

## CHAPTER 6

# Three-dimensional Flows in Axial Turbomachines

*It cost much labour and many days before all these things were brought to perfection. (DEFOE, Robinson Crusoe.)*

### Introduction

IN CHAPTERS 4 and 5 the fluid motion through the blade rows of axial turbomachines was assumed to be two-dimensional in the sense that radial (i.e. spanwise) velocities did not exist. This is a not unreasonable assumption for axial turbomachines of high hub-tip ratio. However, with hub-tip ratios less than about 4/5, radial velocities through a blade row may become appreciable, the consequent redistribution of mass flow (with respect to radius) seriously affecting the outlet velocity profile (and flow angle distribution). It is the temporary imbalance between the strong centrifugal forces exerted on the fluid and radial pressures restoring equilibrium which is responsible for these radial flows. Thus, to an observer travelling with a fluid particle, radial motion will continue until sufficient fluid is transported (radially) to change the pressure distribution to that necessary for equilibrium. The flow in an annular passage in which there is no radial component of velocity, whose streamlines lie in circular, cylindrical surfaces and which is axisymmetric, is commonly known as *radial equilibrium* flow.

An analysis called *the radial equilibrium method*, widely used for three-dimensional design calculations in axial compressors and turbines, is based upon the assumption that any radial flow which may occur, is completed *within* a blade row, the flow *outside* the row then being in radial equilibrium. Figure 6.1 illustrates the nature of this assumption. The other assumption that the flow is axisymmetric implies that the effect of the discrete blades is not transmitted to the flow.

### Theory of radial equilibrium

Consider a small element of fluid of mass  $dm$ , shown in Figure 6.2, of unit depth and subtending an angle  $d\theta$  at the axis, rotating about the axis with tangential velocity,  $c_\theta$  at radius  $r$ . The element is in radial equilibrium so that the pressure forces balance the centrifugal forces;

$$(p + dp)(r + dr)d\theta - prd\theta - (p + \frac{1}{2}dp)drd\theta = dmc_\theta^2/r.$$

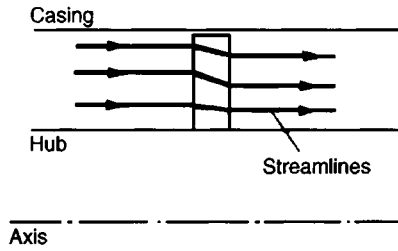


FIG. 6.1. Radial equilibrium flow through a rotor blade row.

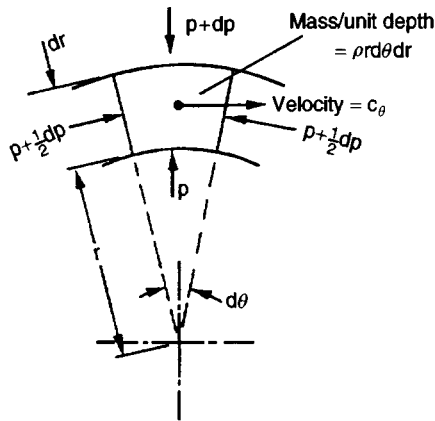


FIG. 6.2. A fluid element in radial equilibrium ( $c_r = 0$ ).

Writing  $dm = \rho r d\theta dr$  and ignoring terms of the second order of smallness the above equation reduces to,

$$\frac{1}{\rho} \frac{dp}{dr} = \frac{c_\theta^2}{r}. \quad (6.1)$$

If the swirl velocity  $c_\theta$  and density are known functions of radius, the radial pressure variation along the blade length can be determined,

$$p_{\text{tip}} - p_{\text{root}} = \int_{\text{root}}^{\text{tip}} \rho c_\theta^2 \frac{dr}{r}. \quad (6.2a)$$

For an incompressible fluid

$$p_{\text{tip}} - p_{\text{root}} = \rho \int_{\text{root}}^{\text{tip}} c_\theta^2 \frac{dr}{r}. \quad (6.2b)$$

The stagnation enthalpy is written (with  $c_r = 0$ )

$$h_0 = h + \frac{1}{2}(c_x^2 + c_\theta^2) \quad (6.3)$$

therefore,

$$\frac{dh_0}{dr} = \frac{dh}{dr} + c_x \frac{dc_x}{dr} + c_\theta \frac{dc_\theta}{dr}. \quad (6.4)$$

The thermodynamic relation  $Tds = dh - (1/\rho)dp$  can be similarly written

$$T \frac{ds}{dr} = \frac{dh}{dr} - \frac{1}{\rho} \frac{dp}{dr}. \quad (6.5)$$

Combining eqns. (6.1), (6.4) and (6.5), eliminating  $dp/dr$  and  $dh/dr$ , the *radial equilibrium equation* may be obtained,

$$\frac{dh_0}{dr} - T \frac{ds}{dr} = c_x \frac{dc_x}{dr} + \frac{c_\theta}{r} \frac{d}{dr}(rc_\theta). \quad (6.6)$$

If the stagnation enthalpy  $h_0$  and entropy  $s$  remain the same at all radii,  $dh_0/dr = ds/dr = 0$ , eqn. (6.6) becomes,

$$c_x \frac{dc_x}{dr} + \frac{c_\theta}{r} \frac{d}{dr}(rc_\theta) = 0. \quad (6.6a)$$

Equation (6.6a) will hold for the flow between the rows of an adiabatic, reversible (ideal) turbomachine in which rotor rows either deliver or receive equal work at all radii. Now if the flow is incompressible, instead of eqn. (6.3) use  $p_0 = p + \frac{1}{2}\rho(c_x^2 + c_\theta^2)$  to obtain

$$\frac{1}{\rho} \frac{dp_0}{dr} = \frac{1}{\rho} \frac{dp}{dr} + c_x \frac{dc_x}{dr} + c_\theta \frac{dc_\theta}{dr}. \quad (6.7)$$

Combining eqns. (6.1) and (6.7), then

$$\frac{1}{\rho} \frac{dp_0}{dr} = c_x \frac{dc_x}{dr} + \frac{c_\theta}{r} \frac{d}{dr}(rc_\theta). \quad (6.8)$$

Equation (6.8) clearly reduces to eqn. (6.6a) in a turbomachine in which equal work is delivered at all radii and the total pressure losses across a row are uniform with radius.

Equation (6.6a) may be applied to two sorts of problem as follows: (i) the design (or indirect) problem – in which the tangential velocity distribution is specified and the axial velocity variation is found, or (ii) the direct problem – in which the swirl angle distribution is specified, the axial and tangential velocities being determined.

## The indirect problem

### 1. Free-vortex flow

This is a flow where the product of radius and tangential velocity remains constant (i.e.  $rc_\theta = K$ , a constant). The term “vortex-free” might be more appropriate as the vorticity (to be precise we mean *axial* vorticity component) is then zero.

Consider an element of an ideal inviscid fluid rotating about some fixed axis, as indicated in Figure. 6.3. The *circulation*  $\Gamma$ , is defined as the line integral of velocity around a curve enclosing an area  $A$ , or  $\Gamma = \oint c ds$ . The *vorticity* at a point is defined as, the limiting value of circulation  $\delta\Gamma$  divided by area  $\delta A$ , as  $\delta A$  becomes vanishingly small. Thus vorticity,  $\omega = d\Gamma/dA$ .

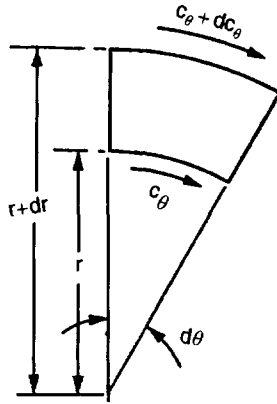


FIG. 6.3. Circulation about an element of fluid.

For the element shown in Figure 6.3,  $c_r = 0$  and

$$\begin{aligned} d\Gamma &= (c_\theta + dc_\theta)(r + dr)d\theta - c_\theta r d\theta \\ &= \left( \frac{dc_\theta}{dr} + \frac{c_\theta}{r} \right) r d\theta dr \end{aligned}$$

ignoring the product of small terms. Thus,  $\omega = d\Gamma/dA = (1/r)d(rc_\theta)/dr$ . If the vorticity is zero,  $d(rc_\theta)/dr$  is also zero and, therefore,  $rc_\theta$  is constant with radius.

Putting  $rc_\theta = \text{constant}$  in eqn. (6.6a), then  $dc_x/dr = 0$  and so  $c_x = a$  constant. This information can be applied to the incompressible flow through a free-vortex compressor or turbine stage, enabling the radial variation in flow angles, reaction and work to be found.

**Compressor stage.** Consider the case of a compressor stage in which  $rc_{\theta 1} = K_1$  before the rotor and  $rc_{\theta 2} = K_2$  after the rotor, where  $K_1, K_2$  are constants. The work done by the rotor on unit mass of fluid is

$$\begin{aligned} \Delta W &= U(c_{\theta 2} - c_{\theta 1}) = \Omega r(K_2/r - K_1/r) \\ &= \text{constant.} \end{aligned}$$

Thus, the work done is equal at all radii.

The relative flow angles (see Figure 5.2) entering and leaving the rotor are

$$\begin{aligned} \tan \beta_1 &= \frac{U}{c_x} - \tan \alpha_1 = \frac{\Omega r - K_1/r}{c_x}, \\ \tan \beta_2 &= \frac{U}{c_x} - \tan \alpha_2 = \frac{\Omega r - K_2/r}{c_x}. \end{aligned}$$

in which  $c_{x1} = c_{x2} = c_x$  for incompressible flow.

In Chapter 5, reaction in an axial compressor is defined by

$$R = \frac{\text{static enthalpy rise in the rotor}}{\text{static enthalpy rise in the stage}}.$$

For a normal stage ( $\alpha_1 = \alpha_3$ ) with  $c_x$  constant across the stage, the reaction was shown to be

$$R = \frac{c_x}{2U}(\tan \beta_1 + \tan \beta_2). \quad (5.11)$$

Substituting values of  $\tan \beta_1$  and  $\tan \beta_2$  into eqn. (5.11), the reaction becomes

$$R = 1 - \frac{k}{r^2}, \quad (6.9)$$

where

$$k = (K_1 + K_2)/(2\Omega).$$

It will be clear that as  $k$  is positive, the reaction increases from root to tip. Likewise, from eqn. (6.1) we observe that as  $c_\theta^2/r$  is always positive (excepting  $c_\theta = 0$ ), so static pressure increases from root to tip. For the free-vortex flow  $rc_\theta = K$ , the static pressure variation is obviously  $p/\rho = \text{constant} - K/(2r^2)$  upon integrating eqn. (6.1).

