Chapter 2

Free Vibration of Single Degree of Freedom Systems

2.1 $\delta_{st} = 5 \times 10^{-3}$ m

$\omega_n = \left(\frac{g}{\delta_{st}}\right)^{1/2} = \left(\frac{9.81}{5 \times 10^{-3}}\right)^{1/2} = 44.2945$ rad/sec $= 7.0497$ Hz

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2.2 $\tau_n = 0.21$ sec $= 2\pi\sqrt{\frac{m}{k}}$, $\sqrt{m} = 0.21\sqrt{k}/2\pi$

(i) $(\tau_n)_{new} = \frac{2\pi\sqrt{m}}{\sqrt{k_{new}}} = \frac{2\pi\sqrt{m}}{\sqrt{1.5\,k}} = \frac{2\pi\left(\frac{0.21\sqrt{k}}{2\pi}\right)}{\sqrt{1.5\,k}} = 0.1715$ sec.

(ii) $(\tau_n)_{new} = \frac{2\pi\sqrt{m}}{\sqrt{k_{new}}} = \frac{2\pi\sqrt{m}}{\sqrt{0.5\,k}} = 2\pi\left(\frac{0.21\sqrt{k}}{2\pi}\right)\frac{1}{\sqrt{0.5\,k}} = 0.2970$ sec.

2.3 $\omega_n = 62.832$ rad/sec $= \sqrt{\frac{k}{m}}$, $\sqrt{m} = \sqrt{k}/62.832$

when spring constant is reduced, ω_n decreases.

$(\omega_n)_{new} = 0.55\,\omega_n = 34.5576$ rad/sec $= \sqrt{\frac{k_{new}}{m_{new}}} = \sqrt{\frac{k-800}{m}}$

$\sqrt{\frac{k-800}{k}} \times 62.836 = 34.5576$, $\sqrt{\frac{k-800}{k}} = 0.55$

$\frac{k-800}{k} = (0.55)^2 = 0.3025$

$k = 1146.9534$ N/m

$\sqrt{m} = \sqrt{k}/62.832$; $m = k/62.832^2 = \frac{1146.9534}{3947.8602}$

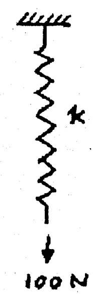
$m = 0.2905$ kg

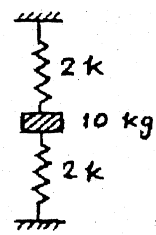
2.4 $k = 100/\left(\frac{10}{1000}\right) = 10000$ N/m

$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{4k}{m}} = \left(\frac{4 \times 10^4}{10}\right)^{1/2}$

$= 63.2456$ rad/sec

$\tau_n = \frac{2\pi}{\omega_n} = \frac{6.2832}{63.2456} = 0.0993$ sec





(2.5) $m = \dfrac{2000}{386.4}$.

Let $\omega_n = 7.5$ rad/sec.

$$\omega_n = \sqrt{\frac{k_{eq}}{m}}$$

$$k_{eq} = m\,\omega_n^2 = \left(\frac{2000}{386.4}\right)(7.5)^2 = 291.1491 \text{ lb/in} = 4\text{ k}$$

where k is the stiffness of the air spring.

Thus $k = \dfrac{291.1491}{4} = 72.7873$ lb/in.

(2.6) $x = A\cos(\omega_n t - \phi_0)$, $\quad \dot{x} = -\omega_n A \sin(\omega_n t - \phi_0)$,

$\ddot{x} = -\omega_n^2 A \cos(\omega_n t - \phi_0)$

(a) $\omega_n A = 0.1$ m/sec ; $\tau_n = \dfrac{2\pi}{\omega_n} = 2$ sec, $\omega_n = 3.1416$ rad/sec

$A = 0.1/\omega_n = 0.03183$ m

(d) $x_0 = x(t=0) = A\cos(-\phi_0) = 0.02$ m

$\cos(-\phi_0) = \dfrac{0.02}{A} = 0.6283$

$\phi_0 = 51.0724°$

(b) $\dot{x}_0 = \dot{x}(t=0) = -\omega_n A \sin(-\phi_0) = -0.1\sin(-51.0724°)$

$= 0.07779$ m/sec

(c) $\ddot{x}\big|_{max} = \omega_n^2 A = (3.1416)^2(0.03183) = 0.314151$ m/sec^2

(2.7) For small angular rotation of bar PQ about P,

$$\frac{1}{2}(k_{12})_{eq}(\theta l_3)^2 = \frac{1}{2}k_1(\theta l_1)^2 + \frac{1}{2}k_2(\theta l_2)^2$$

i.e., $\quad (k_{12})_{eq} = (k_1 l_1^2 + k_2 l_2^2)/l_3^2$

Let k_{eq} = overall spring constant at Q.

$$\frac{1}{k_{eq}} = \frac{1}{(k_{12})_{eq}} + \frac{1}{k_3}$$

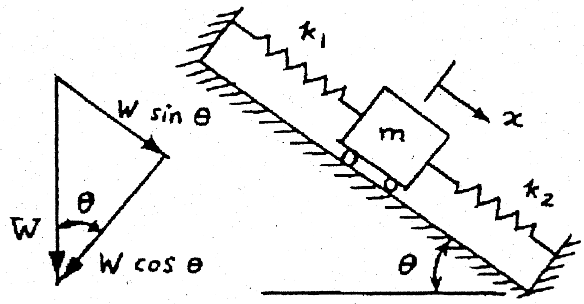
$$k_{eq} = \frac{(k_{12})_{eq}\,k_3}{(k_{12})_{eq} + k_3} = \frac{\left\{k_1\left(\frac{l_1}{l_3}\right)^2 + k_2\left(\frac{l_2}{l_3}\right)^2\right\}k_3}{k_1\left(\frac{l_1}{l_3}\right)^2 + k_2\left(\frac{l_2}{l_3}\right)^2 + k_3}$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k_1 k_2 \ell_1^2 + k_2 k_3 \ell_2^2}{m\,(k_1 \ell_1^2 + k_2 \ell_2^2 + k_3 \ell_3^2)}}$$

2.8 $m = 2000$ kg , $\delta_{st} = 0.02$ m.

$$\omega_n = (g/\delta_{st})^{1/2} = \left(\frac{9.81}{0.02}\right)^{1/2} = 22.1472 \text{ rad/sec}$$

2.9 Let x be measured from the position of mass at which the springs are unstretched.



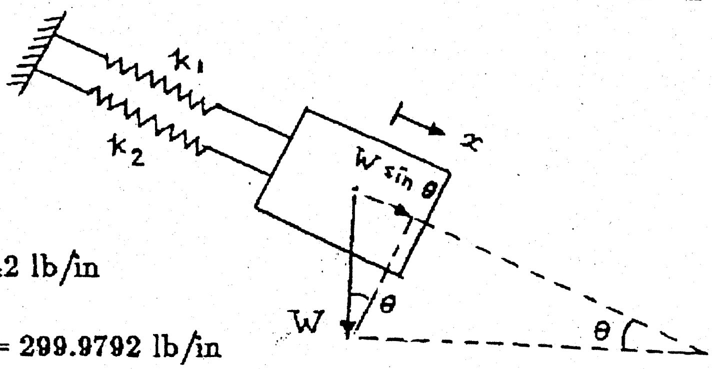
Equation of motion is

$$m\ddot{x} = -k_1(x+\delta_{st}) - k_2(x+\delta_{st}) + W \sin\theta \quad \text{---- } (E_1)$$

where $\delta_{st}(k_1+k_2) = W \sin\theta$.

Thus Eq. (E_1) becomes $m\ddot{x} + (k_1+k_2)\,x = 0 \Rightarrow \omega_n = \sqrt{\dfrac{k_1+k_2}{m}}$.

2.10



$$k_1 = \frac{A_1 E_1}{\ell_1} = \frac{\frac{\pi}{4}(0.05)^2\,(30\,(10^6))}{30\,(12)} = 163.6250 \text{ lb/in}$$

$$k_2 = \frac{A_2 E_2}{\ell_2} = \frac{163.625\,(25)}{30} = 136.3542 \text{ lb/in}$$

$$k_{eq} = k_1 + k_2 = 163.6250 + 136.3542 = 299.9792 \text{ lb/in}$$

Let x be measured from the unstretched length of the springs. The equation of motion is:

$$m\,\ddot{x} = -(k_1 + k_2)\,(x + \delta_{st}) + W \sin\theta$$

where $(k_1 + k_2)\,\delta_{st} = W \sin\theta$

$$\text{i.e.,}\quad m\,\ddot{x} + (k_1 + k_2)\,x = 0$$

Thus the natural frequency of vibration of the cart is given by

$$\omega_n = \sqrt{\frac{k_1 + k_2}{m}} = \sqrt{\frac{299.9792\,(386.4)}{5000}} = 4.8148 \text{ rad/sec}$$

2.11 Weight of electronic chassis = 500 N. To be able to use the unit in a vibratory environment with a frequency range of 0 - 5 Hz, its natural frequency must be away from the frequency of the environment. Let the natural frequency be ω_n = 10 Hz = 62.832 rad/sec. Since

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = 62.832$$

we have

$$k_{eq} = m\,\omega_n^2 = \left(\frac{500}{9.81}\right)(62.832)^2 = 20.1857\,(10^4)\ \text{N/m} \equiv 4\,k$$

so that k = spring constant of each spring = 50,464.25 N/m. For a helical spring,

$$k = \frac{G\,d^4}{8\,n\,D^3}$$

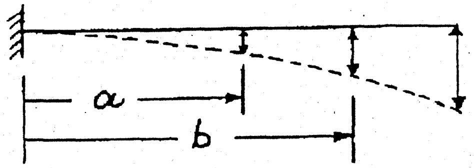
Assuming the material of springs as steel with G = 80 (10^9) Pa, n = 5 and d = 0.005 m, we find

$$k = 50{,}464.25 = \frac{80\,(10^9)\,(0.005)^4}{8\,(5)\,D^3}$$

This gives

$$D^3 = \frac{1250\,(10^{-3})}{50464.25} = 24{,}770.0\,(10^{-9}) \quad \text{or} \quad D = 0.0291492\ \text{m} = 2.91492\ \text{cm}$$

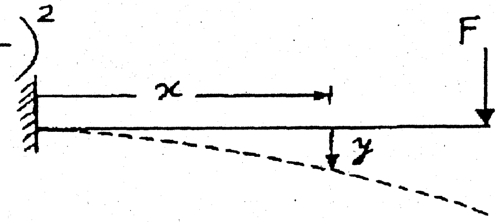
2.12 (i) with springs k_1 and k_2:



Let y_a, y_b, y_l be deflections of beam at distances a, b, l from fixed end.

$$\frac{1}{2}(k_{12})_{eq}\, y_l^2 = \frac{1}{2} k_1 y_a^2 + \frac{1}{2} k_2 y_b^2$$

i.e.,
$$(k_{12})_{eq} = k_1\left(\frac{y_a}{y_l}\right)^2 + k_2\left(\frac{y_b}{y_l}\right)^2$$



$$y = \frac{F x^2}{6EI}(3l - x)$$

$$@\ x = a, \quad y_a = \frac{F a^2}{6EI}(3l - a)$$

$$@\ x = b, \quad y_b = \frac{F b^2}{6EI}(3l - b)$$

$$@\ x = l, \quad y_l = \frac{F l^3}{3EI}$$

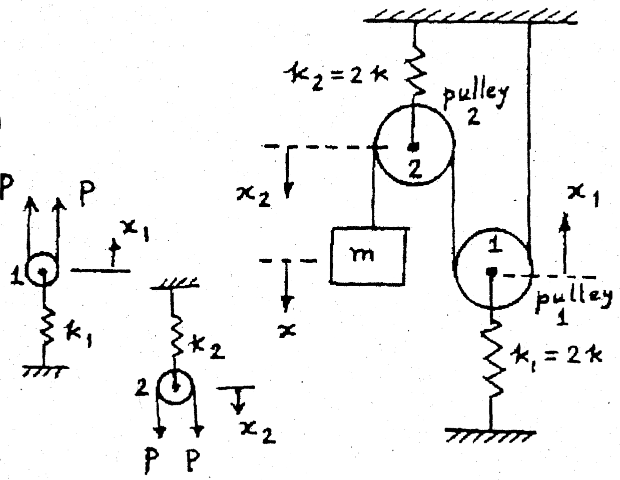
$$\omega_n = \left[\frac{k_1 k_3 \left(\frac{y_a}{y_l}\right)^2 + k_2 k_3 \left(\frac{y_b}{y_l}\right)^2}{m\left\{k_1\left(\frac{y_a}{y_l}\right)^2 + k_2\left(\frac{y_b}{y_l}\right)^2 + k_{beam}\right\}}\right]^{\frac{1}{2}} \quad \text{where } k_{beam} = \frac{3EI}{l^3}$$

$$= \left[\frac{k_1(3EI)\,a^4(3l-a)^2 + k_2(3EI)\,b^4(3l-b)^2}{m\,l^3\left\{k_1 a^4(3l-a)^2 + k_2 b^4(3l-b)^2 + 12\,EI\,l^3\right\}}\right]^{\frac{1}{2}}$$

(ii) without springs k_1 and k_2:

$$\omega_n = \sqrt{\frac{k_{beam}}{m}} = \sqrt{\frac{3EI}{m l^3}}$$

(2.13) Let x_1, x_2 = displacements of pulleys 1, 2



$$x = 2x_1 + 2x_2 \qquad ---- \quad (E_1)$$

Let P = tension in rope.

For equilibrium of pulley 1,

$$2P = k_1 x_1 \qquad ---- \quad (E_2)$$

For equilibrium of pulley 2,

$$2P = k_2 x_2 \qquad ---- \quad (E_3)$$

where $\frac{1}{k_1} = \frac{1}{4k} + \frac{1}{4k} = \frac{1}{2k}$; $k_1 = 2k$

and $k_2 = k + k = 2k$

Combining Eqs. (E_1) to (E_3):

$$x = 2x_1 + 2x_2 = 2\left(\frac{2P}{k_1}\right) + 2\left(\frac{2P}{k_2}\right) = 4P\left(\frac{1}{2k} + \frac{1}{2k}\right) = \frac{4P}{k}$$

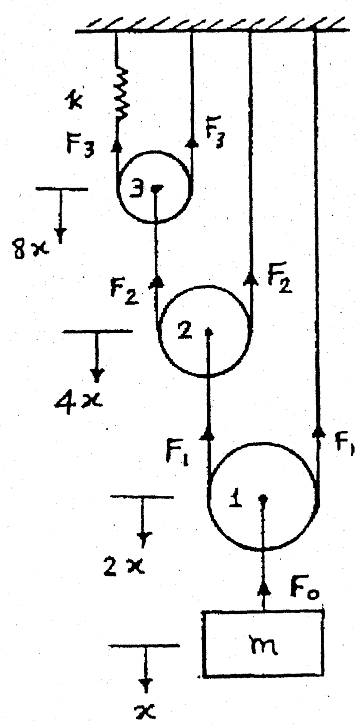
Let k_{eq} = equivalent spring constant of the system:

$$k_{eq} = \frac{P}{x} = \frac{k}{4}$$

Equation of motion of mass m: $\quad m\ddot{x} + k_{eq} x = 0$

$$\therefore \omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k}{4m}}$$

(2.14) For a displacement of x of mass m, pulleys 1, 2 and 3 undergo displacements of 2x, 4x and 8x, respectively. The equation of motion of mass m can be written as



$$m\ddot{x} + F_0 = 0 \qquad (1)$$

where $F_0 = 2F_1 = 4F_2 = 8F_3$ as shown in figure.

Since $F_3 = (8x)\,k$, Eq. (1) can be rewritten as

$$m\ddot{x} + 8F_3 = 8(8k) = 0 \qquad (2)$$

from which we can find

$$\omega_n = \sqrt{\frac{64\,k}{m}} = 8\sqrt{\frac{k}{m}} \qquad (3)$$

2.15 (a) $\omega_n = \sqrt{4k/M}$

(b) $\omega_n = \sqrt{4k/(M+m)}$

Initial conditions:

Velocity of falling mass $m = v = \sqrt{2 g \ell}$ $\left(\because\ v^2 - \overset{0}{u^2} = 2 g \ell\right)$

$x = 0$ at static equilibrium position.

$$x_0 = x(t=0) = -\frac{\text{weight}}{k_{eq}} = -\frac{mg}{4k}$$

Conservation of momentum: $(M+m)\,\dot{x}_0 = m\,v = m\sqrt{2 g \ell}$

$$\dot{x}_0 = \dot{x}(t=0) = \frac{m}{M+m}\sqrt{2 g \ell}$$

Complete solution: $x(t) = A_0 \sin(\omega_n t + \phi_0)$

where $A_0 = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2} = \sqrt{\frac{m^2 g^2}{16 k^2} + \frac{m^2 g \ell}{2k(M+m)}}$

and

$$\phi_0 = \tan^{-1}\left(\frac{x_0 \omega_n}{\dot{x}_0}\right) = \tan^{-1}\left(\frac{-\sqrt{g}}{\sqrt{8 \ell k (M+m)}}\right)$$

2.16 (a) Velocity of anvil $= v = 50$ ft/sec $= 600$ in/sec. $x = 0$ at static equilibrium position.

$$x_0 = x(t=0) = -\frac{\text{weight}}{k_{eq}} = -\frac{m\,g}{4\,k}$$

Conservation of momentum:

$$(M+m)\,\dot{x}_0 = m\,v \quad \text{or} \quad \dot{x}_0 = \dot{x}(t=0) = \frac{m\,v}{M+m}$$

Natural frequency:

$$\omega_n = \sqrt{\frac{4\,k}{M+m}}$$

Complete solution:

$$x(t) = A_0 \sin\,(\omega_n\,t + \phi_0)$$

where

$$A_0 = \left\{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2\right\}^{\frac{1}{2}} = \left\{\frac{m^2\,g^2}{16\,k^2} + \frac{m^2\,v^2}{(M+m)\,4\,k}\right\}^{\frac{1}{2}}$$

and

$$\phi_0 = \tan^{-1}\left(\frac{x_0\,\omega_n}{\dot{x}_0}\right) = \tan^{-1}\left(-\frac{m\,g}{4\,k}\sqrt{\frac{4\,k}{(M+m)}}\,\frac{(M+m)}{m\,v}\right) = \tan^{-1}\left(-\frac{g\sqrt{M+m}}{v\sqrt{4\,k}}\right)$$

Since $v = 600$, $m = 12/386.4$, $M = 100/386.4$, $k = 100$, we find

$$A_0 = \left\{ \left(\frac{12\ (386.4)}{4\ (100)\ (386.4)} \right)^2 + \left(\frac{12\ (600)}{386.4} \right)^2 \frac{386.4}{112\ (400)} \right\}^{\frac{1}{2}} = 1.7308 \text{ in}$$

$$\phi_0 = \tan^{-1} \left(- \frac{386.4 \sqrt{112}}{\sqrt{386.4}\ (600) \sqrt{400}} \right) = \tan^{-1} (-0.01734) = -0.9934 \text{ deg}$$

(b) $x = 0$ at static equilibrium position: $x_0 = x(t{=}0) = 0$. Conservation of momentum gives:

$$M \dot{x}_0 = m v \quad \text{or} \quad \dot{x}_0 = \dot{x}(t{=}0) = \frac{m v}{M}$$

Complete solution:

$$x(t) = A_0 \sin (\omega_n t + \phi_0)$$

where

$$A_0 = \left\{ x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right\}^{\frac{1}{2}} = \left\{ \frac{m^2 v^2 (M)}{M^2\ 4 k} \right\}^{\frac{1}{2}} = \frac{m v}{\sqrt{4 k M}} = \frac{12\ (600) \sqrt{386.4}}{386.4 \sqrt{4\ (100)\ (100)}} = 1.8314 \text{ in}$$

$$\phi_0 = \tan^{-1} \left(\frac{x_0\ \omega_n}{\dot{x}_0} \right) = \tan^{-1} (0) = 0$$

2.17 $k_1 = \dfrac{3 E_1 I_1}{l_1^3}$ (at tip) ; $k_2 = \dfrac{48 E_2 I_2}{l_2^3}$ (at middle)

$k_{eq} = k_1 + k_2$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\left(\frac{3 E_1 I_1}{l_1^3} + \frac{48 E_2 I_2}{l_2^3} \right) \frac{g}{W}}$$

2.18 $$k = \frac{AE}{l} = \frac{\left\{ \frac{\pi}{4} (0.01)^2 \right\} \left\{ 2.07 \times 10^{11} \right\}}{20} = 0.8129 \times 10^6 \text{ N/m}$$

$m = 1000$ kg

$$\omega_n = \sqrt{\frac{k}{m}} = \left(\frac{0.8129 \times 10^6}{1000} \right)^{1/2} = 28.5114 \text{ rad/sec}$$

$\dot{x}_0 = 2$ m/s, $x_0 = 0$ (suddenly stopped while it has velocity)

Period of ensuing vibration $= \tau_n = \dfrac{2\pi}{\omega_n} = \dfrac{2\pi}{28.5114} = 0.2204$ sec

Amplitude $= A = \dot{x}_0 / \omega_n = 2/28.5114 = 0.07015$ m

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2.19 $\omega_n = 2$ Hz $= 12.5664$ rad/sec $= \sqrt{\dfrac{k}{m}}$

$\sqrt{k} = 12.5664 \sqrt{m}$

$$\omega_n' = \sqrt{\frac{k'}{m'}} = \sqrt{\frac{k}{m+1}} = 6.2832 \text{ rad/sec}$$

$$\sqrt{k} = 6.2832\sqrt{m+1}$$
$$= 12.5664\sqrt{m}$$

$$\sqrt{m+1} = 2\sqrt{m}\,, \qquad m = \frac{1}{3}\ \text{kg}$$

$$k = (12.5664)^2\ m = 52.6381\ \text{N/m}$$

2.20 Cable stiffness $= k = \dfrac{A\,E}{\ell} = \dfrac{1}{4}\left[\dfrac{\pi}{4}\,(0.01)^2\right]\, 2.07\,(10^{11}) = 4.0644\,(10^6)\ \text{N/m}$

$$\tau_n = 0.1 = \frac{1}{f_n} = \frac{2\,\pi}{\omega_n}$$

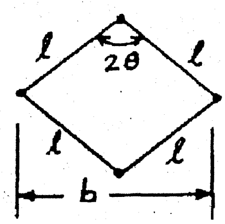
$$\omega_n = \frac{2\,\pi}{0.1} = 20\,\pi = \sqrt{\frac{k}{m}}$$

Hence

$$m = \frac{k}{\omega_n^2} = \frac{4.0644\,(10^6)}{(20\,\pi)^2} = 1029.53\ \text{kg}$$

2.21 $b = 2\,\ell\,\sin\theta$

Neglect masses of links.



(a) $k_{eq} = k\left(\dfrac{4\ell^2 - b^2}{b^2}\right) = k\left(\dfrac{4\ell^2 - 4\ell^2\sin^2\theta}{4\ell^2\sin^2\theta}\right)$

$$= k\left(\frac{\cos^2\theta}{\sin^2\theta}\right)$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k\,g\,\operatorname{cosec}^2\theta}{W}} \qquad \text{(from solution of problem 1.8)}$$

(b) $\omega_n = \sqrt{\dfrac{k\,g}{W}}$ since $k_{eq} = k$.

