Supplements

Supplement 1

Elementary Functions and Their Properties

Throughout Supplement 1 it is assumed that n is a positive integer, unless otherwise specified.

1.1. Trigonometric Functions

► **Simplest relations**

$$\sin^2 x + \cos^2 x = 1, \quad \sin(-x) = -\sin x, \quad \cos(-x) = \cos x,$$
$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad 1 + \tan^2 x = \frac{1}{\cos^2 x}, \quad 1 + \cot^2 x = \frac{1}{\sin^2 x},$$
$$\tan x \cot x = 1, \quad \tan(-x) = -\tan x, \quad \cot(-x) = -\cot x.$$

► **Relations between trigonometric functions of single argument**

$$\sin x = \pm\sqrt{1-\cos^2 x} = \pm\frac{\tan x}{\sqrt{1+\tan^2 x}} = \pm\frac{1}{\sqrt{1+\cot^2 x}},$$
$$\cos x = \pm\sqrt{1-\sin^2 x} = \pm\frac{1}{\sqrt{1+\tan^2 x}} = \pm\frac{\cot x}{\sqrt{1+\cot^2 x}},$$
$$\tan x = \pm\frac{\sin x}{\sqrt{1-\sin^2 x}} = \pm\frac{\sqrt{1-\cos^2 x}}{\cos x} = \frac{1}{\cot x},$$
$$\cot x = \pm\frac{\sqrt{1-\sin^2 x}}{\sin x} = \pm\frac{\cos x}{\sqrt{1-\cos^2 x}} = \frac{1}{\tan x}.$$

► **Reduction formulas**

$$\sin(x \pm n\pi) = (-1)^n \sin x, \qquad \cos(x \pm n\pi) = (-1)^n \cos x,$$
$$\sin\left(x \pm \frac{2n+1}{2}\pi\right) = \pm(-1)^n \cos x, \qquad \cos\left(x \pm \frac{2n+1}{2}\pi\right) = \mp(-1)^n \sin x,$$
$$\tan(x \pm n\pi) = \tan x, \qquad \cot(x \pm n\pi) = \cot x,$$
$$\tan\left(x \pm \frac{2n+1}{2}\pi\right) = -\cot x, \qquad \cot\left(x \pm \frac{2n+1}{2}\pi\right) = -\tan x,$$
$$\sin\left(x \pm \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(\sin x \pm \cos x), \qquad \cos\left(x \pm \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(\cos x \mp \sin x),$$
$$\tan\left(x \pm \frac{\pi}{4}\right) = \frac{\tan x \pm 1}{1 \mp \tan x}, \qquad \cot\left(x \pm \frac{\pi}{4}\right) = \frac{\cot x \mp 1}{1 \pm \cot x}.$$

► Addition formulas

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y, \qquad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}, \qquad \cot(x \pm y) = \frac{1 \mp \tan x \tan y}{\tan x \pm \tan y}.$$

► Addition and subtraction of trigonometric functions

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right),$$
$$\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right),$$
$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right),$$
$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right),$$
$$\sin^2 x - \sin^2 y = \cos^2 y - \cos^2 x = \sin(x+y)\sin(x-y),$$
$$\sin^2 x - \cos^2 y = -\cos(x+y)\cos(x-y),$$
$$\tan x \pm \tan y = \frac{\sin(x \pm y)}{\cos x \cos y}, \quad \cot x \pm \cot y = \frac{\sin(y \pm x)}{\sin x \sin y},$$
$$a\cos x + b\sin x = r\sin(x+\varphi) = r\cos(x-\psi).$$

Here $r = \sqrt{a^2+b^2}$, $\sin\varphi = a/r$, $\cos\varphi = b/r$, $\sin\psi = b/r$, and $\cos\psi = a/r$.

► Products of trigonometric functions

$$\sin x \sin y = \tfrac{1}{2}[\cos(x-y) - \cos(x+y)],$$
$$\cos x \cos y = \tfrac{1}{2}[\cos(x-y) + \cos(x+y)],$$
$$\sin x \cos y = \tfrac{1}{2}[\sin(x-y) + \sin(x+y)].$$

► Powers of trigonometric functions

$$\cos^2 x = \tfrac{1}{2}\cos 2x + \tfrac{1}{2}, \qquad \sin^2 x = -\tfrac{1}{2}\cos 2x + \tfrac{1}{2},$$
$$\cos^3 x = \tfrac{1}{4}\cos 3x + \tfrac{3}{4}\cos x, \qquad \sin^3 x = -\tfrac{1}{4}\sin 3x + \tfrac{3}{4}\sin x,$$
$$\cos^4 x = \tfrac{1}{8}\cos 4x + \tfrac{1}{2}\cos 2x + \tfrac{3}{8}, \qquad \sin^4 x = \tfrac{1}{8}\cos 4x - \tfrac{1}{2}\cos 2x + \tfrac{3}{8},$$
$$\cos^5 x = \tfrac{1}{16}\cos 5x + \tfrac{5}{16}\cos 3x + \tfrac{5}{8}\cos x, \qquad \sin^5 x = \tfrac{1}{16}\sin 5x - \tfrac{5}{16}\sin 3x + \tfrac{5}{8}\sin x,$$

$$\cos^{2n} x = \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1} C_{2n}^k \cos[2(n-k)x] + \frac{1}{2^{2n}}C_{2n}^n,$$
$$\cos^{2n+1} x = \frac{1}{2^{2n}}\sum_{k=0}^{n} C_{2n+1}^k \cos[(2n-2k+1)x],$$
$$\sin^{2n} x = \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1} (-1)^{n-k} C_{2n}^k \cos[2(n-k)x] + \frac{1}{2^{2n}}C_{2n}^n,$$
$$\sin^{2n+1} x = \frac{1}{2^{2n}}\sum_{k=0}^{n} (-1)^{n-k} C_{2n+1}^k \sin[(2n-2k+1)x].$$

Here $C_m^k = \dfrac{m!}{k!\,(m-k)!}$ are binomial coefficients $(0! = 1)$.

► **Trigonometric functions of multiple arguments**

$\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x,$ $\sin 2x = 2\sin x\cos x,$

$\cos 3x = -3\cos x + 4\cos^3 x,$ $\sin 3x = 3\sin x - 4\sin^3 x,$

$\cos 4x = 1 - 8\cos^2 x + 8\cos^4 x,$ $\sin 4x = 4\cos x\,(\sin x - 2\sin^3 x),$

$\cos 5x = 5\cos x - 20\cos^3 x + 16\cos^5 x,$ $\sin 5x = 5\sin x - 20\sin^3 x + 16\sin^5 x,$

$$\cos(2nx) = 1 + \sum_{k=1}^{n}(-1)^k \frac{n^2(n^2-1)\dots[n^2-(k-1)^2]}{(2k)!}4^k\sin^{2k} x,$$

$$\cos[(2n+1)x] = \cos x\left\{1+\sum_{k=1}^{n}(-1)^k\frac{[(2n+1)^2-1][(2n+1)^2-3^2]\dots[(2n+1)^2-(2k-1)^2]}{(2k)!}\sin^{2k} x\right\},$$

$$\sin(2nx) = 2n\cos x\left[\sin x + \sum_{k=1}^{n}(-4)^k\frac{(n^2-1)(n^2-2^2)\dots(n^2-k^2)}{(2k-1)!}\sin^{2k-1} x\right],$$

$$\sin[(2n+1)x] = (2n+1)\left\{\sin x+\sum_{k=1}^{n}(-1)^k\frac{[(2n+1)^2-1][(2n+1)^2-3^2]\dots[(2n+1)^2-(2k-1)^2]}{(2k+1)!}\sin^{2k+1} x\right\},$$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}, \qquad \tan 3x = \frac{3\tan x - \tan^3 x}{1-3\tan^2 x}, \qquad \tan 4x = \frac{4\tan x - 4\tan^3 x}{1-6\tan^2 x+\tan^4 x}.$$

TrigonoTrigonometrmetric functions of half argument

$$\sin^2\frac{x}{2} = \frac{1-\cos x}{2}, \qquad \cos^2\frac{x}{2} = \frac{1+\cos x}{2},$$

$$\tan\frac{x}{2} = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}, \qquad \cot\frac{x}{2} = \frac{\sin x}{1-\cos x} = \frac{1+\cos x}{\sin x},$$

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}, \qquad \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}, \qquad \tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}.$$

► **Euler and de Moivre formulas. Relationship with hyperbolic functions**

$$e^{y+ix} = e^y(\cos x + i\sin x), \quad (\cos x + i\sin x)^n = \cos(nx) + i\sin(nx), \quad i^2 = -1,$$

$$\sin(ix) = i\sinh x, \quad \cos(ix) = \cosh x, \quad \tan(ix) = i\tanh x, \quad \cot(ix) = -i\coth x.$$

► **Differentiation formulas**

$$\frac{d\sin x}{dx} = \cos x, \quad \frac{d\cos x}{dx} = -\sin x, \quad \frac{d\tan x}{dx} = \frac{1}{\cos^2 x}, \quad \frac{d\cot x}{dx} = -\frac{1}{\sin^2 x}.$$

► **Expansion into power series**

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad (|x| < \infty),$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad (|x| < \infty),$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots \qquad (|x| < \pi/2),$$

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \cdots \qquad (|x| < \pi).$$

1.2. Hyperbolic Functions

► **Definitions**

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$$

► **Simplest relations**

$$\cosh^2 x - \sinh^2 x = 1, \qquad \tanh x \coth x = 1,$$
$$\sinh(-x) = -\sinh x, \qquad \cosh(-x) = \cosh x,$$
$$\tanh x = \frac{\sinh x}{\cosh x}, \qquad \coth x = \frac{\cosh x}{\sinh x},$$
$$\tanh(-x) = -\tanh x, \qquad \coth(-x) = -\coth x,$$
$$1 - \tanh^2 x = \frac{1}{\cosh^2 x}, \qquad \coth^2 x - 1 = \frac{1}{\sinh^2 x}.$$

► **Relations between hyperbolic functions of single argument ($x \geq 0$)**

$$\sinh x = \sqrt{\cosh^2 x - 1} = \frac{\tanh x}{\sqrt{1 - \tanh^2 x}} = \frac{1}{\sqrt{\coth^2 x - 1}},$$
$$\cosh x = \sqrt{\sinh^2 x + 1} = \frac{1}{\sqrt{1 - \tanh^2 x}} = \frac{\coth x}{\sqrt{\coth^2 x - 1}},$$
$$\tanh x = \frac{\sinh x}{\sqrt{\sinh^2 x + 1}} = \frac{\sqrt{\cosh^2 x - 1}}{\cosh x} = \frac{1}{\coth x},$$
$$\coth x = \frac{\sqrt{\sinh^2 x + 1}}{\sinh x} = \frac{\cosh x}{\sqrt{\cosh^2 x - 1}} = \frac{1}{\tanh x}.$$

► **Addition formulas**

$$\sinh(x \pm y) = \sinh x \cosh y \pm \sinh y \cosh x, \qquad \cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y,$$
$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}, \qquad \coth(x \pm y) = \frac{\coth x \coth y \pm 1}{\coth y \pm \coth x}.$$

► **Addition and subtraction of hyperbolic functions**

$$\sinh x \pm \sinh y = 2 \sinh\left(\frac{x \pm y}{2}\right) \cosh\left(\frac{x \mp y}{2}\right),$$
$$\cosh x + \cosh y = 2 \cosh\left(\frac{x + y}{2}\right) \cosh\left(\frac{x - y}{2}\right),$$
$$\cosh x - \cosh y = 2 \sinh\left(\frac{x + y}{2}\right) \sinh\left(\frac{x - y}{2}\right),$$
$$\sinh^2 x - \sinh^2 y = \cosh^2 x - \cosh^2 y = \sinh(x + y) \sinh(x - y),$$
$$\sinh^2 x + \cosh^2 y = \cosh(x + y) \cosh(x - y),$$
$$\tanh x \pm \tanh y = \frac{\sinh(x \pm y)}{\cosh x \cosh y}, \qquad \coth x \pm \coth y = \pm\frac{\sinh(x \pm y)}{\sinh x \sinh y}.$$

► **Products of hyperbolic functions**

$$\sinh x \sinh y = \tfrac{1}{2}[\cosh(x + y) - \cosh(x - y)],$$
$$\cosh x \cosh y = \tfrac{1}{2}[\cosh(x + y) + \cosh(x - y)],$$
$$\sinh x \cosh y = \tfrac{1}{2}[\sinh(x + y) + \sinh(x - y)].$$

► **Powers of hyperbolic functions**

$\cosh^2 x = \frac{1}{2}\cosh 2x + \frac{1}{2}$, $\sinh^2 x = \frac{1}{2}\cosh 2x - \frac{1}{2}$,

$\cosh^3 x = \frac{1}{4}\cosh 3x + \frac{3}{4}\cosh x$, $\sinh^3 x = \frac{1}{4}\sinh 3x - \frac{3}{4}\sinh x$,

$\cosh^4 x = \frac{1}{8}\cosh 4x + \frac{1}{2}\cosh 2x + \frac{3}{8}$, $\sinh^4 x = \frac{1}{8}\cosh 4x - \frac{1}{2}\cosh 2x + \frac{3}{8}$,

$\cosh^5 x = \frac{1}{16}\cosh 5x + \frac{5}{16}\cosh 3x + \frac{5}{8}\cosh x$, $\sinh^5 x = \frac{1}{16}\sinh 5x - \frac{5}{16}\sinh 3x + \frac{5}{8}\sinh x$,

$$\cosh^{2n} x = \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1} C_{2n}^k \cosh[2(n-k)x] + \frac{1}{2^{2n}} C_{2n}^n,$$

$$\cosh^{2n+1} x = \frac{1}{2^{2n}}\sum_{k=0}^{n} C_{2n+1}^k \cosh[(2n-2k+1)x],$$

$$\sinh^{2n} x = \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1} (-1)^k C_{2n}^k \cosh[2(n-k)x] + \frac{(-1)^n}{2^{2n}} C_{2n}^n,$$

$$\sinh^{2n+1} x = \frac{1}{2^{2n}}\sum_{k=0}^{n} (-1)^k C_{2n+1}^k \sinh[(2n-2k+1)x].$$

Here C_m^k are binomial coefficients.

► **Hyperbolic functions of multiple arguments**

$\cosh 2x = 2\cosh^2 x - 1$, $\sinh 2x = 2\sinh x \cosh x$,

$\cosh 3x = -3\cosh x + 4\cosh^3 x$, $\sinh 3x = 3\sinh x + 4\sinh^3 x$,

$\cosh 4x = 1 - 8\cosh^2 x + 8\cosh^4 x$, $\sinh 4x = 4\cosh x(\sinh x + 2\sinh^3 x)$,

$\cosh 5x = 5\cosh x - 20\cosh^3 x + 16\cosh^5 x$, $\sinh 5x = 5\sinh x + 20\sinh^3 x + 16\sinh^5 x$.

$$\cosh(nx) = 2^{n-1}\cosh^n x + \frac{n}{2}\sum_{k=0}^{[n/2]} \frac{(-1)^{k+1}}{k+1} C_{n-k-2}^{k-2} 2^{n-2k-2} (\cosh x)^{n-2k-2},$$

$$\sinh(nx) = \sinh x \sum_{k=0}^{[(n-1)/2]} 2^{n-k-1} C_{n-k-1}^k (\cosh x)^{n-2k-1}.$$

Here C_m^k are binomial coefficients and $[A]$ stands for the integer part of a number A.

► **Relationship with trigonometric functions**

$\sinh(ix) = i\sin x$, $\cosh(ix) = \cos x$, $\tanh(ix) = i\tan x$, $\coth(ix) = -i\cot x$, $i^2 = -1$.

► **Differentiation formulas**

$$\frac{d\sinh x}{dx} = \cosh x, \quad \frac{d\cosh x}{dx} = \sinh x, \quad \frac{d\tanh x}{dx} = \frac{1}{\cosh^2 x}, \quad \frac{d\coth x}{dx} = -\frac{1}{\sinh^2 x}.$$

► **Expansion into power series**

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \qquad (|x| < \infty),$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \qquad (|x| < \infty),$$

$$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots \qquad (|x| < \pi/2),$$

$$\coth x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \cdots \qquad (|x| < \pi).$$

1.3. Inverse Trigonometric Functions

► Definitions and some properties

$$\sin(\arcsin x) = x, \qquad \cos(\arccos x) = x,$$
$$\tan(\arctan x) = x, \qquad \cot(\operatorname{arccot} x) = x.$$

Principal values of inverse trigonometric functions are defined by the inequalities

$$-\tfrac{\pi}{2} \le \arcsin x \le \tfrac{\pi}{2}, \quad 0 \le \arccos x \le \pi \qquad (-1 \le x \le 1),$$
$$-\tfrac{\pi}{2} < \arctan x < \tfrac{\pi}{2}, \quad 0 < \operatorname{arccot} x < \pi \qquad (-\infty < x < \infty).$$

► Simplest formulas

$$\arcsin(-x) = -\arcsin x, \qquad \arccos(-x) = \pi - \arccos x,$$
$$\arctan(-x) = -\arctan x, \qquad \operatorname{arccot}(-x) = \pi - \operatorname{arccot} x,$$

$$\arcsin(\sin x) = \begin{cases} x - 2n\pi & \text{if } 2n\pi - \frac{\pi}{2} \le x \le 2n\pi + \frac{\pi}{2}, \\ -x + 2(n+1)\pi & \text{if } (2n+1)\pi - \frac{\pi}{2} \le x \le 2(n+1)\pi + \frac{\pi}{2}, \end{cases}$$

$$\arccos(\cos x) = \begin{cases} x - 2n\pi & \text{if } 2n\pi \le x \le (2n+1)\pi, \\ -x + 2(n+1)\pi & \text{if } (2n+1)\pi \le x \le 2(n+1)\pi, \end{cases}$$

$$\arctan(\tan x) = x - n\pi \quad \text{if} \quad n\pi - \tfrac{\pi}{2} < x < n\pi + \tfrac{\pi}{2},$$
$$\operatorname{arccot}(\cot x) = x - n\pi \quad \text{if} \quad n\pi < x < (n+1)\pi.$$

► Relations between inverse trigonometric functions

$$\arcsin x + \arccos x = \tfrac{\pi}{2}, \qquad \arctan x + \operatorname{arccot} x = \tfrac{\pi}{2};$$

$$\arcsin x = \begin{cases} \arccos\sqrt{1-x^2} & \text{if } 0 \le x \le 1, \\ -\arccos\sqrt{1-x^2} & \text{if } -1 \le x \le 0, \\ \arctan\dfrac{x}{\sqrt{1-x^2}} & \text{if } -1 < x < 1, \\ \operatorname{arccot}\dfrac{\sqrt{1-x^2}}{x} - \pi & \text{if } -1 \le x < 0; \end{cases} \qquad \arccos x = \begin{cases} \arcsin\sqrt{1-x^2} & \text{if } 0 \le x \le 1, \\ \pi - \arcsin\sqrt{1-x^2} & \text{if } -1 \le x \le 0, \\ \arctan\dfrac{\sqrt{1-x^2}}{x} & \text{if } 0 < x \le 1, \\ \operatorname{arccot}\dfrac{x}{\sqrt{1-x^2}} & \text{if } -1 < x < 1; \end{cases}$$

$$\arctan x = \begin{cases} \arcsin\dfrac{x}{\sqrt{1+x^2}} & \text{for any } x, \\ \arccos\dfrac{1}{\sqrt{1+x^2}} & \text{if } x \ge 0, \\ -\arccos\dfrac{1}{\sqrt{1+x^2}} & \text{if } x \le 0, \\ \operatorname{arccot}\dfrac{1}{x} & \text{if } x > 0; \end{cases} \qquad \operatorname{arccot} x = \begin{cases} \arcsin\dfrac{1}{\sqrt{1+x^2}} & \text{if } x > 0, \\ \pi - \arcsin\dfrac{1}{\sqrt{1+x^2}} & \text{if } x < 0, \\ \arctan\dfrac{1}{x} & \text{if } x > 0, \\ \pi + \arctan\dfrac{1}{x} & \text{if } x < 0. \end{cases}$$

► Addition and subtraction of inverse trigonometric functions

$$\arcsin x + \arcsin y = \arcsin\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) \quad \text{for} \quad x^2 + y^2 \le 1,$$
$$\arccos x \pm \arccos y = \pm\arccos\left[xy \mp \sqrt{(1-x^2)(1-y^2)}\right] \quad \text{for} \quad x \pm y \ge 0,$$
$$\arctan x + \arctan y = \arctan\frac{x+y}{1-xy} \quad \text{for} \quad xy < 1,$$
$$\arctan x - \arctan y = \arctan\frac{x-y}{1+xy} \quad \text{for} \quad xy > -1.$$

► **Differentiation formulas**

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}},\quad \frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}},\quad \frac{d}{dx}\arctan x = \frac{1}{1+x^2},\quad \frac{d}{dx}\operatorname{arccot} x = -\frac{1}{1+x^2}.$$

► **Expansion into power series**

$$\arcsin x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots \qquad (|x| < 1),$$
$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad (|x| \le 1),$$
$$\arctan x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots \qquad (|x| > 1).$$

The expansions for $\arccos x$ and $\operatorname{arccot} x$ can be obtained with the aid of the formulas $\arccos x = \frac{\pi}{2} - \arcsin x$ and $\operatorname{arccot} x = \frac{\pi}{2} - \arctan x$.

1.4. Inverse Hyperbolic Functions

► **Relationship with logarithmic function**

$$\operatorname{Arsinh} x = \ln\bigl(x + \sqrt{x^2+1}\,\bigr), \qquad \operatorname{Artanh} x = \frac{1}{2}\ln\frac{1+x}{1-x},$$
$$\operatorname{Arcosh} x = \ln\bigl(x + \sqrt{x^2-1}\,\bigr), \qquad \operatorname{Arcoth} x = \frac{1}{2}\ln\frac{1+x}{x-1};$$
$$\operatorname{Arsinh}(-x) = -\operatorname{Arsinh} x, \qquad \operatorname{Artanh}(-x) = -\operatorname{Artanh} x,$$
$$\operatorname{Arcosh}(-x) = \operatorname{Arcosh} x, \qquad \operatorname{Arcoth}(-x) = -\operatorname{Arcoth} x.$$

► **Relations between inverse hyperbolic functions**

$$\operatorname{Arsinh} x = \operatorname{Arcosh}\sqrt{x^2+1} = \operatorname{Artanh}\frac{x}{\sqrt{x^2+1}},$$
$$\operatorname{Arcosh} x = \operatorname{Arsinh}\sqrt{x^2-1} = \operatorname{Artanh}\frac{\sqrt{x^2-1}}{x},$$
$$\operatorname{Artanh} x = \operatorname{Arsinh}\frac{x}{\sqrt{1-x^2}} = \operatorname{Arcosh}\frac{1}{\sqrt{1-x^2}} = \operatorname{Arcoth}\frac{1}{x}.$$

► **Addition and subtraction of inverse hyperbolic functions**

$$\operatorname{Arsinh} x \pm \operatorname{Arsinh} y = \operatorname{Arsinh}\bigl(x\sqrt{1+y^2} \pm y\sqrt{1+x^2}\,\bigr),$$
$$\operatorname{Arcosh} x \pm \operatorname{Arcosh} y = \operatorname{Arcosh}\Bigl[xy \pm \sqrt{(x^2-1)(y^2-1)}\,\Bigr],$$
$$\operatorname{Arsinh} x \pm \operatorname{Arcosh} y = \operatorname{Arsinh}\Bigl[xy \pm \sqrt{(x^2+1)(y^2-1)}\,\Bigr],$$
$$\operatorname{Artanh} x \pm \operatorname{Artanh} y = \operatorname{Artanh}\frac{x \pm y}{1 \pm xy}, \qquad \operatorname{Artanh} x \pm \operatorname{Arcoth} y = \operatorname{Artanh}\frac{xy \pm 1}{y \pm x}.$$

► **Differentiation formulas**

$$\frac{d}{dx}\operatorname{Arsinh} x = \frac{1}{\sqrt{x^2+1}}, \qquad \frac{d}{dx}\operatorname{Arcosh} x = \frac{1}{\sqrt{x^2-1}},$$
$$\frac{d}{dx}\operatorname{Artanh} x = \frac{1}{1-x^2} \quad (x^2 < 1), \qquad \frac{d}{dx}\operatorname{Arcoth} x = \frac{1}{1-x^2} \quad (x^2 > 1).$$

► **Expansion into power series**

$$\operatorname{Arsinh} x = x - \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots \quad (|x| < 1),$$

$$\operatorname{Artanh} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots \quad (|x| < 1).$$

⦿ References for Supplement 1: H. B. Dwight (1961), M. Arbramowitz and I. A. Stegun (1964), G. A. Korn and T. M. Korn (1968), W. H. Beyer (1991).

Supplement 2

Tables of Indefinite Integrals

2.1. Integrals Containing Rational Functions

► **Integrals containing $a + bx$.***

1. $\displaystyle\int \frac{dx}{a+bx} = \frac{1}{b}\ln|a+bx|.$

2. $\displaystyle\int (a+bx)^n\,dx = \frac{(a+bx)^{n+1}}{b(n+1)}, \quad n \neq -1.$

3. $\displaystyle\int \frac{x\,dx}{a+bx} = \frac{1}{b^2}\bigl(a+bx-a\ln|a+bx|\bigr).$

4. $\displaystyle\int \frac{x^2\,dx}{a+bx} = \frac{1}{b^3}\left[\frac{1}{2}(a+bx)^2 - 2a(a+bx) + a^2\ln|a+bx|\right].$

5. $\displaystyle\int \frac{dx}{x(a+bx)} = -\frac{1}{a}\ln\left|\frac{a+bx}{x}\right|.$

6. $\displaystyle\int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2}\ln\left|\frac{a+bx}{x}\right|.$

7. $\displaystyle\int \frac{x\,dx}{(a+bx)^2} = \frac{1}{b^2}\left(\ln|a+bx| + \frac{a}{a+bx}\right).$

8. $\displaystyle\int \frac{x^2\,dx}{(a+bx)^2} = \frac{1}{b^3}\left(a+bx-2a\ln|a+bx| - \frac{a^2}{a+bx}\right).$

9. $\displaystyle\int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} - \frac{1}{a^2}\ln\left|\frac{a+bx}{x}\right|.$

10. $\displaystyle\int \frac{x\,dx}{(a+bx)^3} = \frac{1}{b^2}\left[-\frac{1}{a+bx} + \frac{a}{2(a+bx)^2}\right].$

► **Integrals containing $a + x$ and $b + x$.**

11. $\displaystyle\int \frac{a+x}{b+x}\,dx = x + (a-b)\ln|b+x|.$

12. $\displaystyle\int \frac{dx}{(a+x)(b+x)} = \frac{1}{a-b}\ln\left|\frac{b+x}{a+x}\right|,$ $a \neq b$. For $a = b$, see integral 2 with $n = -2$.

13. $\displaystyle\int \frac{x\,dx}{(a+x)(b+x)} = \frac{1}{a-b}\bigl(a\ln|a+x| - b\ln|b+x|\bigr).$

14. $\displaystyle\int \frac{dx}{(a+x)(b+x)^2} = \frac{1}{(b-a)(b+x)} + \frac{1}{(a-b)^2}\ln\left|\frac{a+x}{b+x}\right|.$

* Throughout this section, the integration constant C is omitted for brevity.

15. $\displaystyle\int \frac{x\,dx}{(a+x)(b+x)^2} = \frac{b}{(a-b)(b+x)} - \frac{a}{(a-b)^2} \ln\left|\frac{a+x}{b+x}\right|.$

16. $\displaystyle\int \frac{x^2\,dx}{(a+x)(b+x)^2} = \frac{b^2}{(b-a)(b+x)} + \frac{a^2}{(a-b)^2} \ln|a+x| + \frac{b^2-2ab}{(b-a)^2} \ln|b+x|.$

17. $\displaystyle\int \frac{dx}{(a+x)^2(b+x)^2} = -\frac{1}{(a-b)^2}\left(\frac{1}{a+x} + \frac{1}{b+x}\right) + \frac{2}{(a-b)^3} \ln\left|\frac{a+x}{b+x}\right|.$

18. $\displaystyle\int \frac{x\,dx}{(a+x)^2(b+x)^2} = \frac{1}{(a-b)^2}\left(\frac{a}{a+x} + \frac{b}{b+x}\right) + \frac{a+b}{(a-b)^3} \ln\left|\frac{a+x}{b+x}\right|.$

19. $\displaystyle\int \frac{x^2\,dx}{(a+x)^2(b+x)^2} = -\frac{1}{(a-b)^2}\left(\frac{a^2}{a+x} + \frac{b^2}{b+x}\right) + \frac{2ab}{(a-b)^3} \ln\left|\frac{a+x}{b+x}\right|.$

► **Integrals containing $a^2 + x^2$.**

20. $\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a} \arctan \frac{x}{a}.$

21. $\displaystyle\int \frac{dx}{(a^2+x^2)^2} = \frac{x}{2a^2(a^2+x^2)} + \frac{1}{2a^3} \arctan \frac{x}{a}.$

22. $\displaystyle\int \frac{dx}{(a^2+x^2)^3} = \frac{x}{4a^2(a^2+x^2)^2} + \frac{3x}{8a^4(a^2+x^2)} + \frac{3}{8a^5} \arctan \frac{x}{a}.$

23. $\displaystyle\int \frac{dx}{(a^2+x^2)^{n+1}} = \frac{x}{2na^2(a^2+x^2)^n} + \frac{2n-1}{2na^2} \int \frac{dx}{(a^2+x^2)^n}; \quad n = 1, 2, \ldots$

24. $\displaystyle\int \frac{x\,dx}{a^2+x^2} = \frac{1}{2} \ln(a^2+x^2).$

25. $\displaystyle\int \frac{x\,dx}{(a^2+x^2)^2} = -\frac{1}{2(a^2+x^2)}.$

26. $\displaystyle\int \frac{x\,dx}{(a^2+x^2)^3} = -\frac{1}{4(a^2+x^2)^2}.$

27. $\displaystyle\int \frac{x\,dx}{(a^2+x^2)^{n+1}} = -\frac{1}{2n(a^2+x^2)^n}; \quad n = 1, 2, \ldots$

28. $\displaystyle\int \frac{x^2\,dx}{a^2+x^2} = x - a \arctan \frac{x}{a}.$

29. $\displaystyle\int \frac{x^2\,dx}{(a^2+x^2)^2} = -\frac{x}{2(a^2+x^2)} + \frac{1}{2a} \arctan \frac{x}{a}.$

30. $\displaystyle\int \frac{x^2\,dx}{(a^2+x^2)^3} = -\frac{x}{4(a^2+x^2)^2} + \frac{x}{8a^2(a^2+x^2)} + \frac{1}{8a^3} \arctan \frac{x}{a}.$

31. $\displaystyle\int \frac{x^2\,dx}{(a^2+x^2)^{n+1}} = -\frac{x}{2n(a^2+x^2)^n} + \frac{1}{2n} \int \frac{dx}{(a^2+x^2)^n}; \quad n = 1, 2, \ldots$

32. $\displaystyle\int \frac{x^3\,dx}{a^2+x^2} = \frac{x^2}{2} - \frac{a^2}{2} \ln(a^2+x^2).$

33. $\displaystyle\int \frac{x^3\,dx}{(a^2+x^2)^2} = \frac{a^2}{2(a^2+x^2)} + \frac{1}{2} \ln(a^2+x^2).$

34. $\displaystyle\int \frac{x^3\,dx}{(a^2+x^2)^{n+1}} = -\frac{1}{2(n-1)(a^2+x^2)^{n-1}} + \frac{a^2}{2n(a^2+x^2)^n}; \quad n = 2, 3, \ldots$

35. $\displaystyle\int \frac{dx}{x(a^2+x^2)} = \frac{1}{2a^2} \ln \frac{x^2}{a^2+x^2}.$

36. $\displaystyle\int \frac{dx}{x(a^2+x^2)^2} = \frac{1}{2a^2(a^2+x^2)} + \frac{1}{2a^4} \ln \frac{x^2}{a^2+x^2}.$

37. $\displaystyle\int \frac{dx}{x(a^2+x^2)^3} = \frac{1}{4a^2(a^2+x^2)^2} + \frac{1}{2a^4(a^2+x^2)} + \frac{1}{2a^6}\ln\frac{x^2}{a^2+x^2}.$

38. $\displaystyle\int \frac{dx}{x^2(a^2+x^2)} = -\frac{1}{a^2x} - \frac{1}{a^3}\arctan\frac{x}{a}.$

39. $\displaystyle\int \frac{dx}{x^2(a^2+x^2)^2} = -\frac{1}{a^4x} - \frac{x}{2a^4(a^2+x^2)} - \frac{3}{2a^5}\arctan\frac{x}{a}.$

40. $\displaystyle\int \frac{dx}{x^3(a^2+x^2)^2} = -\frac{1}{2a^4x^2} - \frac{1}{2a^4(a^2+x^2)} - \frac{1}{a^6}\ln\frac{x^2}{a^2+x^2}.$

41. $\displaystyle\int \frac{dx}{x^2(a^2+x^2)^3} = -\frac{1}{a^6x} - \frac{x}{4a^4(a^2+x^2)^2} - \frac{7x}{8a^6(a^2+x^2)} - \frac{15}{8a^7}\arctan\frac{x}{a}.$

42. $\displaystyle\int \frac{dx}{x^3(a^2+x^2)^3} = -\frac{1}{2a^6x^2} - \frac{1}{a^6(a^2+x^2)} - \frac{1}{4a^4(a^2+x^2)^2} - \frac{3}{2a^8}\ln\frac{x^2}{a^2+x^2}.$

► **Integrals containing $a^2 - x^2$.**

43. $\displaystyle\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|.$

44. $\displaystyle\int \frac{dx}{(a^2-x^2)^2} = \frac{x}{2a^2(a^2-x^2)} + \frac{1}{4a^3}\ln\left|\frac{a+x}{a-x}\right|.$

45. $\displaystyle\int \frac{dx}{(a^2-x^2)^3} = \frac{x}{4a^2(a^2-x^2)^2} + \frac{3x}{8a^4(a^2-x^2)} + \frac{3}{16a^5}\ln\left|\frac{a+x}{a-x}\right|.$

46. $\displaystyle\int \frac{dx}{(a^2-x^2)^{n+1}} = \frac{x}{2na^2(a^2-x^2)^n} + \frac{2n-1}{2na^2}\int\frac{dx}{(a^2-x^2)^n}; \quad n = 1,\ 2,\ \ldots$

47. $\displaystyle\int \frac{x\,dx}{a^2-x^2} = -\frac{1}{2}\ln|a^2-x^2|.$

48. $\displaystyle\int \frac{x\,dx}{(a^2-x^2)^2} = \frac{1}{2(a^2-x^2)}.$

49. $\displaystyle\int \frac{x\,dx}{(a^2-x^2)^3} = \frac{1}{4(a^2-x^2)^2}.$

50. $\displaystyle\int \frac{x\,dx}{(a^2-x^2)^{n+1}} = \frac{1}{2n(a^2-x^2)^n}; \quad n = 1,\ 2,\ \ldots$

51. $\displaystyle\int \frac{x^2\,dx}{a^2-x^2} = -x + \frac{a}{2}\ln\left|\frac{a+x}{a-x}\right|.$

52. $\displaystyle\int \frac{x^2\,dx}{(a^2-x^2)^2} = \frac{x}{2(a^2-x^2)} - \frac{1}{4a}\ln\left|\frac{a+x}{a-x}\right|.$

53. $\displaystyle\int \frac{x^2\,dx}{(a^2-x^2)^3} = \frac{x}{4(a^2-x^2)^2} - \frac{x}{8a^2(a^2-x^2)} - \frac{1}{16a^3}\ln\left|\frac{a+x}{a-x}\right|.$

54. $\displaystyle\int \frac{x^2\,dx}{(a^2-x^2)^{n+1}} = \frac{x}{2n(a^2-x^2)^n} - \frac{1}{2n}\int\frac{dx}{(a^2-x^2)^n}; \quad n = 1,\ 2,\ \ldots$

55. $\displaystyle\int \frac{x^3\,dx}{a^2-x^2} = -\frac{x^2}{2} - \frac{a^2}{2}\ln|a^2-x^2|.$

56. $\displaystyle\int \frac{x^3\,dx}{(a^2-x^2)^2} = \frac{a^2}{2(a^2-x^2)} + \frac{1}{2}\ln|a^2-x^2|.$

57. $\displaystyle\int \frac{x^3\,dx}{(a^2-x^2)^{n+1}} = -\frac{1}{2(n-1)(a^2-x^2)^{n-1}} + \frac{a^2}{2n(a^2-x^2)^n}; \quad n = 2,\ 3, \ldots$

58. $\displaystyle\int \frac{dx}{x(a^2-x^2)} = \frac{1}{2a^2}\ln\left|\frac{x^2}{a^2-x^2}\right|.$

59. $\displaystyle\int \frac{dx}{x(a^2-x^2)^2} = \frac{1}{2a^2(a^2-x^2)} + \frac{1}{2a^4}\ln\left|\frac{x^2}{a^2-x^2}\right|.$

60. $\displaystyle\int \frac{dx}{x(a^2-x^2)^3} = \frac{1}{4a^2(a^2-x^2)^2} + \frac{1}{2a^4(a^2-x^2)} + \frac{1}{2a^6}\ln\left|\frac{x^2}{a^2-x^2}\right|.$

► **Integrals containing $a^3 + x^3$.**

61. $\displaystyle\int \frac{dx}{a^3+x^3} = \frac{1}{6a^2}\ln\frac{(a+x)^2}{a^2-ax+x^2} + \frac{1}{a^2\sqrt{3}}\arctan\frac{2x-a}{a\sqrt{3}}.$

62. $\displaystyle\int \frac{dx}{(a^3+x^3)^2} = \frac{x}{3a^3(a^3+x^3)} + \frac{2}{3a^3}\int\frac{dx}{a^3+x^3}.$

63. $\displaystyle\int \frac{x\,dx}{a^3+x^3} = \frac{1}{6a}\ln\frac{a^2-ax+x^2}{(a+x)^2} + \frac{1}{a\sqrt{3}}\arctan\frac{2x-a}{a\sqrt{3}}.$

64. $\displaystyle\int \frac{x\,dx}{(a^3+x^3)^2} = \frac{x^2}{3a^3(a^3+x^3)} + \frac{1}{3a^3}\int\frac{x\,dx}{a^3+x^3}.$

65. $\displaystyle\int \frac{x^2\,dx}{a^3+x^3} = \frac{1}{3}\ln|a^3+x^3|.$

66. $\displaystyle\int \frac{dx}{x(a^3+x^3)} = \frac{1}{3a^3}\ln\left|\frac{x^3}{a^3+x^3}\right|.$

67. $\displaystyle\int \frac{dx}{x(a^3+x^3)^2} = \frac{1}{3a^3(a^3+x^3)} + \frac{1}{3a^6}\ln\left|\frac{x^3}{a^3+x^3}\right|.$

68. $\displaystyle\int \frac{dx}{x^2(a^3+x^3)} = -\frac{1}{a^3x} - \frac{1}{a^3}\int\frac{x\,dx}{a^3+x^3}.$

69. $\displaystyle\int \frac{dx}{x^2(a^3+x^3)^2} = -\frac{1}{a^6x} - \frac{x^2}{3a^6(a^3+x^3)} - \frac{4}{3a^6}\int\frac{x\,dx}{a^3+x^3}.$

► **Integrals containing $a^3 - x^3$.**

70. $\displaystyle\int \frac{dx}{a^3-x^3} = \frac{1}{6a^2}\ln\frac{a^2+ax+x^2}{(a-x)^2} + \frac{1}{a^2\sqrt{3}}\arctan\frac{2x+a}{a\sqrt{3}}.$

71. $\displaystyle\int \frac{dx}{(a^3-x^3)^2} = \frac{x}{3a^3(a^3-x^3)} + \frac{2}{3a^3}\int\frac{dx}{a^3-x^3}.$

72. $\displaystyle\int \frac{x\,dx}{a^3-x^3} = \frac{1}{6a}\ln\frac{a^2+ax+x^2}{(a-x)^2} - \frac{1}{a\sqrt{3}}\arctan\frac{2x+a}{a\sqrt{3}}.$

73. $\displaystyle\int \frac{x\,dx}{(a^3-x^3)^2} = \frac{x^2}{3a^3(a^3-x^3)} + \frac{1}{3a^3}\int\frac{x\,dx}{a^3-x^3}.$

74. $\displaystyle\int \frac{x^2\,dx}{a^3-x^3} = -\frac{1}{3}\ln|a^3-x^3|.$

75. $\displaystyle\int \frac{dx}{x(a^3-x^3)} = \frac{1}{3a^3}\ln\left|\frac{x^3}{a^3-x^3}\right|.$

76. $\displaystyle\int \frac{dx}{x(a^3-x^3)^2} = \frac{1}{3a^3(a^3-x^3)} + \frac{1}{3a^6}\ln\left|\frac{x^3}{a^3-x^3}\right|.$

77. $\displaystyle\int \frac{dx}{x^2(a^3-x^3)} = -\frac{1}{a^3x} + \frac{1}{a^3}\int\frac{x\,dx}{a^3-x^3}.$

78. $\displaystyle\int \frac{dx}{x^2(a^3-x^3)^2} = -\frac{1}{a^6x} - \frac{x^2}{3a^6(a^3-x^3)} + \frac{4}{3a^6}\int\frac{x\,dx}{a^3-x^3}.$

► **Integrals containing $a^4 \pm x^4$.**

79. $\displaystyle\int \frac{dx}{a^4+x^4} = \frac{1}{4a^3\sqrt{2}} \ln \frac{a^2+ax\sqrt{2}+x^2}{a^2-ax\sqrt{2}+x^2} + \frac{1}{2a^3\sqrt{2}} \arctan \frac{ax\sqrt{2}}{a^2-x^2}.$

80. $\displaystyle\int \frac{x\,dx}{a^4+x^4} = \frac{1}{2a^2} \arctan \frac{x^2}{a^2}.$

81. $\displaystyle\int \frac{x^2\,dx}{a^4+x^4} = -\frac{1}{4a\sqrt{2}} \ln \frac{a^2+ax\sqrt{2}+x^2}{a^2-ax\sqrt{2}+x^2} + \frac{1}{2a\sqrt{2}} \arctan \frac{ax\sqrt{2}}{a^2-x^2}.$

82. $\displaystyle\int \frac{dx}{a^4-x^4} = \frac{1}{4a^3} \ln\left|\frac{a+x}{a-x}\right| + \frac{1}{2a^3} \arctan \frac{x}{a}.$

83. $\displaystyle\int \frac{x\,dx}{a^4-x^4} = \frac{1}{4a^2} \ln\left|\frac{a^2+x^2}{a^2-x^2}\right|.$

84. $\displaystyle\int \frac{x^2\,dx}{a^4-x^4} = \frac{1}{4a} \ln\left|\frac{a+x}{a-x}\right| - \frac{1}{2a} \arctan \frac{x}{a}.$

85. $\displaystyle\int \frac{dx}{x(a+bx^m)} = \frac{1}{am} \ln\left|\frac{x^m}{a+bx^m}\right|.$

2.2. Integrals Containing Irrational Functions

► **Integrals containing $x^{1/2}$.**

1. $\displaystyle\int \frac{x^{1/2}\,dx}{a^2+b^2x} = \frac{2}{b^2}x^{1/2} - \frac{2a}{b^3} \arctan \frac{bx^{1/2}}{a}.$

