gaseous species. It involves the flow of a liquid along a surface that may be inclined at some angle $\phi \geq 0$.



The flow is sustained by gravity, and the gas species A outside the film is absorbed at the liquid–gas interface. The film is in fully developed laminar flow over the entire plate, such that its velocity components in the y- and z-directions are zero. The mass density of A at $y = 0$ in the liquid is a constant $\rho_{A,o}$ independent of x.

(a) Write the appropriate form of the x-momentum equation for the film. Solve this equation for the distribution of the x velocity component, $u(y)$, in the film. Express your result in terms of δ, g, ϕ, and the liquid properties μ and ρ. Write an expression for the maximum velocity u_{max}.

(b) Obtain an appropriate form of the A species conservation equation for conditions within the film. If it is further assumed that the transport of species A across the gas–liquid interface does not penetrate very far into the film, the position $y = \delta$ may, for all practical purposes, be viewed as $y = \infty$. This condition implies that to a good approximation, $u = u_{max}$ in the region of penetration. Subject to these assumptions, determine an expression for $\rho_A(x, y)$ that applies in the film. *Hint:* This problem is analogous to conduction in a semi-infinite medium with a sudden change in surface temperature.

(c) If a local mass transfer convection coefficient is defined as

$$h_{m,x} \equiv \frac{n''_{A,x}}{\rho_{A,o}}$$

where $n''_{A,x}$ is the local mass flux at the gas–liquid interface, develop a suitable correlation for Sh_x as a function of Re_x and Sc.

(d) Develop an expression for the total gas absorption rate per unit width for a film of length L (kg/s · m).

(e) A water film that is 1 mm thick runs down the inside surface of a vertical tube that is 2 m long and has an inside diameter of 50 mm. An airstream containing ammonia (NH_3) moves through the tube, such that the mass density of NH_3 at the gas–liquid interface (but in the liquid) is 25 kg/m^3. A dilute solution of ammonia in water is formed, and the diffusion coefficient is 2×10^{-9} m^2/s. What is the mass rate of NH_3 removal by absorption?

11S.1 *Log Mean Temperature Difference Method for Multipass and Cross-Flow Heat Exchangers*

Although flow conditions are more complicated in multipass and cross-flow heat exchangers, Equations 11.6, 11.7, 11.14, and 11.15 may still be used if the following modification is made to the log mean temperature difference [1]:

$$\Delta T_{\text{lm}} = F\,\Delta T_{\text{lm,CF}} \tag{11S.1}$$

That is, the appropriate form of ΔT_{lm} is obtained by applying a correction factor to the value of ΔT_{lm} that would be computed *under the assumption of counterflow conditions.* Hence from Equation 11.17, $\Delta T_1 = T_{h,i} - T_{c,o}$ and $\Delta T_2 = T_{h,o} - T_{c,i}$.

Algebraic expressions for the correction factor F have been developed for various shell-and-tube and cross-flow heat exchanger configurations [1–3], and the results may be represented graphically. Selected results are shown in Figures 11S.1 through 11S.4 for common heat exchanger configurations. The notation (T, t) is used to specify the fluid temperatures, with the variable t always assigned to the tube-side fluid. With this convention it does not matter whether the hot fluid or the cold fluid flows through the shell or the tubes.

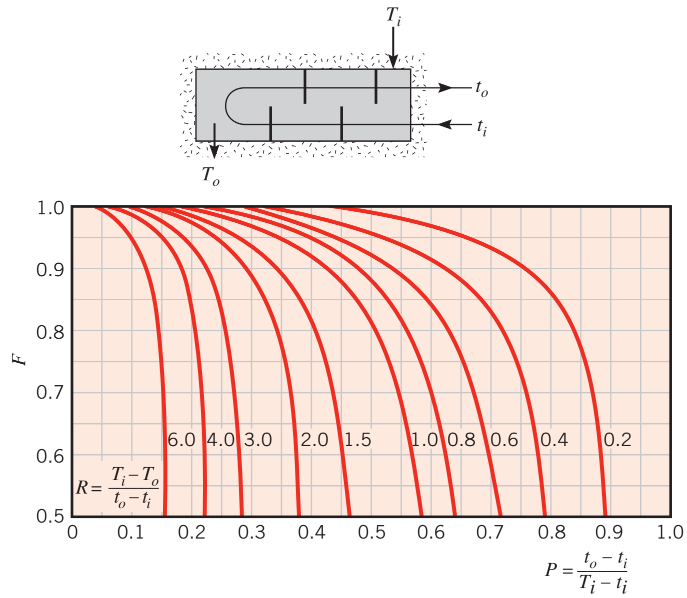


FIGURE 11S.1 Correction factor for a shell-and-tube heat exchanger with one shell and any multiple of two tube passes (two, four, etc. tube passes).