2.22 $y = \sqrt{\ell^2 - (\ell\sin\theta - x)^2} - \ell\cos\theta = \sqrt{\ell^2\cos^2\theta - x^2 + 2\,\ell\,x\sin\theta} - \ell\cos\theta$

$$= \ell\cos\theta\sqrt{1 - \frac{x^2}{\ell^2\cos^2\theta} + \frac{2\,\ell\,x\sin\theta}{\ell^2\cos^2\theta}} - \ell\cos\theta$$

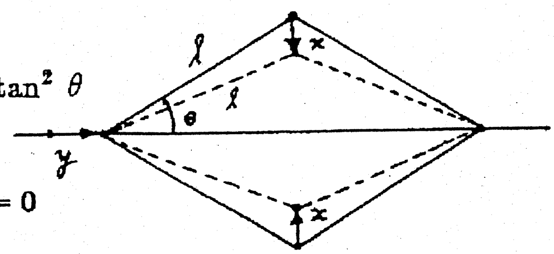
$$\frac{1}{2}\,k_{eq}\,x^2 = \frac{1}{2}\,k_1\,y^2 + \frac{1}{2}\,k_2\,y^2$$

where

$$y \approx \ell\cos\theta\left[1 - \frac{1}{2}\,\frac{x^2}{\ell^2\cos^2\theta} + \frac{1}{2}\,\frac{2\,\ell\,x\sin\theta}{\ell^2\cos^2\theta}\right] - \ell\cos\theta$$
$$\approx \frac{x\sin\theta}{\cos\theta} = x\tan\theta$$

Thus k_{eq} can be expressed as

$$k_{eq} = (k_1 + k_2) \tan^2 \theta$$

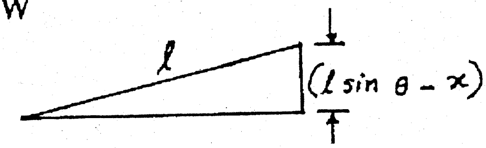


Equation of motion:

$$m \ddot{x} + k_{eq} x = 0$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{(k_1 + k_2) g}{W}} \tan \theta$$



2.23 (a) Neglect masses of rigid links. Let x = displacement of W. Springs are in series.

$$k_{eq} = \frac{k}{2}$$

Equation of motion:

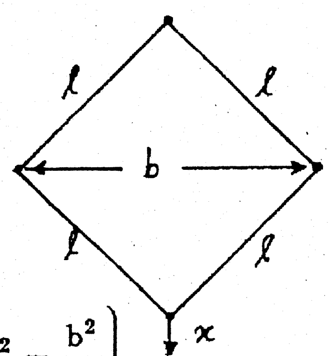
$$m \ddot{x} + k_{eq} x = 0$$

Natual frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k}{2\,m}}$$

(b) Under a displacement of x of mass, each spring will be compressed by an an amount:

$$x_s = x \, \frac{2}{b} \sqrt{\ell^2 - \frac{b^2}{4}}$$



Equivalent spring constant:

$$\frac{1}{2} k_{eq} x^2 = 2 \left(\frac{1}{2} k x_s^2 \right)$$

$$\text{or} \quad k_{eq} = 2 k \left(\frac{x_s}{x} \right)^2 = 2 k \left(\frac{4}{b^2} \right) \left(\ell^2 - \frac{b^2}{4} \right) = \frac{8 k}{b^2} \left(\ell^2 - \frac{b^2}{4} \right)$$

Equation of motion:

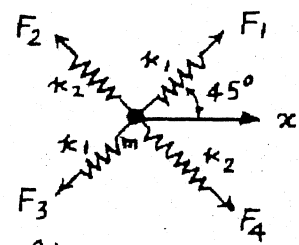
$$m \ddot{x} + k_{eq} x = 0$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{8 k}{b^2 m} \left(\ell^2 - \frac{b^2}{4} \right)}$$

2.24 $F_1 = F_3 = k_1\, x \cos 45°$

$F_2 = F_4 = k_2\, x \cos 135°$



F = force along x $= F_1 \cos 45° + F_2 \cos 135° + F_3 \cos 45° + F_4 \cos 135°$

$$= 2x\,(k_1 \cos^2 45° + k_2 \cos^2 135°)$$

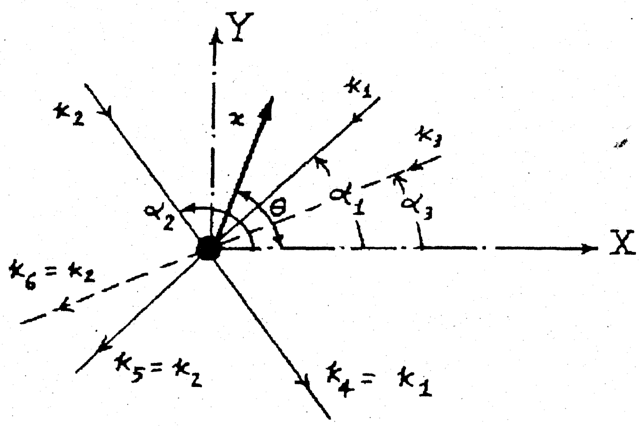
$$k_{eq} = \frac{F}{x} = 2\left(\frac{k_1}{2} + \frac{k_2}{2}\right) = k_1 + k_2$$

Equation of motion: $m\ddot{x} + (k_1 + k_2)\,x = 0$

2.25 Let α_i denote the angle made by i^{th} spring with respect to X axis.

Let x = displacement of mass along the direction defined by θ.

If k_{eq} = equivalent spring constant, the equivalence of potential energies gives



$$\frac{1}{2} k_{eq}\, x^2 = \frac{1}{2} \sum_{i=1}^{6} k_i \left\{ x \cos(\theta - \alpha_i) \right\}^2$$

$$k_{eq} = \sum_{i=1}^{6} k_i \cos^2(\theta - \alpha_i) = \sum_{i=1}^{6} k_i (\cos\theta \cos\alpha_i + \sin\theta \sin\alpha_i)^2$$

$$= \sum_{i=1}^{6} k_i \left(\cos^2\alpha_i \cos^2\theta + \sin^2\alpha_i \sin^2\theta \right) + 2 \sum_{i=1}^{6} \left(\cos\alpha_i \sin\alpha_i \cos\theta \sin\theta \right)$$

Natural frequency $= \omega_n = \sqrt{\dfrac{k_{eq}}{m}}$

For ω_n to be independent of θ, $\displaystyle\sum_{i=1}^{6} k_i \cos^2\alpha_i = \sum_{i=1}^{6} k_i \sin^2\alpha_i$ ... (E_1)

and $\displaystyle\sum_{i=1}^{6} k_i \cos\alpha_i \sin\alpha_i = 0$ ... (E_2)

(E_1) and (E_2) can be rewritten as

$$\sum_{i=1}^{6} k_i \left(\frac{1}{2} + \frac{1}{2}\cos 2\alpha_i \right) = \sum_{i=1}^{6} k_i \left(\frac{1}{2} - \frac{1}{2}\cos 2\alpha_i \right)$$

and $\displaystyle\frac{1}{2} \sum_{i=1}^{6} k_i \sin 2\alpha_i = 0$

i.e. $\displaystyle\sum_{i=1}^{6} k_i \cos 2\alpha_i = 0$ --- (E_3)

and $\displaystyle\sum_{i=1}^{6} k_i \sin 2\alpha_i = 0$ --- (E_4)

In the present example, (E_3) and (E_4) become

$$k_1 \cos 60^\circ + k_2 \cos 240^\circ + k_3 \cos 2\alpha_3 + k_1 \cos 420^\circ + k_2 \cos 600^\circ + k_3 \cos (360^\circ + 2\alpha_3) = 0$$

$$k_1 \sin 60^\circ + k_2 \sin 240^\circ + k_3 \sin 2\alpha_3 + k_1 \sin 420^\circ + k_2 \sin 600^\circ + k_3 \sin (360^\circ + 2\alpha_3) = 0$$

i.e., $\left.\begin{array}{l} k_1 - k_2 + 2 k_3 \cos 2\alpha_3 = 0 \\ \sqrt{3}\, k_1 - \sqrt{3}\, k_2 + 2 k_3 \sin 2\alpha_3 = 0 \end{array}\right\}$; $\begin{array}{l} 2 k_3 \cos 2\alpha_3 = k_2 - k_1 \quad \ldots (E_5) \\ 2 k_3 \sin 2\alpha_3 = \sqrt{3}\,(k_2 - k_1) \ldots (E_6) \end{array}$

Squaring (E_5) and (E_6) and adding,

$$4 k_3^2 = (k_2 - k_1)^2 (1+3)$$

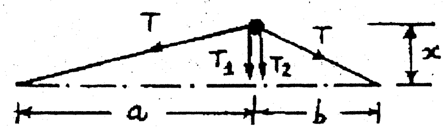
$$\therefore k_3 = \pm (k_2 - k_1) \Rightarrow k_3 = |k_2 - k_1|$$

Dividing (E_6) by (E_5),

$$\tan 2\alpha_3 = \sqrt{3}$$

$$\therefore \alpha_3 = \tfrac{1}{2} \tan^{-1} (\sqrt{3}) = 30^\circ$$

2.26 $T_1 = \frac{x}{a} T, \quad T_2 = \frac{x}{b} T$



(a) $m \ddot{x} + (T_1 + T_2) = 0$

$$m \ddot{x} + \left(\frac{T}{a} + \frac{T}{b}\right) x = 0$$

(b) $\omega_n = \sqrt{\dfrac{\frac{T}{a} + \frac{T}{b}}{m}} = \sqrt{\dfrac{T}{m\,a\,b}(a+b)}$

2.27 $m = \dfrac{160}{386.4} \dfrac{\text{lb-sec}^2}{\text{inch}}, \; k = 10$ lb/inch.

Velocity of jumper as he falls through 200 ft:

$$m\,g\,h = \frac{1}{2} m\,v^2 \quad \text{or} \quad v = \sqrt{2\,g\,h} = \sqrt{2\,(386.4)\,(200\,(12))} = 1{,}361.8811 \text{ in/sec}$$

About static equilibrium position:

$$x_0 = x(t{=}0) = 0, \quad \dot{x}_0 = \dot{x}(t{=}0) = 1{,}361.8811 \text{ in/sec}$$

Response of jumper: $x(t) = A_0 \sin(\omega_n t + \phi_0)$

where

$$A_0 = \left\{ x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2 \right\}^{\frac{1}{2}} = \frac{\dot{x}_0}{\omega_n} = \frac{\dot{x}_0 \sqrt{m}}{\sqrt{k}} = \frac{1361.8811}{\sqrt{10}} \sqrt{\left(\frac{160}{386.4}\right)} = 277.1281 \text{ in}$$

and $\phi_0 = \tan^{-1}\left(\dfrac{x_0\,\omega_n}{\dot{x}_0}\right) = 0$

2.28 The natural frequency of a vibrating rope is given by (see Problem 2.26):

$$\omega_n = \sqrt{\frac{T\,(a+b)}{m\,a\,b}}$$

where T = tension in rope, m = mass, and a and b are lengths of the rope on both sides of the mass. For the given data:

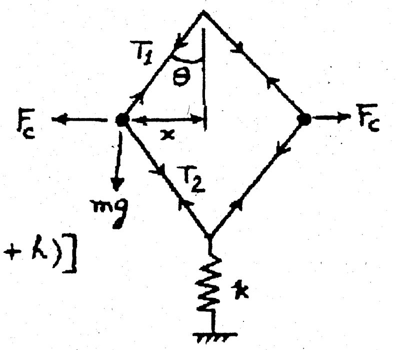
$$10 = \left\{ \frac{T\,(80+160)}{\left(\frac{120}{386.4}\right)(80)\,(160)} \right\}^{\frac{1}{2}} = \sqrt{T\,(0.060375)}$$

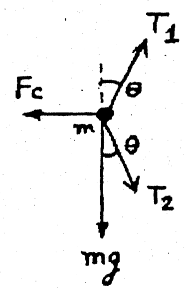
which yields

$$T = \frac{100}{0.060375} = 1{,}656.3147 \text{ lb}$$

2.29 When $\omega = 0$, total vertical height $= 2l + h$

When $\omega \neq 0$, total vertical height $= (2l\cos\theta + h)$





$F_c = m\,x\,\omega^2$

$x = l\sin\theta$

spring force $= k\left[2l + h - (2l\cos\theta + h)\right]$
$= 2kl\,(1 - \cos\theta)$

For vertical equilibrium of mass m,

$$mg + T_2\cos\theta = T_1\cos\theta \quad \text{--- } (E_1)$$

For horizontal equilibrium, $F_c = (T_1 + T_2)\sin\theta$

$$T_2 = (F_c - T_1\sin\theta)/\sin\theta \quad \text{--- } (E_2)$$

From (E_2), (E_1) can be expressed as

$$mg + \left(\frac{F_c - T_1\sin\theta}{\sin\theta}\right)\cos\theta = T_1\cos\theta$$

i.e. $$T_1 = \frac{mg + F_c\cot\theta}{2\cos\theta} = \frac{mg + m\omega^2 l\cos\theta}{2\cos\theta}$$

$$T_2 = \frac{F_c - T_1\sin\theta}{\sin\theta} = \frac{m\,x\,\omega^2 - \frac{mg}{2}\tan\theta - \frac{m\omega^2 l}{2}\sin\theta}{\sin\theta}$$

$$= \frac{m\,l\,\omega^2}{2} - \frac{mg}{2\cos\theta}$$

spring force $= 2kl\,(1 - \cos\theta) = 2\,T_2\cos\theta$
$= m\,l\,\omega^2\cos\theta - mg$

$$\cos\theta = \left(\frac{2kl + mg}{2kl + m\,l\,\omega^2}\right) \quad \text{--- } (E_3)$$

This equation defines the equilibrium position of mass m.
For small oscillations about the equilibrium position,
Newton's second law gives

$$2m\ddot{y} + k y = 0 \quad , \qquad \omega_n = \sqrt{\frac{2k}{m}}$$

2.30 (a) Let P = total spring force, F = centrifugal force acting on each ball. Equilibrium of moments about the pivot of bell crank lever (O) gives:

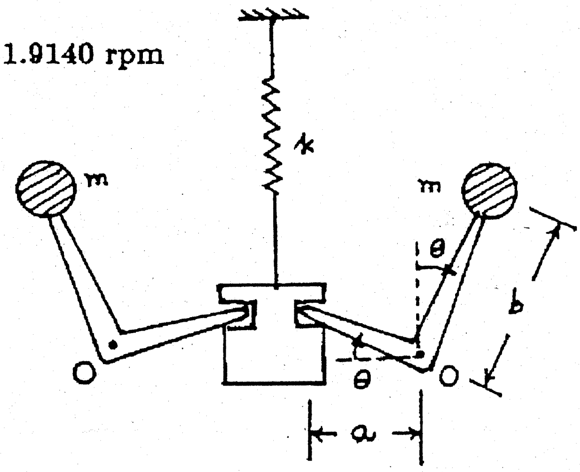
$$F\left(\frac{20}{100}\right) = \frac{P}{2}\left(\frac{12}{100}\right) \tag{1}$$

When $P = 10^4 \left(\frac{1}{100}\right) = 100$ N, and

$$F = m\,r\,\omega^2 = m\,r\left(\frac{2\,\pi\,N}{60}\right)^2 = \frac{25}{9.81}\left(\frac{16}{100}\right)\left(\frac{2\,\pi\,N}{60}\right)^2 = 0.004471\ N^2$$

where N = speed of the governor in rpm. Equation (1) gives:

$$0.004471\ N^2\ (0.2) = \frac{100}{2}\ (0.12) \quad \text{or} \quad N = 81.9140\ \text{rpm}$$



(b) Consider a small displacement of the ball arm about the vertical position. Equilibrium about point O gives:

$$(m\,b^2)\,\ddot{\theta} + (k\ a\ \sin\theta)\ a\ \cos\theta = 0 \tag{2}$$

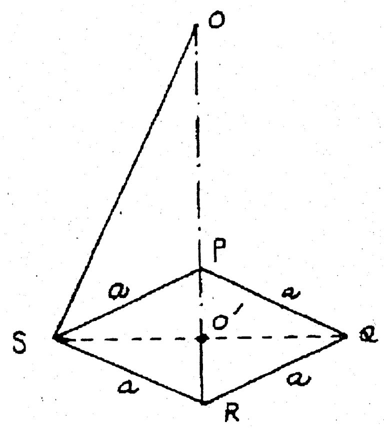
For small vallues of θ, $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, and hence Eq. (2) gives

$$m\,b^2\,\ddot{\theta} + k\,a^2\,\theta = 0$$

from which the natural frequency can be determined as

$$\omega_n = \left\{\frac{k\,a^2}{m\,b^2}\right\}^{\frac{1}{2}} = \left\{(10)^4\left(\frac{0.12}{0.20}\right)^2\frac{9.81}{25}\right\}^{\frac{1}{2}} = 37.5851\ \text{rad/sec}$$

2.31 $SO' = \frac{a}{\sqrt{2}}$, $OO' = h$, $OS = \sqrt{h^2 + \frac{a^2}{2}}$



When each wire stretches by x_s , let the resulting vertical displacement of the platform be x .

$$OS + x_s = \sqrt{(h+x)^2 + \frac{a^2}{2}}$$

$$x_s = \sqrt{h^2 + \frac{a^2}{2}} \left\{ \sqrt{\frac{(h+x)^2 + \frac{a^2}{2}}{h^2 + \frac{a^2}{2}}} - 1 \right\}$$

$$= \sqrt{h^2 + \frac{a^2}{2}} \left[\sqrt{1 + \left\{ \frac{2hx + x^2}{(h^2 + \frac{a^2}{2})} \right\}} - 1 \right]$$

For small x, x^2 is negligible compared to $2hx$ and $\sqrt{1+\theta} \simeq 1 + \frac{\theta}{2}$ and hence

$$x_s = \sqrt{h^2 + \frac{a^2}{2}} \left[1 + \frac{hx}{(h^2 + \frac{a^2}{2})} - 1 \right] = \frac{h}{\sqrt{h^2 + \frac{a^2}{2}}} \cdot x$$

Potential energy equivalence gives

$$\frac{1}{2} k_{eq} x^2 = 4 \left(\frac{1}{2} k \; x_s^2 \right)$$

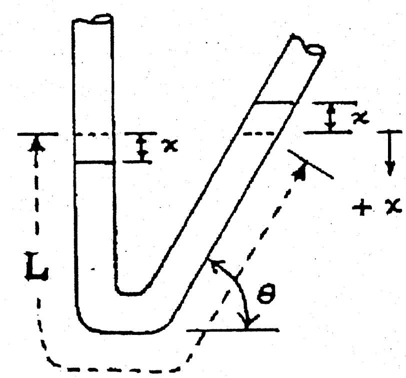
$$k_{eq} = 4k \left(\frac{x_s}{x} \right)^2 = \frac{4kh^2}{(h^2 + \frac{a^2}{2})}$$

Equation of motion of M:

$$M\ddot{x} + k_{eq} x = 0$$

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{(k_{eq}/M)^{1/2}} = \frac{\pi \sqrt{M}}{h} \left(\frac{2h^2 + a^2}{2k} \right)^{1/2}$$

2.32 Equation of motion:



$$m\ddot{x} = \sum F_x$$

i.e., $(L A \rho)\ddot{x} = -2 (A x \rho g)$

i.e., $\ddot{x} + \frac{2g}{L} x = 0$

where A = cross-sectional area of the tube and ρ = density of mercury. Thus the natural frequency is given by:

$$\omega_n = \sqrt{\frac{2g}{L}}$$

2.33 Assume same area of cross section for all segments of the cable. Speed of blades = 300 rpm = 5 Hz = 31.416 rad/sec.

$$\omega_n^2 = \frac{k_{eq}}{m} = (2\ (31.416))^2 = (62.832)^2$$

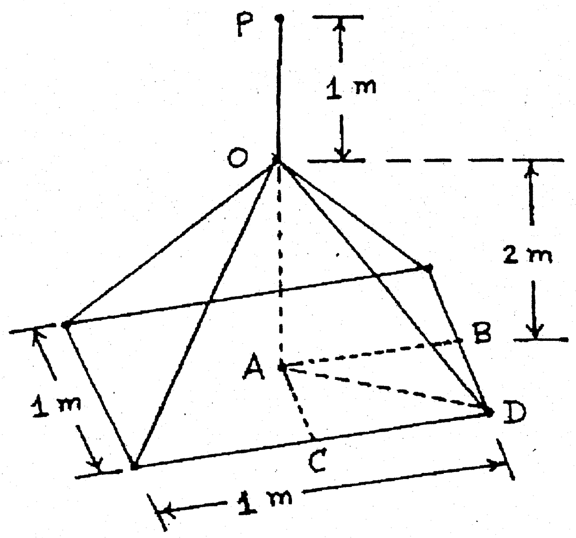
$$k_{eq} = m\ \omega_n^2 = 250\ (62.832)^2 = 98.6965\ (10^4)\ \mathrm{N/m} \qquad (1)$$

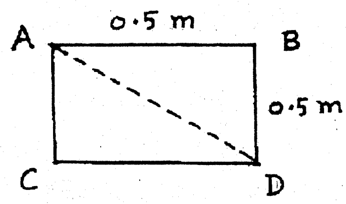
$$AD = \sqrt{0.5^2 + 0.5^2} = 0.7071\ \mathrm{m}\ , \quad OD = \sqrt{2^2 + 0.7071^2} = 2.1213\ \mathrm{m}$$

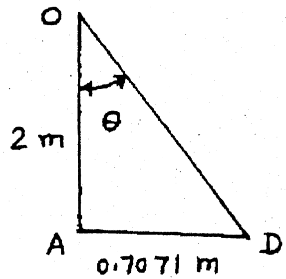
Stiffness of cable segments:

$$k_{PO} = \frac{A\,E}{\ell_{PO}} = \frac{A\ (207)\ (10^9)}{1} = 207\ (10^9)\ A\ \mathrm{N/m}$$

$$K_{OD} = \frac{A\,E}{\ell_{OD}} = \frac{A\ (207)\ (10^9)}{2.1213} = 97.5817\ (10^9)\ A\ \mathrm{N/m}$$







$\theta = \tan^{-1}\left(\frac{0.7071}{2}\right)$
$= 19.4710°$

The total sttiffness of the four inclined cables (k_{ic}) is given by:

$$k_{ic} = 4\ k_{OD}\ \cos^2\theta$$
$$= 4\ (97.5817)\ (10^9)\ A\ \cos^2 19.4710° = 346.9581\ (10^9)\ A\ \mathrm{N/m}$$

