CHAPTER 37
DEFLECTION

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GLOSSARY OF SYMBOLS

a	Dimension
A	Area
b	Dimension
C	Constant
D, d	Diameter
E	Young's modulus
F	Force
G	Shear modulus
I	Second moment of area
J	Second polar moment of area
k	Spring rate
K	Constant
ℓ	Length
M	Moment
N	Number

q	Unit load
Q	Fictitious force
R	Support reaction
T	Torque
U	Strain energy
V	Shear force
w	Unit weight
W	Total weight
x	Coordinate
y	Coordinate
δ	Deflection
θ	Slope, torsional deflection

37.1 STIFFNESS OR SPRING RATE

The *spring rate* (also called *stiffness* or *scale*) of a body or ensemble of bodies is defined as the partial derivative of force (torque) with respect to colinear displacement (rotation). For a helical tension or compression spring,

$$F = \frac{d^4 G y}{8D^3 N} \qquad \text{thus} \qquad k = \frac{\partial F}{\partial y} = \frac{d^4 G}{8D^3 N} \tag{37.1}$$

where D = mean coil diameter
d = wire diameter
N = number of active turns

In a round bar subject to torsion,

$$T = \frac{GJ\theta}{\ell} \qquad \text{thus} \qquad k = \frac{\partial T}{\partial \theta} = \frac{GJ}{\ell} \tag{37.2}$$

and the tensile force in an elongating bar of any cross section is

$$F = \frac{AE\delta}{\ell} \qquad \text{thus} \qquad k = \frac{\partial F}{\partial \delta} = \frac{AE}{\ell} \tag{37.3}$$

If k is constant, as in these cases, then displacement is said to be linear with respect to force (torque). For contacting bodies with all four radii of curvature finite, the approach of the bodies is proportional to load to the two-thirds power, making the spring rate proportional to load to the one-third power. In hydrodynamic film bearings, the partial derivative would be evaluated numerically by dividing a small change in load by the displacement in the direction of the load.

37.2 DEFLECTION DUE TO BENDING

The relations involved in the bending of beams are well known and are given here for reference purposes as follows:

$$\frac{q}{EI}=\frac{d^4y}{dx^4} \tag{37.4}$$

$$\frac{V}{EI}=\frac{d^3y}{dx^3} \tag{37.5}$$

$$\frac{M}{EI}=\frac{d^2y}{dx^2} \tag{37.6}$$

$$\theta=\frac{dy}{dx} \tag{37.7}$$

$$y=f(x) \tag{37.8}$$

These relations are illustrated by the beam of Fig. 37.1. Note that the x axis is *positive* to the right and the y axis is *positive* upward. All quantities—loading, shear force, support reactions, moment, slope, and deflection—have the same sense as y; they are positive if upward, negative if downward.

37.3 PROPERTIES OF BEAMS

Table 37.1 lists a number of useful properties of beams having a variety of loadings. These must all have the same cross section throughout the length, and a linear relation must exist between the force and the deflection. Beams having other loadings can be solved using two or more sets of these relations and the principle of superposition.

In using Table 37.1, remember that the deflection at the center of a beam with off-center loads is usually within 2.5 percent of the maximum value.

37.4 ANALYSIS OF FRAMES

Castigliano's theorem is presented in Chap. 38, and the energy equations needed for its use are listed in Table 38.2. The method can be used to find the deflection at any point of a frame such as the one shown in Fig. 37.2. For example, the deflection δ_C at C in the direction of F_2 can be found using Eq. (38.2) as

$$\delta_C=\frac{\partial U}{\partial F_2} \tag{37.9}$$

where U = the strain energy stored in the entire frame due to all the forces. If the deflection is desired in another direction or at a point where no force is acting, then a fictitious force Q is added to the system at that point and in the direction in which

