CHAPTER 30

INSTABILITIES IN BEAMS AND COLUMNS

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NOTATION

A	Area of cross section
$B(n)$	Arbitrary constants
$c(n)$	Coefficients in series
$c(y), c(z)$	Distance from y and z axis, respectively, to outermost compressive fiber
e	Eccentricity of axial load P
E	Modulus of elasticity of material
$E(t)$	Tangent modulus for buckling outside of elastic range
$F(x)$	A function of x
G	Shear modulus of material
h	Height of cross section
H	Horizontal (transverse) force on column
I	Moment of inertia of cross section
$I(y), I(z)$	Moment of inertia with respect to y and z axis, respectively
J	Torsion constant; polar moment of inertia
k^2	P/EI
K	Effective-length coefficient
$K(0)$	Spring constant for constraining spring at origin

$K(T, 0), K(T, L)$	Torsional spring constants at $x = 0, L$, respectively
l	Developed length of cross section
L	Length of column or beam
L_{eff}	Effective length of column
M, M'	Bending moments
$M(0), M(L), M_{mid}$	Bending moments at $x = 0, L$, and midpoint, respectively
$M(0)_{cr}$	Critical moment for buckling of beam
M_{tr}	Moment due to transverse load
$M(y), M(z)$	Moment about y and z axis, respectively
n	Integer; running index
P	Axial load on column
P_{cr}	Critical axial load for buckling of column
r	Radius of gyration
R	Radius of cross section
s	Running coordinate, measured from one end
t	Thickness of cross section
T	Torque about x axis
x	Axial coordinate of column or beam
y, z	Transverse coordinates and deflections
Y	Initial deflection (crookedness) of column
Y_{tr}	Deflection of beam-column due to transverse load
η	Factor of safety
σ	Stress
ϕ	Angle of twist

As the terms *beam* and *column* imply, this chapter deals with members whose cross-sectional dimensions are small in comparison with their lengths. Particularly, we are concerned with the stability of beams and columns whose axes in the undeformed state are substantially straight. Classically, instability is associated with a state in which the deformation of an idealized, perfectly straight member can become arbitrarily large. However, some of the criteria for stable design which we will develop will take into account the influences of imperfections such as the eccentricity of the axial load and the crookedness of the centroidal axis of the column. The magnitudes of these imperfections are generally not known, but they can be estimated from manufacturing tolerances. For axially loaded columns, the onset of instability is related to the moment of inertia of the column cross section about its minor principal axis. For beams, stability design requires, in addition to the moment of inertia, the consideration of the torsional stiffness.

30.1 EULER'S FORMULA

We will begin with the familiar Euler column-buckling problem. The column is idealized as shown in Fig. 30.1. The top and bottom ends are pinned; that is, the moments at the ends are zero. The bottom pin is fixed against translation; the top pin is free to move in the vertical direction only; and the force P acts along the x axis, which coincides with the centroidal axis in the undeformed state. It is important to keep in mind that the analysis which follows applies only to columns with cross sections and loads that are symmetrical about the xy plane in Fig. 30.1 and satisfy the usual assumptions of linear beam theory. It is particularly important in this connection to keep in mind that this analysis is valid only when the deformation is such that the square of the slope of the tangent at any point on the deflection curve is negligibly small compared to unity (fortunately, this is generally true in design applications). In such a case, the familiar differential equation for the bending of a beam is applicable. Thus,

$$EI\frac{d^2y}{dx^2} = M \tag{30.1}$$

For the column in Fig. 30.1,

$$M = -Py \tag{30.2}$$

We take E and I as constant, and let

$$\frac{P}{EI} = k^2 \tag{30.3}$$

Then we get, from Eqs. (30.1), (30.2), and (30.3),

$$\frac{d^2y}{dx^2} + k^2y = 0 \tag{30.4}$$

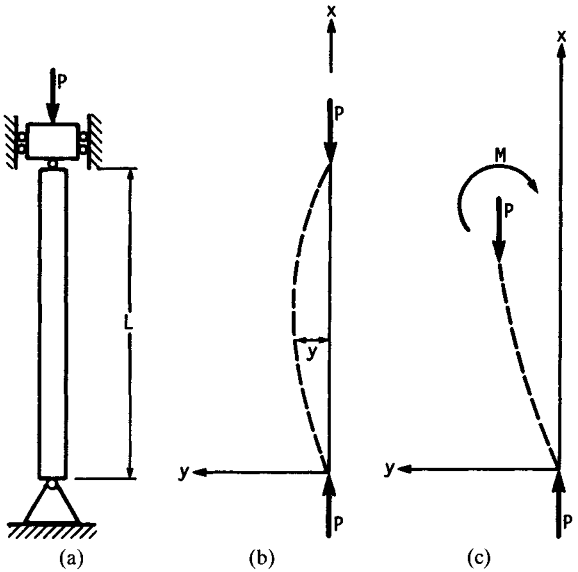

FIGURE 30.1 Deflection of a simply supported column. (*a*) Ideal simply supported column; (*b*) column-deflection curve; (*c*) free-body diagram of deflected segment.

The boundary conditions at $x = 0$ and $x = L$ are

$$y(0) = y(L) = 0 \tag{30.5}$$

In order that Eqs. (30.4) and (30.5) should have a solution $y(x)$ that need not be equal to zero for all values of x, k must take one of the values in Eq. (30.6):

$$k(n) = \frac{n\pi}{L} \qquad n = 1, 2, 3, \ldots \tag{30.6}$$

which means that the axial load P must take one of the values in Eq. (30.7):

$$P(n) = \frac{n^2\pi^2 EI}{L^2} \qquad n = 1, 2, 3, \ldots \tag{30.7}$$

For each value of n, the corresponding nonzero solution for y is

$$y(n) = B(n) \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots \tag{30.8}$$

where $B(n)$ is an arbitrary constant.