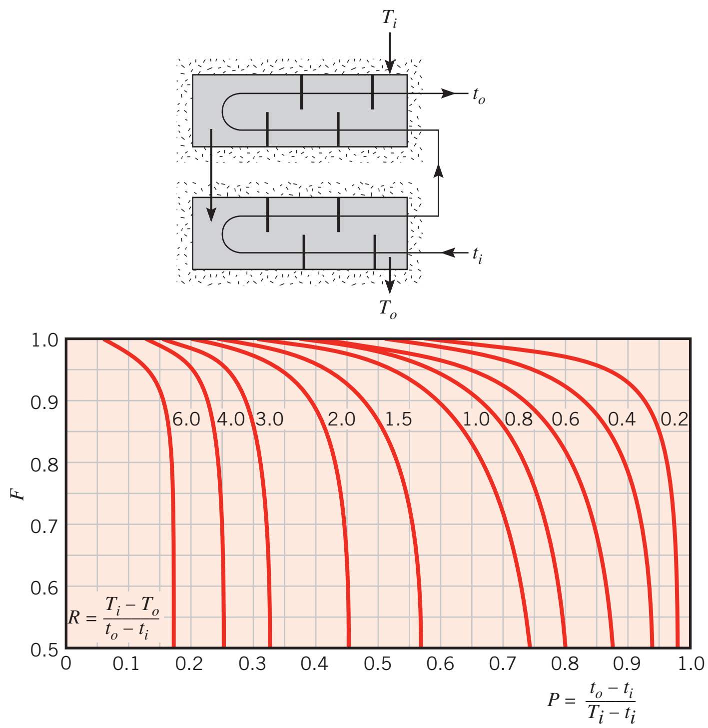


FIGURE 11S.2 Correction factor for a shell-and-tube heat exchanger with two shell passes and any multiple of four tube passes (four, eight, etc. tube passes).

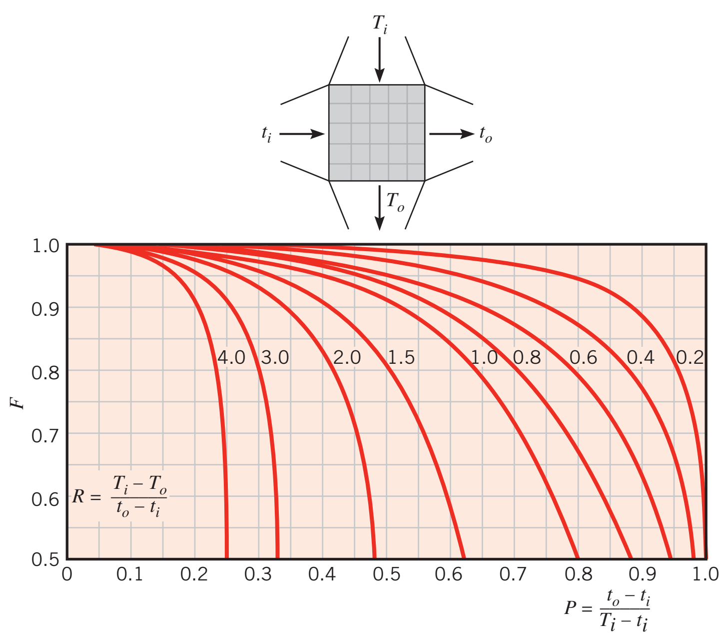


FIGURE 11S.3 Correction factor for a single-pass, cross-flow heat exchanger with both fluids unmixed.

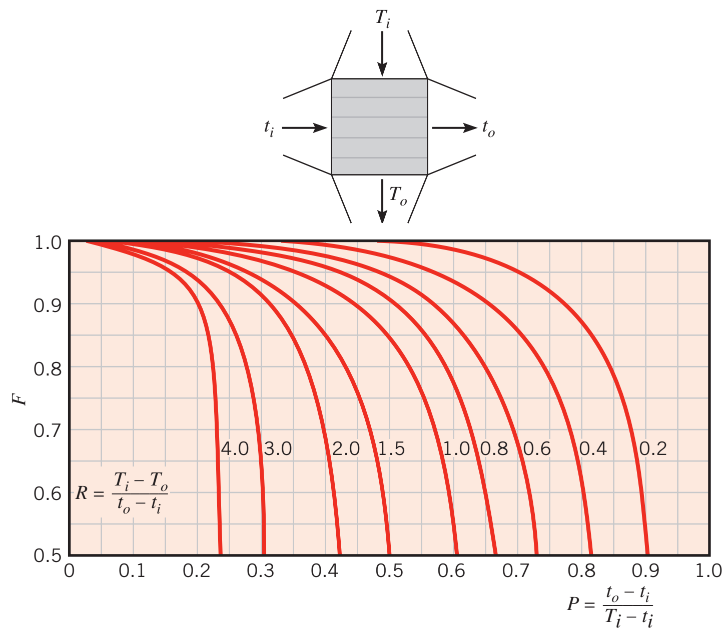


FIGURE 11S.4 Correction factor for a single-pass, cross-flow heat exchanger with one fluid mixed and the other unmixed.

