CHAPTER 5
GEAR TRAINS

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5.1 ORDINARY GEAR TRAINS

Gear trains consist of two or more gears meshed for the purpose of transmitting motion from one axis to another. Ordinary gear trains have axes, relative to the frame, for all gears making up the train. Figure 5.1*a* shows a simple ordinary train in which there is only one gear for each axis. In Fig. 5.1*b*, a compound ordinary train is seen to be one in which two or more gears may rotate about a single axis.

The ratio of the angular velocities of a pair of gears is the inverse of their numbers of teeth. The equations for each mesh in the simple train are

$$n_3 = \frac{N_2}{N_3} n_2 \qquad n_4 = \frac{N_3}{N_4} n_3 \qquad n_5 = \frac{N_4}{N_5} n_4 \tag{5.1}$$

where n is in revolutions per minute (r/min) and N = number of teeth. These equations can be combined to give the velocity ratio of the first gear in the train to the last gear:

$$n_5 = \frac{N_4}{N_5} \frac{N_3}{N_4} \frac{N_2}{N_3} n_2 \tag{5.2}$$

Note that the tooth numbers in the numerator are those of the driving gears, and the tooth numbers in the denominator belong to the driven gears. Gears 3 and 4 both drive and are, in turn, driven. Thus, they are called *idler gears.* Since their tooth numbers cancel, idler gears do not affect the magnitude of the input-output ratio, but they do change directions of rotation. Note the directional arrows in the figure. Idler gears can also produce a saving of space and money. In Fig. 5.2, the simple train of the previous figure has been repeated. In dotted outline is shown a pair of gears on the same center distance as gears 2 and 5 and having the same input-output ratio as the simple train.

Finally, Eq. (5.2) is simplified to become

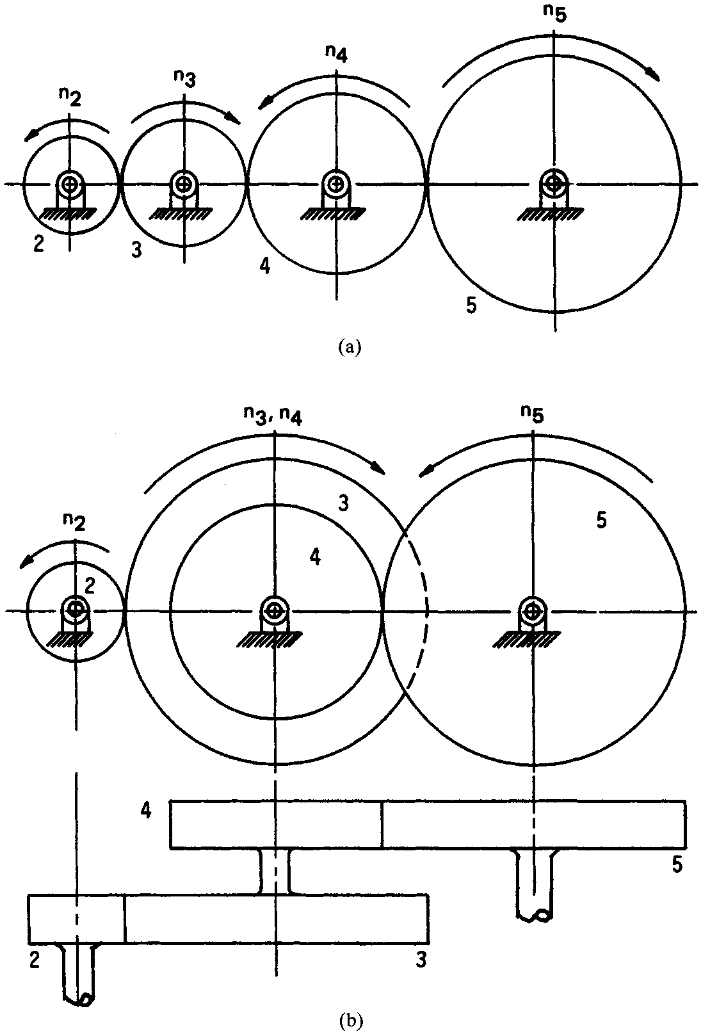

FIGURE 5.1 Ordinary gear trains. (*a*) Simple; (*b*) compound.

$$n_5 = -\frac{N_2}{N_5} n_2 \tag{5.3}$$

where the minus sign is now introduced to indicate contrarotation of the two gears.