2. $\displaystyle\int \frac{x^{3/2}\,dx}{a^2+b^2x} = \frac{2x^{3/2}}{3b^2} - \frac{2a^2x^{1/2}}{b^4} + \frac{2a^3}{b^5} \arctan \frac{bx^{1/2}}{a}.$

3. $\displaystyle\int \frac{x^{1/2}\,dx}{(a^2+b^2x)^2} = -\frac{x^{1/2}}{b^2(a^2+b^2x)} + \frac{1}{ab^3} \arctan \frac{bx^{1/2}}{a}.$

4. $\displaystyle\int \frac{x^{3/2}\,dx}{(a^2+b^2x)^2} = \frac{2x^{3/2}}{b^2(a^2+b^2x)} + \frac{3a^2x^{1/2}}{b^4(a^2+b^2x)} - \frac{3a}{b^5} \arctan \frac{bx^{1/2}}{a}.$

5. $\displaystyle\int \frac{dx}{(a^2+b^2x)x^{1/2}} = \frac{2}{ab} \arctan \frac{bx^{1/2}}{a}.$

6. $\displaystyle\int \frac{dx}{(a^2+b^2x)x^{3/2}} = -\frac{2}{a^2x^{1/2}} - \frac{2b}{a^3} \arctan \frac{bx^{1/2}}{a}.$

7. $\displaystyle\int \frac{dx}{(a^2+b^2x)^2x^{1/2}} = \frac{x^{1/2}}{a^2(a^2+b^2x)} + \frac{1}{a^3b} \arctan \frac{bx^{1/2}}{a}.$

8. $\displaystyle\int \frac{x^{1/2}\,dx}{a^2-b^2x} = -\frac{2}{b^2}x^{1/2} + \frac{2a}{b^3} \ln\left|\frac{a+bx^{1/2}}{a-bx^{1/2}}\right|.$

9. $\displaystyle\int \frac{x^{3/2}\,dx}{a^2-b^2x} = -\frac{2x^{3/2}}{3b^2} - \frac{2a^2x^{1/2}}{b^4} + \frac{a^3}{b^5} \ln\left|\frac{a+bx^{1/2}}{a-bx^{1/2}}\right|.$

10. $\displaystyle\int \frac{x^{1/2}\,dx}{(a^2-b^2x)^2} = \frac{x^{1/2}}{b^2(a^2-b^2x)} - \frac{1}{2ab^3} \ln\left|\frac{a+bx^{1/2}}{a-bx^{1/2}}\right|.$

11. $\displaystyle\int \frac{x^{3/2}\,dx}{(a^2-b^2x)^2} = \frac{3a^2x^{1/2}-2b^2x^{3/2}}{b^4(a^2-b^2x)} - \frac{3a}{2b^5} \ln\left|\frac{a+bx^{1/2}}{a-bx^{1/2}}\right|.$

12. $\displaystyle\int \frac{dx}{(a^2-b^2x)x^{1/2}} = \frac{1}{ab} \ln\left|\frac{a+bx^{1/2}}{a-bx^{1/2}}\right|.$

13. $$\int \frac{dx}{(a^2-b^2x)x^{3/2}} = -\frac{2}{a^2x^{1/2}} + \frac{b}{a^3}\ln\left|\frac{a+bx^{1/2}}{a-bx^{1/2}}\right|.$$

14. $$\int \frac{dx}{(a^2-b^2x)^2x^{1/2}} = \frac{x^{1/2}}{a^2(a^2-b^2x)} + \frac{1}{2a^3b}\ln\left|\frac{a+bx^{1/2}}{a-bx^{1/2}}\right|.$$

► **Integrals containing $(a+bx)^{p/2}$.**

15. $$\int (a+bx)^{p/2}\,dx = \frac{2}{b(p+2)}(a+bx)^{(p+2)/2}.$$

16. $$\int x(a+bx)^{p/2}\,dx = \frac{2}{b^2}\left[\frac{(a+bx)^{(p+4)/2}}{p+4} - \frac{a(a+bx)^{(p+2)/2}}{p+2}\right].$$

17. $$\int x^2(a+bx)^{p/2}\,dx = \frac{2}{b^3}\left[\frac{(a+bx)^{(p+6)/2}}{p+6} - \frac{2a(a+bx)^{(p+4)/2}}{p+4} + \frac{a^2(a+bx)^{(p+2)/2}}{p+2}\right].$$

► **Integrals containing $(x^2+a^2)^{1/2}$.**

18. $$\int (x^2+a^2)^{1/2}\,dx = \frac{1}{2}x(a^2+x^2)^{1/2} + \frac{a^2}{2}\ln\left[x+(x^2+a^2)^{1/2}\right].$$

19. $$\int x(x^2+a^2)^{1/2}\,dx = \frac{1}{3}(a^2+x^2)^{3/2}.$$

20. $$\int (x^2+a^2)^{3/2}\,dx = \frac{1}{4}x(a^2+x^2)^{3/2} + \frac{3}{8}a^2x(a^2+x^2)^{1/2} + \frac{3}{8}a^4\ln\left|x+(x^2+a^2)^{1/2}\right|.$$

21. $$\int \frac{1}{x}(x^2+a^2)^{1/2}\,dx = (a^2+x^2)^{1/2} - a\ln\left|\frac{a+(x^2+a^2)^{1/2}}{x}\right|.$$

22. $$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left[x+(x^2+a^2)^{1/2}\right].$$

23. $$\int \frac{x\,dx}{\sqrt{x^2+a^2}} = (x^2+a^2)^{1/2}.$$

24. $$\int (x^2+a^2)^{-3/2}\,dx = a^{-2}x(x^2+a^2)^{-1/2}.$$

► **Integrals containing $(x^2-a^2)^{1/2}$.**

25. $$\int (x^2-a^2)^{1/2}\,dx = \frac{1}{2}x(x^2-a^2)^{1/2} - \frac{a^2}{2}\ln\left|x+(x^2-a^2)^{1/2}\right|.$$

26. $$\int x(x^2-a^2)^{1/2}\,dx = \frac{1}{3}(x^2-a^2)^{3/2}.$$

27. $$\int (x^2-a^2)^{3/2}\,dx = \frac{1}{4}x(x^2-a^2)^{3/2} - \frac{3}{8}a^2x(x^2-a^2)^{1/2} + \frac{3}{8}a^4\ln\left|x+(x^2-a^2)^{1/2}\right|.$$

28. $$\int \frac{1}{x}(x^2-a^2)^{1/2}\,dx = (x^2-a^2)^{1/2} - a\arccos\left|\frac{a}{x}\right|.$$

29. $$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+(x^2-a^2)^{1/2}\right|.$$

30. $$\int \frac{x\,dx}{\sqrt{x^2-a^2}} = (x^2-a^2)^{1/2}.$$

31. $$\int (x^2-a^2)^{-3/2}\,dx = -a^{-2}x(x^2-a^2)^{-1/2}.$$

► **Integrals containing $(a^2 - x^2)^{1/2}$.**

32. $\int (a^2 - x^2)^{1/2}\,dx = \frac{1}{2}x(a^2 - x^2)^{1/2} + \frac{a^2}{2}\arcsin\frac{x}{a}.$

33. $\int x(a^2 - x^2)^{1/2}\,dx = -\frac{1}{3}(a^2 - x^2)^{3/2}.$

34. $\int (a^2 - x^2)^{3/2}\,dx = \frac{1}{4}x(a^2 - x^2)^{3/2} + \frac{3}{8}a^2x(a^2 - x^2)^{1/2} + \frac{3}{8}a^4\arcsin\frac{x}{a}.$

35. $\int \frac{1}{x}(a^2 - x^2)^{1/2}\,dx = (a^2 - x^2)^{1/2} - a\ln\left|\frac{a + (a^2 - x^2)^{1/2}}{x}\right|.$

36. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a}.$

37. $\int \frac{x\,dx}{\sqrt{a^2 - x^2}} = -(a^2 - x^2)^{1/2}.$

38. $\int (a^2 - x^2)^{-3/2}\,dx = a^{-2}x(a^2 - x^2)^{-1/2}.$

► **Reduction formulas.** The parameters a, b, p, m, and n below can assume arbitrary values, except for those at which denominators vanish in successive applications of a formula. Notation: $w = ax^n + b$.

39. $\int x^m(ax^n + b)^p\,dx = \frac{1}{m + np + 1}\left(x^{m+1}w^p + npb\int x^m w^{p-1}\,dx\right).$

40. $\int x^m(ax^n + b)^p\,dx = \frac{1}{bn(p + 1)}\left[-x^{m+1}w^{p+1} + (m + n + np + 1)\int x^m w^{p+1}\,dx\right].$

41. $\int x^m(ax^n + b)^p\,dx = \frac{1}{b(m + 1)}\left[x^{m+1}w^{p+1} - a(m + n + np + 1)\int x^{m+n} w^p\,dx\right].$

42. $\int x^m(ax^n + b)^p\,dx = \frac{1}{a(m + np + 1)}\left[x^{m-n+1}w^{p+1} - b(m - n + 1)\int x^{m-n} w^p\,dx\right].$

2.3. Integrals Containing Exponential Functions

1. $\int e^{ax}\,dx = \frac{1}{a}e^{ax}.$

2. $\int a^x\,dx = \frac{a^x}{\ln a}.$

3. $\int xe^{ax}\,dx = e^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right).$

4. $\int x^2e^{ax}\,dx = e^{ax}\left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right).$

5. $\int x^ne^{ax}\,dx = e^{ax}\left[\frac{1}{a}x^n - \frac{n}{a^2}x^{n-1} + \frac{n(n-1)}{a^3}x^{n-2} - \cdots + (-1)^{n-1}\frac{n!}{a^n}x + (-1)^n\frac{n!}{a^{n+1}}\right],\quad n = 1,\ 2,\ \ldots$

6. $\int P_n(x)e^{ax}\,dx = e^{ax}\sum_{k=0}^{n}\frac{(-1)^k}{a^{k+1}}\frac{d^k}{dx^k}P_n(x),$ where $P_n(x)$ is an arbitrary nth-degree polynomial.

7. $\int \frac{dx}{a + be^{px}} = \frac{x}{a} - \frac{1}{ap}\ln|a + be^{px}|.$

8. $$\int \frac{dx}{ae^{px}+be^{-px}} = \begin{cases} \dfrac{1}{p\sqrt{ab}} \arctan\left(e^{px}\sqrt{\dfrac{a}{b}}\right) & \text{if } ab>0, \\ \dfrac{1}{2p\sqrt{-ab}} \ln\left(\dfrac{b+e^{px}\sqrt{-ab}}{b-e^{px}\sqrt{-ab}}\right) & \text{if } ab<0. \end{cases}$$

9. $$\int \frac{dx}{\sqrt{a+be^{px}}} = \begin{cases} \dfrac{1}{p\sqrt{a}} \ln \dfrac{\sqrt{a+be^{px}}-\sqrt{a}}{\sqrt{a+be^{px}}+\sqrt{a}} & \text{if } a>0, \\ \dfrac{2}{p\sqrt{-a}} \arctan \dfrac{\sqrt{a+be^{px}}}{\sqrt{-a}} & \text{if } a<0. \end{cases}$$

2.4. Integrals Containing Hyperbolic Functions

► **Integrals containing cosh x.**

1. $$\int \cosh(a+bx)\,dx = \frac{1}{b}\sinh(a+bx).$$
2. $$\int x\cosh x\,dx = x\sinh x - \cosh x.$$
3. $$\int x^2\cosh x\,dx = (x^2+2)\sinh x - 2x\cosh x.$$
4. $$\int x^{2n}\cosh x\,dx = (2n)!\sum_{k=1}^{n}\left[\frac{x^{2k}}{(2k)!}\sinh x - \frac{x^{2k-1}}{(2k-1)!}\cosh x\right].$$
5. $$\int x^{2n+1}\cosh x\,dx = (2n+1)!\sum_{k=0}^{n}\left[\frac{x^{2k+1}}{(2k+1)!}\sinh x - \frac{x^{2k}}{(2k)!}\cosh x\right].$$
6. $$\int x^p\cosh x\,dx = x^p\sinh x - px^{p-1}\cosh x + p(p-1)\int x^{p-2}\cosh x\,dx.$$
7. $$\int \cosh^2 x\,dx = \tfrac{1}{2}x + \tfrac{1}{4}\sinh 2x.$$
8. $$\int \cosh^3 x\,dx = \sinh x + \tfrac{1}{3}\sinh^3 x.$$
9. $$\int \cosh^{2n} x\,dx = C_{2n}^{n}\frac{x}{2^{2n}} + \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1} C_{2n}^{k}\frac{\sinh[2(n-k)x]}{2(n-k)}, \quad n=1,\,2,\,\dots$$
10. $$\int \cosh^{2n+1} x\,dx = \frac{1}{2^{2n}}\sum_{k=0}^{n} C_{2n+1}^{k}\frac{\sinh[(2n-2k+1)x]}{2n-2k+1} = \sum_{k=0}^{n} C_{n}^{k}\frac{\sinh^{2k+1} x}{2k+1}, \quad n=1,\,2,\dots$$
11. $$\int \cosh^p x\,dx = \frac{1}{p}\sinh x\cosh^{p-1} x + \frac{p-1}{p}\int \cosh^{p-2} x\,dx.$$
12. $$\int \cosh ax\cosh bx\,dx = \frac{1}{a^2-b^2}\bigl[a\cosh bx\sinh ax - b\cosh ax\sinh bx\bigr].$$
13. $$\int \frac{dx}{\cosh ax} = \frac{2}{a}\arctan\bigl(e^{ax}\bigr).$$
14. $$\int \frac{dx}{\cosh^{2n} x} = \frac{\sinh x}{2n-1}\left[\frac{1}{\cosh^{2n-1} x} + \sum_{k=1}^{n-1}\frac{2^k(n-1)(n-2)\dots(n-k)}{(2n-3)(2n-5)\dots(2n-2k-1)}\,\frac{1}{\cosh^{2n-2k-1} x}\right],$$
$n=1,\,2,\,\dots$
15. $$\int \frac{dx}{\cosh^{2n+1} x} = \frac{\sinh x}{2n}\left[\frac{1}{\cosh^{2n} x} + \sum_{k=1}^{n-1}\frac{(2n-1)(2n-3)\dots(2n-2k+1)}{2^k(n-1)(n-2)\dots(n-k)}\,\frac{1}{\cosh^{2n-2k} x}\right]$$
$$+\frac{(2n-1)!!}{(2n)!!}\arctan\sinh x, \qquad n=1,\,2,\,\dots$$

16. $$\int \frac{dx}{a+b\cosh x} = \begin{cases} -\dfrac{\operatorname{sign} x}{\sqrt{b^2-a^2}} \arcsin \dfrac{b+a\cosh x}{a+b\cosh x} & \text{if } a^2 < b^2, \\ \dfrac{1}{\sqrt{a^2-b^2}} \ln \dfrac{a+b+\sqrt{a^2-b^2}\,\tanh(x/2)}{a+b-\sqrt{a^2-b^2}\,\tanh(x/2)} & \text{if } a^2 > b^2. \end{cases}$$

► **Integrals containing $\sinh x$.**

17. $$\int \sinh(a+bx)\,dx = \frac{1}{b}\cosh(a+bx).$$

18. $$\int x\sinh x\,dx = x\cosh x - \sinh x.$$

19. $$\int x^2\sinh x\,dx = (x^2+2)\cosh x - 2x\sinh x.$$

20. $$\int x^{2n}\sinh x\,dx = (2n)!\left[\sum_{k=0}^{n}\frac{x^{2k}}{(2k)!}\cosh x - \sum_{k=1}^{n}\frac{x^{2k-1}}{(2k-1)!}\sinh x\right].$$

21. $$\int x^{2n+1}\sinh x\,dx = (2n+1)!\sum_{k=0}^{n}\left[\frac{x^{2k+1}}{(2k+1)!}\cosh x - \frac{x^{2k}}{(2k)!}\sinh x\right].$$

22. $$\int x^p\sinh x\,dx = x^p\cosh x - px^{p-1}\sinh x + p(p-1)\int x^{p-2}\sinh x\,dx.$$

23. $$\int \sinh^2 x\,dx = -\tfrac{1}{2}x + \tfrac{1}{4}\sinh 2x.$$

24. $$\int \sinh^3 x\,dx = -\cosh x + \tfrac{1}{3}\cosh^3 x.$$

25. $$\int \sinh^{2n} x\,dx = (-1)^n C_{2n}^{n}\frac{x}{2^{2n}} + \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1}(-1)^k C_{2n}^{k}\frac{\sinh[2(n-k)x]}{2(n-k)}, \quad n = 1, 2, \ldots$$

26. $$\int \sinh^{2n+1} x\,dx = \frac{1}{2^{2n}}\sum_{k=0}^{n}(-1)^k C_{2n+1}^{k}\frac{\cosh[(2n-2k+1)x]}{2n-2k+1} = \sum_{k=0}^{n}(-1)^{n+k}C_n^k\frac{\cosh^{2k+1} x}{2k+1},$$
$n = 1, 2, \ldots$

27. $$\int \sinh^p x\,dx = \frac{1}{p}\sinh^{p-1} x\cosh x - \frac{p-1}{p}\int \sinh^{p-2} x\,dx.$$

28. $$\int \sinh ax\sinh bx\,dx = \frac{1}{a^2-b^2}\left[a\cosh ax\sinh bx - b\cosh bx\sinh ax\right].$$

29. $$\int \frac{dx}{\sinh ax} = \frac{1}{a}\ln\left|\tanh\frac{ax}{2}\right|.$$

30. $$\int \frac{dx}{\sinh^{2n} x} = \frac{\cosh x}{2n-1}\left[-\frac{1}{\sinh^{2n-1} x} + \sum_{k=1}^{n-1}(-1)^{k-1}\frac{2^k(n-1)(n-2)\ldots(n-k)}{(2n-3)(2n-5)\ldots(2n-2k-1)}\frac{1}{\sinh^{2n-2k-1} x}\right], \quad n = 1, 2, \ldots$$

31. $$\int \frac{dx}{\sinh^{2n+1} x} = \frac{\cosh x}{2n}\left[-\frac{1}{\sinh^{2n} x} + \sum_{k=1}^{n-1}(-1)^{k-1}\frac{(2n-1)(2n-3)\ldots(2n-2k+1)}{2^k(n-1)(n-2)\ldots(n-k)}\frac{1}{\sinh^{2n-2k} x}\right] + (-1)^n\frac{(2n-1)!!}{(2n)!!}\ln\tanh\frac{x}{2}, \quad n = 1, 2, \ldots$$

32. $$\int \frac{dx}{a+b\sinh x} = \frac{1}{\sqrt{a^2+b^2}}\ln\frac{a\tanh(x/2) - b + \sqrt{a^2+b^2}}{a\tanh(x/2) - b - \sqrt{a^2+b^2}}.$$

33. $$\int \frac{Ax+B\sinh x}{a+b\sinh x}\,dx = \frac{B}{b}x + \frac{Ab-Ba}{b\sqrt{a^2+b^2}}\ln\frac{a\tanh(x/2) - b + \sqrt{a^2+b^2}}{a\tanh(x/2) - b - \sqrt{a^2+b^2}}.$$

► **Integrals containing tanh x or coth x.**

34. $\displaystyle\int \tanh x\,dx = \ln\cosh x.$

35. $\displaystyle\int \tanh^2 x\,dx = x - \tanh x.$

36. $\displaystyle\int \tanh^3 x\,dx = -\tfrac{1}{2}\tanh^2 x + \ln\cosh x.$

37. $\displaystyle\int \tanh^{2n} x\,dx = x - \sum_{k=1}^{n} \frac{\tanh^{2n-2k+1} x}{2n-2k+1}, \qquad n = 1, 2, \dots$

38. $\displaystyle\int \tanh^{2n+1} x\,dx = \ln\cosh x - \sum_{k=1}^{n} \frac{(-1)^k C_n^k}{2k\cosh^{2k} x} = \ln\cosh x - \sum_{k=1}^{n} \frac{\tanh^{2n-2k+2} x}{2n-2k+2}, \quad n = 1, 2, \dots$

39. $\displaystyle\int \tanh^p x\,dx = -\frac{1}{p-1}\tanh^{p-1} x + \int \tanh^{p-2} x\,dx.$

40. $\displaystyle\int \coth x\,dx = \ln|\sinh x|.$

41. $\displaystyle\int \coth^2 x\,dx = x - \coth x.$

42. $\displaystyle\int \coth^3 x\,dx = -\tfrac{1}{2}\coth^2 x + \ln|\sinh x|.$

43. $\displaystyle\int \coth^{2n} x\,dx = x - \sum_{k=1}^{n} \frac{\coth^{2n-2k+1} x}{2n-2k+1}, \quad n = 1, 2, \dots$

44. $\displaystyle\int \coth^{2n+1} x\,dx = \ln|\sinh x| - \sum_{k=1}^{n} \frac{C_n^k}{2k\sinh^{2k} x} = \ln|\sinh x| - \sum_{k=1}^{n} \frac{\coth^{2n-2k+2} x}{2n-2k+2}, \quad n = 1, 2, \dots$

45. $\displaystyle\int \coth^p x\,dx = -\frac{1}{p-1}\coth^{p-1} x + \int \coth^{p-2} x\,dx.$

2.5. Integrals Containing Logarithmic Functions

1. $\displaystyle\int \ln ax\,dx = x\ln ax - x.$

2. $\displaystyle\int x\ln x\,dx = \tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2.$

3. $\displaystyle\int x^p\ln ax\,dx = \begin{cases} \dfrac{1}{p+1}x^{p+1}\ln ax - \dfrac{1}{(p+1)^2}x^{p+1} & \text{if } p \neq -1, \\ \tfrac{1}{2}\ln^2 ax & \text{if } p = -1. \end{cases}$

4. $\displaystyle\int (\ln x)^2\,dx = x(\ln x)^2 - 2x\ln x + 2x.$

5. $\displaystyle\int x(\ln x)^2\,dx = \tfrac{1}{2}x^2(\ln x)^2 - \tfrac{1}{2}x^2\ln x + \tfrac{1}{4}x^2.$

6. $\displaystyle\int x^p(\ln x)^2\,dx = \begin{cases} \dfrac{x^{p+1}}{p+1}(\ln x)^2 - \dfrac{2x^{p+1}}{(p+1)^2}\ln x + \dfrac{2x^{p+1}}{(p+1)^3} & \text{if } p \neq -1, \\ \tfrac{1}{3}\ln^3 x & \text{if } p = -1. \end{cases}$

7. $\displaystyle\int (\ln x)^n\,dx = \frac{x}{n+1}\sum_{k=0}^{n}(-1)^k(n+1)n\dots(n-k+1)(\ln x)^{n-k}, \qquad n = 1, 2, \dots$

8. $\int (\ln x)^q\,dx = x(\ln x)^q - q\int (\ln x)^{q-1}\,dx,\ q \neq -1.$

9. $\int x^n(\ln x)^m\,dx = \dfrac{x^{n+1}}{m+1}\sum_{k=0}^{m}\dfrac{(-1)^k}{(n+1)^{k+1}}(m+1)m\dots(m-k+1)(\ln x)^{m-k},\ n, m = 1,\ 2,\ \dots$

10. $\int x^p(\ln x)^q\,dx = \dfrac{1}{p+1}x^{p+1}(\ln x)^q - \dfrac{q}{p+1}\int x^p(\ln x)^{q-1}\,dx, \quad p,\ q \neq -1.$

11. $\int \ln(a+bx)\,dx = \dfrac{1}{b}(ax+b)\ln(ax+b) - x.$

12. $\int x\ln(a+bx)\,dx = \dfrac{1}{2}\left(x^2 - \dfrac{a^2}{b^2}\right)\ln(a+bx) - \dfrac{1}{2}\left(\dfrac{x^2}{2} - \dfrac{a}{b}x\right).$

13. $\int x^2\ln(a+bx)\,dx = \dfrac{1}{3}\left(x^3 - \dfrac{a^3}{b^3}\right)\ln(a+bx) - \dfrac{1}{3}\left(\dfrac{x^3}{3} - \dfrac{ax^2}{2b} + \dfrac{a^2x}{b^2}\right).$

14. $\int \dfrac{\ln x\,dx}{(a+bx)^2} = -\dfrac{\ln x}{b(a+bx)} + \dfrac{1}{ab}\ln\dfrac{x}{a+bx}.$

15. $\int \dfrac{\ln x\,dx}{(a+bx)^3} = -\dfrac{\ln x}{2b(a+bx)^2} + \dfrac{1}{2ab(a+bx)} + \dfrac{1}{2a^2b}\ln\dfrac{x}{a+bx}.$

16. $$\int \frac{\ln x\,dx}{\sqrt{a+bx}} = \begin{cases} \dfrac{2}{b}\left[(\ln x - 2)\sqrt{a+bx} + \sqrt{a}\,\ln\dfrac{\sqrt{a+bx}+\sqrt{a}}{\sqrt{a+bx}-\sqrt{a}}\right] & \text{if } a > 0, \\ \dfrac{2}{b}\left[(\ln x - 2)\sqrt{a+bx} + 2\sqrt{-a}\arctan\dfrac{\sqrt{a+bx}}{\sqrt{-a}}\right] & \text{if } a < 0. \end{cases}$$

17. $\int \ln(x^2+a^2)\,dx = x\ln(x^2+a^2) - 2x + 2a\arctan(x/a).$

18. $\int x\ln(x^2+a^2)\,dx = \frac{1}{2}\left[(x^2+a^2)\ln(x^2+a^2) - x^2\right].$

19. $\int x^2\ln(x^2+a^2)\,dx = \frac{1}{3}\left[x^3\ln(x^2+a^2) - \frac{2}{3}x^3 + 2a^2x - 2a^3\arctan(x/a)\right].$

2.6. Integrals Containing Trigonometric Functions

► **Integrals containing cos x.** Notation: $n = 1,\ 2,\ \dots$

1. $\int \cos(a+bx)\,dx = \dfrac{1}{b}\sin(a+bx).$

2. $\int x\cos x\,dx = \cos x + x\sin x.$

3. $\int x^2\cos x\,dx = 2x\cos x + (x^2-2)\sin x.$

4. $\int x^{2n}\cos x\,dx = (2n)!\left[\sum_{k=0}^{n}(-1)^k\dfrac{x^{2n-2k}}{(2n-2k)!}\sin x + \sum_{k=0}^{n-1}(-1)^k\dfrac{x^{2n-2k-1}}{(2n-2k-1)!}\cos x\right].$

5. $\int x^{2n+1}\cos x\,dx = (2n+1)!\sum_{k=0}^{n}\left[(-1)^k\dfrac{x^{2n-2k+1}}{(2n-2k+1)!}\sin x + \dfrac{x^{2n-2k}}{(2n-2k)!}\cos x\right].$

6. $\int x^p\cos x\,dx = x^p\sin x + px^{p-1}\cos x - p(p-1)\int x^{p-2}\cos x\,dx.$

7. $\int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x.$

8. $\displaystyle\int \cos^3 x\,dx = \sin x - \tfrac{1}{3}\sin^3 x.$

9. $\displaystyle\int \cos^{2n} x\,dx = \frac{1}{2^{2n}} C_{2n}^{n} x + \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} C_{2n}^{k} \frac{\sin[(2n-2k)x]}{2n-2k}.$

10. $\displaystyle\int \cos^{2n+1} x\,dx = \frac{1}{2^{2n}} \sum_{k=0}^{n} C_{2n+1}^{k} \frac{\sin[(2n-2k+1)x]}{2n-2k+1}.$

11. $\displaystyle\int \frac{dx}{\cos x} = \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|.$

12. $\displaystyle\int \frac{dx}{\cos^2 x} = \tan x.$

13. $\displaystyle\int \frac{dx}{\cos^3 x} = \frac{\sin x}{2\cos^2 x} + \frac{1}{2}\ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|.$

14. $\displaystyle\int \frac{dx}{\cos^n x} = \frac{\sin x}{(n-1)\cos^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2} x},\ n > 1.$

15. $\displaystyle\int \frac{x\,dx}{\cos^{2n} x} = \sum_{k=0}^{n-1} \frac{(2n-2)(2n-4)\dots(2n-2k+2)}{(2n-1)(2n-3)\dots(2n-2k+3)} \frac{(2n-2k)x\sin x - \cos x}{(2n-2k+1)(2n-2k)\cos^{2n-2k+1} x}$
$\displaystyle + \frac{2^{n-1}(n-1)!}{(2n-1)!!}\big(x\tan x + \ln|\cos x|\big).$

16. $\displaystyle\int \cos ax\,\cos bx\,dx = \frac{\sin\big[(b-a)x\big]}{2(b-a)} + \frac{\sin\big[(b+a)x\big]}{2(b+a)},\ a \neq \pm b.$

17. $$\int \frac{dx}{a+b\cos x} = \begin{cases} \dfrac{2}{\sqrt{a^2-b^2}} \arctan \dfrac{(a-b)\tan(x/2)}{\sqrt{a^2-b^2}} & \text{if } a^2 > b^2, \\[2ex] \dfrac{1}{\sqrt{b^2-a^2}} \ln\left|\dfrac{\sqrt{b^2-a^2}+(b-a)\tan(x/2)}{\sqrt{b^2-a^2}-(b-a)\tan(x/2)}\right| & \text{if } b^2 > a^2. \end{cases}$$

18. $\displaystyle\int \frac{dx}{(a+b\cos x)^2} = \frac{b\sin x}{(b^2-a^2)(a+b\cos x)} - \frac{a}{b^2-a^2}\int \frac{dx}{a+b\cos x}.$

19. $\displaystyle\int \frac{dx}{a^2+b^2\cos^2 x} = \frac{1}{a\sqrt{a^2+b^2}} \arctan \frac{a\tan x}{\sqrt{a^2+b^2}}.$

20. $$\int \frac{dx}{a^2-b^2\cos^2 x} = \begin{cases} \dfrac{1}{a\sqrt{a^2-b^2}} \arctan \dfrac{a\tan x}{\sqrt{a^2-b^2}} & \text{if } a^2 > b^2, \\[2ex] \dfrac{1}{2a\sqrt{b^2-a^2}} \ln\left|\dfrac{\sqrt{b^2-a^2}-a\tan x}{\sqrt{b^2-a^2}+a\tan x}\right| & \text{if } b^2 > a^2. \end{cases}$$

21. $\displaystyle\int e^{ax}\cos bx\,dx = e^{ax}\left[\frac{b}{a^2+b^2}\sin bx + \frac{a}{a^2+b^2}\cos bx\right].$

22. $\displaystyle\int e^{ax}\cos^2 x\,dx = \frac{e^{ax}}{a^2+4}\left(a\cos^2 x + 2\sin x\cos x + \frac{2}{a}\right).$

23. $\displaystyle\int e^{ax}\cos^n x\,dx = \frac{e^{ax}\cos^{n-1} x}{a^2+n^2}(a\cos x + n\sin x) + \frac{n(n-1)}{a^2+n^2}\int e^{ax}\cos^{n-2} x\,dx.$

► **Integrals containing sin x.** Notation: $n = 1,\ 2,\ \dots$

24. $\displaystyle\int \sin(a+bx)\,dx = -\frac{1}{b}\cos(a+bx).$

25. $\displaystyle\int x\sin x\,dx = \sin x - x\cos x.$

26. $\int x^2 \sin x\,dx = 2x\sin x - (x^2-2)\cos x.$

27. $\int x^3 \sin x\,dx = (3x^2-6)\sin x - (x^3-6x)\cos x.$

28. $\int x^{2n} \sin x\,dx = (2n)!\left[\sum_{k=0}^{n}(-1)^{k+1}\frac{x^{2n-2k}}{(2n-2k)!}\cos x + \sum_{k=0}^{n-1}(-1)^{k}\frac{x^{2n-2k-1}}{(2n-2k-1)!}\sin x\right].$

29. $\int x^{2n+1} \sin x\,dx = (2n+1)!\sum_{k=0}^{n}\left[(-1)^{k+1}\frac{x^{2n-2k+1}}{(2n-2k+1)!}\cos x + (-1)^{k}\frac{x^{2n-2k}}{(2n-2k)!}\sin x\right].$

30. $\int x^p \sin x\,dx = -x^p\cos x + px^{p-1}\sin x - p(p-1)\int x^{p-2}\sin x\,dx.$

31. $\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x.$

32. $\int x\sin^2 x\,dx = \frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x.$

33. $\int \sin^3 x\,dx = -\cos x + \frac{1}{3}\cos^3 x.$

34. $\int \sin^{2n} x\,dx = \frac{1}{2^{2n}}C_{2n}^{n}x + \frac{(-1)^n}{2^{2n-1}}\sum_{k=0}^{n-1}(-1)^k C_{2n}^{k}\frac{\sin[(2n-2k)x]}{2n-2k},$

where $C_m^k = \dfrac{m!}{k!\,(m-k)!}$ are binomial coefficients $(0! = 1)$.

35. $\int \sin^{2n+1} x\,dx = \frac{1}{2^{2n}}\sum_{k=0}^{n}(-1)^{n+k+1}C_{2n+1}^{k}\frac{\cos[(2n-2k+1)x]}{2n-2k+1}.$

36. $\int \frac{dx}{\sin x} = \ln\left|\tan\frac{x}{2}\right|.$

37. $\int \frac{dx}{\sin^2 x} = -\cot x.$

38. $\int \frac{dx}{\sin^3 x} = -\frac{\cos x}{2\sin^2 x} + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right|.$

39. $\int \frac{dx}{\sin^n x} = -\frac{\cos x}{(n-1)\sin^{n-1} x} + \frac{n-2}{n-1}\int\frac{dx}{\sin^{n-2} x},\ n > 1.$

40. $\int \frac{x\,dx}{\sin^{2n} x} = -\sum_{k=0}^{n-1}\frac{(2n-2)(2n-4)\ldots(2n-2k+2)}{(2n-1)(2n-3)\ldots(2n-2k+3)}\,\frac{\sin x + (2n-2k)x\cos x}{(2n-2k+1)(2n-2k)\sin^{2n-2k+1} x}$

$+\ \frac{2^{n-1}(n-1)!}{(2n-1)!!}\big(\ln|\sin x| - x\cot x\big).$

41. $\int \sin ax\ \sin bx\,dx = \frac{\sin[(b-a)x]}{2(b-a)} - \frac{\sin[(b+a)x]}{2(b+a)},\ a \neq \pm b.$

42. $$\int \frac{dx}{a+b\sin x} = \begin{cases} \dfrac{2}{\sqrt{a^2-b^2}}\arctan\dfrac{b+a\tan x/2}{\sqrt{a^2-b^2}} & \text{if } a^2 > b^2, \\ \dfrac{1}{\sqrt{b^2-a^2}}\ln\left|\dfrac{b-\sqrt{b^2-a^2}+a\tan x/2}{b+\sqrt{b^2-a^2}+a\tan x/2}\right| & \text{if } b^2 > a^2. \end{cases}$$

43. $\int \frac{dx}{(a+b\sin x)^2} = \frac{b\cos x}{(a^2-b^2)(a+b\sin x)} + \frac{a}{a^2-b^2}\int\frac{dx}{a+b\sin x}.$

44. $\int \frac{dx}{a^2+b^2\sin^2 x} = \frac{1}{a\sqrt{a^2+b^2}}\arctan\frac{\sqrt{a^2+b^2}\ \tan x}{a}.$

45. $$\int \frac{dx}{a^2 - b^2 \sin^2 x} = \begin{cases} \dfrac{1}{a\sqrt{a^2-b^2}} \arctan \dfrac{\sqrt{a^2-b^2}\,\tan x}{a} & \text{if } a^2 > b^2, \\ \dfrac{1}{2a\sqrt{b^2-a^2}} \ln\left|\dfrac{\sqrt{b^2-a^2}\,\tan x + a}{\sqrt{b^2-a^2}\,\tan x - a}\right| & \text{if } b^2 > a^2. \end{cases}$$

46. $$\int \frac{\sin x\,dx}{\sqrt{1+k^2\sin^2 x}} = -\frac{1}{k}\arcsin\frac{k\cos x}{\sqrt{1+k^2}}.$$

47. $$\int \frac{\sin x\,dx}{\sqrt{1-k^2\sin^2 x}} = -\frac{1}{k}\ln\left|k\cos x + \sqrt{1-k^2\sin^2 x}\,\right|.$$

48. $$\int \sin x\sqrt{1+k^2\sin^2 x}\,dx = -\frac{\cos x}{2}\sqrt{1+k^2\sin^2 x} - \frac{1+k^2}{2k}\arcsin\frac{k\cos x}{\sqrt{1+k^2}}.$$

49. $$\int \sin x\sqrt{1-k^2\sin^2 x}\,dx = -\frac{\cos x}{2}\sqrt{1-k^2\sin^2 x} - \frac{1-k^2}{2k}\ln\left|k\cos x + \sqrt{1-k^2\sin^2 x}\,\right|.$$

50. $$\int e^{ax}\sin bx\,dx = e^{ax}\left[\frac{a}{a^2+b^2}\sin bx - \frac{b}{a^2+b^2}\cos bx\right].$$

51. $$\int e^{ax}\sin^2 x\,dx = \frac{e^{ax}}{a^2+4}\left(a\sin^2 x - 2\sin x\cos x + \frac{2}{a}\right).$$

52. $$\int e^{ax}\sin^n x\,dx = \frac{e^{ax}\sin^{n-1}x}{a^2+n^2}(a\sin x - n\cos x) + \frac{n(n-1)}{a^2+n^2}\int e^{ax}\sin^{n-2}x\,dx.$$

► **Integrals containing $\sin x$ and $\cos x$.**

53. $$\int \sin ax\cos bx\,dx = -\frac{\cos[(a+b)x]}{2(a+b)} - \frac{\cos\left[(a-b)x\right]}{2(a-b)}, \quad a \neq \pm b.$$

54. $$\int \frac{dx}{b^2\cos^2 ax + c^2\sin^2 ax} = \frac{1}{abc}\arctan\left(\frac{c}{b}\tan ax\right).$$

55. $$\int \frac{dx}{b^2\cos^2 ax - c^2\sin^2 ax} = \frac{1}{2abc}\ln\left|\frac{c\tan ax + b}{c\tan ax - b}\right|.$$

56. $$\int \frac{dx}{\cos^{2n}x\sin^{2m}x} = \sum_{k=0}^{n+m-1} C_{n+m-1}^{k}\frac{\tan^{2k-2m+1}x}{2k-2m+1}, \quad n, m = 1,\,2,\,\ldots$$

57. $$\int \frac{dx}{\cos^{2n+1}x\sin^{2m+1}x} = C_{n+m}^{m}\ln|\tan x| + \sum_{k=0}^{n+m} C_{n+m}^{k}\frac{\tan^{2k-2m}x}{2k-2m}, \quad n, m = 1,\,2,\,\ldots$$

► **Reduction formulas.** The parameters p and q below can assume any values, except for those at which the denominators on the right-hand side vanish.

58. $$\int \sin^p x\cos^q x\,dx = -\frac{\sin^{p-1}x\cos^{q+1}x}{p+q} + \frac{p-1}{p+q}\int \sin^{p-2}x\cos^q x\,dx.$$

59. $$\int \sin^p x\cos^q x\,dx = \frac{\sin^{p+1}x\cos^{q-1}x}{p+q} + \frac{q-1}{p+q}\int \sin^p x\cos^{q-2}x\,dx.$$

60. $$\int \sin^p x\cos^q x\,dx = \frac{\sin^{p-1}x\cos^{q-1}x}{p+q}\left(\sin^2 x - \frac{q-1}{p+q-2}\right) + \frac{(p-1)(q-1)}{(p+q)(p+q-2)}\int \sin^{p-2}x\cos^{q-2}x\,dx.$$

61. $$\int \sin^p x\cos^q x\,dx = \frac{\sin^{p+1}x\cos^{q+1}x}{p+1} + \frac{p+q+2}{p+1}\int \sin^{p+2}x\cos^q x\,dx.$$

62. $$\int \sin^p x\cos^q x\,dx = -\frac{\sin^{p+1}x\cos^{q+1}x}{q+1} + \frac{p+q+2}{q+1}\int \sin^p x\cos^{q+2}x\,dx.$$

63. $\displaystyle\int \sin^p x \cos^q x\,dx = -\frac{\sin^{p-1} x \cos^{q+1} x}{q+1} + \frac{p-1}{q+1}\int \sin^{p-2} x \cos^{q+2} x\,dx.$

64. $\displaystyle\int \sin^p x \cos^q x\,dx = \frac{\sin^{p+1} x \cos^{q-1} x}{p+1} + \frac{q-1}{p+1}\int \sin^{p+2} x \cos^{q-2} x\,dx.$

► **Integrals containing tan x and cot x.**

65. $\displaystyle\int \tan x\,dx = -\ln|\cos x|.$

66. $\displaystyle\int \tan^2 x\,dx = \tan x - x.$

67. $\displaystyle\int \tan^3 x\,dx = \tfrac{1}{2}\tan^2 x + \ln|\cos x|.$

68. $\displaystyle\int \tan^{2n} x\,dx = (-1)^n x - \sum_{k=1}^{n} \frac{(-1)^k(\tan x)^{2n-2k+1}}{2n-2k+1}, \quad n = 1, 2, \ldots$

69. $\displaystyle\int \tan^{2n+1} x\,dx = (-1)^{n+1}\ln|\cos x| - \sum_{k=1}^{n} \frac{(-1)^k(\tan x)^{2n-2k+2}}{2n-2k+2}, \quad n = 1, 2, \ldots$

70. $\displaystyle\int \frac{dx}{a+b\tan x} = \frac{1}{a^2+b^2}\bigl(ax + b\ln|a\cos x + b\sin x|\bigr).$

71. $\displaystyle\int \frac{\tan x\,dx}{\sqrt{a+b\tan^2 x}} = \frac{1}{\sqrt{b-a}}\arccos\left(\sqrt{1-\frac{a}{b}}\,\cos x\right), \quad b > a,\ b > 0.$

72. $\displaystyle\int \cot x\,dx = \ln|\sin x|.$

73. $\displaystyle\int \cot^2 x\,dx = -\cot x - x.$

74. $\displaystyle\int \cot^3 x\,dx = -\tfrac{1}{2}\cot^2 x - \ln|\sin x|.$

75. $\displaystyle\int \cot^{2n} x\,dx = (-1)^n x + \sum_{k=1}^{n} \frac{(-1)^k(\cot x)^{2n-2k+1}}{2n-2k+1}, \quad n = 1, 2, \ldots$

76. $\displaystyle\int \cot^{2n+1} x\,dx = (-1)^n\ln|\sin x| + \sum_{k=1}^{n} \frac{(-1)^k(\cot x)^{2n-2k+2}}{2n-2k+2}, \quad n = 1, 2, \ldots$

77. $\displaystyle\int \frac{dx}{a+b\cot x} = \frac{1}{a^2+b^2}\bigl(ax - b\ln|a\sin x + b\cos x|\bigr).$

2.7. Integrals Containing Inverse Trigonometric Functions

1. $\displaystyle\int \arcsin\frac{x}{a}\,dx = x\arcsin\frac{x}{a} + \sqrt{a^2-x^2}.$

2. $\displaystyle\int \left(\arcsin\frac{x}{a}\right)^2 dx = x\left(\arcsin\frac{x}{a}\right)^2 - 2x + 2\sqrt{a^2-x^2}\,\arcsin\frac{x}{a}.$

3. $\displaystyle\int x\arcsin\frac{x}{a}\,dx = \frac{1}{4}(2x^2-a^2)\arcsin\frac{x}{a} + \frac{x}{4}\sqrt{a^2-x^2}.$

4. $\displaystyle\int x^2\arcsin\frac{x}{a}\,dx = \frac{x^3}{3}\arcsin\frac{x}{a} + \frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2}.$

5. $\displaystyle\int \arccos\frac{x}{a}\,dx = x\arccos\frac{x}{a} - \sqrt{a^2-x^2}.$

6. $\displaystyle\int \left(\arccos\frac{x}{a}\right)^2 dx = x\left(\arccos\frac{x}{a}\right)^2 - 2x - 2\sqrt{a^2-x^2}\,\arccos\frac{x}{a}.$

7. $\displaystyle\int x\arccos\frac{x}{a}\,dx = \frac{1}{4}(2x^2-a^2)\arccos\frac{x}{a} - \frac{x}{4}\sqrt{a^2-x^2}.$

8. $\displaystyle\int x^2\arccos\frac{x}{a}\,dx = \frac{x^3}{3}\arccos\frac{x}{a} - \frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2}.$

9. $\displaystyle\int \arctan\frac{x}{a}\,dx = x\arctan\frac{x}{a} - \frac{a}{2}\ln(a^2+x^2).$

10. $\displaystyle\int x\arctan\frac{x}{a}\,dx = \frac{1}{2}(x^2+a^2)\arctan\frac{x}{a} - \frac{ax}{2}.$

11. $\displaystyle\int x^2\arctan\frac{x}{a}\,dx = \frac{x^3}{3}\arctan\frac{x}{a} - \frac{ax^2}{6} + \frac{a^3}{6}\ln(a^2+x^2).$

12. $\displaystyle\int \operatorname{arccot}\frac{x}{a}\,dx = x\operatorname{arccot}\frac{x}{a} + \frac{a}{2}\ln(a^2+x^2).$

13. $\displaystyle\int x\operatorname{arccot}\frac{x}{a}\,dx = \frac{1}{2}(x^2+a^2)\operatorname{arccot}\frac{x}{a} + \frac{ax}{2}.$

14. $\displaystyle\int x^2\operatorname{arccot}\frac{x}{a}\,dx = \frac{x^3}{3}\operatorname{arccot}\frac{x}{a} + \frac{ax^2}{6} - \frac{a^3}{6}\ln(a^2+x^2).$

⊙ References for Supplement 2: H. B. Dwight (1961), I. S. Gradshteyn and I. M. Ryzhik (1980), A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1986, 1988).