Equivalent stiffness of vertical and inclined cables is given by:

$$\frac{1}{k_{eq}} = \frac{1}{k_{PO}} + \frac{1}{k_{ic}}$$

i.e., $$k_{eq} = \frac{k_{PO}\ k_{ic}}{k_{PO} + k_{ic}}$$

$$= \frac{(207\ (10^9)\ A)\ (346.9581\ (10^9)\ A)}{(207\ (10^9)\ A) + (346.9581\ (10^9)\ A)} = 129.6494\ (10^9)\ A\ \mathrm{N/m} \qquad (2)$$

Equating k_{eq} given by Eqs. (1) and (2), we obtain the area of cross section of cables as:

$$A = \frac{98.6965\ (10^4)}{129.6494\ (10^9)} = 7.6126\ (10^{-6})\ \mathrm{m}^2$$

2.34

$$\frac{1}{2\pi}\left\{\frac{k_1}{m}\right\}^{\frac{1}{2}} = 5 \;;\; \frac{k_1}{m} = 4\,(\pi)^2\,(25) = 986.9651$$

$$\frac{1}{2\pi}\left\{\frac{k_1}{m+5000}\right\}^{\frac{1}{2}} = 4.0825 \;;\; \frac{k_1}{m+5000} = 4\,(\pi)^2\,(16.6668) = 657.9822$$

Using $k_1 = \dfrac{A\,E}{\ell_1}$ we obtain

$$\frac{k_1}{m} = \frac{A\,E}{\ell_1\,m} = \frac{A\,(207)\,(10^9)}{2\,m} = 986.9651$$

$$\text{i.e.,}\quad A = 9.5359\,(10^{-9})\,m \tag{1}$$

Also

$$\frac{k_1}{m+5000} = \frac{A\,E}{\ell_1\,(m+5000)} = 657.9822$$

$$\text{i.e.,}\quad \frac{A}{m+5000} = 6.3573\,(10^{-9}) \tag{2}$$

Using Eqs. (1) and (2), we obtain

$$A = 9.5359\,(10^{-9})\,m = 6.3573\,(10^{-9})\,m + 31.7865\,(10^{-6})$$

$$\text{i.e.,}\quad 3.1786\,(10^{-9})\,m = 31.7865\,(10^{-6})$$

$$\text{i.e.,}\quad m = 10000.1573\text{ kg} \tag{3}$$

Equations (1) and (3) yield

$$A = 9.5359\,(10^{-9})\,m = 9.5359\,(10^{-9})\,(10000.1573) = 0.9536\,(10^{-4})\,m^2$$

2.35 Longitudinal Vibration:

Let W_1 = part of weight W carried by length a of shaft

$W_2 = W - W_1$ = weight carried by length b

x = Elongation of length $a = \dfrac{W_1\,a}{AE}$

y = shortening of length $b = \dfrac{(W-W_1)(l-a)}{AE}$

E = Young's modulus

A = area of cross-section $= \pi d^2/4$

Since $x = y$, $\quad W_1 = \dfrac{W(l-a)}{l}$

x = elongation or static deflection of length $a = \dfrac{Wa\,(l-a)}{AE\,l}$

Considering the shaft of length a with end mass W_1/g as a spring-mass system,

$$\omega_n = \sqrt{\frac{g}{x}} = \left(\frac{g\,l\,AE}{Wa\,(l-a)}\right)^{1/2}$$

Transverse vibration:

spring constant of a fixed-fixed beam with off-center load

$$= k = \frac{3EI\,l^3}{a^3\,b^3} = \frac{3EI\,l^3}{a^3\,(l-a)^3}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \left\{ \frac{3EI\,l^3\,g}{W\,a^3\,(l-a)^3} \right\}^{1/2} \quad \text{with} \quad I = (\pi d^4/64)$$

= moment of inertia

Torsional vibration:

If flywheel is given an angular deflection θ, resisting torques offered by lengths a and b are $\frac{GJ\theta}{a}$ and $\frac{GJ\theta}{b}$.

Total resisting torque $= M_t = GJ\left(\frac{1}{a} + \frac{1}{b}\right)\theta$

$$k_t = \frac{M_t}{\theta} = GJ\left(\frac{1}{a} + \frac{1}{b}\right) \quad \text{where} \quad J = \frac{\pi d^4}{32} = \text{polar moment of inertia}$$

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \left\{ \frac{GJ}{J_0}\left(\frac{1}{a} + \frac{1}{b}\right) \right\}^{1/2}$$

where J_0 = mass polar moment of inertia of the flywheel.

(2.36) $m_{eq_{end}}$ = equivalent mass of a uniform beam at the free end (see Problem 2.38) =

$$\frac{33}{140}\,m = \frac{33}{140}\left\{1\,(1)\,(150 \times 12)\,\frac{0.283}{386.4}\right\} = 0.3107$$

Stiffness of tower (beam) at free end:

$$k_b = \frac{3\,E\,I}{L^3} = \frac{3\,(30 \times 10^6)\,(\frac{1}{12}\,(1)\,(1^3))}{(150 \times 12)^3} = 0.001286 \text{ lb/in}$$

Length of each cable:

$$OA = \sqrt{2} = 1.4142 \text{ ft}, \quad OB = \sqrt{2}\;15 = 21.2132 \text{ ft}, \quad AB = OB - OA = 19.7990 \text{ ft}$$

$$TB = \sqrt{TA^2 + AB^2} = \sqrt{100^2 + 19.7990^2} = 101.9412 \text{ ft}$$

$$\tan\theta = \frac{AT}{AB} = \frac{100}{19.7990} = 5.0508\,, \quad \theta = 78.8008^\circ$$

Axial stiffness of each cable:

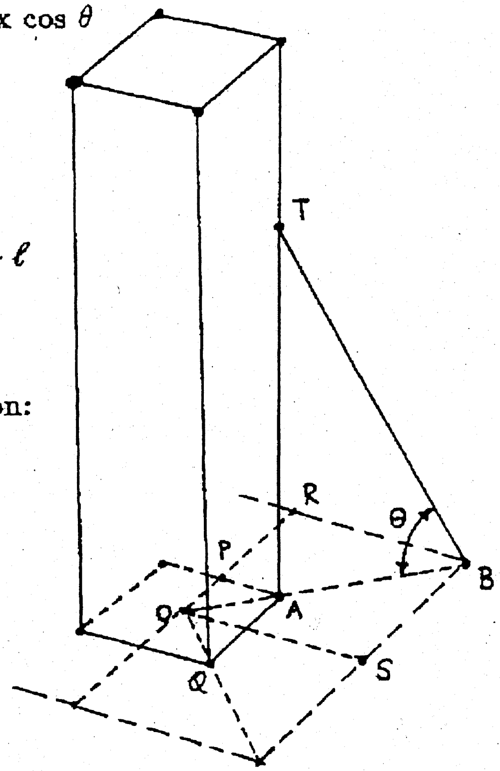
$$k = \frac{A\,E}{\ell} = \frac{(0.5)\,(30 \times 10^6)}{(101.9412 \times 12)} = 12261.971 \text{ lb/in}$$

Axial extension of each cable (y_c) due to a horizontal displacement of x of tower:

$$\ell_1^2 = \ell^2 + x^2 - 2\,\ell\,x\cos(180^\circ - \theta) = \ell^2 + x^2 + 2\,\ell\,x\cos\theta$$

$$\text{or}\quad \ell_1 = \ell\left\{1 + \left(\frac{x}{\ell}\right)^2 + 2\,\frac{x}{\ell}\cos\theta\right\}^{\frac{1}{2}}$$

$$y_c = \ell_1 - \ell \approx \ell\left(1 + \frac{1}{2}\,\frac{x^2}{\ell^2} + \frac{1}{2}\,(2)\,\frac{x}{\ell}\cos\theta\right) - \ell$$
$$= \ell + x\cos\theta - \ell = x\cos\theta$$



Equivalent stiffness of each cable in horizontal direction:

$$\frac{1}{2}\,k\,y_c^2 = \frac{1}{2}\,k_{eq_c}\,x^2 \quad \text{or} \quad k_{eq_c} = k\left(\frac{y_c}{x}\right)^2 = k\cos^2\theta$$

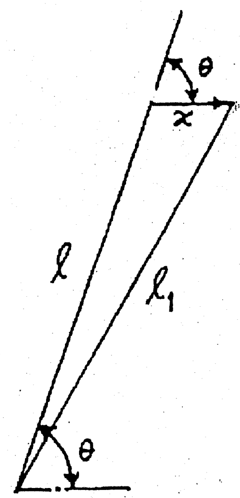
This gives

$$k_{eq_c} = (12261.971)\cos^2 78.8008^\circ = 462.5419 \text{ lb/in}$$

In order to use the relation

$$k_{eq_{end}} = k_b + 4\,k_{eq_c}\left(\frac{y_{L1}}{y_L}\right)^2$$

we find

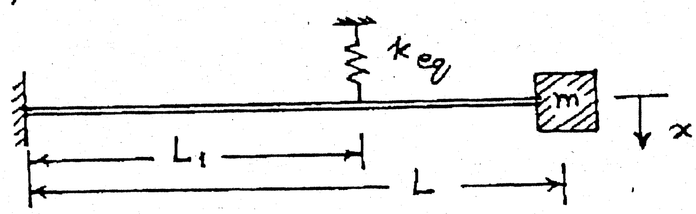


$$\frac{y_{L1}}{y_L} = \left(\frac{F\,L_1^2\,(3\,L - L_1)}{6\,E\,I}\;\frac{3\,E\,I}{F\,L^3}\right) = \frac{L_1^2\,(3\,L - L_1)}{2\,L^3}$$
$$= \frac{100^2\,(3\,(150) - 100)}{2\,(150)^3} = 0.5185$$

Thus

$$k_{eq_{end}} = k_b + 4\,k_{eq_c}\,(0.5185)^2 = 0.001286 + 4\,(462.5419)\,(0.5185)^2$$
$$= 497.4045 \text{ lb/in}$$

Natural frequency:



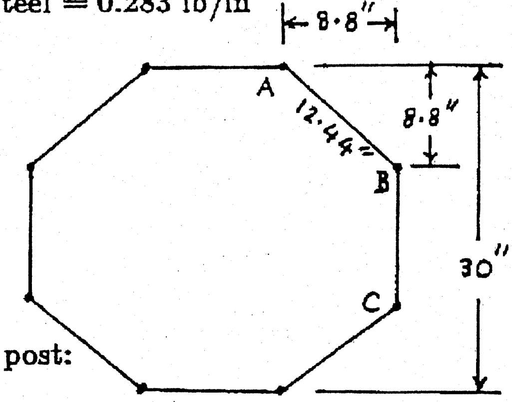
$$\omega_n = \left\{\frac{k_{eq_{end}}}{m_{eq_{end}}}\right\}^{\frac{1}{2}} = \left\{\frac{497.4045}{0.3107}\right\}^{\frac{1}{2}} = 40.0114 \text{ rad/sec}$$

(2.37) Sides of the sign:

$$AB = \sqrt{8.8^2 + 8.8^2} = 12.44 \text{ in} \quad ; \quad BC = 30 - 8.8 - 8.8 = 12.4 \text{ in}$$

$$\text{Area} = 30\,(30) - 4\,(\frac{1}{2}\,(8.8)\,(8.8)) = 745.12 \text{ in}^2$$

$$\text{Thickness} = \frac{1}{8} \text{ in} \quad ; \quad \text{Weight density of steel} = 0.283 \text{ lb/in}^3$$



$$\text{Weight of sign} = (0.283)(\frac{1}{8})(745.12){=}26.64 \text{ lb}$$

$$\text{Weight of sign post} = (72)\,(2)\,(\frac{1}{4})\,(0.283) = 10.19 \text{ lb}$$

Stiffness of sign post (cantilever beam):

$$k = \frac{3\,E\,I}{\ell^3}$$

Area moments of inertia of the cross section of the sign post:

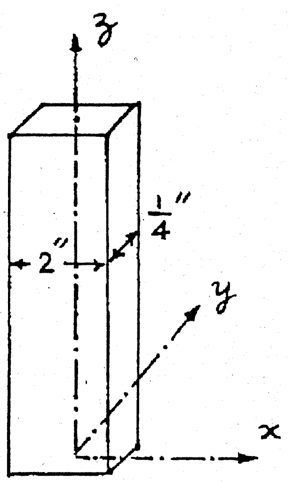
$$I_{xx} = \frac{1}{12}\,(2)\,(\frac{1}{4})^3 = \frac{1}{384} \text{ in}^4$$

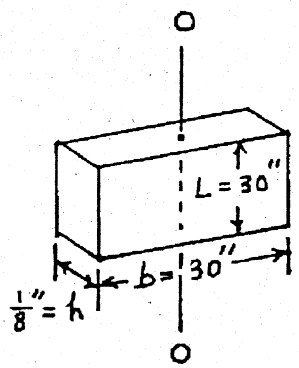
$$I_{yy} = \frac{1}{12}\,(\frac{1}{4})\,(2)^3 = \frac{1}{6} \text{ in}^4$$

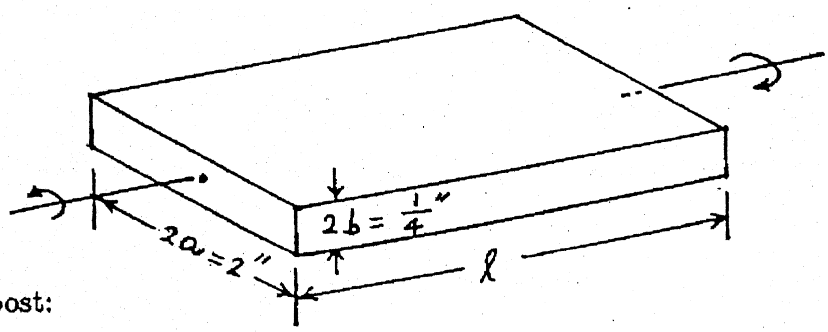
Bending stiffnesses of the sign post:

$$k_{xz} = \frac{3\,E\,I_{yy}}{\ell^3} = \frac{3\,(30\,(10^6))(\frac{1}{6})}{72^3} = 40.1877 \text{ lb/in}$$

$$k_{yz} = \frac{3\,E\,I_{xx}}{\ell^3} = \frac{3\,(30\,(10^6))\,(\frac{1}{384})}{72^3} = 0.6279 \text{ lb/in}$$







Torsional stiffness of the sign post:

$$k_t = 5.33 \frac{a\, b^3}{\ell} G \left\{1 - 0.63 \frac{b}{a}\left(1 - \frac{b^4}{12\, a^4}\right)\right\}$$

(See Ref: N. H. Cook, *Mechanics of Materials for Design*, McGraw-Hill, New York, 1984, p. 342).

Thus

$$k_t = 5.33 \left\{\frac{(1)\,(\frac{1}{8})^3}{72}\right\}(11.5\,(10^6))\left\{1 - (0.63)\,(\frac{1}{8})\left(1 - \frac{(\frac{1}{8})^4}{12\,(1)^4}\right)\right\}$$

$$= 1531.7938 \text{ lb—in/rad}$$

Natural frequency for bending in xz plane:

$$\omega_{xz} = \left\{\frac{k_{xz}}{m}\right\}^{\frac{1}{2}} = \left\{\frac{40.1877}{\left(\frac{26.64}{386.4}\right)}\right\}^{\frac{1}{2}} = 24.1434 \text{ rad/sec}$$

Natural frequency for bending in yz plane:

$$\omega_{yz} = \left\{\frac{k_{yz}}{m}\right\} = \left\{\frac{0.6279}{\left(\frac{26.64}{386.4}\right)}\right\}^{\frac{1}{2}} = 3.0178 \text{ rad/sec}$$

By approximating the shape of the sign as a square of side 30 in (instead of an octagon), we can find its mass moment of inertia as:

$$I_{oo} = \frac{\gamma L}{3}\,(b^3\, h + h^3\, b) = \left(\frac{0.283}{386.4}\right)\left(\frac{30}{3}\right)\left(30^3\,(\frac{1}{8}) + (\frac{1}{8})^3\,(30)\right) = 24.7189$$

Natural torsional frequency:

$$\omega_t = \left\{\frac{k_t}{I_{oo}}\right\}^{\frac{1}{2}} = \left\{\frac{1531.7938}{24.7189}\right\}^{\frac{1}{2}} = 7.8720 \text{ rad/sec}$$

Thus the mode of vibration (close to resonance) is torsion in xy plane.

2.38 (a) Pivoted: Let $l = h$.

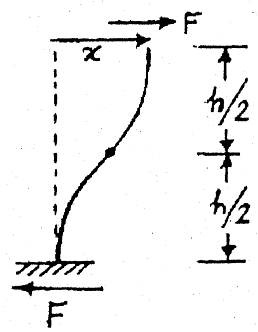
$$k_{eq} = 4\ k_{column} = 4\left(\frac{3EI}{l^3}\right) = \frac{12EI}{l^3}$$

Let $m_{eff\,1}$ = effective mass due to self weight of columns

Equation of motion: $\left(\frac{W}{g} + m_{eff\,1}\right)\ddot{x} + k_{eq}\, x = 0$

Natural frequency of horizontal vibration $= \omega_n = \sqrt{\dfrac{12EI}{l^3\left(\frac{W}{g} + m_{eff\,1}\right)}}$

(b) Fixed:



Since the joint between column and floor does not permit rotation, each column will bend with inflection point at middle.
When force F is applied at ends,

$$x = 2\ \frac{F\,(l/2)^3}{3\,E\,I} = \frac{F\,l^3}{12\,E\,I}$$

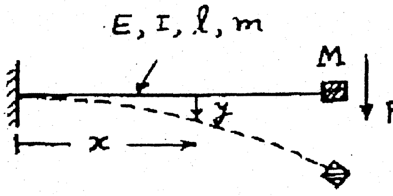
$$k_{column} = \frac{12\,E\,I}{l^3}\ ;\quad k_{eq} = 4\ k_{column} = \frac{48\,E\,I}{l^3}$$

Let $m_{eff\,2}$ = effective mass of each column at top end

Equation of motion: $\left(\frac{W}{g} + m_{eff\,2}\right)\ddot{x} + k_{eq}\ x = 0$

Natural frequency of horizontal vibration $= \omega_n = \sqrt{\dfrac{48\,E\,I}{l^3\left(\frac{W}{g} + m_{eff\,2}\right)}}$

Effective mass (due to self weight):



(a) Let $m_{eff\,1}$ = effective part of mass of beam (m) at end.
Thus vibrating inertia force at end is due to $(M + m_{eff\,1})$.

Assume deflection shape during vibration same as the static deflection shape with a tip load:

$$y(x,t) = Y(x)\ \cos(\omega_n t - \phi) \quad \text{where} \quad Y(x) = \frac{F\,x^2\,(3\,l - x)}{6\,E\,I}.$$

$$Y(x) = \frac{Y_0}{2\,l^3}\ x^2\,(3\,l - x) \quad \text{where} \quad Y_0 = \frac{F\,l^3}{3\,E\,I} = \text{max. tip deflection}$$

$$y(x,t) = \frac{Y_0}{2\,l^3}\,(3\,x^2 l - x^3)\ \cos(\omega_n t - \phi) \qquad (E_1)$$

Max. strain energy of beam = Max. work by force F

$$= \frac{1}{2}\,F\,Y_0 = \frac{3}{2}\ \frac{E\,I}{l^3}\ Y_0^2 \qquad (E_2)$$

Max. kinetic energy due to distributed mass of beam

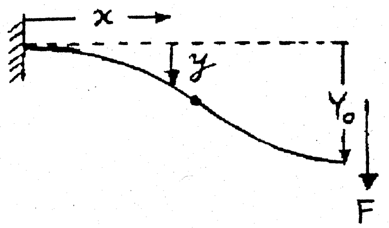
$$= \frac{1}{2}\ \frac{m}{l}\int_0^l \dot{y}^2(x,t)\Big|_{max}\,dx + \frac{1}{2}\left(\dot{y}_{max}\right)^2 M$$

$$= \frac{1}{2}\ \omega_n^2\ Y_0^2\left(\frac{33}{140}\,m\right) + \frac{1}{2}\ \omega_n^2\ Y_0^2\ M \qquad (E_3)$$

$$\therefore\ m_{eff\,1} = \frac{33}{140}\ m = 0.2357\ m$$

(b) Let $Y(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$

$Y(0) = 0, \quad \frac{dY}{dx}(0) = 0, \quad Y(\ell) = Y_0, \quad \frac{dY}{dx}(\ell) = 0$



This leads to $\quad Y(x) = \frac{3\,Y_0}{\ell^2}\, x^2 - \frac{2\,Y_0}{\ell^3}\, x^3$

$$y(x,t) = Y_0 \left(3\,\frac{x^2}{\ell^2} - 2\,\frac{x^3}{\ell^3} \right) \cos(\omega_n t - \phi) \qquad (E_4)$$

$$\text{Maximum strain energy} = \frac{1}{2} EI \int_0^\ell \left(\frac{\partial^2 y}{\partial x^2} \right)^2 dx \,\Big|_{max}$$

$$= \frac{6\,EI\,Y_0^2}{\ell^3} \qquad (E_5)$$

$$\text{Max. kinetic energy} = \frac{1}{2} M\, \omega_n^2\, Y_0^2 + \frac{1}{2}\left(\frac{m}{\ell}\right) Y_0^2\, \omega_n^2 \int_0^\ell \left(\frac{3x^2}{\ell^2} - \frac{2x^3}{\ell^3} \right)^2 dx$$

$$= \frac{1}{2}\, \omega_n^2\, Y_0^2 \left(M + \frac{13}{35}\, m \right) \qquad (E_6)$$

$$\therefore \quad m_{eff\,2} = \frac{13}{35}\, m = 0.3714\, m$$

(2.39) Stiffnesses of segments:

$$A_1 = \frac{\pi}{4}\,(D_1^2 - d_1^2) = \frac{\pi}{4}\,(2^2 - 1.75^2) = 0.7363 \text{ in}^2$$

$$k_1 = \frac{A_1\, E_1}{L_1} = \frac{(0.7363)\,(10^7)}{12} = 61.3583\,(10^4) \text{ lb/in}$$

$$A_2 = \frac{\pi}{4}\,(D_2^2 - d_2^2) = \frac{\pi}{4}\,(1.5^2 - 1.25^2) = 0.5400 \text{ in}^2$$

$$k_2 = \frac{A_2\, E_2}{L_2} = \frac{(0.5400)\,(10^7)}{10} = 54.0\,(10^4) \text{ lb/in}$$

$$A_3 = \frac{\pi}{4}\,(D_3^3 - d_3^2) = \frac{\pi}{4}\,(1^2 - 0.75^2) = 0.3436 \text{ in}^2$$

$$k_3 = \frac{A_3\, E_3}{L_3} = \frac{(0.3436)\,(10^7)}{8} = 42.9516\,(10^4) \text{ lb/in}$$

Equivalent stiffness (springs in series):

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$$

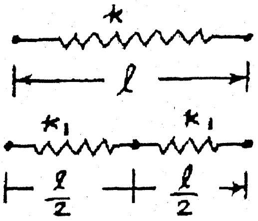
$$= 0.0162977\,(10^{-4}) + 0.0185185\,(10^{-4}) + 0.0232820\,(10^{-4}) = 0.0580982\,(10^{-4})$$

$$\text{or} \quad k_{eq} = 17.2122\,(10^4) \text{ lb/in}$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k_{eq}\, g}{W}} = \sqrt{\frac{17.2122\,(10^4)\,(386.4)}{10}} = 2578.9157 \text{ rad/sec}$$

2.40



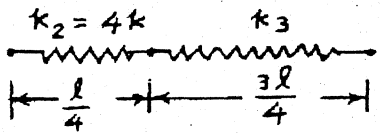
$$\frac{1}{k_{total}} = \frac{1}{k_1} + \frac{1}{k_1}$$

$$k_{total} = \frac{k_1}{2} \equiv k \; ; \; k_1 = 2k$$

$$\tau_n = 2\pi \sqrt{\frac{m}{k_{eq}}}$$

$$0.5 = 2\pi \sqrt{\frac{m}{4k}}$$

$$\sqrt{\frac{m}{k}} = \frac{1}{2\pi}$$



$$\frac{1}{k_{total}} = \frac{1}{k_2} + \frac{1}{k_3} = \frac{1}{4k} + \frac{1}{k_3} = \frac{1}{k}$$

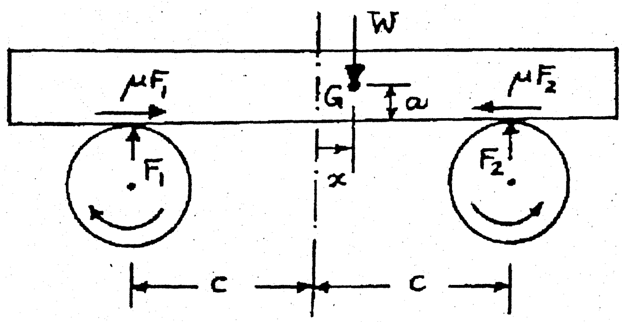
$$k_3 = \frac{4}{3} k$$

$$\tau_n = 2\pi \sqrt{\frac{m}{k_{eq}}} \quad \text{where} \quad k_{eq} = 4k + \frac{4}{3} k = \frac{16}{3} k$$

$$\therefore \tau_n = 2\pi \sqrt{\frac{3m}{16k}} = \frac{2\pi\sqrt{3}}{4} \sqrt{\frac{m}{k}} = \frac{2\pi\sqrt{3}}{4} \left(\frac{1}{2\pi}\right) = 0.4330 \text{ sec}$$

2.41

Let μ = coefficient of friction
x = displacement of C.G. of block
F_1, F_2 = net reactions between roller and block on left and right sides.