An important implication of Figures 11S.1 through 11S.4 is that, *if the temperature change of one fluid is negligible,* either P or R is zero and F is 1. *Hence heat exchanger behavior is independent of the specific configuration.* Such would be the case if one of the fluids underwent a phase change.

EXAMPLE 11S.1

A shell-and-tube heat exchanger must be designed to heat 2.5 kg/s of water from 15 to 85°C. The heating is to be accomplished by passing hot engine oil, which is available at 160°C, through the shell side of the exchanger. The oil is known to provide an average convection coefficient of $h_o = 400\ \text{W/m}^2 \cdot \text{K}$ on the outside of the tubes. Ten tubes pass the water through the shell. Each tube is thin walled, of diameter $D = 25$ mm, and makes eight passes through the shell. If the oil leaves the exchanger at 100°C, what is its flow rate? How long must the tubes be to accomplish the desired heating?

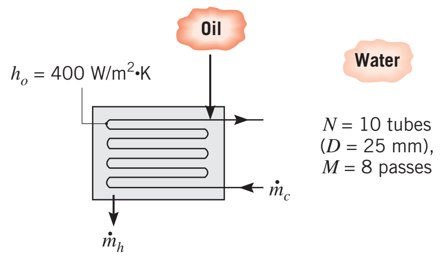
SOLUTION

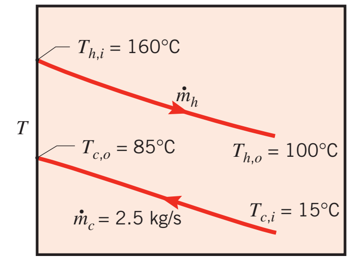
Known: Fluid inlet and outlet temperatures for a shell-and-tube heat exchanger with 10 tubes making eight passes.

Find:

1. Oil flow rate required to achieve specified outlet temperature.
2. Tube length required to achieve specified water heating.

Schematic:





Assumptions:

1. Negligible heat loss to the surroundings and kinetic and potential energy changes.
2. Constant properties.
3. Negligible tube wall thermal resistance and fouling effects.
4. Fully developed water flow in tubes.

Properties: Table A.5, unused engine oil ($\overline{T}_h = 130°\text{C}$): $c_p = 2350$ J/kg·K. Table A.6, water ($\overline{T}_c = 50°\text{C}$): $c_p = 4181$ J/kg·K, $\mu = 548 \times 10^{-6}$ N·s/m^2, $k = 0.643$ W/m·K, $Pr = 3.56$.

Analysis:

1. From the overall energy balance, Equation 11.7b, the heat transfer required of the exchanger is

$$q = \dot{m}_c c_{p,c}(T_{c,o} - T_{c,i}) = 2.5 \text{ kg/s} \times 4181 \text{ J/kg}\cdot\text{K}\,(85 - 15)°\text{C}$$

$$q = 7.317 \times 10^5 \text{ W}$$

Hence, from Equation 11.6b,

$$\dot{m}_h = \frac{q}{c_{p,h}(T_{h,i} - T_{h,o})} = \frac{7.317 \times 10^5 \text{ W}}{2350 \text{ J/kg}\cdot\text{K} \times (160 - 100)°\text{C}} = 5.19 \text{ kg/s} \quad \triangleleft$$

2. The required tube length may be obtained from Equations 11.14 and 11S.1, where

$$q = UAF\,\Delta T_{\text{lm,CF}}$$

From Equation 11.5,

$$U = \frac{1}{(1/h_i) + (1/h_o)}$$

where h_i may be obtained by first calculating Re_D. With $\dot{m}_1 \equiv \dot{m}_c/N = 0.25$ kg/s defined as the water flow rate per tube, Equation 8.6 yields

$$Re_D = \frac{4\dot{m}_1}{\pi D \mu} = \frac{4 \times 0.25 \text{ kg/s}}{\pi (0.025 \text{ m})\, 548 \times 10^{-6} \text{ kg/s}\cdot\text{m}} = 23{,}234$$

Hence the water flow is turbulent, and from Equation 8.60

$$Nu_D = 0.023\, Re_D^{4/5}\, Pr^{0.4} = 0.023(23{,}234)^{4/5}(3.56)^{0.4} = 119$$

$$h_i = \frac{k}{D} Nu_D = \frac{0.643\ \text{W/m}\cdot\text{K}}{0.025\ \text{m}}\, 119 = 3061\ \text{W/m}^2\cdot\text{K}$$

Hence

$$U = \frac{1}{(1/400) + (1/3061)} = 354\ \text{W/m}^2\cdot\text{K}$$

The correction factor F may be obtained from Figure 11S.1, where

$$R = \frac{160 - 100}{85 - 15} = 0.86 \qquad P = \frac{85 - 15}{160 - 15} = 0.48$$