The compound train in Fig. 5.1*b* has the following velocity ratios for the pairs of driver and driven gears:

$$n_3 = -\frac{N_2}{N_3} n_2 \qquad \text{and} \qquad n_5 = -\frac{N_4}{N_5} n_4 \tag{5.4}$$

and, of course, $n_4 = n_3$. Combining the equations yields

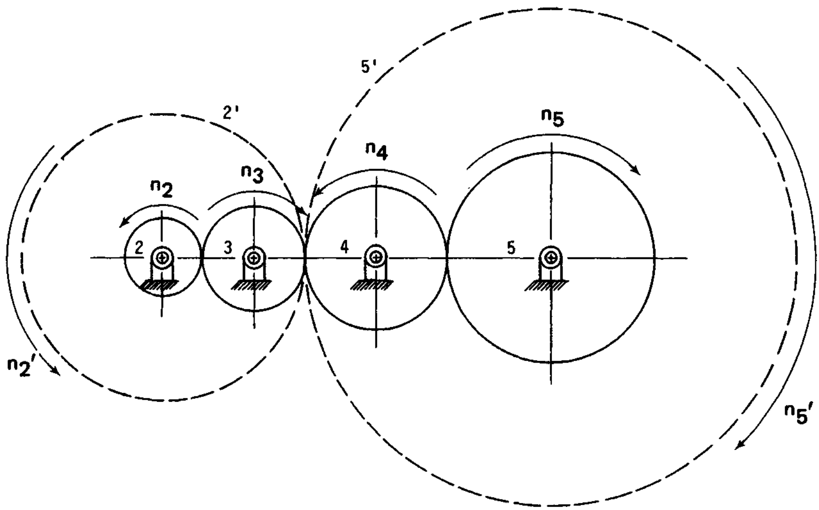

FIGURE 5.2 Gears 2′ and 5′ are required if idler gears are not used.

$$n_5 = \frac{N_2 N_4}{N_3 N_5} n_2 \tag{5.5}$$

and the thing worthy of note here is that the numbers of teeth of all gears constituting a mesh with a compounded pair are required to determine the velocity ratio through the system. Compound gear trains have an advantage over simple gear trains whenever the speed change is large. For example, if a reduction of 12/1 is required, the final gear in a simple train will have a diameter 12 times that of the first gear.

5.2 GEAR TYPE SELECTION

The disposition of the axes to be joined by the gear train often suggests the type of gear to choose. If the axes are parallel, the choices can be spur gears or helical gears. If the axes intersect, bevel gears can be used. If the axes are nonparallel and nonintersecting, then crossed helicals, worm and gear, or hypoid gears will work. In Fig. 5.3, a train having various types of gears is shown. Gears 2 and 3, parallel helical gears, have a speed ratio

$$n_3 = -\frac{N_2}{N_3} n_2 \tag{5.6}$$

Gears 4 and 5, bevel gears, have a speed ratio

$$n_5 = -\frac{N_4}{N_5} n_4 \tag{5.7}$$

Gears 6 and 7, worm and gear, are considered in a slightly different manner. A worm is generally spoken of as having threads, one, two, three or more (see Chap. 12). A

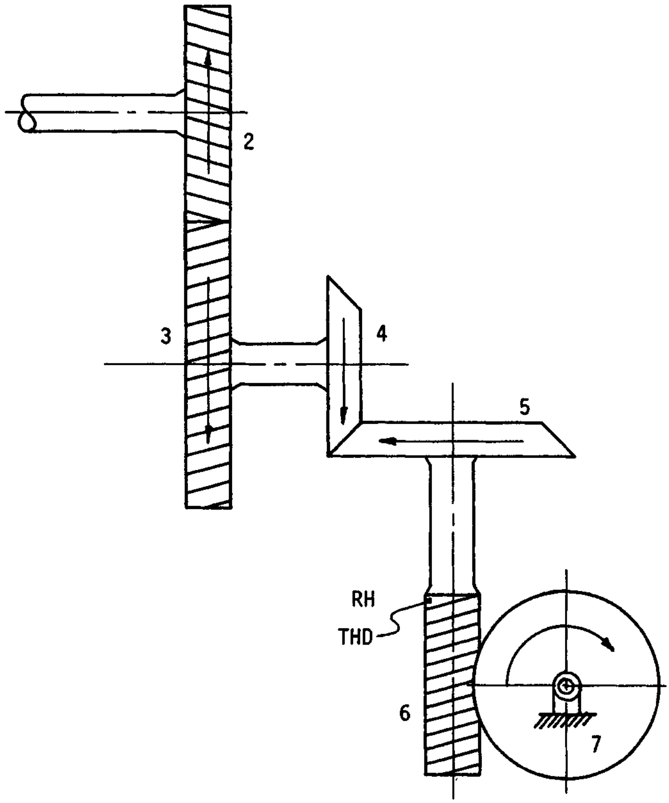

FIGURE 5.3 Various gears used in a train.

worm with one thread would have a lead equal to the pitch of the thread. A worm with two threads would have a lead equal to twice the pitch of the thread. Thus

$$n_7 = \frac{\text{number of threads on 6}}{N_7} n_6 \tag{5.8}$$

Joining Eqs. (5.6), (5.7), and (5.8), we find

$$n_7 = \frac{N_6}{N_7} \frac{N_4}{N_5} \frac{N_2}{N_3} n_2 \tag{5.9}$$

where N_6 represents the number of threads of the worm gear.

To determine the direction of rotation of gear 7, an inversion technique can be used. Fix gear 7 and allow the worm to translate along its axis as it rotates. Here it is necessary to note the hand of the worm, which can be either right or left. In the figure, gear 6 rotates in the same direction as gear 5 and, having a right-hand thread, will move downward (in the drawing). Now, inverting back to the original mechanism, the worm is moved in translation to its proper position, and by doing so, gear 7 is seen to rotate clockwise.