Reactions at left and right due to static load W are $W(c-x)/2c$ and $W(c+x)/2c$, respectively.

M = moment about G due to motion of block $= (\mu F_2 - \mu F_1) a$

Reactions at left and right to balance $M = \frac{M}{2c} = \frac{\mu a}{2c}(F_2 - F_1)$

$$F_1 = \frac{W(c-x)}{2c} - \frac{\mu a}{2c}(F_2 - F_1) \; ; \quad F_2 = \frac{W(c+x)}{2c} + \frac{\mu a}{2c}(F_2 - F_1)$$

Subtraction gives
$$F_2 - F_1 = \frac{Wx}{c} + \frac{\mu a}{c}(F_2 - F_1)$$

i.e.,
$$F_2 - F_1 = \frac{Wx}{c}\left(\frac{c}{c - \mu a}\right) = \frac{Wx}{c - \mu a}$$

Restoring force $= \mu(F_2 - F_1) = \left(\frac{\mu W x}{c - \mu a}\right)$

Equation of motion :
$$\frac{W}{g}\ddot{x} + \frac{\mu W}{(c - \mu a)} x = 0$$

$$\omega_n = \omega = \sqrt{\frac{\mu W g}{W(c - \mu a)}} = \sqrt{\frac{\mu g}{c - \mu a}}$$

Solving this, we get $\quad \omega = [c\omega^2 / (g + a\omega^2)]$

(2.42) From problem 2.41,

Restoring force without springs $= \mu (F_2 - F_1) = \dfrac{\mu W x}{c - \mu a}$

spring restoring force $= 2 k x$

Total restoring force $= \dfrac{\mu W x}{c-\mu a} + 2 k x$

Equation of motion: $\dfrac{W}{g} \ddot{x} + \left(\dfrac{\mu W}{c-\mu a} + 2 k \right) x = 0$

$$\omega_n = \omega = \left\{ \frac{[\mu W + 2 k (c - \mu a)] g}{(c - \mu a) W} \right\}^{1/2}$$

Solution of this equation gives

$$\mu = \left(\frac{\omega^2 W c - 2 k g c}{W g + W \omega^2 a - 2 k g a} \right)$$

(2.43) (a) Natural frequency of vibration of electromagnet (without the automobile):

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{10000.0\ (386.4)}{3000.0}} = 35.8887 \text{ rad/sec}$$

(b) When the automobile is dropped, the electromagnet moves up by a distance (x_0) from its static equilibrium position.

x_0 = static deflection (elongation of cable) under the weight of automobile

$$= \frac{W_{auto}}{k} = \frac{2000}{10000} = 0.2 \text{ in}$$

$\dot{x}_0 = 0$

Resultant motion of electromagnet (+x upwards):

$$x(t) = A_0 \sin (\omega_n t + \phi_0)$$

where

$$A_0 = \left\{ x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right\}^{\frac{1}{2}} = x_0 = 0.2$$

$$\text{and} \quad \phi_0 = \tan^{-1} \left(\frac{x_0 \omega_n}{\dot{x}_0} \right) = \tan^{-1} (\infty) = 90^\circ$$

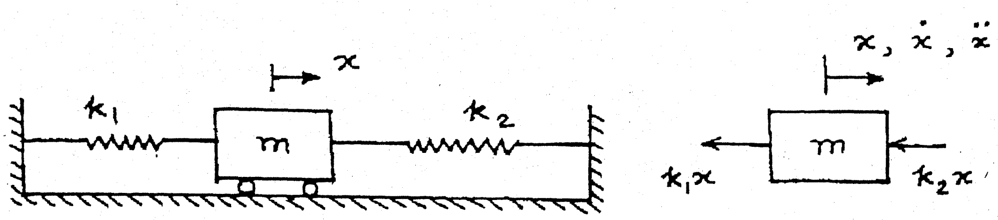
Hence $x(t) = 0.2 \sin (35.8887\ t + 90^\circ) = 0.2 \cos 35.8887\ t$

(c) Maximum x(t):

$$x(t)\big|_{max} = A_0 = 0.2 \text{ in}$$

Maximum tension in cable during motion $= k\ x(t)\big|_{max}$ + Weigh of electromagnet $= 10000\ (0.2) + 3000 = 5{,}000$ lb.

(2.44)



(a) Newton's second law of motion:

$$F(t) = -k_1\, x - k_2\, x = m\, \ddot{x} \quad \text{or} \quad m\, \ddot{x} + (k_1 + k_2)\, x = 0$$

(b) D'Alembert's principle:

$$F(t) - m\, \ddot{x} = 0 \quad \text{or} \quad -k_1\, x - k_2\, x - m\, \ddot{x} = 0$$

$$\text{Thus} \quad m\, \ddot{x} + (k_1 + k_2)\, x = 0$$

(c) Principle of virtual work:

When mass m is given a virtual displacement δx,
Virtual work done by the spring forces = - $(k_1 + k_2)\, x\, \delta x$
Virtual work done by the inertia force = - $(m\, \ddot{x})\, \delta x$
According to the principle of virtual work, the total virtual work done by all forces must be equal to zero:

$$-m\, \ddot{x}\, \delta x - (k_1 + k_2)\, x\, \delta x = 0 \quad \text{or} \quad m\, \ddot{x} + (k_1 + k_2)\, x = 0$$

(d) Principle of conservation of energy:

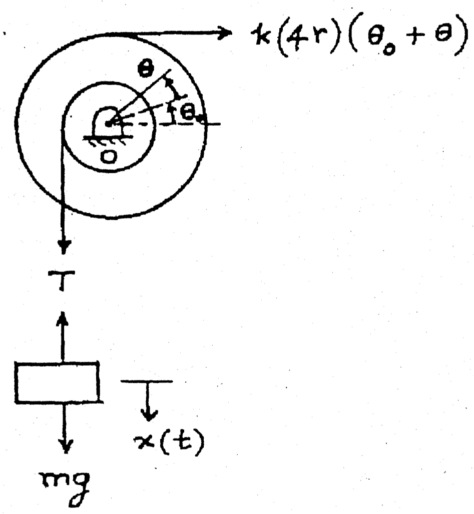
$$T = \text{kinetic energy} = \frac{1}{2}\, m\, \dot{x}^2$$

$$U = \text{strain energy} = \text{potential energy} = \frac{1}{2}\, k_1\, x^2 + \frac{1}{2}\, k_2\, x^2$$

$$T + U = \frac{1}{2}\, m\, \dot{x}^2 + \frac{1}{2}\, (k_1 + k_2)\, x^2 = c = \text{constant}$$

$$\frac{d}{dt}\, (T + U) = 0 \quad \text{or} \quad m\, \ddot{x} + (k_1 + k_2)\, x = 0$$

(2.45)



Equation of motion:

$$\text{Mass m:} \quad m\,g - T = m\,\ddot{x} \tag{1}$$

$$\text{Pulley } J_0: \quad J_0\,\ddot{\theta} = T\,r - k\,4\,r\,(\theta + \theta_0)\,4\,r \tag{2}$$

where θ_0 = angular deflection of the pulley under the weight, mg, given by:

$$m\,g\,r = k\,(4\,r\,\theta_0)\,4\,r \quad \text{or} \quad \theta_0 = \frac{m\,g}{16\,r\,k} \tag{3}$$

Substituting Eqs. (1) and (3) into (2), we obtain

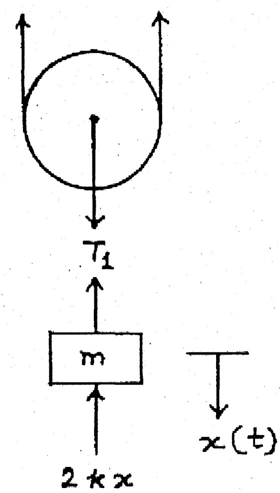
$$J_0\,\ddot{\theta} = (m\,g - m\,\ddot{x})\,r - k\,16\,r^2\,(\theta + \frac{m\,g}{16\,r\,k}) \tag{4}$$

Using $x = r\,\theta$ and $\ddot{x} = r\,\ddot{\theta}$, Eq. (4) becomes

$$(J_0 + m\,r^2)\,\ddot{\theta} + (16\,r^2\,k)\,\theta = 0$$

2.46 Consider the springs connected to the pulleys (by rope) to be in series. Then

$$\frac{1}{k_{eq}} = \frac{1}{k} + \frac{1}{5k} \quad \text{or} \quad k_{eq} = \frac{5}{6}\,k$$



Let the displacement of mass m be x. Then the extension of the rope (springs connected to the pulleys) = 2 x. From the free body diagram, the equation of motion of mass m:

$$m\,\ddot{x} + 2\,k\,x + k_{eq}\,(2\,x) = 0$$

$$\text{or} \quad m\,\ddot{x} + \frac{11}{3}\,k\,x = 0$$

2.47 T = kinetic energy $= T_{mass} + T_{pulley}$

$$= \frac{1}{2}\,m\,\dot{x}^2 + \frac{1}{2}\,J_0\,\dot{\theta}^2 = \frac{1}{2}\,(m\,r^2 + J_0)\,\dot{\theta}^2$$

$$U = \text{potential energy} = \frac{1}{2}\,k\,x_s^2 = \frac{1}{2}\,k\,(4\,r\,\theta)^2 = \frac{1}{2}\,k\,(16\,r^2)\,\theta^2$$

Using $\frac{d}{dt}\,(T + U) = 0$ gives

$$(m\,r^2 + J_0)\,\ddot{\theta} + (16\,r^2\,k)\,\theta = 0$$

2.48 $T = \text{kinetic energy} = \frac{1}{2}\,m\,\dot{x}^2 + \frac{1}{2}\,J_0\,\dot{\theta}^2$

$U = \text{potential energy} = \frac{1}{2}\,k\,x_s^2$

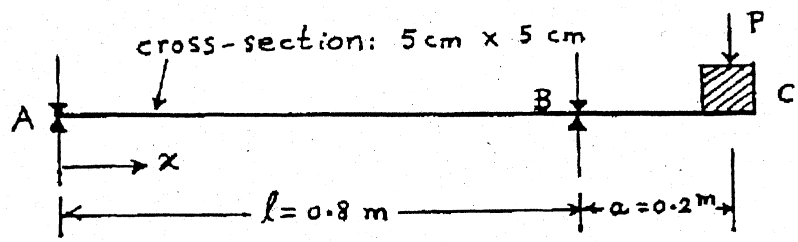
where $\theta = \frac{x}{r}$, x_s = extension of spring $= 4\,r\,\theta = 4\,x$. Hence

$$T = \frac{1}{2}\,(m + \frac{J_0}{r^2})\,\dot{x}^2 \quad ; \quad U = \frac{1}{2}\,(16\,k)\,x^2$$

Using tthe relation $\frac{d}{dt}\,(T + U) = 0$, we obtain the equation of motion of the system as:

$$(m + \frac{J_0}{r^2})\,\ddot{x} + 16\,k\,x = 0$$

(2.49)



Due to a load P at C, deflection at point C is given by (from Appendix B):

$$y(x) = \frac{P\,(x - \ell)}{6\,E\,I\,\ell}\left[a\,(3\,x - \ell) - (x - \ell)^2\right] ; \ell \le x \le \ell + a$$

$$y_C = y(x = \ell + a) = \frac{P\,a^2}{3\,E\,I\,\ell}\,(\ell + a)$$

Moment of inertia of cross section of beam:

$$I = \frac{1}{12}\,(0.05)\,(0.05)^3 = 52.0833\,(10^{-8})\ m^4$$

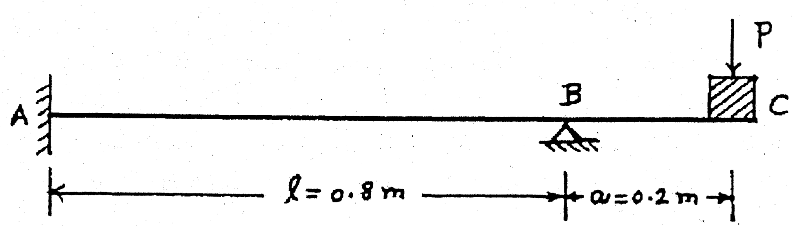
Equivalent stiffness:

$$k_{eq} = \frac{P}{y_C} = \frac{3\,E\,I\,\ell}{a^2\,(\ell + a)} = \frac{3\,(207\,(10^9))\,(52.0833\,(10^{-8}))\,(0.8)}{(0.2)^2\,(0.8 + 0.2)}$$
$$= 6.4687\,(10^6)\ N/m$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{6.4687\,(10^6)}{50}} = 359.6872\ rad/sec$$

(2.50)



From Appendix B, the deflection of fixed-pinned beam with an overhang, due to load P at the free end, is given by:

$$y(x) = \frac{P\,a}{4\,E\,I\,\ell}\left[x^2 - \ell\,x^2 - \left(\frac{2\,\ell}{3\,a} + 1\right)(x-\ell)^3\right]; \quad \ell \le x \le \ell + a$$

Using a = 0.2, ℓ = 0.8, x = a + ℓ = 1.0, and

$$I = \frac{1}{12}\,(0.05)\,(0.05)^3 = 52.0833\,(10^{-8})\ \text{m}^4$$

we obtain

$$y_C = \frac{P\,(0.2)}{4\,(207\,(10^9))\,(52.0833\,(10^{-8}))\,(0.8)}\left[1^2 - 0.8\,(1)^2 - \left(\frac{1.6}{0.6} + 1\right)(0.2)^3\right]$$

$$= P\,(9.895652\,(10^{-8}))$$

$$k_{eq} = \frac{P}{y_C} = 1010.5448\,(10^4)\ \text{N/m}$$

$$\omega_n = \left\{\frac{k_{eq}}{m}\right\}^{\frac{1}{2}} = \left\{\frac{10.1054\,(10^6)}{50}\right\}^{\frac{1}{2}} = 449.5642\ \text{rad/sec}$$

(2.51) The system of Fig. (A) can be drawn in equivalent form as shown in Fig. (B) where both pulleys have the same radius r_1. We notice in Fig. (B) that vibration can take place in only one way with one pulley moving clockwise and the other moving counter clockwise.

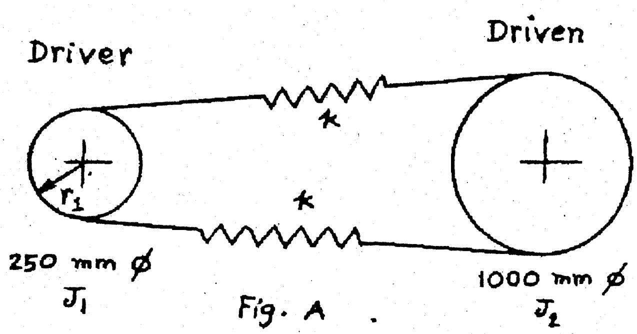


Fig. A

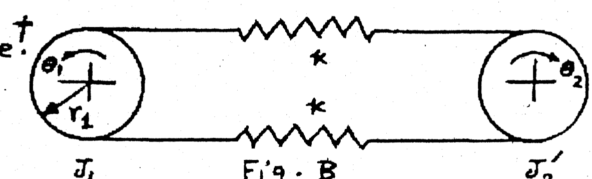


Fig. B

When pulleys rotate in opposite directions, $\frac{\theta_1}{\theta_2} = \frac{J_2}{J_1}$.

The spring force, which has the same value on either pulley is $-k_t(\theta_1+\theta_2)$ where k_t = torsional spring constant of the system. Equation of motion is

$J_1\ddot{\theta}_1 + k_t(\theta_1+\theta_2) = 0$ & $J_2\ddot{\theta}_2 + k_t(\theta_1+\theta_2) = 0$

i.e. $J_1\ddot{\theta}_1 + k_t\left(1+\frac{J_1}{J_2}\right)\theta_1 = 0$ & $J_2\ddot{\theta}_2 + k_t\left(\frac{J_2}{J_1}+1\right)\theta_2 = 0$

$k_t = \frac{\Delta M_t}{\Delta\theta} = \left(\begin{array}{c}\text{force in springs}\\ \text{due to } \Delta\theta\end{array}\right)\frac{r_1}{\Delta\theta}$

$= (2k\,r_1\,\Delta\theta)\frac{r_1}{\Delta\theta} = 2k\,r_1^2$

$= 2k\left(\frac{125}{1000}\right)^2 = k/32$ N-m/rad

∴ Eq. (E_1) gives, for $\omega = 12\pi$ rad,

$k = 454.7935$ N/m.

Either of these equations gives

$$\omega = \left\{ k_t \left(\frac{J_1 + J_2}{J_1 J_2} \right) \right\}^{1/2} \quad --- (E_1)$$

Here $J_1 = 0.2/4 = 0.05$ kg-m^2,

$J_2' = J_2 \text{(speed ratio)}^2 = 0.2 (\frac{1}{4})^2 = 0.0125$ kg-m^2

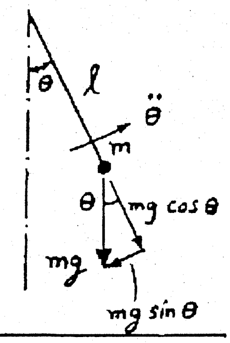
† The other possible motion is rotation of the two pulleys as a whole (as rigid body) in same direction. This will have a natural frequency of zero. See section 5.7.

2.52 $m l \ddot{\theta} + m g \sin \theta = 0$

For small θ, $\quad m l \ddot{\theta} + m g \theta = 0$

$$\omega_n = \sqrt{\frac{g}{l}}$$

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{\frac{9.81}{0.5}}} = 1.4185 \text{ sec}$$



2.53 (a) $\omega_n = \sqrt{\frac{g}{l}}$

(b) $m l^2 \ddot{\theta} + k a^2 \sin\theta + m g l \sin\theta = 0$; $\quad m l^2 \ddot{\theta} + (k a^2 + m g l)\, \theta = 0$

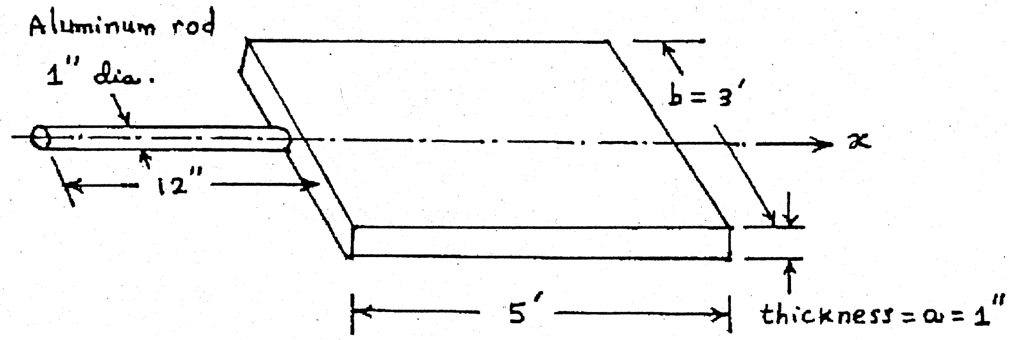
$$\omega_n = \sqrt{\frac{k a^2 + m g l}{m l^2}}$$

(c) $m l^2 \ddot{\theta} + k a^2 \sin\theta - m g l \sin\theta = 0$; $\quad m l^2 \ddot{\theta} + (k a^2 - m g l)\, \theta = 0$

$$\omega_n = \sqrt{\frac{k a^2 - m g l}{m l^2}}$$

configuration (b) has the highest natural frequency.