Supplement 3

Tables of Definite Integrals

Throughout Supplement 3 it is assumed that n is a positive integer, unless otherwise specified.

3.1. Integrals Containing Power-Law Functions

1. $\displaystyle\int_0^\infty \frac{dx}{ax^2+b} = \frac{\pi}{2\sqrt{ab}}.$

2. $\displaystyle\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi\sqrt{2}}{4}.$

3. $\displaystyle\int_0^1 \frac{x^n\,dx}{x+1} = (-1)^n\left[\ln 2 + \sum_{k=1}^{n}\frac{(-1)^k}{k}\right].$

4. $\displaystyle\int_0^\infty \frac{x^{a-1}\,dx}{x+1} = \frac{\pi}{\sin(\pi a)}, \quad 0 < a < 1.$

5. $\displaystyle\int_0^\infty \frac{x^{\lambda-1}\,dx}{(1+ax)^2} = \frac{\pi(1-\lambda)}{a^\lambda \sin(\pi\lambda)}, \quad 0 < \lambda < 2.$

6. $\displaystyle\int_0^1 \frac{dx}{x^2+2x\cos\beta+1} = \frac{\beta}{2\sin\beta}.$

7. $\displaystyle\int_0^1 \frac{\left(x^a+x^{-a}\right)dx}{x^2+2x\cos\beta+1} = \frac{\pi\sin(a\beta)}{\sin(\pi a)\sin\beta}, \quad |a| < 1,\ \beta \neq (2n+1)\pi.$

8. $\displaystyle\int_0^\infty \frac{x^{\lambda-1}\,dx}{(x+a)(x+b)} = \frac{\pi(a^{\lambda-1}-b^{\lambda-1})}{(b-a)\sin(\pi\lambda)}, \quad 0 < \lambda < 2.$

9. $\displaystyle\int_0^\infty \frac{x^{\lambda-1}(x+c)\,dx}{(x+a)(x+b)} = \frac{\pi}{\sin(\pi\lambda)}\left(\frac{a-c}{a-b}a^{\lambda-1} + \frac{b-c}{b-a}b^{\lambda-1}\right), \quad 0 < \lambda < 1.$

10. $\displaystyle\int_0^\infty \frac{x^\lambda\,dx}{(x+1)^3} = \frac{\pi\lambda(1-\lambda)}{2\sin(\pi\lambda)}, \quad -1 < \lambda < 2.$

11. $\displaystyle\int_0^\infty \frac{x^{\lambda-1}\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi\left(b^{\lambda-2}-a^{\lambda-2}\right)}{2\left(a^2-b^2\right)\sin(\pi\lambda/2)}, \quad 0 < \lambda < 4.$

12. $\displaystyle\int_0^1 x^a(1-x)^{1-a}\,dx = \frac{\pi a(1-a)}{2\sin(\pi a)}, \quad -1 < a < 1.$

13. $\displaystyle\int_0^1 \frac{dx}{x^a(1-x)^{1-a}} = \frac{\pi}{\sin(\pi a)}, \quad 0 < a < 1.$

14. $$\int_0^1 \frac{x^a\,dx}{(1-x)^a} = \frac{\pi a}{\sin(\pi a)}, \quad -1 < a < 1.$$

15. $$\int_0^1 x^{p-1}(1-x)^{q-1}\,dx \equiv B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad p, q > 0.$$

16. $$\int_0^1 x^{p-1}(1-x^q)^{-p/q}\,dx = \frac{\pi}{q\sin(\pi p/q)}, \quad q > p > 0.$$

17. $$\int_0^1 x^{p+q-1}(1-x^q)^{-p/q}\,dx = \frac{\pi p}{q^2\sin(\pi p/q)}, \quad q > p.$$

18. $$\int_0^1 x^{q/p-1}(1-x^q)^{-1/p}\,dx = \frac{\pi}{q\sin(\pi/p)}, \quad p > 1,\ q > 0.$$

19. $$\int_0^1 \frac{x^{p-1}-x^{-p}}{1-x}\,dx = \pi\cot(\pi p), \quad |p| < 1.$$

20. $$\int_0^1 \frac{x^{p-1}-x^{-p}}{1+x}\,dx = \frac{\pi}{\sin(\pi p)}, \quad |p| < 1.$$

21. $$\int_0^1 \frac{x^{p}-x^{-p}}{x-1}\,dx = \frac{1}{p} - \pi\cot(\pi p), \quad |p| < 1.$$

22. $$\int_0^1 \frac{x^{p}-x^{-p}}{1+x}\,dx = \frac{1}{p} - \frac{\pi}{\sin(\pi p)}, \quad |p| < 1.$$

23. $$\int_0^1 \frac{x^{1+p}-x^{1-p}}{1-x^2}\,dx = \frac{\pi}{2}\cot\left(\frac{\pi p}{2}\right) - \frac{1}{p}, \quad |p| < 1.$$

24. $$\int_0^1 \frac{x^{1+p}-x^{1-p}}{1+x^2}\,dx = \frac{1}{p} - \frac{\pi}{2\sin(\pi p/2)}, \quad |p| < 1.$$

25. $$\int_0^\infty \frac{x^{p-1}-x^{q-1}}{1-x}\,dx = \pi[\cot(\pi p) - \cot(\pi q)], \quad p, q > 0.$$

26. $$\int_0^1 \frac{dx}{\sqrt{(1+a^2x)(1-x)}} = \frac{2}{a}\arctan a.$$

27. $$\int_0^1 \frac{dx}{\sqrt{(1-a^2x)(1-x)}} = \frac{1}{a}\ln\frac{1+a}{1-a}.$$

28. $$\int_{-1}^1 \frac{dx}{(a-x)\sqrt{1-x^2}} = \frac{\pi}{\sqrt{a^2-1}}, \quad 1 < a.$$

29. $$\int_0^1 \frac{x^n\,dx}{\sqrt{1-x}} = \frac{2\,(2n)!!}{(2n+1)!!}, \quad n = 1, 2, \ldots$$

30. $$\int_0^1 \frac{x^{n-1/2}\,dx}{\sqrt{1-x}} = \frac{\pi\,(2n-1)!!}{(2n)!!}, \quad n = 1, 2, \ldots$$

31. $$\int_0^1 \frac{x^{2n}\,dx}{\sqrt{1-x^2}} = \frac{\pi}{2}\,\frac{1\cdot 3\ldots(2n-1)}{2\cdot 4\ldots(2n)}, \quad n = 1, 2, \ldots$$

32. $$\int_0^1 \frac{x^{2n+1}\,dx}{\sqrt{1-x^2}} = \frac{2\cdot 4\ldots(2n)}{1\cdot 3\ldots(2n+1)}, \quad n = 1, 2, \ldots$$

33. $$\int_0^\infty \frac{x^{\lambda-1}\,dx}{(1+ax)^{n+1}} = (-1)^n\,\frac{\pi C^n_{\lambda-1}}{a^\lambda\sin(\pi\lambda)}, \quad 0 < \lambda < n+1.$$

34. $\displaystyle\int_0^\infty \frac{x^m\,dx}{(a+bx)^{n+1/2}} = 2^{m+1} m!\,\frac{(2n-2m-3)!!}{(2n-1)!!}\,\frac{a^{m-n+1/2}}{b^{m+1}}, \quad a, b > 0,$
$n, m = 1,\ 2,\ \ldots, \quad m < b - \frac{1}{2}.$

35. $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)^n} = \frac{\pi}{2}\,\frac{(2n-3)!!}{(2n-2)!!}\,\frac{1}{a^{2n-1}}, \quad n = 1,\ 2,\ \ldots$

36. $\displaystyle\int_0^\infty \frac{(x+1)^{\lambda-1}}{(x+a)^{\lambda+1}}\,dx = \frac{1-a^{-\lambda}}{\lambda(a-1)}, \quad a > 0.$

37. $\displaystyle\int_0^1 \frac{x^{\lambda-1}\,dx}{(1+ax)(1-x)^\lambda} = \frac{\pi}{(1+a)^\lambda \sin(\pi\lambda)}, \quad 0 < \lambda < 1,\ a > -1.$

38. $\displaystyle\int_0^1 \frac{x^{\lambda-1/2}\,dx}{(1+ax)^\lambda(1-x)^\lambda} = 2\pi^{-1/2}\Gamma\bigl(\lambda+\tfrac{1}{2}\bigr)\Gamma\bigl(1-\lambda\bigr)\cos^{2\lambda} k\,\frac{\sin[(2\lambda-1)k]}{(2\lambda-1)\sin k}, \quad k = \arctan\sqrt{a};$
$-\frac{1}{2} < \lambda < 1,\ a > 0.$

39. $\displaystyle\int_0^\infty \frac{x^{a-1}\,dx}{x^b+1} = \frac{\pi}{b\sin(\pi a/b)}, \quad 0 < a \le b.$

40. $\displaystyle\int_0^\infty \frac{x^{a-1}\,dx}{(x^b+1)^2} = \frac{\pi(a-b)}{b^2\sin[\pi(a-b)/b]}, \quad a < 2b.$

41. $\displaystyle\int_0^\infty \frac{x^{\lambda-1/2}\,dx}{(x+a)^\lambda(x+b)^\lambda} = \sqrt{\pi}\bigl(\sqrt{a}+\sqrt{b}\,\bigr)^{1-2\lambda}\frac{\Gamma(\lambda-1/2)}{\Gamma(\lambda)}, \quad \lambda > 0.$

42. $\displaystyle\int_0^\infty \frac{1-x^a}{1-x^b}\,x^{c-1}\,dx = \frac{\pi\sin A}{b\sin C\sin(A+C)}, \quad A = \frac{\pi a}{b}, \quad C = \frac{\pi c}{b}; \quad a+c < b, \quad c > 0.$

43. $\displaystyle\int_0^\infty \frac{x^{a-1}\,dx}{(1+x^2)^{1-b}} = \tfrac{1}{2}B\bigl(\tfrac{1}{2}a, 1-b-\tfrac{1}{2}a\bigr), \quad \tfrac{1}{2}a+b < 1, \quad a > 0.$

44. $\displaystyle\int_0^\infty \frac{x^{2m}\,dx}{(ax^2+b)^n} = \frac{\pi(2m-1)!!\,(2n-2m-3)!!}{2\,(2n-2)!!\,a^m b^{n-m-1}\sqrt{ab}}, \quad a, b > 0, \quad n > m+1.$

45. $\displaystyle\int_0^\infty \frac{x^{2m+1}\,dx}{(ax^2+b)^n} = \frac{m!\,(n-m-2)!}{2(n-1)!a^{m+1}b^{n-m-1}}, \quad ab > 0, \quad n > m+1 \ge 1.$

46. $\displaystyle\int_0^\infty \frac{x^{\mu-1}\,dx}{(1+ax^p)^\nu} = \frac{1}{pa^{\mu/p}}B\Bigl(\frac{\mu}{p}, \nu-\frac{\mu}{p}\Bigr), \quad p > 0, \quad 0 < \mu < p\nu.$

47. $\displaystyle\int_0^\infty \bigl(\sqrt{x^2+a^2}-x\bigr)^n\,dx = \frac{na^{n+1}}{n^2-1}, \quad n = 2,\ 3,\ \ldots$

48. $\displaystyle\int_0^\infty \frac{dx}{\bigl(x+\sqrt{x^2+a^2}\,\bigr)^n} = \frac{n}{a^{n-1}(n^2-1)}, \quad n = 2,\ 3,\ \ldots$

49. $\displaystyle\int_0^\infty x^m\bigl(\sqrt{x^2+a^2}-x\bigr)^n\,dx = \frac{n\cdot m!\,a^{n+m+1}}{(n-m-1)(n-m+1)\ldots(n+m+1)},$
$n, m = 1,\ 2,\ \ldots, \quad 0 \le m \le n-2$

50. $\displaystyle\int_0^\infty \frac{x^m\,dx}{\bigl(x+\sqrt{x^2+a^2}\,\bigr)^n} = \frac{n\cdot m!}{(n-m-1)(n-m+1)\ldots(n+m+1)a^{n-m-1}}, \quad n = 2,\ 3,\ \ldots$

3.2. Integrals Containing Exponential Functions

1. $\displaystyle\int_0^\infty e^{-ax}\,dx = \frac{1}{a}, \quad a > 0.$

2. $\displaystyle\int_0^1 x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}} - e^{-a}\sum_{k=0}^{n}\frac{n!}{k!}\frac{1}{a^{n-k+1}}, \quad a>0,\ \ n=1,\,2,\,\ldots$

3. $\displaystyle\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}, \quad a>0,\ \ n=1,\,2,\,\ldots$

4. $\displaystyle\int_0^\infty \frac{e^{-ax}}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{a}}, \quad a>0.$

5. $\displaystyle\int_0^\infty x^{\nu-1}e^{-\mu x}\,dx = \frac{\Gamma(\nu)}{\mu^\nu}, \quad \mu,\nu>0.$

6. $\displaystyle\int_0^\infty \frac{dx}{1+e^{ax}} = \frac{\ln 2}{a}.$

7. $\displaystyle\int_0^\infty \frac{x^{2n-1}\,dx}{e^{px}-1} = (-1)^{n-1}\left(\frac{2\pi}{p}\right)^{2n}\frac{B_{2n}}{4n}, \quad n=1,\,2,\,\ldots \quad (B_m \text{ are the Bernoulli numbers}).$

8. $\displaystyle\int_0^\infty \frac{x^{2n-1}\,dx}{e^{px}+1} = (1-2^{1-2n})\left(\frac{2\pi}{p}\right)^{2n}\frac{|B_{2n}|}{4n}, \quad n=1,\,2,\,\ldots$

9. $\displaystyle\int_{-\infty}^\infty \frac{e^{-px}\,dx}{1+e^{-qx}} = \frac{\pi}{q\sin(\pi p/q)}, \quad q>p>0 \ \text{ or } \ 0>p>q.$

10. $\displaystyle\int_0^\infty \frac{e^{ax}+e^{-ax}}{e^{bx}+e^{-bx}}\,dx = \frac{\pi}{2b\cos\left(\dfrac{\pi a}{2b}\right)}, \quad b>a.$

11. $\displaystyle\int_0^\infty \frac{e^{-px}-e^{-qx}}{1-e^{-(p+q)x}}\,dx = \frac{\pi}{p+q}\cot\frac{\pi p}{p+q}, \quad p,q>0.$

12. $\displaystyle\int_0^\infty \left(1-e^{-\beta x}\right)^\nu e^{-\mu x}\,dx = \frac{1}{\beta}B\left(\frac{\mu}{\beta},\nu+1\right).$

13. $\displaystyle\int_0^\infty \exp\left(-ax^2\right)dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}, \quad a>0.$

14. $\displaystyle\int_0^\infty x^{2n+1}\exp\left(-ax^2\right)dx = \frac{n!}{2a^{n+1}}, \quad a>0, \quad n=1,\,2,\,\ldots$

15. $\displaystyle\int_0^\infty x^{2n}\exp\left(-ax^2\right)dx = \frac{1\cdot 3\ldots(2n-1)\sqrt{\pi}}{2^{n+1}a^{n+1/2}}, \quad a>0, \quad n=1,\,2,\,\ldots$

16. $\displaystyle\int_{-\infty}^\infty \exp\left(-a^2x^2 \pm bx\right)dx = \frac{\sqrt{\pi}}{|a|}\exp\left(\frac{b^2}{4a^2}\right).$

17. $\displaystyle\int_0^\infty \exp\left(-ax^2-\frac{b}{x^2}\right)dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left(-2\sqrt{ab}\right), \quad a,b>0.$

18. $\displaystyle\int_0^\infty \exp\left(-x^a\right)dx = \frac{1}{a}\Gamma\left(\frac{1}{a}\right), \quad a>0.$

3.3. Integrals Containing Hyperbolic Functions

1. $\displaystyle\int_0^\infty \frac{dx}{\cosh ax} = \frac{\pi}{2|a|}.$

2. $$\int_0^\infty \frac{dx}{a+b\cosh x} = \begin{cases} \dfrac{2}{\sqrt{b^2-a^2}}\arctan\dfrac{\sqrt{b^2-a^2}}{a+b} & \text{if } |b|>|a|, \\[2ex] \dfrac{1}{\sqrt{a^2-b^2}}\ln\dfrac{a+b+\sqrt{a^2-b^2}}{a+b-\sqrt{a^2+b^2}} & \text{if } |b|<|a|. \end{cases}$$

3. $\displaystyle\int_0^\infty \frac{x^{2n}\,dx}{\cosh ax} = \left(\frac{\pi}{2a}\right)^{2n+1}|E_{2n}|, \quad a > 0.$

4. $\displaystyle\int_0^\infty \frac{x^{2n}}{\cosh^2 ax}\,dx = \frac{\pi^{2n}(2^{2n}-2)}{a(2a)^{2n}}|B_{2n}|, \quad a > 0.$

5. $\displaystyle\int_0^\infty \frac{\cosh ax}{\cosh bx}\,dx = \frac{\pi}{2b\cos\left(\frac{\pi a}{2b}\right)}, \quad b > |a|.$

6. $\displaystyle\int_0^\infty x^{2n}\frac{\cosh ax}{\cosh bx}\,dx = \frac{\pi}{2b}\frac{d^{2n}}{da^{2n}}\frac{1}{\cos\left(\frac{1}{2}\pi a/b\right)}, \quad b > |a|, \quad n = 1, 2, \ldots$

7. $\displaystyle\int_0^\infty \frac{\cosh ax \cosh bx}{\cosh(cx)}\,dx = \frac{\pi}{c}\frac{\cos\left(\frac{\pi a}{2c}\right)\cos\left(\frac{\pi b}{2c}\right)}{\cos\left(\frac{\pi a}{c}\right)+\cos\left(\frac{\pi b}{c}\right)}, \quad c > |a| + |b|.$

8. $\displaystyle\int_0^\infty \frac{x\,dx}{\sinh ax} = \frac{\pi^2}{2a^2}, \quad a > 0.$

9. $\displaystyle\int_0^\infty \frac{dx}{a + b\sinh x} = \frac{1}{\sqrt{a^2+b^2}}\ln\frac{a+b+\sqrt{a^2+b^2}}{a+b-\sqrt{a^2+b^2}}, \quad ab \neq 0.$

10. $\displaystyle\int_0^\infty \frac{\sinh ax}{\sinh bx}\,dx = \frac{\pi}{2b}\tan\left(\frac{\pi a}{2b}\right), \quad b > |a|.$

11. $\displaystyle\int_0^\infty x^{2n}\frac{\sinh ax}{\sinh bx}\,dx = \frac{\pi}{2b}\frac{d^{2n}}{dx^{2n}}\tan\left(\frac{\pi a}{2b}\right), \quad b > |a|, \quad n = 1, 2, \ldots$

12. $\displaystyle\int_0^\infty \frac{x^{2n}}{\sinh^2 ax}\,dx = \frac{\pi^{2n}}{a^{2n+1}}|B_{2n}|, \quad a > 0.$

3.4. Integrals Containing Logarithmic Functions

1. $\displaystyle\int_0^1 x^{a-1}\ln^n x\,dx = (-1)^n n!\,a^{-n-1}, \quad a > 0, \quad n = 1, 2, \ldots$

2. $\displaystyle\int_0^1 \frac{\ln x}{x+1}\,dx = -\frac{\pi^2}{12}.$

3. $\displaystyle\int_0^1 \frac{x^n\ln x}{x+1}\,dx = (-1)^{n+1}\left[\frac{\pi^2}{12} + \sum_{k=1}^{n}\frac{(-1)^k}{k^2}\right], \quad n = 1, 2, \ldots$

4. $\displaystyle\int_0^1 \frac{x^{\mu-1}\ln x}{x+a}\,dx = \frac{\pi a^{\mu-1}}{\sin(\pi\mu)}\left[\ln a - \pi\cot(\pi\mu)\right], \quad 0 < \mu < 1.$

5. $\displaystyle\int_0^1 |\ln x|^\mu\,dx = \Gamma(\mu+1), \quad \mu > -1.$

6. $\displaystyle\int_0^\infty x^{\mu-1}\ln(1+ax)\,dx = \frac{\pi}{\mu a^\mu\sin(\pi\mu)}, \quad -1 < \mu < 0.$

7. $\displaystyle\int_0^1 x^{2n-1}\ln(1+x)\,dx = \frac{1}{2n}\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}, \quad n = 1, 2, \ldots$

8. $\displaystyle\int_0^1 x^{2n}\ln(1+x)\,dx = \frac{1}{2n+1}\left[\ln 4 + \sum_{k=1}^{2n+1}\frac{(-1)^k}{k}\right], \quad n = 0, 1, \ldots$

9. $\displaystyle\int_0^1 x^{n-1/2}\ln(1+x)\,dx = \frac{2\ln 2}{2n+1} + \frac{4(-1)^n}{2n+1}\left[\pi - \sum_{k=0}^{n}\frac{(-1)^k}{2k+1}\right], \quad n = 1, 2, \ldots$

10. $\displaystyle\int_0^\infty \ln\frac{a^2+x^2}{b^2+x^2}\,dx = \pi(a-b), \quad a, b > 0.$

11. $\displaystyle\int_0^\infty \frac{x^{p-1}\ln x}{1+x^q}\,dx = -\frac{\pi^2\cos(\pi p/q)}{q^2\sin^2(\pi p/q)}, \quad 0 < p < q.$

12. $\displaystyle\int_0^\infty e^{-\mu x}\ln x\,dx = -\frac{1}{\mu}(\mathcal{C} + \ln\mu), \quad \mu > 0, \quad \mathcal{C} = 0.5772\ldots$

3.5. Integrals Containing Trigonometric Functions

1. $\displaystyle\int_0^{\pi/2} \cos^{2n} x\,dx = \frac{\pi}{2}\,\frac{1\cdot 3\ldots(2n-1)}{2\cdot 4\ldots(2n)}, \quad n = 1, 2, \ldots$

2. $\displaystyle\int_0^{\pi/2} \cos^{2n+1} x\,dx = \frac{2\cdot 4\ldots(2n)}{1\cdot 3\ldots(2n+1)}, \quad n = 1, 2, \ldots$

3. $\displaystyle\int_0^{\pi/2} x\cos^n x\,dx = -\sum_{k=0}^{m-1}\frac{(n-2k+1)(n-2k+3)\ldots(n-1)}{(n-2k)(n-2k+2)\ldots n}\,\frac{1}{n-2k}$
$$+\begin{cases} \dfrac{\pi}{2}\,\dfrac{(2m-2)!!}{(2m-1)!!} & \text{if } n = 2m-1, \\ \dfrac{\pi^2}{8}\cdot\dfrac{(2m-1)!!}{(2m)!!} & \text{if } n = 2m, \end{cases} \qquad m = 1, 2, \ldots$$

4. $\displaystyle\int_0^\pi \frac{dx}{(a+b\cos x)^{n+1}} = \frac{\pi}{2^n(a+b)^n\sqrt{a^2-b^2}}\sum_{k=0}^{n}\frac{(2n-2k-1)!!\,(2k-1)!!}{(n-k)!\,k!}\left(\frac{a+b}{a-b}\right)^k, \quad a > |b|.$

5. $\displaystyle\int_0^\infty \frac{\cos ax}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2a}}, \quad a > 0.$

6. $\displaystyle\int_0^\infty \frac{\cos ax - \cos bx}{x}\,dx = \ln\left|\frac{b}{a}\right|, \quad ab \neq 0.$

7. $\displaystyle\int_0^\infty \frac{\cos ax - \cos bx}{x^2}\,dx = \tfrac{1}{2}\pi(b-a), \quad a, b \geq 0.$

8. $\displaystyle\int_0^\infty x^{\mu-1}\cos ax\,dx = a^{-\mu}\Gamma(\mu)\cos\left(\tfrac{1}{2}\pi\mu\right), \quad a > 0, \quad 0 < \mu < 1.$

9. $\displaystyle\int_0^\infty \frac{\cos ax}{b^2+x^2}\,dx = \frac{\pi}{2b}e^{-ab}, \quad a, b > 0.$

10. $\displaystyle\int_0^\infty \frac{\cos ax}{b^4+x^4}\,dx = \frac{\pi\sqrt{2}}{4b^3}\exp\left(-\frac{ab}{\sqrt{2}}\right)\left[\cos\left(\frac{ab}{\sqrt{2}}\right) + \sin\left(\frac{ab}{\sqrt{2}}\right)\right], \quad a, b > 0.$

11. $\displaystyle\int_0^\infty \frac{\cos ax}{(b^2+x^2)^2}\,dx = \frac{\pi}{4b^3}(1+ab)e^{-ab}, \quad a, b > 0.$

12. $\displaystyle\int_0^\infty \frac{\cos ax\,dx}{(b^2+x^2)(c^2+x^2)} = \frac{\pi\left(be^{-ac} - ce^{-ab}\right)}{2bc\left(b^2-c^2\right)}, \quad a, b, c > 0.$

13. $\displaystyle\int_0^\infty \cos\left(ax^2\right)dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}, \quad a > 0.$

14. $\int_0^\infty \cos(ax^p)\,dx = \frac{\Gamma(1/p)}{pa^{1/p}} \cos\frac{\pi}{2p}, \quad a > 0, \quad p > 1.$

15. $\int_0^{\pi/2} \sin^{2n} x\,dx = \frac{\pi}{2}\frac{1\cdot 3\ldots(2n-1)}{2\cdot 4\ldots(2n)}, \quad n = 1, 2, \ldots$

16. $\int_0^{\pi/2} \sin^{2n+1} x\,dx = \frac{2\cdot 4\ldots(2n)}{1\cdot 3\ldots(2n+1)}, \quad n = 1, 2, \ldots$

17. $\int_0^\infty \frac{\sin ax}{x}\,dx = \frac{\pi}{2}\operatorname{sign} a.$

18. $\int_0^\infty \frac{\sin^2 ax}{x^2}\,dx = \frac{\pi}{2}|a|.$

19. $\int_0^\infty \frac{\sin ax}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2a}}, \quad a > 0.$

20. $\int_0^\pi x\sin^\mu x\,dx = \frac{\pi^2}{2^{\mu+1}}\frac{\Gamma(\mu+1)}{\left[\Gamma\left(\mu+\frac{1}{2}\right)\right]^2}, \quad \mu > -1.$

21. $\int_0^\infty x^{\mu-1}\sin ax\,dx = a^{-\mu}\Gamma(\mu)\sin\left(\tfrac{1}{2}\pi\mu\right), \quad a > 0, \quad 0 < \mu < 1.$

22. $\int_0^{\pi/2} \frac{\sin x\,dx}{\sqrt{1-k^2\sin^2 x}} = \frac{1}{2k}\ln\frac{1+k}{1-k}.$

23. $\int_0^\infty \sin(ax^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}, \quad a > 0.$

24. $\int_0^\infty \sin(ax^p)\,dx = \frac{\Gamma(1/p)}{pa^{1/p}}\sin\frac{\pi}{2p}, \quad a > 0, \quad p > 1.$

25. $\int_0^{\pi/2} \sin^{2n+1} x\cos^{2m+1} x\,dx = \frac{n!\,m!}{2(n+m+1)!}, \quad n, m = 1, 2, \ldots$

26. $\int_0^{\pi/2} \sin^{p-1} x\cos^{q-1} x\,dx = \tfrac{1}{2}B\left(\tfrac{1}{2}p, \tfrac{1}{2}q\right).$

27. $\int_0^{2\pi} (a\sin x + b\cos x)^{2n}\,dx = 2\pi\frac{(2n-1)!!}{(2n)!!}\left(a^2+b^2\right)^n, \quad n = 1, 2, \ldots$

28. $\int_0^\infty \frac{\sin x\cos ax}{x}\,dx = \begin{cases} \frac{\pi}{2} & \text{if } |a| < 1, \\ \frac{\pi}{4} & \text{if } |a| = 1, \\ 0 & \text{if } 1 < |a|. \end{cases}$

29. $\int_0^\pi \frac{\sin x\,dx}{\sqrt{a^2+1-2a\cos x}} = \begin{cases} 2 & \text{if } 0 \le a \le 1, \\ 2/a & \text{if } 1 < a. \end{cases}$

30. $\int_0^\infty \frac{\tan ax}{x}\,dx = \frac{\pi}{2}\operatorname{sign} a.$

31. $\int_0^{\pi/2} (\tan x)^{\pm\lambda}\,dx = \frac{\pi}{2\cos\left(\frac{1}{2}\pi\lambda\right)}, \quad |\lambda| < 1.$

32. $\int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}, \quad a > 0.$

33. $\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}, \quad a > 0.$

34. $\displaystyle\int_0^\infty \exp\bigl(-ax^2\bigr)\cos bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,\exp\Bigl(-\frac{b^2}{4a}\Bigr).$

35. $\displaystyle\int_0^\infty \cos(ax^2)\cos bx\,dx = \sqrt{\frac{\pi}{8a}}\left[\cos\left(\frac{b^2}{4a}\right) + \sin\left(\frac{b^2}{4a}\right)\right], \quad a,b>0.$

36. $\displaystyle\int_0^\infty (\cos ax + \sin ax)\cos(b^2x^2)\,dx = \frac{1}{b}\sqrt{\frac{\pi}{8}}\exp\left(-\frac{a^2}{2b}\right), \quad a,b>0.$

37. $\displaystyle\int_0^\infty \bigl[\cos ax + \sin ax\bigr]\sin(b^2x^2)\,dx = \frac{1}{b}\sqrt{\frac{\pi}{8}}\exp\Bigl(-\frac{a^2}{2b}\Bigr), \quad a,b>0.$

⦿ References for Supplement 3: H. B. Dwight (1961), I. S. Gradshteyn and I. M. Ryzhik (1980), A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1986, 1988).

Supplement 4

Tables of Laplace Transforms

4.1. General Formulas

No	Original function, $f(x)$	Laplace transform, $\tilde{f}(p) = \int_0^\infty e^{-px} f(x)\,dx$
1	$af_1(x) + bf_2(x)$	$a\tilde{f}_1(p) + b\tilde{f}_2(p)$
2	$f(x/a),\quad a > 0$	$a\tilde{f}(ap)$
3	$\begin{cases} 0 & \text{if } 0 < x < a, \\ f(x-a) & \text{if } a < x, \end{cases}$	$e^{-ap}\tilde{f}(p)$
4	$x^n f(x);\quad n = 1, 2, \ldots$	$(-1)^n \dfrac{d^n}{dp^n} \tilde{f}(p)$
5	$\dfrac{1}{x} f(x)$	$\int_p^\infty \tilde{f}(q)\,dq$
6	$e^{ax} f(x)$	$\tilde{f}(p-a)$
7	$\sinh(ax) f(x)$	$\frac{1}{2}\big[\tilde{f}(p-a) - \tilde{f}(p+a)\big]$
8	$\cosh(ax) f(x)$	$\frac{1}{2}\big[\tilde{f}(p-a) + \tilde{f}(p+a)\big]$
9	$\sin(\omega x) f(x)$	$-\frac{i}{2}\big[\tilde{f}(p-i\omega) - \tilde{f}(p+i\omega)\big],\quad i^2 = -1$
10	$\cos(\omega x) f(x)$	$\frac{1}{2}\big[\tilde{f}(p-i\omega) + \tilde{f}(p+i\omega)\big],\quad i^2 = -1$
11	$f(x^2)$	$\dfrac{1}{\sqrt{\pi}} \int_0^\infty \exp\Big(-\dfrac{p^2}{4t^2}\Big) \tilde{f}(t^2)\,dt$
12	$x^{a-1} f\Big(\dfrac{1}{x}\Big),\quad a > -1$	$\int_0^\infty (t/p)^{a/2} J_a\big(2\sqrt{pt}\,\big) \tilde{f}(t)\,dt$
13	$f(a\sinh x),\quad a > 0$	$\int_0^\infty J_p(at)\tilde{f}(t)\,dt$
14	$f(x+a) = f(x)$ (periodic function)	$\dfrac{1}{1-e^{ap}} \int_0^a f(x)e^{-px}\,dx$
15	$f(x+a) = -f(x)$ (antiperiodic function)	$\dfrac{1}{1+e^{-ap}} \int_0^a f(x)e^{-px}\,dx$
16	$f'_x(x)$	$p\tilde{f}(p) - f(+0)$
17	$f_x^{(n)}(x)$	$p^n\tilde{f}(p) - \sum_{k=1}^{n} p^{n-k} f_x^{(k-1)}(+0)$

No	Original function, $f(x)$	Laplace transform, $\tilde{f}(p)=\int_0^\infty e^{-px}f(x)\,dx$
18	$x^m f_x^{(n)}(x), \quad m\ge n$	$\left(-\frac{d}{dp}\right)^m \left[p^n\tilde{f}(p)\right]$
19	$\frac{d^n}{dx^n}\left[x^m f(x)\right], \quad m\ge n$	$(-1)^m p^n \frac{d^m}{dp^m}\tilde{f}(p)$
20	$\int_0^x f(t)\,dt$	$\frac{\tilde{f}(p)}{p}$
21	$\int_0^x (x-t)f(t)\,dt$	$\frac{1}{p^2}\tilde{f}(p)$
22	$\int_0^x (x-t)^\nu f(t)\,dt, \qquad \nu>-1$	$\Gamma(\nu+1)p^{-\nu-1}\tilde{f}(p)$
23	$\int_0^x e^{-a(x-t)}f(t)\,dt$	$\frac{1}{p+a}\tilde{f}(p)$
24	$\int_0^x \sinh\left[a(x-t)\right]f(t)\,dt$	$\frac{a\tilde{f}(p)}{p^2-a^2}$
25	$\int_0^x \sin\left[a(x-t)\right]f(t)\,dt$	$\frac{a\tilde{f}(p)}{p^2+a^2}$
26	$\int_0^x f_1(t)f_2(x-t)\,dt$	$\tilde{f}_1(p)\tilde{f}_2(p)$
27	$\int_0^x \frac{1}{t}f(t)\,dt$	$\frac{1}{p}\int_p^\infty \tilde{f}(q)\,dq$
28	$\int_x^\infty \frac{1}{t}f(t)\,dt$	$\frac{1}{p}\int_0^p \tilde{f}(q)\,dq$
29	$\int_0^\infty \frac{1}{\sqrt{t}}\sin\left(2\sqrt{xt}\right)f(t)\,dt$	$\frac{\sqrt{\pi}}{p\sqrt{p}}\tilde{f}\left(\frac{1}{p}\right)$
30	$\frac{1}{\sqrt{x}}\int_0^\infty \cos\left(2\sqrt{xt}\right)f(t)\,dt$	$\frac{\sqrt{\pi}}{\sqrt{p}}\tilde{f}\left(\frac{1}{p}\right)$
31	$\int_0^\infty \frac{1}{\sqrt{\pi x}}\exp\left(-\frac{t^2}{4x}\right)f(t)\,dt$	$\frac{1}{\sqrt{p}}\tilde{f}\left(\sqrt{p}\right)$
32	$\int_0^\infty \frac{t}{2\sqrt{\pi x^3}}\exp\left(-\frac{t^2}{4x}\right)f(t)\,dt$	$\tilde{f}\left(\sqrt{p}\right)$
33	$f(x)-a\int_0^x f\left(\sqrt{x^2-t^2}\right)J_1(at)\,dt$	$\tilde{f}\left(\sqrt{p^2+a^2}\right)$
34	$f(x)+a\int_0^x f\left(\sqrt{x^2-t^2}\right)I_1(at)\,dt$	$\tilde{f}\left(\sqrt{p^2-a^2}\right)$

4.2. Expressions With Power-Law Functions

No	Original function, $f(x)$	Laplace transform, $\tilde{f}(p) = \int_0^\infty e^{-px} f(x)\,dx$
1	1	$\dfrac{1}{p}$
2	$\begin{cases} 0 & \text{if } 0 < x < a, \\ 1 & \text{if } a < x < b, \\ 0 & \text{if } b < x. \end{cases}$	$\dfrac{1}{p}\left(e^{-ap} - e^{-bp}\right)$
3	x	$\dfrac{1}{p^2}$
4	$\dfrac{1}{x+a}$	$-e^{ap}\,\mathrm{Ei}(-ap)$
5	$x^n, \qquad n = 1, 2, \ldots$	$\dfrac{n!}{p^{n+1}}$
6	$x^{n-1/2}, \qquad n = 1, 2, \ldots$	$\dfrac{1 \cdot 3 \ldots (2n-1)\sqrt{\pi}}{2^n p^{n+1/2}}$
7	$\dfrac{1}{\sqrt{x+a}}$	$\sqrt{\dfrac{\pi}{p}}\,e^{ap}\,\mathrm{erfc}\left(\sqrt{ap}\right)$
8	$\dfrac{\sqrt{x}}{x+a}$	$\sqrt{\dfrac{\pi}{p}} - \pi\sqrt{a}\,e^{ap}\,\mathrm{erfc}\left(\sqrt{ap}\right)$
9	$(x+a)^{-3/2}$	$2a^{-1/2} - 2(\pi p)^{1/2} e^{ap}\,\mathrm{erfc}\left(\sqrt{ap}\right)$
10	$x^{1/2}(x+a)^{-1}$	$(\pi/p)^{1/2} - \pi a^{1/2} e^{ap}\,\mathrm{erfc}\left(\sqrt{ap}\right)$
11	$x^{-1/2}(x+a)^{-1}$	$\pi a^{-1/2} e^{ap}\,\mathrm{erfc}\left(\sqrt{ap}\right)$
12	$x^\nu, \qquad \nu > -1$	$\Gamma(\nu+1)p^{-\nu-1}$
13	$(x+a)^\nu, \qquad \nu > -1$	$p^{-\nu-1} e^{-ap}\Gamma(\nu+1, ap)$
14	$x^\nu (x+a)^{-1}, \qquad \nu > -1$	$k e^{ap}\Gamma(-\nu, ap), \qquad k = a^\nu \Gamma(\nu+1)$
15	$(x^2 + 2ax)^{-1/2}(x+a)$	$a e^{ap} K_1(ap)$

4.3. Expressions With Exponential Functions

No	Original function, $f(x)$	Laplace transform, $\tilde{f}(p) = \int_0^\infty e^{-px} f(x)\,dx$
1	e^{-ax}	$(p+a)^{-1}$
2	xe^{-ax}	$(p+a)^{-2}$
3	$x^{\nu-1}e^{-ax}, \qquad \nu > 0$	$\Gamma(\nu)(p+a)^{-\nu}$
4	$\dfrac{1}{x}\left(e^{-ax} - e^{-bx}\right)$	$\ln(p+b) - \ln(p+a)$

No	Original function, $f(x)$	Laplace transform, $\tilde{f}(p) = \int_0^\infty e^{-px} f(x)\,dx$
5	$\frac{1}{x^2}\left(1-e^{-ax}\right)^2$	$(p+2a)\ln(p+2a) + p\ln p - 2(p+a)\ln(p+a)$
6	$\exp\left(-ax^2\right), \qquad a>0$	$(\pi b)^{1/2}\exp\left(bp^2\right)\operatorname{erfc}(p\sqrt{b}), \qquad a = \frac{1}{4b}$
7	$x\exp\left(-ax^2\right)$	$2b - 2\pi^{1/2}b^{3/2}p\operatorname{erfc}(p\sqrt{b}), \qquad a = \frac{1}{4b}$
8	$\exp(-a/x), \qquad a \geq 0$	$2\sqrt{a/p}K_1\left(2\sqrt{ap}\right)$
9	$\sqrt{x}\exp(-a/x), \qquad a \geq 0$	$\frac{1}{2}\sqrt{\pi/p^3}\left(1+2\sqrt{ap}\right)\exp\left(-2\sqrt{ap}\right)$
10	$\frac{1}{\sqrt{x}}\exp(-a/x), \qquad a \geq 0$	$\sqrt{\pi/p}\exp\left(-2\sqrt{ap}\right)$
11	$\frac{1}{x\sqrt{x}}\exp(-a/x), \qquad a>0$	$\sqrt{\pi/a}\exp\left(-2\sqrt{ap}\right)$
12	$x^{\nu-1}\exp(-a/x), \qquad a>0$	$2(a/p)^{\nu/2}K_\nu\left(2\sqrt{ap}\right)$
13	$\exp\left(-2\sqrt{ax}\right)$	$p^{-1} - (\pi a)^{1/2}p^{-3/2}e^{a/p}\operatorname{erfc}\left(\sqrt{a/p}\right)$
14	$\frac{1}{\sqrt{x}}\exp\left(-2\sqrt{ax}\right)$	$(\pi/p)^{1/2}e^{a/p}\operatorname{erfc}\left(\sqrt{a/p}\right)$

4.4. Expressions With Hyperbolic Functions

No	Original function, $f(x)$	Laplace transform, $\tilde{f}(p) = \int_0^\infty e^{-px} f(x)\,dx$
1	$\sinh(ax)$	$\frac{a}{p^2-a^2}$
2	$\sinh^2(ax)$	$\frac{2a^2}{p^3-4a^2p}$
3	$\frac{1}{x}\sinh(ax)$	$\frac{1}{2}\ln\frac{p+a}{p-a}$
4	$x^{\nu-1}\sinh(ax), \qquad \nu>-1$	$\frac{1}{2}\Gamma(\nu)\left[(p-a)^{-\nu} - (p+a)^{-\nu}\right]$
5	$\sinh\left(2\sqrt{ax}\right)$	$\frac{\sqrt{\pi a}}{p\sqrt{p}}e^{a/p}$
6	$\sqrt{x}\sinh\left(2\sqrt{ax}\right)$	$\pi^{1/2}p^{-5/2}\left(\frac{1}{2}p+a\right)e^{a/p}\operatorname{erf}\left(\sqrt{a/p}\right) - a^{1/2}p^{-2}$
7	$\frac{1}{\sqrt{x}}\sinh\left(2\sqrt{ax}\right)$	$\pi^{1/2}p^{-1/2}e^{a/p}\operatorname{erf}\left(\sqrt{a/p}\right)$
8	$\frac{1}{\sqrt{x}}\sinh^2\left(\sqrt{ax}\right)$	$\frac{1}{2}\pi^{1/2}p^{-1/2}\left(e^{a/p}-1\right)$
9	$\cosh(ax)$	$\frac{p}{p^2-a^2}$