5.3 PLANETARY GEAR TRAINS

Planetary gear trains, also referred to as *epicyclic gear trains,* are those in which one or more gears orbit about the central axis of the train. Thus, they differ from an ordinary train by having a moving axis or axes. Figure 5.4 shows a basic arrangement that is functional by itself or when used as a part of some more complex system. Gear 2 is called a *sun gear,* gear 4 is a *planet,* link 3 is an *arm,* or *planet carrier,* and gear 5 an internal-toothed *ring gear.*

Planetary gear trains are, fundamentally, two-degree-of-freedom systems. Therefore, two inputs are required before they can be uniquely analyzed. Quite frequently a fixed gear is included in the train. Its velocity is zero, but this zero velocity constitutes one of the input values. Any link in the train shown except the planet can serve as an input or an output link. If, for example, the rotations of link 2 and link 5 were the input values, the rotation of the arm would be the output. The term *link* refers to the individual machine elements comprising a mechanism or linkage, and gear trains are included in this broad array of systems. Each link is paired, or joined, with at least two other links by some form of connection, such as pin points, sliding joints, or

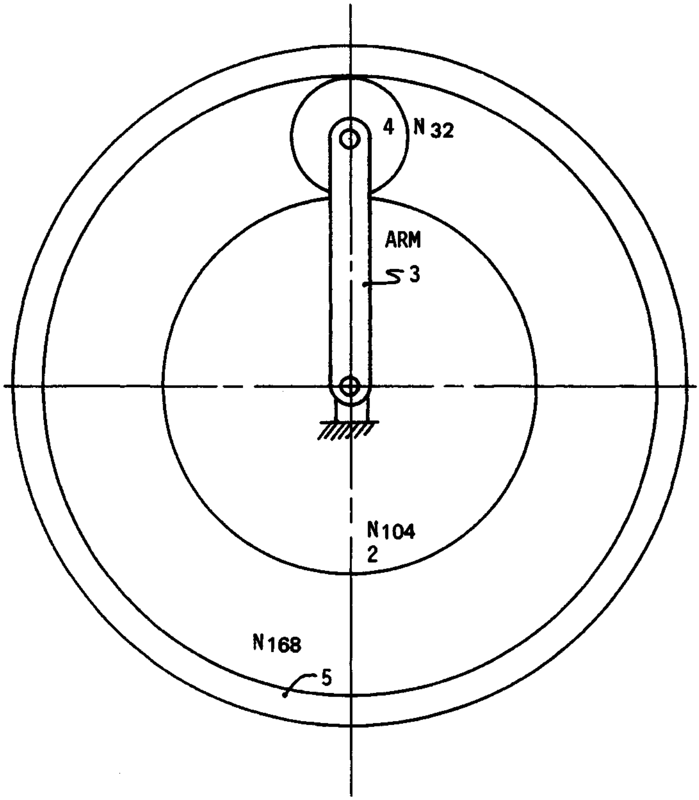

FIGURE 5.4 A basic planetary train.

direct contact, a pairing that is prevalent in cam-and-gear systems. An explanation and an illustration of the *joint* types are found in Refs. [5.1] and [5.2] as well as others (see Chap. 3).

There are several methods for analyzing planetary trains. Among these are instant-centers, formula, and tabular methods. By instant centers, as in Ref. [5.3] and on a face view of the train, draw vectors representing the velocities of the instant centers for which input information is known. Then, by simple graphical construction, the velocity of another center can be found and converted to a rotational speed. Figure 5.5 illustrates this technique.

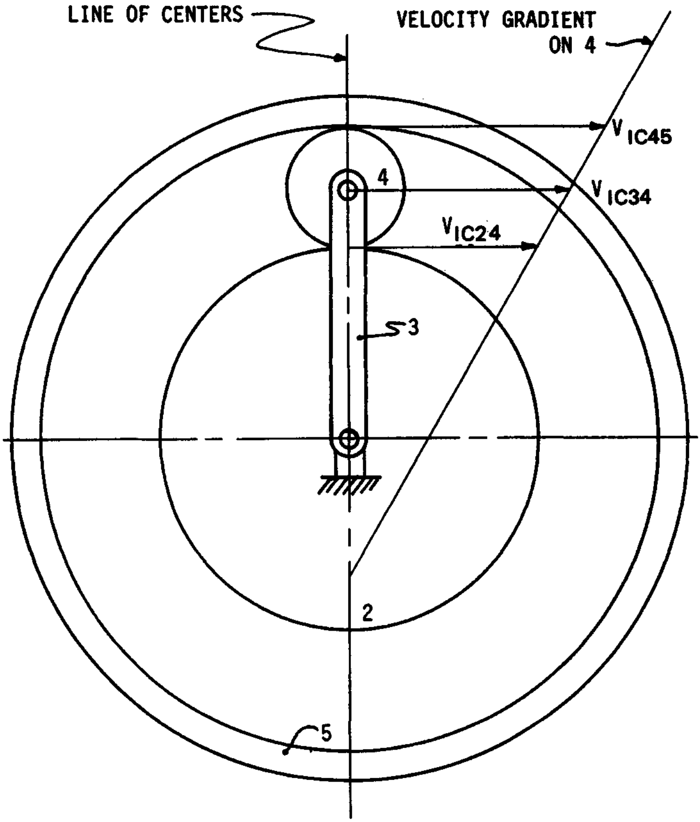

FIGURE 5.5 Instant-centers method of velocity analysis.

