5

Basic Hydrodynamic Equations

5.1 NAVIER–STOKES EQUATIONS

The pressure distribution and load capacity of a hydrodynamic bearing are analyzed and solved by using classical fluid dynamics equations. In a thin fluid film, the viscosity is the most important fluid property determining the magnitude of the pressure wave, while the effect of the fluid inertia (*ma*) is relatively small and negligible. Reynolds (1894) introduced classical hydrodynamic lubrication theory. Although a lot of subsequent research has been devoted to this discipline, Reynolds' equation still forms the basis of most analytical research in hydrodynamic lubrication. The Reynolds equation can be derived from the Navier–Stokes equations, which are the fundamental equations of fluid motion.

The derivation of the Navier–Stokes equations is based on several assumptions, which are included in the list of assumptions (Sec. 4.2) that forms the basis of the theory of hydrodynamic lubrication. An important assumption for the derivation of the Navier–Stokes equations is that there is a linear relationship between the respective components of stress and strain rate in the fluid.

In the general case of three-dimensional flow, there are nine stress components referred to as components of the *stress tensor*. The directions of the stress components are shown in Fig 5-1.

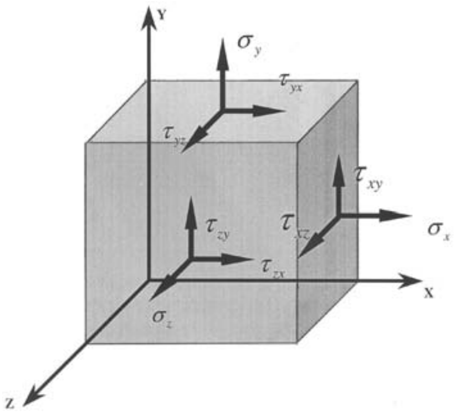

FIG. 5-1 Stress components acting on a rectangular fluid element.

The stress components σ_x, σ_y, σ_z are of tension or compression (if the sign is negative), as shown in Fig. 5-1. However, the mixed components τ_{xy}, τ_{zy}, τ_{xz} are shear stresses parallel to the surfaces.

It is possible to show by equilibrium considerations that the shear components are symmetrical:

$$\tau_{xy} = \tau_{yx}, \qquad \tau_{yz} = \tau_{zy}, \qquad \tau_{xz} = \tau_{zx} \tag{5-1}$$

Due to symmetry, the number of stress components is reduced from nine to six.

In rectangular coordinates the six stress components are

$$\begin{aligned}
\sigma_x &= -p + 2\mu \frac{\partial u}{\partial x} \\
\sigma_y &= -p + 2\mu \frac{\partial v}{\partial y} \\
\sigma_z &= -p + 2\mu \frac{\partial w}{\partial z} \\
\tau_{xy} = \tau_{yx} &= \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \\
\tau_{yz} = \tau_{zy} &= \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\
\tau_{zx} = \tau_{xz} &= \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)
\end{aligned} \tag{5-2}$$

A fluid that can be described by Eq. (5-2) is referred to as *Newtonian fluid*. This equation is based on the assumption of a linear relationship between the stress and strain-rate components. For most lubricants, such a linear relationship is an adequate approximation. However, under extreme conditions, e.g., very high pressure of point or line contacts, this assumption is no longer valid. An assumption that is made for convenience is that the viscosity, μ, of the lubricant is constant. Also, lubrication oils are practically *incompressible*, and this property simplifies the Navier–Stokes equations because the density, ρ, can be assumed to be constant. However, this assumption cannot be applied to air bearings.

Comment. As mentioned earlier, in thin films the velocity component ν is small in comparison to u and w, and two shear components can be approximated as follows:

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \approx \mu\frac{\partial u}{\partial y}$$
$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \approx \mu\frac{\partial w}{\partial y} \tag{5-3}$$

The Navier–Stokes equations are based on the balance of forces acting on a small, infinitesimal fluid element having the shape of a rectangular parallelogram with dimensions dx, dy, and dz, as shown in Fig. 5-1. The force balance is similar to that in Fig. 4-1; however, the general balance of forces is of three dimensions, in the x, y and z directions. The surface forces are the product of stresses, or pressures, and the corresponding areas.

When the fluid is at rest there is a uniform hydrostatic pressure. However, when there is fluid motion, there are deviatoric normal stresses σ'_x, σ'_y, σ'_z that are above the hydrostatic (average) pressure, p. Each of the three normal stresses is the sum of the average pressure, and the deviatoric normal stress (above the average pressure), as follows:

$$\sigma_x = -p + \sigma'_x, \qquad \sigma_y = -p + \sigma'_y, \qquad \sigma_z = -p + \sigma'_z \tag{5-4a}$$

According to Newton's second law, the sum of all forces acting on a fluid element, including surface forces in the form of stresses and body forces such as the gravitational force, is equal to the product of mass and acceleration (ma) of the fluid element. After dividing by the volume of the fluid element, the equations of the force balance become

$$\rho\frac{du}{dt} = X - \frac{\partial p}{\partial x} + \frac{\partial \sigma'_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial_{xz}}{\partial z}$$
$$\rho\frac{dv}{dt} = Y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \sigma'_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \tag{5-4b}$$
$$\rho\frac{dw}{dt} = Z - \frac{\partial p}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma'_z}{\partial z}$$

Here, p is the pressure, u, v, and w are the velocity components in the x, y, and z directions, respectively. The three forces X, Y, Z are the components of a body force, per unit volume, such as the gravity force that is acting on the fluid. According to the assumptions, the fluid density, ρ, and the viscosity, μ, are considered constant. The derivation of the Navier–Stokes equations is included in most fluid dynamics textbooks (e.g., White, 1985).

For an incompressible flow, the continuity equation, which is derived from the conservation of mass, is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{5-5}$$

After substituting the stress components of Eq. (5-2) into Eq. (5-4b), using the continuity equation (5-5) and writing in full the convective time derivative of the acceleration components, the following Navier–Stokes equations in Cartesian coordinates for a Newtonian incompressible and constant-viscosity fluid are obtained

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = X - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{5-6a}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = Y = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \tag{5-6b}$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = Z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \tag{5-6c}$$

The Navier–Stokes equations can be solved for the velocity distribution. The velocity is described by its three components, u, v, and w, which are functions of the location (x, y, z) and time. In general, fluid flow problems have four unknowns: u, v, and w and the pressure distribution, p. Four equations are required to solve for the four unknown functions. The equations are the three Navier–Stokes equations, the fourth equation is the continuity equation (5-5).