No	Original function, $f(x)$	Laplace transform, $\tilde{f}(p) = \int_0^\infty e^{-px} f(x)\,dx$
10	$\cosh^2(ax)$	$\dfrac{p^2 - 2a^2}{p^3 - 4a^2 p}$
11	$x^{\nu-1}\cosh(ax), \qquad \nu > 0$	$\frac{1}{2}\Gamma(\nu)\left[(p-a)^{-\nu} + (p+a)^{-\nu}\right]$
12	$\cosh\left(2\sqrt{ax}\right)$	$\dfrac{1}{p} + \dfrac{\sqrt{\pi a}}{p\sqrt{p}} e^{a/p} \operatorname{erf}\left(\sqrt{a/p}\right)$
13	$\sqrt{x}\cosh\left(2\sqrt{ax}\right)$	$\pi^{1/2} p^{-5/2}\left(\frac{1}{2}p + a\right)e^{a/p}$
14	$\dfrac{1}{\sqrt{x}}\cosh\left(2\sqrt{ax}\right)$	$\pi^{1/2} p^{-1/2} e^{a/p}$
15	$\dfrac{1}{\sqrt{x}}\cosh^2\left(\sqrt{ax}\right)$	$\frac{1}{2}\pi^{1/2} p^{-1/2}\left(e^{a/p} + 1\right)$

4.5. Expressions With Logarithmic Functions

No	Original function, $f(x)$	Laplace transform, $\tilde{f}(p) = \int_0^\infty e^{-px} f(x)\,dx$
1	$\ln x$	$-\dfrac{1}{p}(\ln p + \mathcal{C})$, $\mathcal{C} = 0.5772\ldots$ is the Euler constant
2	$\ln(1 + ax)$	$-\dfrac{1}{p} e^{p/a} \operatorname{Ei}(-p/a)$
3	$\ln(x + a)$	$\dfrac{1}{p}\left[\ln a - e^{ap} \operatorname{Ei}(-ap)\right]$
4	$x^n \ln x, \qquad n = 1, 2, \ldots$	$\dfrac{n!}{p^{n+1}}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln p - \mathcal{C}\right)$, $\mathcal{C} = 0.5772\ldots$ is the Euler constant
5	$\dfrac{1}{\sqrt{x}}\ln x$	$-\sqrt{\pi/p}\left[\ln(4p) + \mathcal{C}\right]$
6	$x^{n-1/2}\ln x, \qquad n = 1, 2, \ldots$	$\dfrac{k_n}{p^{n+1/2}}\left[2 + \frac{2}{3} + \frac{2}{5} + \cdots + \frac{2}{2n-1} - \ln(4p) - \mathcal{C}\right]$, $k_n = 1 \cdot 3 \cdot 5 \ldots (2n-1)\dfrac{\sqrt{\pi}}{2^n}, \quad \mathcal{C} = 0.5772\ldots$
7	$x^{\nu-1}\ln x, \quad \nu > 0$	$\Gamma(\nu)p^{-\nu}\left[\psi(\nu) - \ln p\right]$, $\psi(\nu)$ is the logarithmic derivative of the gamma function
8	$(\ln x)^2$	$\dfrac{1}{p}\left[(\ln x + \mathcal{C})^2 + \frac{1}{6}\pi^2\right], \quad \mathcal{C} = 0.5772\ldots$
9	$e^{-ax}\ln x$	$-\dfrac{\ln(p + a) + \mathcal{C}}{p + a}, \qquad \mathcal{C} = 0.5772\ldots$

4.6. Expressions With Trigonometric Functions

No	Original function, $f(x)$	Laplace transform, $\tilde{f}(p)=\int_0^\infty e^{-px}f(x)\,dx$
1	$\sin(ax)$	$\dfrac{a}{p^2+a^2}$
2	$\lvert\sin(ax)\rvert, \quad a>0$	$\dfrac{a}{p^2+a^2}\coth\left(\dfrac{\pi p}{2a}\right)$
3	$\sin^{2n}(ax), \quad n=1,2,\ldots$	$\dfrac{a^{2n}(2n)!}{p\left[p^2+(2a)^2\right]\left[p^2+(4a)^2\right]\ldots\left[p^2+(2na)^2\right]}$
4	$\sin^{2n+1}(ax), \quad n=1,2,\ldots$	$\dfrac{a^{2n+1}(2n+1)!}{\left[p^2+a^2\right]\left[p^2+3^2a^2\right]\ldots\left[p^2+(2n+1)^2a^2\right]}$
5	$x^n\sin(ax), \quad n=1,2,\ldots$	$\dfrac{n!\,p^{n+1}}{\left(p^2+a^2\right)^{n+1}}\sum_{0\le 2k\le n}(-1)^kC_{n+1}^{2k+1}\left(\dfrac{a}{p}\right)^{2k+1}$
6	$\dfrac{1}{x}\sin(ax)$	$\arctan\left(\dfrac{a}{p}\right)$
7	$\dfrac{1}{x}\sin^2(ax)$	$\frac{1}{4}\ln\left(1+4a^2p^{-2}\right)$
8	$\dfrac{1}{x^2}\sin^2(ax)$	$a\arctan(2a/p)-\frac{1}{4}p\ln\left(1+4a^2p^{-2}\right)$
9	$\sin\left(2\sqrt{ax}\right)$	$\dfrac{\sqrt{\pi a}}{p\sqrt{p}}e^{-a/p}$
10	$\dfrac{1}{x}\sin\left(2\sqrt{ax}\right)$	$\pi\,\mathrm{erf}\left(\sqrt{a/p}\right)$
11	$\cos(ax)$	$\dfrac{p}{p^2+a^2}$
12	$\cos^2(ax)$	$\dfrac{p^2+2a^2}{p\left(p^2+4a^2\right)}$
13	$x^n\cos(ax), \quad n=1,2,\ldots$	$\dfrac{n!\,p^{n+1}}{\left(p^2+a^2\right)^{n+1}}\sum_{0\le 2k\le n+1}(-1)^kC_{n+1}^{2k}\left(\dfrac{a}{p}\right)^{2k}$
14	$\dfrac{1}{x}\left[1-\cos(ax)\right]$	$\frac{1}{2}\ln\left(1+a^2p^{-2}\right)$
15	$\dfrac{1}{x}\left[\cos(ax)-\cos(bx)\right]$	$\dfrac{1}{2}\ln\dfrac{p^2+b^2}{p^2+a^2}$
16	$\sqrt{x}\cos\left(2\sqrt{ax}\right)$	$\frac{1}{2}\pi^{1/2}p^{-5/2}(p-2a)e^{-a/p}$
17	$\dfrac{1}{\sqrt{x}}\cos\left(2\sqrt{ax}\right)$	$\sqrt{\pi/p}\,e^{-a/p}$
18	$\sin(ax)\sin(bx)$	$\dfrac{2abp}{\left[p^2+(a+b)^2\right]\left[p^2+(a-b)^2\right]}$
19	$\cos(ax)\sin(bx)$	$\dfrac{b\left(p^2-a^2+b^2\right)}{\left[p^2+(a+b)^2\right]\left[p^2+(a-b)^2\right]}$

No	Original function, $f(x)$	Laplace transform, $\tilde{f}(p) = \int_0^\infty e^{-px} f(x)\,dx$
20	$\cos(ax)\cos(bx)$	$\dfrac{p\left(p^2+a^2+b^2\right)}{\left[p^2+(a+b)^2\right]\left[p^2+(a-b)^2\right]}$
21	$\dfrac{ax\cos(ax)-\sin(ax)}{x^2}$	$p\arctan\dfrac{a}{x} - a$
22	$e^{bx}\sin(ax)$	$\dfrac{a}{(p-b)^2+a^2}$
23	$e^{bx}\cos(ax)$	$\dfrac{p-b}{(p-b)^2+a^2}$
24	$\sin(ax)\sinh(ax)$	$\dfrac{2a^2p}{p^4+4a^4}$
25	$\sin(ax)\cosh(ax)$	$\dfrac{a\left(p^2+2a^2\right)}{p^4+4a^4}$
26	$\cos(ax)\sinh(ax)$	$\dfrac{a\left(p^2-2a^2\right)}{p^4+4a^4}$
27	$\cos(ax)\cosh(ax)$	$\dfrac{p^3}{p^4+4a^4}$

4.7. Expressions With Special Functions

No	Original function, $f(x)$	Laplace transform, $\tilde{f}(p) = \int_0^\infty e^{-px} f(x)\,dx$
1	$\operatorname{erf}(ax)$	$\dfrac{1}{p}\exp\left(b^2p^2\right)\operatorname{erfc}(bp), \qquad b = \dfrac{1}{2a}$
2	$\operatorname{erf}\left(\sqrt{ax}\right)$	$\dfrac{\sqrt{a}}{p\sqrt{p+a}}$
3	$e^{ax}\operatorname{erf}\left(\sqrt{ax}\right)$	$\dfrac{\sqrt{a}}{\sqrt{p}\,(p-a)}$
4	$\operatorname{erf}\left(\frac{1}{2}\sqrt{a/x}\right)$	$\dfrac{1}{p}\left[1-\exp\left(-\sqrt{ap}\right)\right]$
5	$\operatorname{erfc}\left(\sqrt{ax}\right)$	$\dfrac{\sqrt{p+a}-\sqrt{a}}{p\sqrt{p+a}}$
6	$e^{ax}\operatorname{erfc}\left(\sqrt{ax}\right)$	$\dfrac{1}{p+\sqrt{ap}}$
7	$\operatorname{erfc}\left(\frac{1}{2}\sqrt{a/x}\right)$	$\dfrac{1}{p}\exp\left(-\sqrt{ap}\right)$
8	$\operatorname{Ci}(x)$	$\dfrac{1}{2p}\ln(p^2+1)$

No	Original function, $f(x)$	Laplace transform, $\tilde{f}(p)=\int_0^\infty e^{-px}f(x)\,dx$
9	$\mathrm{Si}(x)$	$\frac{1}{p}\operatorname{arccot} p$
10	$\mathrm{Ei}(-x)$	$-\frac{1}{p}\ln(p+1)$
11	$J_0(ax)$	$\frac{1}{\sqrt{p^2+a^2}}$
12	$J_\nu(ax), \quad \nu>-1$	$\frac{a^\nu}{\sqrt{p^2+a^2}\left(p+\sqrt{p^2+a^2}\right)^\nu}$
13	$x^n J_n(ax), \quad n=1,\,2,\,\ldots$	$1\cdot 3\cdot 5\ldots(2n-1)a^n\left(p^2+a^2\right)^{-n-1/2}$
14	$x^\nu J_\nu(ax), \quad \nu>-\frac{1}{2}$	$2^\nu\pi^{-1/2}\Gamma\left(\nu+\frac{1}{2}\right)a^\nu\left(p^2+a^2\right)^{-\nu-1/2}$
15	$x^{\nu+1}J_\nu(ax), \quad \nu>-1$	$2^{\nu+1}\pi^{-1/2}\Gamma\left(\nu+\frac{3}{2}\right)a^\nu p\left(p^2+a^2\right)^{-\nu-3/2}$
16	$J_0\left(2\sqrt{ax}\right)$	$\frac{1}{p}e^{-a/p}$
17	$\sqrt{x}J_1\left(2\sqrt{ax}\right)$	$\frac{\sqrt{a}}{p^2}e^{-a/p}$
18	$x^{\nu/2}J_\nu\left(2\sqrt{ax}\right), \quad \nu>-1$	$a^{\nu/2}p^{-\nu-1}e^{-a/p}$
19	$I_0(ax)$	$\frac{1}{\sqrt{p^2-a^2}}$
20	$I_\nu(ax), \quad \nu>-1$	$\frac{a^\nu}{\sqrt{p^2-a^2}\left(p+\sqrt{p^2-a^2}\right)^\nu}$
21	$x^\nu I_\nu(ax), \quad \nu>-\frac{1}{2}$	$2^\nu\pi^{-1/2}\Gamma\left(\nu+\frac{1}{2}\right)a^\nu\left(p^2-a^2\right)^{-\nu-1/2}$
22	$x^{\nu+1}I_\nu(ax), \quad \nu>-1$	$2^{\nu+1}\pi^{-1/2}\Gamma\left(\nu+\frac{3}{2}\right)a^\nu p\left(p^2-a^2\right)^{-\nu-3/2}$
23	$I_0\left(2\sqrt{ax}\right)$	$\frac{1}{p}e^{a/p}$
24	$\frac{1}{\sqrt{x}}I_1\left(2\sqrt{ax}\right)$	$\frac{1}{\sqrt{a}}\left(e^{a/p}-1\right)$
25	$x^{\nu/2}I_\nu\left(2\sqrt{ax}\right), \quad \nu>-1$	$a^{\nu/2}p^{-\nu-1}e^{a/p}$
26	$Y_0(ax)$	$-\frac{2}{\pi}\frac{\operatorname{Arsinh}(p/a)}{\sqrt{p^2+a^2}}$
27	$K_0(ax)$	$\frac{\ln\left(p+\sqrt{p^2-a^2}\right)-\ln a}{\sqrt{p^2-a^2}}$

⊙ References for Supplement 4: G. Doetsch (1950, 1956, 1958), H. Bateman and A. Erdélyi (1954), V. A. Ditkin and A. P. Prudnikov (1965).

Supplement 5

Tables of Inverse Laplace Transforms

5.1. General Formulas

No	Laplace transform, $\tilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\tilde{f}(p)\,dp$
1	$\tilde{f}(p+a)$	$e^{-ax}f(x)$
2	$\tilde{f}(ap), \quad a>0$	$\dfrac{1}{a}f\Big(\dfrac{x}{a}\Big)$
3	$\tilde{f}(ap+b), \quad a>0$	$\dfrac{1}{a}\exp\Big(-\dfrac{b}{a}x\Big)f\Big(\dfrac{x}{a}\Big)$
4	$\tilde{f}(p-a)+\tilde{f}(p+a)$	$2f(x)\cosh(ax)$
5	$\tilde{f}(p-a)-\tilde{f}(p+a)$	$2f(x)\sinh(ax)$
6	$e^{-ap}\tilde{f}(p), \quad a\ge 0$	$\begin{cases} 0 & \text{if } 0\le x<a, \\ f(x-a) & \text{if } a<x. \end{cases}$
7	$p\tilde{f}(p)$	$\dfrac{df(x)}{dx}, \quad \text{if } f(+0)=0$
8	$\dfrac{1}{p}\tilde{f}(p)$	$\displaystyle\int_0^x f(t)\,dt$
9	$\dfrac{1}{p+a}\tilde{f}(p)$	$e^{-ax}\displaystyle\int_0^x e^{at}f(t)\,dt$
10	$\dfrac{1}{p^2}\tilde{f}(p)$	$\displaystyle\int_0^x (x-t)f(t)\,dt$
11	$\dfrac{\tilde{f}(p)}{p(p+a)}$	$\dfrac{1}{a}\displaystyle\int_0^x \big[1-e^{a(x-t)}\big]f(t)\,dt$
12	$\dfrac{\tilde{f}(p)}{(p+a)^2}$	$\displaystyle\int_0^x (x-t)e^{-a(x-t)}f(t)\,dt$
13	$\dfrac{\tilde{f}(p)}{(p+a)(p+b)}$	$\dfrac{1}{b-a}\displaystyle\int_0^x \big[e^{-a(x-t)}-e^{-b(x-t)}\big]f(t)\,dt$
14	$\dfrac{\tilde{f}(p)}{(p+a)^2+b^2}$	$\dfrac{1}{b}\displaystyle\int_0^x e^{-a(x-t)}\sin\big[b(x-t)\big]f(t)\,dt$
15	$\dfrac{1}{p^n}\tilde{f}(p), \quad n=1,\,2,\,\dots$	$\dfrac{1}{(n-1)!}\displaystyle\int_0^x (x-t)^{n-1}f(t)\,dt$
16	$\tilde{f}_1(p)\tilde{f}_2(p)$	$\displaystyle\int_0^x f_1(t)f_2(x-t)\,dt$

No	Laplace transform, $\tilde{f}(p)$	Inverse transform, $f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px}\tilde{f}(p)\,dp$
17	$\frac{1}{\sqrt{p}}\tilde{f}\Big(\frac{1}{p}\Big)$	$\int_0^\infty \frac{\cos\big(2\sqrt{xt}\,\big)}{\sqrt{\pi x}} f(t)\,dt$
18	$\frac{1}{p\sqrt{p}}\tilde{f}\Big(\frac{1}{p}\Big)$	$\int_0^\infty \frac{\sin\big(2\sqrt{xt}\,\big)}{\sqrt{\pi t}} f(t)\,dt$
19	$\frac{1}{p^{2\nu+1}}\tilde{f}\Big(\frac{1}{p}\Big)$	$\int_0^\infty (x/t)^\nu J_{2\nu}\big(2\sqrt{xt}\,\big) f(t)\,dt$
20	$\frac{1}{p}\tilde{f}\Big(\frac{1}{p}\Big)$	$\int_0^\infty J_0\big(2\sqrt{xt}\,\big) f(t)\,dt$
21	$\frac{1}{p}\tilde{f}\Big(p+\frac{1}{p}\Big)$	$\int_0^x J_0\big(2\sqrt{xt-t^2}\,\big) f(t)\,dt$
22	$\frac{1}{p^{2\nu+1}}\tilde{f}\Big(p+\frac{a}{p}\Big)$	$\int_0^x \Big(\frac{x-t}{at}\Big)^\nu J_{2\nu}\big(2\sqrt{axt-at^2}\,\big) f(t)\,dt$
23	$\tilde{f}\big(\sqrt{p}\big)$	$\int_0^\infty \frac{t}{2\sqrt{\pi x^3}} \exp\Big(-\frac{t^2}{4x}\Big) f(t)\,dt$
24	$\frac{1}{\sqrt{p}}\tilde{f}\big(\sqrt{p}\big)$	$\frac{1}{\sqrt{\pi x}}\int_0^\infty \exp\Big(-\frac{t^2}{4x}\Big) f(t)\,dt$
25	$\tilde{f}\big(p+\sqrt{p}\big)$	$\frac{1}{2\sqrt{\pi}}\int_0^x \frac{t}{(x-t)^{3/2}} \exp\Big[-\frac{t^2}{4(x-t)}\Big] f(t)\,dt$
26	$\tilde{f}\big(\sqrt{p^2+a^2}\,\big)$	$f(x)-a\int_0^x f\big(\sqrt{x^2-t^2}\,\big) J_1(at)\,dt$
27	$\tilde{f}\big(\sqrt{p^2-a^2}\,\big)$	$f(x)+a\int_0^x f\big(\sqrt{x^2-t^2}\,\big) I_1(at)\,dt$
28	$\frac{\tilde{f}\big(\sqrt{p^2+a^2}\,\big)}{\sqrt{p^2+a^2}}$	$\int_0^x J_0\big(a\sqrt{x^2-t^2}\,\big) f(t)\,dt$
29	$\frac{\tilde{f}\big(\sqrt{p^2-a^2}\,\big)}{\sqrt{p^2-a^2}}$	$\int_0^x I_0\big(a\sqrt{x^2-t^2}\,\big) f(t)\,dt$
30	$\tilde{f}\big(\sqrt{(p+a)^2-b^2}\,\big)$	$e^{-ax}f(x)+be^{-ax}\int_0^x f\big(\sqrt{x^2-t^2}\,\big) I_1(bt)\,dt$
31	$\tilde{f}(\ln p)$	$\int_0^\infty \frac{x^{t-1}}{\Gamma(t)} f(t)\,dt$
32	$\frac{1}{p}\tilde{f}(\ln p)$	$\int_0^\infty \frac{x^t}{\Gamma(t+1)} f(t)\,dt$
33	$\tilde{f}(p-ia)+\tilde{f}(p+ia),\quad i^2=-1$	$2f(x)\cos(ax)$
34	$i\big[\tilde{f}(p-ia)-\tilde{f}(p+ia)\big],\ i^2=-1$	$2f(x)\sin(ax)$
35	$\frac{d\tilde{f}(p)}{dp}$	$-xf(x)$
36	$\frac{d^n\tilde{f}(p)}{dp^n}$	$(-x)^n f(x)$

No	Laplace transform, $\tilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\tilde{f}(p)\,dp$
37	$p^n \dfrac{d^m \tilde{f}(p)}{dp^m}, \quad m \geq n$	$(-1)^m \dfrac{d^n}{dx^n}\left[x^m f(x)\right]$
38	$\displaystyle\int_p^{\infty} \tilde{f}(q)\,dq$	$\dfrac{1}{x} f(x)$
39	$\dfrac{1}{p}\displaystyle\int_0^p \tilde{f}(q)\,dq$	$\displaystyle\int_x^{\infty} \frac{f(t)}{t}\,dt$
40	$\dfrac{1}{p}\displaystyle\int_p^{\infty} \tilde{f}(q)\,dq$	$\displaystyle\int_0^x \frac{f(t)}{t}\,dt$

5.2. Expressions With Rational Functions

No	Laplace transform, $\tilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\tilde{f}(p)\,dp$
1	$\dfrac{1}{p}$	1
2	$\dfrac{1}{p+a}$	e^{-ax}
3	$\dfrac{1}{p^2}$	x
4	$\dfrac{1}{p(p+a)}$	$\dfrac{1}{a}\left(1 - e^{-ax}\right)$
5	$\dfrac{1}{(p+a)^2}$	xe^{-ax}
6	$\dfrac{p}{(p+a)^2}$	$(1 - ax)e^{-ax}$
7	$\dfrac{1}{p^2 - a^2}$	$\dfrac{1}{a}\sinh(ax)$
8	$\dfrac{p}{p^2 - a^2}$	$\cosh(ax)$
9	$\dfrac{1}{(p+a)(p+b)}$	$\dfrac{1}{a-b}\left(e^{-bx} - e^{-ax}\right)$
10	$\dfrac{p}{(p+a)(p+b)}$	$\dfrac{1}{a-b}\left(ae^{-ax} - be^{-bx}\right)$
11	$\dfrac{1}{p^2 + a^2}$	$\dfrac{1}{a}\sin(ax)$
12	$\dfrac{p}{p^2 + a^2}$	$\cos(ax)$
13	$\dfrac{1}{(p+b)^2 + a^2}$	$\dfrac{1}{a}e^{-bx}\sin(ax)$
14	$\dfrac{p}{(p+b)^2 + a^2}$	$e^{-bx}\left[\cos(ax) - \dfrac{b}{a}\sin(ax)\right]$

No	Laplace transform, $\tilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\tilde{f}(p)\,dp$
15	$\dfrac{1}{p^3}$	$\frac{1}{2}x^2$
16	$\dfrac{1}{p^2(p+a)}$	$\dfrac{1}{a^2}\left(e^{-ax}+ax-1\right)$
17	$\dfrac{1}{p(p+a)(p+b)}$	$\dfrac{1}{ab(a-b)}\left(a-b+be^{-ax}-ae^{-bx}\right)$
18	$\dfrac{1}{p(p+a)^2}$	$\dfrac{1}{a^2}\left(1-e^{-ax}-axe^{-ax}\right)$
19	$\dfrac{1}{(p+a)(p+b)(p+c)}$	$\dfrac{(c-b)e^{-ax}+(a-c)e^{-bx}+(b-a)e^{-cx}}{(a-b)(b-c)(c-a)}$
20	$\dfrac{p}{(p+a)(p+b)(p+c)}$	$\dfrac{a(b-c)e^{-ax}+b(c-a)e^{-bx}+c(a-b)e^{-cx}}{(a-b)(b-c)(c-a)}$
21	$\dfrac{p^2}{(p+a)(p+b)(p+c)}$	$\dfrac{a^2(c-b)e^{-ax}+b^2(a-c)e^{-bx}+c^2(b-a)e^{-cx}}{(a-b)(b-c)(c-a)}$
22	$\dfrac{1}{(p+a)(p+b)^2}$	$\dfrac{1}{(a-b)^2}\left[e^{-ax}-e^{-bx}+(a-b)xe^{-bx}\right]$
23	$\dfrac{p}{(p+a)(p+b)^2}$	$\dfrac{1}{(a-b)^2}\left\{-ae^{-ax}+\left[a+b(b-a)x\right]e^{-bx}\right\}$
24	$\dfrac{p^2}{(p+a)(p+b)^2}$	$\dfrac{1}{(a-b)^2}\left[a^2e^{-ax}+b(b-2a-b^2x+abx)e^{-bx}\right]$
25	$\dfrac{1}{(p+a)^3}$	$\frac{1}{2}x^2e^{-ax}$
26	$\dfrac{p}{(p+a)^3}$	$x\left(1-\frac{1}{2}ax\right)e^{-ax}$
27	$\dfrac{p^2}{(p+a)^3}$	$\left(1-2ax+\frac{1}{2}a^2x^2\right)e^{-ax}$
28	$\dfrac{1}{p(p^2+a^2)}$	$\dfrac{1}{a^2}\left[1-\cos(ax)\right]$
29	$\dfrac{1}{p\left[(p+b)^2+a^2\right]}$	$\dfrac{1}{a^2+b^2}\left\{1-e^{-bx}\left[\cos(ax)+\dfrac{b}{a}\sin(ax)\right]\right\}$
30	$\dfrac{1}{(p+a)(p^2+b^2)}$	$\dfrac{1}{a^2+b^2}\left[e^{-ax}+\dfrac{a}{b}\sin(bx)-\cos(bx)\right]$
31	$\dfrac{p}{(p+a)(p^2+b^2)}$	$\dfrac{1}{a^2+b^2}\left[-ae^{-ax}+a\cos(bx)+b\sin(bx)\right]$
32	$\dfrac{p^2}{(p+a)(p^2+b^2)}$	$\dfrac{1}{a^2+b^2}\left[a^2e^{-ax}-ab\sin(bx)+b^2\cos(bx)\right]$
33	$\dfrac{1}{p^3+a^3}$	$\dfrac{1}{3a^2}e^{-ax}-\dfrac{1}{3a^2}e^{ax/2}\left[\cos(kx)-\sqrt{3}\sin(kx)\right]$, $k=\frac{1}{2}a\sqrt{3}$
34	$\dfrac{p}{p^3+a^3}$	$-\dfrac{1}{3a}e^{-ax}+\dfrac{1}{3a}e^{ax/2}\left[\cos(kx)+\sqrt{3}\sin(kx)\right]$, $k=\frac{1}{2}a\sqrt{3}$

► Trigonometric functions of multiple arguments

$\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x,$ $\quad \sin 2x = 2\sin x\cos x,$

$\cos 3x = -3\cos x + 4\cos^3 x,$ $\quad \sin 3x = 3\sin x - 4\sin^3 x,$

$\cos 4x = 1 - 8\cos^2 x + 8\cos^4 x,$ $\quad \sin 4x = 4\cos x\,(\sin x - 2\sin^3 x),$

$\cos 5x = 5\cos x - 20\cos^3 x + 16\cos^5 x,$ $\quad \sin 5x = 5\sin x - 20\sin^3 x + 16\sin^5 x,$

$$\cos(2nx) = 1 + \sum_{k=1}^{n}(-1)^k \frac{n^2(n^2-1)\ldots[n^2-(k-1)^2]}{(2k)!}4^k\sin^{2k} x,$$

$$\cos[(2n+1)x] = \cos x\left\{1+\sum_{k=1}^{n}(-1)^k\frac{[(2n+1)^2-1][(2n+1)^2-3^2]\ldots[(2n+1)^2-(2k-1)^2]}{(2k)!}\sin^{2k} x\right\},$$

$$\sin(2nx) = 2n\cos x\left[\sin x + \sum_{k=1}^{n}(-4)^k\frac{(n^2-1)(n^2-2^2)\ldots(n^2-k^2)}{(2k-1)!}\sin^{2k-1} x\right],$$

$$\sin[(2n+1)x] = (2n+1)\left\{\sin x+\sum_{k=1}^{n}(-1)^k\frac{[(2n+1)^2-1][(2n+1)^2-3^2]\ldots[(2n+1)^2-(2k-1)^2]}{(2k+1)!}\sin^{2k+1} x\right\},$$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}, \qquad \tan 3x = \frac{3\tan x - \tan^3 x}{1-3\tan^2 x}, \qquad \tan 4x = \frac{4\tan x - 4\tan^3 x}{1-6\tan^2 x + \tan^4 x}.$$

► Trigonometric functions of half argument

$$\sin^2\frac{x}{2} = \frac{1-\cos x}{2}, \qquad \cos^2\frac{x}{2} = \frac{1+\cos x}{2},$$

$$\tan\frac{x}{2} = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}, \quad \cot\frac{x}{2} = \frac{\sin x}{1-\cos x} = \frac{1+\cos x}{\sin x},$$

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}, \quad \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}, \quad \tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}.$$

► Euler and de Moivre formulas. Relationship with hyperbolic functions

$$e^{y+ix} = e^y(\cos x + i\sin x), \quad (\cos x + i\sin x)^n = \cos(nx) + i\sin(nx), \quad i^2 = -1,$$

$$\sin(ix) = i\sinh x, \quad \cos(ix) = \cosh x, \quad \tan(ix) = i\tanh x, \quad \cot(ix) = -i\coth x.$$

► Differentiation formulas

$$\frac{d\sin x}{dx} = \cos x, \quad \frac{d\cos x}{dx} = -\sin x, \quad \frac{d\tan x}{dx} = \frac{1}{\cos^2 x}, \quad \frac{d\cot x}{dx} = -\frac{1}{\sin^2 x}.$$

► Expansion into power series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad (|x| < \infty),$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad (|x| < \infty),$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots \qquad (|x| < \pi/2),$$

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \cdots \qquad (|x| < \pi).$$

No	Laplace transform, $\tilde{f}(p)$	Inverse transform, $f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{px} \tilde{f}(p)\, dp$
54	$\frac{1}{(p^2+a^2)^2}$	$\frac{1}{2a^3}\left[\sin(ax) - ax\cos(ax)\right]$
55	$\frac{p}{(p^2+a^2)^2}$	$\frac{1}{2a} x \sin(ax)$
56	$\frac{p^2}{(p^2+a^2)^2}$	$\frac{1}{2a}\left[\sin(ax) + ax\cos(ax)\right]$
57	$\frac{p^3}{(p^2+a^2)^2}$	$\cos(ax) - \frac{1}{2} ax \sin(ax)$
58	$\frac{1}{\left[(p+b)^2+a^2\right]^2}$	$\frac{1}{2a^3} e^{-bx}\left[\sin(ax) - ax\cos(ax)\right]$
59	$\frac{1}{(p^2-a^2)(p^2-b^2)}$	$\frac{1}{a^2-b^2}\left[\frac{1}{a}\sinh(ax) - \frac{1}{b}\sinh(bx)\right]$
60	$\frac{p}{(p^2-a^2)(p^2-b^2)}$	$\frac{\cosh(ax) - \cosh(bx)}{a^2-b^2}$
61	$\frac{p^2}{(p^2-a^2)(p^2-b^2)}$	$\frac{a\sinh(ax) - b\sinh(bx)}{a^2-b^2}$
62	$\frac{p^3}{(p^2-a^2)(p^2-b^2)}$	$\frac{a^2\cosh(ax) - b^2\cosh(bx)}{a^2-b^2}$
63	$\frac{1}{(p^2+a^2)(p^2+b^2)}$	$\frac{1}{b^2-a^2}\left[\frac{1}{a}\sin(ax) - \frac{1}{b}\sin(bx)\right]$
64	$\frac{p}{(p^2+a^2)(p^2+b^2)}$	$\frac{\cos(ax) - \cos(bx)}{b^2-a^2}$
65	$\frac{p^2}{(p^2+a^2)(p^2+b^2)}$	$\frac{-a\sin(ax) + b\sin(bx)}{b^2-a^2}$
66	$\frac{p^3}{(p^2+a^2)(p^2+b^2)}$	$\frac{-a^2\cos(ax) + b^2\cos(bx)}{b^2-a^2}$
67	$\frac{1}{p^n}, \quad n = 1, 2, \ldots$	$\frac{1}{(n-1)!} x^{n-1}$
68	$\frac{1}{(p+a)^n}, \quad n = 1, 2, \ldots$	$\frac{1}{(n-1)!} x^{n-1} e^{-ax}$
69	$\frac{1}{p(p+a)^n}, \quad n = 1, 2, \ldots$	$a^{-n}\left[1 - e^{-ax} e_n(ax)\right], \quad e_n(z) = 1 + \frac{z}{1!} + \cdots + \frac{z^n}{n!}$
70	$\frac{1}{p^{2n}+a^{2n}}, \quad n = 1, 2, \ldots$	$-\frac{1}{na^{2n}} \sum_{k=1}^{n} \exp(a_k x)\left[a_k\cos(b_k x) - b_k\sin(b_k x)\right],$ $a_k = a\cos\varphi_k,\ b_k = a\sin\varphi_k,\ \varphi_k = \frac{\pi(2k-1)}{2n}$
71	$\frac{1}{p^{2n}-a^{2n}}, \quad n = 1, 2, \ldots$	$\frac{1}{na^{2n-1}}\sinh(ax) + \frac{1}{na^{2n}} \sum_{k=2}^{n} \exp(a_k x)$ $\times\left[a_k\cos(b_k x) - b_k\sin(b_k x)\right],$ $a_k = a\cos\varphi_k,\ b_k = a\sin\varphi_k,\ \varphi_k = \frac{\pi(k-1)}{n}$

No	Laplace transform, $\tilde{f}(p)$	Inverse transform, $f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px}\tilde{f}(p)\,dp$
72	$\frac{1}{p^{2n+1}+a^{2n+1}}, \quad n=0,\,1,\,\dots$	$\frac{e^{-ax}}{(2n+1)a^{2n}} - \frac{2}{(2n+1)a^{2n+1}}\sum_{k=1}^{n}\exp(a_k x)$ $\times\bigl[a_k\cos(b_k x) - b_k\sin(b_k x)\bigr]$, $a_k = a\cos\varphi_k,\ b_k = a\sin\varphi_k,\ \varphi_k = \frac{\pi(2k-1)}{2n+1}$
73	$\frac{1}{p^{2n+1}-a^{2n+1}}, \quad n=0,\,1,\,\dots$	$\frac{e^{ax}}{(2n+1)a^{2n}} + \frac{2}{(2n+1)a^{2n+1}}\sum_{k=1}^{n}\exp(a_k x)$ $\times\bigl[a_k\cos(b_k x) - b_k\sin(b_k x)\bigr]$, $a_k = a\cos\varphi_k,\ b_k = a\sin\varphi_k,\ \varphi_k = \frac{2\pi k}{2n+1}$
74	$\frac{Q(p)}{P(p)}$, $P(p) = (p-a_1)\dots(p-a_n)$; $Q(p)$ is a polynomial of degree $\le n-1$; $a_i \ne a_j$ if $i \ne j$	$\sum_{k=1}^{n}\frac{Q(a_k)}{P'(a_k)}\exp(a_k x)$, (the prime stand for the differentiation)
75	$\frac{Q(p)}{P(p)}$, $P(p) = (p-a_1)^{m_1}\dots(p-a_n)^{m_n}$; $Q(p)$ is a polynomial of degree $< m_1+m_2+\dots+m_n-1$; $a_i \ne a_j$ if $i \ne j$	$\sum_{k=1}^{n}\sum_{l=1}^{m_k}\frac{\Phi_{kl}(a_k)}{(m_k-l)!\,(l-1)!}x^{m_k-l}\exp(a_k x)$, $\Phi_{kl}(p) = \frac{d^{l-1}}{dp^{l-1}}\left[\frac{Q(p)}{P_k(p)}\right], \quad P_k(p) = \frac{P(p)}{(p-a_k)^{m_k}}$
76	$\frac{Q(p)+pR(p)}{P(p)}$, $P(p) = (p^2+a_1^2)\dots(p^2+a_n^2)$; $Q(p)$ and $R(p)$ are polynomials of degree $\le 2n-2$; $a_l \ne a_j,\ l \ne j$	$\sum_{k=1}^{n}\frac{Q(ia_k)\sin(a_k x) + a_k R(ia_k)\cos(a_k x)}{a_k P_k(ia_k)}$, $P_m(p) = \frac{P(p)}{p^2+a_m^2}, \quad i^2 = -1$

5.3. Expressions With Square Roots

No	Laplace transform, $\tilde{f}(p)$	Inverse transform, $f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px}\tilde{f}(p)\,dp$
1	$\frac{1}{\sqrt{p}}$	$\frac{1}{\sqrt{\pi x}}$
2	$\sqrt{p-a} - \sqrt{p-b}$	$\frac{e^{bx}-e^{ax}}{2\sqrt{\pi x^3}}$
3	$\frac{1}{\sqrt{p+a}}$	$\frac{1}{\sqrt{\pi x}}e^{-ax}$
4	$\sqrt{\frac{p+a}{p}} - 1$	$\frac{1}{2}ae^{-ax/2}\bigl[I_1\bigl(\tfrac{1}{2}ax\bigr) + I_0\bigl(\tfrac{1}{2}ax\bigr)\bigr]$
5	$\frac{\sqrt{p+a}}{p+b}$	$\frac{e^{-ax}}{\sqrt{\pi x}} + (a-b)^{1/2}e^{-bx}\operatorname{erf}\bigl[(a-b)^{1/2}x^{1/2}\bigr]$

No	Laplace transform, $\tilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px}\tilde{f}(p)\,dp$
6	$\dfrac{1}{p\sqrt{p}}$	$2\sqrt{\dfrac{x}{\pi}}$
7	$\dfrac{1}{(p+a)\sqrt{p+b}}$	$(b-a)^{-1/2}e^{-ax}\operatorname{erf}\left[(b-a)^{1/2}x^{1/2}\right]$
8	$\dfrac{1}{\sqrt{p}\,(p-a)}$	$\dfrac{1}{\sqrt{a}}e^{ax}\operatorname{erf}\left(\sqrt{ax}\right)$
9	$\dfrac{1}{p^{3/2}(p-a)}$	$a^{-3/2}e^{ax}\operatorname{erf}\left(\sqrt{ax}\right) - 2a^{-1}\pi^{-1/2}x^{1/2}$
10	$\dfrac{1}{\sqrt{p}+a}$	$\pi^{-1/2}x^{-1/2} - ae^{a^2x}\operatorname{erfc}\left(a\sqrt{x}\right)$
11	$\dfrac{a}{p\left(\sqrt{p}+a\right)}$	$1 - e^{a^2x}\operatorname{erfc}\left(a\sqrt{x}\right)$
12	$\dfrac{1}{p+a\sqrt{p}}$	$e^{a^2x}\operatorname{erfc}\left(a\sqrt{x}\right)$
13	$\dfrac{1}{\left(\sqrt{p}+\sqrt{a}\right)^2}$	$1 - \dfrac{2}{\sqrt{\pi}}(ax)^{1/2} + (1-2ax)e^{ax}\left[\operatorname{erf}\left(\sqrt{ax}\right) - 1\right]$
14	$\dfrac{1}{p\left(\sqrt{p}+\sqrt{a}\right)^2}$	$\dfrac{1}{a} + \left(2x - \dfrac{1}{a}\right)e^{ax}\operatorname{erfc}\left(\sqrt{ax}\right) - \dfrac{2}{\sqrt{\pi a}}\sqrt{x}$
15	$\dfrac{1}{\sqrt{p}\left(\sqrt{p}+a\right)^2}$	$2\pi^{-1/2}x^{1/2} - 2axe^{a^2x}\operatorname{erfc}\left(a\sqrt{x}\right)$
16	$\dfrac{1}{\left(\sqrt{p}+a\right)^3}$	$\dfrac{2}{\sqrt{\pi}}(a^2x+1)\sqrt{x} - ax(2a^2x+3)e^{a^2x}\operatorname{erfc}\left(a\sqrt{x}\right)$
17	$p^{-n-1/2}, \quad n = 1, 2, \ldots$	$\dfrac{2^n}{1\cdot 3\ldots(2n-1)\sqrt{\pi}}\,x^{n-1/2}$
18	$(p+a)^{-n-1/2}$	$\dfrac{2^n}{1\cdot 3\ldots(2n-1)\sqrt{\pi}}\,x^{n-1/2}e^{-ax}$
19	$\dfrac{1}{\sqrt{p^2+a^2}}$	$J_0(ax)$
20	$\dfrac{1}{\sqrt{p^2-a^2}}$	$I_0(ax)$
21	$\dfrac{1}{\sqrt{p^2+ap+b}}$	$\exp\left(-\tfrac{1}{2}ax\right)J_0\left[\left(b-\tfrac{1}{4}a^2\right)^{1/2}x\right]$
22	$\left(\sqrt{p^2+a^2}-p\right)^{1/2}$	$\dfrac{1}{\sqrt{2\pi x^3}}\sin(ax)$
23	$\dfrac{1}{\sqrt{p^2+a^2}}\left(\sqrt{p^2+a^2}+p\right)^{1/2}$	$\dfrac{\sqrt{2}}{\sqrt{\pi x}}\cos(ax)$
24	$\dfrac{1}{\sqrt{p^2-a^2}}\left(\sqrt{p^2-a^2}+p\right)^{1/2}$	$\dfrac{\sqrt{2}}{\sqrt{\pi x}}\cosh(ax)$

No	Laplace transform, $\tilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\tilde{f}(p)\,dp$
25	$\left(\sqrt{p^2+a^2}+p\right)^{-n}$	$na^{-n}x^{-1}J_n(ax)$
26	$\left(\sqrt{p^2-a^2}+p\right)^{-n}$	$na^{-n}x^{-1}I_n(ax)$
27	$\left(p^2+a^2\right)^{-n-1/2}$	$\dfrac{(x/a)^n J_n(ax)}{1\cdot3\cdot5\ldots(2n-1)}$
28	$\left(p^2-a^2\right)^{-n-1/2}$	$\dfrac{(x/a)^n I_n(ax)}{1\cdot3\cdot5\ldots(2n-1)}$