Hence $F \approx 0.87$. From Equations 11.15 and 11.17, it follows that

$$\Delta T_{\text{lm,CF}} = \frac{(T_{h,i} - T_{c,o}) - (T_{h,o} - T_{c,i})}{\ln[(T_{h,i} - T_{c,o})/(T_{h,o} - T_{c,i})]} = \frac{75 - 85}{\ln(75/85)} = 79.9°\text{C}$$

Hence, since $A = N\pi DL$, where $N = 10$ is the number of tubes,

$$L = \frac{q}{UN\pi DF\Delta T_{\text{lm,CF}}} = \frac{7.317 \times 10^5\ \text{W}}{354\ \text{W/m}^2\cdot\text{K} \times 10\pi(0.025\ \text{m})0.87(79.9°\text{C})}$$

$$L = 37.9\ \text{m}$$ ◁

Comments:

1. With $(L/D) = 37.9\ \text{m}/0.025\ \text{m} = 1516$, the assumption of fully developed conditions throughout the tube is justified.
2. With eight passes, the shell length is approximately $L/M = 4.7$ m.

11S.2 *Compact Heat Exchangers*

As discussed in Section 11.1, *compact heat exchangers* are typically used when a large heat transfer surface area per unit volume is desired and at least one of the fluids is a gas. Many different tubular and plate configurations have been considered, where differences are due primarily to fin design and arrangement. Heat transfer and flow characteristics have been determined for specific configurations and are typically presented in the format of Figures 11S.5 and 11S.6. Heat transfer results are correlated in terms of the Colburn j factor $j_H = St\, Pr^{2/3}$ and the Reynolds number, where both the Stanton ($St = h/Gc_p$) and Reynolds ($Re = GD_h/\mu$) numbers are based on the maximum mass velocity

$$G \equiv \rho V_{\text{max}} = \frac{\rho V A_{\text{fr}}}{A_{\text{ff}}} = \frac{\dot{m}}{A_{\text{ff}}} = \frac{\dot{m}}{\sigma A_{\text{fr}}} \tag{11S.2}$$

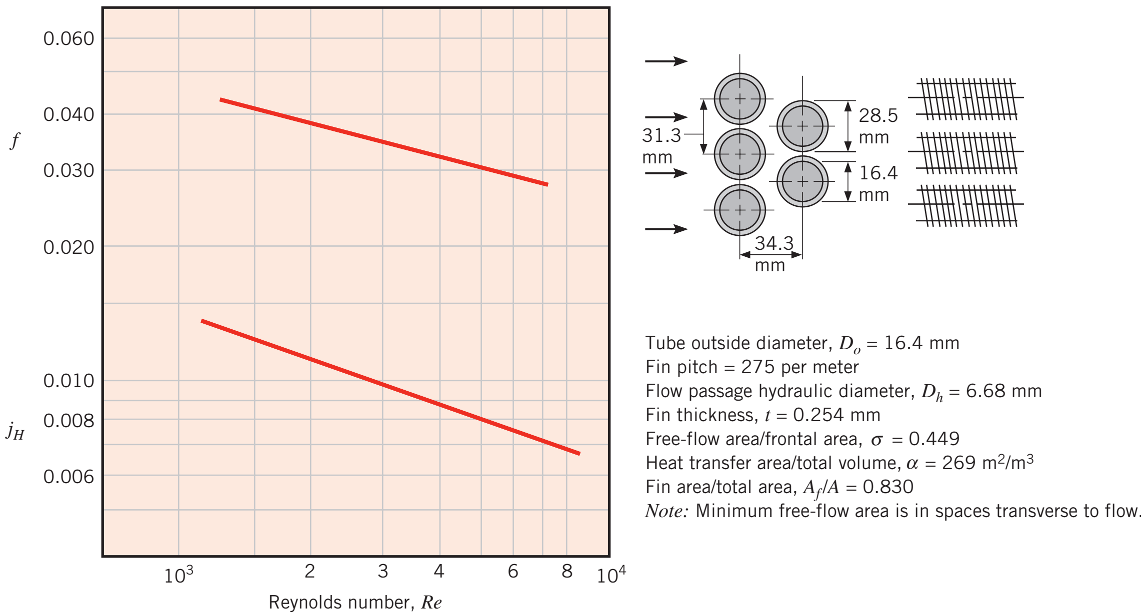


FIGURE 11S.5 Heat transfer and friction factor for a circular tube–circular fin heat exchanger, surface CF-7.0-5/8J from Kays and London [4].

