CHAPTER 3
LINKAGES

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Linkages are mechanical devices that appear very straightforward to both analyze and design. Given proper technique, that is generally the case. The methods described in this chapter reveal the complexity (and, I think, the beauty) of linkages. I have gained significant satisfaction during my 20 years of work with them from both theoretical and functioning hardware standpoints.

3.1 BASIC LINKAGE CONCEPTS

3.1.1 Kinematic Elements

A linkage is composed of rigid-body members, or *links*, connected to one another by rigid kinematic elements, or *pairs*. The nature of those connections as well as the shape of the links determines the kinematic properties of the linkage.

Although many kinematic pairs are conceivable and most do physically exist, only four have general practical use for linkages. In Fig. 3.1, the four cases are seen to include two with one degree of freedom ($f = 1$), one with $f = 2$, and one with $f = 3$. Single-degree-of-freedom pairs constitute joints in planar linkages or spatial linkages. The cylindrical and spherical joints are useful only in spatial linkages.

The links which connect these kinematic pairs are usually binary (two connections) but may be tertiary (three connections) or even more. A commonly used tertiary link is the *bell crank* familiar to most machine designers. Since our primary

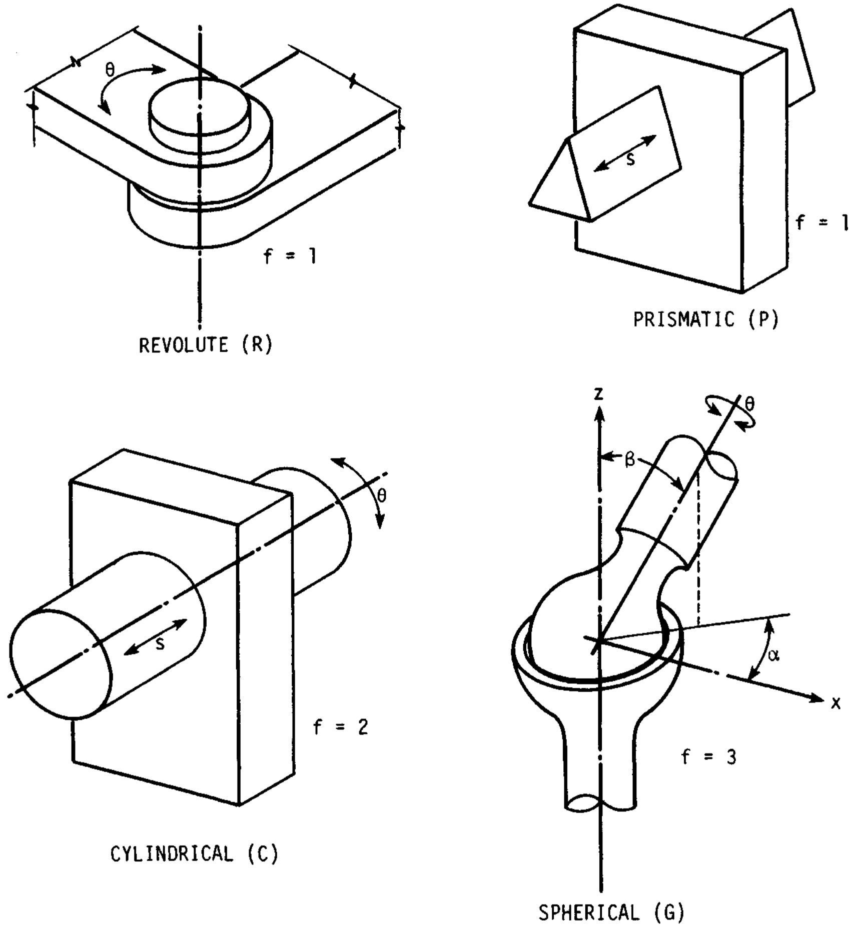

FIGURE 3.1 Kinematic pairs useful in linkage design. The quantity f denotes the number of degrees of freedom.

interest in most linkages is to provide a particular output for a prescribed input, we deal with closed kinematic chains, examples of which are depicted in Fig. 3.2. Considerable work is now under way on robotics, which are basically open chains. Here we restrict ourselves to the closed-loop type. Note that many complex linkages can be created by compounding the simple four-bar linkage. This may not always be necessary once the design concepts of this chapter are applied.

3.1.2 Freedom of Motion

The degree of freedom for a mechanism is expressed by the formula

$$F = \lambda(l - j - 1) + \sum_{i=1}^{j} f_i \tag{3.1}$$

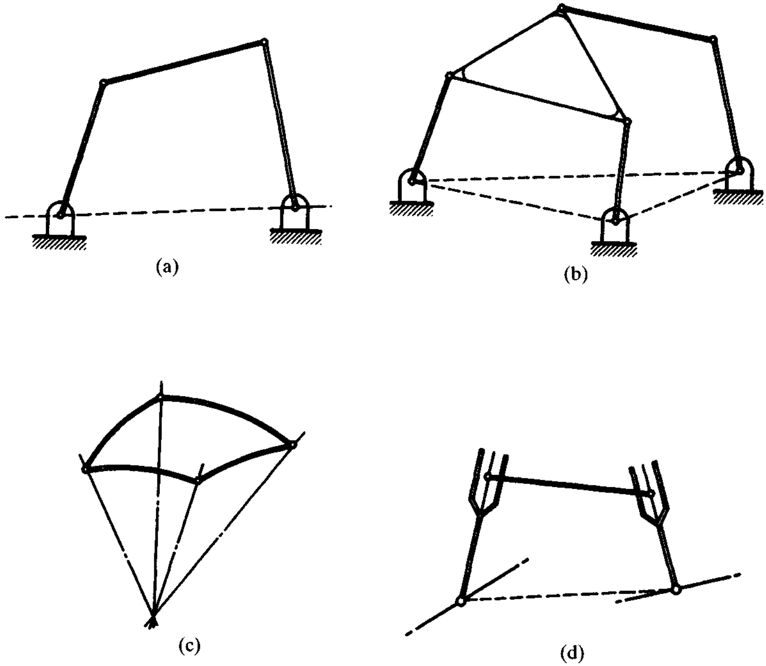

FIGURE 3.2 Closed kinematic chains. (*a*) Planar four-bar linkage; (*b*) planar six-bar linkage; (*c*) spherical four-bar linkage; (*d*) spatial RCCR four-bar linkage.

where l = number of links (fixed link included)
j = number of joints
f_i = f of ith joint
λ = integer
= 3 for plane, spherical, or particular spatial linkages
= 6 for most spatial linkages

Since the majority of linkages used in machines are planar, the particular case for plane mechanisms with one degree of freedom is found to be

$$2j - 3l + 4 = 0 \tag{3.2}$$

Thus, in a four-bar linkage, there are four joints (either revolute or prismatic). For a six-bar linkage, we need seven such joints. A peculiar special case occurs when a sufficient number of links in a plane linkage are parallel, which leads to such special devices as the pantograph.

Considerable theory has evolved over the years about numerous aspects of linkages. It is often of little help in creating usable designs. Among the best references available are Hartenberg and Denavit [3.9], Hall [3.8], Beyer [3.1], Hain [3.7], Rosenauer and Willis [3.10], Shigley and Uicker [3.11], and Tao [3.12].

3.1.3 Number Synthesis

Before you can dimensionally synthesize a linkage, you may need to use *number synthesis,* which establishes the number of links and the number of joints that are required to obtain the necessary mobility. An excellent description of this subject appears in Hartenberg and Denavit [3.9]. The four-bar linkage is emphasized here because of its wide applicability.