5.4. Expressions With Arbitrary Powers

No	Laplace transform, $\tilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\tilde{f}(p)\,dp$
1	$(p+a)^{-\nu}$, $\nu>0$	$\dfrac{1}{\Gamma(\nu)}x^{\nu-1}e^{-ax}$
2	$\left[(p+a)^{1/2}+(p+b)^{1/2}\right]^{-2\nu}$, $\nu>0$	$\dfrac{\nu}{(a-b)^\nu}x^{-1}\exp\left[-\tfrac{1}{2}(a+b)x\right]I_\nu\left[\tfrac{1}{2}(a-b)x\right]$
3	$\left[(p+a)(p+b)\right]^{-\nu}$, $\nu>0$	$\dfrac{\sqrt{\pi}}{\Gamma(\nu)}\left(\dfrac{x}{a-b}\right)^{\nu-1/2}\exp\left(-\dfrac{a+b}{2}x\right)I_{\nu-1/2}\left(\dfrac{a-b}{2}x\right)$
4	$\left(p^2+a^2\right)^{-\nu-1/2}$, $\nu>-\frac{1}{2}$	$\dfrac{\sqrt{\pi}}{(2a)^\nu\Gamma(\nu+\frac{1}{2})}x^\nu J_\nu(ax)$
5	$\left(p^2-a^2\right)^{-\nu-1/2}$, $\nu>-\frac{1}{2}$	$\dfrac{\sqrt{\pi}}{(2a)^\nu\Gamma(\nu+\frac{1}{2})}x^\nu I_\nu(ax)$
6	$p\left(p^2+a^2\right)^{-\nu-1/2}$, $\nu>0$	$\dfrac{a\sqrt{\pi}}{(2a)^\nu\Gamma\left(\nu+\frac{1}{2}\right)}x^\nu J_{\nu-1}(ax)$
7	$p\left(p^2-a^2\right)^{-\nu-1/2}$, $\nu>0$	$\dfrac{a\sqrt{\pi}}{(2a)^\nu\Gamma\left(\nu+\frac{1}{2}\right)}x^\nu I_{\nu-1}(ax)$
8	$\left[(p^2+a^2)^{1/2}+p\right]^{-\nu}=$ $a^{-2\nu}\left[(p^2+a^2)^{1/2}-p\right]^{\nu}$, $\nu>0$	$\nu a^{-\nu}x^{-1}J_\nu(ax)$
9	$\left[(p^2-a^2)^{1/2}+p\right]^{-\nu}=$ $a^{-2\nu}\left[p-(p^2-a^2)^{1/2}\right]^{\nu}$, $\nu>0$	$\nu a^{-\nu}x^{-1}I_\nu(ax)$
10	$p\left[(p^2+a^2)^{1/2}+p\right]^{-\nu}$, $\nu>1$	$\nu a^{1-\nu}x^{-1}J_{\nu-1}(ax)-\nu(\nu+1)a^{-\nu}x^{-2}J_\nu(ax)$
11	$p\left[(p^2-a^2)^{1/2}+p\right]^{-\nu}$, $\nu>1$	$\nu a^{1-\nu}x^{-1}I_{\nu-1}(ax)-\nu(\nu+1)a^{-\nu}x^{-2}I_\nu(ax)$
12	$\dfrac{\left(\sqrt{p^2+a^2}+p\right)^{-\nu}}{\sqrt{p^2+a^2}}$, $\nu>-1$	$a^{-\nu}J_\nu(ax)$
13	$\dfrac{\left(\sqrt{p^2-a^2}+p\right)^{-\nu}}{\sqrt{p^2-a^2}}$, $\nu>-1$	$a^{-\nu}I_\nu(ax)$

5.5. Expressions With Exponential Functions

No	Laplace transform, $\tilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\tilde{f}(p)\,dp$
1	$p^{-1}e^{-ap}, \quad a>0$	$\begin{cases} 0 & \text{if } 0<x<a, \\ 1 & \text{if } a<x. \end{cases}$
2	$p^{-1}\left(1-e^{-ap}\right), \quad a>0$	$\begin{cases} 1 & \text{if } 0<x<a, \\ 0 & \text{if } a<x. \end{cases}$
3	$p^{-1}\left(e^{-ap}-e^{-bp}\right), \quad 0\le a<b$	$\begin{cases} 0 & \text{if } 0<x<a, \\ 1 & \text{if } a<x<b, \\ 0 & \text{if } b<x. \end{cases}$
4	$p^{-2}\left(e^{-ap}-e^{-bp}\right), \quad 0\le a<b$	$\begin{cases} 0 & \text{if } 0<x<a, \\ x-a & \text{if } a<x<b, \\ b-a & \text{if } b<x. \end{cases}$
5	$(p+b)^{-1}e^{-ap}, \quad a>0$	$\begin{cases} 0 & \text{if } 0<x<a, \\ e^{-b(x-a)} & \text{if } a<x. \end{cases}$
6	$p^{-\nu}e^{-ap}, \quad \nu>0$	$\begin{cases} 0 & \text{if } 0<x<a, \\ \dfrac{(x-a)^{\nu-1}}{\Gamma(\nu)} & \text{if } a<x. \end{cases}$
7	$p^{-1}\left(e^{ap}-1\right)^{-1}, \quad a>0$	$f(x)=n \quad \text{if} \quad na<x<(n+1)a; \quad n=0,\,1,\,2,\,\ldots$
8	$e^{a/p}-1$	$\sqrt{\dfrac{a}{x}}\,I_1\big(2\sqrt{ax}\big)$
9	$p^{-1/2}e^{a/p}$	$\dfrac{1}{\sqrt{\pi x}}\cosh\big(2\sqrt{ax}\big)$
10	$p^{-3/2}e^{a/p}$	$\dfrac{1}{\sqrt{\pi a}}\sinh\big(2\sqrt{ax}\big)$
11	$p^{-5/2}e^{a/p}$	$\sqrt{\dfrac{x}{\pi a}}\cosh\big(2\sqrt{ax}\big)-\dfrac{1}{2\sqrt{\pi a^3}}\sinh\big(2\sqrt{ax}\big)$
12	$p^{-\nu-1}e^{a/p}, \quad \nu>-1$	$(x/a)^{\nu/2}I_\nu(2\sqrt{ax})$
13	$1-e^{-a/p}$	$\sqrt{\dfrac{a}{x}}\,J_1\big(2\sqrt{ax}\big)$
14	$p^{-1/2}e^{-a/p}$	$\dfrac{1}{\sqrt{\pi x}}\cos\big(2\sqrt{ax}\big)$
15	$p^{-3/2}e^{-a/p}$	$\dfrac{1}{\sqrt{\pi a}}\sin\big(2\sqrt{ax}\big)$
16	$p^{-5/2}e^{-a/p}$	$\dfrac{1}{2\sqrt{\pi a^3}}\sin\big(2\sqrt{ax}\big)-\sqrt{\dfrac{x}{\pi a}}\cos\big(2\sqrt{ax}\big)$
17	$p^{-\nu-1}e^{-a/p}, \quad \nu>-1$	$(x/a)^{\nu/2}J_\nu(2\sqrt{ax})$
18	$\exp\big(-\sqrt{ap}\big), \quad a>0$	$\dfrac{\sqrt{a}}{2\sqrt{\pi}}\,x^{-3/2}\exp\Big(-\dfrac{a}{4x}\Big)$
19	$p\exp\big(-\sqrt{ap}\big), \quad a>0$	$\dfrac{\sqrt{a}}{8\sqrt{\pi}}\,(a-6x)x^{-7/2}\exp\Big(-\dfrac{a}{4x}\Big)$
20	$\dfrac{1}{p}\exp\big(-\sqrt{ap}\big), \quad a\ge 0$	$\operatorname{erfc}\Big(\dfrac{\sqrt{a}}{2\sqrt{x}}\Big)$

No	Laplace transform, $\tilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\tilde{f}(p)\,dp$
21	$\sqrt{p}\exp\bigl(-\sqrt{ap}\,\bigr), \quad a > 0$	$\dfrac{1}{4\sqrt{\pi}}\,(a-2x)x^{-5/2}\exp\Bigl(-\dfrac{a}{4x}\Bigr)$
22	$\dfrac{1}{\sqrt{p}}\exp\bigl(-\sqrt{ap}\,\bigr), \quad a \ge 0$	$\dfrac{1}{\sqrt{\pi x}}\exp\Bigl(-\dfrac{a}{4x}\Bigr)$
23	$\dfrac{1}{p\sqrt{p}}\exp\bigl(-\sqrt{ap}\,\bigr), \quad a \ge 0$	$\dfrac{2\sqrt{x}}{\sqrt{\pi}}\exp\Bigl(-\dfrac{a}{4x}\Bigr) - \sqrt{a}\,\operatorname{erfc}\Bigl(\dfrac{\sqrt{a}}{2\sqrt{x}}\Bigr)$
24	$\dfrac{\exp\bigl(-k\sqrt{p^2+a^2}\,\bigr)}{\sqrt{p^2+a^2}}, \quad k > 0$	$\begin{cases} 0 & \text{if } 0 < x < k, \\ J_0\bigl(a\sqrt{x^2-k^2}\,\bigr) & \text{if } k < x. \end{cases}$
25	$\dfrac{\exp\bigl(-k\sqrt{p^2-a^2}\,\bigr)}{\sqrt{p^2-a^2}}, \quad k > 0$	$\begin{cases} 0 & \text{if } 0 < x < k, \\ I_0\bigl(a\sqrt{x^2-k^2}\,\bigr) & \text{if } k < x. \end{cases}$

5.6. Expressions With Hyperbolic Functions

No	Laplace transform, $\tilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\tilde{f}(p)\,dp$
1	$\dfrac{1}{p\sinh(ap)}, \quad a > 0$	$f(x) = 2n$ if $a(2n-1) < x < a(2n+1)$; $n = 0, 1, 2, \ldots \ (x > 0)$
2	$\dfrac{1}{p^2\sinh(ap)}, \quad a > 0$	$f(x) = 2n(x-an)$ if $a(2n-1) < x < a(2n+1)$; $n = 0, 1, 2, \ldots \ (x > 0)$
3	$\dfrac{\sinh(a/p)}{\sqrt{p}}$	$\dfrac{1}{2\sqrt{\pi x}}\bigl[\cosh\bigl(2\sqrt{ax}\,\bigr) - \cos\bigl(2\sqrt{ax}\,\bigr)\bigr]$
4	$\dfrac{\sinh(a/p)}{p\sqrt{p}}$	$\dfrac{1}{2\sqrt{\pi a}}\bigl[\sinh\bigl(2\sqrt{ax}\,\bigr) - \sin\bigl(2\sqrt{ax}\,\bigr)\bigr]$
5	$p^{-\nu-1}\sinh(a/p), \quad \nu > -2$	$\frac{1}{2}(x/a)^{\nu/2}\bigl[I_\nu\bigl(2\sqrt{ax}\,\bigr) - J_\nu\bigl(2\sqrt{ax}\,\bigr)\bigr]$
6	$\dfrac{1}{p\cosh(ap)}, \quad a > 0$	$f(x) = \begin{cases} 0 & \text{if } a(4n-1) < x < a(4n+1), \\ 2 & \text{if } a(4n+1) < x < a(4n+3), \end{cases}$ $n = 0, 1, 2, \ldots \ (x > 0)$
7	$\dfrac{1}{p^2\cosh(ap)}, \quad a > 0$	$x - (-1)^n(x-2an)$ if $2n-1 < x/a < 2n+1$; $n = 0, 1, 2, \ldots \ (x > 0)$
8	$\dfrac{\cosh(a/p)}{\sqrt{p}}$	$\dfrac{1}{2\sqrt{\pi x}}\bigl[\cosh\bigl(2\sqrt{ax}\,\bigr) + \cos\bigl(2\sqrt{ax}\,\bigr)\bigr]$
9	$\dfrac{\cosh(a/p)}{p\sqrt{p}}$	$\dfrac{1}{2\sqrt{\pi a}}\bigl[\sinh\bigl(2\sqrt{ax}\,\bigr) + \sin\bigl(2\sqrt{ax}\,\bigr)\bigr]$
10	$p^{-\nu-1}\cosh(a/p), \quad \nu > -1$	$\frac{1}{2}(x/a)^{\nu/2}\bigl[I_\nu\bigl(2\sqrt{ax}\,\bigr) + J_\nu\bigl(2\sqrt{ax}\,\bigr)\bigr]$
11	$\dfrac{1}{p}\tanh(ap), \quad a > 0$	$f(x) = (-1)^{n-1}$ if $2a(n-1) < x < 2an$; $n = 1, 2, \ldots$

No	Laplace transform, $\tilde{f}(p)$	Inverse transform, $f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px}\tilde{f}(p)\,dp$
12	$\frac{1}{p}\coth(ap), \quad a > 0$	$f(x) = (2n-1)$ if $2a(n-1) < x < 2an$; $n = 1, 2, \ldots$
13	$\operatorname{Arcoth}(p/a)$	$\frac{1}{x}\sinh(ax)$

5.7. Expressions With Logarithmic Functions

No	Laplace transform, $\tilde{f}(p)$	Inverse transform, $f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px}\tilde{f}(p)\,dp$
1	$\frac{1}{p}\ln p$	$-\ln x - \mathcal{C}$, $\mathcal{C} = 0.5772\ldots$ is the Euler constant
2	$p^{-n-1}\ln p$	$\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln x - \mathcal{C}\right)\frac{x^n}{n!}$, $\mathcal{C} = 0.5772\ldots$ is the Euler constant
3	$p^{-n-1/2}\ln p$	$k_n\left[2 + \frac{2}{3} + \frac{2}{5} + \cdots + \frac{2}{2n-1} - \ln(4x) - \mathcal{C}\right]x^{n-1/2}$, $k_n = \frac{2^n}{1\cdot 3\cdot 5\ldots(2n-1)\sqrt{\pi}}, \quad \mathcal{C} = 0.5772\ldots$
4	$p^{-\nu}\ln p, \quad \nu > 0$	$\frac{1}{\Gamma(\nu)}x^{\nu-1}\left[\psi(\nu) - \ln x\right], \quad \psi(\nu)$ is the logarithmic derivative of the gamma function
5	$\frac{1}{p}(\ln p)^2$	$(\ln x + \mathcal{C})^2 - \frac{1}{6}\pi^2, \quad \mathcal{C} = 0.5772\ldots$
6	$\frac{1}{p^2}(\ln p)^2$	$x\left[(\ln x + \mathcal{C} - 1)^2 + 1 - \frac{1}{6}\pi^2\right]$
7	$\frac{\ln(p+b)}{p+a}$	$e^{-ax}\left\{\ln(b-a) - \operatorname{Ei}\left[(a-b)x\right]\right\}$
8	$\frac{\ln p}{p^2 + a^2}$	$\frac{1}{a}\cos(ax)\operatorname{Si}(ax) + \frac{1}{a}\sin(ax)\left[\ln a - \operatorname{Ci}(ax)\right]$
9	$\frac{p\ln p}{p^2 + a^2}$	$\cos(ax)\left[\ln a - \operatorname{Ci}(ax)\right] - \sin(ax)\operatorname{Si}(ax)]$
10	$\ln\frac{p+b}{p+a}$	$\frac{1}{x}\left(e^{-ax} - e^{-bx}\right)$
11	$\ln\frac{p^2 + b^2}{p^2 + a^2}$	$\frac{2}{x}\left[\cos(ax) - \cos(bx)\right]$
12	$p\ln\frac{p^2 + b^2}{p^2 + a^2}$	$\frac{2}{x}\left[\cos(bx) + bx\sin(bx) - \cos(ax) - ax\sin(ax)\right]$
13	$\ln\frac{(p+a)^2 + k^2}{(p+b)^2 + k^2}$	$\frac{2}{x}\cos(kx)(e^{-bx} - e^{-ax})$
14	$p\ln\left(\frac{1}{p}\sqrt{p^2 + a^2}\right)$	$\frac{1}{x^2}\left[\cos(ax) - 1\right] + \frac{a}{x}\sin(ax)$
15	$p\ln\left(\frac{1}{p}\sqrt{p^2 - a^2}\right)$	$\frac{1}{x^2}\left[\cosh(ax) - 1\right] - \frac{a}{x}\sinh(ax)$

5.8. Expressions With Trigonometric Functions

No	Laplace transform, $\tilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px}\tilde{f}(p)\,dp$
1	$\dfrac{\sin(a/p)}{\sqrt{p}}$	$\dfrac{1}{\sqrt{\pi x}}\sinh\bigl(\sqrt{2ax}\,\bigr)\sin\bigl(\sqrt{2ax}\,\bigr)$
2	$\dfrac{\sin(a/p)}{p\sqrt{p}}$	$\dfrac{1}{\sqrt{\pi a}}\cosh\bigl(\sqrt{2ax}\,\bigr)\sin\bigl(\sqrt{2ax}\,\bigr)$
3	$\dfrac{\cos(a/p)}{\sqrt{p}}$	$\dfrac{1}{\sqrt{\pi x}}\cosh\bigl(\sqrt{2ax}\,\bigr)\cos\bigl(\sqrt{2ax}\,\bigr)$
4	$\dfrac{\cos(a/p)}{p\sqrt{p}}$	$\dfrac{1}{\sqrt{\pi a}}\sinh\bigl(\sqrt{2ax}\,\bigr)\cos\bigl(\sqrt{2ax}\,\bigr)$
5	$\dfrac{1}{\sqrt{p}}\exp\bigl(-\sqrt{ap}\,\bigr)\sin\bigl(\sqrt{ap}\,\bigr)$	$\dfrac{1}{\sqrt{\pi x}}\sin\Bigl(\dfrac{a}{2x}\Bigr)$
6	$\dfrac{1}{\sqrt{p}}\exp\bigl(-\sqrt{ap}\,\bigr)\cos\bigl(\sqrt{ap}\,\bigr)$	$\dfrac{1}{\sqrt{\pi x}}\cos\Bigl(\dfrac{a}{2x}\Bigr)$
7	$\arctan\dfrac{a}{p}$	$\dfrac{1}{x}\sin(ax)$
8	$\dfrac{1}{p}\arctan\dfrac{a}{p}$	$\operatorname{Si}(ax)$
9	$p\arctan\dfrac{a}{p} - a$	$\dfrac{1}{x^2}\bigl[ax\cos(ax) - \sin(ax)\bigr]$
10	$\arctan\dfrac{2ap}{p^2 + b^2}$	$\dfrac{2}{x}\sin(ax)\cos\bigl(x\sqrt{a^2 + b^2}\,\bigr)$

5.9. Expressions With Special Functions

No	Laplace transform, $\tilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px}\tilde{f}(p)\,dp$
1	$\exp\bigl(ap^2\bigr)\operatorname{erfc}\bigl(p\sqrt{a}\,\bigr)$	$\dfrac{1}{\sqrt{\pi a}}\exp\Bigl(-\dfrac{x^2}{4a}\Bigr)$
2	$\dfrac{1}{p}\exp\bigl(ap^2\bigr)\operatorname{erfc}\bigl(p\sqrt{a}\,\bigr)$	$\operatorname{erf}\Bigl(\dfrac{x}{2\sqrt{a}}\Bigr)$
3	$\operatorname{erfc}\bigl(\sqrt{ap}\,\bigr), \quad a > 0$	$\begin{cases} 0 & \text{if } 0 < x < a, \\ \dfrac{\sqrt{a}}{\pi x\sqrt{x-a}} & \text{if } a < x. \end{cases}$
4	$e^{ap}\operatorname{erfc}\bigl(\sqrt{ap}\,\bigr)$	$\dfrac{\sqrt{a}}{\pi\sqrt{x}\,(x+a)}$
5	$\dfrac{1}{\sqrt{p}}e^{ap}\operatorname{erfc}\bigl(\sqrt{ap}\,\bigr)$	$\dfrac{1}{\sqrt{\pi(x+a)}}$
6	$\operatorname{erf}\bigl(\sqrt{a/p}\,\bigr)$	$\dfrac{1}{\pi x}\sin\bigl(2\sqrt{ax}\,\bigr)$

No	Laplace transform, $\tilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px}\tilde{f}(p)\,dp$
7	$\dfrac{1}{\sqrt{p}}\exp(a/p)\,\mathrm{erf}\bigl(\sqrt{a/p}\,\bigr)$	$\dfrac{1}{\sqrt{\pi x}}\sinh\bigl(2\sqrt{ax}\,\bigr)$
8	$\dfrac{1}{\sqrt{p}}\exp(a/p)\,\mathrm{erfc}\bigl(\sqrt{a/p}\,\bigr)$	$\dfrac{1}{\sqrt{\pi x}}\exp\bigl(-2\sqrt{ax}\,\bigr)$
9	$p^{-a}\gamma(a, bp), \quad a, b > 0$	$\begin{cases} x^{a-1} & \text{if } 0 < x < b, \\ 0 & \text{if } b < x. \end{cases}$
10	$\gamma(a, b/p), \quad a > 0$	$b^{a/2}x^{a/2-1}J_a\bigl(2\sqrt{bx}\,\bigr)$
11	$a^{-p}\gamma(p, a)$	$\exp\bigl(-ae^{-x}\bigr)$
12	$K_0(ap), \quad a > 0$	$\begin{cases} 0 & \text{if } 0 < x < a, \\ (x^2 - a^2)^{-1/2} & \text{if } a < x. \end{cases}$
13	$K_\nu(ap), \quad a > 0$	$\begin{cases} 0 & \text{if } 0 < x < a, \\ \dfrac{\cosh\bigl[\nu\,\mathrm{Arcosh}(x/a)\bigr]}{\sqrt{x^2 - a^2}} & \text{if } a < x. \end{cases}$
14	$K_0\bigl(a\sqrt{p}\bigr)$	$\dfrac{1}{2x}\exp\Bigl(-\dfrac{a^2}{4x}\Bigr)$
15	$\dfrac{1}{\sqrt{p}}K_1\bigl(a\sqrt{p}\,\bigr)$	$\dfrac{1}{a}\exp\Bigl(-\dfrac{a^2}{4x}\Bigr)$

⦿ References for Supplement 5: G. Doetsch (1950, 1956, 1958), H. Bateman and A. Erdélyi (1954), I. I. Hirschman and D. V. Widder (1955), V. A. Ditkin and A. P. Prudnikov (1965).

Supplement 6

Tables of Fourier Cosine Transforms

6.1. General Formulas

No	Original function, $f(x)$	Cosine transform, $\check{f}_c(u) = \int_0^\infty f(x)\cos(ux)\,dx$
1	$af_1(x) + bf_2(x)$	$a\check{f}_{1c}(u) + b\check{f}_{2c}(u)$
2	$f(ax), \quad a > 0$	$\frac{1}{a}\check{f}_c\left(\frac{u}{a}\right)$
3	$x^{2n}f(x), \quad n = 1, 2, \ldots$	$(-1)^n \frac{d^{2n}}{du^{2n}}\check{f}_c(u)$
4	$x^{2n+1}f(ax), \quad n = 0, 1, \ldots$	$(-1)^n \frac{d^{2n+1}}{du^{2n+1}}\check{f}_s(u), \quad \check{f}_s(u) = \int_0^\infty f(x)\sin(xu)\,dx$
5	$f(ax)\cos(bx), \quad a, b > 0$	$\frac{1}{2a}\left[\check{f}_c\left(\frac{u+b}{a}\right) + \check{f}_c\left(\frac{u-b}{a}\right)\right]$

6.2. Expressions With Power-Law Functions

No	Original function, $f(x)$	Cosine transform, $\check{f}_c(u) = \int_0^\infty f(x)\cos(ux)\,dx$
1	$\begin{cases} 1 & \text{if } 0 < x < a, \\ 0 & \text{if } a < x \end{cases}$	$\frac{1}{u}\sin(au)$
2	$\begin{cases} x & \text{if } 0 < x < 1, \\ 2-x & \text{if } 1 < x < 2, \\ 0 & \text{if } 2 < x \end{cases}$	$\frac{4}{u^2}\cos u \sin^2 \frac{u}{2}$
3	$\frac{1}{a+x}, \quad a > 0$	$-\sin(au)\operatorname{si}(au) - \cos(au)\operatorname{Ci}(au)$
4	$\frac{1}{a^2+x^2}, \quad a > 0$	$\frac{\pi}{2a}e^{-au}$ (the integral is understood in the sense of Cauchy principal value)
5	$\frac{1}{a^2-x^2}, \quad a > 0$	$\frac{\pi\sin(au)}{2u}$
6	$\frac{a}{a^2+(b+x)^2} + \frac{a}{a^2+(b-x)^2}$	$\pi e^{-au}\cos(bu)$
7	$\frac{b+x}{a^2+(b+x)^2} + \frac{b-x}{a^2+(b-x)^2}$	$\pi e^{-au}\sin(bu)$
8	$\frac{1}{a^4+x^4}, \quad a > 0$	$\frac{1}{2}\pi a^{-3}\exp\left(-\frac{au}{\sqrt{2}}\right)\sin\left(\frac{\pi}{4} + \frac{au}{\sqrt{2}}\right)$

No	Original function, $f(x)$	Cosine transform, $\check{f}_c(u) = \int_0^\infty f(x)\cos(ux)\,dx$
9	$\dfrac{1}{(a^2+x^2)(b^2+x^2)}, \quad a,b>0$	$\dfrac{\pi}{2}\dfrac{ae^{-bu}-be^{-au}}{ab(a^2-b^2)}$
10	$\dfrac{x^{2m}}{(x^2+a)^{n+1}}$, $n,m=1,\,2,\,\ldots;\ \ n+1>m\ge 0$	$(-1)^{n+m}\dfrac{\pi}{2n!}\dfrac{\partial^n}{\partial a^n}\left(a^{1/\sqrt{m}}e^{-u\sqrt{a}}\right)$
11	$\dfrac{1}{\sqrt{x}}$	$\sqrt{\dfrac{\pi}{2u}}$
12	$\begin{cases}\dfrac{1}{\sqrt{x}} & \text{if } 0<x<a,\\ 0 & \text{if } a<x\end{cases}$	$2\sqrt{\dfrac{\pi}{2u}}\,C(au)$, $C(u)$ is the Fresnel integral
13	$\begin{cases}0 & \text{if } 0<x<a,\\ \dfrac{1}{\sqrt{x}} & \text{if } a<x\end{cases}$	$\sqrt{\dfrac{\pi}{2u}}\left[1-2C(au)\right]$, $C(u)$ is the Fresnel integral
14	$\begin{cases}0 & \text{if } 0<x<a,\\ \dfrac{1}{\sqrt{x-a}} & \text{if } a<x\end{cases}$	$\sqrt{\dfrac{\pi}{2u}}\left[\cos(au)-\sin(au)\right]$
15	$\dfrac{1}{\sqrt{a^2+x^2}}$	$K_0(au)$
16	$\begin{cases}\dfrac{1}{\sqrt{a^2-x^2}} & \text{if } 0<x<a,\\ 0 & \text{if } a<x\end{cases}$	$\dfrac{\pi}{2}J_0(au)$
17	$x^{-\nu}, \quad 0<\nu<1$	$\sin\left(\tfrac{1}{2}\pi\nu\right)\Gamma(1-\nu)u^{\nu-1}$

6.3. Expressions With Exponential Functions

No	Original function, $f(x)$	Cosine transform, $\check{f}_c(u) = \int_0^\infty f(x)\cos(ux)\,dx$
1	e^{-ax}	$\dfrac{a}{a^2+u^2}$
2	$\dfrac{1}{x}\left(e^{-ax}-e^{-bx}\right)$	$\dfrac{1}{2}\ln\dfrac{b^2+u^2}{a^2+u^2}$
3	$\sqrt{x}e^{-ax}$	$\tfrac{1}{2}\sqrt{\pi}\,(a^2+u^2)^{-3/4}\cos\left(\tfrac{3}{2}\arctan\dfrac{u}{a}\right)$
4	$\dfrac{1}{\sqrt{x}}e^{-ax}$	$\sqrt{\dfrac{\pi}{2}}\left[\dfrac{a+(a^2+u^2)^{1/2}}{a^2+u^2}\right]^{1/2}$
5	$x^ne^{-ax}, \quad n=1,\,2,\,\ldots$	$\dfrac{a^{n+1}n!}{(a^2+u^2)^{n+1}}\sum\limits_{0\le 2k\le n+1}(-1)^kC_{n+1}^{2k}\left(\dfrac{u}{a}\right)^{2k}$
6	$x^{n-1/2}e^{-ax}, \quad n=1,\,2,\,\ldots$	$k_nu\dfrac{\partial^n}{\partial a^n}\dfrac{1}{r\sqrt{r-a}}$, where $r=\sqrt{a^2+u^2},\ k_n=(-1)^n\sqrt{\pi/2}$
7	$x^{\nu-1}e^{-ax}$	$\Gamma(\nu)(a^2+u^2)^{-\nu/2}\cos\left(\nu\arctan\dfrac{u}{a}\right)$

No	Original function, $f(x)$	Cosine transform, $\check{f}_{\rm c}(u) = \int_0^\infty f(x)\cos(ux)\,dx$
8	$\dfrac{x}{e^{ax}-1}$	$\dfrac{1}{2u^2} - \dfrac{\pi^2}{2a^2\sinh^2(\pi a^{-1}u)}$
9	$\dfrac{1}{x}\left(\dfrac{1}{2} - \dfrac{1}{x} + \dfrac{1}{e^x-1}\right)$	$-\dfrac{1}{2}\ln\left(1-e^{-2\pi u}\right)$
10	$\exp\left(-ax^2\right)$	$\dfrac{1}{2}\sqrt{\dfrac{\pi}{a}}\exp\left(-\dfrac{u^2}{4a}\right)$
11	$\dfrac{1}{\sqrt{x}}\exp\left(-\dfrac{a}{x}\right)$	$\sqrt{\dfrac{\pi}{2u}}\,e^{-\sqrt{2au}}\left[\cos\left(\sqrt{2au}\right) - \sin\left(\sqrt{2au}\right)\right]$
12	$\dfrac{1}{x\sqrt{x}}\exp\left(-\dfrac{a}{x}\right)$	$\sqrt{\dfrac{\pi}{a}}\,e^{-\sqrt{2au}}\cos\left(\sqrt{2au}\right)$

6.4. Expressions With Hyperbolic Functions

No	Original function, $f(x)$	Cosine transform, $\check{f}_{\rm c}(u) = \int_0^\infty f(x)\cos(ux)\,dx$
1	$\dfrac{1}{\cosh(ax)}, \quad a>0$	$\dfrac{\pi}{2a\cosh\left(\frac{1}{2}\pi a^{-1}u\right)}$
2	$\dfrac{1}{\cosh^2(ax)}, \quad a>0$	$\dfrac{\pi u}{2a^2\sinh\left(\frac{1}{2}\pi a^{-1}u\right)}$
3	$\dfrac{\cosh(ax)}{\cosh(bx)}, \quad \lvert a\rvert<b$	$\dfrac{\pi}{b}\left[\dfrac{\cos\left(\frac{1}{2}\pi ab^{-1}\right)\cosh\left(\frac{1}{2}\pi b^{-1}u\right)}{\cos\left(\pi ab^{-1}\right)+\cosh\left(\pi b^{-1}u\right)}\right]$
4	$\dfrac{1}{\cosh(ax)+\cos b}$	$\dfrac{\pi\sinh\left(a^{-1}bu\right)}{a\sin b\sinh\left(\pi a^{-1}u\right)}$
5	$\exp\left(-ax^2\right)\cosh(bx), \quad a>0$	$\dfrac{1}{2}\sqrt{\dfrac{\pi}{a}}\exp\left(\dfrac{b^2-u^2}{4a}\right)\cos\left(\dfrac{abu}{2}\right)$
6	$\dfrac{x}{\sinh(ax)}$	$\dfrac{\pi^2}{4a^2\cosh^2\left(\frac{1}{2}\pi a^{-1}u\right)}$
7	$\dfrac{\sinh(ax)}{\sinh(bx)}, \quad \lvert a\rvert<b$	$\dfrac{\pi}{2b}\,\dfrac{\sin\left(\pi ab^{-1}\right)}{\cos\left(\pi ab^{-1}\right)+\cosh\left(\pi b^{-1}u\right)}$
8	$\dfrac{1}{x}\tanh(ax), \quad a>0$	$\ln\left[\coth\left(\frac{1}{4}\pi a^{-1}u\right)\right]$

6.5. Expressions With Logarithmic Functions

No	Original function, $f(x)$	Cosine transform, $\check{f}_{\rm c}(u) = \int_0^\infty f(x)\cos(ux)\,dx$
1	$\begin{cases}\ln x & \text{if } 0<x<1,\\ 0 & \text{if } 1<x\end{cases}$	$-\dfrac{1}{u}\,\mathrm{Si}(u)$

No	Original function, $f(x)$	Cosine transform, $\check{f}_c(u) = \int_0^\infty f(x)\cos(ux)\,dx$
2	$\dfrac{\ln x}{\sqrt{x}}$	$-\sqrt{\dfrac{\pi}{2u}}\left[\ln(4u) + \mathcal{C} + \dfrac{\pi}{2}\right]$, $\mathcal{C} = 0.5772\ldots$ is the Euler constant
3	$x^{\nu-1}\ln x, \quad 0 < \nu < 1$	$\Gamma(\nu)\cos\left(\dfrac{\pi\nu}{2}\right)u^{-\nu}\left[\psi(\nu) - \dfrac{\pi}{2}\tan\left(\dfrac{\pi\nu}{2}\right) - \ln u\right]$
4	$\ln\left\lvert\dfrac{a+x}{a-x}\right\rvert, \quad a > 0$	$\dfrac{2}{u}\left[\cos(au)\,\mathrm{Si}(au) - \sin(au)\,\mathrm{Ci}(au)\right]$
5	$\ln\left(1 + a^2/x^2\right), \quad a > 0$	$\dfrac{\pi}{u}\left(1 - e^{-au}\right)$
6	$\ln\dfrac{a^2 + x^2}{b^2 + x^2}, \quad a, b > 0$	$\dfrac{\pi}{u}\left(e^{-bu} - e^{-au}\right)$
7	$e^{-ax}\ln x, \quad a > 0$	$-\dfrac{aC + \frac{1}{2}a\ln(u^2 + a^2) + u\arctan(u/a)}{u^2 + a^2}$
8	$\ln\left(1 + e^{-ax}\right), \quad a > 0$	$\dfrac{a}{2u^2} - \dfrac{\pi}{2u\sinh\left(\pi a^{-1}u\right)}$
9	$\ln\left(1 - e^{-ax}\right), \quad a > 0$	$\dfrac{a}{2u^2} - \dfrac{\pi}{2u}\coth\left(\pi a^{-1}u\right)$

6.6. Expressions With Trigonometric Functions

No	Original function, $f(x)$	Cosine transform, $\check{f}_c(u) = \int_0^\infty f(x)\cos(ux)\,dx$
1	$\dfrac{\sin(ax)}{x}, \quad a > 0$	$\begin{cases} \frac{1}{2}\pi & \text{if } u < a, \\ \frac{1}{4}\pi & \text{if } u = a, \\ 0 & \text{if } u > a \end{cases}$
2	$x^{\nu-1}\sin(ax), \quad a > 0,\ \lvert\nu\rvert < 1$	$\pi\,\dfrac{(u+a)^{-\nu} - \lvert u+a\rvert^{-\nu}\,\mathrm{sign}(u-a)}{4\Gamma(1-\nu)\cos\left(\frac{1}{2}\pi\nu\right)}$
3	$\dfrac{x\sin(ax)}{x^2 + b^2}, \quad a, b > 0$	$\begin{cases} \frac{1}{2}\pi e^{-ab}\cosh(bu) & \text{if } u < a, \\ -\frac{1}{2}\pi e^{-bu}\sinh(ab) & \text{if } u > a \end{cases}$
4	$\dfrac{\sin(ax)}{x(x^2 + b^2)}, \quad a, b > 0$	$\begin{cases} \frac{1}{2}\pi b^{-2}\left[1 - e^{-ab}\cosh(bu)\right] & \text{if } u < a, \\ \frac{1}{2}\pi b^{-2}e^{-bu}\sinh(ab) & \text{if } u > a \end{cases}$
5	$e^{-bx}\sin(ax), \quad a, b > 0$	$\dfrac{1}{2}\left[\dfrac{a+u}{(a+u)^2 + b^2} + \dfrac{a-u}{(a-u)^2 + b^2}\right]$
6	$\dfrac{1}{x}\sin^2(ax), \quad a > 0$	$\dfrac{1}{4}\ln\left\lvert 1 - 4\dfrac{a^2}{u^2}\right\rvert$
7	$\dfrac{1}{x^2}\sin^2(ax), \quad a > 0$	$\begin{cases} \frac{1}{4}\pi(2a - u) & \text{if } u < 2a, \\ 0 & \text{if } u > 2a \end{cases}$
8	$\dfrac{1}{x}\sin\left(\dfrac{a}{x}\right), \quad a > 0$	$\dfrac{\pi}{2}J_0\left(2\sqrt{au}\right)$

No	Original function, $f(x)$	Cosine transform, $\check{f}_{\rm c}(u) = \int_0^\infty f(x)\cos(ux)\,dx$
9	$\dfrac{1}{\sqrt{x}}\sin\bigl(a\sqrt{x}\bigr)\sin\bigl(b\sqrt{x}\bigr),\ \ a,b>0$	$\sqrt{\dfrac{\pi}{u}}\,\sin\Bigl(\dfrac{ab}{2u}\Bigr)\sin\Bigl(\dfrac{a^2+b^2}{4u}-\dfrac{\pi}{4}\Bigr)$
10	$\sin\bigl(ax^2\bigr),\quad a>0$	$\sqrt{\dfrac{\pi}{8a}}\left[\cos\Bigl(\dfrac{u^2}{4a}\Bigr)-\sin\Bigl(\dfrac{u^2}{4a}\Bigr)\right]$
11	$\exp\bigl(-ax^2\bigr)\sin\bigl(bx^2\bigr),\quad a>0$	$\dfrac{\sqrt{\pi}}{(A^2+B^2)^{1/4}}\exp\Bigl(-\dfrac{Au^2}{A^2+B^2}\Bigr)\sin\Bigl(\varphi-\dfrac{Bu^2}{A^2+B^2}\Bigr)$, $A=4a,\ B=4b,\ \varphi=\frac{1}{2}\arctan(b/a)$
12	$\dfrac{1-\cos(ax)}{x},\quad a>0$	$\dfrac{1}{2}\ln\Bigl\lvert 1-\dfrac{a^2}{u^2}\Bigr\rvert$
13	$\dfrac{1-\cos(ax)}{x^2},\quad a>0$	$\begin{cases}\frac{1}{2}\pi(a-u) & \text{if } u<a,\\ 0 & \text{if } u>a\end{cases}$
14	$x^{\nu-1}\cos(ax),\quad a>0,\ 0<\nu<1$	$\frac{1}{2}\Gamma(\nu)\cos\bigl(\frac{1}{2}\pi\nu\bigr)\bigl[\lvert u-a\rvert^{-\nu}+(u+a)^{-\nu}\bigr]$
15	$\dfrac{\cos(ax)}{x^2+b^2},\quad a,b>0$	$\begin{cases}\frac{1}{2}\pi b^{-1}e^{-ab}\cosh(bu) & \text{if } u<a,\\ \frac{1}{2}\pi b^{-1}e^{-bu}\cosh(ab) & \text{if } u>a\end{cases}$
16	$e^{-bx}\cos(ax),\quad a,b>0$	$\dfrac{b}{2}\left[\dfrac{1}{(a+u)^2+b^2}+\dfrac{1}{(a-u)^2+b^2}\right]$
17	$\dfrac{1}{\sqrt{x}}\cos\bigl(a\sqrt{x}\bigr)$	$\sqrt{\dfrac{\pi}{u}}\,\sin\Bigl(\dfrac{a^2}{4u}+\dfrac{\pi}{4}\Bigr)$
18	$\dfrac{1}{\sqrt{x}}\cos\bigl(a\sqrt{x}\bigr)\cos\bigl(b\sqrt{x}\bigr)$	$\sqrt{\dfrac{\pi}{u}}\,\cos\Bigl(\dfrac{ab}{2u}\Bigr)\sin\Bigl(\dfrac{a^2+b^2}{4u}+\dfrac{\pi}{4}\Bigr)$
19	$\exp\bigl(-bx^2\bigr)\cos(ax),\quad b>0$	$\dfrac{1}{2}\sqrt{\dfrac{\pi}{b}}\exp\Bigl(-\dfrac{a^2+u^2}{4b}\Bigr)\cosh\Bigl(\dfrac{au}{2b}\Bigr)$
20	$\cos\bigl(ax^2\bigr),\quad a>0$	$\sqrt{\dfrac{\pi}{8a}}\left[\cos\bigl(\frac{1}{4}a^{-1}u^2\bigr)+\sin\bigl(\frac{1}{4}a^{-1}u^2\bigr)\right]$
21	$\exp\bigl(-ax^2\bigr)\cos\bigl(bx^2\bigr),\quad a>0$	$\dfrac{\sqrt{\pi}}{(A^2+B^2)^{1/4}}\exp\Bigl(-\dfrac{Au^2}{A^2+B^2}\Bigr)\cos\Bigl(\varphi-\dfrac{Bu^2}{A^2+B^2}\Bigr)$, $A=4a,\ B=4b,\ \varphi=\frac{1}{2}\arctan(b/a)$

6.7. Expressions With Special Functions

No	Original function, $f(x)$	Cosine transform, $\check{f}_{\rm c}(u) = \int_0^\infty f(x)\cos(ux)\,dx$
1	$\mathrm{Ei}(-ax)$	$-\dfrac{1}{u}\arctan\Bigl(\dfrac{u}{a}\Bigr)$
2	$\mathrm{Ci}(ax)$	$\begin{cases}0 & \text{if } 0<u<a,\\ -\frac{\pi}{2u} & \text{if } a<u\end{cases}$
3	$\mathrm{si}(ax)$	$-\dfrac{1}{2u}\ln\Bigl\lvert\dfrac{u+a}{u-a}\Bigr\rvert,\quad u\neq a$

No	Original function, $f(x)$	Cosine transform, $\check{f}_c(u) = \int_0^\infty f(x)\cos(ux)\,dx$
4	$J_0(ax), \quad a>0$	$\begin{cases} \dfrac{1}{\sqrt{a^2-u^2}} & \text{if } 0<u<a, \\ 0 & \text{if } a<u \end{cases}$
5	$J_\nu(ax), \quad a>0,\ \nu>-1$	$\begin{cases} \dfrac{\cos\bigl[\nu\arcsin(u/a)\bigr]}{\sqrt{a^2-u^2}} & \text{if } 0<u<a, \\ -\dfrac{a^\nu\sin(\pi\nu/2)}{\xi(u+\xi)^\nu} & \text{if } a<u, \end{cases}$ where $\xi=\sqrt{u^2-a^2}$
6	$\dfrac{1}{x}J_\nu(ax), \quad a>0,\ \nu>0$	$\begin{cases} \nu^{-1}\cos\bigl[\nu\arcsin(u/a)\bigr] & \text{if } 0<u<a, \\ \dfrac{a^\nu\cos(\pi\nu/2)}{\nu\bigl(u+\sqrt{u^2-a^2}\bigr)^\nu} & \text{if } a<u \end{cases}$
7	$x^{-\nu}J_\nu(ax), \quad a>0,\ \nu>-\frac{1}{2}$	$\begin{cases} \dfrac{\sqrt{\pi}\,\bigl(a^2-u^2\bigr)^{\nu-1/2}}{(2a)^\nu\Gamma\bigl(\nu+\frac{1}{2}\bigr)} & \text{if } 0<u<a, \\ 0 & \text{if } a<u \end{cases}$
8	$x^{\nu+1}J_\nu(ax)$, $a>0,\ -1<\nu<-\frac{1}{2}$	$\begin{cases} 0 & \text{if } 0<u<a, \\ \dfrac{2^{\nu+1}\sqrt{\pi}\,a^\nu u}{\Gamma\bigl(-\nu-\frac{1}{2}\bigr)(u^2-a^2)^{\nu+3/2}} & \text{if } a<u \end{cases}$
9	$J_0\bigl(a\sqrt{x}\bigr), \quad a>0$	$\dfrac{1}{u}\sin\Bigl(\dfrac{a^2}{4u}\Bigr)$
10	$\dfrac{1}{\sqrt{x}}J_1\bigl(a\sqrt{x}\bigr), \quad a>0$	$\dfrac{4}{a}\sin^2\Bigl(\dfrac{a^2}{8u}\Bigr)$
11	$x^{\nu/2}J_\nu\bigl(a\sqrt{x}\bigr), \ a>0,\ -1<\nu<\frac{1}{2}$	$\Bigl(\dfrac{a}{2}\Bigr)^\nu u^{-\nu-1}\sin\Bigl(\dfrac{a^2}{4u}-\dfrac{\pi\nu}{2}\Bigr)$
12	$J_0\bigl(a\sqrt{x^2+b^2}\bigr)$	$\begin{cases} \dfrac{\cos\bigl(b\sqrt{a^2-u^2}\bigr)}{\sqrt{a^2-u^2}} & \text{if } 0<u<a, \\ 0 & \text{if } a<u \end{cases}$
13	$Y_0(ax), \quad a>0$	$\begin{cases} 0 & \text{if } 0<u<a, \\ -\dfrac{1}{\sqrt{u^2-a^2}} & \text{if } a<u \end{cases}$
14	$x^\nu Y_\nu(ax), \quad a>0,\ \lvert\nu\rvert<\frac{1}{2}$	$\begin{cases} 0 & \text{if } 0<u<a, \\ -\dfrac{(2a)^\nu\sqrt{\pi}}{\Gamma\bigl(\frac{1}{2}-\nu\bigr)(u^2-a^2)^{\nu+1/2}} & \text{if } a<u \end{cases}$
15	$K_0\bigl(a\sqrt{x^2+b^2}\bigr), \quad a,b>0$	$\dfrac{\pi}{2\sqrt{u^2+a^2}}\exp\bigl(-b\sqrt{u^2+a^2}\bigr)$

⦿ References for Supplement 6: G. Doetsch (1950, 1956, 1958), H. Bateman and A. Erdélyi (1954), V. A. Ditkin and A. P. Prudnikov (1965).