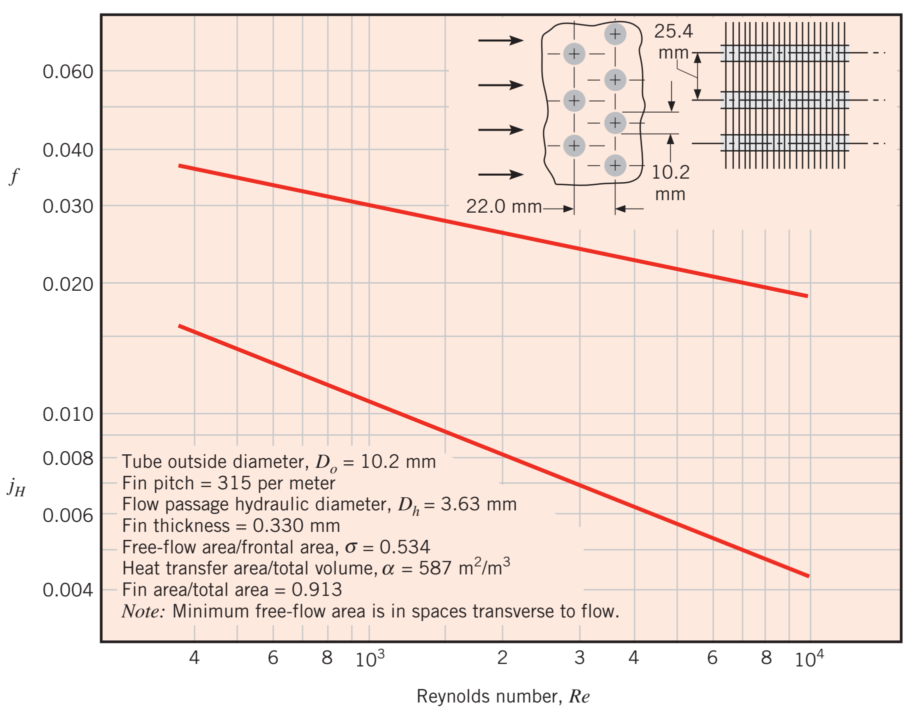


FIGURE 11S.6 Heat transfer and friction factor for a circular tube–continuous fin heat exchanger, surface 8.0-3/8T from Kays and London [4].

The quantity σ is the ratio of the minimum free-flow area of the finned passages (cross-sectional area perpendicular to flow direction), A_{ff}, to the frontal area, A_{fr}, of the exchanger. Values of σ, D_h (the hydraulic diameter of the flow passage), α (the heat transfer surface area per total heat exchanger volume), A_f/A (the ratio of fin to total heat transfer surface area), and other geometrical parameters are listed for each configuration. The ratio A_f/A is used in Equation 11.3 to evaluate the temperature effectiveness η_o. In a design calculation, α would be used to determine the required heat exchanger volume, after the total heat transfer surface area has been found; in a performance calculation it would be used to determine the surface area from knowledge of the heat exchanger volume.

In a compact heat exchanger calculation, empirical information, such as that provided in Figures 11S.5 and 11S.6, would first be used to determine the average convection coefficient of the finned surfaces. The overall heat transfer coefficient would then be determined, and using the ε–NTU method, the heat exchanger design or performance calculations would be performed.

The pressure drop associated with flow across finned-tube banks, such as those of Figures 11S.5 and 11S.6, may be computed from the expression

$$\Delta p = \frac{G^2 v_i}{2}\left[(1+\sigma^2)\left(\frac{v_o}{v_i}-1\right) + f\frac{A}{A_{ff}}\frac{v_m}{v_i}\right] \qquad (11S.3)$$

where v_i and v_o are the fluid inlet and outlet specific volumes and $v_m = (v_i + v_o)/2$. The first term on the right-hand side of Equation 11S.3 accounts for the cumulative effects of pressure change due to inviscid fluid acceleration and deceleration at the exchanger inlet and outlet, respectively. The effects are *reversible*, and if fluid density variations may be neglected ($v_o \simeq v_i$), the term is negligible. The second term accounts for losses due to fluid friction in the heat exchanger core, with fully developed conditions presumed to exist throughout the core. For a prescribed core configuration, the friction factor is known as a function of Reynolds number, as, for example, from Figures 11S.5 and 11S.6; and for a prescribed heat exchanger size, the area ratio may be evaluated from the relation $(A/A_{ff}) = (\alpha V/\sigma A_{fr})$, where V is the total heat exchanger volume.

Equation 11S.3 does not account for irreversible losses due to viscous effects at the inlet and outlet of the heat exchanger. The losses depend on the nature of the ductwork used to transport fluids to and from the heat exchanger core. If the transition between the ductwork and the core occurs with little flow separation, the losses are small. However, if there are abrupt changes between the duct cross-sectional area and the free-flow area of the heat exchanger, separation is pronounced and the attendant losses are large. Inlet and exit losses may be estimated from empirical *contraction* and *expansion coefficients* obtained for a variety of core geometries [4].

The classic work of Kays and London [4] provides Colburn j and friction factor data for many different compact heat exchanger cores, which include flat tube (Figure 11.5*a*) and plate–fin (Figure 11.5*d*, *e*) configurations, as well as other circular tube configurations (Figure 11.5*b*, *c*). Other excellent sources of information are provided by References 5, 6, 7, and 8.