5.2 REYNOLDS HYDRODYNAMIC LUBRICATION EQUATION

Hydrodynamic lubrication involves a thin-film flow, and in most cases the fluid inertia and body forces are very small and negligible in comparison to the viscous forces. Therefore, in a thin-film flow, the inertial terms [all terms on the left side of Eqs. (5.6)] can be disregarded as well as the body forces X, Y, Z. It is well known in fluid dynamics that the ratio of the magnitude of the inertial terms relative to the viscosity terms in Eqs. (5-6) is of the order of magnitude of the

Reynolds number, Re. For a lubrication flow (thin-film flow), Re $\ll$ 1, the Navier–Stokes equations reduce to the following simple form:

$$\frac{\partial p}{\partial x} = \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{5-7a}$$

$$\frac{\partial p}{\partial y} = \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \tag{5-7b}$$

$$\frac{\partial p}{\partial z} = \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \tag{5-7c}$$

These equations indicate that viscosity is the dominant effect in determining the pressure distribution in a fluid film bearing.

The assumptions of classical hydrodynamic lubrication theory are summarized in Chapter 4. The velocity components of the flow in a thin film are primarily u and w in the x and z directions, respectively. These directions are along the fluid film layer (see Fig. 1-2). At the same time, there is a relatively very slow velocity component, v, in the y direction across the fluid film layer. Therefore, the pressure gradient in the y direction in Eq. (5-7b) is very small and can be disregarded.

In addition, Eqs. (5-7a and c) can be further simplified because the order of magnitude of the dimensions of the thin fluid film in the x and z directions is much higher than that in the y direction across the film thickness. The orders of magnitude are

$$\begin{aligned} x &= O(B) \\ y &= O(h) \\ z &= O(L) \end{aligned} \tag{5-8a}$$

Here, the symbol O represents *order of magnitude*. The dimension B is the bearing length along the direction of motion (x direction), and h is an average fluid film thickness. The width L is in the z direction of an inclined slider. In a journal bearing, L is in the axial z direction and is referred to as the *bearing length*.

In hydrodynamic bearings, the fluid film thickness is very small in comparison to the bearing dimensions, $h \ll B$ and $h \ll L$. By use of Eqs. (5-8b), a comparison can be made between the orders of magnitude of the second derivatives of the various terms on the right-hand side of Eq. (5-7a), which are as follows:

$$\frac{\partial^2 u}{\partial y^2} = O\left(\frac{U}{h^2}\right)$$
$$\frac{\partial^2 u}{\partial x^2} = O\left(\frac{U}{B^2}\right) \qquad (5\text{-}8b)$$
$$\frac{\partial^2 u}{\partial z^2} = O\left(\frac{U}{L^2}\right)$$

In conventional finite-length bearings, the ratios of dimensions are of the following orders:

$$\frac{L}{B} = O(1)$$
$$\frac{h}{B} = L(10^{-3}) \qquad (5\text{-}9)$$

Equations (5-8) and (5-9) indicate that the order of the term $\partial^2 u/\partial y^2$ is larger by 10^6, in comparison to the order of the other two terms, $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial z^2$. Therefore, the last two terms can be neglected in comparison to the first one in Eq. (5-7a). In the same way, only the term $\partial^2 w/\partial y^2$ is retained in Eq. (5-7c). According to the assumptions, the pressure gradient across the film thickness, $\partial p/\partial y$, is negligible, and the Navier–Stokes equations reduce to the following two simplified equations:

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2} \quad \text{and} \quad \frac{\partial p}{\partial z} = \mu \frac{\partial^2 w}{\partial y^2} \qquad (5\text{-}10)$$

The first equation is identical to Eq. (4-4), which was derived from first principles in Chapter 4 for an infinitely long bearing. In a long bearing, there is a significant pressure gradient only in the x direction; however, for a finite-length bearing, there is a pressure gradient in the x and z directions, and the two Eqs. (5-10) are required for solving the flow and pressure distributions.

The two Eqs. (5-10) together with the continuity Eq. (5-5) and the boundary condition of the flow are used to derive the Reynolds equation. The derivation of the Reynolds equation is included in several books devoted to the analysis of hydrodynamic lubrication see Pinkus (1966), and Szeri (1980). The Reynolds equation is widely used for solving the pressure distribution of hydrodynamic bearings of finite length. The Reynolds equation for Newtonian incompressible and constant-viscosity fluid in a thin clearance between two rigid surfaces of relative motion is given by

$$\frac{\partial}{\partial x}\left(\frac{h^3}{\mu}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{h^3}{\mu}\frac{\partial p}{\partial z}\right)=6(U_1-U_2)\frac{\partial h}{\partial x}+6\frac{\partial}{\partial x}(U_1+U_2)+12(V_2-V_1) \tag{5-11}$$

The velocity components of the two surfaces that form the film boundaries are shown in Fig. 5-2. The tangential velocity components, U_1 and U_2, in the x direction are of the lower and upper sliding surfaces, respectively (two fluid film boundaries). The normal velocity components, in the y direction, V_1 and V_2, are of the lower and upper boundaries, respectively. In a journal bearing, these components are functions of x (or angle θ) around the journal bearing.

The right side of Eq. (5-11) must be negative in order to result in a positive pressure wave and load capacity. Each of the three terms on the right-hand side of Eq. (5-11) has a physical meaning concerning the generation of the pressure wave. Each term is an action that represents a specific type of relative motion of the surfaces. Each action results in a positive pressure in the fluid film. The various actions are shown in Fig. 5-3. These three actions can be present in a bearing simultaneously, one at a time or in any other combination. The following are the various actions.

Viscous wedge action: This action generates positive pressure wave by dragging the viscous fluid into a converging wedge.