Supplement 7

Tables of Fourier Sine Transforms

7.1. General Formulas

No	Original function, $f(x)$	Sine transform, $\check{f}_{\rm s}(u) = \int_0^\infty f(x)\sin(ux)\,dx$
1	$af_1(x) + bf_2(x)$	$a\check{f}_{1\rm s}(u) + b\check{f}_{2\rm s}(u)$
2	$f(ax), \quad a > 0$	$\frac{1}{a}\check{f}_{\rm s}\left(\frac{u}{a}\right)$
3	$x^{2n}f(x), \quad n = 1, 2, \ldots$	$(-1)^n \frac{d^{2n}}{du^{2n}}\check{f}_{\rm s}(u)$
4	$x^{2n+1}f(ax), \quad n = 0, 1, \ldots$	$(-1)^{n+1}\frac{d^{2n+1}}{du^{2n+1}}\check{f}_{\rm c}(u), \quad \check{f}_{\rm c}(u) = \int_0^\infty f(x)\cos(xu)\,dx$
5	$f(ax)\cos(bx), \quad a, b > 0$	$\frac{1}{2a}\left[\check{f}_{\rm s}\left(\frac{u+b}{a}\right) + F_s\left(\frac{u-b}{a}\right)\right]$

7.2. Expressions With Power-Law Functions

No	Original function, $f(x)$	Sine transform, $\check{f}_{\rm s}(u) = \int_0^\infty f(x)\sin(ux)\,dx$
1	$\begin{cases} 1 & \text{if } 0 < x < a, \\ 0 & \text{if } a < x \end{cases}$	$\frac{1}{u}\left[1 - \cos(au)\right]$
2	$\begin{cases} x & \text{if } 0 < x < 1, \\ 2 - x & \text{if } 1 < x < 2, \\ 0 & \text{if } 2 < x \end{cases}$	$\frac{4}{u^2}\sin u \sin^2\frac{u}{2}$
3	$\frac{1}{x}$	$\frac{\pi}{2}$
4	$\frac{1}{a+x}, \quad a > 0$	$\sin(au)\,\mathrm{Ci}(au) - \cos(au)\,\mathrm{si}(au)$
5	$\frac{x}{a^2+x^2}, \quad a > 0$	$\frac{\pi}{2}e^{-au}$
6	$\frac{1}{x(a^2+x^2)}, \quad a > 0$	$\frac{\pi}{2a^2}\left(1 - e^{-au}\right)$
7	$\frac{a}{a^2+(x-b)^2} - \frac{a}{a^2+(x+b)^2}$	$\pi e^{-au}\sin(bu)$
8	$\frac{x+b}{a^2+(x+b)^2} - \frac{x-b}{a^2+(x-b)^2}$	$\pi e^{-au}\cos(bu)$

No	Original function, $f(x)$	Sine transform, $\check{f}_s(u) = \int_0^\infty f(x)\sin(ux)\,dx$
9	$\dfrac{x}{(x^2+a^2)^n}, \quad a>0,\ n=1,2,\dots$	$\dfrac{\pi u e^{-au}}{2^{2n-2}(n-1)!\,a^{2n-3}} \sum_{k=0}^{n-2} \dfrac{(2n-k-4)!}{k!\,(n-k-2)!}(2au)^k$
10	$\dfrac{x^{2m+1}}{(x^2+a)^{n+1}}$, $n,m=0,1,\dots;\ 0\le m\le n$	$(-1)^{n+m}\dfrac{\pi}{2n!}\dfrac{\partial^n}{\partial a^n}\left(a^m e^{-u\sqrt{a}}\right)$
11	$\dfrac{1}{\sqrt{x}}$	$\sqrt{\dfrac{\pi}{2u}}$
12	$\dfrac{1}{x\sqrt{x}}$	$\sqrt{2\pi u}$
13	$x(a^2+x^2)^{-3/2}$	$uK_0(au)$
14	$\dfrac{\left(\sqrt{a^2+x^2}-a\right)^{1/2}}{\sqrt{a^2+x^2}}$	$\sqrt{\dfrac{\pi}{2u}}\,e^{-au}$
15	$x^{-\nu}, \quad 0<\nu<2$	$\cos\left(\tfrac{1}{2}\pi\nu\right)\Gamma(1-\nu)u^{\nu-1}$

7.3. Expressions With Exponential Functions

No	Original function, $f(x)$	Sine transform, $\check{f}_s(u) = \int_0^\infty f(x)\sin(ux)\,dx$
1	$e^{-ax}, \quad a>0$	$\dfrac{u}{a^2+u^2}$
2	$x^n e^{-ax}, \quad a>0,\ n=1,2,\dots$	$n!\left(\dfrac{a}{a^2+u^2}\right)^{n+1} \sum_{k=0}^{[n/2]} (-1)^k C_{n+1}^{2k+1}\left(\dfrac{u}{a}\right)^{2k+1}$
3	$\dfrac{1}{x}e^{-ax}, \quad a>0$	$\arctan\dfrac{u}{a}$
4	$\sqrt{x}e^{-ax}, \quad a>0$	$\dfrac{\sqrt{\pi}}{2}(a^2+u^2)^{-3/4}\sin\left(\dfrac{3}{2}\arctan\dfrac{u}{a}\right)$
5	$\dfrac{1}{\sqrt{x}}e^{-ax}, \quad a>0$	$\sqrt{\dfrac{\pi}{2}}\,\dfrac{\left(\sqrt{a^2+u^2}-a\right)^{1/2}}{\sqrt{a^2+u^2}}$
6	$\dfrac{1}{x\sqrt{x}}e^{-ax}, \quad a>0$	$\sqrt{2\pi}\left(\sqrt{a^2+u^2}-a\right)^{1/2}$
7	$x^{n-1/2}e^{-ax}, \quad a>0,\ n=1,2,\dots$	$(-1)^n\sqrt{\dfrac{\pi}{2}}\dfrac{\partial^n}{\partial a^n}\left[\dfrac{\left(\sqrt{a^2+u^2}-a\right)^{1/2}}{\sqrt{a^2+u^2}}\right]$
8	$x^{\nu-1}e^{-ax}, \quad a>0,\ \nu>-1$	$\Gamma(\nu)(a^2+u^2)^{-\nu/2}\sin\left(\nu\arctan\dfrac{u}{a}\right)$
9	$x^{-2}\left(e^{-ax}-e^{-bx}\right), \quad a,b>0$	$\dfrac{u}{2}\ln\left(\dfrac{u^2+b^2}{u^2+a^2}\right)+b\arctan\left(\dfrac{u}{b}\right)-a\arctan\left(\dfrac{u}{a}\right)$

No	Original function, $f(x)$	Sine transform, $\check{f}_s(u) = \int_0^\infty f(x)\sin(ux)\,dx$
10	$\dfrac{1}{e^{ax}+1}, \quad a>0$	$\dfrac{1}{2u} - \dfrac{\pi}{2a\sinh(\pi u/a)}$
11	$\dfrac{1}{e^{ax}-1}, \quad a>0$	$\dfrac{\pi}{2a}\coth\left(\dfrac{\pi u}{a}\right) - \dfrac{1}{2u}$
12	$\dfrac{e^{x/2}}{e^x-1}$	$-\frac{1}{2}\tanh(\pi u)$
13	$x\exp\left(-ax^2\right)$	$\dfrac{\sqrt{\pi}}{4a^{3/2}}\,u\exp\left(-\dfrac{u^2}{4a}\right)$
14	$\dfrac{1}{x}\exp\left(-ax^2\right)$	$\dfrac{\pi}{2}\operatorname{erf}\left(\dfrac{u}{2\sqrt{a}}\right)$
15	$\dfrac{1}{\sqrt{x}}\exp\left(-\dfrac{a}{x}\right)$	$\sqrt{\dfrac{\pi}{2u}}\,e^{-\sqrt{2au}}\left[\cos\left(\sqrt{2au}\right)+\sin\left(\sqrt{2au}\right)\right]$
16	$\dfrac{1}{x\sqrt{x}}\exp\left(-\dfrac{a}{x}\right)$	$\sqrt{\dfrac{\pi}{a}}\,e^{-\sqrt{2au}}\sin\left(\sqrt{2au}\right)$

7.4. Expressions With Hyperbolic Functions

No	Original function, $f(x)$	Sine transform, $\check{f}_s(u) = \int_0^\infty f(x)\sin(ux)\,dx$
1	$\dfrac{1}{\sinh(ax)}, \quad a>0$	$\dfrac{\pi}{2a}\tanh\left(\frac{1}{2}\pi a^{-1}u\right)$
2	$\dfrac{x}{\sinh(ax)}, \quad a>0$	$\dfrac{\pi^2\sinh\left(\frac{1}{2}\pi a^{-1}u\right)}{4a^2\cosh^2\left(\frac{1}{2}\pi a^{-1}u\right)}$
3	$\dfrac{1}{x}e^{-bx}\sinh(ax), \quad b>\|a\|$	$\frac{1}{2}\arctan\left(\dfrac{2au}{u^2+b^2-a^2}\right)$
4	$\dfrac{1}{x\cosh(ax)}, \quad a>0$	$\arctan\left[\sinh\left(\frac{1}{2}\pi a^{-1}u\right)\right]$
5	$1-\tanh\left(\frac{1}{2}ax\right), \quad a>0$	$\dfrac{1}{u} - \dfrac{\pi}{a\sinh\left(\pi a^{-1}u\right)}$
6	$\coth\left(\frac{1}{2}ax\right)-1, \quad a>0$	$\dfrac{\pi}{a}\coth\left(\pi a^{-1}u\right) - \dfrac{1}{u}$
7	$\dfrac{\cosh(ax)}{\sinh(bx)}, \quad \|a\|<b$	$\dfrac{\pi}{2b}\dfrac{\sinh\left(\pi b^{-1}u\right)}{\cos\left(\pi ab^{-1}\right)+\cosh\left(\pi b^{-1}u\right)}$
8	$\dfrac{\sinh(ax)}{\cosh(bx)}, \quad \|a\|<b$	$\dfrac{\pi}{b}\dfrac{\sin\left(\frac{1}{2}\pi ab^{-1}\right)\sinh\left(\frac{1}{2}\pi b^{-1}u\right)}{\cos\left(\pi ab^{-1}\right)+\cosh\left(\pi b^{-1}u\right)}$

7.5. Expressions With Logarithmic Functions

No	Original function, $f(x)$	Sine transform, $\check{f}_s(u) = \int_0^\infty f(x)\sin(ux)\,dx$
1	$\begin{cases} \ln x & \text{if } 0 < x < 1, \\ 0 & \text{if } 1 < x \end{cases}$	$\frac{1}{u}\left[\operatorname{Ci}(u) - \ln u - \mathcal{C}\right]$, $\mathcal{C} = 0.5772\ldots$ is the Euler constant
2	$\frac{\ln x}{x}$	$-\frac{1}{2}\pi(\ln u + \mathcal{C})$
3	$\frac{\ln x}{\sqrt{x}}$	$-\sqrt{\frac{\pi}{2u}}\left[\ln(4u) + \mathcal{C} - \frac{\pi}{2}\right]$
4	$x^{\nu-1}\ln x, \quad \lvert\nu\rvert < 1$	$\frac{\pi u^{-\nu}\left[\psi(\nu) + \frac{\pi}{2}\cot\left(\frac{\pi\nu}{2}\right) - \ln u\right]}{2\Gamma(1-\nu)\cos\left(\frac{\pi\nu}{2}\right)}$
5	$\ln\left\lvert\frac{a+x}{a-x}\right\rvert, \quad a > 0$	$\frac{\pi}{u}\sin(au)$
6	$\ln\frac{(x+b)^2+a^2}{(x-b)^2+a^2}, \quad a, b > 0$	$\frac{2\pi}{u}e^{-au}\sin(bu)$
7	$e^{-ax}\ln x, \quad a > 0$	$\frac{a\arctan(u/a) - \frac{1}{2}u\ln(u^2+a^2) - e^{\mathcal{C}}u}{u^2+a^2}$
8	$\frac{1}{x}\ln\left(1 + a^2x^2\right), \quad a > 0$	$-\pi\operatorname{Ei}\left(-\frac{u}{a}\right)$

7.6. Expressions With Trigonometric Functions

No	Original function, $f(x)$	Sine transform, $\check{f}_s(u) = \int_0^\infty f(x)\sin(ux)\,dx$
1	$\frac{\sin(ax)}{x}, \quad a > 0$	$\frac{1}{2}\ln\left\lvert\frac{u+a}{u-a}\right\rvert$
2	$\frac{\sin(ax)}{x^2}, \quad a > 0$	$\begin{cases} \frac{1}{2}\pi u & \text{if } 0 < u < a, \\ \frac{1}{2}\pi a & \text{if } u > a \end{cases}$
3	$x^{\nu-1}\sin(ax), \quad a > 0,\ -2 < \nu < 1$	$\pi\frac{\lvert u-a\rvert^{-\nu} - \lvert u+a\rvert^{-\nu}}{4\Gamma(1-\nu)\sin\left(\frac{1}{2}\pi\nu\right)}, \quad \nu \neq 0$
4	$\frac{\sin(ax)}{x^2+b^2}, \quad a, b > 0$	$\begin{cases} \frac{1}{2}\pi b^{-1}e^{-ab}\sinh(bu) & \text{if } 0 < u < a, \\ \frac{1}{2}\pi b^{-1}e^{-bu}\sinh(ab) & \text{if } u > a \end{cases}$
5	$\frac{\sin(\pi x)}{1-x^2}$	$\begin{cases} \sin u & \text{if } 0 < u < \pi, \\ 0 & \text{if } u > \pi \end{cases}$
6	$e^{-ax}\sin(bx), \quad a > 0$	$\frac{a}{2}\left[\frac{1}{a^2+(b-u)^2} - \frac{1}{a^2+(b+u)^2}\right]$
7	$x^{-1}e^{-ax}\sin(bx), \quad a > 0$	$\frac{1}{4}\ln\frac{(u+b)^2+a^2}{(u-b)^2+a^2}$
8	$\frac{1}{x}\sin^2(ax), \quad a > 0$	$\begin{cases} \frac{1}{4}\pi & \text{if } 0 < u < 2a, \\ \frac{1}{8}\pi & \text{if } u = 2a, \\ 0 & \text{if } u > 2a \end{cases}$

No	Original function, $f(x)$	Sine transform, $\check{f}_s(u) = \int_0^\infty f(x)\sin(ux)\,dx$
9	$\dfrac{1}{x^2}\sin^2(ax), \quad a>0$	$\frac{1}{4}(u+2a)\ln\lvert u+2a\rvert + \frac{1}{4}(u-2a)\ln\lvert u-2a\rvert$ $-\frac{1}{2}u\ln u$
10	$\exp(-ax^2)\sin(bx), \quad a>0$	$\dfrac{1}{2}\sqrt{\dfrac{\pi}{a}}\exp\Bigl(-\dfrac{u^2+b^2}{4a}\Bigr)\sinh\Bigl(\dfrac{bu}{2a}\Bigr)$
11	$\dfrac{1}{x}\sin(ax)\sin(bx), \quad a\ge b>0$	$\begin{cases} 0 & \text{if } 0<u<a-b, \\ \frac{\pi}{4} & \text{if } a-b<u<a+b, \\ 0 & \text{if } a+b<u \end{cases}$
12	$\sin\Bigl(\dfrac{a}{x}\Bigr), \quad a>0$	$\dfrac{\pi\sqrt{a}}{2\sqrt{u}}J_1\bigl(2\sqrt{au}\bigr)$
13	$\dfrac{1}{\sqrt{x}}\sin\Bigl(\dfrac{a}{x}\Bigr), \quad a>0$	$\sqrt{\dfrac{\pi}{8u}}\bigl[\sin\bigl(2\sqrt{au}\bigr)-\cos\bigl(2\sqrt{au}\bigr)+\exp\bigl(-2\sqrt{au}\bigr)\bigr]$
14	$\exp\bigl(-a\sqrt{x}\bigr)\sin\bigl(a\sqrt{x}\bigr), \quad a>0$	$a\sqrt{\dfrac{\pi}{8}}\,u^{-3/2}\exp\Bigl(-\dfrac{a^2}{2u}\Bigr)$
15	$\dfrac{\cos(ax)}{x}, \quad a>0$	$\begin{cases} 0 & \text{if } 0<u<a, \\ \frac{1}{4}\pi & \text{if } u=a, \\ \frac{1}{2}\pi & \text{if } a<u \end{cases}$
16	$x^{\nu-1}\cos(ax), \quad a>0,\ \lvert\nu\rvert<1$	$\dfrac{\pi(u+a)^{-\nu}-\operatorname{sign}(u-a)\lvert u-a\rvert^{-\nu}}{4\Gamma(1-\nu)\cos\bigl(\frac{1}{2}\pi\nu\bigr)}$
17	$\dfrac{x\cos(ax)}{x^2+b^2}, \quad a,b>0$	$\begin{cases} -\frac{1}{2}\pi e^{-ab}\sinh(bu) & \text{if } u<a, \\ \frac{1}{2}\pi e^{-bu}\cosh(ab) & \text{if } u>a \end{cases}$
18	$\dfrac{1-\cos(ax)}{x^2}, \quad a>0$	$\dfrac{u}{2}\ln\Bigl\lvert\dfrac{u^2-a^2}{u^2}\Bigr\rvert+\dfrac{a}{2}\ln\Bigl\lvert\dfrac{u+a}{u-a}\Bigr\rvert$
19	$\dfrac{1}{\sqrt{x}}\cos\bigl(a\sqrt{x}\bigr)$	$\sqrt{\dfrac{\pi}{u}}\cos\Bigl(\dfrac{a^2}{4u}+\dfrac{\pi}{4}\Bigr)$
20	$\dfrac{1}{\sqrt{x}}\cos\bigl(a\sqrt{x}\bigr)\cos\bigl(b\sqrt{x}\bigr), \quad a,b>0$	$\sqrt{\dfrac{\pi}{u}}\cos\Bigl(\dfrac{ab}{2u}\Bigr)\cos\Bigl(\dfrac{a^2+b^2}{4u}+\dfrac{\pi}{4}\Bigr)$

7.7. Expressions With Special Functions

No	Original function, $f(x)$	Sine transform, $\check{f}_s(u) = \int_0^\infty f(x)\sin(ux)\,dx$
1	$\operatorname{erfc}(ax), \quad a>0$	$\dfrac{1}{u}\Bigl[1-\exp\Bigl(-\dfrac{u^2}{4a^2}\Bigr)\Bigr]$
2	$\operatorname{ci}(ax), \quad a>0$	$-\dfrac{1}{2u}\ln\Bigl\lvert 1-\dfrac{u^2}{a^2}\Bigr\rvert$
3	$\operatorname{si}(ax), \quad a>0$	$\begin{cases} 0 & \text{if } 0<u<a, \\ -\frac{1}{2}\pi u^{-1} & \text{if } a<u \end{cases}$

No	Original function, $f(x)$	Sine transform, $\check{f}_{\mathrm{s}}(u) = \int_0^\infty f(x)\sin(ux)\,dx$
4	$J_0(ax), \quad a>0$	$\begin{cases} 0 & \text{if } 0<u<a, \\ \dfrac{1}{\sqrt{u^2-a^2}} & \text{if } a<u \end{cases}$
5	$J_\nu(ax), \quad a>0,\ \nu>-2$	$\begin{cases} \dfrac{\sin\left[\nu\arcsin(u/a)\right]}{\sqrt{a^2-u^2}} & \text{if } 0<u<a, \\ \dfrac{a^\nu\cos(\pi\nu/2)}{\xi(u+\xi)^\nu} & \text{if } a<u, \end{cases}$ where $\xi=\sqrt{u^2-a^2}$
6	$\dfrac{1}{x}J_0(ax), \quad a>0,\ \nu>0$	$\begin{cases} \arcsin(u/a) & \text{if } 0<u<a, \\ \pi/2 & \text{if } a<u \end{cases}$
7	$\dfrac{1}{x}J_\nu(ax), \quad a>0,\ \nu>-1$	$\begin{cases} \nu^{-1}\sin\left[\nu\arcsin(u/a)\right] & \text{if } 0<u<a, \\ \dfrac{a^\nu\sin(\pi\nu/2)}{\nu\left(u+\sqrt{u^2-a^2}\right)^\nu} & \text{if } a<u \end{cases}$
8	$x^\nu J_\nu(ax), \quad a>0,\ -1<\nu<\frac{1}{2}$	$\begin{cases} 0 & \text{if } 0<u<a, \\ \dfrac{\sqrt{\pi}(2a)^\nu}{\Gamma\left(\frac{1}{2}-\nu\right)\left(u^2-a^2\right)^{\nu+1/2}} & \text{if } a<u \end{cases}$
9	$x^{-1}e^{-ax}J_0(bx), \quad a>0$	$\arcsin\left(\dfrac{2u}{\sqrt{(u+b)^2+a^2}+\sqrt{(u-b)^2+a^2}}\right)$
10	$\dfrac{J_0(ax)}{x^2+b^2}, \quad a,b>0$	$\begin{cases} b^{-1}\sinh(bu)K_0(ab) & \text{if } 0<u<a, \\ 0 & \text{if } a<u \end{cases}$
11	$\dfrac{xJ_0(ax)}{x^2+b^2}, \quad a,b>0$	$\begin{cases} 0 & \text{if } 0<u<a, \\ \frac{1}{2}\pi e^{-bu}I_0(ab) & \text{if } a<u \end{cases}$
12	$\dfrac{\sqrt{x}J_{2n+1/2}(ax)}{x^2+b^2}$, $a,b>0, \quad n=0,\,1,\,2,\,\ldots$	$\begin{cases} (-1)^n\sinh(bu)K_{2n+1/2}(ab) & \text{if } 0<u<a, \\ 0 & \text{if } a<u \end{cases}$
13	$\dfrac{x^\nu J_\nu(ax)}{x^2+b^2}$, $a,b>0, \quad -1<\nu<\frac{5}{2}$	$\begin{cases} b^{\nu-1}\sinh(bu)K_\nu(ab) & \text{if } 0<u<a, \\ 0 & \text{if } a<u \end{cases}$
14	$\dfrac{x^{1-\nu}J_\nu(ax)}{x^2+b^2}$, $a,b>0, \quad \nu>-\frac{3}{2}$	$\begin{cases} 0 & \text{if } 0<u<a, \\ \frac{1}{2}\pi b^{-\nu}e^{-bu}I_\nu(ab) & \text{if } a<u \end{cases}$
15	$J_0\left(a\sqrt{x}\right), \quad a>0$	$\dfrac{1}{u}\cos\left(\dfrac{a^2}{4u}\right)$
16	$\dfrac{1}{\sqrt{x}}J_1\left(a\sqrt{x}\right), \quad a>0$	$\dfrac{2}{a}\sin\left(\dfrac{a^2}{4u}\right)$
17	$x^{\nu/2}J_\nu\left(a\sqrt{x}\right)$, $a>0, \quad -2<\nu<\frac{1}{2}$	$\dfrac{a^\nu}{2^\nu u^{\nu+1}}\cos\left(\dfrac{a^2}{4u}-\dfrac{\pi\nu}{2}\right)$

No	Original function, $f(x)$	Sine transform, $\check{f}_{\mathrm{s}}(u) = \int_0^\infty f(x)\sin(ux)\,dx$
18	$Y_0(ax), \quad a > 0$	$\begin{cases} \dfrac{2\arcsin(u/a)}{\pi\sqrt{a^2-u^2}} & \text{if } 0 < u < a, \\ \dfrac{2\left[\ln\left(u-\sqrt{u^2-a^2}\right)-\ln a\right]}{\pi\sqrt{u^2-a^2}} & \text{if } a < u \end{cases}$
19	$Y_1(ax), \quad a > 0$	$\begin{cases} 0 & \text{if } 0 < u < a, \\ -\frac{u}{a\sqrt{u^2-a^2}} & \text{if } a < u \end{cases}$
20	$K_0(ax), \quad a > 0$	$\dfrac{\ln\left(u+\sqrt{u^2+a^2}\right)-\ln a}{\sqrt{u^2+a^2}}$
21	$xK_0(ax), \quad a > 0$	$\dfrac{\pi u}{2(u^2+a^2)^{3/2}}$
22	$x^{\nu+1}K_\nu(ax), \quad a > 0,\ \nu > -\frac{3}{2}$	$\sqrt{\pi}\,(2a)^\nu\Gamma\left(\nu+\frac{3}{2}\right)u(u^2+a^2)^{-\nu-3/2}$

⦿ References for Supplement 7: G. Doetsch (1950, 1956, 1958), H. Bateman and A. Erdélyi (1954), I. I. Hirschman and D. V. Widder (1955), V. A. Ditkin and A. P. Prudnikov (1965).

Supplement 8

Tables of Mellin Transforms

8.1. General Formulas

No	Original function, $f(x)$	Mellin transform, $\hat{f}(s) = \int_0^\infty f(x)x^{s-1}\,dx$
1	$af_1(x) + bf_2(x)$	$a\hat{f}_1(s) + b\hat{f}_2(s)$
2	$f(ax),\ a > 0$	$a^{-s}\hat{f}(s)$
3	$x^a f(x)$	$\hat{f}(s + a)$
4	$f(1/x)$	$\hat{f}(-s)$
5	$f\left(x^\beta\right),\ \ \beta > 0$	$\dfrac{1}{\beta}\hat{f}\left(\dfrac{s}{\beta}\right)$
6	$f\left(x^{-\beta}\right),\ \ \beta > 0$	$\dfrac{1}{\beta}\hat{f}\left(-\dfrac{s}{\beta}\right)$
7	$x^\lambda f\left(ax^\beta\right),\ \ a, \beta > 0$	$\dfrac{1}{\beta}a^{-\frac{s+\lambda}{\beta}}\hat{f}\left(\dfrac{s+\lambda}{\beta}\right)$
8	$x^\lambda f\left(ax^{-\beta}\right),\ \ a, \beta > 0$	$\dfrac{1}{\beta}a^{\frac{s+\lambda}{\beta}}\hat{f}\left(-\dfrac{s+\lambda}{\beta}\right)$
9	$f'_x(x)$	$-(s-1)\hat{f}(s-1)$
10	$xf'_x(x)$	$-s\hat{f}(s)$
11	$f_x^{(n)}(x)$	$(-1)^n\dfrac{\Gamma(s)}{\Gamma(s-n)}\hat{f}(s-n)$
12	$\left(x\dfrac{d}{dx}\right)^n f(x)$	$(-1)^n s^n \hat{f}(s)$
13	$\left(\dfrac{d}{dx}x\right)^n f(x)$	$(-1)^n (s-1)^n \hat{f}(s)$
14	$x^\alpha \int_0^\infty t^\beta f_1(xt) f_2(t)\,dt$	$\hat{f}_1(s+\alpha)\hat{f}_2(1 - s - \alpha + \beta)$
15	$x^\alpha \int_0^\infty t^\beta f_1\left(\dfrac{x}{t}\right) f_2(t)\,dt$	$\hat{f}_1(s+\alpha)\hat{f}_2(s + \alpha + \beta + 1)$

8.2. Expressions With Power-Law Functions

No	Original function, $f(x)$	Mellin transform, $\hat{f}(s) = \int_0^\infty f(x)x^{s-1}\,dx$
1	$\begin{cases} x & \text{if } 0 < x < 1, \\ 2-x & \text{if } 1 < x < 2, \\ 0 & \text{if } 2 < x \end{cases}$	$\begin{cases} \dfrac{2(2^s-1)}{s(s+1)} & \text{if } s \neq 0, \\ 2\ln 2 & \text{if } s = 0, \end{cases}$ $\operatorname{Re} s > -1$
2	$\dfrac{1}{x+a}$, $a > 0$	$\dfrac{\pi a^{s-1}}{\sin(\pi s)}$, $0 < \operatorname{Re} s < 1$
3	$\dfrac{1}{(x+a)(x+b)}$, $a, b > 0$	$\dfrac{\pi\left(a^{s-1} - b^{s-1}\right)}{(b-a)\sin(\pi s)}$, $0 < \operatorname{Re} s < 2$
4	$\dfrac{x+a}{(x+b)(x+c)}$, $b, c > 0$	$\dfrac{\pi}{\sin(\pi s)}\left[\left(\dfrac{b-a}{b-c}\right)b^{s-1} + \left(\dfrac{c-a}{c-b}\right)c^{s-1}\right]$, $0 < \operatorname{Re} s < 1$
5	$\dfrac{1}{x^2+a^2}$, $a > 0$	$\dfrac{\pi a^{s-2}}{2\sin\left(\frac{1}{2}\pi s\right)}$, $0 < \operatorname{Re} s < 2$
6	$\dfrac{1}{x^2+2ax\cos\beta+a^2}$, $a > 0$, $\lvert\beta\rvert < \pi$	$-\dfrac{\pi a^{s-2}\sin\left[\beta(s-1)\right]}{\sin\beta\,\sin(\pi s)}$, $0 < \operatorname{Re} s < 2$
7	$\dfrac{1}{(x^2+a^2)(x^2+b^2)}$, $a, b > 0$	$\dfrac{\pi\left(a^{s-2} - b^{s-2}\right)}{2(b^2-a^2)\sin\left(\frac{1}{2}\pi s\right)}$, $0 < \operatorname{Re} s < 4$
8	$\dfrac{1}{(1+ax)^{n+1}}$, $a > 0$, $n = 1, 2, \ldots$	$\dfrac{(-1)^n\pi}{a^s\sin(\pi s)}C_{s-1}^n$, $0 < \operatorname{Re} s < n+1$
9	$\dfrac{1}{x^n+a^n}$, $a > 0$, $n = 1, 2, \ldots$	$\dfrac{\pi a^{s-n}}{n\sin(\pi s/n)}$, $0 < \operatorname{Re} s < n$
10	$\dfrac{1-x}{1-x^n}$, $n = 2, 3, \ldots$	$\dfrac{\pi\sin(\pi/n)}{n\sin(\pi s/n)\sin\left[\pi(s+1)/n\right]}$, $0 < \operatorname{Re} s < n-1$
11	$\begin{cases} x^\nu & \text{if } 0 < x < 1, \\ 0 & \text{if } 1 < x \end{cases}$	$\dfrac{1}{s+\nu}$, $\operatorname{Re} s > -\nu$
12	$\dfrac{1-x^\nu}{1-x^{n\nu}}$, $n = 2, 3, \ldots$	$\dfrac{\pi\sin(\pi/n)}{n\nu\sin\left(\frac{\pi s}{n\nu}\right)\sin\left[\frac{\pi(s+\nu)}{n\nu}\right]}$, $0 < \operatorname{Re} s < (n-1)\nu$

8.3. Expressions With Exponential Functions

No	Original function, $f(x)$	Mellin transform, $\hat{f}(s) = \int_0^\infty f(x)x^{s-1}\,dx$
1	e^{-ax}, $a > 0$	$a^{-s}\Gamma(s)$, $\operatorname{Re} s > 0$
2	$\begin{cases} e^{-bx} & \text{if } 0 < x < a, \\ 0 & \text{if } a < x, \end{cases}$ $b > 0$	$b^{-s}\gamma(s, ab)$, $\operatorname{Re} s > 0$
3	$\begin{cases} 0 & \text{if } 0 < x < a, \\ e^{-bx} & \text{if } a < x, \end{cases}$ $b > 0$	$b^{-s}\Gamma(s, ab)$
4	$\dfrac{e^{-ax}}{x+b}$, $a, b > 0$	$e^{ab}b^{s-1}\Gamma(s)\Gamma(1-s, ab)$, $\operatorname{Re} s > 0$
5	$\exp\left(-ax^\beta\right)$, $a, \beta > 0$	$\beta^{-1}a^{-s/\beta}\Gamma(s/\beta)$, $\operatorname{Re} s > 0$

No	Original function, $f(x)$	Mellin transform, $\hat{f}(s)=\int_0^\infty f(x)x^{s-1}\,dx$
6	$\exp\left(-ax^{-\beta}\right),\quad a,\beta>0$	$\beta^{-1}a^{s/\beta}\Gamma(-s/\beta),\quad \operatorname{Re}s<0$
7	$1-\exp\left(-ax^{\beta}\right),\quad a,\beta>0$	$-\beta^{-1}a^{-s/\beta}\Gamma(s/\beta),\quad -\beta<\operatorname{Re}s<0$
8	$1-\exp\left(-ax^{-\beta}\right),\quad a,\beta>0$	$-\beta^{-1}a^{s/\beta}\Gamma(-s/\beta),\quad 0<\operatorname{Re}s<\beta$

8.4. Expressions With Logarithmic Functions

No	Original function, $f(x)$	Mellin transform, $\hat{f}(s)=\int_0^\infty f(x)x^{s-1}\,dx$
1	$\begin{cases}\ln x & \text{if } 0<x<a,\\ 0 & \text{if } a<x\end{cases}$	$\dfrac{s\ln a-1}{s^2a^s},\quad \operatorname{Re}s>0$
2	$\ln(1+ax),\quad a>0$	$\dfrac{\pi}{sa^s\sin(\pi s)},\quad -1<\operatorname{Re}s<0$
3	$\ln\lvert 1-x\rvert$	$\dfrac{\pi}{s}\cot(\pi s),\quad -1<\operatorname{Re}s<0$
4	$\dfrac{\ln x}{x+a},\quad a>0$	$\dfrac{\pi a^{s-1}\left[\ln a-\pi\cot(\pi s)\right]}{\sin(\pi s)},\quad 0<\operatorname{Re}s<1$
5	$\dfrac{\ln x}{(x+a)(x+b)},\quad a,b>0$	$\dfrac{\pi\left[a^{s-1}\ln a-b^{s-1}\ln b-\pi\cot(\pi s)(a^{s-1}-b^{s-1})\right]}{(b-a)\sin(\pi s)},$ $0<\operatorname{Re}s<1$
6	$\begin{cases}x^\nu\ln x & \text{if } 0<x<1,\\ 0 & \text{if } 1<x\end{cases}$	$-\dfrac{1}{(s+\nu)^2},\quad \operatorname{Re}s>-\nu$
7	$\dfrac{\ln^2 x}{x+1}$	$\dfrac{\pi^3\left[2-\sin^2(\pi s)\right]}{\sin^3(\pi s)},\quad 0<\operatorname{Re}s<1$
8	$\begin{cases}\ln^{\nu-1}x & \text{if } 0<x<1,\\ 0 & \text{if } 1<x\end{cases}$	$\Gamma(\nu)(-s)^{-\nu},\quad \operatorname{Re}s<0,\ \nu>0$
9	$\ln\left(x^2+2x\cos\beta+1\right),\quad \lvert\beta\rvert<\pi$	$\dfrac{2\pi\cos(\beta s)}{s\sin(\pi s)},\quad -1<\operatorname{Re}s<0$
10	$\ln\left\lvert\dfrac{1+x}{1-x}\right\rvert$	$\dfrac{\pi}{s}\tan\left(\tfrac{1}{2}\pi s\right),\quad -1<\operatorname{Re}s<1$
11	$e^{-x}\ln^n x,\quad n=1,\,2,\,\ldots$	$\dfrac{d^n}{ds^n}\Gamma(s),\quad \operatorname{Re}s>0$

8.5. Expressions With Trigonometric Functions

No	Original function, $f(x)$	Mellin transform, $\hat{f}(s)=\int_0^\infty f(x)x^{s-1}\,dx$
1	$\sin(ax),\quad a>0$	$a^{-s}\Gamma(s)\sin\left(\tfrac{1}{2}\pi s\right),\quad -1<\operatorname{Re}s<1$
2	$\sin^2(ax),\quad a>0$	$-2^{-s-1}a^{-s}\Gamma(s)\cos\left(\tfrac{1}{2}\pi s\right),\quad -2<\operatorname{Re}s<0$
3	$\sin(ax)\sin(bx),\quad a,b>0,\ a\neq b$	$\tfrac{1}{2}\Gamma(s)\cos\left(\tfrac{1}{2}\pi s\right)\left[\lvert b-a\rvert^{-s}-(b+a)^{-s}\right],$ $-2<\operatorname{Re}s<1$

No	Original function, $f(x)$	Mellin transform, $\hat{f}(s)=\int_0^\infty f(x)x^{s-1}\,dx$
4	$\cos(ax), \quad a>0$	$a^{-s}\Gamma(s)\cos\left(\frac{1}{2}\pi s\right), \quad 0<\operatorname{Re} s<1$
5	$\sin(ax)\cos(bx), \quad a,b>0$	$\frac{\Gamma(s)}{2}\sin\left(\frac{\pi s}{2}\right)\left[(a+b)^{-s}+\lvert a-b\rvert^{-s}\operatorname{sign}(a-b)\right]$, $-1<\operatorname{Re} s<1$
6	$e^{-ax}\sin(bx), \quad a>0$	$\frac{\Gamma(s)\sin\left[s\arctan(b/a)\right]}{(a^2+b^2)^{s/2}}, \quad -1<\operatorname{Re} s$
7	$e^{-ax}\cos(bx), \quad a>0$	$\frac{\Gamma(s)\cos\left[s\arctan(b/a)\right]}{(a^2+b^2)^{s/2}}, \quad 0<\operatorname{Re} s$
8	$\begin{cases}\sin(a\ln x) & \text{if } 0<x<1,\\ 0 & \text{if } 1<x\end{cases}$	$-\frac{a}{s^2+a^2}, \quad \operatorname{Re} s>0$
9	$\begin{cases}\cos(a\ln x) & \text{if } 0<x<1,\\ 0 & \text{if } 1<x\end{cases}$	$\frac{s}{s^2+a^2}, \quad \operatorname{Re} s>0$
10	$\arctan x$	$-\frac{\pi}{2s\cos\left(\frac{1}{2}\pi s\right)}, \quad -1<\operatorname{Re} s<0$
11	$\operatorname{arccot} x$	$\frac{\pi}{2s\cos\left(\frac{1}{2}\pi s\right)}, \quad 0<\operatorname{Re} s<1$

8.6. Expressions With Special Functions

No	Original function, $f(x)$	Mellin transform, $\hat{f}(s)=\int_0^\infty f(x)x^{s-1}\,dx$
1	$\operatorname{erfc} x$	$\frac{\Gamma\left(\frac{1}{2}s+\frac{1}{2}\right)}{\sqrt{\pi}\,s}, \quad \operatorname{Re} s>0$
2	$\operatorname{Ei}(-x)$	$-s^{-1}\Gamma(s), \quad \operatorname{Re} s>0$
3	$\operatorname{Si}(x)$	$-s^{-1}\sin\left(\frac{1}{2}\pi s\right)\Gamma(s), \quad -1<\operatorname{Re} s<0$
4	$\operatorname{si}(x)$	$-4s^{-1}\sin\left(\frac{1}{2}\pi s\right)\Gamma(s), \quad -1<\operatorname{Re} s<0$
5	$\operatorname{Ci}(x)$	$-s^{-1}\cos\left(\frac{1}{2}\pi s\right)\Gamma(s), \quad 0<\operatorname{Re} s<1$
6	$J_\nu(ax), \quad a>0$	$\frac{2^{s-1}\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}s\right)}{a^s\Gamma\left(\frac{1}{2}\nu-\frac{1}{2}s+1\right)}, \quad -\nu<\operatorname{Re} s<\frac{3}{2}$
7	$Y_\nu(ax), \quad a>0$	$-\frac{2^{s-1}}{\pi a^s}\Gamma\left(\frac{s}{2}+\frac{\nu}{2}\right)\Gamma\left(\frac{s}{2}-\frac{\nu}{2}\right)\cos\left[\frac{\pi(s-\nu)}{2}\right]$, $\lvert\nu\rvert<\operatorname{Re} s<\frac{3}{2}$
8	$e^{-ax}I_\nu(ax), \quad a>0$	$\frac{\Gamma(1/2-s)\Gamma(s+\nu)}{\sqrt{\pi}\,(2a)^s\Gamma(1+\nu-s)}, \quad -\nu<\operatorname{Re} s<\frac{1}{2}$
9	$K_\nu(ax), \quad a>0$	$\frac{2^{s-2}}{a^s}\Gamma\left(\frac{s}{2}+\frac{\nu}{2}\right)\Gamma\left(\frac{s}{2}-\frac{\nu}{2}\right), \quad \lvert\nu\rvert<\operatorname{Re} s$
10	$e^{-ax}K_\nu(ax), \quad a>0$	$\frac{\sqrt{\pi}\,\Gamma(s-\nu)\Gamma(s+\nu)}{(2a)^s\Gamma(s+1/2)}, \quad \lvert\nu\rvert<\operatorname{Re} s$

⦿ References for Supplement 8: H. Bateman and A. Erdélyi (1954), V. A. Ditkin and A. P. Prudnikov (1965).

Supplement 9

Tables of Inverse Mellin Transforms

See Section 8.1 of Supplement 8 for general formulas.