Calculate V_{IC24} and V_{IC45} from

$$V = r\omega \tag{5.10}$$

where r = radius dimension and ω = angular velocity in radians per second (rad/s). Draw these vectors to scale in the face view of the train. Then V_{IC24} and V_{IC45} will emanate from their instant-center positions. Now draw a straight line through the termini of the velocity vectors.[†] The velocity of IC34 will be a vector perpendicular to the line of centers and having its terminus on the velocity gradient. Determine ω of link 3 by using Eq. (5.10). Thus,

$$V_{IC24} = r_2\omega_2 \qquad \text{and} \qquad V_{IC45} = r_5\omega_5$$

Choose a scale and construct the two vectors. Next, draw the gradient line and construct V_{IC34}. Scale its magnitude and determine n_3 according to

$$n_3 = \frac{V_{IC34}}{2\pi r_3} 60 \tag{5.11}$$

where r_3 = radius of the arm and n_3 is in revolutions per minute.

If gear 5 is fixed, then $V_{IC45} = 0$; using V_{IC24}, connect the terminus of V_{IC24} and IC45 with a straight line, and find V_{IC34} as before. See Fig. 5.6.

[†] This line can be called a *velocity gradient* for link 4.

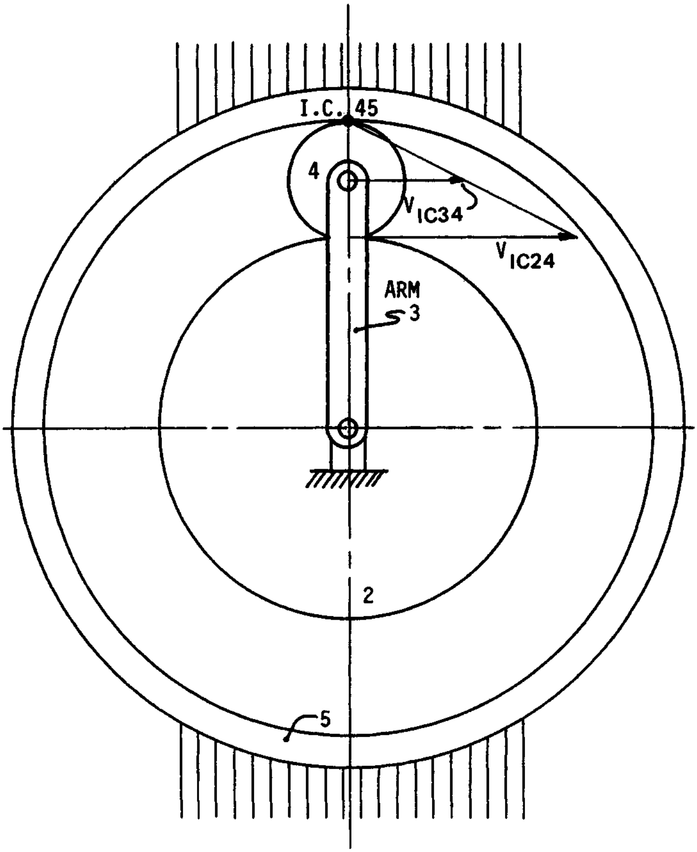

FIGURE 5.6 Gear 5 is fixed.

By formula, the relative-motion equation will establish the velocity of the gears relative to the arm; that is,

$$n_{23} = n_2 - n_3 \tag{5.12}$$

$$n_{53} = n_5 - n_3 \tag{5.13}$$

Then, dividing (5.13) by (5.12), we see that

$$\frac{n_{53}}{n_{23}} = \frac{n_5 - n_3}{n_2 - n_3} \tag{5.14}$$

which represents the ratio of the relative velocity of gear 5 to that of gear 2 with both velocities related to the arm. The right-hand side of the equation is called the *train value.* If the arm should be held fixed, then the ratio of output to input speeds for an ordinary train is obtained.

The equation for train value, which is seen in most references, can be written

$$e = \frac{n_L - n_A}{n_F - n_A} \tag{5.15}$$

where n_F = speed of first gear in train
n_L = speed of last gear in train
n_A = speed of arm

The following example will illustrate the use of Eq. (5.15).

Example 1. Refer to the planetary train of Fig. 5.4. The tooth numbers are $N_2 = 104$, $N_4 = 32$, and $N_5 = 168$. Gear 2 is driven at 250 r/min in a clockwise *negative* direction, and gear 5 is driven at 80 r/min in a counterclockwise *positive* direction. Find the speed and direction of rotation of the arm.