In words, the preceding results state the following: Suppose that we have a perfectly straight prismatic column with constant properties over its entire length. If the column is subjected to a perfectly axial load, there is a set of load values, together with a set of sine-shaped deformation curves for the column axis, such that the applied moment due to the axial load and the resisting internal moment are in equilibrium everywhere along the column, no matter what the amplitude of the sine curve may be. From Eq. (30.7), the smallest load at which such deformation occurs, called the *critical load,* is

$$P_{cr} = \frac{\pi^2 EI}{L^2} \tag{30.9}$$

This is the familiar Euler formula.

30.2 EFFECTIVE LENGTH

Note that the sinusoidal shape of the solution function is determined by the differential equation and does not depend on the boundary conditions. If we can find a segment of a sinusoidal curve that satisfies our chosen boundary conditions and, in turn, we can find some segment of that curve which matches the curve in Fig. 30.1, we can establish a correlation between the two cases. This notion is the basis for the "effective-length" concept. Recall that Eq. (30.9) was obtained for a column with both ends simply supported (that is, the moment is zero at the ends). Figure 30.2 illustrates columns of length L with various idealized end conditions. In each case, there is a multiple of L, KL, which is called the *effective length of the column* L_{eff}, that has a shape which is similar to and behaves like a simply supported column of that length. To determine the critical loads for columns whose end supports may be idealized as shown in Fig. 30.2, we can make use of Eq. (30.9) if we replace L by KL, with the appropriate value of K taken from Fig. 30.2. Particular care has to be taken to distinguish between the case in Fig. 30.2*c*, where both ends of the column are secured against rotation and transverse translation, and the case in Fig. 30.2*e*, where the ends do not rotate, but relative transverse movement of one end of the column with respect to the other end is possible. The effective length in the first case is half that in the second case, so that the critical load in the first case is four times that in the second case. A major difficulty with using the results in Fig. 30.2 is that in real problems a column end is seldom perfectly fixed or perfectly free (even approximately) with regard to translation or rotation. In addition, we must remember that the critical load is inversely proportional to the square of the effective length. Thus a change of 10 percent in L_{eff} will result in a change of about 20 percent in the critical load, so that a fair approximation of the effective length produces an unsatisfactory approximation of the critical load. We will now develop more general results that will allow us to take into account the elasticity of the structure surrounding the column.

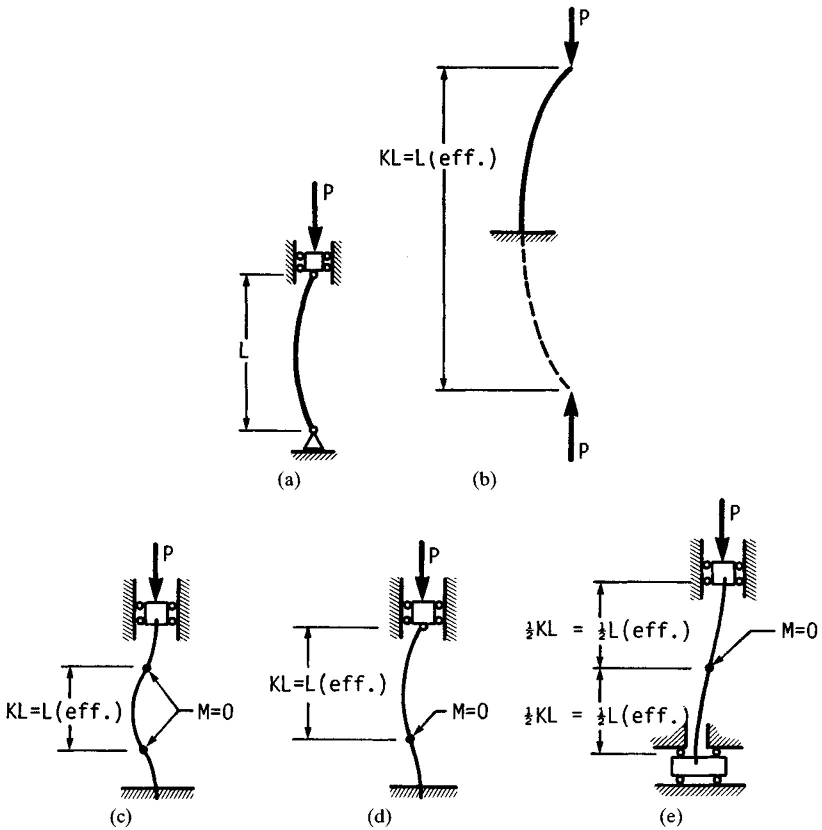

FIGURE 30.2 Effective column lengths for different types of support. (*a*) Simply supported, $K = 1$; (*b*) fixed-free, $K = 2$; (*c*) fixed-fixed, $K = ½$; (*d*) fixed-pinned, $K = 0.707$; (*e*) ends nonrotating, but have transverse translation.