Elastic stretching or compression of the boundary surface: This action generates a positive pressure by compression of the boundary. The compression of the surface reduces the clearance volume and the viscous fluid is squeezed out, resulting in a pressure rise. This action is

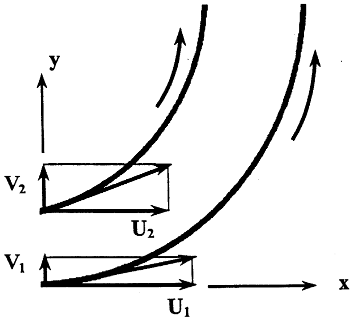

FIG. 5-2 Directions of the velocity components of fluid-film boundaries in the Reynolds equation.

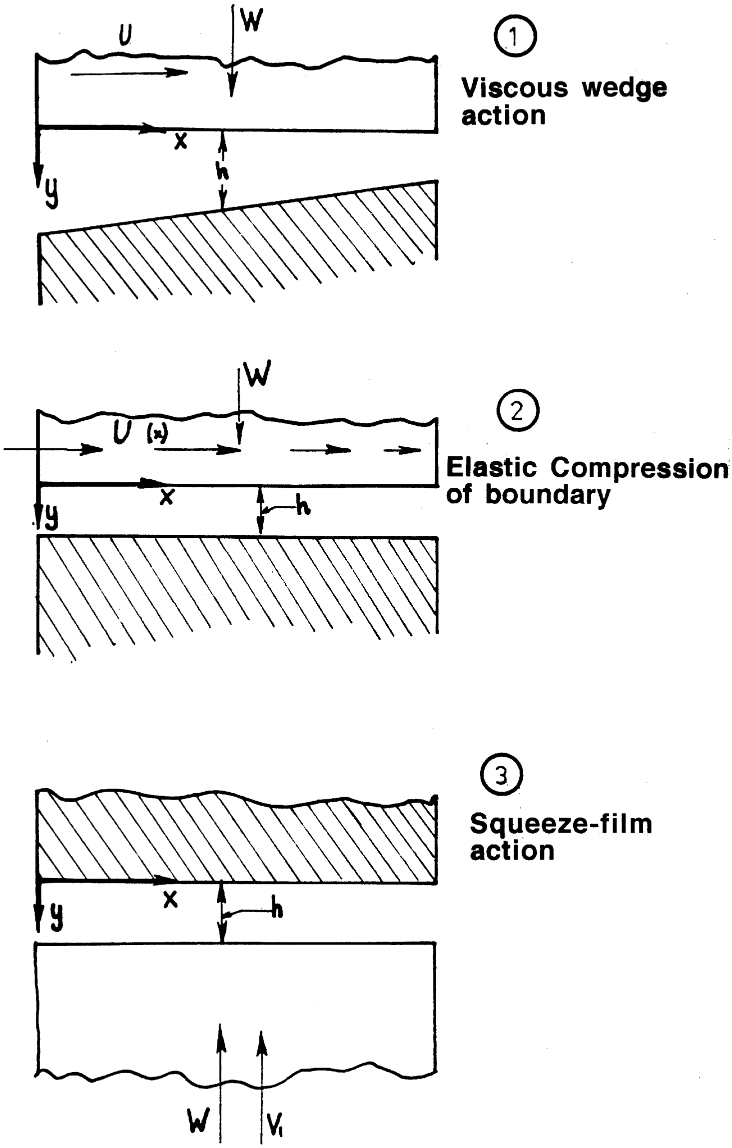

FIG. 5-3 Viscous film actions that result in a positive pressure wave.

negligible in practical rigid bearings. Continuous stretching or compression of the boundaries does not exist in steady-state operation. It can act only as a transient effect, under dynamic condition, for an elastomer bearing material. This action is usually not considered for rigid bearing materials.

Squeeze-film action: The squeezing action generates a positive pressure by reduction of the fluid film volume. The incompressible viscous fluid is squeezed out through the thin clearance. The thin clearance has resistance to the squeeze-film flow, resulting in a pressure buildup to overcome the flow resistance (see Problem 5-3).

In most practical bearings, the surfaces are rigid and there is no stretching or compression action. In that case, the Reynolds equation for an incompressible fluid and constant viscosity reduces to

$$\frac{\partial}{\partial x}\left(\frac{h^3}{\mu}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{h^3}{\mu}\frac{\partial p}{\partial z}\right)=6(U_1-U_2)\frac{\partial h}{\partial x}+12(V_2-V_1) \tag{5-12}$$

As indicated earlier, the two right-hand terms must be negative in order to result in a positive pressure wave. On the right side of the Reynolds equation, the first term of relative sliding motion (U_1-U_2) describes a viscous wedge effect. It requires inclined surfaces, $\partial h/\partial x$, to generate a fluid film wedge action that results in a pressure wave. Positive pressure is generated if the film thickness reduces in the x direction (negative $\partial h/\partial x$).

The second term on the right side of the Reynolds equation describes a squeeze-film action. The difference in the normal velocity (V_2-V_1) represents the motion of surfaces toward each other, referred to as *squeeze-film action*. A positive pressure builds up if (V_2-V_1) is negative and the surfaces are approaching each other. The Reynolds equation indicates that a squeeze-film effect is a viscous effect that can generate a pressure wave in the fluid film, even in the case of parallel boundaries.

It is important to mention that the Reynolds equation is objective, in the sense that the pressure distribution must be independent of the selection of the coordinate system. In Fig. 5-2 the coordinates are stationary and the two surfaces are moving relative to the coordinate system. However, the same pressure distribution must result if the coordinates are attached to one surface and are moving and rotating with it. For convenience, in most problems we select a stationary coordinate system where the x coordinate is along the bearing surface and the y coordinate is normal to this surface. In that case, the lower surface has only a tangential velocity, U_1, and there is no normal component, $V_1=0$.

The value of each of the velocity components of the fluid boundary, U_1, U_2, V_1, V_2, depends on the selection of the coordinate system. Surface velocities in a stationary coordinate system would not be the same as those in a moving coordinate system. However, velocity differences on the right-hand side of the Reynolds equation, which represent relative motion, are independent of the selection of the coordinate system. In journal bearings under dynamic conditions, the journal center is not stationary. The velocity of the center must be considered for the derivation of the right-hand side terms of the Reynolds equation.

