4

Principles of Hydrodynamic Lubrication

4.1 INTRODUCTION

A hydrodynamic plane-slider is shown in Fig. 1-2 and the widely used hydrodynamic journal bearing is shown in Fig. 1-3. Hydrodynamic lubrication is the fluid dynamic effect that generates a lubrication fluid film that completely separates the sliding surfaces. The fluid film is in a thin clearance between two surfaces in relative motion. The hydrodynamic effect generates a hydrodynamic *pressure wave* in the fluid film that results in *load-carrying capacity*, in the sense that the fluid film has sufficient pressure to carry the external load on the bearing. The pressure wave is generated by a wedge of viscous lubricant drawn into the clearance between the two converging surfaces or by a squeeze-film action.

The thin clearance of a plane-slider and a journal bearing has the shape of a thin converging wedge. The fluid adheres to the solid surfaces and is dragged into the converging clearance. High shear stresses drag the fluid into the wedge due to the motion of the solid surfaces. In turn, high pressure must build up in the fluid film before the viscous fluid escapes through the thin clearance. The pressure wave in the fluid film results in a load-carrying capacity that supports the external load on the bearing. In this way, the hydrodynamic film can completely separate the sliding surfaces, and, thus, wear of the sliding surfaces is prevented. Under steady conditions, the hydrodynamic load capacity, W, of a bearing is equal to the external load, F, on the bearing, but it is acting in the opposite direction. The hydrodynamic theory of lubrication solves for the fluid velocity, pressure wave, and resultant load capacity.

Experiments and hydrodynamic analysis indicated that the hydrodynamic load capacity is proportional to the sliding speed and fluid viscosity. At the same time, the load capacity dramatically increases for a thinner fluid film. However, there is a practical limit to how much the bearing designer can reduce the film thickness. A very thin fluid film is undesirable, particularly in machines with vibrations. Whenever the hydrodynamic film becomes too thin, it results in occasional contact of the surfaces, which results in severe wear. Picking the optimum film-thickness is an important decision in the design process; it will be discussed in the following chapters.

Tower (1880) conducted experiments and demonstrated for the first time the existence of a pressure wave in a hydrodynamic journal bearing. Later, Reynolds (1886) derived the classical theory of hydrodynamic lubrication. A large volume of analytical and experimental research work in hydrodynamic lubrication has subsequently followed the work of Reynolds. The classical theory of Reynolds and his followers is based on several assumptions that were adopted to simplify the mathematical derivations, most of which are still applied today. Most of these assumptions are justified because they do not result in a significant deviation from the actual conditions in the bearing. However, some other classical assumptions are not realistic but were necessary to simplify the analysis. As in other disciplines, the introduction of computers permitted complex hydrodynamic lubrication problems to be solved by numerical analysis and have resulted in the numerical solution of such problems under realistic conditions without having to rely on certain inaccurate assumptions.

At the beginning of the twentieth century, only long hydrodynamic journal bearings had been designed. The length was long in comparison to the diameter, $L > D$; long-bearing theory of Reynolds is applicable to such bearings. Later, however, the advantages of a short bearing were recognized. In modern machinery, the bearings are usually short, $L < D$; short-bearing theory is applicable. The advantage of a long bearing is its higher load capacity in comparison to a short bearing. Moreover, the load capacity of a long bearing is even much higher per unit of bearing area. In comparison, the most important advantages of a short bearing that make it widely used are: (a) better cooling due to faster circulation of lubricant; (b) less sensitivity to misalignment; and (c) a compact design.

Simplified models are commonly used in engineering to provide insight and simple design tools. Hydrodynamic lubrication analysis is much simplified if the bearing is assumed to be *infinitely long* or *infinitely short*. But for a finite-length bearing, there is a three-dimensional flow that requires numerical solution by computer. In order to simplify the analysis, long journal bearings, $L > D$, are often solved as infinitely long bearings, while short bearings, $L < D$, are often solved as infinitely short bearings.

4.2 ASSUMPTIONS OF HYDRODYNAMIC LUBRICATION THEORY

The first assumption of hydrodynamic lubrication theory is that the fluid film flow is laminar. The flow is laminar at low Reynolds number (Re). In fluid dynamics, the Reynolds number is useful for estimating the ratio of the inertial and viscous forces. For a fluid film flow, the expression for the Reynolds number is

$$\mathrm{Re} = \frac{U\rho h}{\mu} = \frac{Uh}{\nu} \tag{4-1}$$

Here, h is the average magnitude of the variable film thickness, ρ is the fluid density, μ is the fluid viscosity, and ν is the kinematic viscosity. The transition from laminar to turbulent flow in hydrodynamic lubrication initiates at about $\mathrm{Re} = 1000$, and the flow becomes completely turbulent at about $\mathrm{Re} = 1600$. The Reynolds number at the transition can be lower if the bearing surfaces are rough or in the presence of vibrations. In practice, there are always some vibrations in rotating machinery.

In most practical bearings, the Reynolds number is sufficiently low, resulting in laminar fluid film flow. An example problem is included in Chapter 5, where Re is calculated for various practical cases. That example shows that in certain unique applications, such as where water is used as a lubricant (in certain centrifugal pumps or in boats), the Reynolds number is quite high, resulting in turbulent fluid film flow.

Classical hydrodynamic theory is based on the assumption of a linear relation between the fluid stress and the strain-rate. Fluids that demonstrate such a linear relationship are referred to as *Newtonian fluids* (see Chapter 2). For most lubricants, including mineral oils, synthetic lubricants, air, and water, a linear relationship between the stress and the strain-rate components is a very close approximation. In addition, liquid lubricants are considered to be *incompressible*. That is, they have a negligible change of volume under the usual pressures in hydrodynamic lubrication.

Differential equations are used for theoretical modeling in various disciplines. These equations are usually simplified under certain conditions by disregarding terms of a relatively lower order of magnitude. Order analysis of the various terms of an equation, under specific conditions, is required for determining the most significant terms, which capture the most important effects. A term in an equation can be disregarded and omitted if it is lower by one or several orders of magnitude in comparison to other terms in the same equation. Dimensionless analysis is a useful tool for determining the relative orders of magnitude of the terms in an equation. For example, in fluid dynamics, the dimensionless Reynolds number is a useful tool for estimating the ratio of inertial and viscous forces.

In hydrodynamic lubrication, the fluid film is very thin, and in most practical cases the Reynolds number is low. Therefore, the effect of the inertial forces of the fluid (*ma*) as well as gravity forces (*mg*) are very small and can be neglected in comparison to the dominant effect of the viscous stresses. This assumption is applicable for most practical hydrodynamic bearings, except in unique circumstances.