30.3 GENERALIZATION OF THE PROBLEM

We will begin with a generalization of the case in Fig. 30.2*e*. In Fig. 30.3, the lower end is no longer free to translate, but instead is elastically constrained. The differential equation is

$$EI\frac{d^2y}{dx^2} = M = M(0) + P[y(0) - y] + Hx \tag{30.10}$$

Here H, the horizontal force at the origin, may be expressed in terms of the deflection at the origin $y(0)$ and the constant of the constraining spring $K(0)$:

$$H = -K(0)y(0) \tag{30.11}$$

$M(0)$ is the moment which prevents rotation of the beam at the origin. The moment which prevents rotation of the beam at the end $x = L$ is $M(L)$. The boundary conditions are

$$y(L) = 0 \qquad \frac{dy(L)}{dx} = 0 \qquad \frac{dy(0)}{dx} = 0 \tag{30.12}$$

The rest of the symbols are the same as before. We define k as in Eq. (30.3). As in the case of the simply supported column, Eqs. (30.10), (30.11), and (30.12) have solutions in which $y(x)$ need not be zero for all values of x, but again these solutions occur only for certain values of kL. Here these values of kL must satisfy Eq. (30.13):

$$[2(1 - \cos kL) - kL \sin kL]L^3K(0) + EI(kL)^3 \sin kL = 0 \tag{30.13}$$

The physical interpretation is the same as in the simply supported case. If we denote the lowest value of kL that satisfies Eq. (30.13) by $(kL)_{cr}$, then the column buckling load is given by

$$P_{cr} = \frac{EI(kL)^2_{cr}}{L^2} \tag{30.14}$$

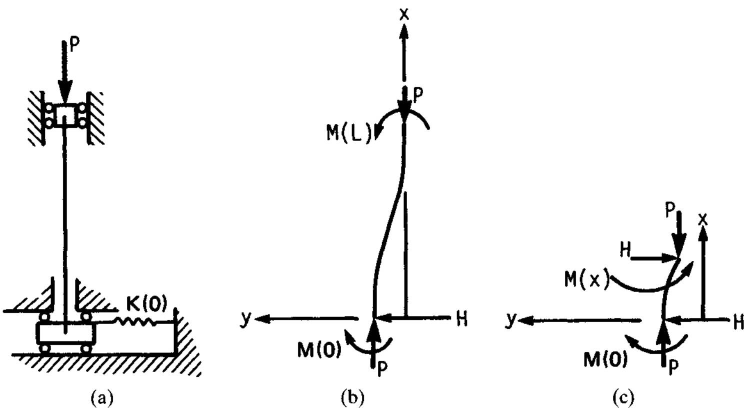

FIGURE 30.3 Column with ends fixed against rotation and an elastic end constraint against transverse deflection. (*a*) Undeflected column; (*b*) deflection curve; (*c*) free-body diagram of deflected segment.

Since the column under consideration here has greater resistance to buckling than the case in Fig. 30.2*e*, the $(kL)_{cr}$ here will be greater than π. We can therefore evaluate Eq. (30.13) beginning with $kL = \pi$ and increasing it slowly until the value of the left side of Eq. (30.13) changes sign. Since $(kL)_{cr}$ lies between the values of kL for which the left side of Eq. (30.13) has opposite signs, we now have bounds on $(kL)_{cr}$. To obtain improved bounds, we take the average of the two bounding values, which we will designate by $(kL)_{av}$. If the value of the left side of Eq. (30.13) obtained by using $(kL)_{av}$ is positive (negative), then $(kL)_{cr}$ lies between $(kL)_{av}$ and that value of kL for which the left side of Eq. (30.13) is negative (positive). This process is continued, using the successive values of $(kL)_{av}$ to obtain improved bounds on $(kL)_{cr}$, until the desired accuracy is obtained.

The last two equations in Eq. (30.12) imply perfect rigidity of the surrounding structure with respect to rotation. A more general result may be obtained by taking into account the elasticity of the surrounding structure in this respect. Suppose that the equivalent torsional spring constants for the surrounding structure are $K(T, 0)$ and $K(T, L)$ at $x = 0$ and $x = L$, respectively. Then Eq. (30.12) is replaced by

$$y(L) = 0$$
$$M(0) = K(T, 0)\,\frac{dy(0)}{dx} \tag{30.15}$$
$$M(L) = -K(T, L)\,\frac{dy(L)}{dx}$$

Proceeding as before, with Eq. (30.15) replacing Eq. (30.12), we obtain the following equation for kL:

$$\left\{\left[\frac{L^3}{EI(kL)^3}\right][K(T,0)+K(T,L)]K(0)-\left[\frac{L^4K(0)K(T,0)K(T,L)}{(EI)^2(kL)^3}\right]\right.$$
$$\left.+\left[\frac{K(0)L^2}{(kL)}\right]+\left[\frac{LK(T,0)K(T,L)}{EI(kL)}\right]-\left[\frac{EI(kL)}{L}\right]\right\}\sin kL$$
$$+\left\{K(T,0)+K(T,L)-\left[\frac{L^3}{EI(kL)^2}\right][K(T,0)+K(T,L)]K(0)\right\}\cos kL$$
$$+2\left[\frac{L^4K(0)K(T,0)+K(T,L)}{(EI)^2(kL)^4}\right](1-\cos kL)=0 \tag{30.16}$$

The lowest value of kL satisfying Eq. (30.16) is the $(kL)_{cr}$ to be substituted in Eq. (30.14) in order to obtain the critical load. Here there is no apparent good guess with which to begin computations. Considering the current accessibility of computers, a convenient approach would be to obtain a plot of the left side of Eq. (30.16) for $0 \le kL < \pi$, and if there is no change in sign, extend the plot up to $kL = 2\pi$, which is the solution for the column with a perfectly rigid surrounding structure (Fig. 30.2*c*).