5.3 WIDE PLANE-SLIDER

The equation of a plane-slider has been derived from first principles in Chapter 4. Here, this equation will be derived from the Reynolds equation and compared to that in Chapter 4.

A plane-slider and its coordinate system are shown in Fig. 1-2. The lower plate is stationary, and the velocity components at the lower wall are $U_1 = 0$ and $V_1 = 0$. At the same time, the velocity at the upper wall is equal to that of the slider. The slider has only a horizontal velocity component, $U_2 = U$, where U is the plane-slider velocity. Since the velocity of the slider is in only the x direction and there is no normal component in the y direction, $V_2 = 0$. After substituting the velocity components of the two surfaces into Eq. (5-11), the Reynolds equation will reduce to the form

$$\frac{\partial}{\partial x}\left(\frac{h^3}{\mu}\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{h^3}{\mu}\frac{\partial p}{\partial z}\right) = 6U\frac{\partial h}{\partial x} \tag{5-13}$$

For a wide bearing, $L \gg B$, we have $\partial p/\partial z \cong 0$, and the second term on the left side of Eq. (5-13) can be omitted. The Reynolds equation reduces to the following simplified form:

$$\frac{\partial}{\partial x}\left(\frac{h^3}{\mu}\frac{\partial p}{\partial x}\right) = 6U\frac{\partial h}{\partial x} \tag{5-14}$$

For a plane-slider, if the x coordinate is in the direction of a converging clearance (the clearance reduces with x), as shown in Fig. 4-4, integration of Eq. (5-14) results in a pressure gradient expression equivalent to that of a hydrodynamic journal bearing or a negative-slope slider in Chapter 4. The following equation is the expression for the pressure gradient for a converging clearance, $\partial h/\partial x < 0$ (negative slope):

$$\frac{dp}{dx} = 6U\mu\frac{h - h_0}{h^3} \qquad \text{for initial } \frac{\partial h}{\partial x} < 0 \text{ (negative slope in Fig. 4-4)} \tag{5-15}$$

The unknown constant h_0 (constant of integration) is determined from additional information concerning the boundary conditions of the pressure wave. The meaning of h_0 is discussed in Chapter 4—it is the film thickness at the point of a peak pressure along the fluid film. In a converging clearance such as a journal bearing near $x = 0$, the clearance slope is negative, $\partial h/\partial x < 0$. The result is that the pressure increases at the start of the pressure wave (near $x = 0$). At that point, the pressure gradient $dp/dx > 0$ because $h > h_0$.

However, if the x coordinate is in the direction of a diverging clearance (the clearance increases with x), as shown in Fig. (1-2), Eq. (5-15) changes its sign and takes the following form:

$$\frac{dp}{dx} = 6U\mu \frac{h_0 - h}{h^3} \qquad \text{for initial } \frac{\partial h}{\partial x} > 0 \text{ (positive slope in Fig. 4-5)} \tag{5-15}$$

5.4 FLUID FILM BETWEEN A FLAT PLATE AND A CYLINDER

A fluid film between a plate and a cylinder is shown in Fig. 5-4. In Chapter 4, the pressure wave for relative sliding is derived, where the cylinder is stationary and a flat plate has a constant velocity in the x direction. In the following example, the previous problem is extended to a combination of rolling and sliding. In this case, the flat plate has a velocity U in the x direction and the cylinder rotates at an angular velocity ω around its stationary center. The coordinate system (x, y) is stationary.

In Sec. 4.8, it is mentioned that there is a significant pressure wave only in the region close to the minimum film thickness. In this region, the slope between the two surfaces (the fluid film boundaries), as well as between the two surface velocities, is of a very small angle α. For a small α, we can approximate that $\cos\alpha \approx 1$.

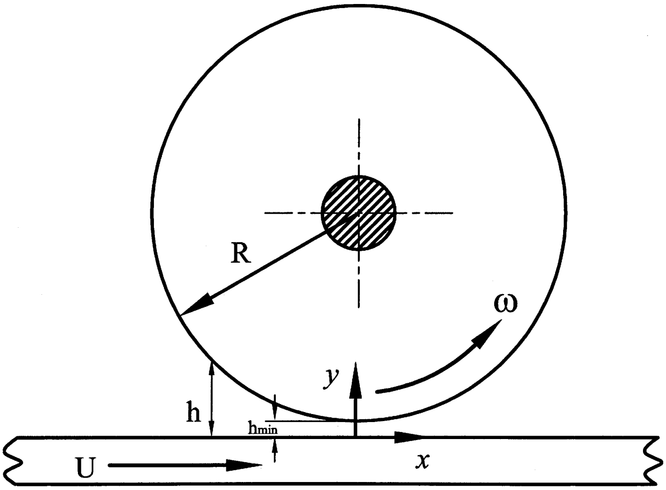

FIG. 5-4 Fluid film between a moving plate and a rotating cylinder.

In Fig. 5-4, the surface velocity of the cylinder is not parallel to the x direction and it has a normal component V_2. The surface velocities on the rotating cylinder surface are

$$U_2 = \omega R \cos\alpha \approx \omega R \qquad V_2 \approx \omega R \frac{\partial h}{\partial x} \tag{5-16}$$

At the same time, on the lower plate there is only velocity U in the x direction and the boundary velocity is

$$U_1 = U \qquad V_1 = 0 \tag{5-17}$$

Substituting Eqs. (5-16) and (5-17) into the right side of Eq. (5-11), yields

$$\begin{aligned} 6(U_1 - U_2)\frac{\partial h}{\partial x} + 12(V_2 - V_1) &= 6(U - \omega R)\frac{\partial h}{\partial x} + 12\omega R \frac{\partial h}{\partial x} \\ &= 6(U + \omega R)\frac{\partial h}{\partial x} \end{aligned} \tag{5-18}$$

The Reynolds equation for a fluid film between a plate and a cylinder becomes

$$\frac{\partial}{\partial x}\left(\frac{h^3}{\mu}\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{h^3}{\mu}\frac{\partial p}{\partial z}\right) = 6(U + \omega R)\frac{\partial h}{\partial x} \tag{5-19}$$

For a long bearing, the pressure gradient in the axial direction is negligible, $\partial p/\partial z \cong 0$. Integration of Eq. (5-19) yields

$$\frac{dp}{dx} = 6\mu(U + \omega R)\frac{h - h_0}{h^3} \tag{5-20}$$

This result indicates that the pressure gradient, the pressure wave, and the load capacity are proportional to the sum of the two surface velocities in the x direction. The sum of the plate and cylinder velocities is $(U + \omega R)$. In the case of pure rolling, $U = \omega R$, the pressure wave, and the load capacity are twice the magnitude of that generated by pure sliding. Pure sliding is when the cylinder is stationary, $\omega = 0$, and only the plate has a sliding velocity U.