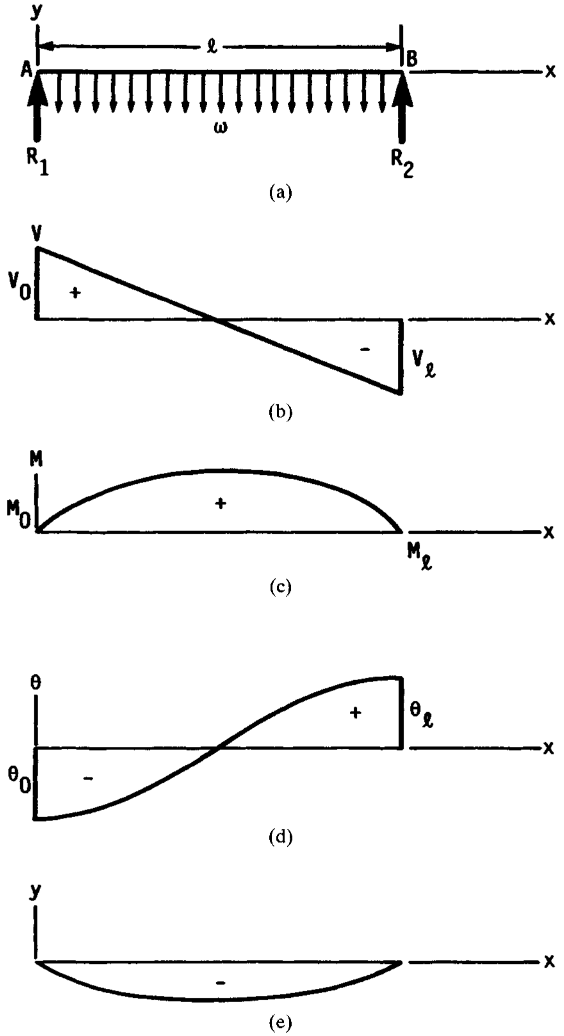


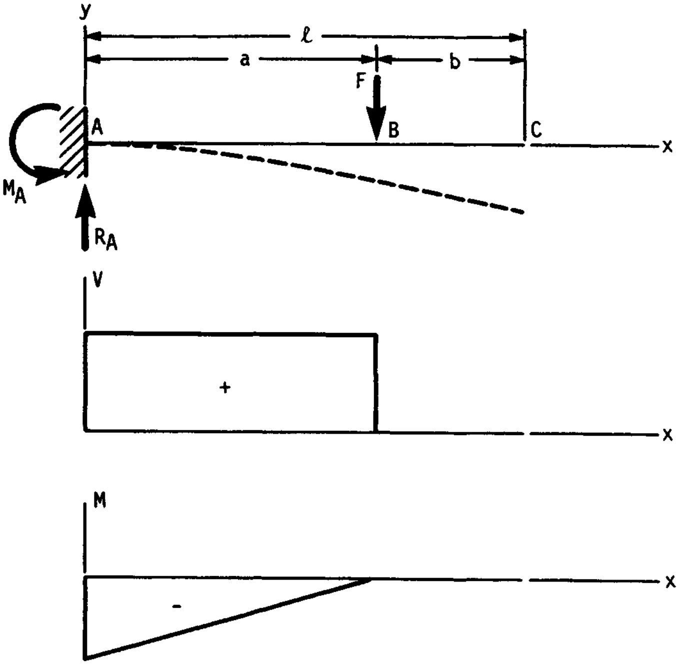
FIGURE 37.1 (*a*) Loading diagram showing beam supported at *A* and *B* with uniform load *w* having units of force per unit length, $R_1 = R_2 = w\ell/2$; (*b*) shear-force diagram showing end conditions; (*c*) moment diagram; (*d*) slope diagram; (*e*) deflection diagram.

the deflection is desired. After the partial derivatives have been found, Q is equated to zero, and the remaining terms give the wanted deflection.

The first step in using the method is to make a force analysis of each member of the frame. If Eq. (*a*) is to be solved, then the numerical values of F_1 and F_2 can be used in the force analysis, but the value of F_2 must *not* be substituted until after each

TABLE 37.1 Properties of Beams

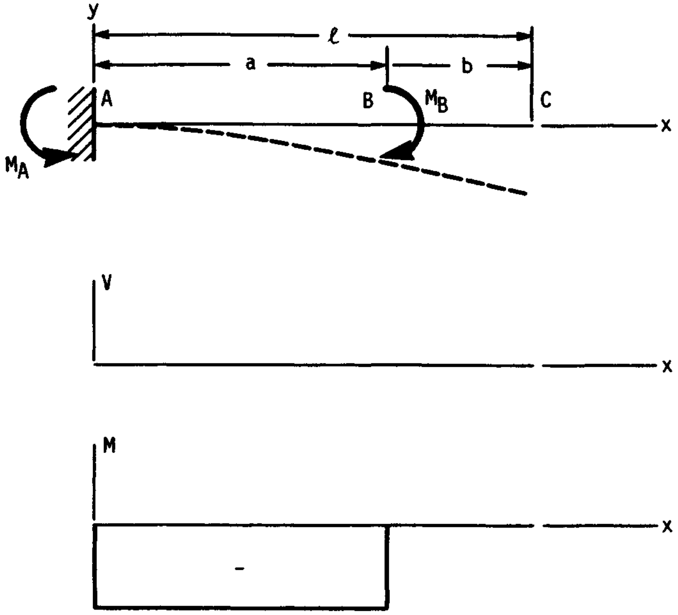
1. Cantilever—intermediate load



$R_A = F \qquad M_A = -Fa$

$y_B = -\dfrac{Fa^3}{3EI} \qquad y_C = -\dfrac{Fa^3}{3EI}\left(1 + \dfrac{3b}{2a}\right)$

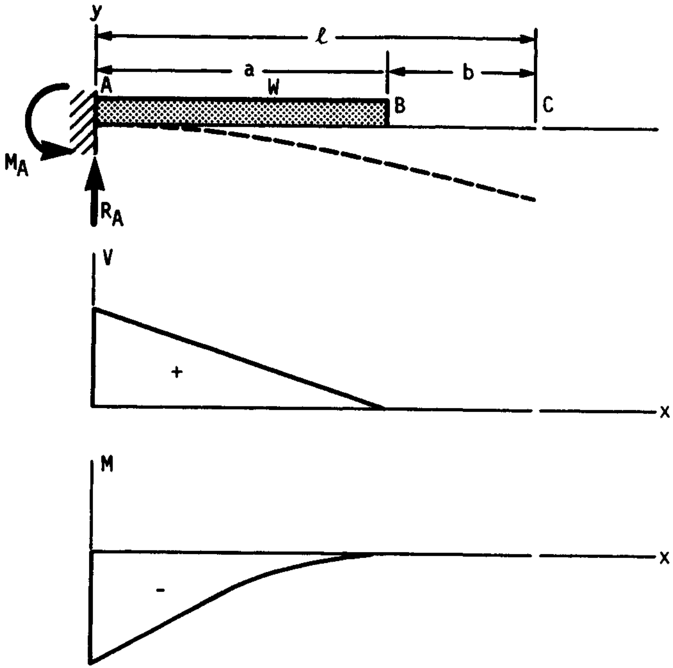
2. Cantilever—intermediate couple



$V = 0 \qquad M_A = M$

$y_B = -\dfrac{Ma^2}{2EI} \qquad y_C = -\dfrac{Ma^2}{2EI}\left(1 + \dfrac{2b}{a}\right)$