Solution. $n_F = n_2 = -250$ r/min $\quad n_L = n_5 = +80$ r/min

$$e = \left(-\frac{N_2}{N_4}\right)\left(\frac{N_4}{N_5}\right) = \left(-\frac{104}{32}\right)\left(\frac{32}{168}\right) = -\frac{13}{21}$$

In Eq. (5.15),

$$-\frac{13}{21} = \frac{80 - n_3}{-250 - n_3} \qquad n_3 = -46.2 \text{ r/min}$$

By tabular method, a table is first formed according to the following:

1. Include a column for any gear centered on the planetary axis.
2. Do not include a column for any gear whose axis of rotation is fixed and different from the planetary axis.
3. A column for the arm is not necessary.
4. The planet, or planets, may be included in a column or not, as preferred.

Gears which fit rule 2 are treated as ordinary gear train elements. They are used as input motions to the planetary system, or they may function as output motions.

The table contains three rows arranged so that each entry in a column will constitute one term of the relative-motion equation

TABLE 5.1 Solution by Tabulation

Step	Gear 2	Gear 5
1. Gears locked	n_3	n_3
2. Arm fixed	$n_2 - n_3$	$-\dfrac{N_2}{N_4}\left(\dfrac{N_4}{N_5}\right)(n_2 - n_3)$
3. Results	n_2	n_5

$$n_y + n_{xy} = n_x \tag{5.16}$$

This is best shown by example. Using the planetary train of the previous example, we form Table 5.1, and the equation from the column for gear 5 is

$$n_3 - \frac{N_2N_4}{N_4N_5}(n_2 - n_3) = n_5$$

Rearranging and canceling N_4, we find

$$n_3\left(1 + \frac{N_2}{N_5}\right) - n_2\frac{N_2}{N_5} = n_5 \tag{5.17}$$

This is the characteristic equation of the planetary train, as shown in Fig. 5.4.

Note that three rotational quantities appear—n_3, n_2, and n_5. There must be two input rotations in order to solve for the output. This is easily done when the input rotations and the tooth numbers are inserted. When a positive sense is assigned to counterclockwise and a negative sense to clockwise rotation, the sign of the output rotation indicates its sense of direction.

Note that planet 4 was not included in the table (it could have been); however, gear 4 served its purpose by acting as an idler to change a direction of rotation. This is evidenced by the presence of a negative sign in the second row of the column for gear 5.

A convenient means of representing a planetary train was shown by Levai [5.4]. Type A of Fig. 5.7*a* shows an edge view of the planetary train first seen in Fig. 5.4. It and the other 11 configurations represent all possible variations for a planetary train. The equations in Table 5.2 are the characteristic equations of the 12 types.

An examination of the equations and their corresponding types reveals that certain ones are identical. Types C and D in Fig. 5.7*b* are identical because of the arrangement of gears. Whereas in type C the meshes of 2 and 4 and of 7 and 8 are external, the input and output meshes are internal in type D. The same relationship can be seen in types G and H in Fig. 5.7*c*. Certain pair types are alike in equation form but differ in sign. Compare types E and K and F and L in Fig. 5.7*b* and *d*, and B and G (or B and H) in Fig. 5.7*a* and *c*.

The speed of a planet gear relative to the frame or relative to the arm may be required. If appreciable speeds and forces are involved, this information will facilitate the selection of bearings. Using type A as an example, set up Table 5.3. Row 2 in the column for gear 4 is the speed of gear 4 relative to the arm, and row 3 in the column for gear 4 is its speed relative to the frame.

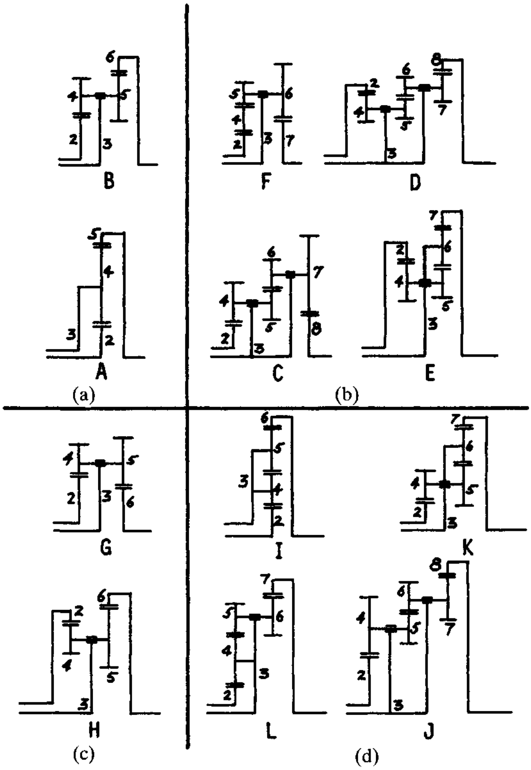

FIGURE 5.7 Twelve variations of planetary trains.

Example 2. Figure 5.8 shows a planetary gear train with input at gear 2. Also, gear 6′ is seen to be part of the frame, in which case its rotation is zero. For $n_2 = 100$ r/min clockwise (negative), find output rotation n_6.