The fluid is assumed to be continuous, in the sense that there is continuity (no sudden change in the form of a step function) in the fluid flow variables, such as shear stresses and pressure distribution. In fact, there are always very small air bubbles in the lubricant that cause discontinuity. However, this effect is usually negligible, unless there is a massive fluid foaming or *fluid cavitation* (formation of bubbles when the vapor pressure is higher than the fluid pressure). In general, classical fluid dynamics is based on the continuity assumption. It is important for mathematical derivations that all functions be continuous and differentiable, such as stress, strain-rate, and pressure functions.

The following are the basic ten assumptions of classical hydrodynamic lubrication theory. The first nine were investigated and found to be justified, in the sense that they result in a negligible deviation from reality for most practical oil bearings (except in some unique circumstances). The tenth assumption however, has been introduced only for the purpose of simplifying the analysis.

Assumptions of classical hydrodynamic lubrication theory

1. The flow is laminar because the Reynolds number, *Re*, is low.
2. The fluid lubricant is continuous, Newtonian, and incompressible.
3. The fluid adheres to the solid surface at the boundary and there is no fluid slip at the boundary; that is, the velocity of fluid at the solid boundary is equal to that of the solid.
4. The velocity component, v, across the thin film (in the y direction) is negligible in comparison to the other two velocity components, u and w, in the x and z directions, as shown in Fig. 1-2.
5. Velocity gradients along the fluid film, in the x and z directions, are small and negligible relative to the velocity gradients across the film because the fluid film is thin, i.e., $du/dy \gg du/dx$ and $dw/dy \gg dw/dz$.
6. The effect of the curvature in a journal bearing can be ignored. The film thickness, h, is very small in comparison to the radius of curvature, R, so the effect of the curvature on the flow and pressure distribution is relatively small and can be disregarded.

7. The pressure, p, across the film (in the y direction) is constant. In fact, pressure variations in the y direction are very small and their effect is negligible in the equations of motion.
8. The force of gravity on the fluid is negligible in comparison to the viscous forces.
9. Effects of fluid inertia are negligible in comparison to the viscous forces. In fluid dynamics, this assumption is usually justified for low-Reynolds-number flow.

These nine assumptions are justified for most practical hydrodynamic bearings. In contrast, the following additional tenth assumption has been introduced only for simplification of the analysis.

10. The fluid viscosity, μ, is constant.

It is well known that temperature varies along the hydrodynamic film, resulting in a variable viscosity. However, in view of the significant simplification of the analysis, most of the practical calculations are still based on the assumption of a constant equivalent viscosity that is determined by the average fluid film temperature. The last assumption can be applied in practice because it has already been verified that reasonably accurate results can be obtained for regular hydrodynamic bearings by considering an equivalent viscosity. The average temperature is usually determined by averaging the temperature of the bearing inlet and outlet lubricant. Various other methods have been suggested to calculate the equivalent viscosity.

A further simplification of the analysis can be obtained for very long and very short bearings. If a bearing is very long, the flow in the axial direction (z direction) can be neglected, and the three-dimensional flow reduces to a much simpler two-dimensional flow problem that can yield a closed form of analytical solution.

A long journal bearing is where the bearing length is much larger than its diameter, $L \gg D$, and a short journal bearing is where $L \ll D$. If $L \gg D$, the bearing is assumed to be infinitely long; if $L \ll D$, the bearing is assumed to be infinitely short.

For a journal bearing whose length L and diameter D are of a similar order of magnitude, the analysis is more complex. This three-dimensional flow analysis is referred to as a *finite-length bearing analysis*. Computer-aided numerical analysis is commonly applied to solve for the finite bearing. The results are summarized in tables that are widely used for design purposes (see Chapter 8).

4.3 HYDRODYNAMIC LONG BEARING

The coordinates of a long hydrodynamic journal bearing are shown in Fig. 4-1. The velocity components of the fluid flow, u, v, and w are in the x, y, and z directions, respectively. A journal bearing is long if the bearing length, L, is much larger than its diameter, D. A plane-slider (see Fig. 1-2) is long if the bearing width, L, in the z direction is much larger than the length, B, in the x direction (the direction of the sliding motion), or $L \gg B$.

In addition to the ten classical assumptions, there is an additional assumption for a long bearing—it can be analyzed as an infinitely long bearing. The pressure gradient in the z direction (axial direction) can be neglected in comparison to the pressure gradient in the x direction (around the bearing). The pressure is assumed to be constant along the z direction, resulting in two-dimensional flow, $w = 0$.

In fact, in actual long bearings there is a side flow from the bearing edge, in the z direction, because the pressure inside the bearing is higher than the ambient pressure. This side flow is referred to as an *end effect*. In addition to flow, there are other end effects, such as capillary forces. But for a long bearing, these effects are negligible in comparison to the constant pressure along the entire length.

4.4 DIFFERENTIAL EQUATION OF FLUID MOTION

The following analysis is based on first principles. It does not use the Navier–Stokes equations or the Reynolds equation and does not require in-depth knowledge of fluid dynamics. The following self-contained derivation can help in understanding the physical concepts of hydrodynamic lubrication.

An additional merit of a derivation that does not rely on the Navier–Stokes equations is that it allows extending the theory to applications where the Navier–Stokes equations do not apply. An example is lubrication with non-Newtonian fluids, which cannot rely on the classical Navier–Stokes equations because they

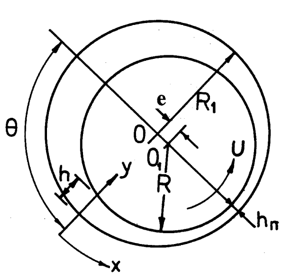

FIG. 4-1 Coordinates of a long journal bearing.

assume the fluid is Newtonian. Since the following analysis is based on first principles, a similar derivation can be applied to non-Newtonian fluids (see Chapter 19).

The following hydrodynamic lubrication analysis includes a derivation of the differential equation of fluid motion and a solution for the flow and pressure distribution inside a fluid film. The boundary conditions of the velocity and the conservation of mass (or the equivalent conservation of volume for an incompressible flow) are considered for this derivation.

The equation of the fluid motion is derived by considering the balance of forces acting on a small, infinitesimal fluid element having the shape of a rectangular parallelogram of dimensions dx and dy, as shown in Fig. 4-2. This elementary fluid element inside the fluid film is shown in Fig. 1-2. The derivation is for a two-dimensional flow in the x and y directions. In an infinitely long bearing, there is no flow or pressure gradient in the z direction. Therefore, the third dimension of the parallelogram (in the z direction) is of unit length (1).