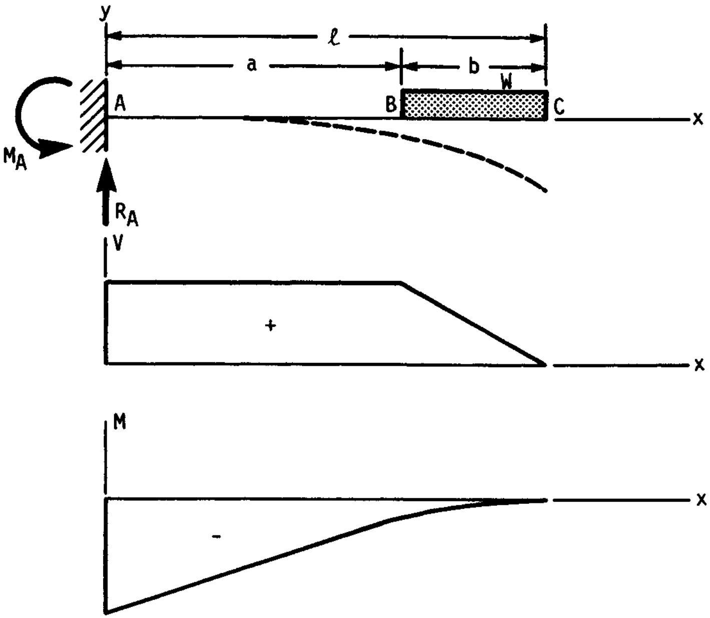
3. Cantilever—distributed load



$$R_A = W \qquad M_A = -\frac{Wa}{2}$$

$$y_B = -\frac{Wa^3}{8EI} \qquad y_C = -\frac{Wa^3}{8EI}\left(1 + \frac{4b}{3a}\right)$$

4. Cantilever—partial distributed load

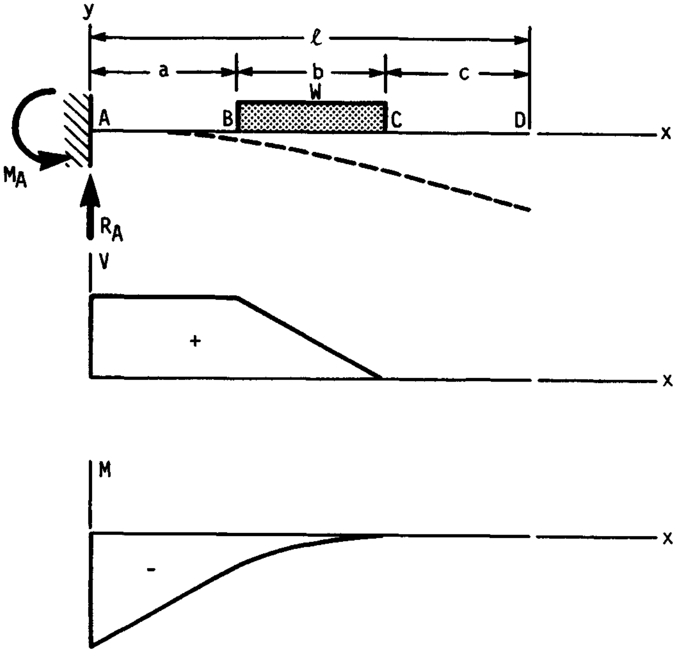


$$R_A = W \qquad M_A = -W\left(a + \frac{b}{2}\right)$$

$$y_C = -\frac{W}{24EI}(8a^3 + 18a^2b + 12ab^2 + 3b^3)$$

TABLE 37.1 Properties of Beams (*Continued*)

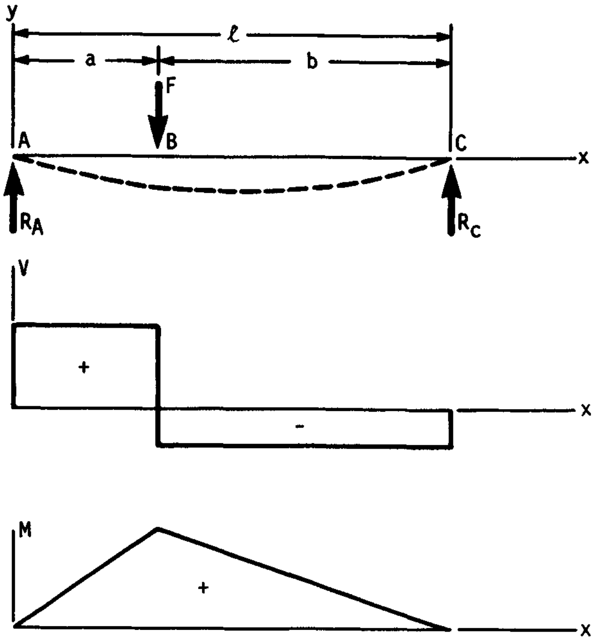
5. Cantilever—partial distributed load



$$R_A = W \qquad M_A = -W\left(a + \frac{b}{2}\right)$$

$$y_D = -\frac{W}{24EI}(8a^3 + 18a^2b + 12ab^2 + 3b^3 + 12a^2c + 12abc + 4b^2c)$$

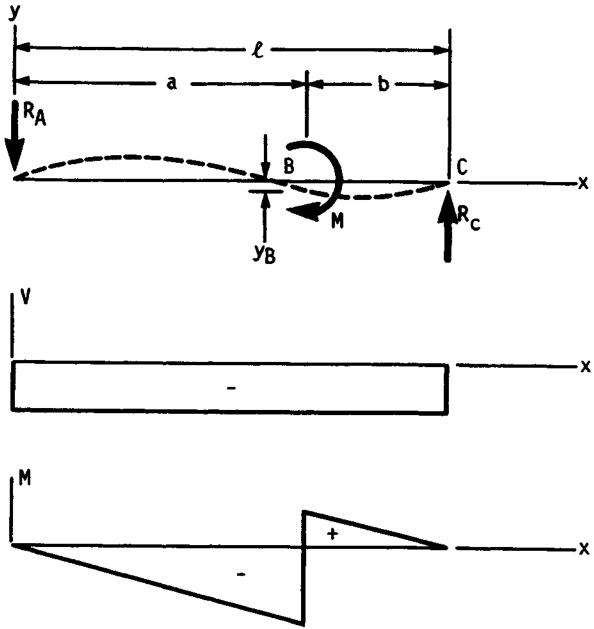
6. Simple support—intermediate load



$$R_A = \frac{Fb}{\ell} \qquad R_B = \frac{Fa}{\ell} \qquad M_B = \frac{Fab}{\ell}$$

At center $y = -\frac{F\ell^3}{48EI}\left(\frac{3a}{\ell} - \left(\frac{4a}{\ell}\right)^3\right]$