Solution. Gears 2, 4, 5, and 6 and arm 3 form a type B planetary train:

$$n_3\left(1 + \frac{N_2N_5'}{N_4N_6'}\right) - n_2\,\frac{N_2N_6'}{N_4N_6'} = n_6'$$

Solving for n_3 yields

$$n_3\left(1 + \frac{1}{8}\right) - (-100)\left(\frac{1}{8}\right) = 0$$

$$n_3 = -\frac{100}{9}\ \text{r/min}$$

For type G:

$$n_3\left(1 - \frac{N_2N_5}{N_4N_6}\right) + \left(n_2\,\frac{N_2N_5}{N_4N_6}\right) = n_6$$

Then we solve type G for n_6:

TABLE 5.2 Characteristic Equations for 12 Planetary Trains of Fig. 5.7

Type	Equation
A	$n_3\left(1+\frac{N_2}{N_5}\right)-n_2\frac{N_2}{N_5}=n_5$
B	$n_3\left(1+\frac{N_2N_5}{N_4N_6}\right)-n_2\frac{N_2N_5}{N_4N_6}=n_6$
C	$n_3\left(1+\frac{N_2N_5N_7}{N_4N_6N_8}\right)-n_2\frac{N_2N_5N_7}{N_4N_6N_8}=n_8$
D	$n_3\left(1+\frac{N_2N_5N_7}{N_4N_6N_8}\right)-n_2\frac{N_2N_5N_7}{N_4N_6N_8}=n_8$
E	$n_3\left(1+\frac{N_2N_5}{N_4N_7}\right)-n_2\frac{N_2N_5}{N_4N_7}=n_7$
F	$n_3\left(1+\frac{N_2N_6}{N_5N_7}\right)-n_2\frac{N_2N_6}{N_5N_7}=n_7$
G	$n_3\left(1-\frac{N_2N_5}{N_4N_6}\right)+n_2\frac{N_2N_5}{N_4N_6}=n_6$
H	$n_3\left(1-\frac{N_2N_5}{N_4N_6}\right)+n_2\frac{N_2N_5}{N_4N_6}=n_6$
I	$n_3\left(1-\frac{N_2}{N_6}\right)+n_2\frac{N_2}{N_6}=n_6$
J	$n_3\left(1-\frac{N_2N_5N_7}{N_4N_6N_8}\right)+n_2\frac{N_2N_5N_7}{N_4N_6N_8}=n_8$
K	$n_3\left(1-\frac{N_2N_5}{N_4N_7}\right)+n_2\frac{N_2N_5}{N_4N_7}=n_7$
L	$n_3\left(1-\frac{N_2N_6}{N_5N_7}\right)+n_2\frac{N_2N_6}{N_5N_7}=n_7$

TABLE 5.3 Solution of Type A Train

Step	Gear 2	Gear 4	Gear 5
1. Gears locked	n_3	n_3	n_3
2. Arm fixed	$n_3 - n_2$	$-\frac{N_2}{N_4}(n_3 - n_2)$	$-\frac{N_2}{N_5}(n_3 - n_2)$
3. Results	n_3	n_4	n_5

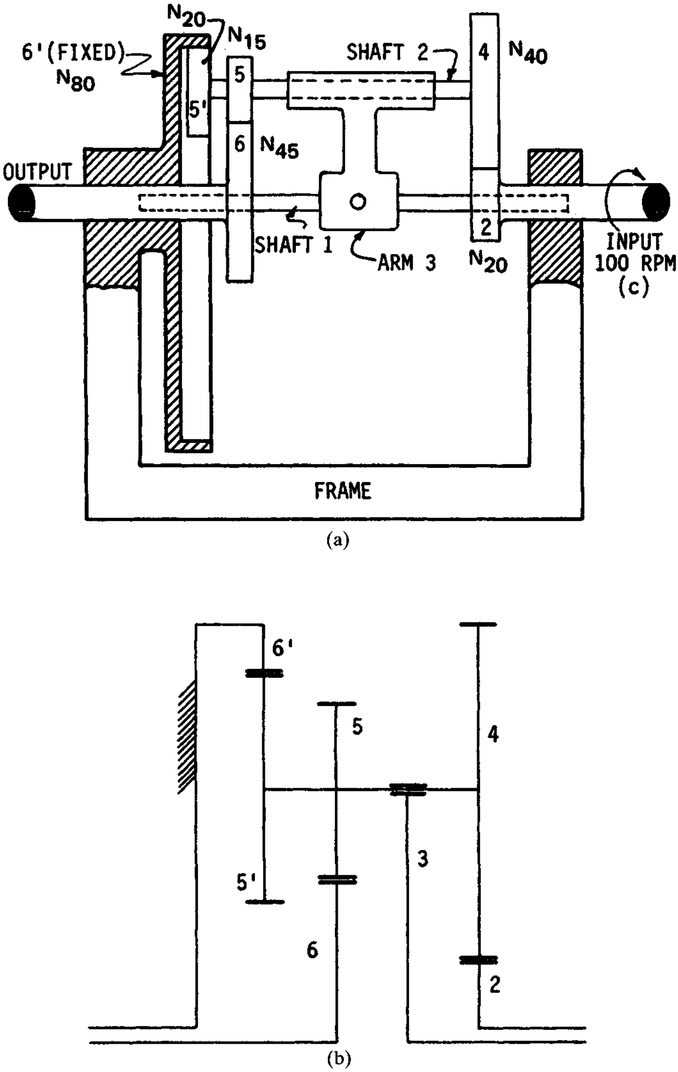

FIGURE 5.8 (*a*) View of a gear train and (*b*) its symbolic notation.