30.4 MODIFIED BUCKLING FORMULAS

The critical-load formulas developed above provide satisfactory values of the allowable load for very slender columns for which buckling, as manifested by unacceptably large deformation, will occur within the elastic range of the material. For more massive columns, the deformation enters the plastic region (where strain increases more rapidly with stress) prior to the onset of buckling. To take into account this change in the stress-strain relationship, we modify the Euler formula. We define the *tangent modulus* $E(t)$ as the slope of the tangent to the stress-strain curve at a given strain. Then the modified formulas for the critical load are obtained by substituting $E(t)$ for E in Eq. (30.9) and Eq. (30.13) plus Eq. (30.14) or Eq. (30.16) plus Eq. (30.14). This will produce a more accurate prediction of the buckling load. However, this may not be the most desirable design approach. In general, a design which will produce plastic deformation under the operating load is undesirable. Hence, for a column which will undergo plastic deformation prior to buckling, the preferred design-limiting criterion is the onset of plastic deformation, not the buckling.

30.5 STRESS-LIMITING CRITERION

We will now develop a design criterion which will enable us to use the yield strength as the upper bound for acceptable design regardless of whether the stress at the onset of yielding precedes or follows buckling. Here we follow Ref. [30.1]. This approach has the advantage of providing a single bounding criterion that holds irrespective of the mode of failure. We begin by noting that, in general, real columns will have some imperfection, such as crookedness of the centroidal axis or eccentricity of the axial load. Figure 30.4 shows the difference between the behavior of an ideal, perfectly straight column subjected to an axial load, in which case we obtain a distinct critical point, and the behavior of a column with some imperfection.

It is clear from Fig. 30.4 that the load-deflection curve for an imperfect column has no distinct critical point. Instead, it has two distinct regions. For small axial loads, the deflection increases slowly with load. When the load is approaching the critical value obtained for a perfect column, a small increment in load produces a large change in deflection. These two regions are joined by a "knee." Thus the advent of buckling in a real column corresponds to the entry of the column into the second, above-the-knee, load-deflection region. A massive column will reach the stress at the yield point prior to buckling, so that the yield strength will be the limiting criterion for the maximum allowable load. A slender column will enter the above-the-

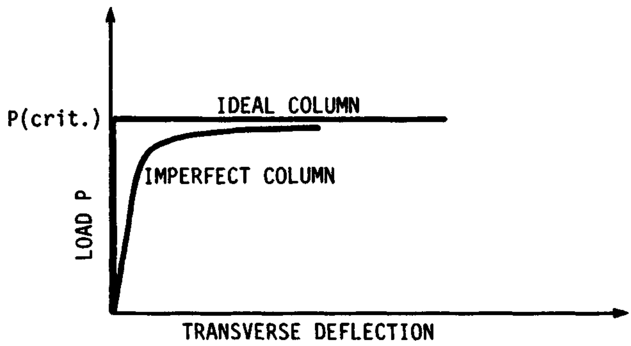

FIGURE 30.4 Typical load-deflection curves for ideal and real columns.

knee region prior to reaching the stress at the yield point, but once in the above-the-knee region, it requires only a small increment in load to produce a sufficiently large increase in deflection to reach the yield point. Thus the corresponding yield load may be used as an adequate approximation of the buckling load for a slender column as well. Hence the yield strength provides an adequate design bound for both massive and slender columns. It is also important to note that, in general, columns found in applications are sufficiently massive that the linear theory developed here is valid within the range of deflection that is of interest.

Application of Eq. (30.1) to a simply supported imperfect column with constant properties over its length yields a modification of Eq. (30.4). Thus,

$$\frac{d^2y}{dx^2} + k^2y = k^2(e - Y) \tag{30.17}$$

where e = eccentricity of the axial load P (taken as positive in the positive y direction) and Y = initial deflection (crookedness) of the unloaded column. The x axis is taken through the end points of the centroidal axis, so that Eq. (30.5) still holds and Y is zero at the end points. Note that the functions in the right side of Eq. (30.8) form a basis for a trigonometric (Fourier) series, so that any function of interest may be expressed in terms of such a series. Thus we can write

$$Y = \sum_{n=1}^{\infty} c(n) \sin \frac{n\pi x}{L} \tag{30.18}$$

where

$$c(n) = \frac{2}{L}\int_0^L Y(x) \sin \frac{n\pi x}{L}\, dx \tag{30.19}$$

The solution for the deflection y in Eq. (15.17) is given by

$$y = P\sum_{n=1}^{\infty}\left[c(n) - \frac{4e}{n\pi}\right]\frac{\sin (n\pi x/L)}{[EI(n\pi/L)^2 - P]} \tag{30.20}$$

The maximum deflection y_{max} of a simply supported column will usually (except for cases with a pronounced and asymmetrical initial deformation or antisymmetrical load eccentricity) occur at the column midpoint. A good approximation (probably within 10 percent) of y_{max} in the above-the-knee region that may be used in deflection-limited column design is given by the coefficient of the first term in Eq. (30.20):

$$y_{max} = \frac{P[c(1) - (4e/\pi)]}{EI(\pi/L)^2 - P} \tag{30.21}$$

The maximum bending moment will also usually occur at the column midpoint and at incipient yielding is closely approximated by

$$M_{max} = P\left\{e - Y_{mid} + \left[\frac{4e}{\pi} - c(1)\right]\frac{P}{[EI(\pi/L)^2 - P]}\right\} \tag{30.22}$$