**EXAMPLE 6.1.** An axial flow compressor stage is designed to give free-vortex tangential velocity distributions for all radii before and after the rotor blade row. The tip diameter is constant and 1.0 m; the hub diameter is 0.9 m and constant for the stage. At the rotor tip the flow angles are as follows

- Absolute inlet angle,  $\alpha_1 = 30^\circ$ .
- Relative inlet angle,  $\beta_1 = 60^\circ$ .
- Absolute outlet angle,  $\alpha_2 = 60^\circ$ .
- Relative outlet angle,  $\beta_2 = 30^\circ$ .

Determine,

- (i) the axial velocity;
- (ii) the mass flow rate;
- (iii) the power absorbed by the stage;
- (iv) the flow angles at the hub;
- (v) the reaction ratio of the stage at the hub;

given that the rotational speed of the rotor is 6000 rev/min and the gas density is  $1.5 \text{ kg/m}^3$  which can be assumed constant for the stage. It can be further assumed that stagnation enthalpy and entropy are constant before and after the rotor row for the purpose of simplifying the calculations.

**Solution.** (i) The rotational speed,  $\Omega = 2\pi N/60 = 628.4 \text{ rad/s}$ .

Therefore blade tip speed,  $U_t = \Omega r_t = 314.2 \text{ m/s}$  and blade speed at hub,  $U_h = \Omega r_h = 282.5 \text{ m/s}$ .

From the velocity diagram for the stage (e.g. Figure 5.2), the blade tip speed is

$$U_t = c_x(\tan 60^\circ + \tan 30^\circ) = c_x(\sqrt{3} + 1/\sqrt{3}).$$

Therefore  $c_x = 136 \text{ m/s}$ , constant at all radii by eqn. (6.6a).

(ii) The rate of mass flow,  $\dot{m} = \pi(r_t^2 - r_h^2)\rho c_x$

$$= \pi(0.5^2 - 0.45^2)1.5 \times 136 = 30.4 \text{ kg/s.}$$

(iii) The power absorbed by the stage,

$$\begin{aligned}\dot{W}_c &= \dot{m}U_t(c_{\theta 2t} - c_{\theta 1t}) \\ &= \dot{m}U_t c_x (\tan \alpha_{2t} - \tan \alpha_{1t}) \\ &= 30.4 \times 314.2 \times 136(\sqrt{3} - 1/\sqrt{3}) \\ &= 1.5 \text{ MW.}\end{aligned}$$

(iv) At inlet to the rotor tip,

$$c_{\theta 1t} = c_x \tan \alpha_1 = 136/\sqrt{3} = 78.6 \text{ m/s.}$$

The absolute flow is a free-vortex,  $rc_\theta = \text{constant}$ .

Therefore  $c_{\theta 1h} = c_{\theta 1t}(r_t/r_h) = 78.6 \times 0.5/0.45 = 87.3 \text{ m/s}$ .

At outlet to the rotor tip,

$$c_{\theta 2t} = c_x \tan \alpha_2 = 136 \times \sqrt{3} = 235.6 \text{ m/s.}$$

Therefore  $c_{\theta 2h} = c_{\theta 2t}(r_t/r_h) = 235.6 \times 0.5/0.45 = 262 \text{ m/s}$ .

The flow angles at the hub are,

$$\tan \alpha_1 = c_{\theta 1h}/c_x = 87.3/136 = 0.642,$$

$$\tan \beta_1 = U_h/c_x - \tan \alpha_1 = 1.436,$$

$$\tan \alpha_2 = c_{\theta 2h}/c_x = 262/136 = 1.928,$$

$$\tan \beta_2 = U_h/c_x - \tan \alpha_2 = 0.152.$$

Thus  $\alpha_1 = 32.75^\circ$ ,  $\beta_1 = 55.15^\circ$ ,  $\alpha_2 = 62.6^\circ$ ,  $\beta_2 = 8.64^\circ$  at the hub.

(v) The reaction at the hub can be found by several methods. With eqn. (6.9)

$$R = 1 - k/r^2$$

and noticing that, from symmetry of the velocity triangles,

$$R = 0.5 \text{ at } r = r_t, \text{ then } k = 0.5r_t^2.$$

Therefore  $R_h = 1 - 0.5(0.5/0.45)^2 = 0.382$ .

The velocity triangles will be asymmetric and similar to those in Figure 5.4(b).

The simplicity of the flow under free-vortex conditions is, superficially, very attractive to the designer and many compressors have been designed to conform to this flow. (Constant (1945, 1953) may be consulted for an account of early British compressor design methods.) Figure 6.4 illustrates the variation of fluid angles and Mach numbers of a typical compressor stage designed for free-vortex flow. Characteristic of this flow are the large fluid deflections near the inner wall and high Mach numbers near the outer wall, both effects being deleterious to efficient performance.

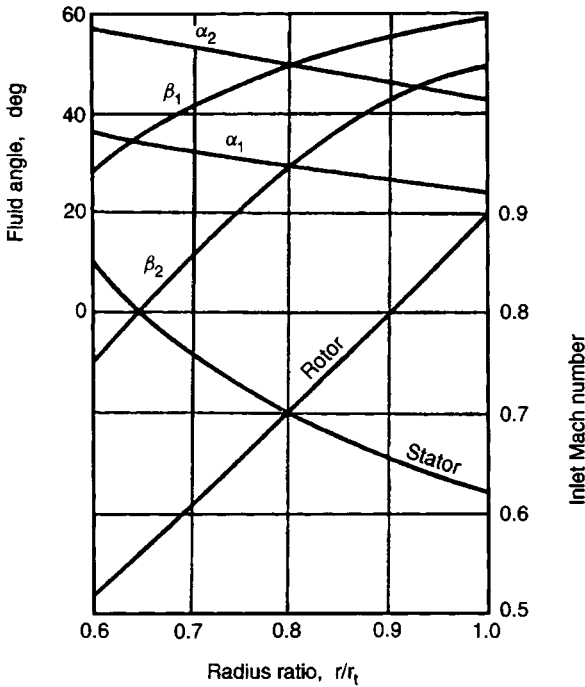


FIG. 6.4. Variation of fluid angles and Mach numbers of a free-vortex compressor stage with radius (adapted from Howell 1945).

A further serious disadvantage is the large amount of rotor twist from root to tip which adds to the expense of blade manufacture.

Many types of vortex design have been proposed to overcome some of the disadvantages set by free-vortex design and several of these are compared by Horlock (1958). Radial equilibrium solutions for the work and axial velocity distributions of some of these vortex flows in an axial compressor stage are given below.

## 2. Forced vortex

This is sometimes called “solid-body” rotation because  $c_\theta$  varies directly with  $r$ . At entry to the rotor assume  $h_{01}$  is constant and  $c_{\theta 1} = K_1 r$ .

With eqn. (6.6a)

$$\frac{d}{dr} \left( \frac{c_{x1}^2}{2} \right) = -K_1 \frac{d}{dr} (K_1 r^2)$$

and, after integrating,

$$c_{x1}^2 = \text{constant} - 2K_1^2 r^2. \quad (6.10)$$

After the rotor  $c_{\theta 2} = K_2 r$  and  $h_{02} - h_{01} = U(c_{\theta 2} - c_{\theta 1}) = \Omega(K_2 - K_1)r^2$ . Thus, as the work distribution is non-uniform, the radial equilibrium equation in the form

eqn. (6.6) is required for the flow after the rotor.

$$\frac{dh_{02}}{dr} = 2\Omega(K_2 - K_1)r = \frac{d}{dr} \left( \frac{c_{x2}^2}{2} \right) + K_2 \frac{d}{dr} (K_2 r^2).$$

After rearranging and integrating

$$c_{x2}^2 = \text{constant} - 2[K_2^2 - \Omega(K_2 - K_1)]r^2. \quad (6.11)$$

The constants of integration in eqns. (6.10) and (6.11) can be found from the continuity of mass flow, i.e.

$$\frac{\dot{m}}{2\pi\rho} = \int_{r_h}^{r_t} c_{x1} r dr = \int_{r_h}^{r_t} c_{x2} r dr, \quad (6.12)$$

which applies to the assumed incompressible flow.

### 3. General whirl distribution

The tangential velocity distribution is given by

$$c_{\theta 1} = ar^n - b/r \text{ (before rotor)}, \quad (6.13a)$$

$$c_{\theta 2} = ar^n + b/r \text{ (after rotor)}. \quad (6.13b)$$

The distribution of work for all values of the index  $n$  is constant with radius so that if  $h_{01}$  is uniform,  $h_{02}$  is also uniform with radius. From eqns. (6.13)

$$\Delta W = h_{02} - h_{01} = U(c_{\theta 2} - c_{\theta 1}) = 2b\Omega. \quad (6.14)$$

Selecting different values of  $n$  gives several of the tangential velocity distributions commonly used in compressor design. With  $n = 0$ , or zero power blading, it leads to the so-called “exponential” type of stage design (included as an exercise at the end of this chapter). With  $n = 1$ , or *first power blading*, the stage design is called (incorrectly, as it transpires later) “constant reaction”.

*First power stage design.* For a given stage temperature rise the discussion in Chapter 5 would suggest the choice of 50% reaction at all radii for the highest stage efficiency. With swirl velocity distributions

$$c_{\theta 1} = ar - b/r, \quad c_{\theta 2} = ar + b/r \quad (6.15)$$

before and after the rotor respectively, and rewriting the expression for reaction, eqn. (5.11), as

$$R = 1 - \frac{c_x}{2U} (\tan \alpha_1 + \tan \alpha_2), \quad (6.16)$$

then, using eqn. (6.15),

$$R = 1 - a/\Omega = \text{constant}. \quad (6.17)$$

Implicit in eqn. (6.16) is the assumption that the axial velocity across the rotor remains constant which, of course, is tantamount to ignoring radial equilibrium.

The axial velocity *must* change in crossing the rotor row so that eqn. (6.17) is only a crude approximation at the best. Just how crude is this approximation will be indicated below.