3.2 MOBILITY CRITERION

In any given four-bar linkage, selection of any link to be the crank may result in its inability to fully rotate. This is not always necessary in practical mechanisms. A criterion for determining whether any link might be able to rotate 360° exists. Refer to Fig. 3.3, where l, s, p, and q are defined. *Grubler's criterion* states that

$$l + s < p + q \tag{3.3}$$

If the criterion is not satisfied, only double-rocker linkages are possible. When it is satisfied, choice of the shortest link as driver will result in a crank-rocker linkage; choice of any of the other three links as driver will result in a drag link or a double-rocker mechanism.

A significant majority of the mechanisms that I have designed in industry are the double-rocker type. Although they do not possess some theoretically desirable characteristics, they are useful for various types of equipment.

3.3 ESTABLISHING PRECISION POSITIONS

In designing a mechanism with a certain number of required precision positions, you will be faced with the problem of how to space them. In many practical situations, there will be no choice, since particular conditions must be satisfied.

If you do have a choice, Chebychev spacing should be used to reduce the structural error. Figure 3.4 shows how to space four positions within a prescribed interval [3.9]. I have found that the end-of-interval points can be used instead of those just inside with good results.

3.4 PLANE FOUR-BAR LINKAGE

3.4.1 Basic Parameters

The apparently simple four-bar linkage is actually an incredibly sophisticated device which can perform wonders once proper design techniques are known and used. Figure 3.5 shows the parameters required to define the general case. Such a linkage can be used for three types of motion:

1. *Crank-angle coordination.* Motion of driver link b causes prescribed motion of link d.
2. *Path generation.* Motion of driver link b causes point C to move along a prescribed path.
3. *Motion generation.* Movement of driver link b causes line CD to move in a prescribed planar motion.

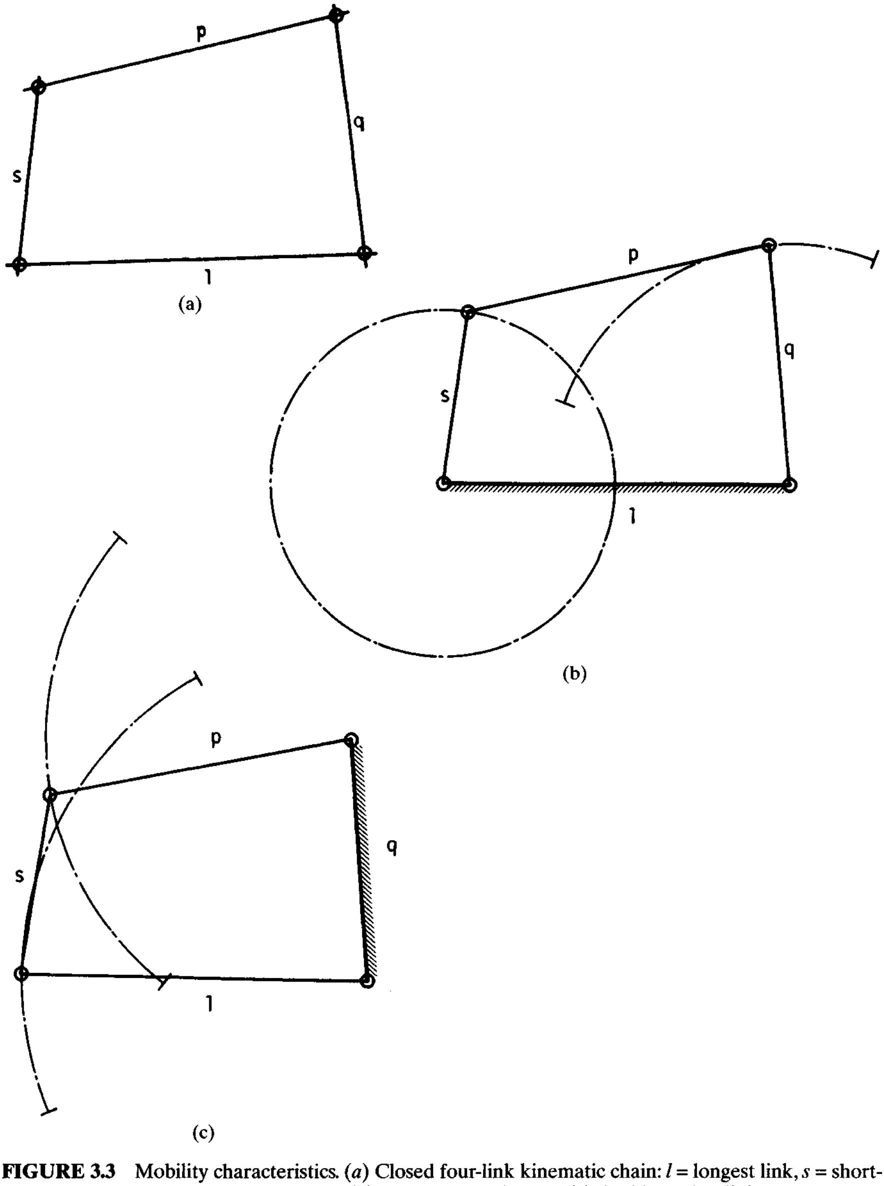

FIGURE 3.3 Mobility characteristics. (*a*) Closed four-link kinematic chain: l = longest link, s = shortest link, p, q = intermediate-length links; (*b*) crank-rocker linkage; (*c*) double-rocker linkage.

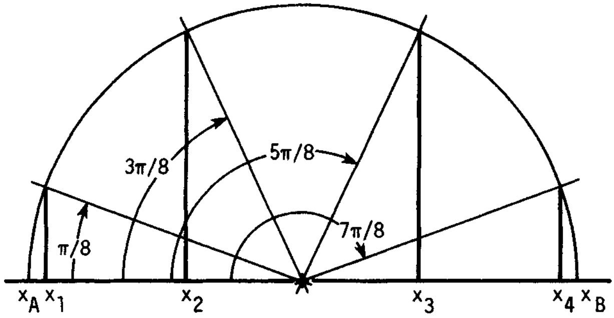

FIGURE 3.4 Four-precision-point spacing (Chebychev)

$x_1 = x_A + 0.0381(x_B - x_A)$ $x_2 = x_A + 0.3087(x_B - x_A)$

$x_3 = x_A + 0.6913(x_B - x_A)$ $x_4 = x_A + 0.9619(x_B - x_A)$

In general, for n precision points

$$x_j = \frac{1}{2}(x_A + x_B) - \frac{1}{2}(x_B - x_A)\cos\frac{\pi(2j-1)}{2n} \qquad j = 1, 2, \ldots, n$$

3.4.2 Kinematic Inversion

A very useful concept in mechanism design is that by inverting the motion, new interesting characteristics become evident. By imagining yourself attached to what is actually a moving body, you can determine various properties, such as the location of a joint which connects that body to its neighbor. This technique has been found useful in many industrial applications, such as the design of the four-bar automobile window regulator ([3.6]).

3.4.3 Velocity Ratio

At times the velocity of the output will need to be controlled as well as the corresponding position. When the motion of the input crank and the output crank is coordinated, it is an easy matter to establish the velocity ratio ω_d/ω_b. When you extend line AB in Fig. 3.5 until it intersects the line through the fixed pivots O_A and O_B in a point S, you find that

$$\frac{\omega_d}{\omega_b} = \frac{O_A S}{O_A O_B + O_A S} \tag{3.4}$$

Finding the linear velocity of a point on the coupler is not nearly as straightforward. A very good approximation is to determine the travel distance along the path of the point during a particular motion of the crank.