2.54



$$m = \text{mass of a panel} = (5 \times 12)\,(3 \times 12)\,(1)\,(\frac{0.283}{386.4}) = 1.5820$$

$$J_0 = \text{mass moment of inertia of panel about x-axis} = \frac{m}{12}\,(a^2 + b^2)$$

$$= \frac{1.5820}{12}\,(1^2 + 36^2) = 170.9878$$

$$I_0 = \text{polar moment of inertia of rod} = \frac{\pi}{32}\, d^4 = \frac{\pi}{32}\,(1)^4 = 0.098175 \text{ in}^4$$

$$k_t = \frac{G\,I_0}{\ell} = \frac{(3.8\,(10^6))\,(0.098175)}{12} = 3.1089\,(10^4)\text{ lb–in/rad}$$

$$\omega_n = \left\{\frac{k_t}{J_0}\right\}^{\frac{1}{2}} = \left\{\frac{3.1089\,(10^4)}{170.9878}\right\}^{\frac{1}{2}} = 13.4841\text{ rad/sec}$$

2.55 I_0 = polar moment of inertia of cross section of shaft AB

$$= \frac{\pi}{32}\,d^4 = \frac{\pi}{32}\,(1)^4 = 0.098175\text{ in}^4$$

k_t = torsional stiffness of shaft AB $= \dfrac{G\,I_0}{\ell}$

$$= \frac{(12\,(10^6))\,(0.098175)}{6} = 19.635\,(10^4)\text{ lb–in/rad}$$

J_0 = mass moment of inertia of the three blades about y–axis

$$= 3\,J_0\,|_{PQ} = 3\left(\frac{1}{3}\,m\,\ell^2\right) = m\,\ell^2 = \left(\frac{2}{386.4}\right)(12)^2 = 0.7453$$

Torsional natural frequency:

$$\omega_n = \left\{\frac{k_t}{J_0}\right\}^{\frac{1}{2}} = \left\{\frac{19.635\,(10^4)}{0.7453}\right\}^{\frac{1}{2}} = 513.2747\text{ rad/sec}$$

2.56 J_0 = mass moment of inertia of the ring = 1.0 kg–m^2.

I_{os} = polar moment of inertia of the cross section of steel shaft

$$= \frac{\pi}{32}\,(d_{os}^4 - d_{is}^4) = \frac{\pi}{4}\,(0.05^4 - 0.04^4) = 36.2266\,(10^{-8})\text{ m}^4$$

I_{ob} = polar moment of inertia of cross section of brass shaft

$$= \frac{\pi}{32}\,(d_{ob}^4 - d_{ib}^4) = \frac{\pi}{32}\,(0.04^4 - 0.03^4) = 17.1806\,(10^{-8})\text{ m}^4$$

k_{ts} = torsional stiffness of steel shaft

$$= \frac{G_s\,I_{os}}{\ell} = \frac{(80\,(10^9))\,(36.2266\,(10^{-8}))}{2} = 14490.64\text{ N–m/rad}$$

k_{tb} = torsional stiffness of brass shaft

$$= \frac{G_b\,I_{ob}}{\ell} = \frac{(40\,(10^9))\,(17.1806\,(10^{-8}))}{2} = 3436.12\text{ N–m/rad}$$

$k_{t_{eq}} = k_{ts} + k_{tb} = 17{,}926.76$ N–m/rad

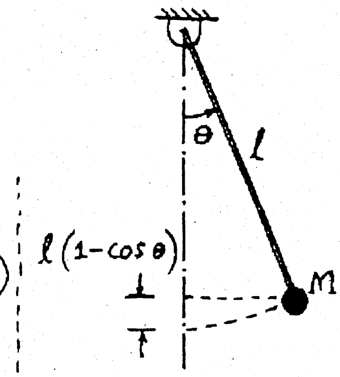
Torsional natural frequency:

$$\omega_n = \sqrt{\frac{k_{t_{eq}}}{J_0}} = \sqrt{\frac{17926.76}{1}} = 133.8908\text{ rad/sec}$$

Natural time period:

$$\tau_n = \frac{2\,\pi}{\omega_n} = \frac{2\,\pi}{133.8908} = 0.04693\text{ sec}$$

(2.57) Kinetic energy of system is



$$T = T_{rod} + T_{bob} = \frac{1}{2}\left(\frac{1}{3} m l^2\right)\dot{\theta}^2 + \frac{1}{2} M l^2 \dot{\theta}^2$$

Potential energy of system is
(since mass of the rod acts through its center)

$$U = U_{rod} + U_{bob} = \frac{1}{2} m g l (1-\cos\theta) + \frac{1}{2} M g l (1-\cos\theta)$$

Equation of motion:

$$\frac{d}{dt}(T+U) = 0$$

i.e. $$\left(M+\frac{m}{3}\right) l^2 \ddot{\theta} + \left(M+\frac{m}{2}\right) g l \sin\theta = 0$$

For small angles, $$\ddot{\theta} + \frac{\left(M+\frac{m}{2}\right)}{\left(M+\frac{m}{3}\right)} \frac{g}{l} \cdot \theta = 0$$

$$\omega_n = \sqrt{\frac{\left(M+\frac{m}{2}\right) g}{\left(M+\frac{m}{3}\right) l}}$$

(2.58) For the shaft, $$J = \frac{\pi d^4}{32} = \frac{\pi (0.05)^4}{32} = 61.3594 \times 10^{-8} \text{ m}^4$$

$$k_t = \frac{GJ}{l} = \frac{(0.793 \times 10^{11})(61.3594 \times 10^{-8})}{2} = 24329.002 \text{ N-m/rad}$$

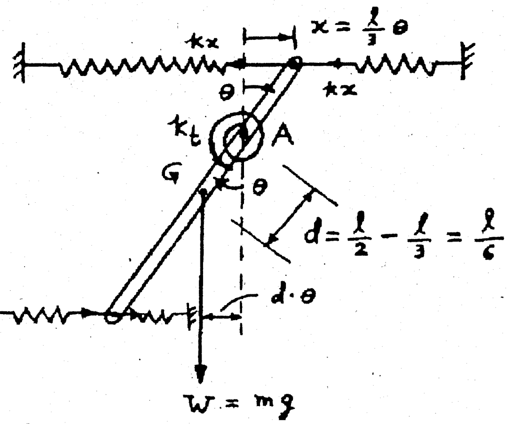
For the disc,

$$J_0 = \frac{MD^2}{8} = \left(\rho \frac{\pi D^2}{4} h\right)\frac{D^2}{8} = \frac{\rho \pi D^4 h}{32}$$

$$= \frac{(7.83 \times 10^3)\, \pi\, (1)^4 (0.1)}{32} = 76.8710 \text{ kg-m}^2$$

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \left(\frac{24329.002}{76.8710}\right)^{1/2} = 17.7902 \text{ rad/sec}$$

(2.59) Equation of motion



$$J_A \ddot{\theta} = - W d\theta - 2k\left(\frac{l}{3}\theta\right)\frac{l}{3} - 2k\left(\frac{2l}{3}\theta\right)\frac{2l}{3} - k_t \theta$$

where

$$J_A = J_G + m d^2 = \frac{1}{12} m l^2 + m \frac{l^2}{36} = \frac{1}{9} m l^2$$

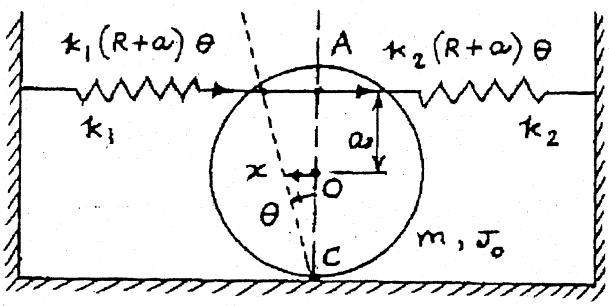
$$\therefore \frac{m l^2}{9}\ddot{\theta} + \left(mgd + 2k\frac{l^2}{9} + \frac{8k l^2}{9} + k_t\right)\theta = 0$$

$$\omega_n = \sqrt{\frac{\left(mgd + \frac{2}{9}kl^2 + \frac{8}{9}kl^2 + k_t\right) 9}{m l^2}} = \sqrt{\frac{9mgd + 10kl^2 + 9k_t}{m l^2}}$$

For given data,

$$\omega_n = \sqrt{\frac{9(10)(9.81)(5/6) + 10\,(2000)\,(5)^2 + 9\,(1000)}{10\,(5)^2}} = 45.1547 \ \frac{rad}{sec}$$

(2.60) $J_o = \frac{1}{2}\, m\, R^2$, $\quad J_C = \frac{1}{2}\, m\, R^2 + m\, R^2$



Let angular displacement $= \theta$

Equation of motion:

$$J_c\, \ddot{\theta} + k_1 (R+a)^2\, \theta + k_2\, (R+a)^2\, \theta = 0$$

$$\omega_n = \sqrt{\frac{(k_1 + k_2)\,(R+a)^2}{J_c}} = \sqrt{\frac{(k_1 + k_2)(R+a)^2}{1.5\ m\ R^2}} \qquad (E_1)$$

Equation (E_1) shows that ω_n increases with the value of a.

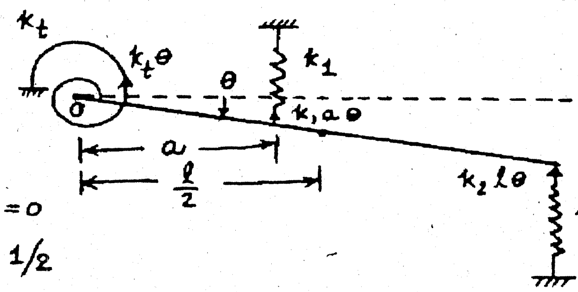
$\therefore$ ω_n will be maximum when $a = R$.

(2.61) Net g acting on the pendulum $= 9.81 - 5 = 4.81\ m/sec^2 = g_n$

$$\omega_n = \sqrt{\frac{g_n}{l}} = \sqrt{\frac{4.81}{5}} = 3.1016\ rad/sec$$

$$\tau_n = 2\pi/\omega_n = 2.0258\ sec$$

(2.62) Equation of motion:



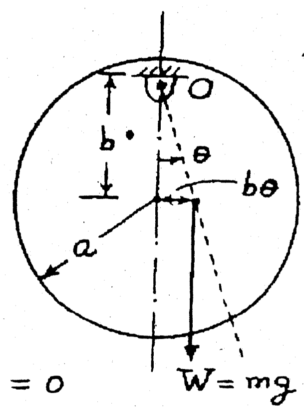
$$J_o\, \ddot{\theta} = -k_t\theta - (k_1 a\,\theta)a - (k_2\, l\,\theta)\, l$$

where $J_o = \frac{1}{12} m l^2 + m \left(\frac{l}{2}\right)^2 = \frac{1}{3} m l^2$

$$\therefore\ \frac{1}{3} m l^2\, \ddot{\theta} + (k_t + k_1 a^2 + k_2 l^2)\,\theta = 0$$

$$\omega_n = \left\{ \frac{3\,(k_t + k_1 a^2 + k_2 l^2)}{m l^2} \right\}^{1/2}$$

(2.63) $J_o = J_G + m b^2 = \frac{1}{2} m a^2 + m b^2$



Equation of motion:

$$J_o\, \ddot{\theta} + m g\, b\, \theta = 0$$

$$\omega_n = \sqrt{\frac{m g b}{J_o}} = \sqrt{\frac{2 g\, b}{a^2 + 2 b^2}}$$

$$\frac{\partial \omega_n}{\partial b} = \frac{1}{2} \left(\frac{2 g b}{a^2 + 2 b^2} \right)^{-\frac{1}{2}} \left\{ \frac{(a^2 + 2b^2)(2g) - 2 g b\,(4b)}{(a^2 + 2b^2)^2} \right\} = 0$$

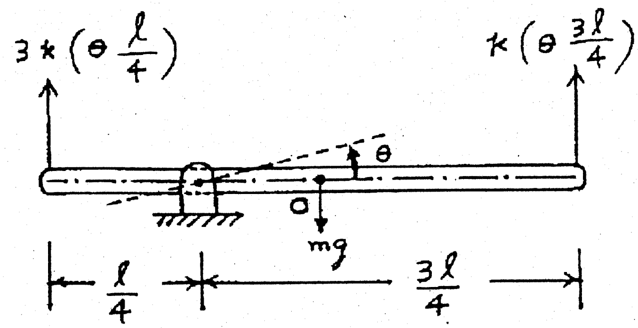
i.e., $b = \pm \frac{a}{\sqrt{2}}$

$$\omega_n \Big|_{b=+a/\sqrt{2}} = \sqrt{\frac{2g \frac{a}{\sqrt{2}}}{a^2 + 2(a^2/2)}} = \sqrt{\frac{g}{\sqrt{2}\, a}}$$

$b = -a/\sqrt{2}$ gives imaginary value for ω_n.

Since $\omega_n = 0$ when $b = 0$, we have $\omega_n \big|_{max}$ at $b = \frac{a}{\sqrt{2}}$.

2.64



Let θ be measured from static equilibrium position so that gravity force need not be considered.

(a) Newton's second law of motion:

$$J_0\, \ddot{\theta} = -3\,k\,(\theta \frac{\ell}{4})\, \frac{\ell}{4} - k\,(\theta\, \frac{3\,\ell}{4})\,(\frac{3\,\ell}{4}) \quad \text{or} \quad J_0\, \ddot{\theta} + \frac{3}{4}\, k\, \ell^2\, \theta = 0$$

(b) D'Alembert's principle:

$$M(t) - J_0\, \ddot{\theta} = 0 \quad \text{or} \quad -3\,k\,(\theta \frac{\ell}{4})\,(\frac{\ell}{4}) - k\,(\theta\, \frac{3\,\ell}{4})\,(\frac{3\,\ell}{4}) - J_0\, \ddot{\theta} = 0$$

$$\text{or} \quad J_0\, \ddot{\theta} + \frac{3}{4}\, k\, \ell^2\, \theta = 0$$

(c) Principle of virtual work:

Virtual work done by spring force:

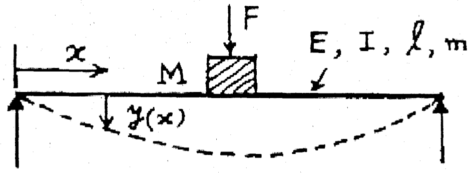
$$\delta W_s = -3\,k\,(\theta\, \frac{\ell}{4})\,(\frac{\ell}{4}\, \delta\theta) - k\,(\theta\, \frac{3\,\ell}{4})\,(\frac{3\,\ell}{4}\, \delta\theta)$$

Virtual work done by inertia moment $= -\,(J_0\, \ddot{\theta})\, \delta\theta$

Setting total virtual work done by all forces/moments equal to zero, we obtain

$$J_0\, \ddot{\theta} + \frac{3}{4}\, k\, \ell^2\, \theta = 0$$

2.65 Let m_{eff} = effective part of mass of beam (m) at middle. Thus vibratory inertia force at middle is due to $(M + m_{eff})$. Assume a deflection shape: $y(x,t) = Y(x) \cos(\omega_n t - \phi)$ where $Y(x)$ = static deflection shape due to load at middle given by:



$$Y(x) = Y_0 \left(3\,\frac{x}{\ell} - 4\,\frac{x^3}{\ell^3} \right) \;;\; 0 \le x \le \frac{\ell}{2}$$

where Y_0 = maximum deflection of the beam at middle $= \dfrac{F\,\ell^3}{48\,E\,I}$

Maximum strain energy of beam = maximum work done by force F $= \dfrac{1}{2}\,F\,Y_0$.

Maximum kinetic energy due to distributed mass of beam:

$$= 2\left\{ \frac{1}{2}\,\frac{m}{\ell} \int_0^{\ell/2} \dot{y}^2(x,t)\,\big|_{max}\,dx \right\} + \frac{1}{2}\left(\dot{y}_{max}\right)^2 M$$

$$= \frac{m\,\omega_n^2}{\ell} \int_0^{\ell/2} Y^2(x)\,dx + \frac{1}{2}\,\omega_n^2\,Y_{max}^2\,M$$

$$= \frac{m\,\omega_n^2}{\ell} \int_0^{\ell/2} Y_0^2 \left(\frac{9\,x^2}{\ell^2} + 16\,\frac{x^6}{\ell^6} - 24\,\frac{x^4}{\ell^4} \right) dx + \frac{1}{2}\,Y_0^2\,M\,\omega_n^2$$

$$= \frac{m\,\omega_n^2\,Y_0^2}{\ell} \left[\frac{9}{\ell^2}\,\frac{x^3}{3} + \frac{16}{\ell^6}\,\frac{x^7}{7} - \frac{24}{\ell^4}\,\frac{x^5}{5} \right] \Big|_0^{\ell/2} + \frac{1}{2}\,Y_0^2\,M\,\omega_n^2$$

$$= \frac{1}{2}\,Y_0^2\,\omega_n^2 \left(\frac{17}{35}\,m + M \right)$$

This shows that $m_{eff} = \dfrac{17}{35}\,m = 0.4857\ m$

(2.66) For small angular rotation of bar PQ about P,

$$\frac{1}{2}\,(k_{12})_{eq}\,(\theta\,l_3)^2 = \frac{1}{2}\,k_1\,(\theta\,l_1)^2 + \frac{1}{2}\,k_2\,(\theta\,l_2)^2$$

$$(k_{12})_{eq} = \frac{k_1\,l_1^2 + k_2\,l_2^2}{l_3^2}$$

Since $(k_{12})_{eq}$ and k_3 are in series,

$$k_{eq} = \frac{(k_{12})_{eq}\;k_3}{(k_{12})_{eq} + k_3} = \frac{k_1\,k_3\,l_1^2 + k_2\,k_3\,l_2^2}{k_1\,l_1^2 + k_2\,l_2^2 + k_3\,l_3^2}$$

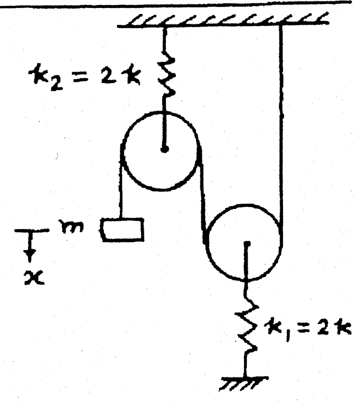
T = kinetic energy $= \frac{1}{2}\,m\,\dot{x}^2$, U = potential energy $= \frac{1}{2}\,k_{eq}\,x^2$

If $x = X\cos\omega_n t$,

$T_{max} = \frac{1}{2}\,m\,\omega_n^2\,X^2$, $U_{max} = \frac{1}{2}\,k_{eq}\,X^2$

$T_{max} = U_{max}$ gives $\qquad \omega_n = \sqrt{\dfrac{k_1 k_3 l_1^2 + k_2 k_3 l_2^2}{m(k_1 l_1^2 + k_2 l_2^2 + k_3 l_3^2)}}$

(2.67) When mass m moves by x, spring k_1 deflects by $x/4$.



T = kinetic energy $= \frac{1}{2} m (\dot{x})^2$

U = potential energy $= 2\left\{\frac{1}{2}(2k)\left(\frac{x}{4}\right)^2\right\}$

$= \frac{1}{8} k x^2$

For harmonic motion,

$$T_{max} = \frac{1}{2} m \omega_n^2 X^2, \qquad U_{max} = \frac{1}{8} k X^2$$

$T_{max} = U_{max}$ gives $\qquad \omega_n = \sqrt{\dfrac{k}{4m}}$

(2.68) Refer to the figure of solution of problem 2.24.

$T = \frac{1}{2} m \dot{x}^2 \quad , \qquad U = \frac{1}{2}\left[2k_1 (x \cos 45^\circ)^2 + 2k_2 (x \cos 135^\circ)^2\right]$

$\qquad = \frac{1}{2}(k_1 + k_2) x^2$

For harmonic motion,

$$T_{max} = \frac{1}{2} m \omega_n^2 X^2, \qquad U_{max} = \frac{1}{2}(k_1 + k_2) X^2$$

$T_{max} = U_{max}$ gives $\qquad \omega_n = \sqrt{\dfrac{k_1 + k_2}{m}}$

(2.69) Kinetic energy (K.E.) $= \frac{1}{2} m \dot{x}^2$

Potential energy (P.E.) $= \frac{1}{2} T_1 x + \frac{1}{2} T_2 x$ = work done in displacing mass m by distance x against the total force (tension) of $T_1 + T_2$.

$T_1 = \frac{x}{a} T, \qquad T_2 = \frac{x}{b} T \qquad$ from solution of problem 2.18

Max. K.E. $= \frac{1}{2} m \omega_n^2 X^2, \qquad$ Max. P.E. $= \frac{1}{2} T\left(\frac{1}{a} + \frac{1}{b}\right) X^2$

Max. K.E. = Max. P.E. gives $\qquad \omega_n = \sqrt{\dfrac{T(a+b)}{m a b}} = \sqrt{\dfrac{T l}{m a (l-a)}}$

(2.70) $T = K.E. = \frac{1}{2} J_A \dot{\theta}^2 = \frac{1}{2}(J_G + m d^2)\dot{\theta}^2 = \frac{1}{2}\left(\frac{1}{12} m l^2 + m \frac{l^2}{36}\right)\dot{\theta}^2$

$\qquad = \frac{1}{2}\left(\frac{m l^2}{9}\right)\dot{\theta}^2$

$U = P.E. = m g d (1 - \cos\theta) + 2\left(\frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2\right) + \frac{1}{2} k_t \theta^2$

with $\cos\theta \simeq 1 - \frac{1}{2}\theta^2, \quad x_1 = \frac{l}{3}\theta$ and $x_2 = \frac{2l}{3}\theta$

$$U = mg\,\frac{l}{6}\,\frac{\theta^2}{2} + k\,\frac{l^2}{9}\,\theta^2 + k\,\frac{4l^2}{9}\,\theta^2 + \frac{1}{2}\,k_t\,\theta^2$$

$$T_{max} = \frac{1}{2}\left(\frac{ml^2}{9}\right)\Theta^2\,, \qquad U_{max} = \frac{1}{2}\,\frac{mgl}{6}\,\Theta^2 + \frac{1}{2}\left(\frac{10\,k\,l^2}{9}\right)\Theta^2 + \frac{1}{2}\,k_t\,\Theta^2$$

$T_{max} = U_{max}$ gives

$$\omega_n = \sqrt{\frac{mg\,\frac{l}{6} + \frac{10\,k\,l^2}{9} + k_t}{\frac{ml^2}{9}}} = 45.1547\ \frac{\text{rad}}{\text{sec}} \text{ for given data}$$

(2.71) Refer to the figure in the solution of problem 2.62.

$$T = \frac{1}{2}\,J_0\,\dot{\theta}^2$$

$$U = \frac{1}{2}\,k_t\,\theta^2 + \frac{1}{2}\,k_1\,(\theta a)^2 + \frac{1}{2}\,k_2\,(\theta\,l)^2$$

For $\theta(t) = \Theta\cos\omega_n t$,

$$T_{max} = \frac{1}{2}\,J_0\,\omega_n^2\,\Theta^2\,, \qquad U_{max} = \frac{1}{2}\left(k_t + k_1 a^2 + k_2 l^2\right)\Theta^2$$

$T_{max} = U_{max}$ gives

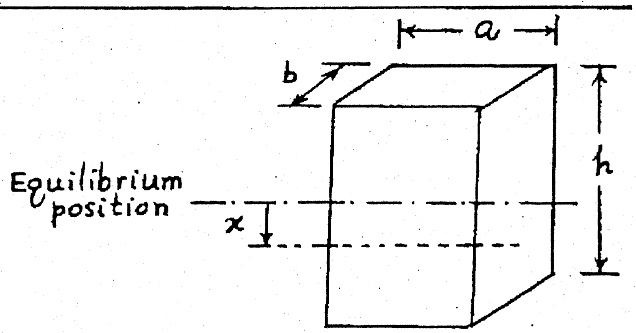
$$\omega_n = \sqrt{\frac{k_t + k_1 a^2 + k_2 l^2}{J_0}} = \sqrt{\frac{3\,(k_t + k_1 a^2 + k_2 l^2)}{m\,l^2}}$$

since $J_0 = m l^2/3$.