The unknown constant h_0, (constant of integration) is the film thickness at the point of a peak pressure, and it can be solved from the boundary conditions of the pressure wave.

5.5 TRANSITION TO TURBULENCE

For the estimation of the Reynolds number, Re, the average radial clearance, C, is taken as the average film thickness. The Reynolds number for the flow inside the clearance of a hydrodynamic journal bearing is

$$\mathrm{Re} = \frac{U\rho C}{\mu} = \frac{UC}{\nu} \tag{5-21}$$

Here, U is the journal surface velocity, as shown in Fig. 1-2 and ν is the kinematic viscosity $\nu = \mu/\rho$. In most cases, hydrodynamic lubrication flow involves low Reynolds numbers. There are other examples of thin-film flow in fluid mechanics, such as the boundary layer, where Re is low.

The flow in hydrodynamic lubrication is laminar at low Reynolds numbers. Experiments in journal bearings indicate that the transition from laminar to turbulent flow occurs between Re = 1000 and Re = 1600. The value of the Reynolds number at the transition to turbulence is not the same in all cases. It depends on the surface finish of the rotating surfaces as well as the level of vibrations in the bearing. The transition is gradual: Turbulence starts to develop at about Re = 1000; and near Re = 1600, full turbulent behavior is maintained. In hydrodynamic bearings, turbulent flow is undesirable because it increases the friction losses. Viscous friction in turbulent flow is much higher in comparison to laminar flow. The effect of the turbulence is to increase the apparent viscosity; that is, the bearing performance is similar to that of a bearing having laminar flow and much higher lubricant viscosity.

In journal bearings, Taylor vortexes can develop at high Reynolds numbers. The explanation for the initiation of Taylor vortexes involves the centrifugal forces in the rotating fluid inside the bearing clearance. At high Re, the fluid film becomes unstable because the centrifugal forces are high relative to the viscous resistance. Theory indicates that in concentric cylinders, Taylor vortexes would develop only if the inner cylinder is rotating relative to the outer, stationary cylinder.

This instability gives rise to vortexes (Taylor vortexes) in the fluid film. Taylor (1923) published his classical work on the theory of stability between rotating cylinders. According to this theory, a stable laminar flow in a journal bearing is when the Reynolds number is below the following ratio:

$$\mathrm{Re} < 41\left(\frac{R}{C}\right)^{1/2} \tag{5-22}$$

In journal bearings, the order of R/C is 1000; therefore, the limit of the laminar flow is Re = 1300, which is between the two experimental values of Re = 1000 and Re = 1600, mentioned earlier. If the clearance C were reduced, it would extend the Re limit for laminar flow. The purpose of the following example problem is to illustrate the magnitude of the Reynolds number for common journal bearings with various lubricants.

In addition to Taylor vortexes, transition to turbulence can be initiated due to high-Reynolds-number flow, in a similar way to instability in the flow between two parallel plates.

Example Problem 5-1

Calculation of the Reynolds Number

The value of the Reynolds number, Re, is considered for a common hydrodynamic journal bearing with various fluid lubricants. The journal diameter is $d = 50$ mm; the radial clearance ratio is $C/R = 0.001$. The journal speed is 10,000 RPM. Find the Reynolds number for each of the following lubricants, and determine if Taylor vortices can occur.

a. The lubricant is mineral oil, SAE 10, and its operating temperature is 70°C. The lubricant density is $\rho = 860$ kg/m^3.
b. The lubricant is air, its viscosity is $\mu = 2.08 \times 10^{-5}$ N-s/m^2, and its density is $\rho = 0.995$ kg/m^3.
c. The lubricant is water, its viscosity is $\mu = 4.04 \times 10^{-4}$ N-s/m^2, and its density is $\rho = 978$ kg/m^3.
d. For mineral oil, SAE 10, at 70°C (in part a) find the journal speed at which instability, in the form of Taylor vortices, initiates.

Solution

The journal bearing data is as follows:

Journal speed, $N = 10{,}000$ RPM
Journal diameter $d = 50$ mm, $R = 25 \times 10^{-3}$, and $C/R = 0.001$
$C = 25 \times 10^{-6}$ m

The journal surface velocity is calculated from

$$U = \frac{\pi d N}{60} = \frac{\pi(0.050 \times 10{,}000)}{60} = 26.18\,\text{m/s}$$

a. For estimation of the Reynolds number, the average clearance C is used as the average film thickness, and Re is calculated from (5-22):

$$\text{Re} = \frac{U\rho C}{\mu} < 41\left(\frac{R}{C}\right)^{1/2}$$

The critical Re for Taylor vortices is

$$\text{Re (critical)} = 41\left(\frac{R}{C}\right)^{1/2} = 41 \times (1000)^{0.5} = 1300$$

For SAE 10 oil, the lubricant viscosity (from Fig. 2-2) and density are:

Viscosity: $\mu = 0.01$ N-s/m^2
Density: $\rho = 860$ kg/m^3

The Reynolds number is

$$\text{Re} = \frac{U\rho C}{\mu} = \frac{26.18 \times 860 \times 25 \times 10^{-6}}{0.01} = 56.3 \text{ (laminar flow)}$$

This example shows that a typical journal bearing lubricated by mineral oil and operating at relatively high speed is well within the laminar flow region.