Assuming constant stagnation enthalpy at entry to the stage, integrating eqn. (6.6a), the axial velocity distributions before and after the rotor are

$$c_{x1}^2 = \text{constant} - 4a\left(\frac{1}{2}ar^2 - b \ln r\right), \quad (6.18a)$$

$$c_{x2}^2 = \text{constant} - 4a\left(\frac{1}{2}ar^2 + b \ln r\right), \quad (6.18b)$$

More conveniently, these expressions can be written non-dimensionally as,

$$\left(\frac{c_{x1}}{U_t}\right)^2 = A_1 - \left(\frac{2a}{\Omega}\right)^2 \left[ \frac{1}{2} \left(\frac{r}{r_t}\right)^2 - \frac{b}{ar_t^2} \ln \left(\frac{r}{r_t}\right) \right], \quad (6.19a)$$

$$\left(\frac{c_{x2}}{U_t}\right)^2 = A_2 - \left(\frac{2a}{\Omega}\right)^2 \left[ \frac{1}{2} \left(\frac{r}{r_t}\right)^2 + \frac{b}{ar_t^2} \ln \left(\frac{r}{r_t}\right) \right], \quad (6.19b)$$

in which  $U_t = \Omega r_t$  is the tip blade speed. The constants  $A_1, A_2$  are not entirely arbitrary as the continuity equation, eqn. (6.12), must be satisfied.

**EXAMPLE 6.2.** As an illustration consider a single stage of an axial-flow air compressor of hub-tip ratio 0.4 with a nominally constant reaction (i.e. according to eqn. (6.17)) of 50%. Assuming incompressible, inviscid flow, a blade tip speed of 300 m/s, a blade tip diameter of 0.6 m, and a stagnation temperature rise of 16.1°C, determine the radial equilibrium values of axial velocity before and after the rotor. The axial velocity far upstream of the rotor at the casing is 120 m/s. Take  $C_p$  for air as 1.005 kJ/(kg°C).

**Solution:** The constants in eqn. (6.19) can be easily determined. From eqn. (6.17)

$$2a/\Omega = 2(1 - R) = 1.0.$$

Combining eqns. (6.14) and (6.17)

$$\begin{aligned} \frac{b}{ar_t^2} &= \frac{\Delta W}{2\Omega^2(1 - R)r_t^2} = \frac{C_p \cdot \Delta T_0}{2U_t^2(1 - R)} \\ &= \frac{1005 \times 16.1}{300^2} = 0.18. \end{aligned}$$

The inlet axial velocity distribution is completely specified and the constant  $A_1$  solved. From eqn. (6.19a)

$$\left(\frac{c_{x1}}{U_t}\right)^2 = A_1 - \left[\frac{1}{2}\left(\frac{r}{r_t}\right)^2 - 0.18 \ln(r/r_t)\right].$$

At  $r = r_t$ ,  $c_{x1}/U_t = 0.4$  and hence  $A_1 = 0.66$ .

Although an explicit solution for  $A_2$  can be worked out from eqn. (6.19b) and eqn. (6.12), it is far speedier to use a semigraphical procedure. For an arbitrarily selected value of  $A_2$ , the distribution of  $c_{x2}/U_t$  is known. Values of  $(r/r_t) \cdot (c_{x2}/U_t)$

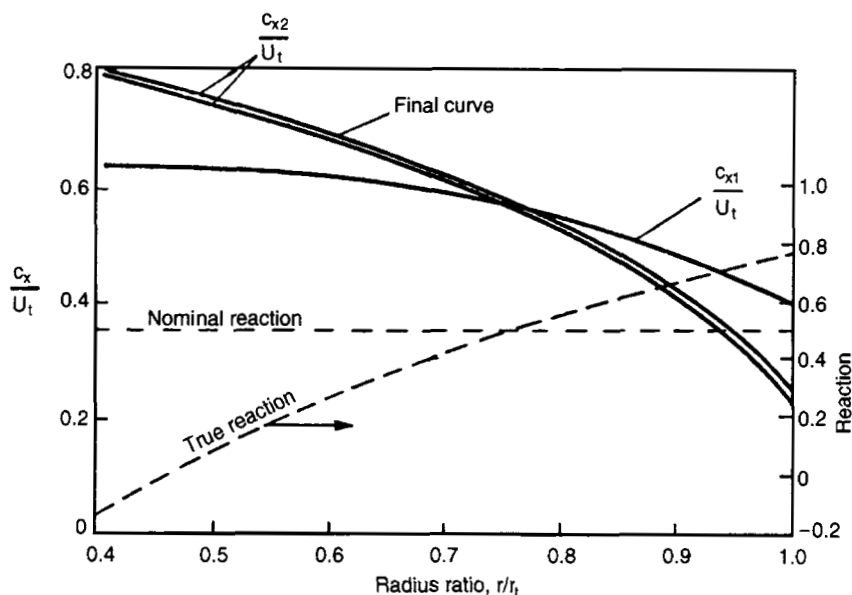


FIG. 6.5. Solution of exit axial-velocity profile for a first power stage.

and  $(r/r_t) \cdot (c_{x1}/U_t)$  are plotted against  $r/r_t$  and the areas under these curves compared. New values of  $A_2$  are then chosen until eqn. (6.12) is satisfied. This procedure is quite rapid and normally requires only two or three attempts to give a satisfactory solution. Figure 6.5 shows the final solution of  $c_{x2}/U_t$  obtained after three attempts. The solution is,

$$\left(\frac{c_{x2}}{U_t}\right)^2 = 0.56 - \left[ \frac{1}{2} \left(\frac{r}{r_t}\right)^2 + 0.18 \ln \left(\frac{r}{r_t}\right) \right].$$

It is illuminating to calculate the actual variation in reaction taking account of the change in axial velocity. From eqn. (5.10c) the true reaction across a normal stage is,

$$R' = \frac{w_1^2 - w_2^2}{2U(c_{\theta 2} - c_{\theta 1})}.$$

From the velocity triangles, Figure 5.2,

$$w_1^2 - w_2^2 = (w_{\theta 1} + w_{\theta 2})(w_{\theta 1} - w_{\theta 2}) + (c_{x1}^2 - c_{x2}^2).$$

As  $w_{\theta 1} + w_{\theta 2} = 2U - (c_{\theta 1} + c_{\theta 2})$  and  $w_{\theta 1} - w_{\theta 2} = c_{\theta 2} - c_{\theta 1}$ ,

$$R' = 1 - \frac{c_{\theta 1} + c_{\theta 2}}{2U} + \frac{c_{x1}^2 - c_{x2}^2}{2U(c_{\theta 2} - c_{\theta 1})}.$$

For the first power swirl distribution, eqn. (6.15),

$$R' = 1 - \frac{a}{\Omega} + \frac{c_{x1}^2 - c_{x2}^2}{4\Omega b}.$$

From the radial equilibrium solution in eqns. (6.19), after some rearrangement,

$$\frac{c_{x1}^2 - c_{x2}^2}{4\Omega b} = \frac{A_1 - A_2}{2\psi_t} + \left(\frac{2a}{\Omega}\right) \ln\left(\frac{r}{r_t}\right),$$

where

$$\psi_t = \frac{\Delta W}{U_t^2} = \frac{C_p \Delta T_0}{\Omega^2 r_t^2}.$$

In the above example,  $1 - a/\Omega = \frac{1}{2}$ ,  $\psi_t = 0.18$

$$R' = 0.778 + \ln(r/r_t).$$

The true reaction variation is shown in Figure 6.5 and it is evident that eqn. (6.17) is *invalid* as a result of axial velocity changes.

## The direct problem

The flow angle variation is specified in the direct problem and the radial equilibrium equation enables the solution of  $c_x$  and  $c_\theta$  to be found. The general radial equilibrium equation can be written in the form

$$\begin{aligned} \frac{dh_0}{dr} - T \frac{ds}{dr} &= \frac{c_\theta^2}{r} + c \frac{dc}{dr} \\ &= \frac{c^2 \sin^2 \alpha}{r} + c \frac{dc}{dr}, \end{aligned} \quad (6.20)$$

as  $c_\theta = c \sin \alpha$ .

If both  $dh_0/dr$  and  $ds/dr$  are zero, eqn. (6.20) integrated gives

$$\log c = - \int \sin^2 \alpha \frac{dr}{r} + \text{constant}$$

or, if  $c = c_m$  at  $r = r_m$ , then

$$\frac{c}{c_m} = \exp \left( - \int_{r_m}^r \sin^2 \alpha \frac{dr}{r} \right). \quad (6.21)$$

If the flow angle  $\alpha$  is held constant, eqn. (6.21) simplifies still further,

$$\frac{c}{c_m} = \frac{c_x}{c_{xm}} = \frac{c_\theta}{c_{\theta m}} = \left( \frac{r}{r_m} \right)^{-\sin^2 \alpha} \quad (6.22)$$

The vortex distribution represented by eqn. (6.22) is frequently employed in practice as untwisted blades are relatively simple to manufacture.

The general solution of eqn. (6.20) can be found by introducing a suitable *integrating factor* into the equation. Multiplying throughout by  $\exp[2 \int \sin^2 \alpha dr/r]$  it follows that

$$\frac{d}{dr} \left\{ c^2 \exp \left[ 2 \int \sin^2 \alpha dr/r \right] \right\} = 2 \left( \frac{dh_0}{dr} - T \frac{ds}{dr} \right) \exp \left[ 2 \int \sin^2 \alpha dr/r \right].$$

After integrating and inserting the limit  $c = c_m$  at  $r = r_m$ , then

$$\begin{aligned} c^2 \exp \left[ 2 \int^r \sin^2 \alpha dr/r \right] - c_m^2 \exp \left[ 2 \int^{r_m} \sin^2 \alpha dr/r \right] \\ = 2 \int_{r_m}^r \left( \frac{dh_0}{dr} - T \frac{ds}{dr} \right) \exp \left[ 2 \int \sin^2 \alpha dr/r \right] dr. \end{aligned} \quad (6.23)$$

Particular solutions of eqn. (6.23) can be readily obtained for simple radial distributions of  $\alpha$ ,  $h_0$  and  $s$ . Two solutions are considered here in which both  $2dh_0/dr = kc_m^2/r_m$  and  $ds/dr = 0$ ,  $k$  being an arbitrary constant

(i) Let  $a = 2 \sin^2 \alpha$ . Then  $\exp[2 \int \sin^2 \alpha dr/r] = r^a$  and, hence

$$\left( \frac{c}{c_m} \right)^2 \left( \frac{r}{r_m} \right)^a = 1 + \frac{k}{1+a} \left[ \left( \frac{r}{r_m} \right)^{1+a} - 1 \right]. \quad (6.23a)$$

Equation (6.22) is obtained immediately from this result with  $k = 0$ .