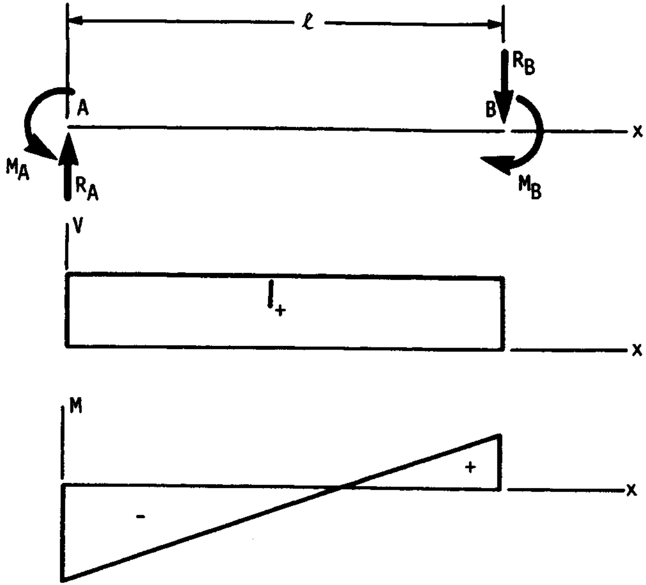
7. Simple support—intermediate couple



$R_A = -R_C = -\frac{M}{\ell}$ $\quad M_{AB} = R_A x$

$y_B = -\frac{Mab}{3EI\ell}(a - b) \quad a > b$

8. Simple support—end moments



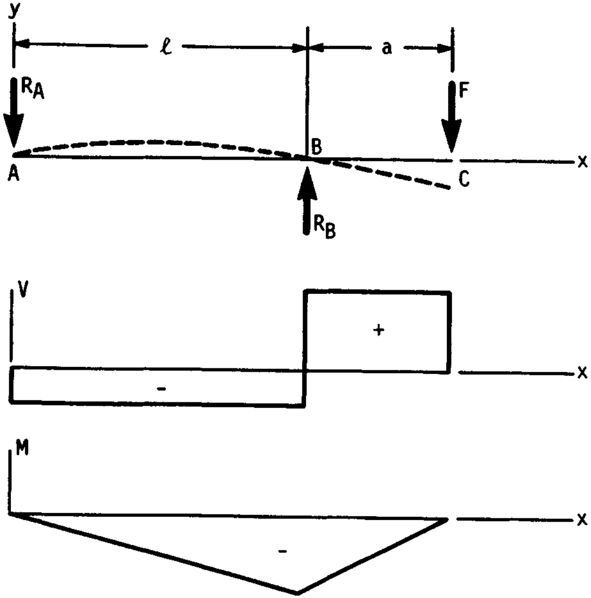
M_B positive, M_A negative, $|M_A| > |M_B|$

$R_A = -R_B = -\frac{M_A + M_B}{\ell}$

when $|M_A| = |M_B|$ $\quad y_{\max} = \frac{M\ell^2}{8EI}$

TABLE 37.1 Properties of Beams (*Continued*)

9. Simple support—overhung load

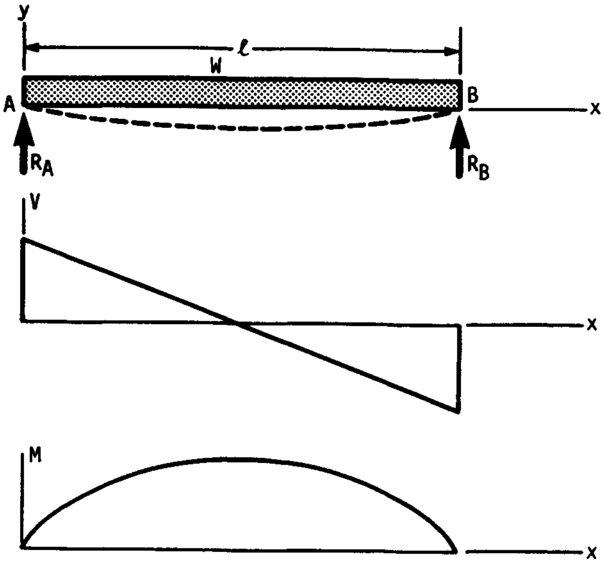


$R_A = -\frac{Fa}{\ell} \qquad R_B = \frac{F}{\ell}(\ell + a)$

$M_B = -Fa$

$y_C = -\frac{Fa^2}{3EI}(\ell + a)$

10. Simple support— uniform loading

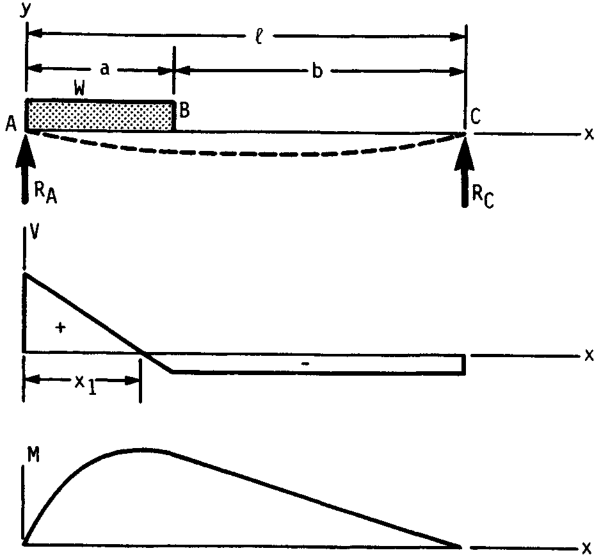


$R_A = R_B = \frac{W}{2} \qquad M_{\max} = \frac{W\ell}{8}$

$y_{\max} = -\frac{5W\ell^3}{384EI}$

TABLE 37.1 Properties of Beams (*Continued*)