3.4.4 Torque Ratio

Because of the conservation of energy, the following relationship holds:

$$T_b d\phi = T_d d\psi \tag{3.5}$$

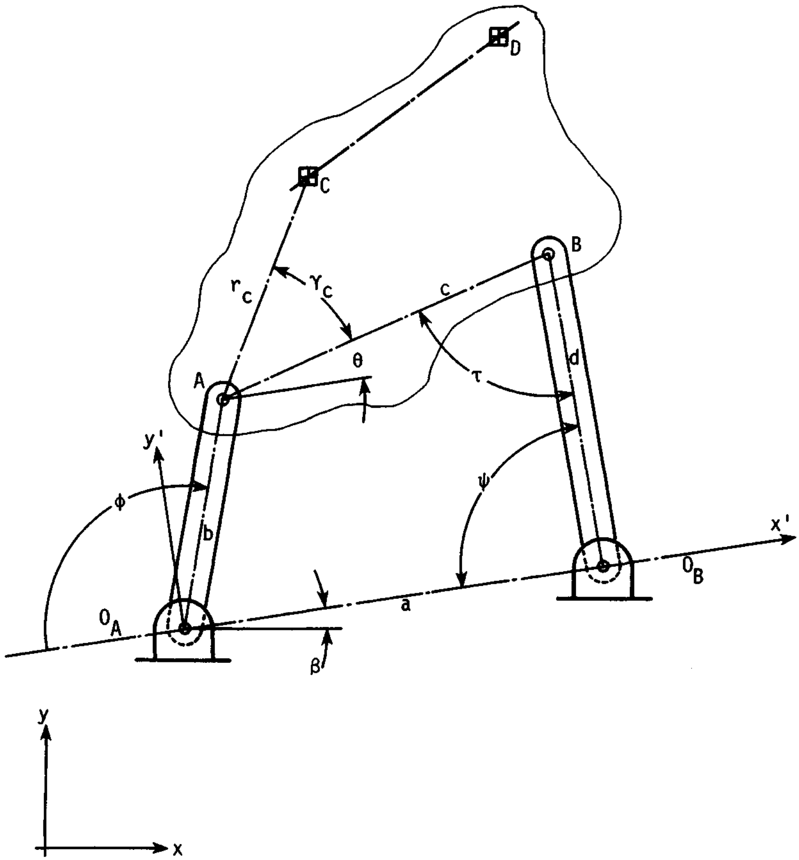

FIGURE 3.5 General four-bar linkage in a plane.

Since both sides of (3.5) can be divided by *dt,* we have, after some rearranging,

$$n = \frac{T_b}{T_d} = \frac{d\psi/dt}{d\phi/dt} = \frac{\omega_d}{\omega_b} \tag{3.6}$$

The torque ratio n is thus the inverse of the velocity ratio. Quite a few mechanisms that I have designed have made significant use of torque ratios.

3.4.5 Transmission Angle

For the four-bar linkage of Fig. 3.5, the transmission angle τ occurs between the coupler and the driven link. This angle should be as close to 90° as possible. Useful linkages for motion generation have been created with τ approaching 20°. When a crank rocker is being designed, you should try to keep $45° < \tau < 135°$. Double-rocker or drag link mechanisms usually have other criteria which are more significant than the transmission angle.

3.5 PLANE OFFSET SLIDER-CRANK LINKAGE

A variation of the four-bar linkage which is often seen occurs when the output link becomes infinitely long and the path of point B is a straight line. Point B becomes the slider of the slider-crank linkage. Although coupler b could have the characteristics shown in Fig. 3.6, it is seldom used in practice. Here we are interested in the motion of point B while crank a rotates. In general, the path of point B does not pass through the fixed pivot O_A, but is offset by dimension ε. An obvious example of the degenerate case ($\varepsilon \equiv 0$) is the piston crank in an engine.

The synthesis of this linkage is well described by Hartenberg and Denavit [3.9]. I have used the method many times after programming it for the digital computer.

3.6 KINEMATIC ANALYSIS OF THE PLANAR FOUR-BAR LINKAGE

3.6.1 Position Geometry

Refer to Fig. 3.7, where the parameters are defined. Given the link lengths a, b, c, and d and the crank position angle ϕ, the angular position of coupler c is

$$\theta = \pi - (\tau + \psi) \tag{3.7}$$

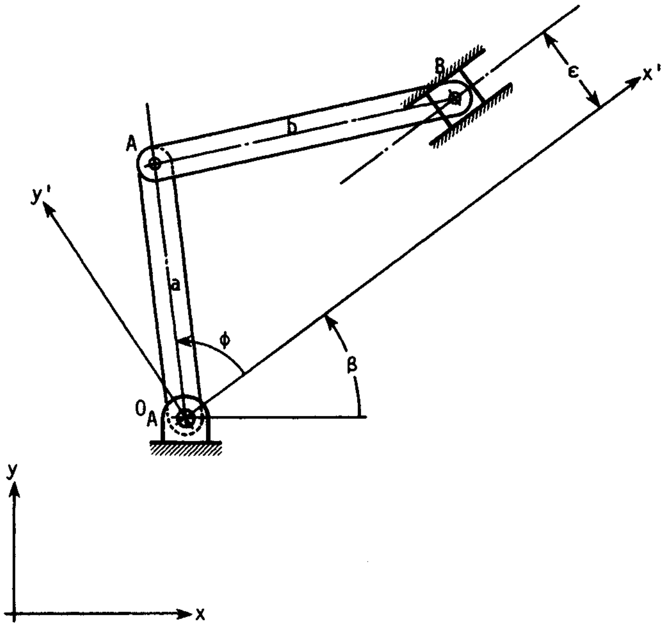

FIGURE 3.6 General offset slider-crank linkage.

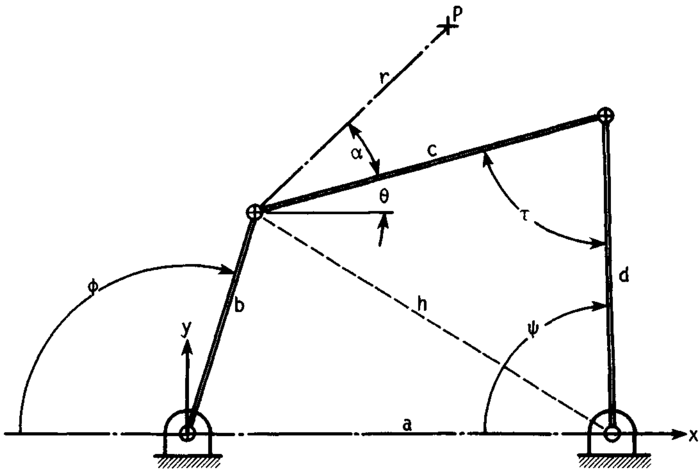

FIGURE 3.7 Parameters for analysis of a four-bar linkage.