(ii) Let  $br/r_m = 2 \sin^2 \alpha$ . Then,

$$c^2 \exp(br/r_m) - c_m^2 \exp(b) = (kc_m^2/r_m) \int_{r_m}^r \exp(br/r_m) dr$$

and eventually,

$$\left( \frac{c}{c_m} \right)^2 = \frac{k}{b} + \left( 1 - \frac{k}{b} \right) \exp \left[ b \left( 1 - \frac{r}{r_m} \right) \right]. \quad (6.23b)$$

## Compressible flow through a fixed blade row

In the blade rows of high-performance gas turbines, fluid velocities approaching, or even exceeding, the speed of sound are quite normal and compressibility effects may no longer be ignored. A simple analysis is outlined below for the inviscid flow of a perfect gas through a *fixed* row of blades which, nevertheless, can be extended to the flow through moving blade rows.

The radial equilibrium equation, eqn. (6.6), applies to *compressible* flow as well as incompressible flow. With constant stagnation enthalpy and constant entropy, a free-vortex flow therefore implies uniform axial velocity downstream of a blade row, regardless of any *density* changes incurred in passing through the blade row. In fact, for high-speed flows there *must* be a density change in the blade row which implies a streamline shift as shown in Figure 6.1. This may be illustrated by considering the free-vortex flow of a perfect gas as follows. In radial equilibrium,

$$\frac{1}{\rho} \frac{dp}{dr} = \frac{c_\theta^2}{r} = \frac{K^2}{r^3} \text{ with } c_\theta = K/r.$$

For reversible adiabatic flow of a perfect gas,  $\rho = Ep^{1/\gamma}$ , where  $E$  is constant. Thus

$$\int p^{-1/\gamma} dp = EK^2 \int r^{-3} dr + \text{constant},$$

therefore

$$p = \left[ \text{constant} - \left( \frac{\gamma - 1}{2\gamma} \right) \frac{EK^2}{r^2} \right]^{\gamma/(\gamma-1)} \quad (6.24)$$

For this free-vortex flow the pressure, and therefore the density also, must be larger at the casing than at the hub. The density difference from hub to tip may be appreciable in a high-velocity, high-swirl angle flow. If the fluid is without swirl at entry to the blades the density will be uniform. Therefore, from continuity of mass flow there must be a redistribution of fluid in its passage across the blade row to compensate for the changes in density. Thus, for this blade row, the continuity equation is,

$$\dot{m} = \rho_1 A_1 c_{x1} = 2\pi c_{x2} \int_{r_h}^{r_t} \rho_2 r dr, \quad (6.25)$$

where  $\rho_2$  is the density of the swirling flow, obtainable from eqn. (6.24).

## Constant specific mass flow

Although there appears to be no evidence that the redistribution of the flow across blade rows is a source of inefficiency, it has been suggested by Horlock (1966) that the radial distribution of  $c_\theta$  for each blade row is chosen so that the product of axial velocity and density is constant with radius, i.e.

$$d\dot{m}/dA = \rho c_x = \rho c \cos \alpha = \rho_m c_m \cos \alpha_m = \text{constant} \quad (6.26)$$

where subscript  $m$  denotes conditions at  $r = r_m$ . This *constant specific mass flow design* is the logical choice when radial equilibrium theory is applied to compressible flows as the assumption that  $c_r = 0$  is then likely to be realised.

Solutions may be determined by means of a simple numerical procedure and, as an illustration of one method, a turbine stage is considered here. It is convenient to assume that the stagnation enthalpy is uniform at nozzle entry, the entropy is constant throughout the stage and the fluid is a perfect gas. At nozzle exit under these conditions the equation of radial equilibrium, eqn. (6.20), can be written as

$$dc/c = -\sin^2 \alpha dr/r. \quad (6.27)$$

From eqn. (6.1), nothing that at constant entropy the acoustic velocity  $a = \sqrt{(dp/d\rho)}$ ,

$$\frac{1}{\rho} \frac{dp}{dr} = \frac{1}{\rho} \left( \frac{dp}{d\rho} \right) \left( \frac{d\rho}{dr} \right) = \frac{a^2}{\rho} \frac{d\rho}{dr} = \frac{c^2}{r} \sin^2 \alpha, \\ \therefore d\rho/\rho = M^2 \sin^2 \alpha dr/r \quad (6.28)$$

where the flow Mach number

$$M = c/a = c/\sqrt{(\gamma RT)}. \quad (6.28a)$$

The isentropic relation between temperature and density for a perfect gas is

$$T/T_m = (\rho/\rho_m)^{\gamma-1}$$

which after logarithmic differentiation gives

$$dT/T = (\gamma - 1)d\rho/\rho. \quad (6.29)$$

Using the above set of equations the procedure for determining the nozzle exit flow is as follows. Starting at  $r = r_m$ , values of  $c_m$ ,  $\alpha_m$ ,  $T_m$  and  $\rho_m$  are assumed to be known. For a small finite interval  $\Delta r$ , the changes in velocity  $\Delta c$ , density  $\Delta\rho$ , and temperature  $\Delta T$  can be computed using eqns. (6.27), (6.28) and (6.29) respectively. Hence, at the new radius  $r = r_m + \Delta r$  the velocity  $c = c_m + \Delta c$ , the density  $\rho = \rho_m + \Delta\rho$  and temperature  $T = T_m + \Delta T$  are obtained. The corresponding flow angle  $\alpha$  and Mach number  $M$  can now be determined from eqns. (6.26) and (6.28a) respectively. Thus, all parameters of the problem are known at radius  $r = r_m + \Delta r$ . This procedure is repeated for further increments in radius to the casing and again from the mean radius to the hub.

Figure 6.6 shows the distributions of flow angle and Mach number computed with this procedure for a turbine nozzle blade row of 0.6 hub/tip radius ratio. The input data used was  $\alpha_m = 70.4^\circ$  and  $M = 0.907$  at the mean radius. Air was assumed at a stagnation pressure of 859 kPa and a stagnation temperature of 465 K. A remarkable feature of these results is the almost uniform swirl angle which is obtained.

With the nozzle exit flow fully determined the flow at rotor outlet can now be computed by a similar procedure. The procedure is a little more complicated than that for the nozzle row because the specific work done by the rotor is not uniform with radius. Across the rotor, using the notation of Chapter 4,

$$h_{o2} - h_{o3} = U(c_{\theta 2} + c_{\theta 3}) \quad (6.30)$$

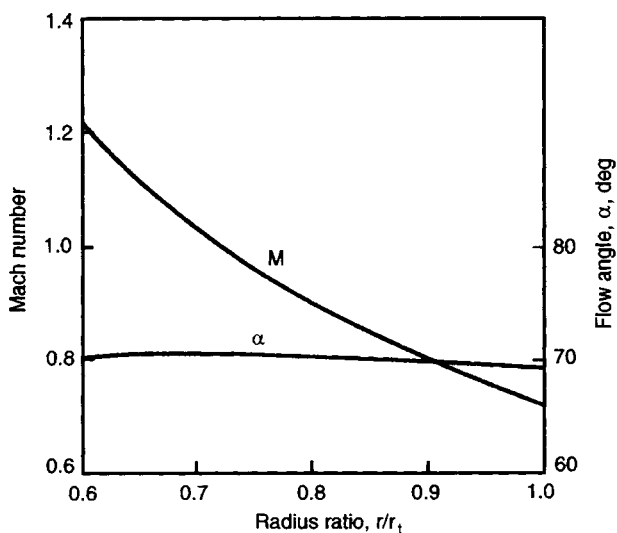


FIG. 6.6. Flow angle and Mach number distributions with radius of a nozzle blade row designed for constant specific mass flow.

and hence the gradient in stagnation enthalpy after the rotor is

$$dh_{o3}/dr = -d[U(c_{\theta 2} + c_{\theta 3})]/dr = -d(Uc_{\theta 2})/dr - d(Uc_3 \sin \alpha_3)/dr.$$

After differentiating the last term,

$$-dh_o = d(Uc_{\theta 2}) + U(c \sin \alpha dr/r + \sin \alpha dc + c \cos \alpha d\alpha) \quad (6.30a)$$

the subscript 3 having now been dropped.

From eqn. (6.20) the radial equilibrium equation applied to the rotor exit flow is

$$dh_o = c^2 \sin^2 \alpha dr/r + c dc. \quad (6.30b)$$

After logarithmic differentiation of  $\rho c \cos \alpha = \text{constant}$ ,

$$d\rho/\rho + dc/c = \tan \alpha d\alpha. \quad (6.31)$$

Eliminating successively  $dh_o$  between eqns. (6.30a) and (6.30b),  $d\rho/\rho$  between eqns. (6.28) and (6.31) and finally  $d\alpha$  from the resulting equations gives

$$\frac{dc}{c} \left(1 + \frac{c_\theta}{U}\right) = -\sin^2 \alpha \left\{ \frac{d(rc_\theta)}{rc_\theta} + \left(1 + \frac{c_\theta}{U} + M_x^2\right) \frac{dr}{r} \right\} \quad (6.32)$$

where  $M_x = M \cos \alpha = c \cos \alpha / \sqrt{(\gamma RT)}$  and the static temperature

$$\begin{aligned} T &= T_3 = T_{o3} - c_3^2/(2C_p) \\ &= T_{o2} - [U(c_{\theta 2} + c_{\theta 3}) + \frac{1}{2}c_3^2]/C_p. \end{aligned} \quad (6.33)$$

The verification of eqn. (6.32) is left as an exercise for the diligent student.

Provided that the exit flow angle  $\alpha_3$  at  $r = r_m$  and the mean rotor blade speeds are specified, the velocity distribution, etc., at rotor exit can be readily computed from these equations.

## Off-design performance of a stage

A turbine stage is considered here although, with some minor modifications, the analysis can be made applicable to a compressor stage.