b. The Reynolds number for air as lubricant is calculated as follows:

$$\text{Re} = \frac{U\rho C}{\mu} = \frac{26.18 \times 0.995 \times 25 \times 10^{-6}}{2.08 \times 10^{-5}}$$
$$= 31.3 \text{ (laminar flow)}$$

c. The Reynolds number for water as lubricant is calculated as follows:

$$\text{Re} = \frac{U\rho C}{\mu} = \frac{26.18 \times 978 \times 25 \times 10^{-6}}{4.04 \times 10^{-4}}$$
$$= 1584 \text{ (turbulent flow)}$$

The kinematic viscosity of water is low relative to that of oil or air. This results in relatively high Re and turbulent flow in journal bearings. In centrifugal pumps or bearings submerged in water in ships, there are design advantages in using water as a lubricant. However, this example indicates that water lubrication often involves turbulent flow.

d. The calculation of journal speed where instability in the form of Taylor vortices initiates is obtained from

$$\text{Re} = \frac{U\rho C}{\mu} = 41\left(\frac{R}{C}\right)^{1/2} = 41 \times (1000)^{0.5} = 1300$$

Surface velocity U is derived as unknown in the following equation:

$$1300 = \frac{U\rho C}{\mu} = \frac{U \times 860 \times 25 \times 10^{-6}}{0.01} \Rightarrow U = 604.5 \text{ m/s}$$

and the surface velocity at the transition to Taylor instability is

$$U = \frac{\pi d N}{60} = \frac{\pi(0.050)N}{60} = 604.5 \text{ m/s}$$

The journal speed N where instability in the form of Taylor vortices initiates is solved from the preceding equation:

$$N = 231{,}000 \text{ RPM}$$

This speed is above the range currently applied in journal bearings.

Example Problem 5-2

Short Plane-Slider

Derive the equation of the pressure wave in a short plane-slider. The assumption of an infinitely short bearing can be applied where the width L (in the z direction) is very short relative to the length B ($L \ll B$). In practice, an infinitely short bearing can be assumed where $L/B = O(10^{-1})$.

Solution

Order-of-magnitude considerations indicate that in an infinitely short bearing, dp/dx is very small and can be neglected in comparison to dp/dz. In that case, the first term on the left side of Eq. (5.12) is small and can be neglected in comparison to the second term. This omission simplifies the Reynolds equation to the following form:

$$\frac{\partial}{\partial z}\left(\frac{h^3}{\mu}\frac{\partial p}{\partial z}\right) = -6U\frac{\partial h}{\partial x} \tag{5-23}$$

Double integration results in the following parabolic pressure distribution, in the z direction:

$$p = -\frac{3\mu U}{h^3}\frac{dh}{dx}z^2 + C_1 z + C_2 \tag{5-24}$$

The two constants of integration can be obtained from the boundary conditions of the pressure wave. At the two ends of the bearing, the pressure is equal to atmospheric pressure, $p = 0$. These boundary conditions can be written as

$$\text{at } z = \pm\frac{L}{2}: \quad p = 0 \tag{5-25}$$

The following expression for the pressure distribution in a short plane-slider (a function of x and z) is obtained:

$$p(x, z) = -3\mu U\left(\frac{L^2}{4} - z^2\right)\frac{h'}{h^3} \tag{5-26}$$

Here, $h' = \partial h/\partial x$. In the case of a plane-slider, $\partial h/\partial x = -\tan a$, the slope of the plane-slider.

Comment. For a short bearing, the result indicates discontinuity of the pressure wave at the front and back ends of the plane-slider (at $h = h_1$ and $h = h_2$). In fact, the pressure at the front and back ends increases gradually, but this has only a small effect on the load capacity. This deviation from the actual pressure wave is similar to the edge effect in an infinitely long bearing.

5.6 CYLINDRICAL COORDINATES

There are many problems that are conveniently described in cylindrical coordinates, and the Navier–Stokes equations in cylindrical coordinates are useful for that purpose. The three coordinates r, ϕ, z are the radial, tangential, and axial coordinates, respectively, v_r, v_ϕ, v_z are the velocity components in the respective directions. For hydrodynamic lubrication of thin films, the inertial terms are disregarded and the three Navier–Stokes equations for an incompressible, Newtonian fluid in cylindrical coordinates are as follows:

$$\begin{aligned}
\frac{\partial p}{\partial r} &= \mu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial^2 v_r}{\partial r} - \frac{v_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_\phi}{\partial \phi^2} - \frac{2}{r^2}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial^2 v_r}{\partial z^2}\right) \\
\frac{1}{r}\frac{\partial p}{\partial \phi} &= \mu\left(\frac{\partial^2 v_\phi}{\partial r^2} + \frac{1}{r}\frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \phi} + \frac{\partial^2 v_\phi}{\partial z^2}\right) \\
\frac{\partial p}{\partial z} &= \mu\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \phi^2} + \frac{\partial^2 v_z}{\partial z^2}\right)
\end{aligned} \tag{5-27}$$

Here, v_r, v_ϕ, v_z are the velocity components in the radial, tangential, and vertical directions r, ϕ, and z, respectively. The constant density is ρ, the variable pressure is p, and the constant viscosity is μ.

The equation of continuity in cylindrical coordinates is

$$\left(\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}\right) = 0 \tag{5-28}$$

In cylindrical coordinates, the six stress components are

$$\begin{aligned}
\sigma_r &= -p + 2\mu\frac{\partial v_r}{\partial r} \\
\sigma_\phi &= -p + 2\mu\left(\frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r}\right) \\
\sigma_z &= -\rho + 2\mu\frac{\partial v_z}{\partial z} \\
\tau_{rz} &= \mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right) \\
\tau_{r\phi} &= \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \phi}\right] \\
\tau_{\phi z} &= \mu\left(\frac{\partial v_\phi}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \phi}\right)
\end{aligned} \tag{5-29}$$

5.7 SQUEEZE-FILM FLOW

An example of the application of cylindrical coordinates to the squeeze-film between two parallel, circular, concentric disks is shown in Fig. 5-5. The fluid film between the two discs is very thin. The disks approach each other at a certain speed. This results in a squeezing of the viscous thin film and in a radial pressure distribution in the clearance and a load capacity that resists the motion of the moving disk.