The pressure in the x direction and the shear stress, τ, in the y direction are shown in Fig. 4-2. The stresses are subject to continuous variations. A relation between the pressure and shear-stress gradients is derived from the balance of forces on the fluid element. The forces are the product of stresses, or pressures, and the corresponding areas. The fluid inertial force (ma) is very small and is therefore neglected in the classical hydrodynamic theory (see assumptions listed earlier), allowing the derivation of the following force equilibrium equation in a similar way to a static problem:

$$(\tau + d\tau)dx \cdot 1 - \tau\, dx \cdot 1 = (p + dp) \cdot dy \cdot 1 = p\, dy \cdot 1 \tag{4-2}$$

Equation (4-2) reduces to

$$d\tau\, dx = dp\, dy \tag{4-3}$$

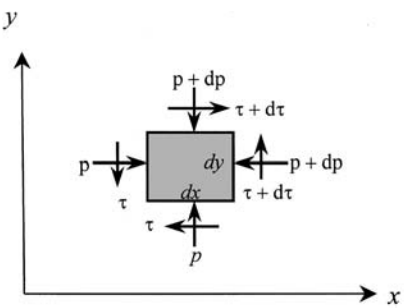

FIG. 4-2 Balance of forces on an infinitesimal fluid element.

After substituting the full differential expression $d\tau = (\partial\tau/\partial y)dy$ in Eq. (4-3) and substituting the equation $\tau = \mu\,(\partial u/\partial y)$ for the shear stress, Eq. (4-3) takes the form of the following differential equation:

$$\frac{dp}{dx} = \mu \frac{\partial^2 u}{\partial y^2} \tag{4-4}$$

A partial derivative is used because the velocity, u, is a function of x and y. Equation (4-4), is referred to as the *equation of fluid motion*, because it can be solved for the velocity distribution, u, in a thin fluid film of a hydrodynamic bearing.

Comment. In fact, it is shown in Chapter 5 that the complete equation for the shear stress is $\tau = \mu(du/dy + dv/dx)$. However, according to our assumptions, the second term is very small and is neglected in this derivation.

4.5 FLOW IN A LONG BEARING

The following simple solution is limited to a fluid film of steady geometry. It means that the geometry of the fluid film does not vary with time relative to the coordinate system, and it does not apply to time-dependent fluid film geometry such as a bearing under dynamic load. A more universal approach is possible by using the Reynolds equation (see Chapter 6). The Reynolds equation applies to all fluid films, including time-dependent fluid film geometry.

Example Problems 4-1

Journal Bearing

In Fig. 4-1, a journal bearing is shown in which the bearing is stationary and the journal turns around a stationary center. Derive the equations for the fluid velocity and pressure gradient.

The variable film thickness is due to the journal eccentricity. In hydrodynamic bearings, $h = h(x)$ is the variable film thickness around the bearing. The coordinate system is attached to the stationary bearing, and the journal surface has a constant velocity, $U = \omega R$, in the x direction.

Solution

The coordinate x is along the bearing surface curvature. According to the assumptions, the curvature is disregarded and the flow is solved as if the boundaries were a straight line.

Equation (4-4) can be solved for the velocity distribution, $u = u(x, y)$. Following the assumptions, variations of the pressure in the y direction are negligible (Assumption 6), and the pressure is taken as a constant across the film thickness because the fluid film is thin. Therefore, in two-dimensional flow of a long bearing, the pressure is a function of x only. In order to simplify the solution for the velocity, u, the following substitution is made in Eq. (4-4):

$$2m(x) = \frac{1}{\mu}\frac{dp}{dx} \tag{4-5}$$

where $m(x)$ is an unknown function of x that must be solved in order to find the pressure distribution. Equation 4-4 becomes

$$\frac{\partial u^2}{\partial y^2} = 2m(x) \tag{4-6}$$

Integrating Eq. (4-6) twice yields the following expression for the velocity distribution, u, across the fluid film (n and k are integration constants):

$$u = my^2 + ny + k \tag{4-7}$$

Here, m, n, and k are three unknowns that are functions of x only. Three equations are required to solve for these three unknowns. Two equations are obtained from the two boundary conditions of the flow at the solid surfaces, and the third equation is derived from the continuity condition, which is equivalent to the conservation of mass of the fluid (or conservation of volume for incompressible flow).

The fluid adheres to the solid wall (no slip condition), and the fluid velocity at the boundaries is equal to that of the solid surface. In a journal bearing having a stationary bearing and a rotating journal at surface speed $U = \omega R$ (see Fig. 4-1), the boundary conditions are

$$\begin{array}{ll} \text{at } y = 0: & u = 0 \\ \text{at } y = h(x): & u = U\cos\alpha \approx U \end{array} \tag{4-8}$$

The slope between the tangential velocity U and the x direction is very small; therefore, $\cos\alpha \approx 1$, and we can assume that at $y = h(x)$, $u \approx U$.

The third equation, which is required for the three unknowns, m, n, and k, is obtained from considerations of conservation of mass. For an infinitely long bearing, there is no flow in the axial direction, z; therefore, the amount of mass flow through each cross section of the fluid film is constant (the cross-sectional plane is normal to the x direction). Since the fluid is incompressible, the volume flow rate is also constant at any cross section. The constant-volume flow rate, q, per unit of bearing length is obtained by integration of the velocity component, u, along the film thickness, as follows:

$$q = \int_0^h u\,dy = \text{constant} \tag{4-9}$$

Equation (4-9) is applicable only for a steady fluid film geometry that does not vary with time.

The pressure wave around the journal bearing is shown in Fig. 1-3. At the peak of the pressure wave, $dp/dx = 0$, and the velocity distribution, $u = u(y)$, at that point is linear according to Eq. (4-4). The linear velocity distribution in a simple shear flow (in the absence of pressure gradient) is shown in Fig. 4-3. If the film thickness at the peak pressure point is $h = h_0$, the flow rate, q, per unit length is equal to the area of the velocity distribution triangle:

$$q = \frac{Uh_0}{2} \tag{4-10}$$

The two boundary conditions of the velocity as well as the conservation of mass condition form the following three equations, which can be solved for m, n and k:

$$\begin{aligned} 0 &= m0^2 + n0 + k \Rightarrow k = 0 \\ U &= mh^2 + nh \\ \frac{Uh_0}{2} &= \int_0^h (my^2 + ny)\, dy \end{aligned} \tag{4-11}$$

After solving for m, n, and k and substituting these values into Eq. (4-7), the following equation for the velocity distribution is obtained:

$$u = 3U\left(\frac{1}{h^2} - \frac{h_0}{h^3}\right)y^2 + U\left(\frac{3h_0}{h^2} - \frac{2}{h}\right)y \tag{4-12}$$

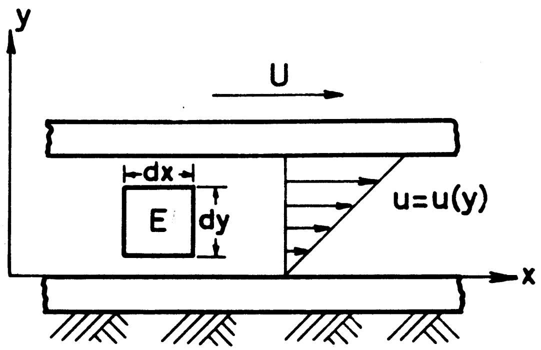

FIG. 4-3 Linear velocity distribution for a simple shear flow (no pressure gradient).