EXAMPLE 11S.2

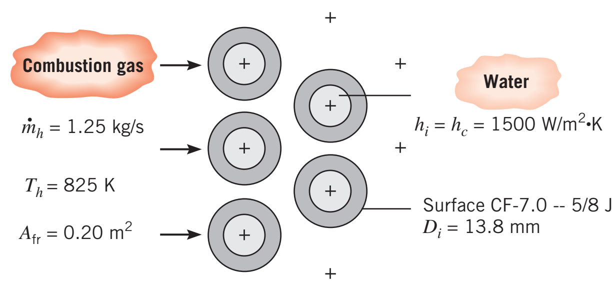
Consider a finned-tube, compact heat exchanger having the core configuration of Figure 11S.5. The core is fabricated from aluminum, and the tubes have an inside diameter of 13.8 mm. In a waste heat recovery application, water flow through the tubes provides an inside convection coefficient of $h_i = 1500\ \mathrm{W/m^2 \cdot K}$, while combustion gases at 1 atm and 825 K are in cross flow over the tubes. If the gas flow rate is 1.25 kg/s and the frontal area is 0.20 m^2, what is the gas-side overall heat transfer coefficient? If a water flow rate of 1 kg/s is to be heated from 290 to 370 K, what is the required heat exchanger volume?

Solution

Known: Compact heat exchanger geometry, gas-side flow rate and temperature, and water-side convection coefficient. Water flow rate and inlet and outlet temperatures.

Find: Gas-side overall heat transfer coefficient. Heat exchanger volume.

Schematic:



Assumptions:

1. Gas has properties of atmospheric air at an assumed mean temperature of 700 K.
2. Fouling is negligible.

Properties: Table A.1, aluminum ($T \approx 300$ K): $k = 237$ W/m$\cdot$K. Table A.4, air ($p = 1$ atm, $\overline{T} = 700$ K): $c_p = 1075$ J/kg$\cdot$K, $\mu = 338.8 \times 10^{-7}$ N$\cdot$s/m^2, $Pr = 0.695$. Table A.6, water ($\overline{T} = 330$ K): $c_p = 4184$ J/kg$\cdot$K.

Analysis: Referring to Equation 11.1b, the combustion gas and the water are the hot and cold fluids, respectively. Hence, neglecting fouling effects and acknowledging that the tube inner surface is not finned ($\eta_{o,c} = 1$), the overall heat transfer coefficient based on the gas-(hot) side surface area is given by

$$\frac{1}{U_h} = \frac{1}{h_c(A_c/A_h)} + A_h R_w + \frac{1}{\eta_{o,h} h_h}$$

where A_h and A_c are the total gas-side (hot) and water-side (cold) surface areas, respectively. If the fin thickness is assumed to be negligible, it is readily shown that

$$\frac{A_c}{A_h} \approx \frac{D_i}{D_o}\left(1 - \frac{A_{f,h}}{A_h}\right)$$

where $A_{f,h}$ is that portion of the total gas-side area associated with the fins. The approximation is valid to within 10%, and for the heat exchanger core conditions (Figure 11S.5)

$$\frac{A_c}{A_h} \approx \frac{13.8}{16.4}(1 - 0.830) = 0.143$$

Obtaining the wall conduction resistance from Equation 3.33, it follows that

$$A_h R_w = \frac{\ln(D_o/D_i)}{2\pi Lk/A_h} = \frac{D_i \ln(D_o/D_i)}{2k(A_c/A_h)}$$

Hence

$$A_h R_w = \frac{(0.0138 \text{ m}) \ln(16.4/13.8)}{2(237 \text{ W/m} \cdot \text{K})(0.143)} = 3.51 \times 10^{-5} \text{ m}^2 \cdot \text{K/W}$$

The gas-side convection coefficient may be obtained by first using Equation 11S.2 to evaluate the mass velocity:

$$G = \frac{\dot{m}}{\sigma A_{\text{fr}}} = \frac{1.25 \text{ kg/s}}{0.449 \times 0.20 \text{ m}^2} = 13.9 \text{ kg/s} \cdot \text{m}^2$$

Hence

$$Re = \frac{13.9 \text{ kg/s} \cdot \text{m}^2 \times 6.68 \times 10^{-3} \text{ m}}{338.8 \times 10^{-7} \text{ kg/s} \cdot \text{m}} = 2740$$

and from Figure 11S.5, $j_H \approx 0.010$. Hence

$$h_h \approx 0.010 \frac{Gc_p}{Pr^{2/3}} = 0.010 \frac{(13.9 \text{ kg/s} \cdot \text{m}^2)(1075 \text{ J/kg} \cdot \text{K})}{(0.695)^{2/3}}$$

$$= 190 \text{ W/m}^2 \cdot \text{K}$$

To obtain the hot-side temperature effectiveness from Equation 11.3, the fin efficiency must first be determined from Figure 3.20. With $r_{2c} = 14.38$ mm, $r_{2c}/r_1 = 1.75$, $L_c = 6.18$ mm, $A_p = 1.57 \times 10^{-6}$ m^2, and $L_c^{3/2}(h_h/kA_p)^{1/2} = 0.34$, it follows that $\eta_f \approx 0.89$. Hence