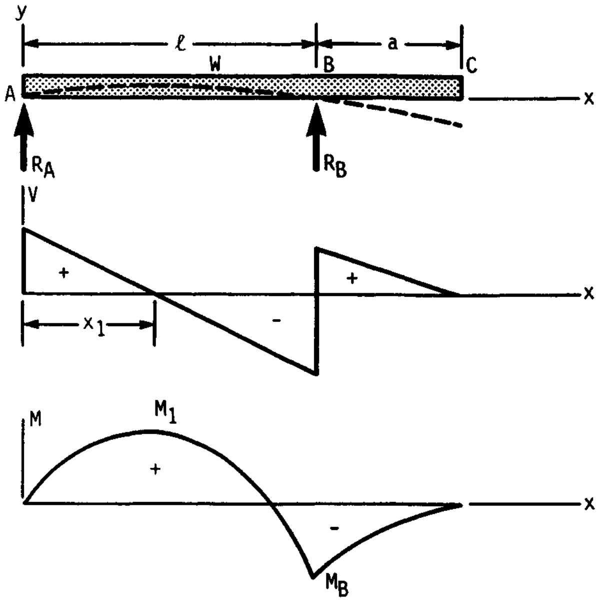
11. Simple support—partial uniform loading



$$R_A = \frac{W}{2\ell}(2\ell - a) \qquad R_B = \frac{Wa}{2\ell} \qquad x_1 = \frac{a}{2\ell}(2\ell - a)$$

$$\text{At center } y = -\frac{Wa}{48EI}(a^2 + 2\ell^2) \qquad a < \frac{\ell}{2}$$

12. Simple support—uniform loading, overhung

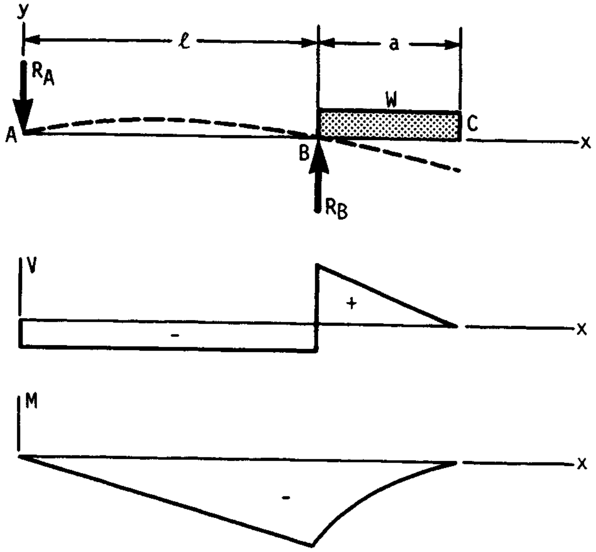


$$R_A = \frac{W}{2\ell}(\ell - a) \qquad R_B = \frac{W}{2\ell}(\ell + a) \qquad x_1 = \frac{1}{2\ell}(\ell^2 - a^2)$$

$$M_1 = \frac{W}{8\ell^2}(\ell + a)(\ell - a)^2 \qquad M_B = -\frac{Wa^2}{2(\ell + a)}$$

$$y_C = \frac{Wa}{24EI}(3a^2 + a\ell - \ell^2)$$

13. Simple support—overhung uniform load

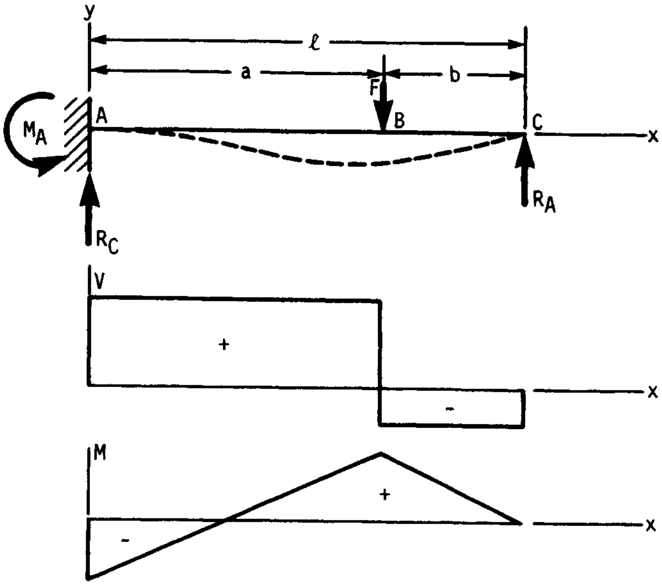


$$R_A = -\frac{Wa}{2\ell} \qquad R_B = \frac{W}{2\ell}(2\ell + a) \qquad M_{\max} = -\frac{Wa}{2}$$

$$y_{\max} = \frac{0.032 Wa\ell^2}{EI} \quad \text{between supports}$$

$$y_C = -\frac{Wa^2}{24EI}(4\ell + 3a)$$

14. Fixed and simple support—intermediate load

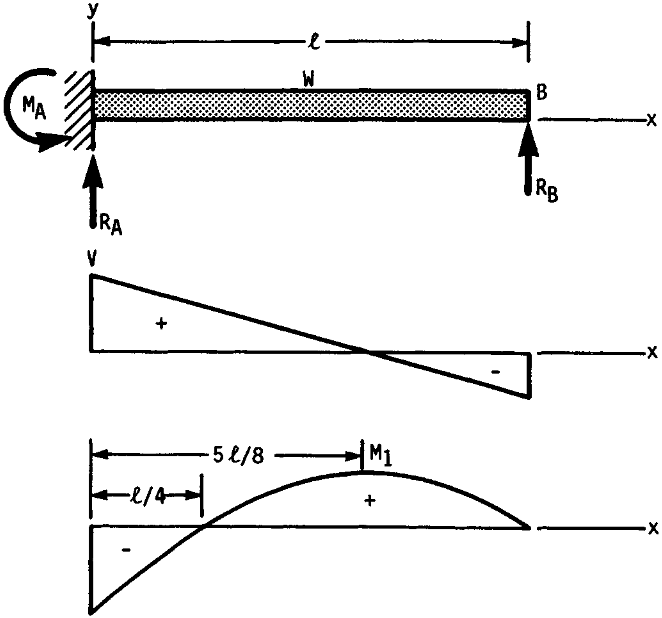


$$R_A = \frac{Fb}{2\ell^3}(3\ell^2 - b^2) \qquad R_C = \frac{Fa^2}{2\ell^3}(3\ell - a)$$