9.1. Expressions With Power-Law Functions

No	Direct transform, $\hat{f}(s)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{\sigma-i\infty}^{\sigma+i\infty} \hat{f}(s)x^{-s}\,ds$
1	$\dfrac{1}{s}$, $\operatorname{Re} s > 0$	$\begin{cases} 1 & \text{if } 0 < x < 1, \\ 0 & \text{if } 1 < x \end{cases}$
2	$\dfrac{1}{s}$, $\operatorname{Re} s < 0$	$\begin{cases} 0 & \text{if } 0 < x < 1, \\ -1 & \text{if } 1 < x \end{cases}$
3	$\dfrac{1}{s+a}$, $\operatorname{Re} s > -a$	$\begin{cases} x^a & \text{if } 0 < x < 1, \\ 0 & \text{if } 1 < x \end{cases}$
4	$\dfrac{1}{s+a}$, $\operatorname{Re} s < -a$	$\begin{cases} 0 & \text{if } 0 < x < 1, \\ -x^a & \text{if } 1 < x \end{cases}$
5	$\dfrac{1}{(s+a)^2}$, $\operatorname{Re} s > -a$	$\begin{cases} -x^a \ln x & \text{if } 0 < x < 1, \\ 0 & \text{if } 1 < x \end{cases}$
6	$\dfrac{1}{(s+a)^2}$, $\operatorname{Re} s < -a$	$\begin{cases} 0 & \text{if } 0 < x < 1, \\ x^a \ln x & \text{if } 1 < x \end{cases}$
7	$\dfrac{1}{(s+a)(s+b)}$, $\operatorname{Re} s > -a, -b$	$\begin{cases} \dfrac{x^a - x^b}{b-a} & \text{if } 0 < x < 1, \\ 0 & \text{if } 1 < x \end{cases}$
8	$\dfrac{1}{(s+a)(s+b)}$, $-a < \operatorname{Re} s < -b$	$\begin{cases} \dfrac{x^a}{b-a} & \text{if } 0 < x < 1, \\ \dfrac{x^b}{b-a} & \text{if } 1 < x \end{cases}$
9	$\dfrac{1}{(s+a)(s+b)}$, $\operatorname{Re} s < -a, -b$	$\begin{cases} 0 & \text{if } 0 < x < 1, \\ \dfrac{x^b - x^a}{b-a} & \text{if } 1 < x \end{cases}$
10	$\dfrac{1}{(s+a)^2 + b^2}$, $\operatorname{Re} s > -a$	$\begin{cases} \dfrac{1}{b} x^a \sin\left(b \ln \dfrac{1}{x}\right) & \text{if } 0 < x < 1, \\ 0 & \text{if } 1 < x \end{cases}$
11	$\dfrac{s+a}{(s+a)^2 + b^2}$, $\operatorname{Re} s > -a$	$\begin{cases} x^a \cos(b \ln x) & \text{if } 0 < x < 1, \\ 0 & \text{if } 1 < x \end{cases}$

No	Direct transform, $\hat{f}(s)$	Inverse transform, $f(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \hat{f}(s)x^{-s}\,ds$
12	$\sqrt{s^2-a^2}-s, \quad \operatorname{Re} s > \lvert a\rvert$	$\begin{cases} -\dfrac{a}{\ln x} I_1(-a\ln x) & \text{if } 0<x<1, \\ 0 & \text{if } 1<x \end{cases}$
13	$\sqrt{\dfrac{s+a}{s-a}}-1, \quad \operatorname{Re} s > \lvert a\rvert$	$\begin{cases} aI_0(-a\ln x)+aI_1(-a\ln x) & \text{if } 0<x<1, \\ 0 & \text{if } 1<x \end{cases}$
14	$(s+a)^{-\nu}, \quad \operatorname{Re} s > -a,\ \nu>0$	$\begin{cases} \dfrac{1}{\Gamma(\nu)} x^a(-\ln x)^{\nu-1} & \text{if } 0<x<1, \\ 0 & \text{if } 1<x \end{cases}$
15	$s^{-1}(s+a)^{-\nu}$, $\operatorname{Re} s>0,\ \operatorname{Re} s>-a,\ \nu>0$	$\begin{cases} a^{-\nu}\big[\Gamma(\nu)\big]^{-1}\gamma(\nu,-a\ln x) & \text{if } 0<x<1, \\ 0 & \text{if } 1<x \end{cases}$
16	$s^{-1}(s+a)^{-\nu}$, $-a<\operatorname{Re} s<0,\ \nu>0$	$\begin{cases} -a^{-\nu}\big[\Gamma(\nu)\big]^{-1}\Gamma(\nu,-a\ln x) & \text{if } 0<x<1, \\ -a^{-\nu} & \text{if } 1<x \end{cases}$
17	$(s^2-a^2)^{-\nu}, \quad \operatorname{Re} s>\lvert a\rvert,\ \nu>0$	$\begin{cases} \dfrac{\sqrt{\pi}\,(-\ln x)^{\nu-1/2} I_{\nu-1/2}(-a\ln x)}{\Gamma(\nu)(2a)^{\nu-1/2}} & \text{if } 0<x<1, \\ 0 & \text{if } 1<x \end{cases}$
18	$(a^2-s^2)^{-\nu}, \quad \operatorname{Re} s<\lvert a\rvert,\ \nu>0$	$\begin{cases} \dfrac{(-\ln x)^{\nu-1/2} K_{\nu-1/2}(-a\ln x)}{\sqrt{\pi}\,\Gamma(\nu)(2a)^{\nu-1/2}} & \text{if } 0<x<1, \\ \dfrac{(\ln x)^{\nu-1/2} K_{\nu-1/2}(a\ln x)}{\sqrt{\pi}\,\Gamma(\nu)(2a)^{\nu-1/2}} & \text{if } 1<x \end{cases}$

9.2. Expressions With Exponential and Logarithmic Functions

No	Direct transform, $\hat{f}(s)$	Inverse transform, $f(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \hat{f}(s)x^{-s}\,ds$
1	$\exp(as^2), \quad a>0$	$\dfrac{1}{2\sqrt{\pi a}}\exp\Big(-\dfrac{\ln^2 x}{4a}\Big)$
2	$s^{-\nu}e^{-a/s}, \quad \operatorname{Re} s>0;\ a,\nu>0$	$\begin{cases} \Big\lvert\dfrac{a}{\ln x}\Big\rvert^{\frac{1-\nu}{2}} J_{\nu-1}\big(2\sqrt{a\lvert\ln x\rvert}\big) & \text{if } 0<x<1, \\ 0 & \text{if } 1<x \end{cases}$
3	$\exp\big(-\sqrt{as}\,\big), \quad \operatorname{Re} s>0,\ a>0$	$\begin{cases} \dfrac{(a/\pi)^{1/2}}{2\lvert\ln x\rvert^{3/2}}\exp\Big(-\dfrac{a}{4\lvert\ln x\rvert}\Big) & \text{if } 0<x<1, \\ 0 & \text{if } 1<x \end{cases}$
4	$\dfrac{1}{s}\exp\big(-a\sqrt{s}\,\big), \quad \operatorname{Re} s>0$	$\begin{cases} \operatorname{erfc}\Big(\dfrac{a}{2\sqrt{\lvert\ln x\rvert}}\Big) & \text{if } 0<x<1, \\ 0 & \text{if } 1<x \end{cases}$
5	$\dfrac{1}{s}\big[\exp\big(-a\sqrt{s}\,\big)-1\big], \quad \operatorname{Re} s>0$	$\begin{cases} -\operatorname{erf}\Big(\dfrac{a}{2\sqrt{\lvert\ln x\rvert}}\Big) & \text{if } 0<x<1, \\ 0 & \text{if } 1<x \end{cases}$

No	Direct transform, $\hat{f}(s)$	Inverse transform, $f(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \hat{f}(s) x^{-s}\, ds$
6	$\sqrt{s} \exp\left(-\sqrt{as}\right), \quad \operatorname{Re} s > 0$	$\begin{cases} \dfrac{a - 2\lvert\ln x\rvert}{4\sqrt{\pi \lvert\ln x\rvert^5}} \exp\left(-\dfrac{a}{4\lvert\ln x\rvert}\right) & \text{if } 0 < x < 1, \\ 0 & \text{if } 1 < x \end{cases}$
7	$\dfrac{1}{\sqrt{s}} \exp\left(-\sqrt{as}\right), \quad \operatorname{Re} s > 0$	$\begin{cases} \dfrac{1}{\sqrt{\pi \lvert\ln x\rvert}} \exp\left(-\dfrac{a}{4\lvert\ln x\rvert}\right) & \text{if } 0 < x < 1, \\ 0 & \text{if } 1 < x \end{cases}$
8	$\ln \dfrac{s + a}{s + b}, \quad \operatorname{Re} s > -a, -b$	$\begin{cases} \dfrac{x^a - x^b}{\ln x} & \text{if } 0 < x < 1, \\ 0 & \text{if } 1 < x \end{cases}$
9	$s^{-\nu} \ln s, \quad \operatorname{Re} s > 0,\ \nu > 0$	$\begin{cases} \lvert\ln x\rvert^{\nu - 1} \dfrac{\psi(\nu) - \ln \lvert\ln x\rvert}{\Gamma(\nu)} & \text{if } 0 < x < 1, \\ 0 & \text{if } 1 < x \end{cases}$

9.3. Expressions With Trigonometric Functions

No	Direct transform, $\hat{f}(s)$	Inverse transform, $f(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \hat{f}(s) x^{-s}\, ds$
1	$\dfrac{\pi}{\sin(\pi s)}, \quad 0 < \operatorname{Re} s < 1$	$\dfrac{1}{x + 1}$
2	$\dfrac{\pi}{\sin(\pi s)}, \quad -n < \operatorname{Re} s < 1 - n,$ $n = \ldots, -1, 0, 1, 2, \ldots$	$(-1)^n \dfrac{x^n}{x + 1}$
3	$\dfrac{\pi^2}{\sin^2(\pi s)}, \quad 0 < \operatorname{Re} s < 1$	$\dfrac{\ln x}{x - 1}$
4	$\dfrac{\pi^2}{\sin^2(\pi s)}, \quad n < \operatorname{Re} s < n + 1,$ $n = \ldots, -1, 0, 1, 2, \ldots$	$\dfrac{\ln x}{x^n (x - 1)}$
5	$\dfrac{2\pi^3}{\sin^3(\pi s)}, \quad 0 < \operatorname{Re} s < 1$	$\dfrac{\pi^2 + \ln^2 x}{x + 1}$
6	$\dfrac{2\pi^3}{\sin^3(\pi s)}, \quad n < \operatorname{Re} s < n + 1,$ $n = \ldots, -1, 0, 1, 2, \ldots$	$\dfrac{\pi^2 + \ln^2 x}{(-x)^n (x + 1)}$
7	$\sin\left(s^2/a\right), \quad a > 0$	$\dfrac{1}{2}\sqrt{\dfrac{a}{\pi}} \sin\left(\tfrac{1}{4} a \lvert\ln x\rvert^2 - \tfrac{1}{4}\pi\right)$
8	$\dfrac{\pi}{\cos(\pi s)}, \quad -\tfrac{1}{2} < \operatorname{Re} s < \tfrac{1}{2}$	$\dfrac{\sqrt{x}}{x + 1}$
9	$\dfrac{\pi}{\cos(\pi s)}, \quad n - \tfrac{1}{2} < \operatorname{Re} s < n + \tfrac{1}{2}$ $n = \ldots, -1, 0, 1, 2, \ldots$	$(-1)^n \dfrac{x^{1/2 - n}}{x + 1}$
10	$\dfrac{\cos(\beta s)}{s \cos(\pi s)}, \quad -1 < \operatorname{Re} s < 0,\ \lvert\beta\rvert < \pi$	$\dfrac{1}{2\pi} \ln(x^2 + 2x \cos\beta + 1)$

No	Direct transform, $\hat{f}(s)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \hat{f}(s)x^{-s}\,ds$
11	$\cos\left(s^2/a\right), \quad a > 0$	$\dfrac{1}{2}\sqrt{\dfrac{a}{\pi}}\,\cos\left(\frac{1}{4}a\lvert\ln x\rvert^2 - \frac{1}{4}\pi\right)$
12	$\arctan\left(\dfrac{a}{s+b}\right), \quad \operatorname{Re} s > -b$	$\begin{cases} \dfrac{x^b}{\lvert\ln x\rvert}\sin\left(a\lvert\ln x\rvert\right) & \text{if } 0 < x < 1, \\ 0 & \text{if } 1 < x \end{cases}$

9.4. Expressions With Special Functions

No	Direct transform, $\hat{f}(s)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \hat{f}(s)x^{-s}\,ds$
1	$\Gamma(s), \quad \operatorname{Re} s > 0$	e^{-x}
2	$\Gamma(s), \quad -1 < \operatorname{Re} s < 0$	$e^{-x} - 1$
3	$\sin\left(\frac{1}{2}\pi s\right)\Gamma(s), \quad -1 < \operatorname{Re} s < 1$	$\sin x$
4	$\sin(as)\Gamma(s)$, $\operatorname{Re} s > -1,\ \lvert a\rvert < \dfrac{\pi}{2}$	$\exp(-x\cos a)\sin(x\sin a)$
5	$\cos\left(\frac{1}{2}\pi s\right)\Gamma(s), \quad 0 < \operatorname{Re} s < 1$	$\cos x$
6	$\cos\left(\frac{1}{2}\pi s\right)\Gamma(s), \quad -2 < \operatorname{Re} s < 0$	$-2\sin^2(x/2)$
7	$\cos(as)\Gamma(s), \quad \operatorname{Re} s > 0,\ \lvert a\rvert < \dfrac{\pi}{2}$	$\exp(-x\cos a)\cos(x\sin a)$
8	$\dfrac{\Gamma(s)}{\cos(\pi s)}, \quad 0 < \operatorname{Re} s < \dfrac{1}{2}$	$e^x \operatorname{erfc}\left(\sqrt{x}\right)$
9	$\Gamma(a+s)\Gamma(b-s)$, $-a < \operatorname{Re} s < b,\ a+b > 0$	$\Gamma(a+b)x^a(x+1)^{-a-b}$
10	$\Gamma(a+s)\Gamma(b+s)$, $\operatorname{Re} s > -a, -b$	$2x^{(a+b)/2}K_{a-b}\left(2\sqrt{x}\right)$
11	$\dfrac{\Gamma(s)}{\Gamma(s+\nu)}, \quad \operatorname{Re} s > 0,\ \nu > 0$	$\begin{cases} \dfrac{(1-x)^{\nu-1}}{\Gamma(\nu)} & \text{if } 0 < x < 1, \\ 0 & \text{if } 1 < x \end{cases}$
12	$\dfrac{\Gamma(1-\nu-s)}{\Gamma(1-s)}$, $\operatorname{Re} s < 1-\nu,\ \nu > 0$	$\begin{cases} 0 & \text{if } 0 < x < 1, \\ \dfrac{(x-1)^{\nu-1}}{\Gamma(\nu)} & \text{if } 1 < x \end{cases}$
13	$\dfrac{\Gamma(s)}{\Gamma(\nu-s+1)}$, $0 < \operatorname{Re} s < \dfrac{\nu}{2} + \dfrac{3}{4}$	$x^{-\nu/2}J_\nu\left(2\sqrt{x}\right)$
14	$\dfrac{\Gamma(s+\nu)\Gamma(s-\nu)}{\Gamma(s+1/2)}, \quad \operatorname{Re} s > \lvert\nu\rvert$	$\pi^{-1/2}e^{-x/2}K_\nu(x/2)$

No	Direct transform, $\hat{f}(s)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \hat{f}(s)x^{-s}\,ds$
15	$\dfrac{\Gamma(s+\nu)\Gamma(1/2-s)}{\Gamma(1+\nu-s)}$, $-\nu < \operatorname{Re} s < \frac{1}{2}$	$\pi^{1/2}e^{-x/2}I_\nu(x/2)$
16	$\psi(s+a)-\psi(s+b)$, $\operatorname{Re} s > -a, -b$	$\begin{cases} \dfrac{x^b-x^a}{1-x} & \text{if } 0<x<1, \\ 0 & \text{if } 1<x \end{cases}$
17	$\Gamma(s)\psi(s)$, $\operatorname{Re} s > 0$	$e^{-x}\ln x$
18	$\Gamma(s,a)$, $a>0$	$\begin{cases} 0 & \text{if } 0<x<a, \\ e^{-x} & \text{if } a<x \end{cases}$
19	$\Gamma(s)\Gamma(1-s,a)$, $\operatorname{Re} s > 0$, $a>0$	$(x+1)^{-1}e^{-a(x+1)}$
20	$\gamma(s,a)$, $\operatorname{Re} s > 0$, $a>0$	$\begin{cases} e^{-x} & \text{if } 0<x<a, \\ 0 & \text{if } a<x \end{cases}$
21	$J_0\left(a\sqrt{b^2-s^2}\right)$, $a>0$	$\begin{cases} 0 & \text{if } 0<x<e^{-a}, \\ \dfrac{\cos\left(b\sqrt{a^2-\ln^2 x}\right)}{\pi\sqrt{a^2-\ln^2 x}} & \text{if } e^{-a}<x<e^a, \\ 0 & \text{if } e^a<x \end{cases}$
22	$s^{-1}I_0(s)$, $\operatorname{Re} s > 0$	$\begin{cases} 1 & \text{if } 0<x<e^{-1}, \\ \pi^{-1}\arccos(\ln x) & \text{if } e^{-1}<x<e, \\ 0 & \text{if } e<x \end{cases}$
23	$I_\nu(s)$, $\operatorname{Re} s > 0$	$\begin{cases} -\dfrac{2^\nu\sin(\pi\nu)}{\pi F(x)\sqrt{\ln^2 x-1}} & \text{if } 0<x<e^{-1}, \\ \dfrac{\cos\left[\nu\arccos(\ln x)\right]}{\pi\sqrt{1-\ln^2 x}} & \text{if } e^{-1}<x<e, \\ 0 & \text{if } e<x, \end{cases}$ $F(x) = \left(\sqrt{-1-\ln x}+\sqrt{1-\ln x}\right)^{2\nu}$
24	$s^{-1}I_\nu(s)$, $\operatorname{Re} s > 0$	$\begin{cases} \dfrac{2^\nu\sin(\pi\nu)}{\pi\nu F(x)} & \text{if } 0<x<e^{-1}, \\ \dfrac{\sin\left[\nu\arccos(\ln x)\right]}{\pi\nu} & \text{if } e^{-1}<x<e, \\ 0 & \text{if } e<x, \end{cases}$ $F(x) = \left(\sqrt{-1-\ln x}+\sqrt{1-\ln x}\right)^{2\nu}$
25	$s^{-\nu}I_\nu(s)$, $\operatorname{Re} s > -\frac{1}{2}$	$\begin{cases} 0 & \text{if } 0<x<e^{-1}, \\ \dfrac{(1-\ln^2 x)^{\nu-1/2}}{\sqrt{\pi}\,2^\nu\Gamma(\nu+1/2)} & \text{if } e^{-1}<x<e, \\ 0 & \text{if } e<x \end{cases}$
26	$s^{-1}K_0(s)$, $\operatorname{Re} s > 0$	$\begin{cases} \operatorname{Arcosh}(-\ln x) & \text{if } 0<x<e^{-1}, \\ 0 & \text{if } e^{-1}<x \end{cases}$
27	$s^{-1}K_1(s)$, $\operatorname{Re} s > 0$	$\begin{cases} \sqrt{\ln^2 x-1} & \text{if } 0<x<e^{-1}, \\ 0 & \text{if } e^{-1}<x \end{cases}$

No	Direct transform, $\hat{f}(s)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{\sigma-i\infty}^{\sigma+i\infty} \hat{f}(s)x^{-s}\,ds$
28	$K_\nu(s), \quad \operatorname{Re} s > 0$	$\begin{cases} \dfrac{\cosh\left[\nu \operatorname{Arcosh}(-\ln x)\right]}{\sqrt{\ln^2 x - 1}} & \text{if } 0 < x < e^{-1}, \\ 0 & \text{if } e^{-1} < x \end{cases}$
29	$s^{-1}K_\nu(s), \quad \operatorname{Re} s > 0$	$\begin{cases} \dfrac{1}{\nu}\sinh\left[\nu \operatorname{Arcosh}(-\ln x)\right] & \text{if } 0 < x < e^{-1}, \\ 0 & \text{if } e^{-1} < x \end{cases}$
30	$s^{-\nu}K_\nu(s), \quad \operatorname{Re} s > 0,\ \nu > -\frac{1}{2}$	$\begin{cases} \dfrac{\sqrt{\pi}\,(\ln^2 x - 1)^{\nu-1/2}}{2^\nu\Gamma(\nu+1/2)} & \text{if } 0 < x < e^{-1}, \\ 0 & \text{if } e^{-1} < x \end{cases}$

⦿ References for Supplement 9: H. Bateman and A. Erdélyi (1954), V. A. Ditkin and A. P. Prudnikov (1965).

Supplement 10

Special Functions and Their Properties

Throughout Supplement 10 it is assumed that n is a positive integer, unless otherwise specified.

10.1. Some Symbols and Coefficients

► **Factorial**

$$0! = 1! = 1, \quad n! = 1 \cdot 2 \cdot 3 \dots (n-1)n, \qquad n = 2, 3, \dots,$$

$$(2n)!! = 2 \cdot 4 \cdot 6 \dots (2n-2)(2n) = 2^n n!,$$

$$(2n+1)!! = 1 \cdot 3 \cdot 5 \dots (2n-1)(2n+1) = \frac{2^{n+1}}{\sqrt{\pi}} \Gamma\left(n + \frac{3}{2}\right),$$

$$n!! = \begin{cases} (2k)!! & \text{if } n = 2k, \\ (2k+1)!! & \text{if } n = 2k+1, \end{cases} \qquad 0!! = 1.$$

► **Binomial coefficients**

$$C_n^k = \frac{n!}{k!(n-k)!}, \qquad \text{where} \quad k = 1, \dots, n,$$

$$C_a^k = (-1)^k \frac{(-a)_k}{k!} = \frac{a(a-1)\dots(a-k+1)}{k!}, \qquad \text{where} \quad k = 1, 2, \dots$$

General case:

$$C_a^b = \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)}, \quad \text{where } \Gamma(x) \text{ is the gamma function.}$$

Properties:

$$C_a^0 = 1, \quad C_n^k = 0 \quad \text{for } k = -1, -2, \dots \text{ or } k > n,$$

$$C_a^{b+1} = \frac{a}{b+1} C_{a-1}^b = \frac{a-b}{b+1} C_a^b, \quad C_a^b + C_a^{b+1} = C_{a+1}^{b+1},$$

$$C_{-1/2}^n = \frac{(-1)^n}{2^{2n}} C_{2n}^n = (-1)^n \frac{(2n-1)!!}{(2n)!!},$$

$$C_{1/2}^n = \frac{(-1)^{n-1}}{n 2^{2n-1}} C_{2n-2}^{n-1} = \frac{(-1)^{n-1}}{n} \frac{(2n-3)!!}{(2n-2)!!},$$

$$C_{n+1/2}^{2n+1} = (-1)^n 2^{-4n-1} C_{2n}^n, \quad C_{2n+1/2}^n = 2^{-2n} C_{4n+1}^{2n},$$

$$C_n^{1/2} = \frac{2^{2n+1}}{\pi C_{2n}^n}, \quad C_n^{n/2} = \frac{2^{2n}}{\pi} C_n^{(n-1)/2}.$$

► **Pochhammer symbol** $(k = 1, 2, \ldots)$

$$(a)_n = a(a+1)\ldots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} = (-1)^n \frac{\Gamma(1-a)}{\Gamma(1-a-n)},$$
$$(a)_0 = 1, \quad (a)_{n+k} = (a)_n (a+n)_k, \quad (n)_k = \frac{(n+k-1)!}{(n-1)!},$$
$$(a)_{-n} = \frac{\Gamma(a-n)}{\Gamma(a)} = \frac{(-1)^n}{(1-a)_n}, \quad \text{where } a \neq 1, \ldots, n;$$
$$(1)_n = n!, \quad (1/2)_n = 2^{-2n} \frac{(2n)!}{n!}, \quad (3/2)_n = 2^{-2n} \frac{(2n+1)!}{n!},$$
$$(a+mk)_{nk} = \frac{(a)_{mk+nk}}{(a)_{mk}}, \quad (a+n)_n = \frac{(a)_{2n}}{(a)_n}, \quad (a+n)_k = \frac{(a)_k (a+k)_n}{(a)_n}.$$

► **Bernoulli numbers, B_n**
Definition:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

The numbers:

$$B_0 = 1, \quad B_1 = -\tfrac{1}{2}, \quad B_2 = \tfrac{1}{6}, \quad B_4 = -\tfrac{1}{30}, \quad B_6 = \tfrac{1}{42}, \quad B_8 = -\tfrac{1}{30}, \quad B_{10} = \tfrac{5}{66}, \quad \ldots,$$
$$B_{2m+1} = 0 \quad \text{for} \quad m = 1, 2, \ldots$$

10.2. Error Functions and Integral Exponent

► **Error function and complementary error function (probability integrals)**
Definitions:

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\,dt, \qquad \operatorname{erfc} x = 1 - \operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp(-t^2)\,dt.$$

Expansion of erf x into series in powers of x as $x \to 0$:

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(k)!(2k+1)} = \frac{2}{\sqrt{\pi}} \exp\bigl(-x^2\bigr) \sum_{k=0}^{\infty} \frac{2^k x^{2k+1}}{2k+1)!!}.$$

Asymptotic expansion of erfc x as $x \to \infty$:

$$\operatorname{erfc} x = \frac{1}{\sqrt{\pi}} \exp\bigl(-x^2\bigr) \left[\sum_{m=0}^{M-1} (-1)^m \frac{\left(\frac{1}{2}\right)_m}{x^{2m+1}} + O\bigl(|x|^{-2M-1}\bigr) \right], \qquad M = 1, 2, \ldots$$

► **Integral exponent**
Definition:

$$\operatorname{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t}\,dt \qquad \text{for} \quad x < 0,$$
$$\operatorname{Ei}(x) = \lim_{\varepsilon \to +0} \left(\int_{-\infty}^{-\varepsilon} \frac{e^t}{t}\,dt + \int_{\varepsilon}^{x} \frac{e^t}{t}\,dt \right) \quad \text{for} \quad x > 0.$$

Other integral representations:

$$\operatorname{Ei}(-x) = -e^{-x}\int_0^\infty \frac{x\sin t + t\cos t}{x^2+t^2}\,dt \quad \text{for} \quad x>0,$$
$$\operatorname{Ei}(-x) = e^{-x}\int_0^\infty \frac{x\sin t - t\cos t}{x^2+t^2}\,dt \quad \text{for} \quad x<0,$$
$$\operatorname{Ei}(-x) = -x\int_1^\infty e^{-xt}\ln t\,dt \qquad \text{for} \quad x>0.$$

Expansion into series in powers of x as $x \to 0$:

$$\operatorname{Ei}(x) = \begin{cases} \mathcal{C} + \ln(-x) + \displaystyle\sum_{k=1}^\infty \frac{x^k}{k\cdot k!} & \text{if } x<0, \\ \mathcal{C} + \ln x + \displaystyle\sum_{k=1}^\infty \frac{x^k}{k\cdot k!} & \text{if } x>0, \end{cases}$$

where $\mathcal{C} = 0.5572\ldots$ is the Euler constant.

Asymptotic expansion as $x \to \infty$:

$$\operatorname{Ei}(-x) = e^{-x}\sum_{k=1}^n (-1)^k \frac{(k-1)!}{x^k} + R_n, \qquad R_n < \frac{n!}{x^n}.$$

► **Integral logarithm**

Definition:

$$\operatorname{li}(x) = \begin{cases} \displaystyle\int_0^x \frac{dt}{\ln t} = \operatorname{Ei}(\ln x) & \text{if } 0<x<1, \\ \displaystyle\lim_{\varepsilon\to+0}\left(\int_0^{1-\varepsilon}\frac{dt}{\ln t} + \int_{1+\varepsilon}^x \frac{dt}{\ln t}\right) & \text{if } x>1. \end{cases}$$

For small x,

$$\operatorname{li}(x) \approx \frac{x}{\ln(1/x)}.$$

Asymptotic expansion as $x \to 1$:

$$\operatorname{li}(x) = \mathcal{C} + \ln|\ln x| + \sum_{k=1}^\infty \frac{\ln^k x}{k\cdot k!}.$$

10.3. Integral Sine and Integral Cosine. Fresnel Integrals

► **Integral sine**

Definition:

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt, \qquad \operatorname{si}(x) = -\int_x^\infty \frac{\sin t}{t}\,dt = \operatorname{Si}(x) - \frac{\pi}{2}.$$

Specific values:

$$\operatorname{Si}(0) = 0, \quad \operatorname{Si}(\infty) = \frac{\pi}{2}, \quad \operatorname{si}(\infty) = 0.$$

Properties:

$$\operatorname{Si}(-x) = -\operatorname{Si}(x), \quad \operatorname{si}(x) + \operatorname{si}(-x) = -\pi, \quad \lim_{x\to-\infty}\operatorname{si}(x) = -\pi.$$

Expansion into series in powers of x as $x \to 0$:

$$\operatorname{Si}(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{2k-1}}{(2k-1)\,(2k-1)!}.$$

Asymptotic expansion as $x \to \infty$:

$$\operatorname{si}(x) = -\cos x \left[\sum_{m=0}^{M-1} \frac{(-1)^m (2m)!}{x^{2m+1}} + O\big(|x|^{-2M-1}\big) \right] + \sin x \left[\sum_{m=1}^{N-1} \frac{(-1)^m (2m-1)!}{x^{2m}} + O\big(|x|^{-2N}\big) \right],$$

where $M, N = 1, 2, \ldots$

► **Integral cosine**

Definition:

$$\operatorname{Ci}(x) = -\int_x^{\infty} \frac{\cos t}{t}\,dt = \mathcal{C} + \ln x + \int_0^x \frac{\cos t - 1}{t}\,dt, \qquad \mathcal{C} = 0.5572\ldots$$

Expansion into series in powers of x as $x \to 0$:

$$\operatorname{Ci}(x) = C + \ln x + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{2k\,(2k)!}.$$

Asymptotic expansion as $x \to \infty$:

$$\operatorname{Ci}(x) = \cos x \left[\sum_{m=1}^{M-1} \frac{(-1)^m (2m-1)!}{x^{2m}} + O\big(|x|^{-2M}\big) \right] + \sin x \left[\sum_{m=0}^{N-1} \frac{(-1)^m (2m)!}{x^{2m+1}} + O\big(|x|^{-2N-1}\big) \right],$$

where $M, N = 1, 2, \ldots$

► **Fresnel integrals**

Definitions:

$$S(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\sin t}{\sqrt{t}}\,dt = \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{x}} \sin t^2, dt,$$

$$C(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\cos t}{\sqrt{t}}\,dt = \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{x}} \cos t^2\,dt.$$

Expansion into series in powers of x as $x \to 0$:

$$S(x) = \sqrt{\frac{2}{\pi}}\, x \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(4k+3)\,(2k+1)!},$$

$$C(x) = \sqrt{\frac{2}{\pi}}\, x \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(4k+1)\,(2k)!}.$$

Asymptotic expansion as $x \to \infty$:

$$S(x) = \frac{1}{2} - \frac{\cos x}{\sqrt{2\pi x}} P(x) - \frac{\sin x}{\sqrt{2\pi x}} Q(x),$$

$$C(x) = \frac{1}{2} + \frac{\sin x}{\sqrt{2\pi x}} P(x) - \frac{\cos x}{\sqrt{2\pi x}} Q(x),$$

$$P(x) = 1 - \frac{1 \cdot 3}{(2x)^2} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{(2x)^4} - \cdots, \quad Q(x) = \frac{1}{2x} - \frac{1 \cdot 3 \cdot 5}{(2x)^3} + \cdots.$$

10.4. Gamma Function. Beta Function

► **Definition. Integral representations**

The gamma function, $\Gamma(z)$, is an analytic function of the complex argument z everywhere, except for the points $z = 0, -1, -2, \ldots$

For $\operatorname{Re} z > 0$,

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt.$$

For $-(n+1) < \operatorname{Re} z < -n$, where $n = 0, 1, 2, \ldots,$

$$\Gamma(z) = \int_0^\infty \left[e^{-t} - \sum_{m=0}^{n} \frac{(-1)^m}{m!} \right] t^{z-1}\, dt.$$

► **Euler formula**

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\ldots(z+n)} \qquad (z \neq 0, -1, -2, \ldots).$$

► **Simplest properties**

$$\Gamma(z+1) = z\Gamma(z), \quad \Gamma(n+1) = n!, \quad \Gamma(1) = \Gamma(2) = 1.$$

► **Symmetry formulas**

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z\sin(\pi z)}, \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$
$$\Gamma\left(\frac{1}{2}+z\right)\Gamma\left(\frac{1}{2}-z\right) = \frac{\pi}{\cos(\pi z)}.$$

► **Multiple argument formulas**

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right),$$
$$\Gamma(3z) = \frac{3^{3z-1/2}}{2\pi}\Gamma(z)\Gamma\left(z+\frac{1}{3}\right)\Gamma\left(z+\frac{2}{3}\right),$$
$$\Gamma(nz) = (2\pi)^{(1-n)/2} n^{nz-1/2} \prod_{k=0}^{n-1} \Gamma\left(z+\frac{k}{n}\right).$$

► **Fractional values of the argument**

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \qquad \Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n}(2n-1)!!,$$
$$\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}, \quad \Gamma\left(\frac{1}{2}-n\right) = (-1)^n \frac{2^n\sqrt{\pi}}{(2n-1)!!}.$$

► **Asymptotic expansion (Stirling formula)**

$$\Gamma(z) = \sqrt{2\pi}\, e^{-z} z^{z-1/2} \left[1 + \tfrac{1}{12} z^{-1} + \tfrac{1}{288} z^{-2} + O(z^{-3})\right] \qquad (|\arg| z < \pi).$$

▶ **Logarithmic derivative of the gamma function**

Definition:

$$\psi(z) = \frac{\ln\Gamma(z)}{dz} = \frac{\Gamma'_z(z)}{\Gamma(z)}.$$

Functional relations:

$$\begin{aligned}
&\psi(z) - \psi(1+z) = -\frac{1}{z},\\
&\psi(z) - \psi(1-z) = -\pi\cot(\pi z),\\
&\psi(z) - \psi(-z) = -\pi\cot(\pi z) - \frac{1}{z},\\
&\psi\left(\tfrac{1}{2}+z\right) - \psi\left(\tfrac{1}{2}-z\right) = \pi\tan(\pi z),\\
&\psi(mz) = \ln m + \frac{1}{m}\sum_{k=0}^{m-1}\psi\left(z+\frac{k}{m}\right).
\end{aligned}$$

Integral representations ($\operatorname{Re} z > 0$):

$$\begin{aligned}
&\psi(z) = \int_0^\infty \left[e^{-t} - (1+t)^{-z}\right]t^{-1}\,dt,\\
&\psi(z) = \ln z + \int_0^\infty \left[t^{-1} - (1-e^{-t})^{-1}\right]e^{-tz}\,dt,\\
&\psi(z) = -\mathcal{C} + \int_0^1 \frac{1-t^{z-1}}{1-t}\,dt,
\end{aligned}$$

where $\mathcal{C} = -\psi(1) = 0.5572\ldots$ is the Euler constant.

Values for integer argument:

$$\psi(1) = -\mathcal{C}, \qquad \psi(n) = -\mathcal{C} + \sum_{k=1}^{n-1} k^{-1} \quad (n = 2, 3, \ldots)$$

▶ **Beta function**

Definition:

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt,$$

where $\operatorname{Re} x > 0$ and $\operatorname{Re} y > 0$.

Relationship with the gamma function:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

10.5. Incomplete Gamma Function

▶ **Definitions. Integral representations**

$$\begin{aligned}
&\gamma(\alpha,x) = \int_0^x e^{-t}t^{\alpha-1}\,dt, \qquad \operatorname{Re}\alpha > 0,\\
&\Gamma(\alpha,x) = \int_x^\infty e^{-t}t^{\alpha-1}\,dt = \Gamma(\alpha) - \gamma(\alpha,x).
\end{aligned}$$

► **Recurrent formulas**

$$\gamma(\alpha+1,x)=\alpha\gamma(\alpha,x)-x^{\alpha}e^{-x},$$
$$\Gamma(\alpha+1,x)=\alpha\Gamma(\alpha,x)+x^{\alpha}e^{-x}.$$

► **Asymptotic expansions as $x \to 0$:**

$$\gamma(\alpha,x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{\alpha+n}}{n!\,(\alpha+n)},$$
$$\Gamma(\alpha,x)=\Gamma(\alpha)-\sum_{n=0}^{\infty}\frac{(-1)^n x^{\alpha+n}}{n!\,(\alpha+n)}.$$

► **Asymptotic expansions as $x \to \infty$:**

$$\gamma(\alpha,x)=\Gamma(\alpha)-x^{\alpha-1}e^{-x}\left[\sum_{m=0}^{M-1}\frac{(1-\alpha)_m}{(-x)^m}+O\bigl(|x|^{-M}\bigr)\right],$$
$$\Gamma(\alpha,x)=x^{\alpha-1}e^{-x}\left[\sum_{m=0}^{M-1}\frac{(1-\alpha)_m}{(-x)^m}+O\bigl(|x|^{-M}\bigr)\right]\quad\bigl(-\tfrac{3}{2}\pi<\arg x<\tfrac{3}{2}\bigr).$$

► **Integral functions related to the gamma function:**

$$\operatorname{erf}x=\frac{1}{\sqrt{\pi}}\gamma\Bigl(\frac{1}{2},\,x^2\Bigr),\quad \operatorname{erfc}x=\frac{1}{\sqrt{\pi}}\Gamma\Bigl(\frac{1}{2},\,x^2\Bigr),\quad \operatorname{Ei}(-x)=-\Gamma(0,x).$$

► **Incomplete beta function:**

$$B_x(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}\,dt,$$

where $\operatorname{Re}x>0$ and $\operatorname{Re}y>0$.

10.6. Bessel Functions

► **Definition and basic formulas**

The Bessel function of the first kind, $J_\nu(x)$, and the Bessel function of the second kind, $Y_\nu(x)$ (also called the Neumann function), are solutions of the Bessel equation

$$x^2y''_{xx}+xy'_x+(x^2-\nu^2)y=0$$

and are defined by the formulas

$$J_\nu(x)=\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)},\quad Y_\nu(x)=\frac{J_\nu(x)\cos\pi\nu-J_{-\nu}(x)}{\sin\pi\nu}.\tag{1}$$

The formula for $Y_\nu(x)$ is valid for $\nu\neq 0,\pm1,\pm2,\ldots$ (the cases $\nu\neq 0,\pm1,\pm2,\ldots$ are discussed in what follows).

The general solution of the Bessel equation has the form $Z_\nu(x)=C_1J_\nu(x)+C_2Y_\nu(x)$ and is called the cylinder function.

The Bessel functions possess the properties

$$2\nu Z_\nu(x) = x[Z_{\nu-1}(x) + Z_{\nu+1}(x)],$$
$$\frac{d}{dx} Z_\nu(x) = \frac{1}{2}[Z_{\nu-1}(x) - Z_{\nu+1}(x)] = \pm\left[\frac{\nu}{x} Z_\nu(x) - Z_{\nu\pm1}(x)\right],$$
$$\frac{d}{dx}[x^\nu Z_\nu(x)] = x^\nu Z_{\nu-1}(x), \qquad \frac{d}{dx}[x^{-\nu} Z_\nu(x)] = -x^{-\nu} Z_{\nu+1}(x),$$
$$\left(\frac{1}{x}\frac{d}{dx}\right)^n [x^\nu J_\nu(x)] = x^{\nu-n} J_{\nu-n}(x), \qquad \left(\frac{1}{x}\frac{d}{dx}\right)^n [x^{-\nu} J_\nu(x)] = (-1)^n x^{-\nu-n} J_{\nu+n}(x),$$
$$J_{-n}(x) = (-1)^n J_n(x), \quad Y_{-n}(x) = (-1)^n Y_n(x), \qquad n = 0, 1, 2, \ldots$$

► **The Bessel functions for $\nu = \pm n \pm \frac{1}{2}$; $n = 0, 1, \ldots$**

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x,$$
$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{1}{x}\sin x - \cos x\right), \qquad J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(-\frac{1}{x}\cos x - \sin x\right),$$

$$J_{n+1/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\sin\left(x - \frac{n\pi}{2}\right)\sum_{k=0}^{[n/2]} \frac{(-1)^k (n+2k)!}{(2k)!\,(n-2k)!\,(2x)^{2k}} + \cos\left(x - \frac{n\pi}{2}\right)\sum_{k=0}^{[(n-1)/2]} \frac{(-1)^k (n+2k+1)!}{(2k+1)!\,(n-2k-1)!\,(2x)^{2k+1}}\right],$$
$$J_{-n-1/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\cos\left(x + \frac{n\pi}{2}\right)\sum_{k=0}^{[n/2]} \frac{(-1)^k (n+2k)!}{(2k)!\,(n-2k)!\,(2x)^{2k}} - \sin\left(x + \frac{n\pi}{2}\right)\sum_{k=0}^{[(n-1)/2]} \frac{(-1)^k (n+2k+1)!}{(2k+1)!\,(n-2k-1)!\,(2x)^{2k+1}}\right],$$

$$Y_{1/2}(x) = -\sqrt{\frac{2}{\pi x}} \cos x, \qquad Y_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x,$$
$$Y_{n+1/2}(x) = (-1)^{n+1} J_{-n-1/2}(x), \qquad Y_{-n-1/2}(x) = (-1)^n J_{n+1/2}(x).$$

► **The Bessel functions for $\nu = \pm n$; $n = 0, 1, 2, \ldots$**

Let $\nu = n$ be an arbitrary integer. The relations

$$J_{-n}(x) = (-1)^n J_n(x), \quad Y_{-n}(x) = (-1)^n Y_n(x)$$

are valid. The function $J_n(x)$ is given by the first formula in (1) with $\nu = n$, and $Y_n(x)$ can be obtained from the second formula in (1) by proceeding to the limit $\nu \to n$. For nonnegative n, $Y_n(x)$ can be represented in the form

$$Y_n(x) = \frac{2}{\pi} J_n(x) \ln\frac{x}{2} - \frac{1}{\pi}\sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}\left(\frac{2}{x}\right)^{n-2k} - \frac{1}{\pi}\sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{2}\right)^{n+2k} \frac{\psi(k+1) + \psi(n+k+1)}{k!\,(n+k)!},$$

where $\psi(1) = -\mathcal{C}$, $\psi(n) = -\mathcal{C} + \sum_{k=1}^{n-1} k^{-1}$, $\mathcal{C} = 0.5572\ldots$ is the Euler constant, $\psi(x) = [\ln\Gamma(x)]'_x$ is the logarithmic derivative of the gamma function.

► **Wronskians and similar formulas**

$$W(J_\nu, J_{-\nu}) = -\frac{2}{\pi x}\sin(\pi\nu), \qquad W(J_\nu, Y_\nu) = \frac{2}{\pi x},$$

$$J_\nu(x)J_{-\nu+1}(x) + J_{-\nu}(x)J_{\nu-1}(x) = \frac{2\sin(\pi\nu)}{\pi x}, \qquad J_\nu(x)Y_{\nu+1}(x) - J_{\nu+1}(x)Y_\nu(x) = -\frac{2}{\pi x}.$$

Here the notation $W(f, g) = fg'_x - f'_x g$ is used.