The driven link d will be at angle

$$\psi = \cos^{-1} \frac{h^2 + a^2 - b^2}{2hb} + \cos^{-1} \frac{h^2 + d^2 - c^2}{2hd} \tag{3.8}$$

where

$$h^2 = a^2 + b^2 + 2ab \cos \phi \tag{3.9}$$

The transmission angle τ will be

$$\tau = \cos^{-1} \frac{c^2 + d^2 - a^2 - b^2 - 2ab \cos \phi}{2cd} \tag{3.10}$$

A point on coupler P has coordinates

$$\begin{aligned} P_x &= -b \cos \phi + r \cos (\theta + \alpha) \\ P_y &= b \sin \phi + r \sin (\theta + \alpha) \end{aligned} \tag{3.11}$$

3.6.2 Velocity and Acceleration

The velocity of the point on the coupler can be expressed as

$$\begin{aligned} \frac{dP_x}{dt} &= b \frac{d\phi}{dt} \sin \phi - r \frac{d\theta}{dt} \sin (\theta + \alpha) \\ \frac{dP_y}{dt} &= b \frac{d\phi}{dt} \cos \phi + r \frac{d\theta}{dt} \cos (\theta + \alpha) \end{aligned} \tag{3.12}$$

As you can see, the mathematics gets very complicated very rapidly. If you need to establish velocity and acceleration data, consult Ref. [3.1], [3.7], or [3.11]. Computer analysis is based on the closed vector loop equations of C. R. Mischke, developed at Pratt Institute in the late 1950s. See [3.19], Chap. 4.

3.6.3 Dynamic Behavior

Since all linkages have clearances in the joints as well as mass for each link, high-speed operation of a four-bar linkage can cause very undesirable behavior. Methods for solving these problems are very complex. If you need further data, refer to numerous theoretical articles originally presented at the American Society of Mechanical Engineers (ASME) mechanism conferences. Many have been published in ASME journals.

3.7 DIMENSIONAL SYNTHESIS OF THE PLANAR FOUR-BAR LINKAGE: MOTION GENERATION

3.7.1 Two Positions of a Plane

The line A_iB_i defines a plane (Fig. 3.8) which is to be the coupler of the linkage to be designed. When two positions are defined, you can determine a particular point, called the *pole* (in this case P_{12}, since the motion goes from position 1 to position 2). The significance of the pole is that it is the point about which the motion of the body is a simple rotation; the pole is seen to be the intersection of the perpendicular bisectors of A_1A_2 and B_1B_2.

A four-bar linkage can be created by choosing any point on a_1a_2 as O_A and any reasonable point on b_1b_2 as O_B. Note that you do not have a totally arbitrary choice for the fixed pivots, even for this elementary case. There are definite limitations, since the four-bar linkage must produce continuous motion between all positions. When a fully rotating crank is sought, the Grubler criterion must be adhered to. For double-rocker mechanisms, the particular link lengths still have definite criteria to meet. You have to check these for every four-bar linkage that you design.

3.7.2 Three Positions of a Plane

When three positions of a plane are specified by the location of line *CD*, as shown in Fig. 3.9, it is possible to construct the center of a circle through C_1, C_2, and C_3 and through D_1, D_2, and D_3. This is only one of an infinite combination of links that can be attached to the moving body containing line *CD*. If the path of one end of line *CD* lies on a circle, then the other end can describe points on a coupler path which correspond to particular rotation angles of the crank (Fig. 3.10); that is a special case of the motion generation problem.

The general three-position situation describes three poles P_{12}, P_{13}, and P_{23} which form a *pole triangle.* You will find this triangle useful since its interior angles ($\theta_{12}/2$ in Fig. 3.9) define precise geometric relationships between the fixed and moving pivots of links which can be attached to the moving body defined by line *CD*. Examples of this geometry are shown in Fig. 3.11, where you can see that

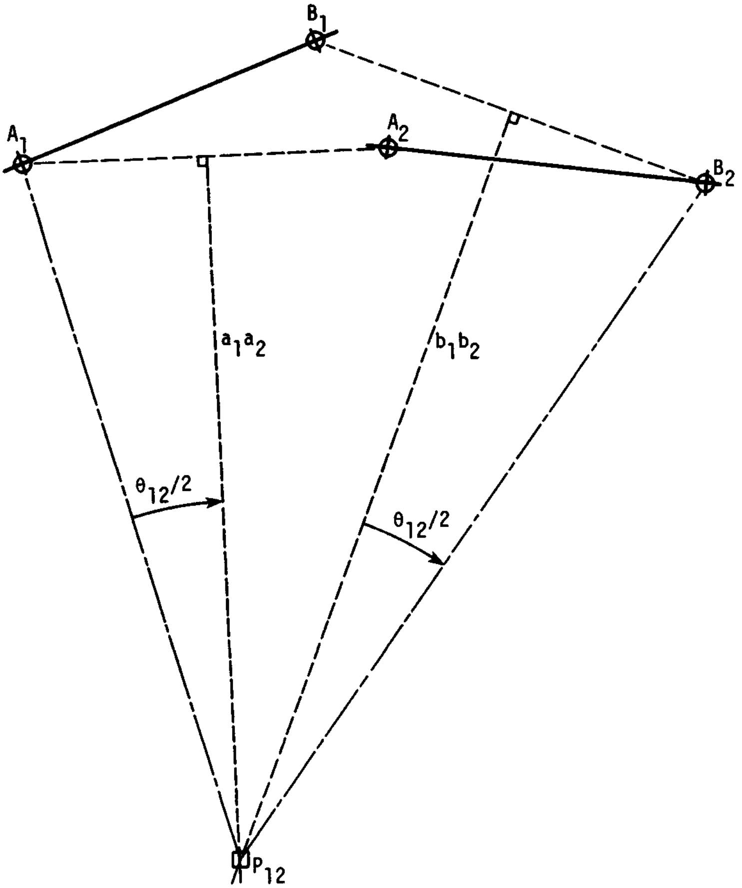

FIGURE 3.8 Two positions of a plane: definition of pole P_{12}.

$$\angle P_{13}P_{12}P_{23} \equiv \angle A_1P_{12}O_A \equiv \angle B_1P_{12}O_B \tag{3.13}$$

The direction in which these angles are measured is critical. For three positions, you may thus choose the fixed or the moving pivot and use this relationship to establish the location of the corresponding moving or fixed pivot, since it is also true that

$$\angle P_{12}P_{13}P_{23} = \angle A_1P_{13}O_A = \angle B_1P_{13}O_B \tag{3.14}$$

The intersection of two such lines (Fig. 3.12) is the required pivot point. Note that the lines defined by the pole triangle relationships extend in both directions from the pole; thus a pivot-point angle may appear to be ±180° from that defined within the triangle. This is perfectly valid.

It is important to observe that arbitrary choices for pivot locations are available when three positions, or less, of the moving plane are specified.

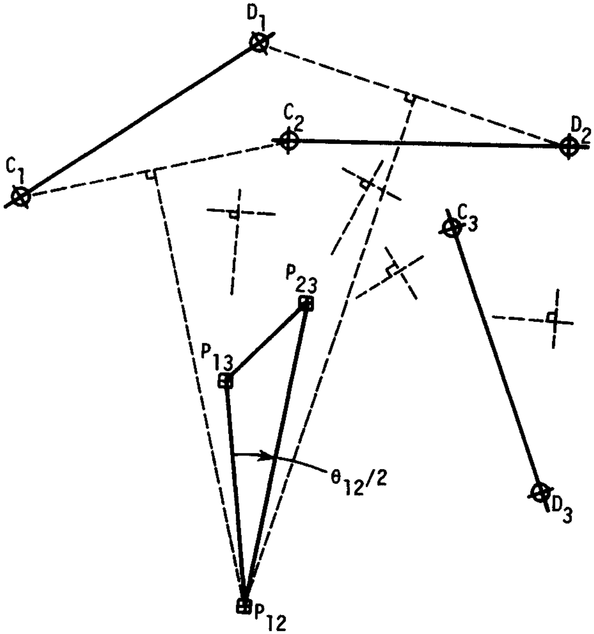

FIGURE 3.9 Three positions of a plane: definition of the pole triangle $P_{12}P_{13}P_{23}$.