(2.72) When prism is displaced by x from equilibrium position, the weight of oil displaced

$= \rho_0\,g\,ab\,x$ = restoring force

Mass of prism $= m = \rho_w\,abh$



Equation of motion:

$$m\ddot{x} + \text{restoring force} = 0$$

$$\rho_w\,abh\,\ddot{x} + \rho_0\,g\,ab\,x = 0$$

$$\omega_n = \sqrt{\frac{\rho_0\,g\,ab}{\rho_w\,abh}} = \sqrt{\frac{\rho_0\,g}{\rho_w\,h}} \qquad (E_1)$$

Since ω_n is independent of cross-section of the prism, ω_n remains same even for a circular wooden prism.

(2.73)

$$T = \frac{1}{2}\,m\,\dot{x}^2 + \frac{1}{2}\,J_0\,\dot{\theta}^2 = \frac{1}{2}\left(m\,R^2 + \frac{1}{2}\,m\,R^2\right)\dot{\theta}^2$$

since $x = R\,\theta$ and $J_0 = \frac{1}{2}\,m\,R^2$.

$$U = \frac{1}{2}\,k_1\,x_1^2 + \frac{1}{2}\,k_2\,x_1^2 = \frac{1}{2}\,(k_1 + k_2)\,(R + a)^2\,\theta^2$$

where $x_1 = (R + a)\,\theta$. Using $\frac{d}{dt}(T + U) = 0$, we obtain

$$\left(\frac{3}{2}\, m R^2\right) \ddot{\theta} + (k_1 + k_2)\,(R + a)^2\, \theta = 0$$

(2.74) Let x(t) be measured from static equilibrium position of mass. T = kinetic energy of the system:

$$T = \frac{1}{2}\, m\, \dot{x}^2 + \frac{1}{2}\, J_0\, \dot{\theta}^2 = \frac{1}{2}\left(m + \frac{J_0}{r^2}\right) \dot{x}^2$$

since $\dot{\theta} = \frac{\dot{x}}{r}$ = angular velocity of pulley. U = potential energy of the system:

$$U = \frac{1}{2}\, k\, y^2 = \frac{1}{2}\, k\,(16\, x^2)$$

since $y = \theta\,(4\, r) = 4\, x$ = deflection of spring. $\frac{d}{dt}(T + U) = 0$ leads to:

$$m\, \ddot{x} + \frac{J_0}{r^2}\, \ddot{x} + 16\, k\, x = 0$$

This gives the natural frequency:

$$\omega_n = \sqrt{\frac{16\, k\, r^2}{m\, r^2 + J_0}}$$

(2.75) For pendulum, $\omega_n = \sqrt{g/l}$ in vaccum $= 0.5$ Hz $= \pi$ rad/sec

$$l = g/\pi^2 = 9.81/\pi^2 = 0.9940 \text{ m}$$

$\omega_d = \omega_n \sqrt{1 - \zeta^2}$ in viscous medium $= 0.45$ Hz $= 0.9\,\pi$ rad/sec

$$\zeta^2 = \frac{\omega_n^2 - \omega_d^2}{\omega_n^2} = \pi^2\left(\frac{1 - 0.81}{1}\right) = 1.8752$$

$$\zeta = 1.3694$$

Equation of motion: $m l^2\, \ddot{\theta} + c_t\, \dot{\theta} + m g l\, \theta = 0$

$$c_{ct} = 2\,(m l^2)\, \omega_n = 2\,(1)\,(0.994)^2 (\pi) = 6.2080$$

Since $\zeta = \frac{c_t}{c_{ct}} = 1.3694$, $c_t = 8.5013$ N-m-sec/rad.

(2.76) From Eq. (2.85),

$$\ln\left(\frac{x_j}{x_{j+1}}\right) = \ln(18) \Rightarrow \frac{2\pi\zeta}{\sqrt{1 - \zeta^2}} = 2.8904$$

$$\zeta = \left\{\frac{(2.8904)^2}{(2.8904)^2 + 4\,\pi^2}\right\}^{\frac{1}{2}} = 0.4179$$

(a) If damping is doubled, $\zeta_{new} = 0.8358$

$$\ln\left(\frac{x_j}{x_{j+1}}\right) = \frac{2\pi\,\zeta_{new}}{\sqrt{1-\zeta_{new}^2}} = \frac{2\pi(0.8358)}{\sqrt{1-(0.8358)^2}} = 9.5656$$

$$\therefore \quad \frac{x_j}{x_{j+1}} = 14265.362$$

(b) If damping is halved, $\zeta = 0.2090$

$$\ln\left(\frac{x_j}{x_{j+1}}\right) = \frac{2\pi\,\zeta_{new}}{\sqrt{1-\zeta_{new}^2}} = \frac{2\pi(0.2090)}{\sqrt{1-(0.2090)^2}} = 1.3428$$

$$\therefore \quad \frac{x_j}{x_{j+1}} = 3.8296$$

2.77 $x(t) = X\, e^{-\zeta\omega_n t} \sin\omega_d t$ where $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$

For maximum or minimum of $x(t)$,

$$\frac{dx}{dt} = X\, e^{-\zeta\omega_n t}\left(-\zeta\omega_n \sin\omega_d t + \omega_d \cos\omega_d t\right) = 0$$

As $e^{-\zeta\omega_n t} \neq 0$ for finite t,

$$-\zeta\omega_n \sin\omega_d t + \omega_d \cos\omega_d t = 0$$

i.e. $\quad \tan\omega_d t = \sqrt{1-\zeta^2}/\zeta$

Using the relation

$$\sin\omega_d t = \pm\frac{\tan\omega_d t}{\sqrt{1+\tan^2\omega_d t}} = \frac{\pm\left(\sqrt{1-\zeta^2}/\zeta\right)}{\sqrt{1+\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)^2}} = \pm\sqrt{1-\zeta^2}$$

we obtain

$$\sin\omega_d t = \sqrt{1-\zeta^2}, \quad \cos\omega_d t = \zeta$$

and

$$\sin\omega_d t = -\sqrt{1-\zeta^2}, \quad \cos\omega_d t = -\zeta$$

$$\frac{d^2x}{dt^2} = X\, e^{-\zeta\omega_n t}\left[\zeta^2\omega_n^2 \sin\omega_d t - 2\zeta\omega_n\omega_d \cos\omega_d t - \omega_d^2 \sin\omega_d t\right]$$

When $\sin\omega_d t = \sqrt{1-\zeta^2}$ and $\cos\omega_d t = \zeta$,

$$\frac{d^2x}{dt^2} = -X\, e^{-\zeta\omega_n t}\,\omega_n^2\sqrt{1-\zeta^2} < 0$$

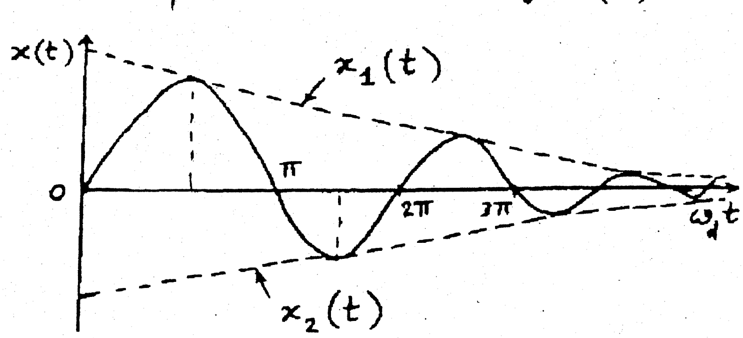
$\therefore \quad \sin\omega_d t = \sqrt{1-\zeta^2}$ corresponds to maximum of $x(t)$.

When $\sin\omega_d t = -\sqrt{1-\zeta^2}$ and $\cos\omega_d t = -\zeta$,

$$\frac{d^2x}{dt^2} = X\, e^{-\zeta\omega_n t}\, \omega_n^2 \sqrt{1-\zeta^2} \quad > 0$$

$\therefore \sin \omega_d t = -\sqrt{1-\zeta^2}$ corresponds to minimum of $x(t)$.

Enveloping curves:



Let the curve passing through the maximum (or minimum) points be

$$x(t) = C\, e^{-\zeta\omega_n t}$$

For maximum points, $t_{max} = \dfrac{\sin^{-1}\left(\sqrt{1-\zeta^2}\right)}{\omega_d}$

and

$$C\, e^{-\zeta\omega_n t_{max}} = X\, e^{-\zeta\omega_n t_{max}} \sin \omega_d t_{max}$$

i.e.

$$C = X\sqrt{1-\zeta^2}$$

$$\therefore x_1(t) = X\sqrt{1-\zeta^2}\; e^{-\zeta\omega_n t}$$

Similarly for minimum points, $t_{min} = \dfrac{\sin^{-1}\left(-\sqrt{1-\zeta^2}\right)}{\omega_d}$

and

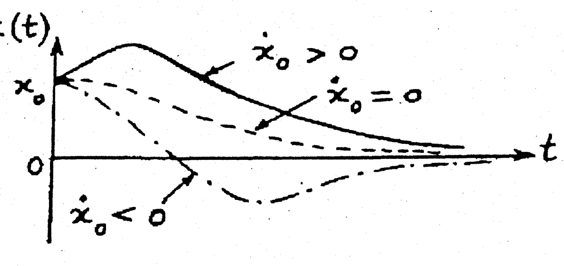
$$C\, e^{-\zeta\omega_n t_{min}} = X\, e^{-\zeta\omega_n t_{min}} \sin \omega_d t_{min}$$

i.e.

$$C = -X\sqrt{1-\zeta^2}$$

$$\therefore x_2(t) = -X\sqrt{1-\zeta^2}\; e^{-\zeta\omega_n t}$$

(2.78) $x(t) = \left[x_0 + (\dot{x}_0 + \omega_n x_0)t\right] e^{-\omega_n t}$ ---- (E_1)



For $x_0 > 0$, graph of Eq. (E_1) is shown for different $\dot{x}_0$.

We assume $\dot{x}_0 > 0$ as it is the only case that gives a maximum.

For maximum of $x(t)$,

$$\frac{dx}{dt} = e^{-\omega_n t}\left\{-(\dot{x}_0 + \omega_n x_0)\,\omega_n t + \dot{x}_0\right\} = 0$$

$$t_m = \frac{\dot{x}_0}{\omega_n(\dot{x}_0 + \omega_n x_0)} \quad ---- (E_2)$$

$$\frac{d^2x}{dt^2} = -e^{-\omega_n t}\left\{2\omega_n \dot{x}_0 + \omega_n^2 x_0 - \omega_n^2(\dot{x}_0 + \omega_n x_0)t\right\} \quad ---- (E_3)$$

(E_2) and (E_3) give

$$\left.\frac{d^2x}{dt^2}\right|_{t=t_m} = -e^{-\omega_n t_m}\left\{2\omega_n \dot{x}_0 + \omega_n^2 x_0 - \omega_n^2(\dot{x}_0 + \omega_n x_0)t_m\right\}$$

$$= -e^{-\omega_n\left(\frac{\dot{x}_0}{\omega_n(\dot{x}_0 + \omega_n x_0)}\right)}\left\{\omega_n \dot{x}_0 + \omega_n^2 x_0\right\} \quad ---- (E_4)$$

For $x_0 > 0$ and $\dot{x}_0 > 0$, $\left.\frac{d^2x}{dt^2}\right|_{t_m} < 0$

Hence t_m given by Eq. (E_2) corresponds to a maximum of $x(t)$.

$$x\Big|_{t=t_m} = \left\{x_0 + (\dot{x}_0 + \omega_n x_0)\frac{\dot{x}_0}{\omega_n(\dot{x}_0 + \omega_n x_0)}\right\} e^{-\omega_n t_m}$$

$$= \left(x_0 + \frac{\dot{x}_0}{\omega_n}\right) e^{-\left(\frac{\dot{x}_0}{\dot{x}_0 + \omega_n x_0}\right)} \quad ---- (E_5)$$

(2.79) Equation (2.92) can be expressed as $\delta = \frac{1}{m} \ln\left(\frac{x_0}{x_m}\right)$

For half cycle, $m = \frac{1}{2}$ and hence $\delta = 2 \ln\left(\frac{x_0}{x_{\frac{1}{2}}}\right) = 2 \ln\left(\frac{1}{0.15}\right)$

$= 3.7942$

Necessary damping ratio ζ_0 is

$$\zeta_0 = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = \frac{3.7942^2}{\sqrt{4\pi^2 + 3.7942^2}}$$

$= 0.5169$

(a)

If $\zeta = \frac{3}{4}\zeta_0 = 0.3877$, the overshoot can be determined by finding δ from Eq. (2.85):

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi(0.3877)}{\sqrt{1 - 0.3877^2}} = 2.6427 = 2 \ln\left(\frac{x_0}{x_{\frac{1}{2}}}\right)$$

$$\ln\left(\frac{x_0}{x_{\frac{1}{2}}}\right) = 1.32135$$

$$x_{\frac{1}{2}} = x_0 / e^{1.32135} = 0.266775\ x_0$$

$\therefore$ overshoot is 26.6775%

(b)

If $\zeta = \frac{5}{4}\zeta_0 = 0.6461$, δ is given by

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi(0.6461)}{\sqrt{1-(0.6461)^2}} = 5.3189 = 2 \ln\left(\frac{x_0}{x_{\frac{1}{2}}}\right)$$

$$\frac{x_0}{x_{\frac{1}{2}}} = 14.2888 , \qquad x_{\frac{1}{2}} = 0.0700\, x_0$$

$\therefore$ overshoot = 7%

2.80 (i) (a) Viscous damping, (b) Coulomb damping.

(iii) (a) $\tau_d = 0.2$ sec, $f_d = 5$ Hz, $\omega_d = 31.416$ rad/sec.
(b) $\tau_n = 0.2$ sec, $f_n = 5$ Hz, $\omega_n = 31.416$ rad/sec.

(ii) (a) $\frac{x_i}{x_{i+1}} = e^{\varsigma \omega_n \tau_d}$

$$\ln\left(\frac{x_i}{x_{i+1}}\right) = \ln 2 = 0.6931 = \frac{2\pi\varsigma}{\sqrt{1-\varsigma^2}}.sp$$

or $39.9590\,\varsigma^2 = 0.4804$ or $\varsigma = 0.1096$

Since $\omega_d = \omega_n\sqrt{1-\varsigma^2}$, we find

$$\omega_n = \frac{\omega_d}{\sqrt{1-\varsigma^2}} = \frac{31.416}{\sqrt{0.98798}} = 31.6065 \text{ rad/sec}$$

$$k = m\,\omega_n^2 = \left(\frac{500}{9.81}\right)(31.6065)^2 = 5.0916\,(10^4) \text{ N/m}$$

$$\varsigma = \frac{c}{c_c} = \frac{c}{2\,m\,\omega_n}$$

Hence $c = 2\,m\,\omega_n\,\varsigma = 2\,(\frac{500}{9.81})\,(31.6065)\,(0.1096) = 353.1164$ N–s/m

(b) From Eq. (2.116):

$$k = m\,\omega_n^2 = \frac{500}{9.81}(31.416)^2 = 5.0304\,(10^4) \text{ N/m}$$

Using N = W = 500 N,

$$\mu = \frac{0.002\,k}{4\,W} = \frac{(0.002)\,(5.0304\,(10^4))}{4\,(500)} = 0.0503$$

2.81 (a) $c_c = 2\sqrt{k\,m} = 2\sqrt{5000 \times 50} = 1000$ N-s/m

(b) $c = c_c/2 = 500$ N-s/m

$$\omega_d = \omega_n\sqrt{1-\varsigma^2} = \sqrt{\frac{k}{m}}\sqrt{1-\left(\frac{c}{c_c}\right)^2} = \sqrt{\frac{5000}{50}}\sqrt{1-\left(\frac{1}{2}\right)^2}$$

$= 8.6603$ rad/sec

(c) From Eq. (2.85), $\delta = \frac{2\pi}{\omega_d}\left(\frac{c}{2m}\right) = \frac{2\pi}{8.6603}\left(\frac{500}{2\times 50}\right)$

$= 3.6276$

2.82 $m = 2000$ kg, $v = \dot{x}_0 = 10$ m/sec, $k = 40{,}000$ N/m

$c = 20{,}000$ N-sec/m

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{40000}{2000}} = 4.4721 \text{ rad/sec}$$

$$c_c = 2\sqrt{km} = 25,298.221 \text{ N-sec/m}$$

$$\zeta = c/c_c = 0.7906$$

$$\omega_d = \omega_n \sqrt{1-\zeta^2} = 4.4721 \sqrt{1-(0.7906)^2} = 2.7384 \text{ rad/sec}$$

$$\tau_d = 2\pi/\omega_d = 2.2945 \text{ sec}$$

(a) For $x_0 = 0$ and $\dot{x}_0 = 10$ m/sec, Eq. (2.72) gives

$$x(t) = e^{-\zeta\omega_n t} \frac{\dot{x}_0}{\omega_n\sqrt{1-\zeta^2}} \sin \omega_n \sqrt{1-\zeta^2}\, t$$

At x_{max}, $\omega_n t \simeq \frac{\pi}{2}$ and $\sin \omega_n \sqrt{1-\zeta^2}\, t \simeq 1$

$$\therefore x_{max} \simeq e^{-0.7906(\pi/2)} \cdot \left(\frac{10}{2.7384}\right) \cdot (1) = 1.0548 \text{ m}$$

(b) $t = \tau_d/4 = 2.2945/4 = 0.5736$ sec.

2.83 $\omega_n = 200$ cycles/min $= 20.944$ rad/sec, $\omega_d = 180$ cycles/min $= 18.8496 \frac{\text{rad}}{\text{sec}}$

$J_0 = 0.2$ kg-m^2

Since $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$, $\zeta = \sqrt{1-\left(\frac{\omega_d}{\omega_n}\right)^2} = \sqrt{1-\left(\frac{18.8496}{20.944}\right)^2} = 0.4359$

$$= \frac{c_t}{(c_t)_{cri}} = \frac{c_t}{2 J_0 \omega_n}$$

$$c_t = 2 J_0 \omega_n \zeta = 2(0.2)(20.944)(0.4359)$$

$$= 3.6518 \text{ N-m-s/rad}$$

Eq. (2.72) can be used to obtain $\theta(t)$ for $\dot{\theta}_0 = 0$, $\theta_0 = 2^\circ = 0.03491$ rad and $t = \tau_d = \frac{2\pi}{\omega_d} = 0.3333$ sec,

$$\theta(t) = e^{-\zeta\omega_n t}\, \theta_0 \left\{ \cos \omega_d t + \frac{\zeta\omega_n}{\omega_d} \sin \omega_d t \right\}$$

$$= e^{-(0.4359)(20.944)(0.3333)} (0.03491) \left\{ \cos 18.8496 \times 0.3333 + \frac{0.4359 \times 20.944}{18.8496} \sin 18.8496 \times 0.3333 \right\}$$

$$= 0.001665 \text{ rad} = 0.09541^\circ$$

2.84 Assume that the bicycle and the boy fall as a rigid body by 5 cm at point A. Thus the mass (m_{eq}) will be subjected to an initial downward displacement of 5 cm (t = 0 assumed at point A):

$$x_0 = 0.05 \text{ m}, \ \dot{x}_0 = 0$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{(50000)(9.81)}{800}} = 24.7614 \text{ rad/sec}$$

$$c_c = 2\,m\,\omega_n = 2\left(\frac{800}{9.81}\right)(24.7614) = 4038.5566 \text{ N-s/m}$$

$$\zeta = \frac{c}{c_c} = \frac{1000.0}{4038.5566} = 0.2476 \text{ (underdamping)}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 24.7614\sqrt{1-0.2476^2} = 23.9905 \text{ rad/sec}$$

Response of the system:

$$x(t) = X\,e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

where $X = \left\{x_0^2 + \left(\frac{\dot{x}_0 + \zeta\,\omega_n\,x_0}{\omega_d}\right)^2\right\}^{\frac{1}{2}}$

$$= \left\{(0.05)^2 + \left(\frac{(0.2476)\,(24.7614)\,(0.05)}{23.9905}\right)^2\right\}^{\frac{1}{2}} = 0.051607 \text{ m}$$

and $\phi = \tan^{-1}\left(\frac{x_0\,\omega_d}{\dot{x}_0 + \zeta\,\omega_n\,x_0}\right) = \tan^{-1}\left(\frac{0.05\,(23.9905)}{0.2476\,(24.7614)\,(0.05)}\right) = 75.6645°$

Thus the displacement of the boy (positive downward) in vertical direction is given by

$$x(t) = 0.051607\,e^{-6.1309\,t}\sin(23.9905\,t + 75.6645°) \text{ m}$$

2.85 Reduction in amplitude of viscously damped free vibration in one cycle = 0.5 in.

$$\frac{x_1}{x_2} = \frac{6.0}{5.5} = 1.0909;\quad \ln\frac{x_1}{x_2} = 0.08701 = \frac{2\,\pi\,\zeta}{\sqrt{1-\zeta^2}}$$

i.e., $0.007571\,(1-\zeta^2) = 39.478602\,\zeta^2$ or $\zeta = 0.013847$

2.86 $\tau_d = 0.2 \text{ sec} = \frac{2\pi}{\omega_d}$, $\quad \omega_d = 31.416 \text{ rad/sec}$

From Eq. (2.92) $\quad \delta = \frac{1}{50}\ln 10 = 0.04605$

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = \frac{0.04605}{\sqrt{(2\pi)^2 + 0.04605^2}} = 0.007329$$

When damping is neglected,

$$\omega_n = \omega_d/\sqrt{1-\zeta^2} = 31.417 \text{ rad/sec};\quad \tau_n = \frac{2\pi}{\omega_n} = 0.19999 \text{ sec}$$

Proportional decrease in period $= \left(\frac{0.2 - 0.19999}{0.2}\right) = 0.00005$

2.87 For critically damped system, Eq. (2.80) gives

$$x(t) = \{x_0 + (\dot{x}_0 + \omega_n x_0)\,t\}\,e^{-\omega_n t} \qquad (E_1)$$

$$\dot{x}(t) = e^{-\omega_n t}\{\dot{x}_0 - \dot{x}_0\,\omega_n t - \omega_n^2\,x_0 t\} \qquad (E_2)$$

Let t_m = time at which $x = x_{max}$ and $\dot{x} = 0$ occur.
Here $x_0 = 0$ and $\dot{x}_0$ = initial recoil velocity. By setting $\dot{x}(t) = 0$, Eq. (E_2) gives

$$t_m = \frac{\dot{x}_0}{\omega_n(\dot{x}_0 + \omega_n x_0)} = \frac{\dot{x}_0}{\omega_n \dot{x}_0} = \frac{1}{\omega_n} \qquad (E_3)$$

With Eq. (E_3) for t_m and $x_0 = 0$, (E_1) gives

$$x_{max} = \dot{x}_0 \, t_m \, e^{-\omega_n t_m} = \frac{\dot{x}_0 \, e^{-1}}{\omega_n}$$

i.e. $\dot{x}_0 = \omega_n \, x_{max} \, e = \omega_n (0.5)(2.7183)$ $\qquad (E_4)$

Using $\dot{x}_0 = 10$ m/sec, $\quad \omega_n = 10/(0.5 * 2.7183) = 7.3575 \frac{rad}{sec}$

When mass of gun is 500 kg,
the stiffness of the spring is

$$k = \omega_n^2 \, m = (7.3575)^2 (500) = 27{,}066.403 \text{ N/m}$$

(2.88) $k = 5000$ N/m , $\quad c_c = 0.2$ N-s/mm $= 200$ N-s/m

$= 2\sqrt{k\,m} = 2\sqrt{5000\,m}$

$m = 2$ kg

$\omega_n = \sqrt{k/m} = \sqrt{5000/2} = 50$ rad/sec

Logarithmic decrement $= \delta = \dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}} = 2.0$

i.e., $\zeta = \frac{c}{c_c} = 0.3033$ and $c = 0.3033\,(0.2) = 60.66$ N-s/m

Assuming $x_0 = 0$ and $\dot{x}_0 = 1$ m/s,

$$x(t) = e^{-\zeta\omega_n t} \frac{\dot{x}_0}{\omega_n\sqrt{1-\zeta^2}} \sin\sqrt{1-\zeta^2}\,\omega_n t$$

For x_{max}, $\omega_n t \approx \pi/2$ and $\sin\sqrt{1-\zeta^2}\,\omega_n t \approx 1$

$$\therefore \quad x_{max} \simeq e^{-0.3033(\pi/2)} \frac{1}{50\sqrt{1-0.3033^2}} (1) = 0.01303 \text{ m}$$

(2.89) For an overdamped system, Eq. (2.81) gives

$$x(t) = e^{-\zeta\omega_n t}\left(C_1 e^{\omega_d t} + C_2 e^{-\omega_d t}\right) \qquad (E_1)$$

Using the relations $\quad e^{\pm x} = \cosh x \pm \sinh x \qquad (E_2)$

Eq. (E_1) can be rewritten as

$$x(t) = e^{-\zeta\omega_n t}\left(C_3 \cosh\omega_d t + C_4 \sinh\omega_d t\right) \qquad (E_3)$$

where $C_3 = C_1 + C_2$ and $C_4 = C_1 - C_2$.