From the value of m, the expression for the pressure gradient, dp/dx, is solved [see Eq. (4-5)]:

$$\frac{dp}{dx} = 6U\mu\frac{h - h_0}{h^3} \tag{4-13}$$

Equation (4-13) still contains an unknown constant, h_0, which is the film thickness at the peak pressure point. This will be solved from additional information about the pressure wave. Equation (4-13) can be integrated for the pressure wave.

Example Problem 4-2

Inclined Plane Slider

As discussed earlier, a steady-fluid-film geometry (relative to the coordinates) must be selected for a simple derivation of the pressure gradient. The second example is of an inclined plane-slider having a configuration as shown in Fig. 4-4. This example is of a converging viscous wedge similar to that of a journal bearing; however, the lower part is moving in the x direction and the upper plane is stationary while the coordinates are stationary. This bearing configuration is selected because the geometry of the clearance (and fluid film) does not vary with time relative to the coordinate system.

Find the velocity distribution and the equation for the pressure gradient in the inclined plane-slider shown in Fig 4-4.

Solution

In this case, the boundary conditions are:

at $y = 0$: $\quad u = U$
at $y = h(x)$: $\quad u = 0$

In this example, the lower boundary is moving and the upper part is stationary. The coordinates are stationary, and the geometry of the fluid film does not vary

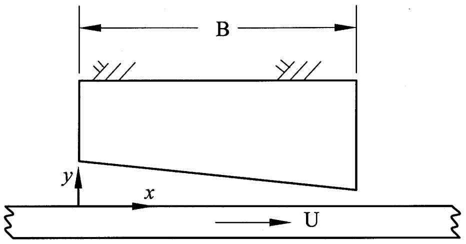

FIG. 4-4 Inclined plane-slider (converging flow in the x direction).

relative to the coordinates. In this case, the flow rate is constant, in a similar way to that of a journal bearing. This flow rate is equal to the area of the velocity distribution triangle at the point of peak pressure, where the clearance thickness is h_0. The equation for the constant flow rate is

$$q = \int_0^h u\, dy = \frac{h_0 U}{2}$$

The two boundary conditions of the velocity and the constant flow-rate condition form the three equations for solving for m, n, and k:

$$U = m0^2 + n0 + k \quad \Rightarrow \quad k = U$$

$$0 = mh^2 + nh + U$$

$$\frac{Uh_0}{2} = \int_0^h (my^2 + ny + U)dy$$

After solving for m, n, and k and substituting these values into Eq. (4-7), the following equation for the velocity distribution is obtained:

$$u = 3U\left(\frac{1}{h^2} - \frac{h_0}{h^3}\right)y^2 + U\left(\frac{3h_0}{h^2} - \frac{4}{h}\right)y + U$$

From the value of m, an identical expression to Eq. (4-13) for the pressure gradient, dp/dx, is obtained for $\partial h/\partial x < 0$ (a converging slope in the x direction):

$$\frac{dp}{dx} = 6U\mu\frac{h - h_0}{h^3} \qquad \text{for } \frac{\partial h}{\partial x} < 0 \text{ (negative slope)}$$

This equation applies to a converging wedge where the coordinate x is in the direction of a converging clearance. It means that the clearance reduces along x as shown in Fig. 4-4.

In a converging clearance near $x = 0$, the clearance slope is negative, $\partial h/\partial x < 0$. This means that the pressure increases near $x = 0$. At that point, $h > h_0$, resulting in $dp/dx > 0$.

If we reverse the direction of the coordinate x, the expression for the pressure gradient would have an opposite sign:

$$\frac{dp}{dx} = 6U\mu\frac{h_0 - h}{h^3} \qquad \text{for } \frac{\partial h}{\partial x} > 0 \text{ (positive slope)}$$

This equation applies to a plane-slider, as shown in Fig 4-5, where the coordinate x is in the direction of increasing clearance. The unknown constant, h_0, will be determined from the boundary conditions of the pressure wave.

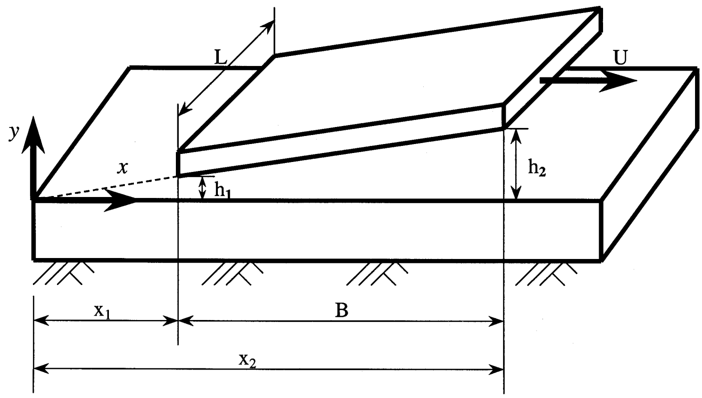

FIG. 4-5 Inclined plane-slider (x coordinate in the direction of a diverging clearance).

4.6 PRESSURE WAVE

4.6.1 Journal Bearing

The pressure wave along the x direction is solved by integration of Eq. (4-13). The two unknowns, h_0, and the integration constant are solved from the two boundary conditions of the pressure wave. In a plane-slider, the locations at the start and end of the pressure wave are used as pressure boundary conditions. These locations are not obvious when the clearance is converging and diverging, such as in journal bearings, and other boundary conditions of the pressure wave are used for solving h_0. Integrating the pressure gradient, Eq. (4-13), results in the following equation for a journal bearing:

$$p = 6\mu U \int_0^x \frac{h - h_0}{h^3}\, dx + p_0 \qquad \text{(4-14a)}$$

Here, the pressure p_0 represents the pressure at $x = 0$. In a journal bearing, the lubricant is often fed into the clearance through a hole in the bearing at $x = 0$. In that case, p_0 is the supply pressure.

4.6.2 Plane-Slider

In the case of an inclined slider, p_0 is the atmospheric pressure. Pressure is commonly measured with reference to atmospheric pressure (gauge pressure), resulting in $p_0 = 0$ for an inclined slider.

The pressure wave, $p(x)$, can be solved for any bearing geometry, as long as the film thickness, $h = h(x)$, is known. The pressure wave can be solved by analytical or numerical integration. The analytical integration of complex functions has been a challenge in the past. However, the use of computers makes numerical integration a relatively easy task.

An inclined plane slider is shown in Fig. 4-5, where the inclination angle is α. The fluid film is equivalent to that in Fig. 4-4, although the x is in the opposite direction, $\partial h/\partial x > 0$, and the pressure wave is

$$p = 6\mu U \int_{x_1}^{x} \frac{h_0 - h}{h^3}\, dx \tag{4-14b}$$

In order to have concise equations, the slope of the plane-slider is substituted by $a = \tan\alpha$, and the variable film thickness is given by the function

$$h(x) = ax \tag{4-15}$$

Here, x is measured from the point of intersection of the plane-slider and the bearing surface. The minimum and maximum film thicknesses are h_1 and h_2, respectively, as shown in Fig. 4-5.