For a thin film, further simplification of the Navier–Stokes equations is similar to that in the derivation of Eq. (5-10). The pressure is assumed to be constant across the film thickness (in the z direction), and the dimension of z is much smaller than that of r or $r\phi$. For a problem of radial symmetry, the Navier–Stokes equations reduce to the following:

$$\frac{dp}{dr} = \mu \frac{\partial^2 v_r}{\partial z^2} \tag{5-30}$$

Here, v_r is the fluid velocity in the radial direction.

Example Problem 5-3

A fluid is squeezed between two parallel, circular, concentric disks, as shown in Fig. 5-5. The fluid film between the two discs is very thin. The upper disk has

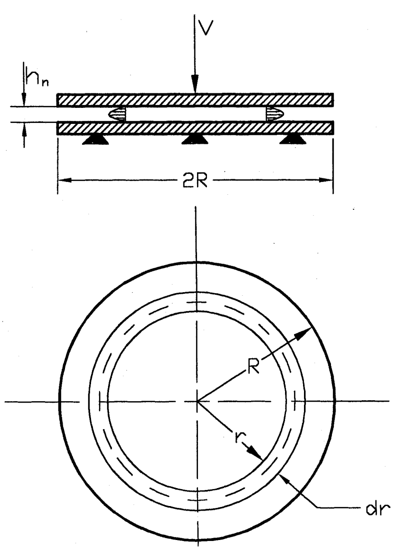

FIG. 5-5 Squeeze-film flow between two concentric, parallel disks.

velocity V, toward the lower disk, and this squeezes the fluid so that it escapes in the radial direction. Derive the equations for the radial pressure distribution in the thin film and the resultant load capacity.

Solution

The first step is to solve for the radial velocity distribution, v_r, in the fluid film. In a similar way to the hydrodynamic lubrication problem in the previous chapter, we can write the differential equation in the form

$$\frac{\partial^2 v_r}{\partial z^2} = \frac{dp}{dr}\frac{1}{\mu} = 2m$$

Here, m is a function of r only; $m = m(r)$ is a substitution that represents the pressure gradient. By integration the preceding equation twice, the following parabolic distribution of the radial velocity is obtained:

$$v_r = mz^2 + nz + k$$

Here, m, n, and k are functions of r only. These functions are solved by the boundary conditions of the radial velocity and the conservation of mass as well as fluid volume for incompressible flow.

The two boundary conditions of the radial velocity are

$$\begin{aligned} &\text{at } z = 0: \quad v_r = 0 \\ &\text{at } z = h: \quad v_r = 0 \end{aligned}$$

In order to solve for the three unknowns m, n and k, a third equation is required; this is obtained from the fluid continuity, which is equivalent to the conservation of mass.

Let us consider a control volume of a disk of radius r around the center of the disk. The downward motion of the upper disk, at velocity V, reduces the volume of the fluid per unit of time. The fluid is incompressible, and the reduction of volume is equal to the radial flow rate Q out of the control volume. The flow rate Q is the product of the area of the control volume πr^2 and the downward velocity V:

$$Q = \pi r^2 V$$

The same flow rate Q must apply in the radial direction through the boundary of the control volume. The flow rate Q is obtained by integration of the radial velocity distribution of the film radial velocity, v_r, along the z direction, multiplied by the circumference of the control volume ($2\pi r$). The flow rate Q becomes

$$Q = 2\pi r \int_0^h v_r \, dz$$

Since the fluid is incompressible, the flow rate of the fluid escaping from the control volume is equal to the flow rate of the volume displaced by the moving disk:

$$V\pi r^2 = 2\pi r \int_0^h v_r \, dz$$

Use of the boundary conditions and the preceding continuity equation allows the solution of m, n, and k. The solution for the radial velocity distribution is

$$v_r = \frac{3rV}{h}\left(\frac{z^2}{h^2} - \frac{z}{h}\right)$$

and the pressure gradient is

$$\Rightarrow \frac{dp}{dr} = -6\mu V \frac{r}{h^3}$$

The negative sign means that the pressure is always decreasing in the r direction. The pressure gradient is a linear function of the radial distance r, and this function can be integrated to solve for the pressure wave:

$$p = \int dp = -\frac{6\mu V}{h^3}\int r \, dr = -\frac{3\mu V}{h^3} r^2 + C$$

Here, C is a constant of integration that is solved by the boundary condition that states that at the outside edge of the disks, the fluid pressure is equal to atmospheric pressure, which can be considered to be zero:

$$\text{at } r = R: \quad p = 0$$

Substituting in the preceding equation yields:

$$0 = -\frac{3\mu V}{h^3} R^2 + C \Rightarrow C = \frac{3\mu V}{h^3} R^2$$

The equation for the radial pressure distribution is therefore

$$\Rightarrow p = -\frac{3\mu V}{h^3} r^2 + \left(\frac{3\mu V}{h^3} R^2\right) = \frac{3\mu V}{h^3}(R^2 - r^2)$$

The pressure has its maximum value at the center of the disk radius:

$$P_{max} = \frac{3\mu V}{h^3}[R^2 - (0)^2] = \frac{3\mu V}{h^3} R^2$$

The parabolic pressure distribution is shown in Fig. 5-6.

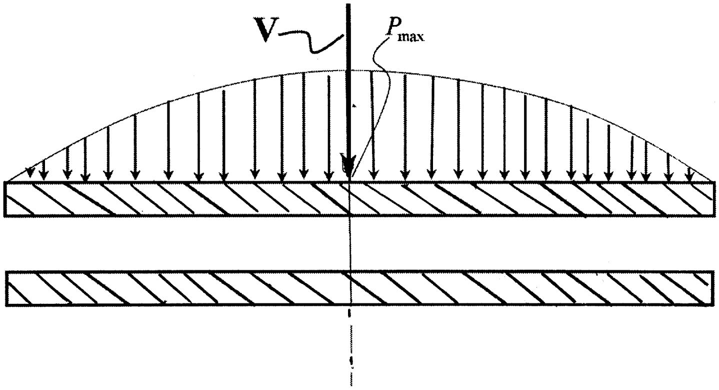

FIG. 5-6 Radial pressure distribution in squeeze-film flow.