Assuming the flow is at constant entropy, apply the radial equilibrium equation, eqn. (6.6), to the flow on both sides of the rotor, then

$$\frac{dh_{o3}}{dr} = \frac{dh_{o2}}{dr} - \Omega \frac{d}{dr}(rc_{\theta 2} + rc_{\theta 3}) = c_{x3} \frac{dc_{x3}}{dr} + \frac{c_{\theta 3}}{r} \frac{d}{dr}(rc_{\theta 3}).$$

Therefore

$$c_{x2} \frac{dc_{x2}}{dr} + \left(\frac{c_{\theta 2}}{r} - \Omega\right) \frac{d}{dr}(rc_{\theta 2}) = c_{x3} \frac{dc_{x3}}{dr} + \left(\frac{c_{\theta 3}}{r} + \Omega\right) \frac{d}{dr}(rc_{\theta 3}).$$

Substituting  $c_{\theta 3} = c_{x3} \tan \beta_3 - \Omega r$  into the above equation, then, after some simplification,

$$\begin{aligned} c_{x2} \frac{dc_{x2}}{dr} + \left(\frac{c_{\theta 2}}{r} - \Omega\right) \frac{d}{dr}(rc_{\theta 2}) &= c_{x3} \frac{dc_{x3}}{dr} + \frac{c_{x3}}{r} \tan \beta_3 \frac{d}{dr}(rc_{x3} \tan \beta_3) \\ &\quad - 2\Omega c_{x3} \tan \beta_3. \end{aligned} \quad (6.34)$$

In a particular problem the quantities  $c_{x2}$ ,  $c_{\theta2}$ ,  $\beta_3$  are known functions of radius and  $\Omega$  can be specified. Equation (6.34) is thus a first order differential equation in which  $c_{x3}$  is unknown and may best be solved, in the general case, by numerical iteration. This procedure requires a guessed value of  $c_{x3}$  at the hub and, by applying eqn. (6.34) to a small interval of radius  $\Delta r$ , a new value of  $c_{x3}$  at radius  $r_h + \Delta r$  is found. By repeating this calculation for successive increments of radius a complete velocity profile  $c_{x3}$  can be determined. Using the continuity relation

$$\int_{r_h}^{r_t} c_{x3} r dr = \int_{r_h}^{r_t} c_{x2} r dr,$$

this initial velocity distribution can be integrated and a new, more accurate, estimate of  $c_{x3}$  at the hub then found. Using this value of  $c_{x3}$  the step-by-step procedure is repeated as described and again checked by continuity. This iterative process is normally rapidly convergent and, in most cases, three cycles of the calculation enables a sufficiently accurate exit velocity profile to be found.

The off-design performance may be obtained by making the approximation that the rotor relative exit angle  $\beta_3$  and the nozzle exit angle  $\alpha_2$  remain constant at a particular radius with a change in mass flow. This approximation is not unrealistic as cascade data (see Chapter 3) suggest that fluid angles at outlet from a blade row alter very little with change in incidence up to the stall point.

Although any type of flow through a stage may be successfully treated using this method, rather more elegant solutions in closed form can be obtained for a few special cases. One such case is outlined below for a free-vortex turbine stage whilst other cases are already covered by eqns. (6.21)–(6.23).

## Free-vortex turbine stage

Suppose, for simplicity, a free-vortex stage is considered where, at the design point, the flow at rotor exit is completely axial (i.e. without swirl). At stage entry the flow is again supposed completely axial and of constant stagnation enthalpy  $h_{01}$ . Free-vortex conditions prevail at entry to the rotor,  $rc_{\theta2} = rc_{x2} \tan \alpha_2 = \text{constant}$ . The problem is to find how the axial velocity distribution at rotor exit varies as the mass flow is altered away from the design value.

At off-design conditions the relative rotor exit angle  $\beta_3$  is assumed to remain equal to the value  $\beta^*$  at the design mass flow (\* denotes design conditions). Thus, referring to the velocity triangles in Figure 6.7, at off-design conditions the swirl velocity  $c_{\theta3}$  is evidently non-zero,

$$\begin{aligned} c_{\theta3} &= c_{x3} \tan \beta_3 - U \\ &= c_{x3} \tan \beta_3^* - \Omega r. \end{aligned} \quad (6.35)$$

At the design condition,  $c_{\theta3}^* = 0$  and so

$$c_{x3}^* \tan \beta_3^* = \Omega r. \quad (6.36)$$

Combining eqns. (6.35) and (6.36)

$$c_{\theta3} = \Omega r \left( \frac{c_{x3}}{c_{x3}^*} - 1 \right). \quad (6.37)$$

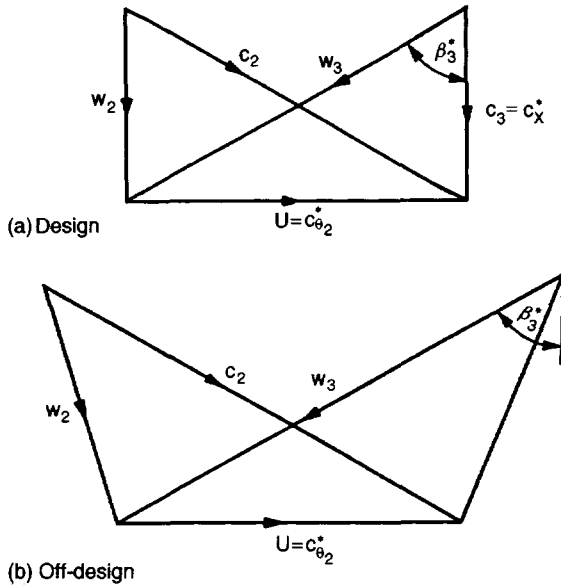


FIG. 6.7. Design and off-design velocity triangles for a free-vortex turbine stage.

The radial equilibrium equation at rotor outlet gives

$$\frac{dh_{03}}{dr} = c_{x3} \frac{dc_{x3}}{dr} + \frac{c_{\theta 3}}{r} \frac{d}{dr}(rc_{\theta 3}) = -\Omega \frac{d}{dr}(rc_{\theta 3}), \quad (6.38)$$

after combining with eqn. (6.33), noting that  $dh_{02}/dr = 0$  and that  $(d/dr)(rc_{\theta 2}) = 0$  at all mass flows. From eqn. (6.37),

$$\Omega + \frac{c_{\theta 3}}{r} = \Omega \frac{c_{x3}}{c_{x3}^*}, \quad rc_{\theta 3} = \Omega r^2 \left( \frac{c_{x3}}{c_{x3}^*} - 1 \right),$$

which when substituted into eqn. (6.38) gives,

$$-\frac{dc_{x3}}{dr} = \frac{\Omega^2}{c_{x3}^*} \left[ 2r \left( \frac{c_{x3}}{c_{x3}^*} - 1 \right) + \frac{r^2}{c_{x3}^*} \frac{dc_{x3}}{dr} \right].$$

After rearranging,

$$\frac{dc_{x3}}{c_{x3} - c_{x3}^*} = \frac{-d(\Omega^2 r^2)}{(c_{x3}^{*2} + \Omega^2 r^2)}. \quad (6.39)$$

Equation (6.39) is immediately integrated in the form

$$\frac{c_{x3} - c_{x3}^*}{c_{x3m} - c_{x3}^*} = \frac{c_{x3}^{*2} + \Omega^2 r_m^2}{c_{x3}^{*2} + \Omega^2 r^2} \quad (6.40)$$

where  $c_{x3} = c_{x3m}$  at  $r = r_m$ . Equation (6.40) is more conveniently expressed in a non-dimensional form by introducing flow coefficients  $\phi = c_{x3}/U_m$ ,  $\phi^* = c_{x3}^*/U_m$

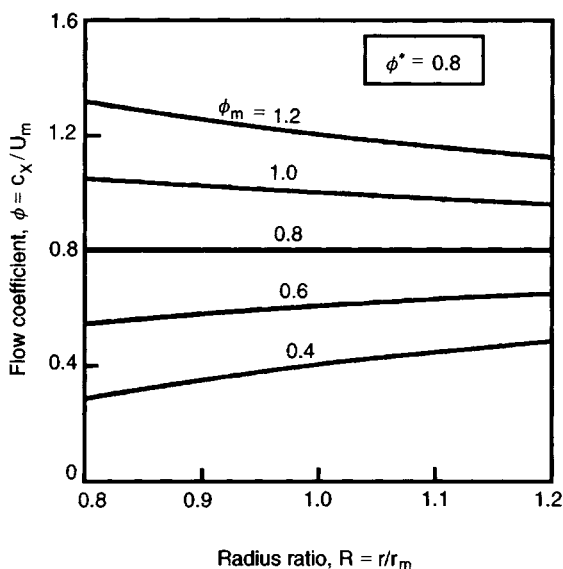


FIG. 6.8. Off-design rotor exit flow coefficients.

and  $\phi_m = c_{x3m}/U_m$ . Thus,

$$\frac{\phi/\phi^* - 1}{\phi_m/\phi^* - 1} = \frac{\phi^{*2} + 1}{\phi^{*2} + (r/r_m)^2}, \quad (6.40a)$$

If  $r_m$  is the mean radius then  $c_{x3m} \doteq c_{x1}$  and, therefore,  $\phi_m$  provides an approximate measure of the overall flow coefficient for the machine (N.B.  $c_{x1}$  is uniform).

The results of this analysis are shown in Figure 6.8 for a representative design flow coefficient  $\phi^* = 0.8$  at several different off-design flow coefficients  $\phi_m$ , with  $r/r_m = 0.8$  at the hub and  $r/r_m = 1.2$  at the tip. It is apparent for values of  $\phi_m < \phi^*$ , that  $c_{x3}$  increases from hub to tip; conversely for  $\phi_m > \phi^*$ ,  $c_{x3}$  decreases towards the tip.

The foregoing analysis is only a special case of the more general analysis of free-vortex turbine and compressor flows (Horlock and Dixon 1966) in which rotor exit swirl,  $rc_{\theta 3}$  is constant (at design conditions), is included. However, from Horlock and Dixon, it is quite clear that even for fairly large values of  $\alpha_{3m}^*$ , the value of  $\phi$  is little different from the value found when  $\alpha_3^* = 0$ , all other factors being equal. In Figure 6.8 values of  $\phi$  are shown when  $\alpha_{3m}^* = 31.4^\circ$  at  $\phi_m = 0.4$  ( $\phi^* = 0.8$ ) for comparison with the results obtained when  $\alpha_3^* = 0$ .

It should be noted that the rotor efflux flow at off-design conditions is *not* a free vortex.