The immediately preceding analysis deals with the bending moment about the z axis (normal to the paper). Clearly, a similar analysis can be made with regard to bending about the y axis (Fig. 30.1). Unlike the analysis of the perfect column, where it is merely a matter of finding the buckling load about the weaker axis, in the present approach the effects about the two axes interact in a manner familiar from analysis of an eccentrically loaded short strut. We now use the familiar expression for combining direct axial stresses and bending stresses about two perpendicular axes. Since there is no ambiguity, we will suppress the negative sign associated with compressive stress:

$$\sigma = \frac{P}{A} + \frac{M(z)c(y)}{I(z)} + \frac{M(y)c(z)}{I(y)} \tag{30.23}$$

where $c(y)$ and $c(z)$ = perpendicular distances from the z axis and y axis, respectively (these axes meet the x axis at the cross-section centroid at the origin), to the outermost fiber in compression; A = cross-sectional area of the column; and σ = total compressive stress in the fiber which is farthest removed from both the y and z axes. For an elastic design limited by yield strength, σ is replaced by the yield strength; $M(z)$ in the right side of Eq. (30.23) is the magnitude of the right side of Eq. (30.22); and $M(y)$ is an expression similar to Eq. (30.22) in which the roles of the y and z axes interchange.

Usually, in elastic design, the yield strength is divided by a chosen factor of safety η to get a permissible or allowable stress. In problems in which the stress increases linearly with the load, dividing the yield stress by the factor of safety is equivalent to multiplying the load by the factor of safety. However, in the problem at hand, it is clear from the preceding development that the stress is not a linear function of the axial load and that we are interested in the behavior of the column as it enters the above-the-knee region in Fig. 30.4. Here it is necessary to multiply the applied axial load by the desired factor of safety. The same procedure applies in introducing a factor of safety in the critical-load formulas previously derived.

Example 1. We will examine the design of a nominally straight column supporting a nominally concentric load. In such a case, a circular column cross section is the most reasonable choice, since there is no preferred direction. For this case, Eq. (30.23) reduces to

$$\sigma = \frac{P}{A} + \frac{Mc}{I} \tag{1}$$

For simplicity, we will suppose that the principal imperfection is due to the eccentric location of the load and that the column crookedness effect need not be taken into account, so that Eq. (30.22) reduces to

$$M_{\max} = P\left\{e + \left(\frac{4e}{\pi}\right)\frac{P}{[EI(\pi/L)^2 - P]}\right\} \tag{2}$$

Note that for a circular cross section of radius R, the area and moment of inertia are, respectively,

$$A = \pi R^2 \qquad \text{and} \qquad I = \frac{\pi R^4}{4} = \frac{A^2}{4\pi} \tag{3}$$

We will express the eccentricity of the load as a fraction of the cross-section radius. Thus,

$$e = \varepsilon R \tag{4}$$

Then we have, from Eqs. (1) through (4),

$$\sigma_{\text{allow}} = \frac{P}{A} + \frac{4P\varepsilon}{A}\left\{1 + \left(\frac{4}{\pi}\right)\frac{P}{[(EA^2)/(4\pi)](\pi/L)^2 - P}\right\} \tag{5}$$

Usually P and L are given, σ and E are the properties of chosen material, and ε is determined from the clearances, tolerances, and kinematics involved, so that Eq. (5) is reduced to a cubic in A.

At the moment, however, we are interested in comparing the allowable nominal column stress P/A with the allowable stress of the material σ_{allow} for columns of different lengths. Keeping in mind that the radius of gyration r of a circular cross section of geometric radius R is $R/2$, we will define

$$\frac{R}{2} = r$$

$$\frac{\sigma_{\text{allow}}}{P/A} = p \tag{6}$$

$$\frac{E}{\sigma_{\text{allow}}} = q$$

Then Eq. (5) may be written as

$$p = 1 + 4\varepsilon + \frac{16\varepsilon}{\pi[\pi^2 pq(r/L)^2 - 1]} \tag{7}$$

The first term on the right side of Eq. (7) is due to direct compressive stress; the second term is due to the bending moment produced by the load eccentricity; the third term is due to the bending moment arising from the column deflection. When ε is small, p will be close to unity unless the denominator in the third term on the right side of Eq. (7) becomes small—that is, the moment due to the column deflection becomes large. The ratio L/r, whose reciprocal appears in the denominator of the third term, is called the *slenderness ratio.* Equation (7) may be rewritten as a quadratic in p. Thus,

$$\pi^2 q\left(\frac{r}{L}\right)^2 p^2 - \left[(1 + 4\varepsilon)\pi^2 q\left(\frac{r}{L}\right)^2 + 1\right]p + (1 + 4\varepsilon) - \frac{16\varepsilon}{\pi} = 0 \tag{8}$$

We will take for q the representative value of 1000 and tabulate $1/p$ for a number of values of L/r and ε. To compare the value of $1/p$ obtained from Eq. (8) with the corresponding result from Euler's formula, we will designate the corresponding result obtained by Euler's formula as $1/p_{\text{cr}}$ and recast Eq. (30.9) as

$$\frac{1}{p_{\text{cr}}} = \pi^2 q\left(\frac{r}{L}\right)^2 \tag{9}$$

To interpret the results in Table 30.1, note that the quantities in the second and third columns of the table are proportional to the allowable loads calculated from the respective equations. As expected, the Euler formula is completely inapplicable when L/r is 50. Also, as expected, the allowable load decreases as the eccentricity increases. However, the effect of eccentricity on the allowable load decreases as the slenderness ratio L/r increases. Hence when L/r is 250, the Euler buckling load, which is the limiting case for which the eccentricity is zero, is only about 2 percent higher than when the eccentricity is 2 percent.