3.7.3 Four Positions of a Moving Plane

When four positions are required, appropriate pivot-point locations are precisely defined by theories generated by Professor Burmester in Germany during the 1880s. His work [3.2] is the next step in using the poles of motion. When you define four positions of a moving plane containing line *CD* as shown in Fig. 3.13, six poles are defined:

$$P_{12} \quad P_{13} \quad P_{14} \quad P_{23} \quad P_{24} \quad P_{34}$$

By selecting opposite poles (P_{12}, P_{34} and P_{13}, P_{24}), you obtain a quadrilateral with significant geometric relationships. For practical purposes, this opposite-pole quadrilateral is best used to establish a locus of points which are the fixed pivots of links that can be attached to the moving body so that it can occupy the four prescribed positions. This locus is known as the *center-point curve* (Fig. 3.14) and can be found as follows:

1. Establish the perpendicular bisector of the two sides $P_{12}P_{24}$ and $P_{13}P_{34}$.
2. Determine points M and M' such that

$$\angle P_{12}MQ_2 \equiv \angle P_{13}M'Q_3$$

3. With M as center and MP_{12} as radius, create circle k. With M' as center and $M'P_{13}$ as radius, create circle k'.
4. The intersections of circles k and k' (shown as c_0 and c_0' in Fig. 3.14) are center points with the particular property that the link whose fixed pivot is c_0 or c_0' has a

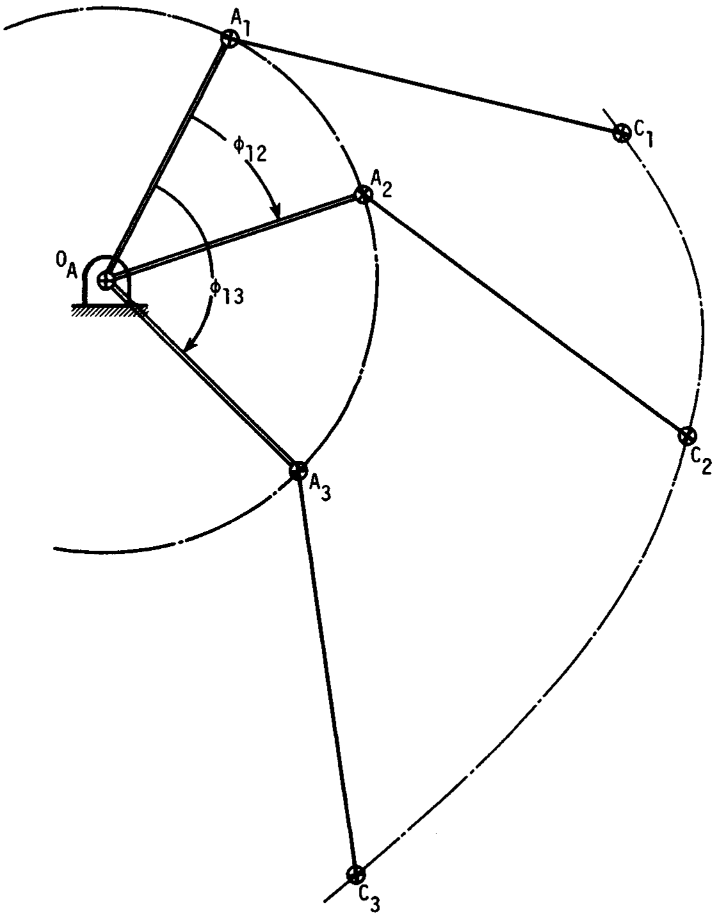

FIGURE 3.10 Path generation as a special case of motion generation.

total rotation angle twice the value defined by the angle(s) in step 2. The magnitude and direction of the link angle ϕ_{14} are defined in the figure.

Note that this construction can produce two, one, or no intersection points. Thus some link rotations are not possible. Depending on how many angles you want to investigate, there will still be plenty of choices. I have found it most convenient to solve the necessary analytic geometry and program it for the digital computer; as many accurate results as desired are easily determined.

Once a center point has been established, the corresponding moving pivot (circle point) can be established. For the first position of the moving body, you need to use the pole triangle $P_{12}P_{13}P_{23}$ angles to establish two lines whose intersection will be the circle point. In Fig. 3.15, the particular angles are

$$\angle P_{13}P_{12}P_{23} \equiv \angle c_1 P_{12} c_0$$

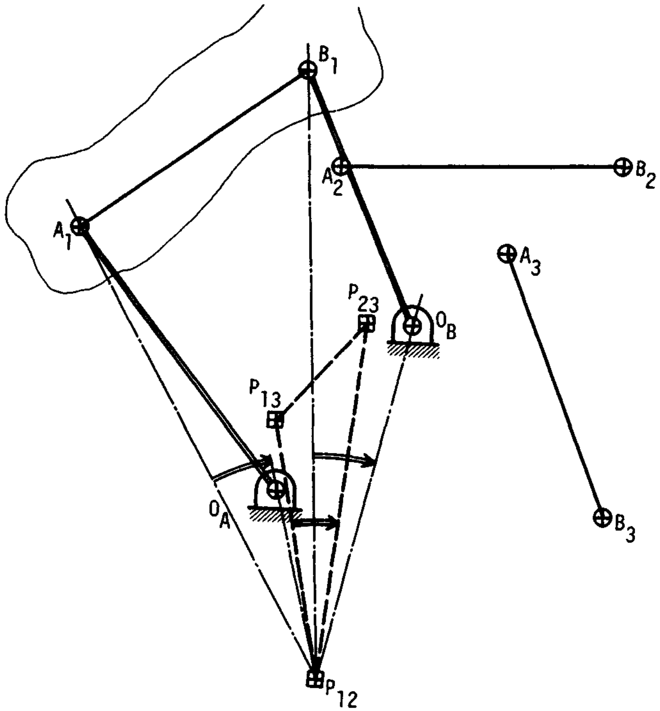

FIGURE 3.11 Geometric relationship between pole triangle angle(s) and location of link fixed and moving pivot points.

and

$$360° - \angle P_{23}P_{13}P_{12} \equiv \angle c_1P_{13}c_0$$

The second equality could also be written

$$\angle P_{23}P_{13}P_{12} \equiv \angle c_1P_{13}c_0 \pm 180°$$

A locus of points thus defined can be created as shown in Fig. 3.16. Each point on the circle-point curve corresponds to a particular point on the center-point curve. Some possible links are defined in Fig. 3.16; each has a known first-to-fourth-position rotation angle. Only those links whose length and/or pivot locations are within prescribed limits need to be retained.