## Actuator disc approach

In the radial equilibrium design method it was assumed that all radial motion took place within the blade row. However, in most turbomachines of low hub-tip ratio, appreciable radial velocities can be measured outside the blade row. Figure 6.9,

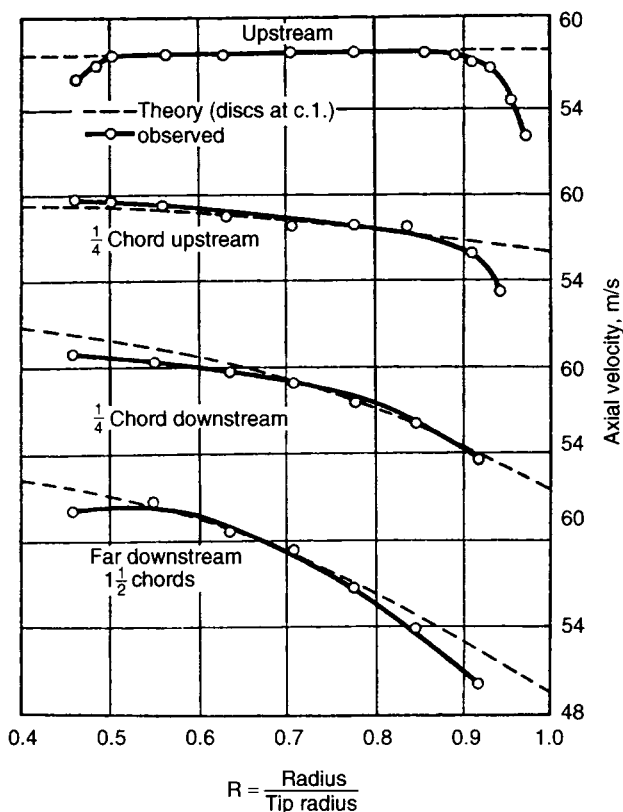
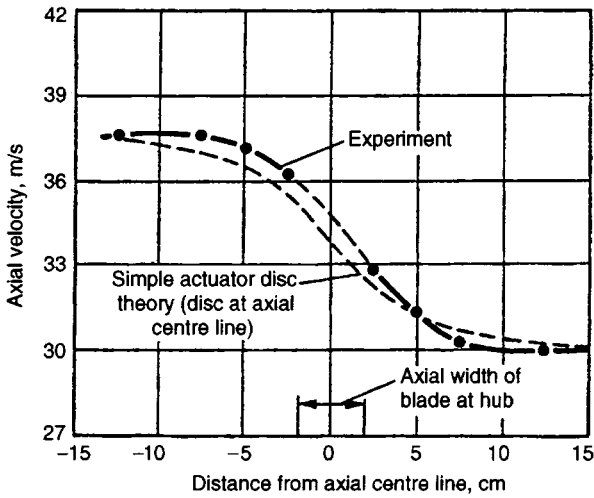
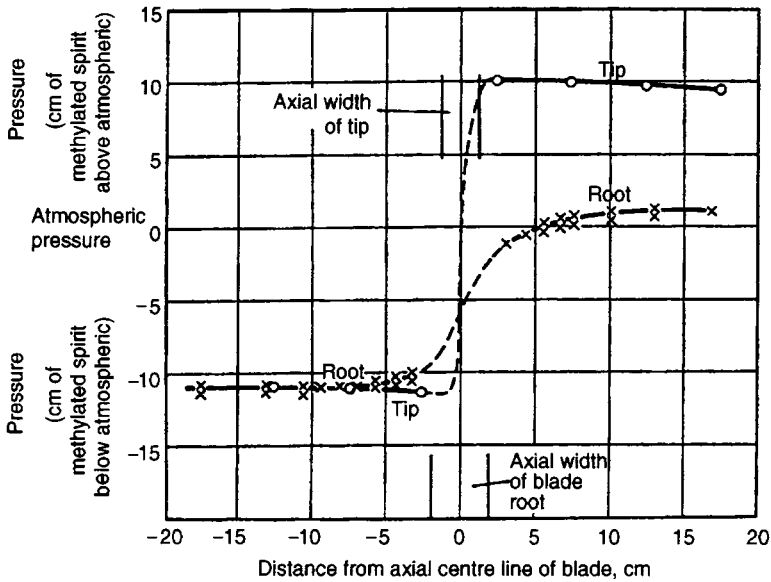


FIG. 6.9. Variation of the distribution in axial velocity through a row of guide vanes (adapted from Hawthorne and Horlock 1962).

taken from a review paper by Hawthorne and Horlock (1962), shows the distribution of the axial velocity component at various axial distances upstream and downstream of an isolated row of stationary inlet guide vanes. This figure clearly illustrates the appreciable redistribution of flow in regions outside of the blade row and that radial velocities must exist in these regions. For the flow through a single row of rotor blades, the variation in pressure (near the hub and tip) and variation in axial velocity (near the hub) both as functions of axial position, are shown in Figure 6.10, also taken from Hawthorne and Horlock. Clearly, radial equilibrium is not established entirely within the blade row.

A more accurate form of three-dimensional flow analysis than radial equilibrium theory is obtained with the *actuator disc* concept. The idea of an actuator disc is quite old and appears to have been first used in the theory of propellers; it has since evolved into a fairly sophisticated method of analysing flow problems in turbomachinery. To appreciate the idea of an actuator disc, imagine that the axial width of each blade row is shrunk while, at the same time, the space-chord ratio, the blade angles and overall length of machine are maintained constant. As the deflection through each blade row for a given incidence is, apart from Reynolds number and Mach number effects (cf. Chapter 3 on cascades), fixed by the cascade



(b)

FIG. 6.10. (a) Pressure variation in the neighbourhood of a rotating blade row. (b) Axial velocity at the hub in the neighbourhood of a rotating blade row (adapted from Hawthorne and Horlock 1962).

geometry, a blade row of reduced width may be considered to affect the flow in exactly the same way as the original row. In the limit as the axial width vanishes, the blade row becomes, conceptually, a *plane discontinuity* of tangential velocity – the actuator disc. Note that while the tangential velocity undergoes an abrupt change in direction, the axial and radial velocities are continuous across the disc.

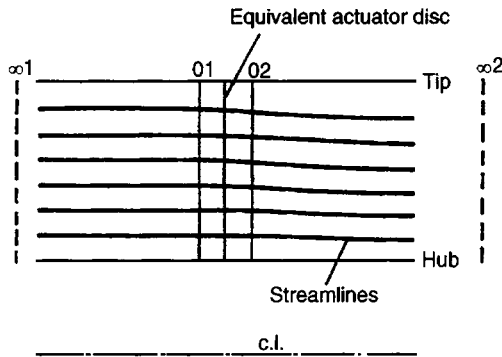


FIG. 6.11. The actuator disc assumption (after Horlock 1958).

An isolated actuator disc is depicted in Figure 6.11 with radial equilibrium established at fairly large axial distances from the disc. An approximate solution to the velocity fields upstream and downstream of the actuator can be found in terms of the axial velocity distributions, *far upstream* and *far downstream* of the disc. The detailed analysis exceeds the scope of this book, involving the solution of the equations of motion, the equation of continuity and the satisfaction of boundary conditions at the walls and disc. The form of the approximate solution is of considerable interest and is quoted below.

For convenience, conditions far upstream and far downstream of the disc are denoted by subscripts  $\infty 1$  and  $\infty 2$  respectively (Figure 6.11). Actuator disc theory proves that at the disc ( $x = 0$ ), at any given radius, the axial velocity is equal to the *mean* of the axial velocities at  $\infty 1$  and  $\infty 2$  at the *same* radius, or

$$c_{x01} = c_{x02} = \frac{1}{2}(c_{x\infty 1} + c_{x\infty 2}). \quad (6.41)$$

Subscripts 01 and 02 denote positions immediately upstream and downstream respectively of the actuator disc. Equation (6.41) is known as the *mean-value rule*.

In the downstream flow field ( $x \geq 0$ ), the *difference* in axial velocity at some position ( $x, r_A$ ) to that at position ( $x = \infty, r_A$ ) is conceived as a velocity perturbation. Referring to Figure 6.12, the axial velocity perturbation at the disc ( $x = 0, r_A$ ) is

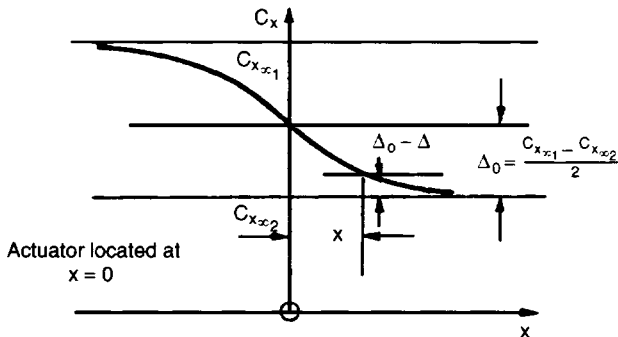


FIG. 6.12. Variation in axial velocity with axial distance from the actuator disc.

denoted by  $\Delta_0$  and at position  $(x, r_A)$  by  $\Delta$ . The important result of actuator disc theory is that velocity perturbations *decay exponentially* away from the disc. This is also true for the upstream flow field ( $x \leq 0$ ). The result obtained for the decay rate is

$$\Delta/\Delta_0 = 1 - \exp[\mp \pi x/(r_t - r_h)], \quad (6.42)$$

where the minus and plus signs above apply to the flow regions  $x \geq 0$  and  $x \leq 0$  respectively. Equation (6.42) is often called the *settling-rate rule*. Since  $c_{x1} = c_{x01} + \Delta$ ,  $c_{x2} = c_{x02} - \Delta$  and noting that  $\Delta_0 = \frac{1}{2}(c_{x01} - c_{x02})$ , eqns. (6.41) and (6.42) combine to give,

$$c_{x1} = c_{x01} - \frac{1}{2}(c_{x01} - c_{x02}) \exp[\pi x/(r_t - r_h)], \quad (6.43a)$$

$$c_{x2} = c_{x02} + \frac{1}{2}(c_{x01} - c_{x02}) \exp[-\pi x/(r_t - r_h)]. \quad (6.43b)$$

At the disc,  $x = 0$ , eqns. (6.43) reduce to eqn. (6.41). It is of particular interest to note, in Figures 6.9 and 6.10, how closely isolated actuator disc theory compares with experimentally derived results.

## Blade row interaction effects

The spacing between consecutive blade rows in axial turbomachines is usually sufficiently small for mutual flow interactions to occur between the rows. This interference may be calculated by an extension of the results obtained from isolated actuator disc theory. As an illustration, the simplest case of two actuator discs situated a distance  $\delta$  apart from one another is considered. The extension to the case of a large number of discs is given in Hawthorne and Harlock (1962).