► **Integral representations**

The functions J_ν and Y_ν can be represented in the form of definite integrals (for $x > 0$):

$$\pi J_\nu(x) = \int_0^\pi \cos(x\sin\theta - \nu\theta)\,d\theta - \sin\pi\nu\int_0^\infty \exp(-x\sinh t - \nu t)\,dt,$$

$$\pi Y_\nu(x) = \int_0^\pi \sin(x\sin\theta - \nu\theta)\,d\theta - \int_0^\infty (e^{\nu t} + e^{-\nu t}\cos\pi\nu)e^{-x\sinh t}\,dt.$$

For $|\nu| < \frac{1}{2}$, $x > 0$,

$$J_\nu(x) = \frac{2^{1+\nu}x^{-\nu}}{\pi^{1/2}\Gamma(\frac{1}{2}-\nu)}\int_1^\infty \frac{\sin(xt)\,dt}{(t^2-1)^{\nu+1/2}},$$

$$Y_\nu(x) = -\frac{2^{1+\nu}x^{-\nu}}{\pi^{1/2}\Gamma(\frac{1}{2}-\nu)}\int_1^\infty \frac{\cos(xt)\,dt}{(t^2-1)^{\nu+1/2}}.$$

For $\nu > -\frac{1}{2}$,

$$J_\nu(x) = \frac{2(x/2)^\nu}{\pi^{1/2}\Gamma(\frac{1}{2}+\nu)}\int_0^{\pi/2}\cos(x\cos t)\sin^{2\nu}t\,dt \quad \text{(Poisson's formula).}$$

For $\nu = 0$, $x > 0$,

$$J_0(x) = \frac{2}{\pi}\int_0^\infty \sin(x\cosh t)\,dt, \qquad Y_0(x) = -\frac{2}{\pi}\int_0^\infty \cos(x\cosh t)\,dt.$$

For integer $\nu = n = 0, 1, 2, \ldots,$

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(nt - x\sin t)\,dt \quad \text{(Bessel's formula),}$$

$$J_{2n}(x) = \frac{2}{\pi}\int_0^{\pi/2}\cos(x\sin t)\cos(2nt)\,dt,$$

$$J_{2n+1}(x) = \frac{2}{\pi}\int_0^{\pi/2}\sin(x\sin t)\sin[(2n+1)t]\,dt.$$

► **Integrals with Bessel functions**

$$\int_0^x x^\lambda J_\nu(x)\,dx = \frac{x^{\lambda+\nu+1}}{2^\nu(\lambda+\nu+1)\,\Gamma(\nu+1)}F\Big(\frac{\lambda+\nu+1}{2},\ \frac{\lambda+\nu+3}{2},\ \nu+1;\ -\frac{x^2}{4}\Big), \qquad \mathrm{Re}(\lambda+\nu) > -1,$$

where $F(a, b, c; x)$ is the hypergeometric series (see Section 10.9 of this supplement),

$$\int_0^x x^\lambda Y_\nu(x)\,dx = -\frac{\cos(\nu\pi)\Gamma(-\nu)}{2^\nu\pi(\lambda+\nu+1)}x^{\lambda+\nu+1}F\Big(\frac{\lambda+\nu+1}{2},\ \nu+1,\ \frac{\lambda+\nu+3}{2},\ -\frac{x^2}{4}\Big)$$
$$-\frac{2^\nu\Gamma(\nu)}{\lambda-\nu+1}x^{\lambda-\nu+1}F\Big(\frac{\lambda-\nu+1}{2},\ 1-\nu,\ \frac{\lambda-\nu+3}{2},\ -\frac{x^2}{4}\Big), \qquad \mathrm{Re}\,\lambda > |\mathrm{Re}\,\nu| - 1.$$

► **Asymptotic expansions as $|x| \to \infty$**

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\left\{\cos\left(\frac{4x - 2\nu\pi - \pi}{4}\right)\left[\sum_{m=0}^{M-1}(-1)^m(\nu, 2m)(2x)^{-2m} + O(|x|^{-2M})\right]\right.$$
$$\left. - \sin\left(\frac{4x - 2\nu\pi - \pi}{4}\right)\left[\sum_{m=0}^{M-1}(-1)^m(\nu, 2m+1)(2x)^{-2m-1} + O(|x|^{-2M-1})\right]\right\},$$
$$Y_\nu(x) = \sqrt{\frac{2}{\pi x}}\left\{\sin\left(\frac{4x - 2\nu\pi - \pi}{4}\right)\left[\sum_{m=0}^{M-1}(-1)^m(\nu, 2m)(2x)^{-2m} + O(|x|^{-2M})\right]\right.$$
$$\left. + \cos\left(\frac{4x - 2\nu\pi - \pi}{4}\right)\left[\sum_{m=0}^{M-1}(-1)^m(\nu, 2m+1)(2x)^{-2m-1} + O(|x|^{-2M-1})\right]\right\},$$

where $(\nu, m) = \dfrac{1}{2^{2m}m!}(4\nu^2 - 1)(4\nu^2 - 3^2)\ldots[4\nu^2 - (2m-1)^2] = \dfrac{\Gamma(\frac{1}{2} + \nu + m)}{m!\,\Gamma(\frac{1}{2} + \nu - m)}$.

For nonnegative integer n and large x,

$$\sqrt{\pi x}\, J_{2n}(x) = (-1)^n(\cos x + \sin x) + O(x^{-2}),$$
$$\sqrt{\pi x}\, J_{2n+1}(x) = (-1)^{n+1}(\cos x - \sin x) + O(x^{-2}).$$

► **Asymptotic for large ν ($\nu \to \infty$).**

$$J_\nu(x) \to \frac{1}{\sqrt{2\pi\nu}}\left(\frac{ex}{2\nu}\right)^\nu, \quad Y_\nu(x) \to -\sqrt{\frac{2}{\pi\nu}}\left(\frac{ex}{2\nu}\right)^{-\nu},$$

where x is fixed,

$$J_\nu(\nu) \to \frac{2^{1/3}}{3^{2/3}\Gamma(2/3)}\frac{1}{\nu^{1/3}}, \quad Y_\nu(\nu) \to -\frac{2^{1/3}}{3^{1/6}\Gamma(2/3)}\frac{1}{\nu^{1/3}}.$$

► **Zeros of Bessel functions**

Each of the functions $J_\nu(x)$ and $Y_\nu(x)$ has infinitely many real zeros (for real ν). All zeros are simple, possibly except for the point $x = 0$.

The zeros γ_m of $J_0(x)$, i.e., the roots of the equation $J_0(\gamma_m) = 0$, are approximately given by

$$\gamma_m = 2.4 + 3.13\,(m-1) \qquad (m = 1, 2, \ldots),$$

with maximum error 0.2%.

► **Hankel functions (Bessel functions of the third kind)**

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z), \quad H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z), \qquad i^2 = -1.$$

10.7. Modified Bessel Functions

► **Definitions. Basic formulas**

The modified Bessel functions of the first kind, $I_\nu(x)$, and the second kind, $K_\nu(x)$ (also called the Macdonald function), of order ν are solutions of the modified Bessel equation

$$x^2y''_{xx} + xy'_x - (x^2 + \nu^2)y = 0$$

and are defined by the formulas

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+\nu}}{k!\,\Gamma(\nu+k+1)}, \qquad K_\nu(x) = \frac{\pi}{2}\,\frac{I_{-\nu} - I_\nu}{\sin \pi\nu},$$

(see below for $K_\nu(x)$ with $\nu = 0, 1, 2, \dots$).

The modified Bessel functions possess the properties

$$K_{-\nu}(x) = K_\nu(x); \qquad I_{-n}(x) = (-1)^n I_n(x), \quad n = 0, 1, 2, \dots$$
$$2\nu I_\nu(x) = x[I_{\nu-1}(x) - I_{\nu+1}(x)], \qquad 2\nu K_\nu(x) = -x[K_{\nu-1}(x) - K_{\nu+1}(x)],$$
$$\frac{d}{dx} I_\nu(x) = \frac{1}{2}[I_{\nu-1}(x) + I_{\nu+1}(x)], \qquad \frac{d}{dx} K_\nu(x) = -\frac{1}{2}[K_{\nu-1}(x) + K_{\nu+1}(x)].$$

► **Modified Bessel functions for $\nu = \pm n \pm \frac{1}{2}$, where $n = 0, 1, 2, \dots$**

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\,\sinh x, \qquad I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\,\cosh x,$$
$$I_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(-\frac{1}{x}\sinh x + \cosh x\right), \quad I_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(-\frac{1}{x}\cosh x + \sinh x\right),$$
$$I_{n+1/2}(x) = \frac{1}{\sqrt{2\pi x}}\left[e^x \sum_{k=0}^{n} \frac{(-1)^k (n+k)!}{k!\,(n-k)!\,(2x)^k} - (-1)^n e^{-x} \sum_{k=0}^{n} \frac{(n+k)!}{k!\,(n-k)!\,(2x)^k}\right],$$
$$I_{-n-1/2}(x) = \frac{1}{\sqrt{2\pi x}}\left[e^x \sum_{k=0}^{n} \frac{(-1)^k (n+k)!}{k!\,(n-k)!\,(2x)^k} + (-1)^n e^{-x} \sum_{k=0}^{n} \frac{(n+k)!}{k!\,(n-k)!\,(2x)^k}\right],$$
$$K_{\pm 1/2}(x) = \sqrt{\frac{\pi}{2x}}\,e^{-x}, \qquad K_{\pm 3/2}(x) = \sqrt{\frac{\pi}{2x}}\left(1 + \frac{1}{x}\right)e^{-x},$$
$$K_{n+1/2}(x) = K_{-n-1/2}(x) = \sqrt{\frac{\pi}{2x}}\,e^{-x} \sum_{k=0}^{n} \frac{(n+k)!}{k!\,(n-k)!\,(2x)^k}.$$

► **Modified Bessel functions $\nu = n$, where $n = 0, 1, 2, \dots$**

If $\nu = n$ is a nonnegative integer, then

$$K_n(x) = (-1)^{n+1} I_n(x) \ln\frac{x}{2} + \frac{1}{2}\sum_{m=0}^{n-1} (-1)^m \left(\frac{x}{2}\right)^{2m-n} \frac{(n-m-1)!}{m!}$$
$$+ \frac{1}{2}(-1)^n \sum_{m=0}^{\infty} \left(\frac{x}{2}\right)^{n+2m} \frac{\psi(n+m+1) + \psi(m+1)}{m!\,(n+m)!}; \quad n = 0, 1, 2, \dots,$$

where $\psi(z)$ is the logarithmic derivative of the gamma function; for $n = 0$, the first sum is dropped.

► **Wronskians and similar formulas.**

$$W(I_\nu, I_{-\nu}) = -\frac{2}{\pi x}\sin(\pi\nu), \quad W(I_\nu, K_\nu) = -\frac{1}{x},$$
$$I_\nu(x) I_{-\nu+1}(x) - I_{-\nu}(x) I_{\nu-1}(x) = -\frac{2\sin(\pi\nu)}{\pi x}, \quad I_\nu(x) K_{\nu+1}(x) + I_{\nu+1}(x) K_\nu(x) = \frac{1}{x},$$

where $W(f, g) = f g'_x - f'_x g$.

► **Integral representations.**

The functions $I_\nu(x)$ and $K_\nu(x)$ can be represented in terms of definite integrals:

$$I_\nu(x) = \frac{x^\nu}{\pi^{1/2}2^\nu\Gamma(\nu+\frac{1}{2})}\int_{-1}^{1}\exp(-xt)(1-t^2)^{\nu-1/2}\,dt \qquad (x>0,\ \nu>-\tfrac{1}{2}),$$

$$K_\nu(x) = \int_0^\infty \exp(-x\cosh t)\cosh(\nu t)\,dt \qquad (x>0),$$

$$K_\nu(x) = \frac{1}{\cos\left(\frac{1}{2}\pi\nu\right)}\int_0^\infty \cos(x\sinh t)\cosh(\nu t)\,dt \qquad (x>0,\ -1<\nu<1),$$

$$K_\nu(x) = \frac{1}{\sin\left(\frac{1}{2}\pi\nu\right)}\int_0^\infty \sin(x\sinh t)\sinh(\nu t)\,dt \qquad (x>0,\ -1<\nu<1).$$

For integer $\nu = n$,

$$I_n(x) = \frac{1}{\pi}\int_0^\pi \exp(x\cos t)\cos(nt)\,dt \qquad (n=0,1,2,\dots),$$

$$K_0(x) = \int_0^\infty \cos(x\sinh t)\,dt = \int_0^\infty \frac{\cos(xt)}{\sqrt{t^2+1}}\,dt \qquad (x>0).$$

► **Integrals with modified Bessel functions**

$$\int_0^x x^\lambda I_\nu(x)\,dx = \frac{x^{\lambda+\nu+1}}{2^\nu(\lambda+\nu+1)\Gamma(\nu+1)}F\left(\frac{\lambda+\nu+1}{2},\ \frac{\lambda+\nu+3}{2},\ \nu+1;\ \frac{x^2}{4}\right), \qquad \mathrm{Re}(\lambda+\nu)>-1,$$

where $F(a,b,c;x)$ is the hypergeometric series (see Section 10.9 of this supplement),

$$\int_0^x x^\lambda K_\nu(x)\,dx = \frac{2^{\nu-1}\Gamma(\nu)}{\lambda-\nu+1}x^{\lambda-\nu+1}F\left(\frac{\lambda-\nu+1}{2},\ 1-\nu,\ \frac{\lambda-\nu+3}{2},\ \frac{x^2}{4}\right)$$
$$+\frac{2^{-\nu-1}\Gamma(-\nu)}{\lambda+\nu+1}x^{\lambda+\nu+1}F\left(\frac{\lambda+\nu+1}{2},\ 1+\nu,\ \frac{\lambda+\nu+3}{2},\ \frac{x^2}{4}\right), \qquad \mathrm{Re}\,\lambda > |\mathrm{Re}\,\nu|-1.$$

► **Asymptotic expansions as $x \to \infty$**

$$I_\nu(x) = \frac{e^x}{\sqrt{2\pi x}}\left\{1+\sum_{m=1}^{M}(-1)^m\frac{(4\nu^2-1)(4\nu^2-3^2)\dots[4\nu^2-(2m-1)^2]}{m!\,(8x)^m}\right\},$$

$$K_\nu(x) = \sqrt{\frac{\pi}{2x}}\,e^{-x}\left\{1+\sum_{m=1}^{M}\frac{(4\nu^2-1)(4\nu^2-3^2)\dots[4\nu^2-(2m-1)^2]}{m!\,(8x)^m}\right\}.$$

The terms of the order of $O(x^{-M-1})$ are omitted in the braces.

10.8. Degenerate Hypergeometric Functions

► **Definitions. Basic Formulas**

The degenerate hypergeometric functions $\Phi(a,b;x)$ and $\Psi(a,b;x)$ are solutions of the degenerate hypergeometric equation

$$xy''_{xx}+(b-x)y'_x-ay=0.$$

TABLE S1
Special cases of the Kummer function $\Phi(a,b;z)$

a	b	z	Φ	Conventional notation
a	a	x	e^x	
1	2	$2x$	$\frac{1}{x}e^x \sinh x$	
a	$a+1$	$-x$	$ax^{-a}\gamma(a,x)$	Incomplete gamma function $\gamma(a,x)=\int_0^x e^{-t}t^{a-1}\,dt$
$\frac{1}{2}$	$\frac{3}{2}$	$-x^2$	$\frac{\sqrt{\pi}}{2}\operatorname{erf} x$	Error function $\operatorname{erf} x=\frac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)\,dt$
$-n$	$\frac{1}{2}$	$\frac{x^2}{2}$	$\frac{n!}{(2n)!}\left(-\frac{1}{2}\right)^{-n}H_{2n}(x)$	Hermite polynomials $H_n=(-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)$, $n=0,1,2,\dots$
$-n$	$\frac{3}{2}$	$\frac{x^2}{2}$	$\frac{n!}{(2n+1)!}\left(-\frac{1}{2}\right)^{-n}H_{2n+1}(x)$	
$-n$	b	x	$\frac{n!}{(b)_n}L_n^{(b-1)}(x)$	Laguerre polynomials $L_n^{(\alpha)}(x)=\frac{e^x x^{-\alpha}}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^{n+\alpha}\right)$, $\alpha=b-1$, $(b)_n=b(b+1)\dots(b+n-1)$
$\nu+\frac{1}{2}$	$2\nu+1$	$2x$	$\Gamma(1+\nu)e^x\left(\frac{x}{2}\right)^{-\nu}I_\nu(x)$	Modified Bessel functions $I_\nu(x)$
$n+1$	$2n+2$	$2x$	$\Gamma\left(n+\frac{3}{2}\right)e^x\left(\frac{x}{2}\right)^{-n-\frac{1}{2}}I_{n+\frac{1}{2}}(x)$	

In the case $b \neq 0, -1, -2, -3, \dots$, the function $\Phi(a,b;x)$ can be represented as Kummer's series:

$$\Phi(a,b;x) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k}{(b)_k}\frac{x^k}{k!},$$

where $(a)_k = a(a+1)\dots(a+k-1)$, $(a)_0 = 1$.

Table S1 presents some special cases when Φ can be expressed in terms of simpler functions.

The function $\Psi(a,b;x)$ is defined as follows:

$$\Psi(a,b;x) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)}\Phi(a,b;x) + \frac{\Gamma(b-1)}{\Gamma(a)}x^{1-b}\Phi(a-b+1,\ 2-b;\ x).$$

► **Some transformations and linear relations**

Kummer transformation:

$$\Phi(a,b;x) = e^x\Phi(b-a,b;-x), \qquad \Psi(a,b;x) = x^{1-b}\Psi(1+a-b,2-b;x).$$

Linear relations for Φ:

$$(b-a)\Phi(a-1,b;x)+(2a-b+x)\Phi(a,b;x)-a\Phi(a+1,b;x)=0,$$
$$b(b-1)\Phi(a,b-1;x)-b(b-1+x)\Phi(a,b;x)+(b-a)x\Phi(a,b+1;x)=0,$$
$$(a-b+1)\Phi(a,b;x)-a\Phi(a+1,b;x)+(b-1)\Phi(a,b-1;x)=0,$$
$$b\Phi(a,b;x)-b\Phi(a-1,b;x)-x\Phi(a,b+1;x)=0,$$
$$b(a+x)\Phi(a,b;x)-(b-a)x\Phi(a,b+1;x)-ab\Phi(a+1,b;x)=0,$$
$$(a-1+x)\Phi(a,b;x)+(b-a)\Phi(a-1,b;x)-(b-1)\Phi(a,b-1;x)=0.$$

Linear relations for Ψ:

$$\Psi(a-1,b;x)-(2a-b+x)\Psi(a,b;x)+a(a-b+1)\Psi(a+1,b;x)=0,$$
$$(b-a-1)\Psi(a,b-1;x)-(b-1+x)\Psi(a,b;x)+x\Psi(a,b+1;x)=0,$$
$$\Psi(a,b;x)-a\Psi(a+1,b;x)-\Psi(a,b-1;x)=0,$$
$$(b-a)\Psi(a,b;x)-x\Psi(a,b+1;x)+\Psi(a-1,b;x)=0,$$
$$(a+x)\Psi(a,b;x)+a(b-a-1)\Psi(a+1,b;x)-x\Psi(a,b+1;x)=0,$$
$$(a-1+x)\Psi(a,b;x)-\Psi(a-1,b;x)+(a-c+1)\Psi(a,b-1;x)=0.$$

► **Differentiation formulas and Wronskian**

Differentiation formulas:

$$\frac{d}{dx}\Phi(a,b;x)=\frac{a}{b}\Phi(a+1,b+1;x),\qquad \frac{d^n}{dx^n}\Phi(a,b;x)=\frac{(a)_n}{(b)_n}\Phi(a+n,b+n;x),$$
$$\frac{d}{dx}\Psi(a,b;x)=-a\Psi(a+1,b+1;x),\qquad \frac{d^n}{dx^n}\Psi(a,b;x)=(-1)^n(a)_n\Psi(a+n,b+n;x).$$

Wronskian:

$$W(\Phi,\Psi)=\Phi\Psi'_x-\Phi'_x\Psi=-\frac{\Gamma(b)}{\Gamma(a)}x^{-b}e^x.$$

► **Degenerate hypergeometric functions for $n = 0, 1, \ldots$**

$$\Psi(a,n+1;x)=\frac{(-1)^{n-1}}{n!\,\Gamma(a-n)}\biggl\{\Phi(a,n{+}1;x)\ln x$$
$$+\sum_{r=0}^{\infty}\frac{(a)_r}{(n+1)_r}\bigl[\psi(a+r)-\psi(1+r)-\psi(1+n+r)\bigr]\frac{x^r}{r!}\biggr\}+\frac{(n-1)!}{\Gamma(a)}\sum_{r=0}^{n-1}\frac{(a-n)_r}{(1-n)_r}\frac{x^{r-n}}{r!},$$

where $n = 0, 1, 2, \ldots$ (the last sum is dropped for $n = 0$), $\psi(z) = [\ln\Gamma(z)]'_z$ is the logarithmic derivative of the gamma function,

$$\psi(1)=-\mathcal{C},\qquad \psi(n)=-\mathcal{C}+\sum_{k=1}^{n-1}k^{-1},$$

where $\mathcal{C} = 0.5572\ldots$ is the Euler constant.

If $b < 0$, then the formula

$$\Psi(a,b;x)=x^{1-b}\Psi(a-b+1,\ 2-b;\ x)$$

is valid for any x.

For $b \neq 0, -1, -2, -3, \ldots$, the general solution of the degenerate hypergeometric equation can be represented in the form

$$y=C_1\Phi(a,b;x)+C_2\Psi(a,b;x),$$

and for $b = 0, -1, -2, -3, \ldots$, in the form

$$y=x^{1-b}\bigl[C_1\Phi(a-b+1,\ 2-b;\ x)+C_2\Psi(a-b+1,\ 2-b;\ x)\bigr].$$

► **Integral representations**

$$\Phi(a,b;x) = \frac{\Gamma(b)}{\Gamma(a)\,\Gamma(b-a)} \int_0^1 e^{xt} t^{a-1} (1-t)^{b-a-1}\,dt \qquad (\text{for } b > a > 0),$$
$$\Psi(a,b;x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-xt} t^{a-1} (1+t)^{b-a-1}\,dt \qquad (\text{for } a > 0,\ x > 0),$$

where $\Gamma(a)$ is the gamma function.

► **Integrals with degenerate hypergeometric functions**

$$\int \Phi(a,b;x)\,dx = \frac{b-1}{a-1}\Psi(a-1,b-1;x) + C,$$
$$\int \Psi(a,b;x)\,dx = \frac{1}{1-a}\Psi(a-1,b-1;x) + C,$$
$$\int x^n \Phi(a,b;x)\,dx = n! \sum_{k=1}^{n+1} \frac{(-1)^{k+1}(1-b)_k x^{n-k+1}}{(1-a)_k (n-k+1)!} \Phi(a-k,b-k;x) + C,$$
$$\int x^n \Psi(a,b;x)\,dx = n! \sum_{k=1}^{n+1} \frac{(-1)^{k+1} x^{n-k+1}}{(1-a)_k (n-k+1)!} \Psi(a-k,b-k;x) + C.$$

► **Asymptotic expansion as $|x| \to \infty$.**

$$\Phi(a,b;x) = \frac{\Gamma(b)}{\Gamma(a)} e^x x^{a-b} \left[\sum_{n=0}^{N} \frac{(b-a)_n (1-a)_n}{n!} x^{-n} + \varepsilon \right], \qquad x > 0,$$
$$\Phi(a,b;x) = \frac{\Gamma(b)}{\Gamma(b-a)} (-x)^{-a} \left[\sum_{n=0}^{N} \frac{(a)_n (a-b+1)_n}{n!} (-x)^{-n} + \varepsilon \right], \qquad x < 0,$$
$$\Psi(a,b;x) = x^{-a} \left[\sum_{n=0}^{N} (-1)^n \frac{(a)_n (a-b+1)_n}{n!} x^{-n} + \varepsilon \right], \qquad -\infty < x < \infty,$$

where $\varepsilon = O(x^{-N-1})$.

10.9. Hypergeometric Functions

► **Definition**

The hypergeometric functions $F(\alpha,\beta,\gamma;x)$ is a solution the Gaussian hypergeometric equation

$$x(x-1)y''_{xx} + [(\alpha+\beta+1)x - \gamma]y'_x + \alpha\beta y = 0.$$

For $\gamma \neq 0, -1, -2, -3, \ldots$, the function $F(\alpha,\beta,\gamma;x)$ can be expressed in terms of the hypergeometric series:

$$F(\alpha,\beta,\gamma;x) = 1 + \sum_{k=1}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k} \frac{x^k}{k!}, \qquad (\alpha)_k = \alpha(\alpha+1)\ldots(\alpha+k-1),$$

which certainly converges for $|x| < 1$.

Table S2 shows some special cases when F can be expressed in term of elementary functions.

TABLE S2

Some special cases when the hypergeometric function $F(\alpha,\beta,\gamma;z)$ can be expressed in terms of elementary functions

α	β	γ	z	F
$-n$	β	γ	x	$\sum_{k=0}^{n} \frac{(-n)_k(\beta)_k}{(\gamma)_k} \frac{x^k}{k!}$, where $n = 1, 2, \ldots$
$-n$	β	$-n-m$	x	$\sum_{k=0}^{n} \frac{(-n)_k(\beta)_k}{(-n-m)_k} \frac{x^k}{k!}$, where $n = 1, 2, \ldots$
α	β	β	x	$(1-x)^{-\alpha}$
α	$\alpha+\frac{1}{2}$	$\frac{1}{2}$	x^2	$\frac{1}{2}\left[(1+x)^{-2\alpha}+(1-x)^{-2\alpha}\right]$
α	$\alpha+\frac{1}{2}$	$\frac{3}{2}$	x^2	$\frac{(1+x)^{1-2\alpha}-(1-x)^{1-2\alpha}}{2x(1-2\alpha)}$
α	$-\alpha$	$\frac{1}{2}$	$-x^2$	$\frac{1}{2}\left[\left(\sqrt{1+x^2}+x\right)^{2\alpha}+\left(\sqrt{1+x^2}-x\right)^{2\alpha}\right]$
α	$1-\alpha$	$\frac{1}{2}$	$-x^2$	$\frac{\left(\sqrt{1+x^2}+x\right)^{2\alpha-1}+\left(\sqrt{1+x^2}-x\right)^{2\alpha-1}}{2\sqrt{1+x^2}}$
α	$\alpha-\frac{1}{2}$	$2\alpha-1$	x	$2^{2\alpha-2}\left(1+\sqrt{1-x}\right)^{2-2\alpha}$
α	$1-\alpha$	$\frac{3}{2}$	$\sin^2 x$	$\frac{\sin[(2\alpha-1)x]}{(\alpha-1)\sin(2x)}$
α	$2-\alpha$	$\frac{3}{2}$	$\sin^2 x$	$\frac{\sin[(2\alpha-2)x]}{(\alpha-1)\sin(2x)}$
α	$1-\alpha$	$\frac{1}{2}$	$\sin^2 x$	$\frac{\cos[(2\alpha-1)x]}{\cos x}$
α	$\alpha+1$	$\frac{1}{2}\alpha$	x	$(1+x)(1-x)^{-\alpha-1}$
α	$\alpha+\frac{1}{2}$	$2\alpha+1$	x	$\left(\frac{1+\sqrt{1-x}}{2}\right)^{-2\alpha}$
α	$\alpha+\frac{1}{2}$	2α	x	$\frac{1}{\sqrt{1-x}}\left(\frac{1+\sqrt{1-x}}{2}\right)^{1-2\alpha}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	x^2	$\frac{1}{x}\arcsin x$
$\frac{1}{2}$	1	$\frac{3}{2}$	$-x^2$	$\frac{1}{x}\arctan x$
1	1	2	$-x$	$\frac{1}{x}\ln(x+1)$
$\frac{1}{2}$	1	$\frac{3}{2}$	x^2	$\frac{1}{2x}\ln\frac{1+x}{1-x}$
$n+1$	$n+m+1$	$n+m+l+2$	x	$\frac{(-1)^m(n+m+l+1)!}{n!\,l!\,(n+m)!\,(m+l)!}\frac{d^{n+m}}{dx^{n+m}}\left\{(1-x)^{m+l}\frac{d^lF}{dx^l}\right\}$, $F=-\frac{\ln(1-x)}{x}$, $n,m,l=0,1,2,\ldots$

► **Basic properties**

The function F possesses the following properties:

$$F(\alpha,\beta,\gamma;x) = F(\beta,\alpha,\gamma;x),$$
$$F(\alpha,\beta,\gamma;x) = (1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\ \gamma-\beta,\ \gamma;\ x),$$
$$F(\alpha,\beta,\gamma;x) = (1-x)^{-\alpha}F\Big(\alpha,\ \gamma-\beta,\ \gamma;\ \frac{x}{x-1}\Big),$$
$$\frac{d^n}{dx^n}F(\alpha,\beta,\gamma;x) = \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}F(\alpha+n,\ \beta+n,\ \gamma+n;\ x).$$

If γ is not an integer, then the general solution of the hypergeometric equation can be written in the form

$$y = C_1F(\alpha,\beta,\gamma;x) + C_2x^{1-\gamma}F(\alpha-\gamma+1,\ \beta-\gamma+1,\ 2-\gamma;\ x).$$

► **Integral representations**

For $\gamma > \beta > 0$, the hypergeometric function can be expressed in terms of a definite integral:

$$F(\alpha,\beta,\gamma;x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tx)^{-\alpha}\,dt,$$

where $\Gamma(\beta)$ is the gamma function.

See M. Abramowitz and I. Stegun (1979) and H. Bateman and A. Erdélyi (1973, Vol. 1) for more detailed information about hypergeometric functions.

10.10. Legendre Functions

► **Definitions. Basic formulas**

The associated Legendre functions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ of the first and the second kind are linearly independent solutions of the Legendre equation:

$$(1-z^2)y''_{zz} - 2zy'_z + [\nu(\nu+1) - \mu^2(1-z^2)^{-1}]y = 0,$$

where the parameters ν and μ and the variable z can assume arbitrary real or complex values.

For $|1-z| < 2$, the formulas

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)}\Big(\frac{z+1}{z-1}\Big)^{\mu/2}F\Big(-\nu,\ 1+\nu,\ 1-\mu,\ \frac{1-z}{2}\Big),$$
$$Q_\nu^\mu(z) = A\Big(\frac{z-1}{z+1}\Big)^{\frac{\mu}{2}}F\Big(-\nu,\ 1+\nu,\ 1+\mu,\ \frac{1-z}{2}\Big) + B\Big(\frac{z+1}{z-1}\Big)^{\frac{\mu}{2}}F\Big(-\nu,\ 1+\nu,\ 1-\mu,\ \frac{1-z}{2}\Big),$$
$$A = e^{i\mu\pi}\frac{\Gamma(-\mu)\,\Gamma(1+\nu+\mu)}{2\,\Gamma(1+\nu-\mu)},\qquad B = e^{i\mu\pi}\frac{\Gamma(\mu)}{2},\qquad i^2 = -1,$$

are valid, where $F(a,b,c;z)$ is the hypergeometric series (see (see Section 10.9 of this supplement).

For $|z| > 1$,

$$P_\nu^\mu(z) = \frac{2^{-\nu-1}\Gamma(-\frac{1}{2}-\nu)}{\sqrt{\pi}\,\Gamma(-\nu-\mu)}z^{-\nu+\mu-1}(z^2-1)^{-\mu/2}F\Big(\frac{1+\nu-\mu}{2},\ \frac{2+\nu-\mu}{2},\ \frac{2\nu+3}{2},\ \frac{1}{z^2}\Big)$$
$$+\ \frac{2^{\nu}\Gamma(\frac{1}{2}+\nu)}{\sqrt{\pi}\,\Gamma(1+\nu-\mu)}z^{\nu+\mu}(z^2-1)^{-\mu/2}F\Big(-\frac{\nu+\mu}{2},\ \frac{1-\nu-\mu}{2},\ \frac{1-2\nu}{2},\ \frac{1}{z^2}\Big),$$
$$Q_\nu^\mu(z) = e^{i\pi\mu}\frac{\sqrt{\pi}\,\Gamma(\nu+\mu+1)}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})}z^{-\nu-\mu-1}(z^2-1)^{\mu/2}F\Big(\frac{2+\nu+\mu}{2},\ \frac{1+\nu+\mu}{2},\ \frac{2\nu+3}{2},\ \frac{1}{z^2}\Big).$$

The functions $P_\nu(z) \equiv P_\nu^0(z)$ and $Q_\nu(z) \equiv Q_\nu^0(z)$ are called the *Legendre functions.*

The modified associated Legendre functions, on the cut $z = x$, $-1 < x < 1$, of the real axis are defined by the formulas

$$P_\nu^\mu(x) = \tfrac{1}{2}\left[e^{\frac{1}{2}i\mu\pi}P_\nu^\mu(x+i0) + e^{-\frac{1}{2}i\mu\pi}P_\nu^\mu(x-i0)\right],$$
$$Q_\nu^\mu(x) = \tfrac{1}{2}e^{-i\mu\pi}\left[e^{-\frac{1}{2}i\mu\pi}Q_\nu^\mu(x+i0) + e^{\frac{1}{2}i\mu\pi}Q_\nu^\mu(x-i0)\right].$$

► Trigonometric expansions

For $-1 < x < 1$, the modified associated Legendre functions can be represented in the form trigonometric series:

$$P_\nu^\mu(\cos\theta) = \frac{2^{\mu+1}}{\sqrt{\pi}}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})}(\sin\theta)^\mu \sum_{k=0}^{\infty}\frac{(\frac{1}{2}+\mu)_k(1+\nu+\mu)_k}{k!\,(\nu+\frac{3}{2})_k}\sin[(2k+\nu+\mu+1)\theta],$$
$$Q_\nu^\mu(\cos\theta) = \sqrt{\pi}\,2^\mu\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})}(\sin\theta)^\mu \sum_{k=0}^{\infty}\frac{(\frac{1}{2}+\mu)_k(1+\nu+\mu)_k}{k!\,(\nu+\frac{3}{2})_k}\cos[(2k+\nu+\mu+1)\theta],$$

where $0 < \theta < \pi$.

► Some relations

$$P_\nu^\mu(z) = P_{-\nu-1}^\mu(z), \quad P_\nu^n(z) = \frac{\Gamma(\nu+n+1)}{\Gamma(\nu-n+1)}P_\nu^{-n}(z), \qquad n = 0, 1, 2, \dots$$
$$Q_\nu^\mu(z) = \frac{\pi}{2\sin(\mu\pi)}e^{i\pi\mu}\left[P_\nu^\mu(z) - \frac{\Gamma(1+\nu+\mu)}{\Gamma(1+\nu-\mu)}P_\nu^{-\mu}(z)\right].$$

For $0 < x < 1$,

$$P_\nu^\mu(-x) = P_\nu^\mu(x)\cos[\pi(\nu+\mu)] - 2\pi^{-1}Q_\nu^\mu(x)\sin[\pi(\nu+\mu)],$$
$$Q_\nu^\mu(-x) = -Q_\nu^\mu(x)\cos[\pi(\nu+\mu)] - \tfrac{1}{2}\pi P_\nu^\mu(x)\sin[\pi(\nu+\mu)].$$

For $-1 < x < 1$,

$$P_{\nu+1}^\mu(x) = \frac{2\nu+1}{\nu-\mu+1}xP_\nu^\mu(x) - \frac{\nu+\mu}{\nu-\mu+1}P_{\nu-1}^\mu(x).$$

Wronskians:

$$W(P_\nu, Q_\nu) = \frac{1}{1-x^2}, \quad W(P_\nu^\mu, Q_\nu^\mu) = \frac{k}{1-x^2}, \qquad k = 2^{2\mu}\frac{\Gamma\left(\frac{\nu+\mu+1}{2}\right)\Gamma\left(\frac{\nu+\mu+2}{2}\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)\Gamma\left(\frac{\nu-\mu+2}{2}\right)}.$$

For $n = 0, 1, 2, \dots$,

$$P_\nu^n(x) = (-1)^n(1-x^2)^{n/2}\frac{d^n}{dx^n}P_\nu(x), \quad Q_\nu^n(x) = (-1)^n(1-x^2)^{n/2}\frac{d^n}{dx^n}Q_\nu(x).$$

► Legendre polynomials

The Legendre polynomials $P_n(x)$ and the Legendre functions $Q_n(x)$ are defined by the formulas

$$P_n(x) = \frac{1}{n!\,2^n}\frac{d^n}{dx^n}(x^2-1)^n, \quad Q_n(x) = \frac{1}{2}P_n(x)\ln\frac{1+x}{1-x} - \sum_{m=1}^{n}\frac{1}{m}P_{m-1}(x)P_{n-m}(x).$$

The polynomials $P_n = P_n(x)$ can be calculated recursively using the relations

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad \dots, \quad P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x).$$

The first three functions $Q_n = Q_n(x)$ have the form

$$Q_0(x) = \frac{1}{2} \ln \frac{1+x}{1-x}, \quad Q_1(x) = \frac{x}{2} \ln \frac{1+x}{1-x} - 1, \quad Q_2(x) = \frac{3x^2-1}{4} \ln \frac{1+x}{1-x} - \frac{3}{2}x.$$

The polynomials $P_n(x)$ have the implicit representation

$$P_n(x) = 2^{-n} \sum_{m=0}^{[n/2]} (-1)^m C_n^m C_{2n-2m}^n x^{n-2m},$$

where $[A]$ is the integer part of a number A.

All zeros of $P_n(x)$ are real and lie on the interval $-1 < x < +1$; the functions $P_n(x)$ form an orthogonal system on the interval $-1 \le x \le +1$, with

$$\int_{-1}^{+1} P_n(x) P_m(x)\, dx = \begin{cases} 0 & \text{if } n \ne m, \\ \dfrac{2}{2n+1} & \text{if } n = m. \end{cases}$$

The generating function is

$$\frac{1}{\sqrt{1 - 2sx + s^2}} = \sum_{n=0}^{\infty} P_n(x) s^n \qquad (|s| < 1).$$

► **Integral representations**

For $n = 0, 1, 2, \dots,$

$$P_\nu^n(z) = \frac{\Gamma(\nu+n+1)}{\pi\Gamma(\nu+1)} \int_0^\pi \left(z + \cos t \sqrt{z^2-1}\right)^\nu \cos(nt)\, dt, \quad \operatorname{Re} z > 0,$$

$$Q_\nu^n(z) = (-1)^n \frac{\Gamma(\nu+n+1)}{2^{\nu+1}\Gamma(\nu+1)} (z^2-1)^{-n/2} \int_0^\pi (z + \cos t)^{n-\nu-1} (\sin t)^{2\nu+1}\, dt, \quad \operatorname{Re} \nu > -1,$$

Note that $z \ne x$, $-1 < x < 1$, in the latter formula.

10.11. Orthogonal Polynomials

All zeros of each of the orthogonal polynomials $\mathcal{P}_n(x)$ considered in this section are real and simple. The zeros of the polynomials $\mathcal{P}_n(x)$ and $\mathcal{P}_{n+1}(x)$ are alternating.

► **Legendre polynomials**

The Legendre polynomials $P_n = P_n(x)$ satisfy the equation

$$(1 - x^2) y''_{xx} - 2x y'_x + n(n+1) y = 0.$$

They are outlined in Section 10.10 of this supplement.

► **Laguerre polynomials**

The Laguerre polynomials $L_n = L_n(x)$ satisfy the equation

$$xy''_{xx} + (1-x)y'_x + ny = 0$$

and are defined by the formulas

$$L_n(x) = e^x \frac{d^n}{dx^n}\bigl(x^n e^{-x}\bigr)(-1)^n\Bigl[x^n - n^2x^{n-1} + \frac{n^2(n-1)^2}{2!}x^{n-2} + \cdots\Bigr].$$

The first four polynomials have the form

$$L_0 = 1, \quad L_1 = -x+1, \quad L_2 = x^2 - 4x + 2, \quad L_3 = -x^3 + 9x^2 - 18x + 6.$$

To calculate L_n for $n \ge 2$, one can use the recurrent formulas

$$L_{n+1}(x) = (2n+1-x)L_n(x) - n^2 L_{n-1}(x).$$

The functions $L_n(x)$ form an orthogonal system on the interval $0 < x < \infty$, with

$$\int_0^\infty e^{-x} L_n(x)L_m(x)\,dx = \begin{cases} 0 & \text{if } n \neq m, \\ (n!)^2 & \text{if } n = m. \end{cases}$$

The associated Laguerre polynomials of degree $n-k$ and order k are given by

$$L_n^k(x) = \frac{d^k}{dx^k} L_n(x).$$

These satisfy the differential equation

$$xy''_{xx} + (k+1-x)y'_x + (n-k)y = 0,$$

where $n = 1, 2, \ldots$ and $k = 0, 1, 2, \ldots$

The generating function is

$$\frac{1}{1-s}\exp\Bigl(-\frac{sx}{1-s}\Bigr) = \sum_{n=0}^\infty L_n(x)\frac{s^n}{n!}.$$

► **Chebyshev polynomials**

The Chebyshev polynomials $T_n = T_n(x)$ satisfy the equation

$$(1-x^2)y''_{xx} - xy'_x + n^2 y = 0 \qquad (1)$$

and are defined by the formulas

$$\begin{aligned} T_n(x) = \cos(n \arccos x) &= \frac{(-2)^n n!}{(2n)!}\sqrt{1-x^2}\,\frac{d^n}{dx^n}\bigl[(1-x^2)^{n-\frac{1}{2}}\bigr] \\ &= \frac{n}{2}\sum_{m=0}^{[n/2]}(-1)^m \frac{(n-m-1)!}{m!\,(n-2m)!}(2x)^{n-2m} \quad (n = 0, 1, 2, \ldots), \end{aligned}$$

where $[A]$ stands for the integer part of a number A.

The first four polynomials are

$$T_0 = 1, \quad T_1 = x, \quad T_2 = 2x^2 - 1, \quad T_3 = 4x^3 - 3x.$$

The recurrent formulas:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad n \geq 2.$$

The functions $T_n(x)$ form an orthogonal system on the interval $-1 < x < +1$, with

$$\int_{-1}^{+1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\,dx = \begin{cases} 0 & \text{if } n \neq m, \\ \frac{1}{2}\pi & \text{if } n = m \neq 0, \\ \pi & \text{if } n = m = 0. \end{cases}$$

The Chebyshev functions of the second kind,

$$U_0(x) = \arcsin x,$$

$$U_n(x) = \sin(n \arcsin x) = \frac{\sqrt{1-x^2}}{n}\frac{dT_n(x)}{dx} \quad (n = 1, 2, \dots),$$

just as the Chebyshev polynomials, also satisfy the differential equation (1).

The generating function is

$$\frac{1 - sx}{1 - 2sx + s^2} = \sum_{n=0}^{\infty} T_n(x)s^n \qquad (|s| < 1).$$

► **Hermite polynomial**

The Hermite polynomial $H_n = H_n(x)$ satisfies the equation

$$y''_{xx} - 2xy'_x + 2ny = 0$$

and is defined by the formulas

$$H_n(x) = (-1)^n \exp\left(x^2\right)\frac{d^n}{dx^n}\exp\left(-x^2\right).$$

The first four polynomials are

$$H_0 = 1, \quad H_1 = x, \quad H_2 = 4x^2 - 2, \quad H_3 = 8x^3 - 12x.$$

The recurrent formulas:

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \qquad n \geq 2.$$

The functions $H_n(x)$ form an orthogonal system on the interval $-\infty < x < \infty$, with

$$\int_{-\infty}^{\infty} \exp\left(-x^2\right) H_n(x)H_m(x)\,dx = \begin{cases} 0 & \text{if } n \neq m, \\ \sqrt{\pi}\,2^n n! & \text{if } n = m. \end{cases}$$

The Hermite functions $\psi_n(x)$ are introduced by the formula $\psi_n(x) = \exp\left(-\frac{1}{2}x^2\right)H_n(x)$, where $n = 0, 1, 2, \dots$

The generating function:

$$\exp\left(-s^2 + 2sx\right) = \sum_{n=0}^{\infty} H_n(x)\frac{s^n}{n!}.$$

► **Jacobi polynomials**

The Jacobi polynomials $P_n^{\alpha,\beta} = P_n^{\alpha,\beta}(x)$ satisfy the equation

$$(1 - x^2)y''_{xx} + \left[\beta - \alpha - (\alpha + \beta + 2)x\right]y'_x + n(n + \alpha + \beta + 1)y = 0$$

and are defined by the formulas

$$P_n^{\alpha,\beta} = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right] = 2^{-n}\sum_{m=0}^{n} C_{n+\alpha}^{m} C_{n+\beta}^{n-m}(x-1)^{n-m}(x+1)^m,$$

where C_b^a are binomial coefficients.

⊙ References for Supplement 10: H. Bateman and A. Erdélyi (1953, 1955), M. Abramowitz and I. A. Stegun (1964).