TABLE 30.1 Influence of Eccentricity and Slenderness Ratio on Allowable Load

ε	L/r	p^{-1}	p_{cr}^{-1}
0.02	50	0.901	3.95
	150	0.408	0.439
	250	0.155	0.158
0.05	50	0.791	3.95
	150	0.374	0.439
	250	0.151	0.158
0.10	50	0.665	3.95
	150	0.333	0.439
	250	0.145	0.158

30.6 BEAM-COLUMN ANALYSIS

A member that is subjected to both a transverse load and an axial load is frequently called a *beam-column*. To apply the immediately preceding stress-limiting criterion to a beam-column, we first determine the moment distribution, say, M_{tr}, and the corresponding deflection, say, Y_{tr}, resulting from the transverse load acting alone. Suppose that the transverse load is symmetrical about the column midpoint, and let $Y_{tr,mid}$ and $M_{tr,mid}$ be the values of Y_{tr} and M_{tr} at the column midpoint. Then the only modifications necessary in the preceding development are to replace Y by $Y + Y_{tr}$ in Eqs. (30.17) and (30.20), and to replace Y_{mid} by $Y_{mid} + Y_{tr,mid}$ and add $M_{tr,mid}$ on the right side of Eq. (30.23). If the transverse load is not symmetrical, then it is necessary to determine the maximum moment by using an approach which will now be developed.

Note that, at any point, x, the moment about the z axis is

$$M(z) = P(e - y - Y) + M(z)_{tr} \tag{30.24}$$

where Y includes the deflection due to $M(z)_{tr}$. $M(y)$ has the same form as Eq. (30.24), but with the roles of y and z interchanged. The maximum stress for any given value of x is given by Eq. (30.23). We seek to apply this equation at that value of x which yields the maximum value of σ. A method that is reasonably efficient in locating a minimum or maximum to any desired accuracy is the golden-section search. However, this method is limited to finding the minimum (maximum) of a unimodal function, that is, a function which has only one minimum (maximum) in the interval in which the search is conducted. We therefore have to conduct some exploratory calculation to find the stress at, say, a dozen points on the beam-column in order to locate the unimodal interval of interest within which to apply the golden-section search. The actual number of exploratory calculations will depend on the individual case. For example, in a simply supported case with a unimodal transverse moment, there is clearly only one maximum. But, in general, we must check enough points to be sure that a potential maximum is not overlooked.

The golden-section search procedure is as follows: Suppose that we seek the minimum value of $F(x)$ in Fig. 30.5 within the interval D (note that if we sought a maximum in Fig. 30.5, we would have to conduct two searches). We locate two points $x(1)$ and $x(2)$. The first is $0.382D$ from the left end of the interval; the second is $0.382D$

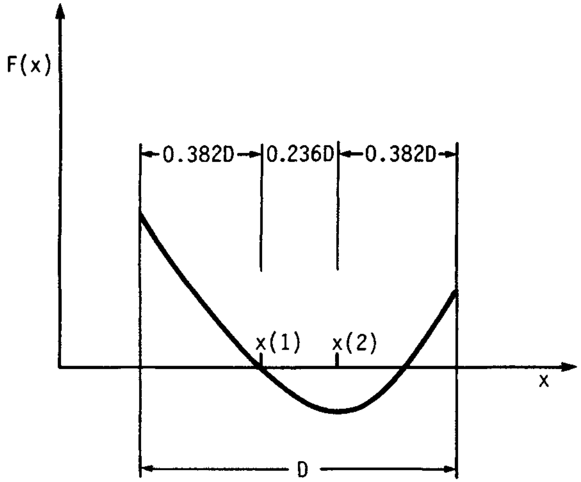

FIGURE 30.5 Finding min $F(x)$ in the interval D.

from the right end. We evaluate $F(x)$ at the chosen points. If the value of $F(x)$ at $x(1)$ is algebraically smaller than at $x(2)$, then we eliminate the subinterval between $x(2)$ and the right end of the search interval. In the oppositive event, we eliminate the subinterval between $x(1)$ and the left end. The distance of $0.236D$ between $x(1)$ and $x(2)$ equals 0.382 of the new search interval, which is $0.618D$ overall. Thus we have already located one point at 0.382 of the new search interval, and we have determined the value of $F(x)$ at that point. We need only locate a point that is 0.382 from the opposite end of the new search interval, evaluate $F(x)$ at that point, and go through the elimination procedure, as before, to further reduce the search interval. The process is repeated until the entire remaining interval is small and the value of $F(x)$ at the two points at which it is evaluated is the same within the desired accuracy. The preceding method may run into difficulty if the function $F(x)$ is flat, that is, if $F(x)$ does not vary over a substantial part of the search interval or if $F(x)$ happens to have the same value at the two points that are compared with each other. The first case is not likely to occur in the types of problems under consideration here. In the second case, we calculate $F(x)$ at another point, near one of the two points of comparison, and use the newly chosen point in place of the nearby original point. Reference [30.2] derives the golden-section search strategy and provides equations (p. 289) for predicting the number of function evaluations required to attain a specified fractional reduction of search interval or an absolute final search interval size.

30.7 APPROXIMATE METHOD

The reason why we devoted so much attention to uniform prismatic column problems is that their solution is analytically simple, so that we could obtain the results directly. In most other cases, we have to be satisfied with approximate formulations for computer (or programmable calculator) calculation of the solution. The finite-element method is the approximation method most widely used in engineering at the present time to reduce problems dealing with continuous systems, such as beams and columns, to sets of algebraic equations that can be solved on a digital computer. When there is a single independent variable involved (as in our case), the interval of interest of the independent variable is divided into a set of subintervals called *finite elements.* Within each subinterval, the solution is represented by an arc that is defined by a simple function, usually a polynomial of low degree. The curve resulting from the connected arc segments should have a certain degree of smoothness (for the problems under discussion, the deflection curve and its first derivative should be continuous) and should approximate the solution. An effective method of obtaining a good approximation to the solution is based on the mechanical energy involved in the deformation process.