Consider each disc in turn as though it were in isolation. Referring to Figure 6.13, disc *A*, located at  $x = 0$ , changes the far upstream velocity  $c_{x01}$  to  $c_{x02}$  far downstream. Let us suppose for simplicity that the effect of disc *B*, located at  $x = \delta$ , exactly cancels the effect of disc *A* (i.e. the velocity far upstream of disc *B* is  $c_{x02}$  which changes to  $c_{x01}$  far downstream). Thus, for disc *A* in isolation,

$$c_x = c_{x01} - \frac{1}{2}(c_{x01} - c_{x02}) \exp\left[\frac{-\pi |x|}{H}\right], \quad x \leq 0, \quad (6.44)$$

$$c_x = c_{x02} + \frac{1}{2}(c_{x01} - c_{x02}) \exp\left[\frac{-\pi |x|}{H}\right], \quad x \geq 0, \quad (6.45)$$

where  $|x|$  denotes modulus of  $x$  and  $H = r_t - r_h$ .

For disc *B* in isolation,

$$c_x = c_{x02} - \frac{1}{2}(c_{x02} - c_{x01}) \exp\left[\frac{-\pi |x - \delta|}{H}\right], \quad x \leq \delta, \quad (6.46)$$

$$c_x = c_{x01} + \frac{1}{2}(c_{x02} - c_{x01}) \exp\left[\frac{-\pi |x - \delta|}{H}\right], \quad x \geq \delta. \quad (6.47)$$

Now the combined effect of the two discs is most easily obtained by extracting from the above four equations the velocity perturbations appropriate to a given

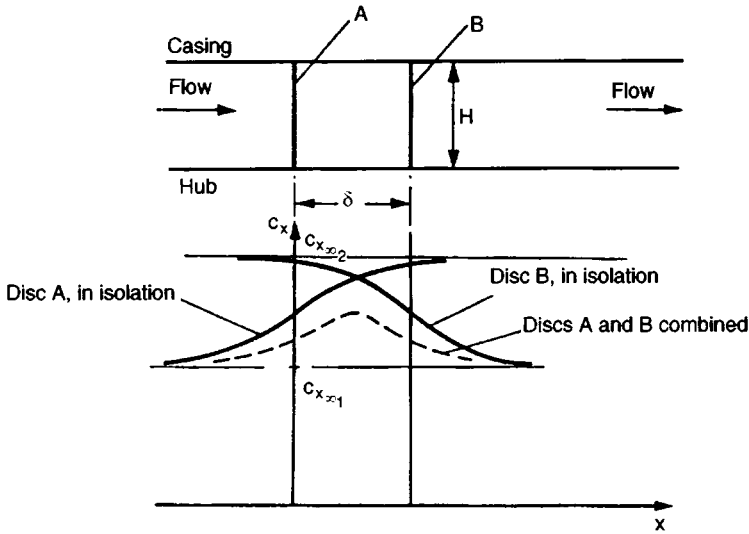


FIG. 6.13. Interaction between two closely spaced actuator discs.

region and adding these to the related radial equilibrium velocity. For  $x \leq 0$ , and to  $c_{x\infty 1}$  the perturbation velocities from eqns. (6.44) and (6.46).

$$c_x = c_{x\infty 1} - \frac{1}{2}(c_{x\infty 1} - c_{x\infty 2}) \left\{ \exp \left[ \frac{-\pi |x|}{H} \right] - \exp \left[ \frac{-\pi |x - \delta|}{H} \right] \right\}. \quad (6.48)$$

For the region  $0 \leq x \leq \delta$ ,

$$c_x = c_{x\infty 2} + \frac{1}{2}(c_{x\infty 1} - c_{x\infty 2}) \left\{ \exp \left[ \frac{-\pi |x|}{H} \right] + \exp \left[ \frac{-\pi |x - \delta|}{H} \right] \right\}. \quad (6.49)$$

For the region  $x \geq \delta$ ,

$$c_x = c_{x\infty 1} + \frac{1}{2}(c_{x\infty 1} - c_{x\infty 2}) \left\{ \exp \left[ \frac{-\pi |x|}{H} \right] - \exp \left[ \frac{-\pi |x - \delta|}{H} \right] \right\}. \quad (6.50)$$

Figure 6.13 indicates the variation of axial velocity when the two discs are regarded as *isolated* and when they are *combined*. It can be seen from the above equations that as the gap between these two discs is increased, so the perturbations tend to vanish. Thus in turbomachines where  $\delta/r$ , is fairly small (e.g. the front stages of aircraft axial compressors or the rear stages of condensing steam turbines), interference effects are strong and one can infer that the simpler radial equilibrium analysis is then inadequate.

## Computer-aided methods of solving the through-flow problem

Although actuator disc theory has given a better understanding of the complicated meridional (the radial-axial plane) through-flow problem in turbomachines of simple

geometry and flow conditions, its application to the design of axial-flow compressors has been rather limited. The extensions of actuator disc theory to the solution of the complex three-dimensional, compressible flows in compressors with varying hub and tip radii and non-uniform total pressure distributions were found to have become too unwieldy in practice. In recent years advanced computational methods have been successfully evolved for predicting the meridional compressible flow in turbomachines with flared annulus walls.

Reviews of numerical methods used to analyse the flow in turbomachines have been given by Gostelow *et al.* (1969), Japikse (1976), Macchi (1985) and Whitfield and Baines (1990) among many others. The literature on computer-aided methods of solving flow problems is now extremely extensive and no attempt is made here to summarise the progress. The real flow in a turbomachine is three-dimensional, unsteady, viscous and is usually compressible, if not transonic or even supersonic. According to Macchi the solution of the full equations of motion with the actual boundary conditions of the turbomachine is still beyond the capabilities of the most powerful modern computers. The best fully three-dimensional methods available are still only simplifications of the real flow.

### Through-flow methods

In any of the so-called *through-flow* methods the equations of motion to be solved are simplified. First, the flow is taken to be steady in both the absolute and relative frames of reference. Secondly, outside of the blade rows the flow is assumed to be axisymmetric, which means that the effects of wakes from an upstream blade row are understood to have “mixed out” so as to give uniform circumferential conditions. Within the blade rows the effects of the blades themselves are modelled by using a passage averaging technique or an equivalent process. Clearly, with these major assumptions, solutions obtained with these through-flow methods can be only approximations to the real flow. As a step beyond this Stow (1985) has outlined the ways, supported by equations, of including the viscous flow effects into the flow calculations.

Three of the most widely used techniques for solving through-flow problems are:

- (1) Streamline curvature, which is based on an iterative procedure, is described in some detail by Macchi (1985) and earlier by Smith (1966). It is the oldest and most widely used method for solving the through-flow problem in axial-flow turbomachines and has with the intrinsic capability of being able to handle variously shaped boundaries with ease. The method is widely used in the gas turbine industry.
- (2) Matrix through-flow or finite difference solutions (Marsh 1968), where computations of the radial equilibrium flow field are made at a number of axial locations *within* each blade row as well as at the leading and trailing edges and outside of the blade row. An illustration of a typical computing mesh for a single blade row is shown in Figure 6.14.
- (3) Time-marching (Denton 1985), where the computation starts from some assumed flow field and the governing equations are marched forward with time. The method, although slow because of the large number of iterations needed to reach a convergent solution, can be used to solve both subsonic and supersonic

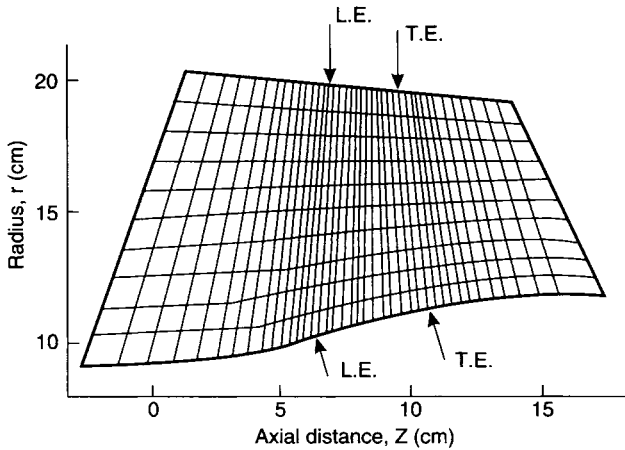


FIG. 6.14. Typical computational mesh for a single blade row (adapted from Macchi 1985).

flow. With the present design trend towards highly loaded blade rows, which can include patches of supersonic flow, this design method has considerable merit.

All three methods solve the same equations of fluid motion, energy and state for an axisymmetric flow through a turbomachine with varying hub and tip radii and therefore lead to the same solution. In the first method the equation for the meridional velocity  $c_m = (c_r^2 + c_x^2)^{1/2}$  in a plane (at  $x = x_a$ ) contain terms involving both the slope and curvature of the meridional streamlines which are estimated by using a polynomial curve-fitting procedure through points of equal stream function on neighbouring planes at  $(x_a - dx)$  and  $(x_a + dx)$ . The major source of difficulty is in accurately estimating the curvature of the streamlines. In the second method a grid of calculating points is formed on which the stream function is expressed as a quasi-linear equation. A set of corresponding finite difference equations are formed which are then solved at all mesh points of the grid. A more detailed description of these methods is rather beyond the scope and intention of the present text.

## Secondary flows

No account of three-dimensional motion in axial turbomachines would be complete without giving, at least, a brief description of secondary flow. When a fluid particle possessing *rotation* is turned (e.g. by a cascade) its axis of rotation is deflected in a manner analogous to the motion of a gyroscope, i.e. in a direction perpendicular to the direction of turning. The result of turning the rotation (or vorticity) vector is the formation of *secondary flows*. The phenomenon must occur to some degree in all turbomachines but is particularly in evidence in axial-flow compressors because of the thick boundary layers on the annulus walls. This case has been discussed in some detail by Horlock (1958), Preston (1953), Carter (1948) and many other writers.