Differentiating (E_3),

$$\dot{x}(t) = e^{-\zeta\omega_n t}\left[C_3\,\omega_d \sinh\omega_d t + C_4\,\omega_d \cosh\omega_d t\right]$$
$$-\zeta\omega_n\, e^{-\zeta\omega_n t}\left[C_3 \cosh\omega_d t + C_4 \sinh\omega_d t\right] \qquad (E_4)$$

Initial conditions $x(t=0) = x_0$ and $\dot{x}(t=0) = \dot{x}_0$ with (E_3) and (E_4) give

$$C_3 = x_0\,, \quad C_4 = \left(\dot{x}_0 + \zeta\omega_n x_0\right)/\omega_d \qquad (E_5)$$

Thus (E_3) becomes

$$x(t) = x_0\, e^{-\zeta\omega_n t}\left(\cosh\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sinh\omega_d t\right)$$
$$+ \frac{\dot{x}_0}{\omega_d}\, e^{-\zeta\omega_n t}\sinh\omega_d t \qquad (E_6)$$

(i) when $\dot{x}_0 = 0$, Eq. (E_6) gives

$$x(t) = x_0\, e^{-\zeta\omega_n t}\left(\cosh\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sinh\omega_d t\right) \qquad (E_7)$$

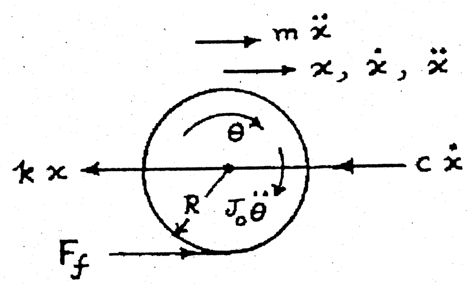
since $e^{-\zeta\omega_n t}$, $\cosh\omega_d t$, $\frac{\zeta\omega_n}{\omega_d}$ and $\sinh\omega_d t$ do not change sign (always positive) and $e^{-\zeta\omega_n t}$ approaches zero with increasing t, $x(t)$ will not change sign.

(ii) when $x_0 = 0$, Eq. (E_6) gives

$$x(t) = \frac{\dot{x}_0}{\omega_d}\, e^{-\zeta\omega_n t}\sinh\omega_d t \qquad (E_8)$$

Here also, ω_d, $e^{-\zeta\omega_n t}$ and $\sinh\omega_d t$ do not change sign (always positive) and $e^{-\zeta\omega_n t}$ approaches zero with increasing t, $x(t)$ will not change sign.

(2.90) Newton's second law of motion:



$$\sum F = m\ddot{x} = -kx - c\dot{x} + F_f \qquad (1)$$
$$\sum M = J_0\ddot{\theta} = -F_f R \qquad (2)$$

where F_f = friction force.

Using $J_0 = \frac{mR^2}{2}$ and $\ddot{\theta} = \frac{\ddot{x}}{R}$, Eq. (2) gives

$$F_f = -\frac{1}{2R}\left(mR^2\right)\frac{\ddot{x}}{R} = -\frac{1}{2}m\ddot{x} \qquad (3)$$

Substitution of Eq. (3) into (1) yields:

$$\frac{3}{2}\, m\, \ddot{x} + c\, \dot{x} + k\, x = 0 \qquad (4)$$

The undamped natural frequency is: $\omega_n = \sqrt{\frac{2\,k}{3\,m}}$ (5)

(2.91) Newton's second law of motion: (measuring x from static equilibrium position of cylinder)

$$\sum F = m\, \ddot{x} = -\, k\, x - c\, \dot{x} - k\, x + F_f \qquad (1)$$
$$\sum M = J_0\, \ddot{\theta} = -\, F_f\, R \qquad (2)$$

where F_f = friction force. Using $J_0 = \frac{1}{2}\, m\, R^2$ and $\ddot{\theta} = \frac{\ddot{x}}{R}$, Eq. (2) gives

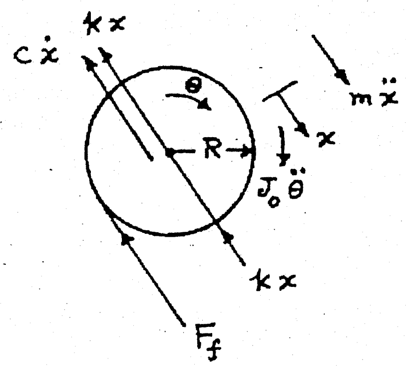
$$F_f = -\, \frac{1}{2}\, m\, \ddot{x} \qquad (3)$$

Substitution of Eq. (3) into (1) gives

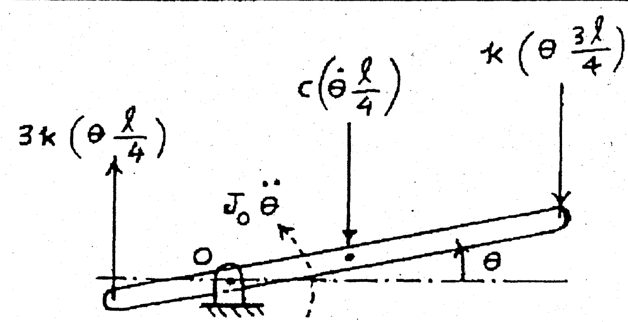
$$\frac{3}{2}\, m\, \ddot{x} + c\, \dot{x} + 2\, k\, x = 0 \qquad (4)$$

Undamped natural frequency of the system:

$$\omega_n = \sqrt{\frac{4\,k}{3\,m}} \qquad (4)$$



(2.92)



Consider a small angular displacement of the bar θ about its static equilibrium position. Newton's second law gives:

$$\sum M = J_0\, \ddot{\theta} = -k \left(\theta\, \frac{3\,\ell}{4}\right) \left(\frac{3\,\ell}{4}\right) - c \left(\dot{\theta}\, \frac{\ell}{4}\right) \left(\frac{\ell}{4}\right) - 3\, k \left(\theta\, \frac{\ell}{4}\right) \left(\frac{\ell}{4}\right)$$

$$\text{i.e.,} \quad J_0\, \ddot{\theta} + \frac{c\, \ell^2}{16}\, \dot{\theta} + \frac{3}{4}\, k\, \ell^2\, \theta = 0$$

where $J_0 = \frac{7}{48}\, m\, \ell^2$. The undamped natural frequency of torsional vibration is given by:

$$\omega_n = \sqrt{\frac{3\, k\, \ell^2}{4\, J_0}} = \sqrt{\frac{36\, k}{7\, m}}$$

(2.93) Let δx = virtual displacement given to cylinder. Virtual work done by various forces:

Inertia forces: $\delta W_i = -(J_0\, \ddot{\theta})\, (\delta\theta) - (m\, \ddot{x})\, \delta x = -(J_0\, \ddot{\theta})\, (\frac{\delta x}{R}) - (m\, \ddot{x})\, \delta x$

Spring force: $\delta W_s = -(k\, x)\, \delta x$
Damping force: $\delta W_d = -(c\, \dot{x})\, \delta x$
By setting the sum of virtual works equal to zero, we obtain:

$$-\frac{J_0}{R}\left(\frac{\ddot{x}}{R}\right) - m\, \ddot{x} - k\, x - c\, \dot{x} = 0 \quad \text{or} \quad \frac{3}{2}\, m\, \ddot{x} + c\, \dot{x} + k\, x = 0$$

(2.94) Let δx = virtual displlacement given to cyllinder from its static equillibrium position. Virtualll works done by various forces:

Inertia forces: $\delta W_i = -(J_0\, \ddot{\theta})\, \delta\theta - (m\, \ddot{x})\, \delta\theta = -(J_0\, \frac{\ddot{x}}{R})\, (\frac{\delta x}{R}) - (m\, \ddot{x})\, \delta x$

Spring force: $\delta W_s = -(k\, x)\, \delta x - (k\, x)\, \delta x = -2\, k\, x\, \delta x$

Damping force: $\delta W_d = -(c\, \dot{x})\, \delta x$
By setting the sum of virtual works equal to zero, we find

$$-\frac{J_0}{R}\, \frac{\ddot{x}}{R} - m\, \ddot{x} - 2\, k\, x - c\, \dot{x} = 0 \tag{1}$$

Using $J_0 = \frac{1}{2}\, m\, R^2$, Eq. (1) can be rewritten as

$$\frac{3}{2}\, m\, \ddot{x} + c\, \dot{x} + 2\, k\, x = 0 \tag{2}$$

(2.95) See figure given in the solution of Problem 2.92. Let $\delta\theta$ be virtuall angular displacement given to the bar about its static equilibrium position. Virtual works done by various forces:
Inertia force: $\delta W_i = -(J_0\, \ddot{\theta})\, \delta\theta$
Spring forces:

$$\delta W_s = -(k\, \theta\, \frac{3\, \ell}{4})\, (\frac{3\, \ell}{4}\, \delta\theta) - (3\, k\, \theta\, \frac{\ell}{4})\, (\frac{\ell}{4}\, \delta\theta) = -\left(\frac{3}{4}\, k\, \ell^2\, \theta\right) \delta\theta$$

Damping force: $\delta W_d = -(c\, \dot{\theta}\, \frac{\ell}{4})\, (\frac{\ell}{4}\, \delta\theta)$

By setting the sum of virtual works equal to zero, we get the equation of motion as:

$$J_0\, \ddot{\theta} + c\, \frac{\ell^2}{16}\, \dot{\theta} + \frac{3}{4}\, k\, \ell^2\, \theta = 0$$

(2.96) See solution of Problem 2.72. When wooden prism is given a displacement x, equation of motion becomes: $m\ddot{x}$ + restoring force = 0
where m = mass of prism = 40 kg and restoring force = weight of fluid displaced $= \rho_0\, g\, a\, b\, x = \rho_0\,(9.81)\,(0.4)\,(0.6)\,x = 2.3544\,\rho_0\,x$ where ρ_0 is the density of the fluid. Thus the equation of motion becomes:

$$40\,\ddot{x} + 2.3544\,\rho_0\,x = 0$$

$$\text{Natural frequency} = \omega_n = \sqrt{\frac{2.3544\,\rho_0}{40}}$$

$$\text{Since } \tau_n = \frac{2\,\pi}{\omega_n} = 0.5, \text{we find}$$

$$\omega_n = \frac{2\,\pi}{0.5} = 4\,\pi = \sqrt{\frac{2.3544\,\rho_0}{40}}$$

Hence $\rho_0 = 2682.8816$ kg/m^3.

(2.97) Let x = displacement of mass and P = tension in rope on the left of mass. Equations of motion:

$$\sum F = m\,\ddot{x} = -k\,x - P \quad \text{or} \quad P = -m\,\ddot{x} - k\,x \tag{1}$$

$$\sum M = J_0\,\ddot{\theta} = P\,r_2 - c\,(\dot{\theta}\,r_1) \tag{2}$$

Using Eq. (1) in (2), we obtain

$$-(m\,\ddot{x} + k\,x)\,r_2 - c\,\dot{\theta}\,r_1 = J_0\,\ddot{\theta} \tag{3}$$

With $x = \theta\,r_2$, Eq. (3) can be written as:

$$(J_0 + m\,r_2^2)\,\ddot{\theta} + c\,r_1\,\dot{\theta} + k\,r_2^2\,\theta = 0 \tag{4}$$

For given data, Eq. (4) becomes

$$[5 + 10\,(0.25)^2]\,\ddot{\theta} + c\,(0.1)\,\dot{\theta} + k\,(0.25)^2\,\theta = 0$$

$$\text{or} \quad 5.625\,\ddot{\theta} + 0.1\,c\,\dot{\theta} + 0.0625\,k\,\theta = 0 \tag{5}$$

Since amplitude is reduced by 80% in 10 cycles,

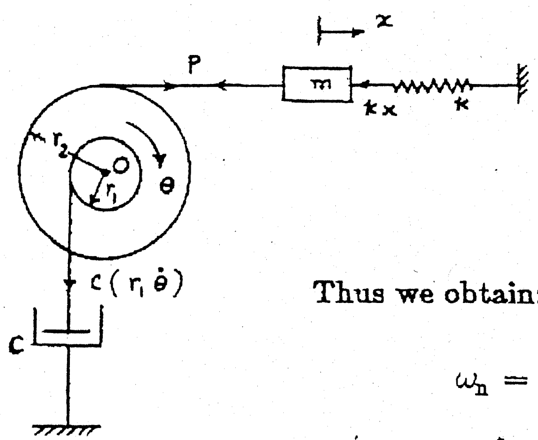
$$\frac{x_1}{x_{11}} = \frac{1.0}{0.2} = 5 = e^{10\,\zeta\,\omega_n\,\tau_d}$$

$$\ln\frac{x_1}{x_{11}} = \ln 5 = 1.6094 = 10\,\zeta\,\omega_n\,\tau_d \tag{6}$$

Since the natural frequency (assumed to be undamped torsional vibration frequency) is 5 Hz, $\omega_n = 2\,\pi\,(5) = 31.416$ rad/sec. Also

$$\tau_d = \frac{1}{f_d} = \frac{2\,\pi}{\omega_d} = \frac{2\,\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{0.2}{\sqrt{1-\zeta^2}} \tag{7}$$

Eq. (6) gives



$$1.6094 = 10\,\varsigma\,(31.416)\left(\frac{0.2}{\sqrt{1-\varsigma^2}}\right) = \frac{62.832\,\varsigma}{\sqrt{1-\varsigma^2}}$$

i.e., $\sqrt{1-\varsigma^2} = \dfrac{62.832}{1.6094}\varsigma = 39.0406\,\varsigma$

i.e., $\varsigma = 0.02561$

Thus we obtain:

$$\omega_n = \sqrt{\frac{0.0625\,k}{5.625}} = 31.416 \quad \text{or} \quad k = 8.8827\,(10^4)\ \text{N/m}$$

$$\varsigma = 0.02561 = \frac{c}{c_c} = \frac{c}{2\,m_{eq}\,\omega_n} = \frac{0.10\,c}{2\,(5.625)\,(31.416)}$$

or $c = 90.5134$ N–s/m

2.98 $m = 20$ kg, $k = 4000$ N/m

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{20}} = 14.1421 \text{ rad/sec}$$

Amplitudes of successive cycles : 50, 45, 40, 35 mm

Amplitudes of successive cycles diminish by 5 mm $= 5\times10^{-3}$ m

System has Coulomb damping.

$$\frac{4\mu N}{k} = 5 * 10^{-3} \Rightarrow \mu N = \left\{\frac{(5\times10^{-3})(4000)}{4}\right\} = 5 \text{ N}$$

= damping force

Frequency of damped vibration = 14.1421 rad/sec.

2.99 $m = 20$ kg, $k = 10000$ N/m, $\dfrac{4\mu N}{k} = \dfrac{150-100}{4}$ mm $= 12.5\times10^{-3}$ m

$$\mu = \frac{(12.5\times10^{-3})(10\,000)}{4\,(20\times9.81)} = 0.1593$$

$$\text{Time elapsed} = 4\,\tau_n = 4\times\frac{2\pi}{\omega_n} = 8\pi\sqrt{\frac{m}{k}} = 1.124 \text{ sec}$$

2.100 $m = 10$ kg, $k = 3000$ N/m, $\mu = 0.12$, $X = 100$ mm

$$\frac{4\mu N}{k} = \frac{4\,(0.12)\,(10\times9.81)}{3000} = 0.0157 \text{ m} = 15.7 \text{ mm}$$

As $6\left(\dfrac{4\mu N}{k}\right) = 94.2$ mm, mass comes to rest at $(100-94.2) = 5.8$ mm

---

2.101 $mg = 25$ N, $k = 1000$ N/m, damping force = constant

Mass released with $x_0 = 10$ cm and $\dot{x}_0 = 0$.

Static deflection of spring due to self weight of mass $= \dfrac{25}{1000}$

$= 0.025$ m

at $t = 0$: $x = 0.1$ m, $\dot{x} = 0$

$x_0 = 0.1$

$$x_1 = x_0 - 2\frac{\mu N}{k} , \quad x_2 = x_0 - \frac{4\mu N}{k}$$

$$x_3 = a_0 - \frac{6\mu N}{k} , \quad x_4 = x_0 - \frac{8\mu N}{k} = 0$$

i.e., $x_0 = \frac{8\mu N}{k} = 0.1$

Magnitude of damping force $= \mu N = \frac{x_0 k}{8} = \frac{(0.1)(1000)}{8}$
$= 12.5$ N

(2.102) $m = 20$ kg, $k = 10{,}000$ N/m, $\mu N = 50$ N, $x_0 = 0.05$ m

(a) Number of half cycles elapsed before mass comes to rest (r) is given by:

$$r \geq \left\{ \frac{x_0 - \frac{\mu N}{k}}{2\frac{\mu N}{k}} \right\} = \frac{0.05 - \left(\frac{50}{10000}\right)}{2\left(\frac{50}{10\,000}\right)} = 4.5$$

$\therefore r = 5$

(b) Time elapsed before mass comes to rest:

$$t_p = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{20}{10000}} = 0.2810 \text{ sec}$$

Time taken $= (2.5 \text{ cycles})\, t_p = 0.7025$ sec

(c) Final extension of spring after 5 half-cycles:

$$x_5 = 0.05 - 5\left(\frac{2\mu N}{k}\right) = 0.05 - 5\left(2 * \frac{50}{10\,000}\right) = 0$$

(displacement from static equilibrium position $= 0$)

But static deflection $= \frac{mg}{k} = \frac{20 * 9.81}{10\,000} = 0.01962$ m

$\therefore$ Final extension of spring $= 1.9620$ cm.

(2.103) (a) Equation of motion for angular oscillations of pendulum:

$$J_0\ddot{\theta} + mgl\sin\theta \pm mg\mu\frac{d}{2}\cos\theta = 0$$

For small angles, $\ddot{\theta} + \frac{mgl}{J_0}\left(\theta \pm \frac{\mu d}{2l}\right) = 0$

This shows that the angle of swing decreases by $\left(\frac{\mu d}{2l}\right)$ in each quarter cycle.

(b) For motion from right to left:

$$\theta(t) = A_1\cos\omega_n t + A_2\sin\omega_n t + \frac{\mu d}{2l}$$

where $\omega_n = \sqrt{\frac{mgl}{J_0}}$.

Let $\theta(t=0) = \theta_0$ and $\dot{\theta}(t=0) = 0$. Then $A_1 = \theta_0 - \frac{\mu d}{2l}$, $A_2 = 0$

$$\theta(t) = \left(\theta_0 - \frac{\mu d}{2 l}\right) \cos \omega_n t + \frac{\mu d}{2 l}$$

For motion from left to right:

$$\theta(t) = A_3 \cos \omega_n t + A_4 \sin \omega_n t - \frac{\mu d}{2 l}$$

At $\omega_n t = \pi$, $\theta = -\theta_0 + \frac{2 \mu d}{2 l}$, $\dot{\theta} = 0$ from previous solution.

$$A_3 = \theta_0 - \frac{3 \mu d}{2 l} , \quad A_4 = 0$$

$$\theta(t) = \left(\theta_0 - \frac{3 \mu d}{2 l}\right) \cos \omega_n t - \frac{\mu d}{2 l}$$

(c) The motion ceases when $\left(\theta_0 - n \frac{4 \mu d}{2 l}\right) < \frac{\mu d}{2 l}$

where n denotes the number of cycles.

2.104 $x(t) = X \sin \omega t$ (under sinusoidal force $F_0 \sin \omega t$)

Damping force $= \mu N$

Total displacement per cycle $= 4 X$

Energy dissipated per cycle $= \Delta W = 4 \mu N X$ (E_1)

If c_{eq} = equivalent viscous damping constant, energy dissipated per cycle is given by Eq. (2.98):

$$\Delta W = \pi c_{eq} \omega X^2 \qquad (E_2)$$

Equating (E_1) and (E_2) gives

$$c_{eq} = \frac{4 \mu N X}{\pi \omega X^2} = \frac{4 \mu N}{\pi \omega X} \qquad (E_3)$$

2.105 Due to viscous damping:

$$\delta = \ln \left(\frac{X_m}{X_{m+1}}\right) \simeq 2 \pi \zeta$$

z_1 = percent decrease in amplitude per cycle at X_m

$$= 100 \left(\frac{X_m - X_{m+1}}{X_m}\right) = 100 \left(1 - \frac{X_{m+1}}{X_m}\right) = 100 \left(1 - e^{-2\pi\zeta}\right)$$

Due to Coulomb damping:

z_2 = percent decrease in amplitude per cycle at X_m

$$= 100 \left(\frac{X_m - X_{m+1}}{X_m}\right) = 100 \left(\frac{4 \mu N}{k X_m}\right)$$

When both types of damping are present:

$$z_1 + z_2 \Big|_{X_m = 20 \text{ mm}} = 2 \ ; \quad z_1 + z_2 \Big|_{X_m = 10 \text{ mm}} = 3$$

i.e.,

$$100\left(1-e^{-2\pi\zeta}\right)+\frac{400}{0.02}\left(\frac{\mu N}{k}\right)=2$$

$$100\left(1-e^{-2\pi\zeta}\right)+\frac{400}{0.01}\left(\frac{\mu N}{k}\right)=3$$

The solution of these equations gives

$$50\left(1-e^{-2\pi\zeta}\right)=0.5 \quad \text{and} \quad \frac{\mu N}{k}=0.5\times10^{-6}\ \text{m}$$

(1.106) Coulomb damping.