$$M_A = \frac{Fb}{2\ell^2}(b^2 - \ell^2) \qquad M_B = \frac{Fa^2b}{2\ell^3}(3\ell - a)$$

$$y_B = \frac{Fba^2}{12EI\ell^3}(3b^2\ell - 3\ell^3 + 3a\ell^2 - ab^2)$$

15. Fixed and simple support—uniform load

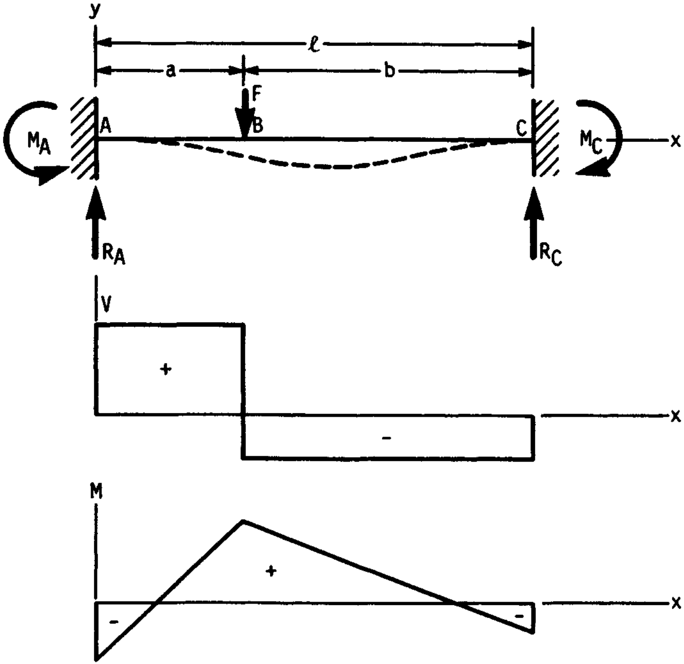


$R_A = \frac{5W}{8} \qquad R_B = \frac{3W}{8}$

$M_A = -\frac{W\ell}{8} \qquad M_1 = \frac{9W\ell}{128}$

$y_{\max} = -\frac{W\ell^3}{185EI}$

16. Fixed supports—intermediate load



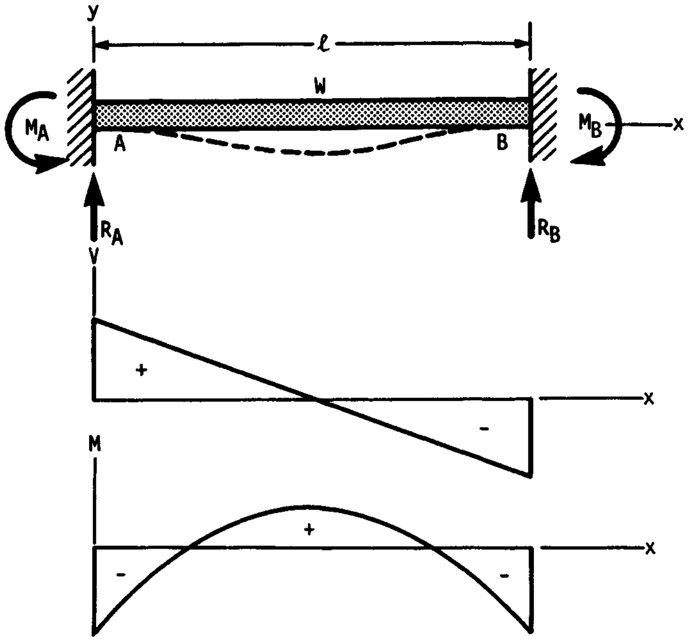
$R_A = \frac{Fb^2}{\ell^3}(3a + b) \qquad R_B = \frac{Fa^2}{\ell^3}(3b + a)$

$M_A = -\frac{Fab^2}{\ell^2} \qquad M_B = \frac{Fab^2}{\ell^3}(3a + b - \ell)$

$M_C = -\frac{Fa^2b}{\ell^2} \qquad y_B = \frac{Fa^3b^2}{6EI\ell^3}(3a + b - 3\ell)$

TABLE 37.1 Properties of Beams (*Continued*)

17. Fixed supports—uniform load



$$R_A = R_B = \frac{W}{2} \qquad M_A = M_B = -\frac{W\ell}{12}$$

$$\text{At } x = \frac{\ell}{2} \qquad M = \frac{W\ell}{24}$$

$$y_{\max} = -\frac{W\ell^3}{384EI}$$

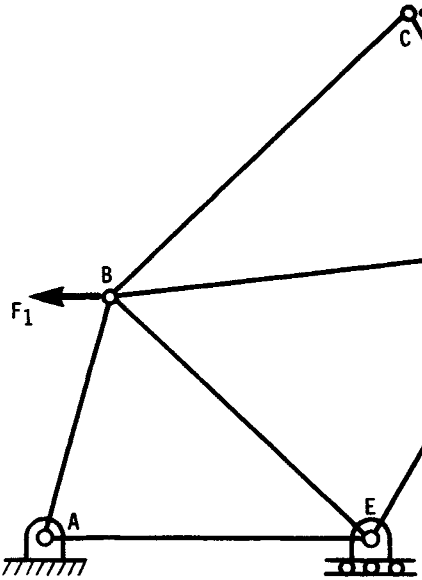


FIGURE 37.2 Frame loaded by three forces.

member has been analyzed and the partial derivatives obtained. The following example demonstrates the technique.

Example 1. Find the downward deflection of point D of the frame shown in Fig. 37.3.

Solution. A force analysis of the system gives an upward reaction at E of $R_E = 225 + 3F_2$. The reaction at A is downward and is $R_A = 75 - 2F_2$.