In order to solve the pressure distribution in any converging fluid film, Eq. (4-14) is integrated after substituting the value of h according to Eq. (4-15). After integration, there are two unknowns: the constant h_0 in Eq. (4-10) and the constant of integration, p_o. The two unknown constants are solved for the two boundary conditions of the pressure wave. At each end of the inclined plane, the pressure is equal to the ambient (atmospheric) pressure, $p = 0$. The boundary conditions are:

$$\begin{aligned} &\text{at } h = h_1: \quad p = 0 \\ &\text{at } h = h_2: \quad p = 0 \end{aligned} \tag{4-16}$$

The solution can be analytically performed in closed form or by numerical integration (see Appendix B). The numerical integration involves iterations to find h_0. Hydrodynamic lubrication equations require frequent use of computer programming to perform the trial-and-error iterations. An example of a numerical integration is shown in Example Problem 4-4.

Analytical Solution. For an infinitely long plane-slider, $L \gg B$, analytical integration results in the following pressure wave along the x direction (between $x = h_1/a$ and $x = h_2/a$):

$$p = \frac{6\mu U}{a^3} \frac{(h_1 - ax)(ax - h_2)}{(h_1 + h_2)x^2} \tag{4-17}$$

At the boundaries $h = h_1$ and $h = h_2$, the pressure is zero (atmospheric pressure).

4.7 PLANE-SLIDER LOAD CAPACITY

Once the pressure wave is solved, it is possible to integrate it again to solve for the bearing load capacity, W. For a plane-slider, the integration for the load capacity is according to the following equation:

$$W = L \int_{x_1}^{x_2} p \, dx \tag{4-18}$$

The foregoing integration of the pressure wave can be derived analytically, in closed form. However, in many cases, the derivation of an analytical solution is too complex, and a computer program can perform a numerical integration. It is beneficial for the reader to solve this problem numerically, and writing a small computer program for this purpose is recommended.

An analytical solution for the load capacity is obtained by substituting the pressure in Eq. (4-17) into Eq. (4-18) and integrating in the boundaries between $x_1 = h_1/a$ and $x_2 = h_2/a$. The final analytical expression for the load capacity in a plane-slider is as follows:

$$W = \frac{6\mu U L B^2}{h_2^2} \left(\frac{1}{\beta - 1} \right)^2 \left[\ln \beta - \frac{2(\beta - 1)}{\beta + 1} \right] \tag{4-19}$$

where β is the ratio of the maximum and minimum film thickness, h_2/h_1. A similar derivation can be followed for nonflat sliders, such as in the case of a slider having a parabolic surface in Problem 4-2, at the end of this chapter.

4.8 VISCOUS FRICTION FORCE IN A PLANE-SLIDER

The friction force, F_f, is obtained by integrating the shear stress, τ over any cross-sectional area along the fluid film. For convenience, a cross section is selected along the bearing stationary wall, $y = 0$. The shear stress at the wall, τ_w, at $y = 0$ can be obtained via the following equation:

$$\tau_w = \mu \frac{du}{dy} \Big|_{(y=0)} \tag{4-20}$$

The velocity distribution can be substituted from Eq. (4-12), and after differentiation of the velocity function according to Eq. (4-20), the shear stress at the wall, $y = 0$, is given by

$$\tau_w = \mu U \left(\frac{3h_0}{h^2} - \frac{2}{h} \right) \tag{4-21}$$

The friction force, F_f, for a long plane-slider is obtained by integration of the shear stress, as follows:

$$F_f = L \int_{x_1}^{x_2} \tau \, dx \tag{4-22}$$

4.8.1 Friction Coefficient

The bearing friction coefficient, f, is defined as the ratio of the friction force to the bearing load capacity:

$$f = \frac{F_f}{W} \tag{4-23}$$

An important objective of a bearing design is to minimize the friction coefficient. The friction coefficient is usually lower with a thinner minimum film thickness, h_n (in a plane-slider, $h_n = h_1$). However, if the minimum film thickness is too low, it involves the risk of severe wear between the surfaces. Therefore, the design involves a compromise between a low-friction requirement and a risk of severe wear. Determination of the desired minimum film thickness, h_n, requires careful consideration. It depends on the surface finish of the sliding surfaces and the level of vibrations and disturbances in the machine. This part of the design process is discussed in the following chapters.

4.9 FLOW BETWEEN TWO PARALLEL PLATES

Example Problem 4-3

Derive the equation of the pressure gradient in a unidirectional flow inside a thin clearance between two stationary parallel plates as shown in Fig. 4-6. The flow is parallel, in the x direction only. The constant clearance between the plates is h_0, and the rate of flow is Q, and the x axis is along the center of the clearance.

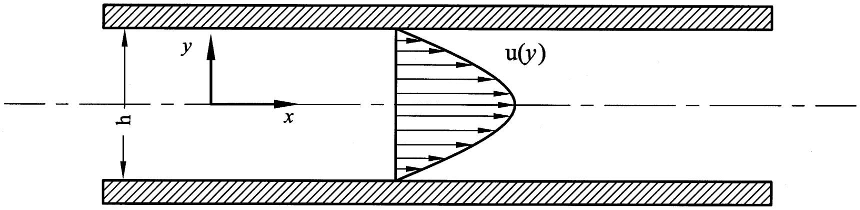

FIG. 4-6 Flow between two parallel plates.

Solution

In a similar way to the solution for hydrodynamic bearing, the parallel flow in the x direction is derived from Eq. (4-4), repeated here as Eq. (4-24):

$$\frac{dp}{dx} = \mu \frac{\partial^2 u}{\partial y^2} \tag{4-24}$$

This equation can be rewritten as

$$\frac{\partial u^2}{\partial y^2} = \frac{1}{\mu}\frac{dp}{dx} \tag{4-25}$$

The velocity profile is solved by a double integration. Integrating Eq. (4-25) twice yields the expression for the velocity u:

$$u = \frac{1}{2\mu}\frac{dp}{dx} y^2 + ny + k \tag{4-26}$$

Here, n and k are integration constants obtained from the two boundary conditions of the flow at the solid surfaces (no-slip condition).