Now that the pressure distribution has been solved, the load capacity is obtained by the following integration:

$$W = \int dW = \int_A p dA = \int_0^R \frac{3\mu V}{h^3}(R^2 - r^2) 2\pi r dr$$

$$= \frac{6\pi\mu V}{h^3}\left[R^2 \int_0^R r\,dr - \int_0^R r^3\,dr\right]$$

$$\therefore \Rightarrow W = \frac{3\pi\mu V R^4}{2h^3}$$

The load capacity equation indicates that a squeeze-film arrangement can act as a damper that resists the squeezing motion. The load capacity increases dramatically as the film thickness becomes thinner, thus preventing the disks from coming into contact. Theoretically, at $h = 0$ the load capacity is approaching infinity. In practice, there is surface roughness and there will be contact by a finite force.

Example Problem 5-4

Two parallel circular disks of 30-mm diameter, as shown in Fig. 5-5, operate as a damper. The clearance is full of SAE 30 oil at a temperature of 50°C. The damper is subjected to a shock load of 7000 N.

a. Find the film thickness h if the load causes a downward speed of the upper disk of 10 m/s.
b. What is the maximum pressure developed due to the impact of the load?

Solution

a. The viscosity of the lubricant is obtained from the viscosity–temperature chart:

$$\mu_{SAE30@50^\circ C} = 5.5 \times 10^{-2}\ \text{N-s/m}^2$$

The film thickness is derived from the load capacity equation:

$$W = \frac{3\pi\mu V R^4}{2h^3}$$

The instantaneous film thickness, when the disk speed is 10 m/s, is

$$h = \left(\frac{3\pi \times 5.5 \times 10^{-2} \times 10 \times 15 \times 10^{-3}}{2 \times 7000}\right)^{\frac{1}{3}}$$

$$h = 0.266\ \text{mm}$$

b. The maximum pressure at the center is

$$p_{\max} = \frac{3\mu V}{h^3} R^2 = \frac{3(5.5 \times 10^{-2}\ \text{N-s/m}^2)(10\ \text{ms})(15 \times 10^{-3}\ \text{m})^2}{(0.266 \times 10^{-3}\ \text{m})^3}$$
$$= 1.97 \times 10^7\ \text{Pa} = 19.7\ \text{MPa}$$

Problems

5-1 Two long cylinders of radii R_1 and R_2, respectively, have parallel centrelines, as shown in Fig. 5-7. The cylinders are submerged in fluid and are rotating in opposite directions at angular speeds of ω_1 and ω_2, respectively. The minimum clearance between the cylinders is h_n. If the fluid viscosity is μ, derive the Reynolds equation, and write the expression for the pressure gradient around the minimum clearance.

The equation of the clearance is

$$h(x) = h_n + \frac{x^2}{2R_{eq}}$$

For calculation of the variable clearance between two cylinders having a convex contact, the equation for the equivalent radius R_{eq} is

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$$

5-2 Two long cylinders of radii R_1 and R_2, respectively, in concave contact

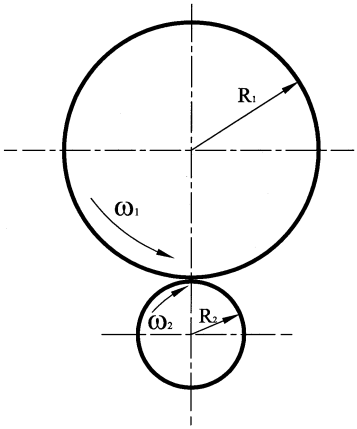

FIG. 5-7 Two parallel cylinders, convex contact.

are shown in Fig. 5-8. The cylinders have parallel centerlines and are in rolling contact. The angular speed of the large (external) cylinder is ω, and the angular speed of the small (internal) cylinder is such that there is rolling without sliding. There is a small minimum clearance between the cylinders, h_n. If the fluid viscosity is μ, derive the Reynolds equation, and write the expression for the pressure gradient around the minimum clearance. For a concave contact (Fig. 5-8) the equivalent radius R_{eq} is

$$\frac{1}{R_{eq}} = \frac{1}{R_1} - \frac{1}{R_2}$$

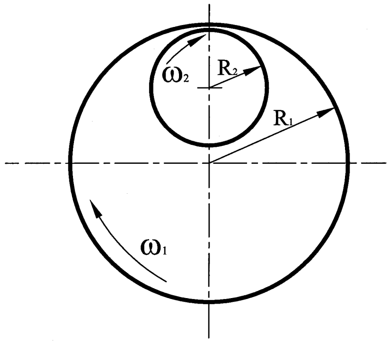

FIG. 5-8 Two parallel cylinders, concave contact.

5-3 The journal diameter of a hydrodynamic bearing is $d = 100$ mm, and the radial clearance ratio is $C/R = 0.001$. The journal speed is $N = 20{,}000$ RPM. The lubricant is mineral oil, SAE 30, at 70°C. The lubricant density at the operating temperature is $\rho = 860\ \text{kg/m}^3$.

a. Find the Reynolds number.
b. Find the journal speed, N, where instability in the form of Taylor vortices initiates.

5-4 Two parallel circular disks of 100-mm diameter have a clearance of 1 mm between them. Under load, the downward velocity of the upper disk is 2 m/s. At the same time, the lower disk is stationary. The clearance is full of SAE 10 oil at a temperature of 60°C.

a. Find the load on the upper disk that results in the instantaneous velocity of 2 m/s.
b. What is the maximum pressure developed due to that load?

5-5 Two parallel circular disks (see Fig. 5-5) of 200-mm diameter have a clearance of $h = 2$ mm between them. The load on the upper disk is 200 N. The lower disk is stationary, and the upper disk has a downward velocity V. The clearance is full of oil, SAE 10, at a temperature of 60°C.

a. Find the downward velocity of the upper disk at that instant.
b. What is the maximum pressure developed due to that load?