30.8 INSTABILITY OF BEAMS

Beams that have rectangular cross sections with the thickness much smaller than the depth are prone to instability involving rotation of the beam cross section about the beam axis. This tendency to instability arises because such beams have low resistance to torsion about their axes. In preparation for the analysis of this problem, recall that for a circular cylindrical member of length L and radius R, subjected to an axial torque T, the angle of twist ϕ is given by

$$\phi = \frac{TL}{JG} \tag{30.25}$$

where G = shear modulus and J = polar moment of inertia of the cross section. For noncircular cross sections, the form of the right side of Eq. (30.25) does not change; the only change is in the expression for J, the torsion constant of the cross section. If the thickness of the cross section, to be denoted by t, is small and does not vary much, then J is given by

$$3J = \int_0^l t^3\, ds \tag{30.26}$$

where s = running coordinate measured from one end of the cross section and l = total developed length of the cross section. Thus, for a rectangular cross section of depth h,

$$J = \frac{ht^3}{3} \tag{30.27}$$

Clearly, J decreases rapidly as t decreases. To study the effect of this circumstance on beam stability, we will examine the deformation of a beam with rectangular cross section subjected to end moments $M(0)$, taking into account rotation of the cross section about the beam axis. The angle of rotation ϕ is assumed to be small, so that $\sin \phi$ may be replaced by ϕ and $\cos \phi$ by unity.

The equations of static equilibrium may be written from Fig. 30.6, where the moments are shown as vectors, using the right-hand rule:

$$\begin{aligned} -M'(z) + M(0) &= 0 \\ M'(y) + M(0)\phi &= 0 \\ \frac{dT}{dx} + \frac{dM'(z)}{dx} &= 0 \end{aligned} \tag{30.28}$$

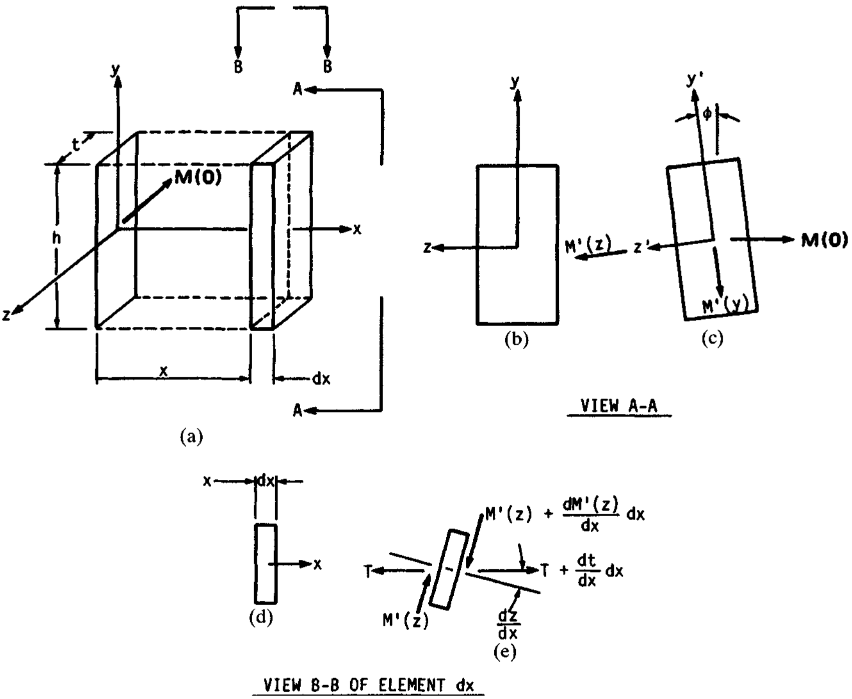

FIGURE 30.6 Instability of beams. (*a*) Segment of a beam with applied moment $M(0)$ at the ends; (*b*) cross section of undeformed beam; (*c*) cross section after onset of beam instability; (*d*) top view of undeformed differential beam element; (*e*) the element after onset of instability.

These, combined with Eq. (30.25) and the standard moment-curvature relation as given by Eq. (30.1), lead to the defining differential equation (30.29) for the angle ϕ:

$$GJ\frac{d^2\phi}{dx^2} + M^2(0)\phi\left[\frac{1}{EI(y)} - \frac{1}{EI(z)}\right] = 0 \tag{30.29}$$

Suppose that the end faces of the beam are fixed against rotation about the x axis. Then the boundary conditions are

$$\phi(0) = \phi(L) = 0 \tag{30.30}$$

Noting the similarity between Eq. (30.4) with its boundary conditions Eq. (30.5) and Eq. (30.29) with its boundary conditions Eq. (30.30), the similarity of the solutions is clear. Thus we obtain the expression for $M(0)_{cr}$:

$$[M(0)_{cr}]^2 = \frac{\pi^2 E}{L^2} GJ\left[\frac{I(z)I(y)}{I(z) - I(y)}\right] \tag{30.31}$$

It may be seen from Eq. (30.31) that if the torsion constant J is small, the critical moment is small. In addition, it may be seen from the bracketed term in Eq. (30.31) that as the cross section approaches a square shape, the denominator becomes small, so that the critical moment becomes very large. The disadvantage of a square cross section is, of course, well known. To demonstrate it explicitly, we rewrite the expression for stress in a beam with rectangular cross section t by h:

$$\sigma = \frac{M(h/2)}{th^3/12} \tag{30.32}$$

in the form

$$th = \frac{6M}{\sigma h} \tag{30.33}$$

Thus, for given values of M and σ, the cross-sectional area required decreases as we increase h. Hence the role of Eq. (30.31) is to define the constraint on the maximum allowable depth-to-thickness ratio. The situation is similar for flanged beams. Here we have obtained the results for a simple problem to illustrate the disadvantage involved and the caution necessary in designing beams with thin-walled open cross sections. Implicit in this is the advantage of using, when possible, closed cross sections, such as box beams, which have a high torsional stiffness.