The two intermediate positions of the link can be determined by establishing the location of the moving pivot (circle point) in the second and third positions of the moving body. Since the positions lie on the arc with center at the fixed pivot (center point) a and radius aa', it is easy to determine the link rotation angles as

$$\phi_{12} = \angle A_1O_AA_2 \qquad \phi_{13} = \angle A_1O_AA_3$$

Linkages need to be actuated or driven by one of the links. Knowing the three rotation angles allows you to choose a drive link which has the desired proportions

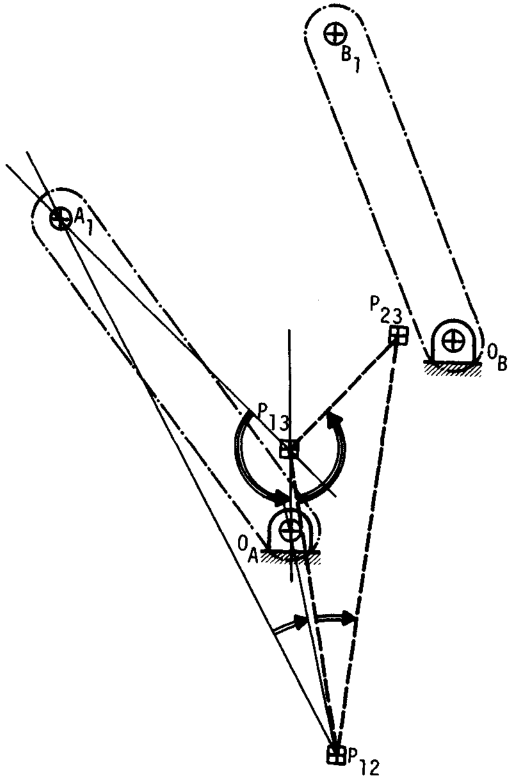

FIGURE 3.12 Determining the moving or fixed pivot by using the pole triangle.

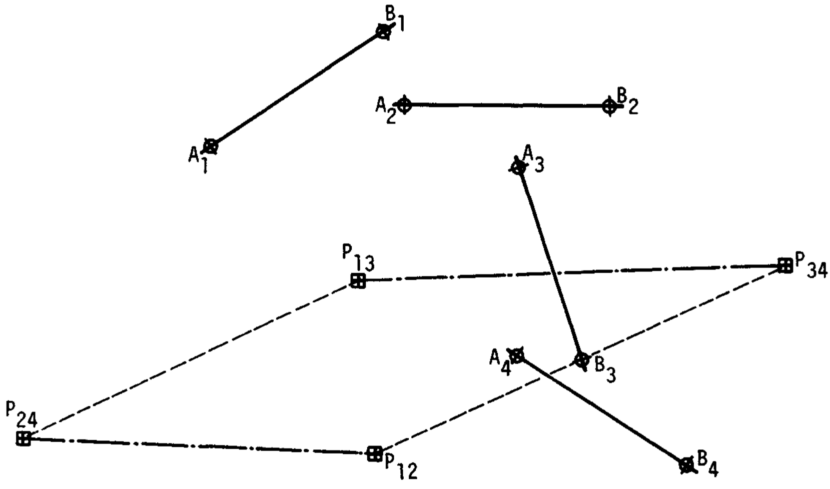

FIGURE 3.13 Four positions of a plane: definition of the opposite-pole quadrilateral formed by lines $P_{13}P_{24}$ and $P_{12}P_{34}$.

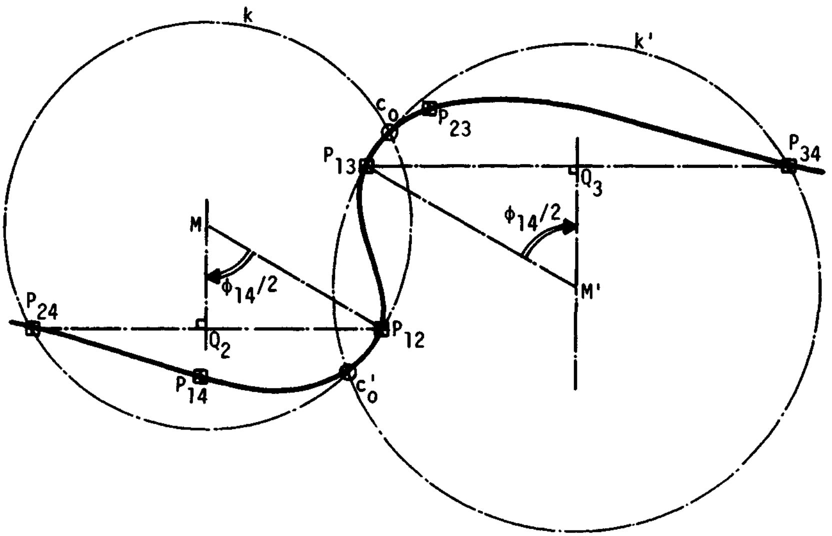

FIGURE 3.14 Determination of points on the center-point curve.

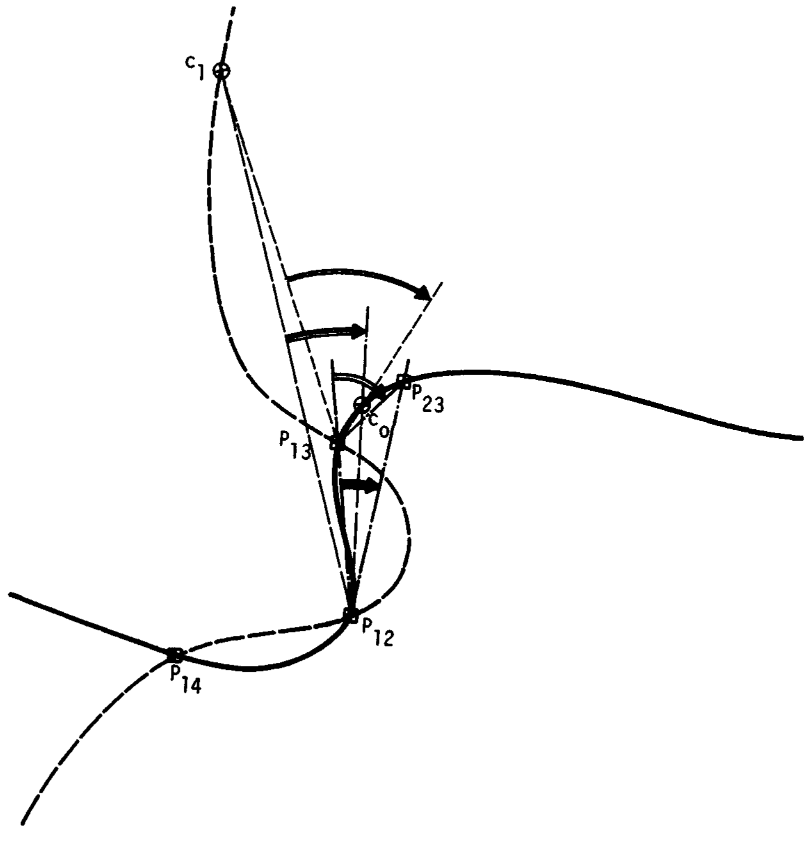

FIGURE 3.15 Determination of a circle point corresponding to a particular center point.