$$-\frac{100}{9}\left(1-\frac{1}{6}\right)+(-100)\left(\frac{1}{6}\right)=n_6$$

$$n_6 = -25.93 \text{ r/min}$$

Example 3. Figure 5.9 shows a type I planetary train, Ref. [5.2]. Here, if $n_2 = 100$ r/min clockwise and $n_3 = 200$ r/min clockwise, both considered negative, determine n_4, n_5, and n_6.

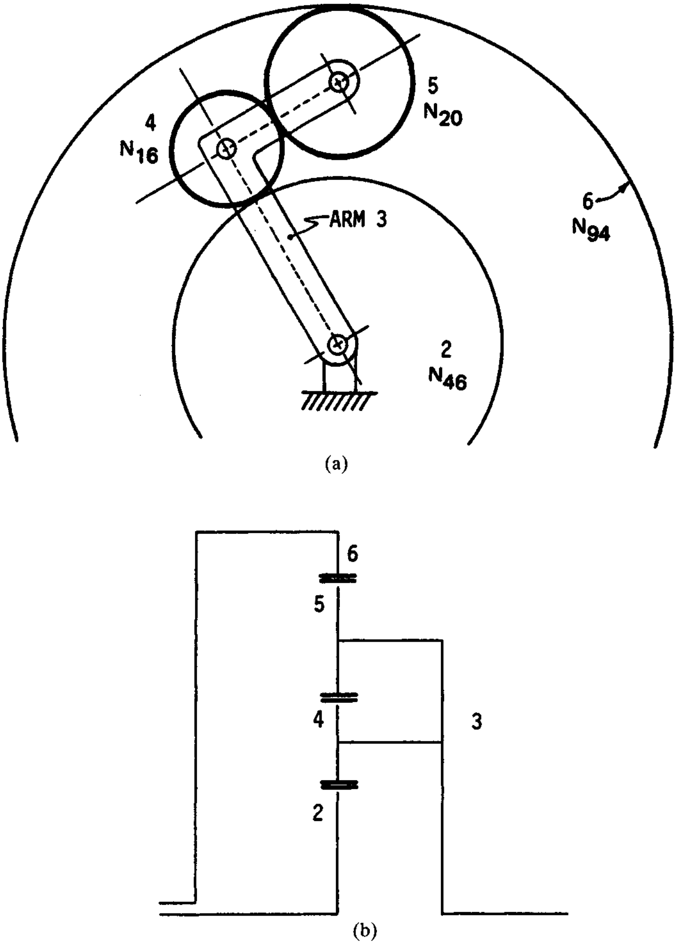

FIGURE 5.9 (*a*) Planetary train; (*b*) symbolic notation.

TABLE 5.4 Solution of Type I Train

Step	Gear 2	Gear 4	Gear 5	Gear 6
1	n_3	n_3	n_3	n_3
2	$n_2 - n_3$	$-\frac{N_2}{N_4}(n_2 - n_3)$	$+\frac{N_2}{N_5}(n_2 - n_3)$	$+\frac{N_2}{N_6}(n_2 - n_3)$
3	n_2	n_4	n_5	n_6

Solution. To determine the angular speeds for the planet, form Table 5.4. The speed of gear 4 can be found by writing the equation in the column for gear 4. Thus,

$$n_3\left(1+\frac{N_2}{N_4}\right)-n_2\frac{N_2}{N_4}=n_4$$

$$-200\left(1+\frac{46}{16}\right)-(-100)\left(\frac{46}{16}\right)=n_4$$

$$n_4=-487.5 \text{ r/min}$$

For gear 5,

$$n_3\left(1-\frac{N_2}{N_5}\right)+n_2\frac{N_2}{N_5}=n_5$$

$$-200\left(1-\frac{46}{20}\right)+(-100)\left(\frac{46}{20}\right)=n_5$$

$$n_5=+30 \text{ r/min}$$

For gear 6,

$$n_3\left(1-\frac{N_2}{N_6}\right)+n_2\frac{N_2}{N_6}=n_6$$

$$-200\left(1-\frac{46}{94}\right)+(-100)\left(\frac{46}{94}\right)=n_6$$

$$n_6=-151 \text{ r/min}$$

5.4 DIFFERENTIAL TRAINS

Differential gear trains are useful as mechanical computing devices. In Fig. 5.10, if ω_4 and ω_6 are input angular velocities and V_A and V_B are the resulting linear velocities of points A and B, respectively, then the velocity of point C on the carrier is

$$V_C=\frac{V_A+V_B}{2} \tag{5.18}$$

The differential gear train also finds application in the wheel-axle system of an automobile. The planet carrier rotates at the same speed as the wheels when the automobile is traveling in a straight line. When the car goes into a curve, however, the inside wheel rotates at a lesser speed than the outside wheel because of the differential gear action. This prevents tire drag along the road during a turn.