The strain energy for member CE is

$$U_{CE} = \frac{R_A^2 \ell}{2AE} \tag{1}$$

The partial deflection is taken with respect to F_2 because the deflection at D in the direction of F_2 is desired. Thus

$$\frac{\partial U_{CE}}{\partial F_2} = \frac{2R_A \ell}{2AE} \frac{\partial R_A}{\partial F_2} \tag{2}$$

Also,

$$\frac{\partial R_A}{\partial F_2} = -2$$

Thus Eq. (2) becomes

$$\frac{\partial U_{CE}}{\partial F_2} = \frac{(75 - 2F_2)(30)}{0.2E} (-2) = \frac{37\,500}{E} \tag{3}$$

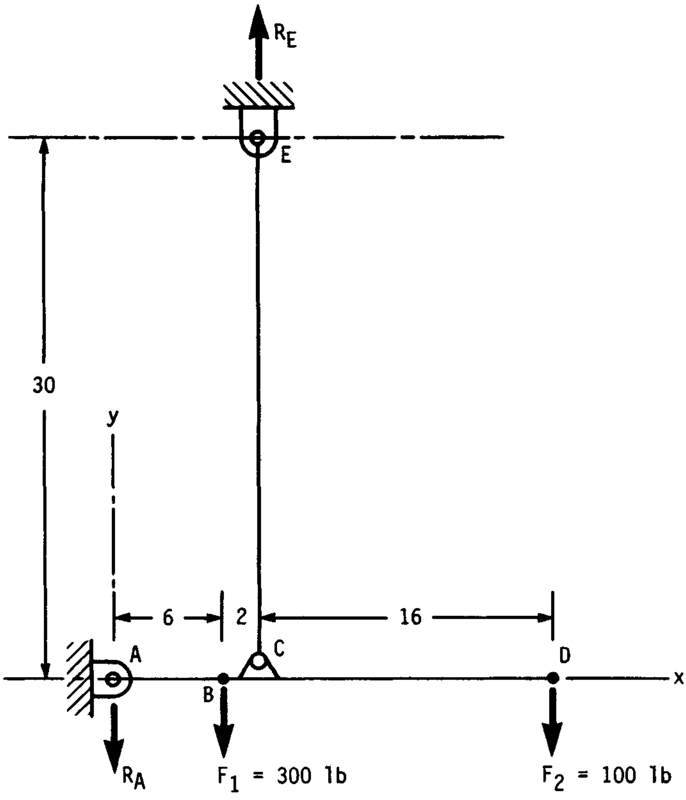


FIGURE 37.3 Frame loaded by two forces. Dimensions in inches: $A_{CE} = 0.20$ in^2; $I_{AD} = 0.18$ in^4; $E = 30 \times 10^6$ psi.

Note that we were able to substitute the value of F_2 in Eq. (3) because the partial derivative had been taken.

The strain energy stored in member *ABCD* will have to be computed in three parts because of the change in direction of the bending moment diagram at points *B* and *C*. For part *AB*, the moment is

$$M_{AB} = R_A x = (75 - 2F_2)x$$

The strain energy is

$$U_{AB} = \int_0^6 \frac{M_{AB}^2}{2EI}\,dx \tag{4}$$

Taking the partial derivative with respect to F_2 as before gives

$$\frac{\partial U_{AB}}{\partial F_2} = \int_0^6 \frac{2M_{AB}}{2EI}\,\frac{\partial M_{AB}}{\partial F_2}\,dx \tag{5}$$

But

$$\frac{\partial M_{AB}}{\partial F_2} = -2x \tag{6}$$

Therefore, Eq. (5) may be written

$$\frac{\partial U_{AB}}{\partial F_2} = \frac{1}{EI}\int_0^6 x(75 - 2F_2)(-2x)\,dx$$

$$= \frac{1}{0.18E}\int_0^6 250x^2\,dx = \frac{100\,000}{E} \tag{7}$$

where the value of F_2 again has been substituted after taking the partial derivative.

For section *BC*, we have

$$M_{BC} = R_A x - F_1(x - 6) = 1800 - 225x - 2F_2 x$$

$$\frac{\partial M_{BC}}{\partial F_2} = -2x$$

$$\frac{\partial U_{BC}}{\partial F_2} = \int_6^8 \frac{2M_{BC}}{2EI}\frac{\partial M_{BC}}{\partial F_2}\,dx$$

$$= \frac{1}{EI}\int_6^8 (1800 - 225x - 2F_2 x)(-2x)\,dx$$

$$= \frac{1}{0.18E}\int_6^8 (-3600x + 850x^2)\,dx = \frac{145\,926}{E}$$

Finally, section *CD* yields

$$M_{CD} = -(24 - x)F_2 \qquad \frac{\partial M_{CD}}{\partial F_2} = -(24 - x)$$

$$\frac{\partial U_{CD}}{\partial F_2} = \int_8^{24} \frac{2M_{CD}}{2EI}\frac{\partial M_{CD}}{\partial F_2}\,dx$$

$$= \frac{1}{EI}\int_8^{24} F_2(24 - x)^2\,dx$$

$$= \frac{1}{0.18E}\int_8^{24} (57\,600 - 4800x + 100x^2)\,dx$$

$$= \frac{758\,519}{E}$$

Then

$$y_D = \frac{\partial U_{CE}}{\partial F_2} + \frac{\partial U_{AB}}{\partial F_2} + \frac{\partial U_{BC}}{\partial F_2} + \frac{\partial U_{CD}}{\partial F_2}$$

$$= \frac{1}{30(10)^6}(37\,500 + 100\,000 + 145\,926 + 758\,519)$$

$$= 0.0347 \text{ in} \qquad \text{(when rounded)}$$

37.4.1 Redundant Members

A frame consisting of one or more redundant members is statically indeterminate because the use of statics is not sufficient to determine all the reactions. In this case, Castigliano's theorem can be used first to determine these reactions and second to determine the desired deflection.

Let R_1, R_2, and R_3 be a set of three indeterminate reactions. The deflection at the supports must be zero, and so Castigliano's theorem can be written three times. Thus

$$\frac{\partial U}{\partial R_1}=0 \qquad \frac{\partial U}{\partial R_2}=0 \qquad \frac{\partial U}{\partial R_3}=0 \tag{37.10}$$

and so the number of equations to be solved is the same as the number of indeterminate reactions.