The boundary conditions at the wall of the two plates are:

$$\text{at } y = \pm\frac{h_0}{2}: \quad u = 0 \tag{4-27}$$

The flow is symmetrical, and the solution for n and k is

$$n = 0, \qquad k = -m\left(\frac{h_0}{2}\right)^2 \tag{4-28}$$

The flow equation becomes

$$u = \frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - \frac{h^2}{4}\right) \tag{4-29}$$

The parabolic velocity distribution is shown in Fig. 4-6. The pressure gradient is obtained from the conservation of mass. For a parallel flow, there is no flow in the z direction. For convenience, the y coordinate is measured from the center of the clearance. The constant-volume flow rate, Q, is obtained by integrating the velocity component, u, along the film thickness, as follows:

$$Q = 2L \int_0^{h/2} u \, dy \tag{4-30}$$

Here, L is the width of the parallel plates, in the direction normal to the flow (in the z direction). Substituting the flow in Eq. (4-29) into Eq. (4-30) and integrating yields the expression for the pressure gradient as a function of flow rate, Q:

$$\frac{dp}{dx} = -\frac{12\mu}{bh_0^3}Q \tag{4-31}$$

This equation is useful for the hydrostatic bearing calculations in Chapter 10. The negative sign means that a negative pressure slope in the x direction is required for a flow in the same direction.

4.10 FLUID FILM BETWEEN A CYLINDER AND A FLAT PLATE

There are important applications of a full fluid film at the rolling contact of a cylinder and a flat plate and at the contact of two parallel cylinders. Examples are cylindrical rolling bearings, cams, and gears. A very thin fluid film that separates the surfaces is shown in Fig. 4-7. In this example, the cylinder is stationary and the flat plate has a velocity U in the x direction. In Chapter 6, this problem is extended to include rolling motion of the cylinder over the plate.

The problem of a cylinder and a flat plate is a special case of the general problem of contact between two parallel cylinders. By using the concept of equivalent radius (see Chapter 12), the equations for a cylinder and a flat plate can be extended to that for two parallel cylinders.

Fluid films at the contacts of rolling-element bearings and gear teeth are referred to as *elastohydrodynamic* (EHD) films. The complete analysis of a fluid film in actual rolling-element bearings and gear teeth is quite complex. Under load, the high contact pressure results in a significant elastic deformation of the contact surfaces as well as a rise of viscosity with pressure (see Chapter 12).

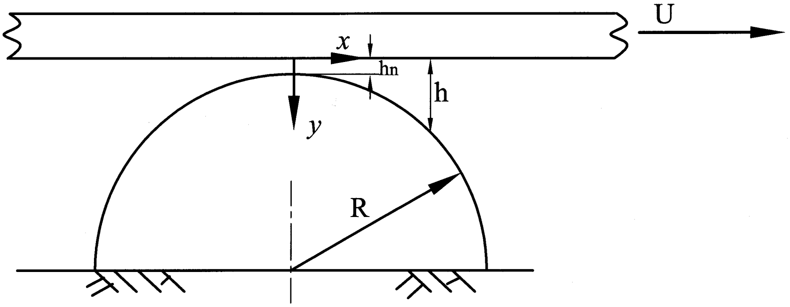

FIG. 4.7 Fluid film between a cylinder and a flat-plate.

However, the following problem is for a light load where the solid surfaces are assumed to be rigid and the viscosity is constant.

The following problem considers a plate and a cylinder with a minimum clearance, h_{min}. In it we consider the case of a light load, where the elastic deformation is very small and can be disregarded (cylinder and plate are assumed to be rigid). In addition, the values of maximum and minimum pressures are sufficiently low, and there is no fluid cavitation. The viscosity is assumed to be constant. The cylinder is stationary, and the flat plate has a velocity U in the x direction as shown in Fig. 4-7.

The cylinder is long in comparison to the film length, and the long-bearing analysis can be applied.

4.10.1 Film Thickness

The film thickness in the clearance between a flat plate and a cylinder is given by

$$h(\theta) = h_{min} + R(1 - \cos\theta) \qquad (4\text{-}32)$$

where θ is a cylinder angle measured from the minimum film thickness at $x = 0$.

Since the minimum clearance, h_{min}, is very small (relative to the cylinder radius), the pressure is generated only at a very small region close to the minimum film thickness, where $x \ll R$, or $x/R \ll 1$.

For a small ratio of x/R, the equation of the clearance, h, can be approximated by a parabolic equation. The following expression is obtained by expanding Eq. (4-32) for h into a Taylor series and truncating terms that include powers higher than $(x/R)^2$. In this way, the expression for the film thickness h can be approximated by

$$h(x) = h_{min} + \frac{x^2}{2R} \qquad (4\text{-}33a)$$

4.10.2 Pressure Wave

The pressure wave can be derived from the expression for the pressure gradient, dp/dx, in Eq. (4-13). The equation is

$$\frac{dp}{dx} = 6\mu U \frac{h_0 - h}{h^3}$$

The unknown h_0 can be replaced by the unknown x_0 according to the equation

$$h_0(x) = h_{min} + \frac{x_0^2}{2R} \qquad (4\text{-}33b)$$

After substituting the value of h according to Eqs. (4-33), the solution for the pressure wave can be obtained by the following integration:

$$p(x) = 24\mu U R^2 \int_{-\infty}^{x} \frac{x_0^2 - x^2}{(2Rh_{\min} + x^2)^3}\, dx \tag{4-34}$$

The unknown x_0 is solved by the following boundary conditions of the pressure wave:

$$\begin{aligned} &\text{at } x = -\infty: \quad p = 0 \\ &\text{at } x = \infty: \quad\;\; p = 0 \end{aligned} \tag{4-35}$$

For numerical integration, the infinity can be replaced by a relatively large finite value, where pressure is very small and can be disregarded.

Remark: The result is an antisymmetrical pressure wave (on the two sides of the minimum film thickness), and there will be no resultant load capacity. In actual cases, the pressures are high, and there is a cavitation at the diverging side. A solution that considers the cavitation with realistic boundary conditions is presented in Chapter 6.

4.11 SOLUTION IN DIMENSIONLESS TERMS

If we perform a numerical integration of Eq. (4-34) for solving the pressure wave, the solution would be limited to a specific bearing geometry of cylinder radius R and minimum clearance $h_{\min}$. The numerical integration must be repeated for a different bearing geometry.

For a universal solution, there is obvious merit to performing a solution in dimensionless terms. For conversion to dimensionless terms, we normalize the x coordinate by dividing it by $\sqrt{2Rh_{\min}}$ and define a dimensionless coordinate as

$$\bar{x} = \frac{x}{\sqrt{2Rh_{\min}}} \tag{4-36}$$

In addition, a dimensionless clearance ratio is defined:

$$\bar{h} = \frac{h}{h_{\min}} \tag{4-37}$$

The equation for the variable clearance ratio as a function of the dimensionless coordinate becomes

$$\bar{h} = 1 + \bar{x}^2 \tag{4-38}$$

Let us recall that the unknown h_0 is the fluid film thickness at the point of peak pressure. It is often convenient to substitute it by the location of the peak pressure, x_0, and the dimensionless relation then is

$$\bar{h}_0 = 1 + \bar{x}_0^2 \tag{4-39}$$

In addition, if the dimensionless pressure is defined as

$$\bar{p} = \frac{h_{\min}^2}{\sqrt{2Rh_{\min}}} \frac{1}{6\mu U} p \tag{4-40}$$

then the following integration gives the dimensionless pressure wave:

$$\bar{p} = \int d\bar{p} = \int_{-\infty}^{\bar{x}} \frac{\bar{x}^2 - \bar{x}_0^2}{(1 + \bar{x}^2)^3} \, d\bar{x} \tag{4-41}$$

For a numerical integration of the pressure wave according to Eq. (4-41), the boundary $\bar{x} = -\infty$ is replaced by a relatively large finite dimensionless value, where pressure is small and can be disregarded, such as $\bar{x} = -4$. An example of numerical integration is given in Example Problem 4.