$$\eta_{o,h} = 1 - \frac{A_f}{A}(1 - \eta_f) = 1 - 0.830(1 - 0.89) = 0.91$$

We then obtain

$$\frac{1}{U_h} = \left(\frac{1}{1500 \times 0.143} + 3.51 \times 10^{-5} + \frac{1}{0.91 \times 190}\right) \text{m}^2 \cdot \text{K/W}$$

$$\frac{1}{U_h} = (4.66 \times 10^{-3} + 3.51 \times 10^{-5} + 5.78 \times 10^{-3}) = 0.010 \text{ m}^2 \cdot \text{K/W}$$

or

$$U_h = 100 \text{ W/m}^2 \cdot \text{K} \quad \triangleleft$$

With $C_c = \dot{m}_c c_{p,c} = 1 \text{ kg/s} \times 4184 \text{ J/kg} \cdot \text{K} = 4184$ W/K, the heat exchanger must be large enough to transfer heat in the amount

$$q = C_c(T_{c,o} - T_{c,i}) = 4184 \text{ W/K } (370 - 290) \text{ K} = 3.35 \times 10^5 \text{ W}$$

With $C_h = \dot{m}_h c_{p,h} = 1.25 \text{ kg/s} \times 1075 \text{ J/kg} \cdot \text{K} = 1344$ W/K, the minimum heat capacity rate corresponds to the hot fluid and the maximum possible heat transfer rate is

$$q_{\max} = C_{\min}(T_{h,i} - T_{c,i}) = 1344 \text{ W/K } (825 - 290) \text{ K} = 7.19 \times 10^5 \text{ W}$$

It follows that

$$\varepsilon = \frac{q}{q_{\max}} = \frac{3.35 \times 10^5 \text{ W}}{7.19 \times 10^5 \text{ W}} = 0.466$$

Hence, with $(C_{\min}/C_{\max}) = 0.321$, Figure 11.14 (cross-flow heat exchanger with both fluids unmixed) yields

$$\text{NTU} = \frac{U_h A_h}{C_{\min}} \approx 0.65$$

The required gas-side heat transfer surface area is then

$$A_h = \frac{0.65 \times 1344\ \text{W/K}}{100\ \text{W/m}^2 \cdot \text{K}} = 8.7\ \text{m}^2$$

With the gas-side surface area per unit heat exchanger volume corresponding to $\alpha = 269\ \text{m}^2/\text{m}^3$ (Figure 11S.5), the required heat exchanger volume is

$$V = \frac{A_h}{\alpha} = \frac{8.7\ \text{m}^2}{269\ \text{m}^2/\text{m}^3} = 0.032\ \text{m}^3 \quad \triangleleft$$

Comments:

1. The effect of the tube wall thermal conduction resistance is negligible, while contributions due to the cold- and hot-side convection resistances are comparable.
2. Knowledge of the heat exchanger volume yields the heat exchanger length in the gas-flow direction, $L = V/A_{\text{fr}} = 0.032\ \text{m}^3/0.20\ \text{m}^2 = 0.16\ \text{m}$, from which the number of tube rows in the flow direction may be determined.

$$N_L \approx \frac{L - D_f}{S_L} + 1 = \frac{(160 - 28.5)\ \text{mm}}{34.3\ \text{mm}} + 1 = 4.8 \approx 5$$

3. The temperature of the gas leaving the heat exchanger is

$$T_{h,o} = T_{h,i} - \frac{q}{C_h} = 825\ \text{K} - \frac{3.35 \times 10^5\ \text{W}}{1344\ \text{W/K}} = 576\ \text{K}$$

Hence the assumption of $\overline{T}_h = 700$ K is excellent.

4. From Figure 11S.5, the friction factor is $f \approx 0.033$. With $(A/A_{\text{ff}}) = (\alpha V/\sigma A_{\text{fr}}) = (269 \times 0.032/0.449 \times 0.20) = 96$, $v_i(825\ \text{K}) = 2.37\ \text{m}^3/\text{kg}$, $v_o(576\ \text{K}) = 1.65\ \text{m}^3/\text{kg}$, and $v_m = 2.01\ \text{m}^3/\text{kg}$, Equation 11S.3 yields a pressure drop of

$$\Delta p = \frac{(13.9\ \text{kg/s} \cdot \text{m}^2)^2\ (2.37\ \text{m}^3/\text{kg})}{2} [(1 + 0.202)(0.696 - 1) + 0.033 \times 96 \times 0.848]$$

$$\Delta p = 530\ \text{kg/s}^2 \cdot \text{m} = 530\ \text{N/m}^2$$

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Problems

Log Mean Temperature Difference Method

11S.1 Solve Problem 11.9 using the LMTD method.

11S.2 Solve Problem 11.10 using the LMTD method.

11S.3 Solve Problem 11.14 using the LMTD method.

11S.4 Solve Problem 11.15 using the LMTD method.