Example 4. See Ref. [5.2], page 329. The tooth numbers for the automotive differential shown in Fig. 5.11 are $N_2 = 17, N_3 = 54, N_4 = 11$, and $N_5 = N_6 = 16$. The drive shaft turns at 1200 r/min. What is the speed of the right wheel if it is jacked up and the left wheel is resting on the road surface?

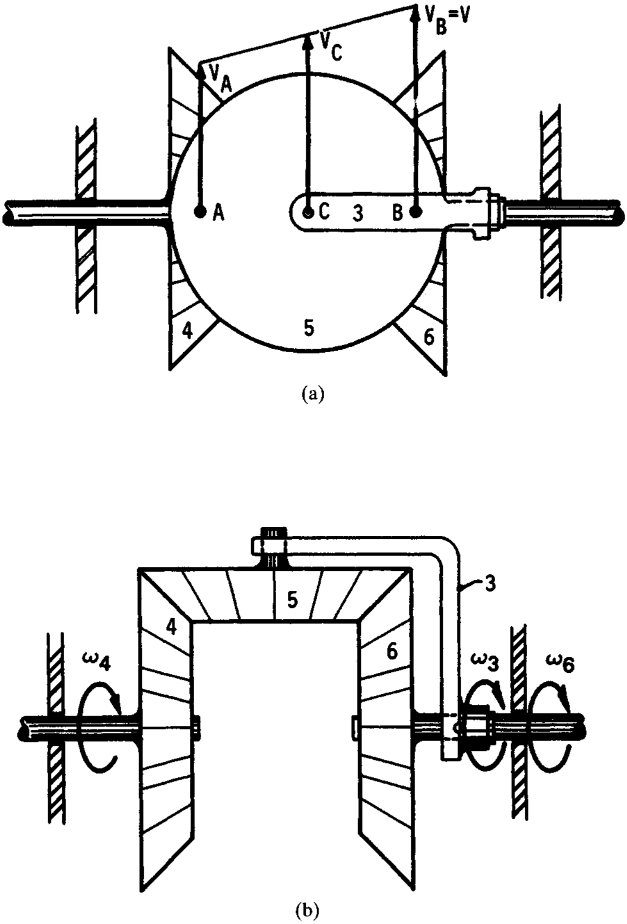

FIGURE 5.10 (*a*) Top and (*b*) front views of a bevel-gear differential used as a mechanical averaging linkage. Point A is the pitch point of gears 4 and 5. Point B is the pitch point of gears 5 and 6.

Solution. The planet carrier, gear 3, is rotating according to the following equation:

$$n_3 = \frac{N_2}{N_3} n_2 = \frac{17}{54}(1200) = 377.78 \text{ r/min}$$

Since the r/min of the left wheel is zero, the pitch point of gears 4 and 5 has a linear velocity twice that of the pin which supports the planet. Therefore, the r/min of the right wheel is twice that of the planet, or

$$n_6 = 2n_3 = 755.56 \text{ r/min}$$

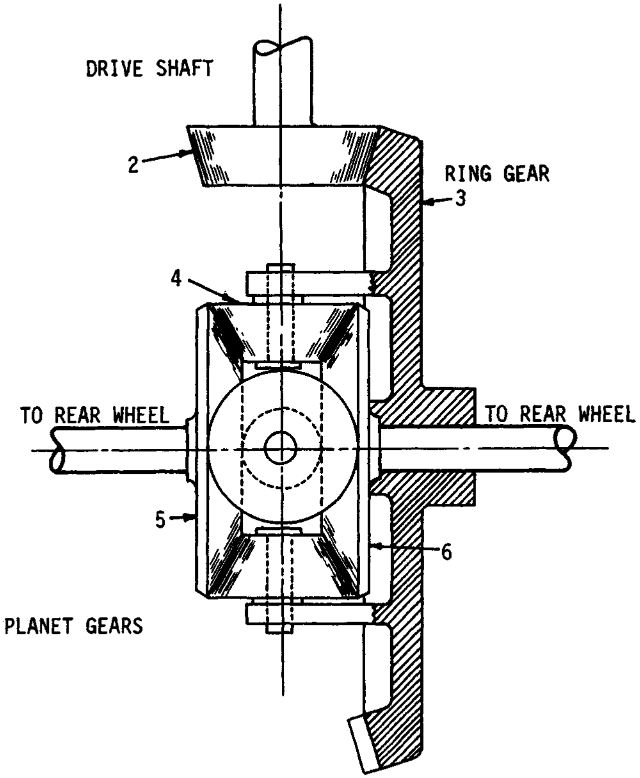

FIGURE 5.11 Schematic drawing of a bevel-gear automotive differential.

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