In setting up Eqs. (37.10), *do not* substitute the numerical value of the particular force corresponding to the desired deflection. This force symbol must appear in the reaction equations because the partial derivatives must be taken with respect to this force when the deflection is found. The method is illustrated by the following example.

Example 2. Find the downward deflection at point D of the frame shown in Fig. 37.4.

Solution. Choose R_B as the statically indeterminate reaction. A static force analysis then gives the remaining reactions as

$$R_A = R_C = 0.625(F - R_B) \tag{1}$$

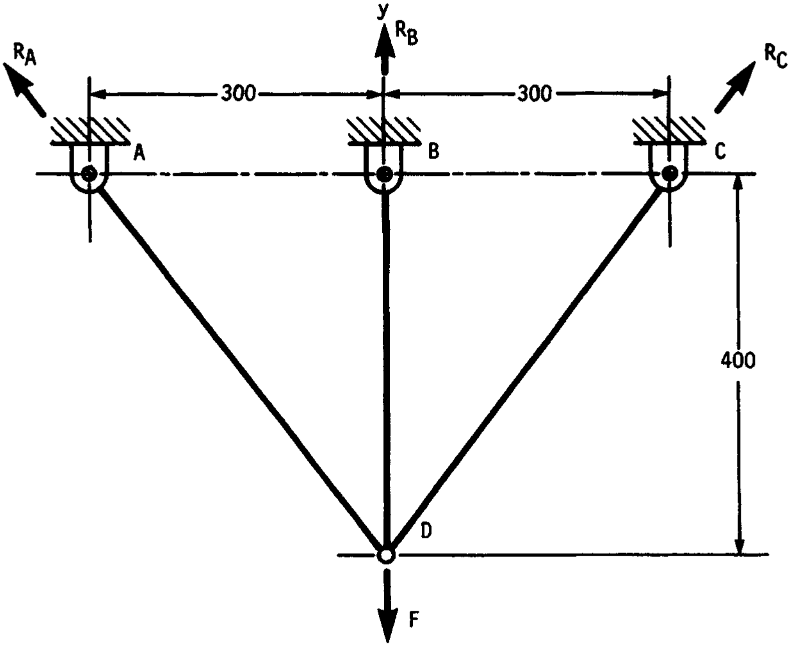


FIGURE 37.4 Frame loaded by a single force. Dimensions in millimeters: $A_{AD} = A_{CD} = 2\ \text{cm}^2$, $A_{BD} = 1.2\ \text{cm}^2$, $E = 207$ GPa, $F = 20$ kN.

The frame consists only of tension members, so the strain energy in each member is

$$U_{AD} = U_{DC} = \frac{R_A^2 \ell_{AD}}{2A_{AD}E} \qquad U_{BD} = \frac{R_B^2 \ell_{BD}}{2A_{BD}E} \tag{2}$$

Using Eq. (37.10), we now write

$$0 = \frac{\partial U}{\partial R_B} = \frac{2R_A \ell_{AD}}{A_{AD}E} \frac{\partial R_A}{\partial R_B} + \frac{R_B \ell_{BD}}{A_{BD}E} \frac{\partial R_B}{\partial R_B} \tag{3}$$

Equation (1) gives $\partial R_A/\partial R_B = -0.625$. Also, $\partial R_B/\partial R_B = 1$. Substituting numerical values in Eq. (3), except for F, gives

$$\frac{2(0.625)(F - R_B)(500)(-0.625)}{2(207)} + \frac{R_B(400)(1)}{1.2(207)} = 0 \tag{4}$$

Solving gives $R_B = 0.369F$. Therefore, from Eq. (1), $R_A = R_C = 0.394F$. This completes the solution of the case of the redundant member. The next problem is to find the deflection at D.

Using Eq. (2), again we write

$$y_D = \frac{\partial U}{\partial F} = \frac{2R_A \ell_{AD}}{A_{AD}E} \frac{\partial R_A}{\partial F} + \frac{R_B \ell_{BD}}{A_{BD}E} \frac{\partial R_B}{\partial F} \tag{5}$$

For use in this equation, we note that $\partial R_A/\partial F = 0.394$ and $\partial R_B/\partial F = 0.369$. Having taken the derivatives, we can now substitute the numerical value of F. Thus Eq. (5) becomes[†]

[†] In general, when using metric quantities, prefixed units are chosen so as to produce number strings of not more than four members. Thus some preferred units in SI are MPa (N/mm^2) for stress, GPa for modulus of elasticity, mm for length, and, say, cm^4 for second moment of area.

People are sometimes confused when they encounter an equation containing a number of mixed units. Suppose we wish to solve a deflection equation of the form

$$y = \frac{64F\ell^3}{3\pi d^4 E}$$

where $F = 1.30$ kN, $\ell = 300$ mm, $d = 2.5$ cm, and $E = 207$ GPa. Form the equation into two parts, the first containing the numbers and the second containing the prefixes. This converts everything to base units, including the result. Thus,

$$y = \frac{64(1.30)(300)^3}{3\pi(2.5)^4(207)} \frac{(\text{kilo})(\text{milli})^3}{(\text{centi})^4(\text{giga})}$$

Now compute the numerical value of the first part and substitute the prefix values in the second. This gives

$$y = (29.48 \times 10^3)\left[\frac{10^3(10^{-3})^3}{(10^{-2})^4(10^9)}\right] = 29.48 \times 10^{-4} \text{ m}$$

$$= 2.948 \text{ mm}$$

Note that we multiplied the result by 10^3 mm/m to get the answer in millimeters. When this approach is used with Eq. (5), it is found that the result must be multiplied by $(10)^{-2}$ to get y in millimeters.

$$y_D = \left\{ \frac{2[0.394(20)](500)(0.394)}{2(207)} + \frac{[0.369(20)](400)(0.369)}{1.2(207)} \right\} 10^{-2}$$

$$= 0.119 \text{ mm}$$

If care is taken to refrain from substituting numerical values for reactions or forces until after partial derivatives are taken, Castigliano's theorem is applicable to statically indeterminate frames containing redundant members.