11S.5 Solve Problem 11.25 using the LMTD method.

11S.6 Solve Problem 11.32 using the LMTD method.

11S.7 Solve Problem 11.49 using the LMTD method.

11S.8 Solve Problem 11.54 using the LMTD method.

11S.9 Solve Problem 11.58 using the LMTD method.

11S.10 Solve Problem 11.67 using the LMTD method.

11S.11 Solve Problem 11.85 using the LMTD method.

Compact Heat Exchangers

11S.12 Consider the compact heat exchanger conditions of Example 11S.2. After extended use, fouling factors of 0.0005 and 0.001 $m^2 \cdot K/W$ are associated with the water- and gas-side conditions, respectively. What is the gas-side overall heat transfer coefficient?

11S.13 Consider the heat exchanger core geometry and frontal area prescribed in Example 11S.2. The exchanger must heat 2 kg/s of water from 300 to 350 K, using 1.25 kg/s of combustion gases entering at 700 K. Using the overall heat transfer coefficient determined in the example, find the required heat exchanger volume, assuming single-pass operation. What is the number of tube rows N_L in the longitudinal (gas-flow) direction? If the velocity of water flowing through the tubes is 100 mm/s, what is the number of tube rows N_T in the transverse direction? What is the required tube length?

11S.14 Consider the conditions of Example 11S.2, but with the continuous fin arrangement of Figure 11S.6 used in lieu of the circular fins of Figure 11S.5. The heat exchanger core is fabricated from aluminum, and the tubes have an inside diameter of 8.2 mm. An inside convection coefficient of 1500 $W/m^2 \cdot K$ may again be assumed for water flow through the tubes, with combustion gases at 1 atm and 825 K in cross flow over the tubes. For a gas flow rate of 1.25 kg/s and a frontal area of 0.20 m^2, what is the gas-side overall heat transfer coefficient? If water at a flow rate of 1 kg/s is to be heated from 290 to 370 K, what is the required heat exchanger volume? *Hint:* Estimate the fin efficiency by assuming a hypothetical circular fin of radius $r_2 = 15.8$ mm for each tube. You may use the aluminum and fluid properties provided in Example 11S.2.

11S.15 A cooling coil consists of a bank of aluminum ($k = 237$ $W/m \cdot K$) finned tubes having the core configuration of Figure 11S.5 and an inner diameter of 13.8 mm. The tubes are installed in a plenum whose square cross section is 0.4 m on a side, thereby providing a frontal area of 0.16 m^2. Atmospheric air at 1.5 kg/s is in cross flow over the tubes, while saturated refrigerant-134a at 1 atm experiences evaporation in the tubes. If the air enters at 37°C and its exit temperature must not exceed 17°C, what is the minimum allowable number of tube rows in the flow direction? A convection coefficient of 5000 $W/m^2 \cdot K$ is associated with evaporation in the tubes.

11S.16 A cooling coil consists of a bank of aluminum ($k = 237$ $W/m \cdot K$) finned tubes having the core configuration of Figure 11S.5 and an inner diameter of 13.8 mm. The tubes are installed in a plenum whose square cross section is 0.4 m on a side, thereby providing a frontal area of 0.16 m^2. Atmospheric air at 1.5 kg/s is in cross flow over the tubes, while saturated refrigerant-134a at 1 atm passes through the tubes. There are four rows of tubes in the airflow direction. If the air enters at 37°C, what is its exit temperature? A convection coefficient of 5000 $W/m^2 \cdot K$ is associated with evaporation in the tubes.

11S.17 A steam generator consists of a bank of stainless steel ($k = 15$ $W/m \cdot K$) tubes having the core configuration of Figure 11S.5 and an inner diameter of 13.8 mm. The tubes are installed in a plenum whose square cross section is 0.6 m on a side, thereby providing a frontal area of 0.36 m^2. Combustion gases, whose properties may be approximated as those of atmospheric air, enter the plenum at 900 K and pass in cross flow over the tubes at 3 kg/s. If saturated water enters the tubes at a pressure of 2.455 bars and a flow rate of 0.5 kg/s, how many tube rows are required to provide saturated steam at the tube outlet? A convection coefficient of 10,000 $W/m^2 \cdot K$ is associated with boiling in the tubes.

11S.18 A steam generator consists of a bank of stainless steel ($k = 15$ $W/m \cdot K$) tubes having the core configuration of Figure 11S.5 and an inner diameter of 13.8 mm. The tubes are installed in a plenum whose square cross section is 0.6 m on a side, thereby providing a frontal area of 0.36 m^2. Combustion gases, whose properties may be approximated as those of atmospheric air, enter the plenum at 900 K and pass in cross flow over the tubes at 3 kg/s. There are 11 rows of tubes in the gas flow direction. If saturated water at 2.455 bar experiences boiling in the tubes, what is the gas exit temperature? A convection coefficient of 10,000 $W/m^2 \cdot K$ is associated with boiling in the tubes.

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