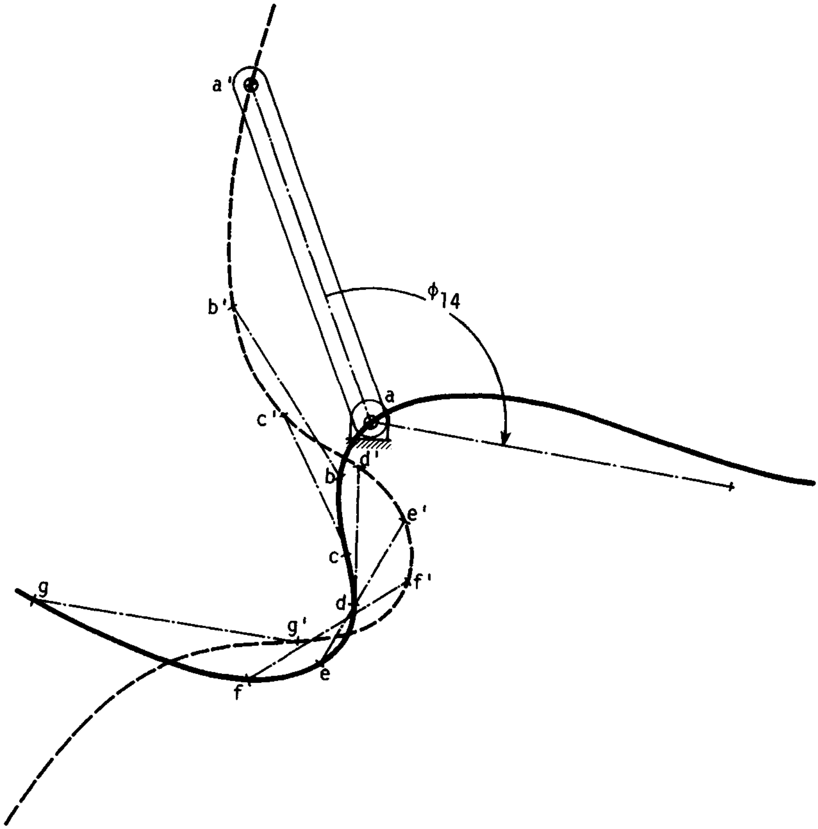

FIGURE 3.16 Some of the links which can be attached to the plane containing *CD*.

of motion. Proper care in selection of the two links will result in a smooth-running four-bar linkage.

3.7.4 Five Positions of a Plane

It would seem desirable to establish as many precision positions as possible. You can choose two sets of four positions (for example, 1235 and 1245) from which the Burmester curves can be created. The intersections (up to six) of those two center-point curves are the only fixed pivots which can be used to guide the moving body through the five positions. Since those pivots and/or link lengths have virtually always been outside the prescribed limits, I never use five-position synthesis.

3.7.5 Available Computer Programs

Two general-purpose planar linkage synthesis programs have been created: KINSYN ([3.17]) and LINCAGES ([3.18]). They involve the fundamentals described in this section and can be valuable when time is limited. I have found it more advantageous to create my own design and analysis programs, since the general programs almost always need to be supplemented by routines that define the particular problem at hand.

3.8 DIMENSIONAL SYNTHESIS OF THE PLANAR FOUR-BAR LINKAGE: CRANK-ANGLE COORDINATION

Many mechanical movements in linkages depend on the angular position of the output crank. In general, you will have to design the four-bar linkage so that a prescribed input crank rotation will produce the desired output crank rotation. Significant work was performed in an attempt to generate functions ([3.9]) using the four-bar linkage until the advent of the microcomputer. Although it is seldom necessary to utilize the function capability, you will find many applications for crank-angle coordination. Two methods are possible: geometric and analytical.

3.8.1 Geometric Synthesis

In a manner similar to that for motion generation (Sec. 3.7), the concept of the pole is once again fundamental. Here, however, it is a *relative pole,* since it defines relative motions. Suppose that you need to coordinate the rotation angles ϕ_{12} for the crank (input) and ψ_{12} for the follower (output). Refer to Fig. 3.17, where the following steps have been drawn:

1. Establish convenient locations for the fixed pivots O_A and O_B.
2. Draw an extended fixed link O_AO_B.
3. With O_A as vertex, set off a line ℓ at angle $-\phi_{12}/2$ (half rotation angle, opposite direction).

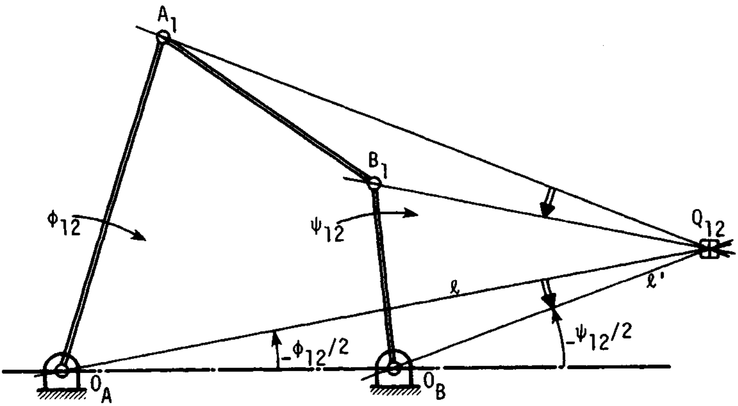

FIGURE 3.17 Crank-angle coordination: definition of relative pole Q_{12}.

4. With O_B as vertex, set off a line ℓ' at angle $-\psi_{12}/2$ (half rotation angle, opposite direction).
5. The intersection of ℓ and ℓ' is the relative pole Q_{12}.
6. Using Q_{12} as the vertex, set off the angle

$$\angle A_1Q_{12}B_1 = \angle O_AQ_{12}O_B$$

in any convenient location, such as that shown.

When only two positions are required, you may choose A_1 and B_1 anywhere on the respective sides of the angle drawn in step 6. For three positions, two relative poles Q_{12} and Q_{13} are used. You may arbitrarily choose either A_1 or B_1, but the other pivot must be found geometrically. Figure 3.18 shows the necessary constructions.

3.8.2 Analytical Synthesis

Although four-bar linkages had been studied analytically for about 100 years, it was not until 1953 that Ferdinand Freudenstein [3.4] derived the now classic relationship

$$R_1 \cos \phi - R_2 \cos \psi + R_3 = \cos (\phi - \psi) \tag{3.15}$$

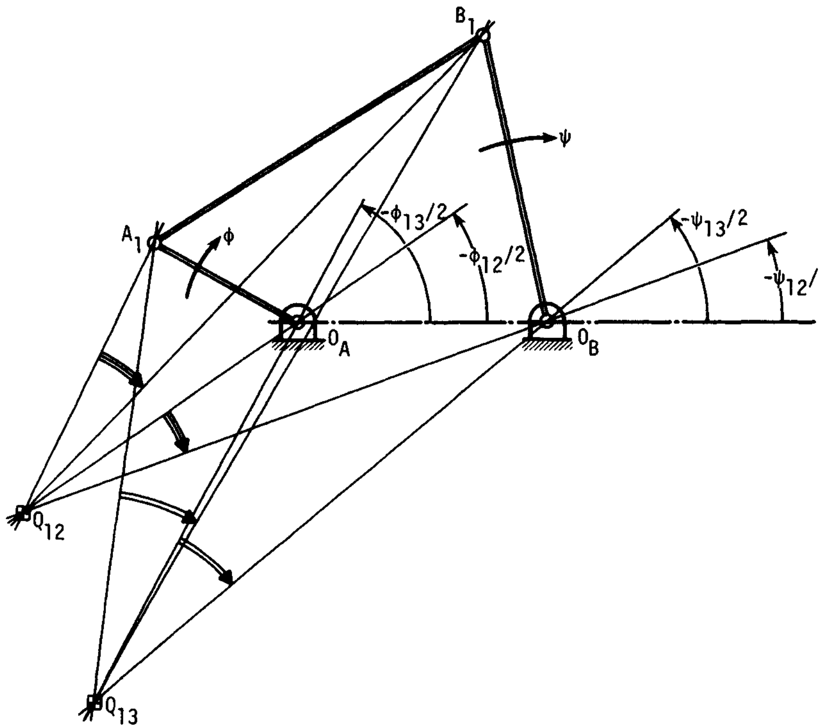

FIGURE 3.18 Geometric construction method for three crank-angle position coordination.

where

$$R_1 = \frac{a}{d} \qquad R_2 = \frac{a}{b} \qquad R_3 = \frac{b^2 - c^2 + d^2 + a^2}{2bd}$$

These link lengths are described in Fig. 3.7. With Eq. (3.15) you can establish a significant variety of linkage requirements. The first derivative of the Freudenstein equation is

$$(R_1 \sin \phi)\left(\frac{d\phi}{dt}\right) - (R_2 \sin \psi)\left(\frac{d\psi}{dt}\right) = \left(\frac{d\phi}{dt} - \frac{d\psi}{dt}\right) \sin (\phi - \psi) \tag{3.16}$$

which provides a relationship for the velocity or torque ratio. By using the relationship in Sec. 3.4.4, Eq. (3.16) becomes

$$R_1 \sin \phi - nR_2 \sin \psi = (1 - n) \sin (\phi - \psi) \tag{3.17}$$

where n is the torque or velocity ratio. A further derivative which would deal with accelerations has never been useful to me. If the need arises, see Ref. [3.9].