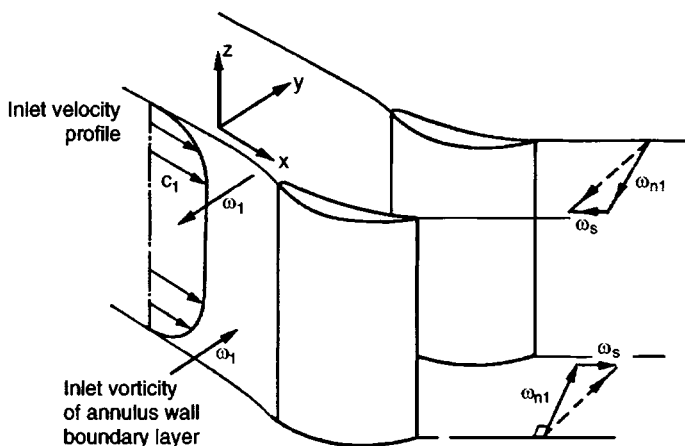


FIG. 6.15. Secondary vorticity produced by a row of guide vanes.

Consider the flow at inlet to the guide vanes of a compressor to be completely axial and with a velocity profile as illustrated in Figure 6.15. This velocity profile is non-uniform as a result of friction between the fluid and the wall; the vorticity of this boundary layer is normal to the approach velocity  $c_1$  and of magnitude

$$\omega_1 = \frac{dc_1}{dz}, \quad (6.51)$$

where  $z$  is distance from the wall.

The direction of  $\omega_1$  follows from the right-hand screw rule and it will be observed that  $\omega_1$  is in opposite directions on the two annulus walls. This vector is turned by the cascade, thereby generating *secondary vorticity* parallel to the outlet stream direction. If the deflection angle  $\epsilon$  is not large, the magnitude of the secondary vorticity  $\omega_s$  is, approximately,

$$\omega_s = -2\epsilon \frac{dc_1}{dz}. \quad (6.52)$$

A swirling motion of the cascade exit flow is associated with the vorticity  $\omega_s$ , as shown in Figure 6.16, which is in opposite directions for the two wall boundary layers. This secondary flow will be the *integrated* effect of the distribution of secondary vorticity along the blade length.

Now if the variation of  $c_1$  with  $z$  is known or can be predicted, then the distribution of  $\omega_s$  along the blade can be found using eqn. (6.52). By considering the secondary flow to be small perturbation of the two-dimensional flow from the vanes, the flow angle distribution can be calculated using a series solution developed by Hawthorne (1955). The actual analysis lies outside the scope (and purpose) of this book, however. Experiments on cascade show excellent agreement with these calculations provided there are but small viscous effects and no flow separations. Such a comparison has been given by Horlock (1963) and a typical result is shown in Figure 6.17. It is clear that the flow is *overturned* near the walls and *underturned*

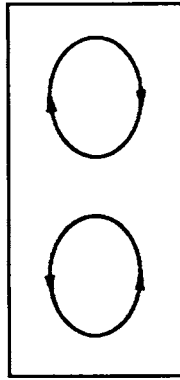


FIG. 6.16. Secondary flows at exit from a blade passage (viewed in upstream direction).

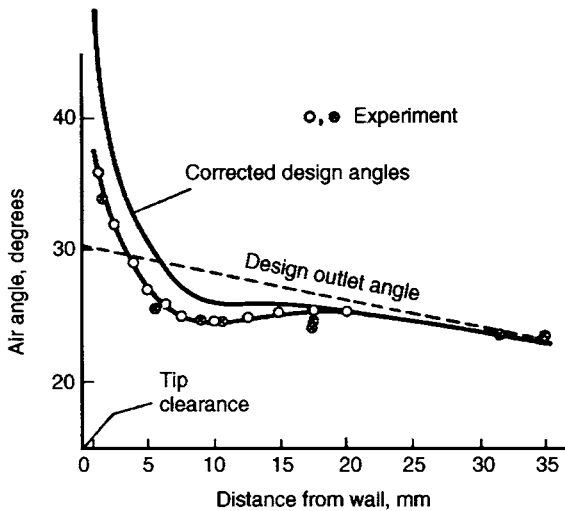


FIG. 6.17. Exit air angle from inlet guide vanes (adapted from Horlock 1963).

some distance away from the walls. It is known that this overturning is a source of inefficiency in compressors as it promotes stalling at the blade extremities.

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## Problems

1. Derive the radial equilibrium equation for an incompressible fluid flowing with axisymmetric swirl through an annular duct.

Air leaves the inlet guide vanes of an axial flow compressor in radial equilibrium and with a free-vortex tangential velocity distribution. The absolute static pressure and static temperature at the hub, radius 0.3 m, are 94.5 kPa and 293 K respectively. At the casing, radius 0.4 m, the absolute static pressure is 96.5 kPa. Calculate the flow angles at exit from the vanes at the hub and casing when the inlet absolute stagnation pressure is 101.3 kPa. Assume the fluid to be inviscid and incompressible. (Take  $R = 0.287 \text{ kJ}/(\text{kg}^\circ\text{C})$  for air.).

2. A gas turbine stage has an initial absolute pressure of 350 kPa and a temperature of  $565^\circ\text{C}$  with negligible initial velocity. At the mean radius, 0.36 m, conditions are as follows:

|                                      |         |
|--------------------------------------|---------|
| Nozzle exit flow angle               | 68 deg  |
| Nozzle exit absolute static pressure | 207 kPa |
| Stage reaction                       | 0.2     |

Determine the flow coefficient and stage loading factor at the mean radius and the reaction at the hub, radius 0.31 m, at the design speed of 8000 rev/min, given that stage is to have a

free vortex swirl at this speed. You may assume that losses are absent. Comment upon the results you obtain.

(Take  $C_p = 1.148 \text{ kJ/(kg}^\circ\text{C)}$  and  $\gamma = 1.33$ .)

3. Gas enters the nozzles of an axial flow turbine stage with uniform total pressure at a uniform velocity  $c_1$  in the axial direction and leaves the nozzles at a constant flow angle  $\alpha_2$  to the axial direction. The absolute flow leaving the rotor  $c_3$  is completely axial at all radii.

Using radial equilibrium theory and assuming no losses in total pressure show that

$$(c_3^2 - c_1^2)/2 = U_m c_{\theta m2} \left[ 1 - \left( \frac{r}{r_m} \right)^{\cos^2 \alpha_2} \right]$$

where  $U_m$  is the mean blade speed,

$c_{\theta m2}$  is the tangential velocity component at nozzle exit at the mean radius  $r = r_m$ .

(Note: The approximate  $c_3 = c_1$  at  $r = r_m$  is used to derive the above expression.)

4. Gas leaves an untwisted turbine nozzle at an angle  $\alpha$  to the axial direction and in radial equilibrium. Show that the variation in axial velocity from root to tip, assuming total pressure is constant, is given by

$$c_x r^{\sin^2 \alpha} = \text{constant.}$$

Determine the axial velocity at a radius of 0.6 m when the axial velocity is 100 m/s at a radius of 0.3 m. The outlet angle  $\alpha$  is 45 deg.

5. The flow at the entrance and exit of an axial-flow compressor rotor is in radial equilibrium. The distributions of the tangential components of absolute velocity with radius are:

$$c_{\theta 1} = ar - b/r, \text{ before the rotor,}$$

$$c_{\theta 2} = ar + b/r, \text{ after the rotor,}$$

where  $a$  and  $b$  are constants. What is the variation of work done with radius? Deduce expressions for the axial velocity distributions before and after the rotor, assuming incompressible flow theory and that the radial gradient of stagnation pressure is zero.

At the mean radius,  $r = 0.3 \text{ m}$ , the stage loading coefficient,  $\psi = \Delta W/U_t^2$  is 0.3, the reaction ratio is 0.5 and the mean axial velocity is 150 m/s. The rotor speed is 7640 rev/min. Determine the rotor flow inlet and outlet angles at a radius of 0.24 m given that the hub-tip ratio is 0.5. Assume that at the mean radius the axial velocity remained unchanged ( $c_{x1} = c_{x2}$  at  $r = 0.3 \text{ m}$ ).

(Note:  $\Delta W$  is the specific work and  $U_t$  the blade tip speed.)

6. An axial-flow turbine stage is to be designed for free-vortex conditions at exit from the nozzle row and for zero swirl at exit from the rotor. The gas entering the stage has a stagnation temperature of 1000 K, the mass flow rate is 32 kg/s, the root and tip diameters are 0.56 m and 0.76 m respectively, and the rotor speed is 8000 rev/min. At the rotor tip the stage reaction is 50% and the axial velocity is constant at 183 m/s. The velocity of the gas entering the stage is equal to that leaving.

Determine:

- (i) the maximum velocity leaving the nozzles;
- (ii) the maximum absolute Mach number in the stage;
- (iii) the root section reaction;
- (iv) the power output of the stage;
- (v) the stagnation and static temperatures at stage exit.

(Take  $R = 0.287 \text{ kJ/(kg}^\circ\text{C)}$  and  $C_p = 1.147 \text{ kJ/(kg}^\circ\text{C)}$ .)

7. The rotor blades of an axial-flow turbine stage are 100 mm long and are designed to receive gas at an incidence of 3 deg from a nozzle row. A free-vortex whirl distribution is to be maintained between nozzle exit and rotor entry. At rotor exit the absolute velocity is 150 m/s in the axial direction at all radii. The deviation is 5 deg for the rotor blades and zero for the nozzle blades at all radii. At the hub, radius 200 mm, the conditions are as follows:

|                          |         |
|--------------------------|---------|
| Nozzle outlet angle      | 70 deg  |
| Rotor blade speed        | 180 m/s |
| Gas speed at nozzle exit | 450 m/s |

Assuming that the axial velocity of the gas is constant across the stage, determine

- (i) the nozzle outlet angle at the tip;
- (ii) the rotor blade inlet angles at hub and tip;
- (iii) the rotor blade outlet angles at hub and tip;
- (iv) the degree of reaction at root and tip.

Why is it essential to have a *positive* reaction in a turbine stage?

8. The rotor and stator of an isolated stage in an axial-flow turbomachine are to be represented by two actuator discs located at axial positions  $x = 0$  and  $x = \delta$  respectively. The hub and tip diameters are constant and the hub-tip radius ratio  $r_h/r_t$  is 0.5. The rotor disc considered on its own has an axial velocity of 100 m/s far upstream and 150 m/s downstream at a constant radius  $r = 0.75r_t$ . The stator disc in isolation has an axial velocity of 150 m/s far upstream and 100 m/s far downstream at radius  $r = 0.75r_t$ . Calculate and plot the axial velocity variation between  $-0.5 \leq x/r_t \leq 0.6$  at the given radius for each actuator disc in isolation and for the combined discs when

- (i)  $\delta = 0.1r_t$ , (ii)  $\delta = 0.25r_t$ , (iii)  $\delta = r_t$ .