In a similar way, for a numerical solution of the unknown x_0 and the load capacity, it is possible to replace infinity with a finite number, for example, the following mathematical boundary conditions of the pressure wave of a full fluid film between a cylinder and a plane:

$$\begin{aligned} &\text{at } \bar{x} = -\infty: \quad \bar{p} = 0 \\ &\text{at } \bar{x} = \infty: \quad \bar{p} = 0 \end{aligned} \tag{4-42a}$$

These conditions are replaced by practical numerical boundary conditions:

$$\begin{aligned} &\text{at } \bar{x} = -4: \quad \bar{p} = 0 \\ &\text{at } \bar{x} = +4: \quad \bar{p} = 0 \end{aligned} \tag{4-42b}$$

These practical numerical boundary conditions do not introduce a significant error because a significant hydrodynamic pressure is developed only near the minimum fluid film thickness, at $\bar{x} = 0$.

Example Problem 4-4

Ice Sled

An ice sled is shown in Fig. 4-8. On the left-hand side, there is a converging clearance that is formed by the geometry of a quarter of a cylinder. A flat plate continues the curved cylindrical shape. The flat part of the sled is parallel to the flat ice. It is running parallel over the flat ice on a thin layer of water film of a constant thickness h_0.

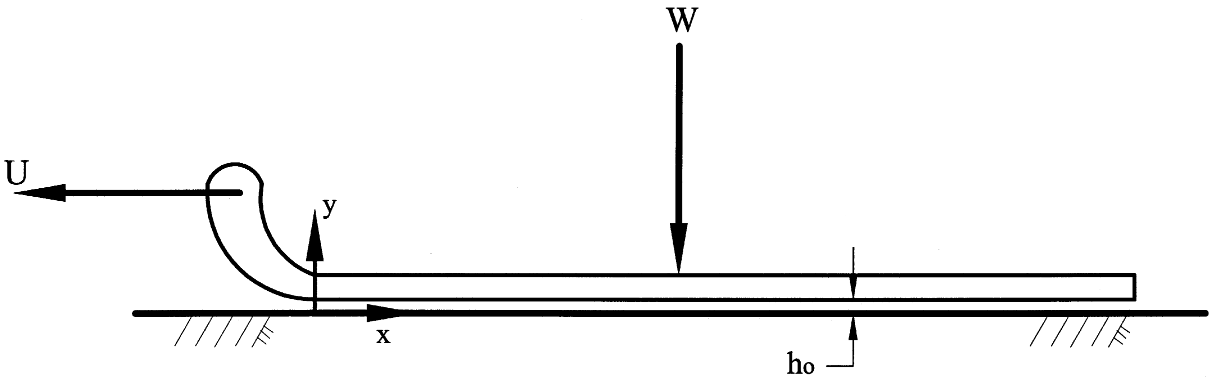

FIG. 4-8 Ice sled.

Derive and plot the pressure wave at the entrance region of the fluid film and under the flat ice sled as it runs over the ice at velocity U. Derive the equation of the sled load capacity.

Solution

The long-bearing approximation is assumed for the sled similar to the converging slope in Fig. 4-4. The fluid film equation for an infinitely long bearing is

$$\frac{dp}{dx} = 6\mu U \frac{h - h_0}{h^3}$$

In the parallel region, $h = h_0$, and it follows from the foregoing equation that

$$\frac{dp}{dx} = 0 \qquad \text{(along the parallel region of constant clearance)}$$

In this case, $h_0 = h_{min}$, and the equation for the variable clearance at the converging region is

$$h(x) = h_0 + \frac{x^2}{2R}$$

This means that for a wide sled, the pressure is constant within the parallel region. This is correct only if $L \gg B$, where L is the bearing width (in the z direction) and B is along the sled in the x direction.

Fluid flow in the converging region generates the pressure, which is ultimately responsible for supporting the load of the sled. Through the adhesive force of viscous shear, the fluid is dragged into the converging clearance, creating the pressure in the parallel region.

Substituting $h(x)$ into the pressure gradient equation yields

$$\frac{dp}{dx} = 6\mu U \frac{\left(h_0 + \frac{x^2}{2R}\right) - h_0}{\left(h_0 + \frac{x^2}{2R}\right)^3}$$

or

$$\frac{dp}{dx} = 24\mu U R^2 \frac{x^2}{(2Rh_0 + x^2)^3}$$

Applying the limits of integration and the boundary condition, we get the following for the pressure distribution:

$$p(x) = 24\mu U R^2 \int_{-\infty}^{x} \frac{x^2}{(2Rh_0 + x^2)^3}\, dx$$

This equation can be integrated analytically or numerically. For numerical integration, since a significant pressure is generated only at a low x value, the infinity boundary of integration is replaced by a finite magnitude.

Conversion to a Dimensionless Equation. As discussed earlier, there is an advantage in solving the pressure distribution in dimensionless terms. A regular pressure distribution curve is limited to the specific bearing data of given radius R and clearance h_0. The advantage of a dimensionless curve is that it is universal and applies to any bearing data. For conversion of the pressure gradient to dimensionless terms, we normalize x by dividing by $\sqrt{2Rh_0}$ and define dimensionless terms as follows:

$$\bar{x} = \frac{x}{\sqrt{2Rh_0}}, \qquad \bar{h} = \frac{h}{h_0}, \qquad \bar{h} = 1 + \bar{x}^2$$

Converting to dimensionless terms, the pressure gradient equation gets the form

$$\frac{dp}{dx} = \frac{6\mu U}{h_0^2} \frac{\bar{h} - 1}{(1 + \bar{x}^2)^3}$$

Here, $h_0 = h_{\min}$ is the minimum film thickness at $x = 0$. Substituting for the dimensionless clearance and rearranging yields

$$\frac{h_0^2}{\sqrt{2Rh_0} 6\mu U} dp = \frac{\bar{x}^2}{(1 + \bar{x}^2)^3} \, dx$$

The left-hand side of this equation is defined as the dimensionless pressure. Dimensionless pressure is equal to the following integral:

$$\bar{p} = \frac{h_0^2}{\sqrt{2Rh_0}} \frac{1}{6\mu U} \int_0^p dp = \int_{-\infty}^{x} \frac{\bar{x}^2}{(1 + \bar{x}^2)^3} \, d\bar{x}$$

Numerical Integration. The dimensionless pressure is solved by an analytical or numerical integration of the preceding function within the specified boundaries (see Appendix B). The pressure p_0 under the flat plate is obtained by integration to the limit $x = 0$:

$$\bar{p} = \frac{h_0^2}{\sqrt{2Rh_0}} \frac{1}{6\mu U} \, p_0 = \sum_{-3}^{x} \frac{\bar{x}_i^2}{(1 + \bar{x}_i^2)^3} \, \Delta x_i$$

The pressure is significant only near $x = 0$. Therefore, for the numerical integration, a finite number replaces infinity. The resulting pressure distribution is shown in Fig. 4-9.