(a) Natural frequency $= \omega_n = \frac{2\pi}{\tau_n} = \frac{2\pi}{1} = 6.2832$ rad/sec. Reduction in amplitude in each cycle:

$$= \frac{4\,\mu\,N}{k} = 4\,\mu\,g\,\frac{m}{k} = \frac{4\,\mu\,g}{\omega_n^2} = 4\,\mu\left(\frac{9.81}{6.2832^2}\right)$$

$$= 0.9940\,\mu = \frac{0.5}{100} = 0.005\ \text{m}$$

Kinetic coefficient of friction $= \mu = 0.00503$

(b) Number of half-cycles executed (r) is:

$$r \geq \frac{\left(x_0 - \frac{\mu N}{k}\right)}{\left(\frac{2\,\mu\,N}{k}\right)} = \frac{\left(x_0 - \frac{\mu g}{\omega_n^2}\right)}{\left(\frac{2\,\mu\,g}{\omega_n^2}\right)}$$

$$\geq \frac{\left(0.1 - \frac{0.00503\,(9.81)}{6.2832^2}\right)}{\left(\frac{2\,(0.00503)\,(9.81)}{6.2832^2}\right)}$$

$$\geq 39.5032$$

$$\geq 40$$

Thus the block stops oscillating after 20 cycles.

(2.107) Friction force $= \mu\,N = 0.2\ (5) = 1$ N. $k = \frac{25}{0.10} = 250$ N/m. Reduction in amplitude in each cycle $= \frac{4\,\mu\,N}{k} = \frac{4\,(1)}{250} = 0.016$ m. Number of half-cycles executed before the motion ceases (r):

$$r \geq \left(\frac{x_0 - \frac{\mu N}{k}}{\frac{2\,\mu\,N}{k}}\right) = \frac{0.1 - 0.004}{0.008} \geq 12$$

Thus after 6 cycles, the mass stops at a distance of 0.1 - 6 (0.016) = 0.004 m from the unstressed position of the spring.

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{250\,(9.81)}{5}} = 22.1472 \text{ rad/sec}$$

$$\tau_n = \frac{2\,\pi}{\omega_n} = 0.2837 \text{ sec}$$

Thus total time of vibration $= 6\,\tau_n = 1.7022$ sec.

(2.108) Energy dissipated in each full load cycle is given by the area enclosed by the hysteresis loop.

The area can be found by counting the squares enclosed by the hysteresis loop. In Fig. 2.99, the number of squares is ≈ 33. Since each square $= \frac{100 \times 1}{1000} = 0.1$ N-m, the energy dissipated in a cycle is

$$\Delta W = 33 \times 0.1 = 3.3 \text{ N-m} = \pi\, k\, \beta\, X^2$$

Since the maximum deflection $= X = 4.3$ mm, and the slope of the force-deflection curve is

$$k \simeq \frac{1800 \text{ N}}{11 \text{ mm}} = 1.6364 \times 10^5 \text{ N/m},$$

the hysteresis damping constant β is given by

$$\beta = \frac{\Delta W}{\pi\, k\, X^2} = \frac{3.3}{\pi\,(1.6364 \times 10^5)\,(0.0043)^2} = 0.3472$$

$\delta = \pi \beta =$ logarithmic decrement $= \pi\,(0.3472) = 1.0908$

Equivalent viscous damping ratio $= \zeta_{eq} = \beta/2 = 0.1736$.

(2.109) $\frac{X_j}{X_{j+1}} = \frac{2+\pi\beta}{2-\pi\beta} = 1.1$, $\beta = 0.03032$

$c_{eq} = \beta\,\sqrt{m k} = 0.03032\,\sqrt{1 \times 2} = 0.04288$ N-s/m

$\Delta W = \pi\, k\, \beta\, X^2 = \pi\,(2)\,(0.03032)\left(\frac{10}{1000}\right)^2 = 19.05 \times 10^{-6}$ N-m

---

(2.110) Logarithmic decrement $= \delta = \ln\left(\frac{X_j}{X_{j+1}}\right) \simeq \pi\beta$

For n cycles, $\delta = \frac{1}{n}\ln\left(\frac{X_0}{X_n}\right) \simeq \pi\beta$

$$\frac{1}{100}\ln\left(\frac{30}{20}\right) = 0.004055 = \pi\beta$$

$\beta = 0.001291$

(2.111) Torque = 2×10^{-3} N-m

angle = 50° = 80 divisions

For a torsional system, Eq. (2.84) gives

$$\frac{\theta_1}{\theta_2} = e^{\zeta \omega_n \tau_d} \qquad (E_1)$$

(b) For one cycle, $\tau_d = 2$ sec and (E_1) gives

$$\frac{80}{5} = e^{2 \zeta \omega_n} \quad \text{or} \quad \zeta \omega_n = \frac{1}{2} \ln(16) = 1.3863 \qquad (E_2)$$

Since $\tau_d = \dfrac{2\pi}{\sqrt{\omega_n^2 - \zeta^2 \omega_n^2}}$,

$$\omega_n^2 = \frac{(2\pi)^2}{\tau_d^2} + \zeta^2 \omega_n^2 = \frac{4\pi^2}{4} + 1.3863^2 = 11.7915$$

i.e., $\omega_n = 3.4339$ rad/sec $\qquad (E_3)$

(d) Since angular displacement of rotor under applied torque

$= 50^\circ = 0.8727$ rad,

$$k_t = \text{torque/angular displacement} = 2 \times 10^{-3} / 0.8727$$
$$= 2.2917 \times 10^{-3} \text{ N-m/rad} \qquad (E_4)$$

(a) Mass moment of inertia of rotor is

$$J_0 = \frac{k_t}{\omega_n^2} = 2.2917 \times 10^{-3} / 11.7915 = 1.9436 \times 10^{-4} \text{ N-m-s}^2 \qquad (E_5)$$

(c) $c_t = 2 J_0 \zeta \omega_n \qquad (E_6)$

Eqs. (E_2) and (E_3) give $\zeta = \dfrac{\zeta \omega_n}{\omega_n} = \dfrac{1.3863}{3.4339} = 0.4037$

Eq. (E_6) gives $c_t = 5.3887 \times 10^{-4}$ N-m-s/rad.

(2.112) The input data for Program 2 and results are given below.

```
C      THE FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
       DIMENSION X(50),XD(50),XDD(50),T(50)
       DATA M,K,C,X0,XD0,N,DELT/
     2 4.0,2500.0,0.0,0.1,-10.0,50,0.01/
C      END OF PROBLEM-DEPENDENT DATA
-------------------------------------------
FREE VIBRATION ANALYSIS
OF A SINGLE DEGREE OF FREEDOM SYSTEM

DATA

M     = 0.40000000E+01
K     = 0.25000000E+04
C     = 0.00000000E+00
X0    = 0.10000000E+00
XD0   =-0.10000000E+02
N     =    50
DELT  = 0.99999998E-02


SYSTEM IS UNDAMPED


RESULTS:
```

I	TIME(I)	X(I)	XD(I)	XDD(I)
1	0.100000E-01	-0.207036E-02	-0.103076E+02	0.129397E+01
2	0.200000E-01	-0.104012E+00	-0.997438E+01	0.650075E+02
3	0.300000E-01	-0.199487E+00	-0.902097E+01	0.124679E+03
4	0.400000E-01	-0.282558E+00	-0.750667E+01	0.176599E+03
5	0.500000E-01	-0.348062E+00	-0.552565E+01	0.217539E+03
6	0.600000E-01	-0.391924E+00	-0.320107E+01	0.244953E+03
7	0.700000E-01	-0.411419E+00	-0.677464E+00	0.257137E+03
8	0.800000E-01	-0.405334E+00	0.188826E+01	0.253334E+03
9	0.900000E-01	-0.374047E+00	0.433659E+01	0.233779E+03
10	0.100000E+00	-0.319503E+00	0.651528E+01	0.199690E+03
⋮				
46	0.460000E+00	0.398512E+00	-0.264442E+01	-0.249070E+03
47	0.470000E+00	0.359954E+00	-0.502704E+01	-0.224971E+03
48	0.480000E+00	0.299016E+00	-0.709711E+01	-0.186885E+03
49	0.490000E+00	0.219486E+00	-0.872591E+01	-0.137179E+03
50	0.500000E+00	0.126310E+00	-0.981218E+01	-0.789439E+02

(2.113) The problem-dependent data and results are given (given by Program 2).

```
C      THE FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
       DIMENSION X(50),XD(50),XDD(50),T(50)
       DATA M,K,C,X0,XD0,N,DELT/
     2 4.0,2500.0,100.0,0.1,-10.0,50,0.01/
C      END OF PROBLEM-DEPENDENT DATA
-------------------------------------------
FREE VIBRATION ANALYSIS
OF A SINGLE DEGREE OF FREEDOM SYSTEM
```

```
DATA

M     = 0.40000000E+01
K     = 0.25000000E+04
C     = 0.10000000E+03
X0    = 0.10000000E+00
XD0   =-0.10000000E+02
N     =    50
DELT  = 0.99999998E-02

SYSTEM IS UNDER DAMPED


RESULTS:
```

I	TIME(I)	X(I)	XD(I)	XDD(I)
1	0.100000E-01	-0.957279E-02	0.807168E+01	-0.195809E+03
2	0.200000E-01	0.613786E-01	0.612586E+01	-0.191508E+03
3	0.300000E-01	0.113259E+00	0.427345E+01	-0.177623E+03
4	0.400000E-01	0.147433E+00	0.259570E+01	-0.157038E+03
5	0.500000E-01	0.165937E+00	0.114627E+01	-0.132367E+03
6	0.600000E-01	0.171219E+00	-0.456091E-01	-0.105872E+03
7	0.700000E-01	0.165914E+00	-0.971334E+00	-0.794127E+02
8	0.800000E-01	0.152654E+00	-0.163885E+01	-0.544377E+02
9	0.900000E-01	0.133930E+00	-0.206856E+01	-0.319924E+02
10	0.100000E+00	0.111980E+00	-0.228942E+01	-0.127520E+02
⋮				
45	0.450000E+00	-0.113169E-03	-0.310021E-01	0.845783E+00
46	0.460000E+00	-0.381386E-03	-0.227080E-01	0.806065E+00
47	0.470000E+00	-0.569294E-03	-0.149992E-01	0.730789E+00
48	0.480000E+00	-0.684317E-03	-0.817052E-02	0.631961E+00
49	0.490000E+00	-0.736251E-03	-0.240284E-02	0.520228E+00
50	0.500000E+00	-0.736195E-03	0.222121E-02	0.404592E+00

2.114 The problem-dependent data (to be used in Program 2) and output are given.

```
C     THE FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION X(50),XD(50),XDD(50),T(50)
      DATA M,K,C,X0,XD0,N,DELT/
     2 4.0,2500.0,200.0,0.1,-10.0,50,0.01/
C     END OF PROBLEM-DEPENDENT DATA
```

```
-----------------------------------------
FREE VIBRATION ANALYSIS
OF A SINGLE DEGREE OF FREEDOM SYSTEM

DATA

M     = 0.40000000E+01
K     = 0.25000000E+04
C     = 0.20000000E+03
X0    = 0.10000000E+00
XD0   =-0.10000000E+02
N     =    50
DELT  = 0.99999998E-02
```

SYSTEM IS CRITICALLY DAMPED

RESULTS:

I	TIME(I)	X(I)	XD(I)	XDD(I)
1	0.100000E-01	0.194700E-01	-0.632776E+01	0.304219E+03
2	0.200000E-01	-0.303265E-01	-0.379082E+01	0.208495E+03
3	0.300000E-01	-0.590458E-01	-0.206660E+01	0.140234E+03
4	0.400000E-01	-0.735759E-01	-0.919699E+00	0.919699E+02
5	0.500000E-01	-0.787888E-01	-0.179066E+00	0.581963E+02
6	0.600000E-01	-0.780956E-01	0.278913E+00	0.348641E+02
7	0.700000E-01	-0.738539E-01	0.543044E+00	0.190065E+02
8	0.800000E-01	-0.676676E-01	0.676676E+00	0.845846E+01
9	0.900000E-01	-0.606045E-01	0.724619E+00	0.164687E+01
10	0.100000E+00	-0.533553E-01	0.718244E+00	-0.256515E+01
⋮				
46	0.460000E+00	-0.339359E-04	0.772422E-03	-0.174112E-01
47	0.470000E+00	-0.270210E-04	0.616356E-03	-0.139296E-01
48	0.480000E+00	-0.215048E-04	0.491539E-03	-0.111364E-01
49	0.490000E+00	-0.171069E-04	0.391783E-03	-0.889736E-02
50	0.500000E+00	-0.136023E-04	0.312109E-03	-0.710396E-02

2.115 *The problem-dependent data (to be used in Program 2) and results are given.*

```
C     THE FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION X(50),XD(50),XDD(50),T(50)
      DATA M,K,C,X0,XD0,N,DELT/
     2 4.0,2500.0,400.0,0.1,-10.0,50,0.01/
C     END OF PROBLEM-DEPENDENT DATA
```

FREE VIBRATION ANALYSIS
OF A SINGLE DEGREE OF FREEDOM SYSTEM

DATA

```
M    = 0.40000000E+01
K    = 0.25000000E+04
C    = 0.40000000E+03
X0   = 0.10000000E+00
XD0  =-0.10000000E+02
N    =    50
DELT = 0.99999998E-02
```

SYSTEM IS OVER DAMPED

RESULTS:

I	TIME(I)	X(I)	XD(I)	XDD(I)

1	0.100000E-01	0.351455E-01	-0.390559E+01	0.368593E+03
2	0.200000E-01	0.990550E-02	-0.151007E+01	0.144816E+03
3	0.300000E-01	0.230870E-03	-0.569458E+00	0.568015E+02
4	0.400000E-01	-0.333730E-02	-0.201042E+00	0.221900E+02
5	0.500000E-01	-0.451879E-02	-0.576071E-01	0.858495E+01
6	0.600000E-01	-0.477582E-02	-0.257606E-02	0.324249E+01
7	0.700000E-01	-0.468266E-02	0.177700E-01	0.114966E+01
8	0.800000E-01	-0.446433E-02	0.245564E-01	0.334569E+00
9	0.900000E-01	-0.420854E-02	0.260878E-01	0.215607E-01
10	0.100000E+00	-0.394902E-02	0.256257E-01	-0.944359E-01
⋮				
46	0.460000E+00	-0.354993E-03	0.237800E-02	-0.159296E-01
47	0.470000E+00	-0.331992E-03	0.222393E-02	-0.148975E-01
48	0.480000E+00	-0.310481E-03	0.207983E-02	-0.139322E-01
49	0.490000E+00	-0.290364E-03	0.194507E-02	-0.130295E-01
50	0.500000E+00	-0.271551E-03	0.181905E-02	-0.121853E-01

2.116 The equations for the natural frequencies of vibration were derived in Problem 2.35.

Operating speed of turbine is:

$$\omega_o = (2400)\frac{2\pi}{60} = 251.328 \text{ rad/sec}$$

Thus we need to satisfy:

$$\omega_n\Big|_{axial} = \left\{\frac{g\, l\, A\, E}{W\, a\,(l-a)}\right\}^{1/2} \geq \omega_o \qquad (E_1)$$

$$\omega_n\Big|_{transverse} = \left\{\frac{3\, E\, I\, l^3\, g}{W\, a^3\,(l-a)^3}\right\}^{1/2} \geq \omega_o \qquad (E_2)$$

$$\omega_n\Big|_{circumferential} = \left\{\frac{G\, J}{J_o}\left(\frac{1}{a}+\frac{1}{l-a}\right)\right\}^{1/2} \geq \omega_o \qquad (E_3)$$

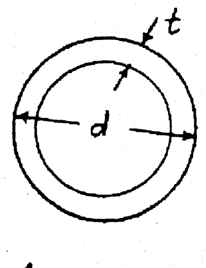
where $A = \frac{\pi d^2}{4}$, $W = 1000 \times 9.81 = 9810$ N,

$I = \frac{\pi d^4}{64}$, $J = \frac{\pi d^2}{32}$, $J_o = 500$ kg-m^2,

and $E = 207 \times 10^9$ N/m^2, $G = 79.3 \times 10^9$ N/m^2 (for steel).

The unknowns d, l and a can be determined to satisfy the inequalities (E_1), (E_2) and (E_3) using a trial and error procedure.

(2.117) From solution of problem 2.38, the requirements can be stated as:



$$\omega_n \Big|_{\text{pivot ends}} = \sqrt{\frac{12\,EI}{l^3\left(\frac{W}{g} + m_{eff\,1}\right)}} \geq \omega_0 \qquad (E_1)$$

where $E = 30 \times 10^6$ psi and $I = \frac{\pi}{64}\left[d^4 - (d-2t)^4\right]$

$$\omega_n \Big|_{\text{fixed ends}} = \sqrt{\frac{48\,EI}{l^3\left(\frac{W}{g} + m_{eff\,2}\right)}} \geq \omega_0 \qquad (E_2)$$

with

$m_{eff\,1} = (0.2357\ m)$, $m_{eff\,2} = (0.3714\ m)$,

m = mass of each column $= \frac{\pi}{4}\left[d^2 - (d-2t)^2\right]\frac{l\rho}{g}$,

$\rho = 0.283$ lb/in^3, $g = 386.4$ in/sec^2,

l = length of column = 96 in.,

W = weight of floor = 4000 lb,

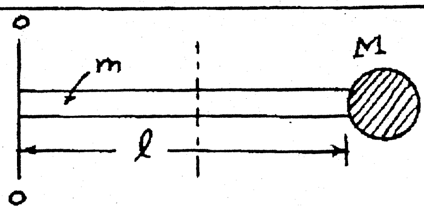
$$\underset{\sim}{W} = \text{weight of columns} = 4\left\{\frac{\pi}{4}\left[d^2 - (d-2t)^2\right] l \rho\right\} \qquad (E_3)$$

Frequency limit $= \omega_0 = 50 \times 2\pi = 314.16$ rad/sec.

Problem: Find d and t such that $\underset{\sim}{W}$ given by Eq. (E_3) is minimized while satisfying the inequalities (E_1) and (E_2).

This problem can be solved either by graphical optimization or by using a trial and error procedure.

(2.118) $J_0 = \frac{ml^2}{12} + \frac{ml^2}{4} + Ml^2 = \frac{1}{3}ml^2 + Ml^2$ --- (E_1)



(i) Viscous damping:

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \left(\frac{k_t}{\frac{1}{3}ml^2 + Ml^2}\right)^{\frac{1}{2}} \quad \text{---} \ (E_2)$$

$$(c_t)_{cri} = 2\,J_0\,\omega_n = 2\sqrt{J_0\,k_t} \quad \text{---} \ (E_3)$$

For critical damping, Eq. (2.80) gives

$$\theta(t) = \left\{\theta_0 + (\dot{\theta}_0 + \omega_n\,\theta_0)\,t\right\} e^{-\omega_n t} \quad \text{---} \ (E_4)$$

For $\theta_0 = 75^\circ = 1.309$ rad and $\dot{\theta}_0 = 0$,

$$\theta(t) = \left(1.309 + 1.309\, \omega_n t\right) e^{-\omega_n t} \quad --- (E_5)$$

For $\theta = 5^\circ = 0.08727$ rad, Eq. (E_5) becomes

$$0.08727 = 1.309 \left(1 + \omega_n t\right) e^{-\omega_n t} \quad --- (E_6)$$

Let time to return = 2 sec. Then Eq. (E_6) gives

$$0.08727 = 1.309 \left(1 + 2\omega_n\right) e^{-2\omega_n} \quad --- (E_7)$$

Solve (E_7) by trial and error to find ω_n. Then choose the values of m, M and k_t to get the desired value of ω_n. Find the damping constant $(c_t)_{cri}$ using Eq. (E_3).

(ii) Coulomb damping:

(a) Follow the procedure of part(i) to find the value of ω_n.

(b) Derive expression for the equivalent torsional viscous damping constant $(c_t)_{eq}$ for Coulomb damping. This expression, for small amounts of damping, is

$$(c_t)_{eq} = \left\{ 4\, T_d / \pi\, \omega_n\, \Theta \right\} \quad --- (E_8)$$

where T_d = friction (damping) torque, and Θ = amplitude of angular oscillations.

(c) If $(c_t)_{eq}$ is to be equal to $(c_t)_{cri} = 2\sqrt{J_0\, k_t}$, we find

$$T_d = \frac{\pi\, \omega\, \Theta}{4} \left(2\sqrt{J_0\, k_t}\right) \quad --- (E_9)$$

(2.119) Let x = vertical displacement of the mass (lunar excursion module), x_s = resulting deflection of each inclined leg (spring). From equivalence of potential energy, we find:
k_{eq_1} = stiffness of each leg in vertical direction $= k \cos^2 \alpha$
Hence for the four legs, the equivalent stiffness in vertical direction is:

$$k_{eq} = 4\ k \cos^2 \alpha$$

Similarly, the equivalent damping coefficient of the four legs in vertical direction is:

$$c_{eq} = 4\ c \cos^2 \alpha$$

where c = damping constant of each leg (in axial motion). Modeling the system as a single degree of freedom system, the equation of motion is:

$$m_{eq}\ \ddot{x} + c_{eq}\ \dot{x} + k_{eq}\ x = 0$$

and the damped period of vibration is:

$$\tau_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{2\pi}{\sqrt{\frac{k_{eq}}{m_{eq}}}\sqrt{1-\left(\frac{c_{eq}^2}{4\ k_{eq}\ m_{eq}}\right)}}$$

Using m_{eq} = 2000 kg, $k_{eq} = 4\ k \cos^2 \alpha$, $c_{eq} = 4\ c \cos^2 \alpha$, and $\alpha = 20°$, the values of k and c can be determined (by trial and error) so as to achieve a value of τ_d between 1 s and 2 s. Once k and c are known, the spring (helical) and damper (viscous) can be designed suitably.

(2.120) Assume no damping. Neglect masses of telescoping boom and strut. Find stiffness of telescoping boom in vertical direction (see Example 2.2). Find the equivalent stiffness of telescoping boom together with the strut in vertical direction. Model the system as a single degree of freedom system with natural time period:

$$\tau_n = \frac{2\pi}{\omega_n} = 2\pi \sqrt{\frac{m_{eq}}{k_{eq}}}$$

Using τ_n = 1 s and $m_{eq} = \left(\frac{W_c + W_f}{g}\right) = \frac{300}{386.4}$, determine the axial stiffness of the strut (k_s). Once k_s is known, the cross section of the strut (A_s) can be found from:

$$k_s = \frac{A_s\ E_s}{\ell_s}$$

with $E_s = 30\ (10^6)$ psi and ℓ_s = length of strut (known).