Since the problem is one of crank-angle coordination, there are potentially five unknowns (R_1, R_2, R_3, ϕ_1, and ψ_1) which you could determine. Combinations of ϕ_{1j}, ψ_{1j}, and n_j may be specified such that a series of equations of the form

$$R_1 \cos (\phi_1 + \phi_{1j}) - R_2 \cos (\psi_1 + \psi_{1j}) + R_3 = \cos (\phi_1 + \phi_{1j} - \psi_1 - \psi_{1j}) \tag{3.18}$$

and

$$R_1 \sin (\phi_1 + \phi_{1j}) - n_j R_2 \sin (\psi_1 + \psi_{1j}) = (1 - n_j) \sin ((\phi_1 + \phi_{1j}) - \psi_1 - \psi_{1j}) \tag{3.19}$$

can be set up and solved. The nonlinear characteristic makes the solution complicated. Results for certain cases may be found in [3.9] and [3.11]. I have found it most useful in a digital computer program to vary ϕ_1 over the range 0 to π in four simultaneous equations. This produces loci for the moving pivot-point locations which go through the relative poles and are reminiscent of Burmester curves. The two sets of four conditions likely to be of practical interest are as follows:

1. Specify crank rotations $\phi_{12}, \phi_{13}, \phi_{14}, \psi_{12}, \psi_{13}$, and ψ_{14}.
2. Specify crank rotations and velocity or torque ratios $\phi_{12}, n_1, \psi_{12}$, and n_2.

3.9 POLE-FORCE METHOD

An extremely useful scheme for determining static balancing forces in a plane linkage was developed by Hain [3.7] and popularized by Tao [3.12]. Although it is potentially useful for design, I have used it primarily to analyze the requirements for counterbalance springs.

Statically balancing the force on the coupler of a four-bar linkage is a problem often encountered. The solution requires knowledge of the forces and/or torques acting on the four-bar linkage as well as determination of the instantaneous centers. Refer to Fig. 3.19*a,* in which the following constructions occur:

1. The intersection T_1 of forces F_{ab} and F_{ac} is found.
2. The intersection of the coupler (extended) with force F_{ab} is S_{ab} and with F_{ac} is S_{ac}.

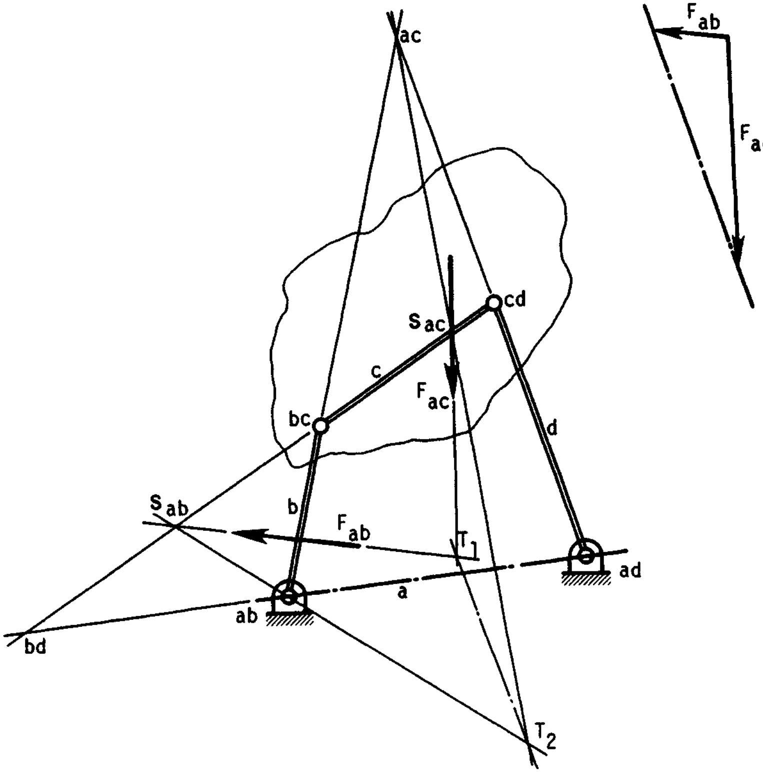

FIGURE 3.19 Pole-force method for balancing a force on the coupler of a four-bar linkage.

3. Determine lines $S_{ab}(ab)$ and $S_{ac}(ac)$; their intersection is T_2.
4. Line T_1T_2 closes the pole-force triangle, which is transferred to Fig. 3.19*b*.
5. The magnitude of F_{ab} required to balance the coupler force F_{ac} is easily found.

Many other cases, any of which you might encounter in practice, are shown by Tao [3.12].

3.10 SPATIAL LINKAGES

Most practical linkages have motion entirely in a plane or possibly in two parallel planes with duplicated mechanisms such as those in a backhoe or a front loader. Design procedures for some elementary types of spatial four-bar linkage have been created (Refs. [3.9] and [3.11]), principally for the RGGR type (Fig. 3.20).

Three principal mathematical methods for writing the loop-closure equation are vectors ([3.3]), dual-number quaternions ([3.14]), and matrices ([3.13]). These techniques have evolved into general-purpose computer programs such as IMP ([3.16])

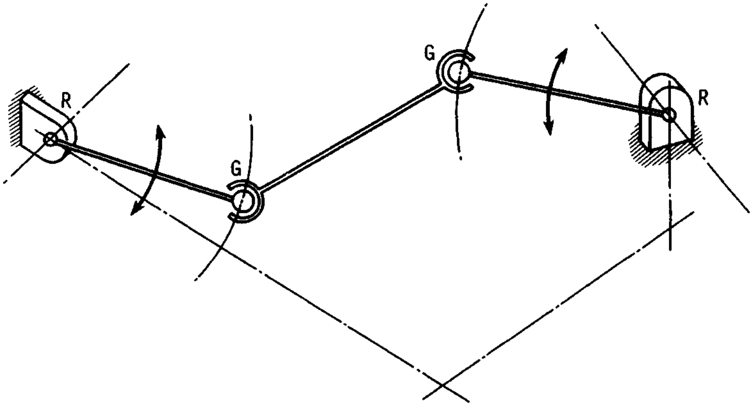

FIGURE 3.20 An RGGR spatial linkage; R designates a revolute joint, and G designates a spherical joint.

and ADAMS and DRAM ([3.5]); they will make your spatial linkage analysis much easier. With such tools available, you can design complex spatial mechanisms by iterative analysis.

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3.16 *IMP, the Integrated Mechanisms Program,* developed by Uicker at the University of Wisconsin.

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