Comparison with Analytical Integration. The maximum pressure at $x = 0$ as well as along the constant clearance, $x > 0$, can also be solved by analytical

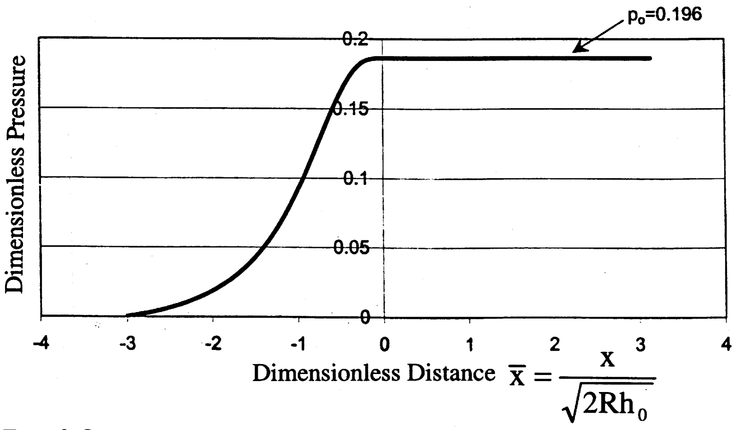

FIG. 4-9 Dimensionless pressure wave.

integration of the following equation:

$$\bar{p}_0 = \int_{-\infty}^{0} \frac{\bar{x}^2}{(1+\bar{x}^2)^3}\, d\bar{x}$$

Using integration tables, the following integral solution is obtained:

$$\bar{p} = -\frac{x}{4(1+x^2)^2} + \frac{x}{8(1+x^2)} + \frac{\arctan(x)}{8}\bigg|_{\infty}^{0} = \frac{\pi}{2 \times 8} = 0.196$$

This result is equal to that obtained by a numerical integration.

Load Capacity. The first step is to find the pressure p_0 from the dimensionless pressure wave, which is equal to 0.196. The converging entrance area is small in comparison to the area under the flat plate. Neglecting the pressure in the entrance region, the equation for the load capacity, W, becomes

$$W = p_0 BL$$

Here, B and L are the dimensions of the flat-plate area.

Calculation of Film Thickness. When the load capacity of the sled W is known, it is possible to solve for the film thickness, $h_0 = h_{\min}$. Substituting in the equation, $W = p_0 BL$, the equation of the constant pressure p_0 under the flat plate as a function of the clearance h_0 can allow us to solve for the constant film thickness. The constant pressure p_0 is derived from its dimensionless counterpart:

$$\bar{p}_0 = \frac{h_0^2}{\sqrt{2Rh_0}} \frac{1}{6\mu U} p_0 = 0.196$$

If the load, cylinder radius, water viscosity, and sled velocity are known, it is possible to solve for the film thickness, h_0.

Example Problem 4-5

Derive the equation for the pressure gradient of a journal bearing if the journal and bearing are both rotating around their stationary centers. The surface velocity of the bearing bore is $U_j, = \omega_j R$, and the surface velocity of the journal is $U_b = \omega_b R_1$.

Solution

Starting from Eq. (4-4):

$$\frac{dp}{dx} = \mu \frac{\partial^2 u}{\partial y^2}$$

and integrating twice (in a similar way to a stationary bearing) yields

$$u = my^2 + ny + k$$

However, the boundary conditions and continuity conditions are as follows:

$$\begin{aligned} &\text{at } y = 0: && u = \omega_b R_1 \\ &\text{at } y = h(x): && u = \omega_j R \end{aligned}$$

In this case, the constant-volume flow rate, q, per unit of bearing length at the point of peak pressure is

$$q = \int_0^h u\, dy = \frac{(\omega_b R_1 + \omega_j R) h_0}{2}$$

Solving for m, n and k and substituting in a similar way to the previous problem while also assuming $R_1 \approx R$, the following equation for the pressure gradient is obtained:

$$\frac{dp}{dx} = 6R(\omega_b + \omega_j)\mu \frac{h - h_0}{h^3}$$

Problems

4-1 A long plane-slider, $L = 200$ mm and $B = 100$ mm, is sliding at velocity of 0.3 m/s. The minimum film thickness is $h_1 = 0.005$ mm and the maximum film thickness is $h_2 = 0.010$ mm. The fluid is SAE 30, and the operating temperature of the lubricant is assumed a constant 30°C.

a. Assume the equation for an infinitely long bearing, and use numerical integration to solve for the pressure wave (use trial and error to solve for x_0). Plot a curve of the pressure distribution $p = p(x)$.

b. Use numerical integration to find the load capacity. Compare this to the load capacity obtained from Eq. (4-19).

c. Find the friction force and the friction coefficient.

4-2 A slider is machined to have a parabolic surface. The slider has a horizontal velocity of 0.3 m/s. The minimum film $h_{min} = 0.020$ mm, and the clearance varies with x according to the following equation:

$$h = 0.020 + 0.01x^2$$

The slider velocity is $U = 0.5$ m/s. The length $L = 300$ mm and the width in the sliding direction $B = 100$ mm. The lubricant is SAE 40 and the temperature is assumed constant, $T = 40°C$. Assume the equation for an infinitely long bearing.

a. Use numerical integration and plot the dimensionless pressure distribution, $p = p(x)$.

b. Use numerical integration to find the load capacity.

c. Find the friction force and the friction coefficient.

4-3 A blade of a sled has the geometry shown in the Figure 4-8. The sled is running over ice on a thin layer of water film. The total load (weight of the sled and person) is 1500 N. The sled velocity is 20 km/h, the radius of the inlet curvature is 30 cm, and the sled length $B = 30$ cm, and width is $L = 100$ cm. The viscosity of water is $\mu = 1.792 \times 10^{-3}$ N-s/m^2.

Find the film thickness ($h_0 = h_{min}$) of the thin water layer shown in Fig. 4-8.

Direction: The clearance between the plate and disk is $h = h_{min} + x^2/2R$, and assume that $p = 0$ at $x = R$.