As we have seen, when the applied moment is constant over the entire length of the beam, the problem of definition and its solution have the same form as for the column-buckling problem. We can also have similar types of boundary conditions. The boundary conditions used in Eq. (30.30) correspond to a simply supported column. If dy/dx and dz/dx are equal to zero at $x = 0$ for all y and z, then $d\phi/dx$ is equal to zero at $x = 0$. If this condition is combined with $\phi(0) = 0$, then we have the equivalent of a clamped column end. Hence we can use here the concept of equivalent beam length in the same manner as we used the equivalent column length before. In case a beam is subjected to transverse loads, so that the applied moment varies with x, the problem is more complex. For the proportioning of flanged beams, Ref. [30.3] should be used as a guide. This reference deals with structural applications, so that the size range of interest dealt with is different from the size range of interest in machine design. But the underlying principles of beam stability are the same, and the proportioning of the members should be similar.

Example 2. We will examine the design of a beam of length L and rectangular cross section t by h. The beam is subjected to an applied moment M (we will not use any modifying symbols here, since there is no ambiguity), which is constant over the length of the beam. As noted previously, the required cross-sectional area th will decrease as h is increased. We take the allowable stress in the material to be σ. The calculated stress in the beam is not to exceed this value. Thus,

$$\sigma \geq \frac{M(h/2)}{th^3/12} \qquad \sigma \geq \frac{6M}{th^2} \qquad th \geq \frac{6M}{h\sigma} \tag{1}$$

We want the cross-sectional area th as small as possible. Hence h should be as large as possible. We can, therefore, replace the inequality in Eq. (1) by the equality

$$th = \frac{6M}{h\sigma} \tag{2}$$

The maximum value of h that we can use is subject to a constraint based on Eq. (30.31). We will use a factor of safety η in this connection. Thus,

$$(\eta M)^2 \leq \frac{\pi^2 E}{L^2} GJ\left[\frac{I(z)I(y)}{I(z) - I(y)}\right] \tag{3}$$

Here

$$I(z) = \frac{th^3}{12} \qquad I(y) = \frac{ht^3}{12} \qquad J = \frac{ht^3}{3} \tag{4}$$

Using Eq. (4), we may write Eq. (3) as

$$(\eta M)^2 \le \frac{\pi^2 EG}{(6L)^2}\left(\frac{t}{h}\right)^2\left[\frac{1}{1-(t/h)^2}\right](th)^4 \tag{5}$$

or, from Eq. (2),

$$(\eta M)^2 \le \frac{\pi^2 EG}{(6L)^2}\left(\frac{6M}{h^3\sigma}\right)^2\left[\frac{1}{1-[(6M/(h^3\sigma)]^2}\right]\left(\frac{6M}{h\sigma}\right)^4 \tag{6}$$

Since we seek to minimize th and maximize h, it may be seen from Eqs. (5) and (6) that the inequality sign may be replaced by the equality sign in those two equations. In Eq. (6), h is the only unspecified quantity. Further, since the square of t/h may be expected to be small compared to unity, we can obtain substantially simpler approximations of reasonable accuracy. As a first step, we have

$$\frac{1}{1-(t/h)^2} = 1 + \left(\frac{t}{h}\right)^2 + \left(\frac{t}{h}\right)^4 + \cdots \tag{7}$$

If we retain only the first two terms in the right side of Eq. (7), we have

$$\eta M = \frac{\pi(EG)^{1/2}}{6L}\left[1 + \left(\frac{6M}{h^3\sigma}\right)^2\right]\left(\frac{6M}{h^3\sigma}\right)\left(\frac{6M}{h\sigma}\right)^2 \tag{8}$$

If we also neglect the square of t/h in comparison with unity, we obtain

$$h = \left[\frac{\pi(6M)^2(EG)^{1/2}}{\eta L\sigma^3}\right]^{1/5} \tag{9}$$

as a reasonable first approximation. Thus if we take the factor of safety η as 1.5, we have, for a steel member with $E = 30$ Mpsi, $G = 12$ Mpsi, and $\sigma = 30$ kpsi,

$$h = \left[\frac{\pi(36)[(30\times 10^6)(12\times 10^6)]^{1/2}}{(1.5)(30\,000)}\,\frac{M^2}{L}\right]^{1/5} = 8.62\left(\frac{M^2}{L}\right)^{1/5} \tag{10}$$

This is a reasonable approximation to the optimal height of the beam cross section. It may also be used as a starting point for an iterative solution to the exact expression, Eq. (6). For the purpose of iteration, we rewrite Eq. (6) as

$$h = \left\{\frac{\pi(6M)^2}{\eta L\sigma^3}\left[\frac{EG}{1-[(6M)/(h^3\sigma)]^2}\right]^{1/2}\right\}^{1/5} \tag{11}$$

The value of h obtained from Eq. (9) is substituted into the right side of Eq. (11). The resultant value of h thus obtained is then resubstituted into the right side of Eq. (11); the iterative process is continued until the computed value of h coincides with the value substituted into the right side to the desired degree of accuracy. Having determined h, we can determine t from Eq. (2).

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