

## Chapter 2

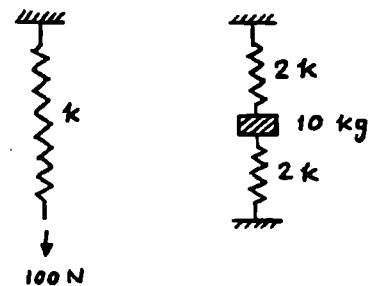
### Free Vibration of Single Degree of Freedom Systems

2.1  $\delta_{st} = 5 \times 10^{-3} \text{ m}$   
 $\omega_n = \left(\frac{g}{\delta_{st}}\right)^{1/2} = \left(\frac{9.81}{5 \times 10^{-3}}\right)^{1/2} = 44.2945 \text{ rad/sec} = 7.0497 \text{ Hz}$

2.2  $\tau_n = 0.21 \text{ sec} = 2\pi \sqrt{\frac{m}{k}}$  ,  $\sqrt{m} = 0.21 \sqrt{k}/2\pi$   
 (i)  $(\tau_n)_{new} = \frac{2\pi \sqrt{m}}{\sqrt{k_{new}}} = \frac{2\pi \sqrt{m}}{\sqrt{1.5k}} = \frac{2\pi \left(\frac{0.21 \sqrt{k}}{2\pi}\right)}{\sqrt{1.5k}} = 0.1715 \text{ sec.}$   
 (ii)  $(\tau_n)_{new} = \frac{2\pi \sqrt{m}}{\sqrt{k_{new}}} = \frac{2\pi \sqrt{m}}{\sqrt{0.5k}} = 2\pi \left(\frac{0.21 \sqrt{k}}{2\pi}\right) \frac{1}{\sqrt{0.5k}} = 0.2970 \text{ sec.}$

2.3  $\omega_n = 62.832 \text{ rad/sec} = \sqrt{\frac{k}{m}}$  ,  $\sqrt{m} = \sqrt{k}/62.832$   
 When spring constant is reduced,  $\omega_n$  decreases.  
 $(\omega_n)_{new} = 0.55 \omega_n = 34.5576 \text{ rad/sec} = \sqrt{\frac{k_{new}}{m_{new}}} = \sqrt{\frac{k-800}{m}}$   
 $\sqrt{\frac{k-800}{k}} \times 62.836 = 34.5576$  ,  $\sqrt{\frac{k-800}{k}} = 0.55$   
 $\frac{k-800}{k} = (0.55)^2 = 0.3025$   
 $k = 1146.9534 \text{ N/m}$   
 $\sqrt{m} = \sqrt{k}/62.832$  ;  $m = k/62.832^2 = \frac{1146.9534}{3947.8602}$   
 $m = 0.2905 \text{ kg}$

2.4  $k = 100 / \left(\frac{10}{1000}\right) = 10000 \text{ N/m}$   
 $\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{4k}{m}} = \left(\frac{4 \times 10^4}{10}\right)^{1/2}$   
 $= 63.2456 \text{ rad/sec}$   
 $\tau_n = \frac{2\pi}{\omega_n} = \frac{6.2832}{63.2456} = 0.0993 \text{ sec}$



2.5

Unit weight: 9000 N  
 Natural frequency 5 rad/s ~ 10 rad/s

Solution:

$$m = \frac{9000}{9.8}$$

Let  $\omega_n = 7.5$  rad/sec

$$\omega_n = \sqrt{\frac{k_{eq}}{m}}$$

$$k_{eq} = m\omega_n^2 = \left[ \frac{9000}{9.8} \right] (7.5)^2 = 51658.2 \text{ N/m} = 4 \text{ k}$$

where k is the stiffness of the air spring.

$$\text{Thus } k = \frac{51658.2}{4} = 12914.5 \text{ N/m}$$

2.6

$$x = A \cos(\omega_n t - \phi_0) \quad , \quad \dot{x} = -\omega_n A \sin(\omega_n t - \phi_0) \quad ,$$

$$\ddot{x} = -\omega_n^2 A \cos(\omega_n t - \phi_0)$$

$$(a) \quad \omega_n A = 0.1 \text{ m/sec} \quad ; \quad \tau_n = \frac{2\pi}{\omega_n} = 2 \text{ sec} \quad , \quad \omega_n = 3.1416 \text{ rad/sec}$$

$$A = 0.1 / \omega_n = 0.03183 \text{ m}$$

$$(d) \quad x_0 = x(t=0) = A \cos(-\phi_0) = 0.02 \text{ m}$$

$$\cos(-\phi_0) = \frac{0.02}{A} = 0.6283$$

$$\phi_0 = 51.0724^\circ$$

$$(b) \quad \dot{x}_0 = \dot{x}(t=0) = -\omega_n A \sin(-\phi_0) = -0.1 \sin(-51.0724^\circ)$$

$$= 0.07779 \text{ m/sec}$$

$$(c) \quad \ddot{x}|_{\max} = \omega_n^2 A = (3.1416)^2 (0.03183) = 0.314151 \text{ m/sec}^2$$

2.7 For small angular rotation of bar PQ about P,

$$\frac{1}{2} (k_{12})_{eq} (\theta l_3)^2 = \frac{1}{2} k_1 (\theta l_1)^2 + \frac{1}{2} k_2 (\theta l_2)^2$$

$$\text{i.e., } (k_{12})_{eq} = (k_1 l_1^2 + k_2 l_2^2) / l_3^2$$

Let  $k_{eq}$  = overall spring constant at Q.

$$\frac{1}{k_{eq}} = \frac{1}{(k_{12})_{eq}} + \frac{1}{k_3}$$

$$k_{eq} = \frac{(k_{12})_{eq} k_3}{(k_{12})_{eq} + k_3} = \frac{\left\{ k_1 \left( \frac{l_1}{l_3} \right)^2 + k_2 \left( \frac{l_2}{l_3} \right)^2 \right\} k_3}{k_1 \left( \frac{l_1}{l_3} \right)^2 + k_2 \left( \frac{l_2}{l_3} \right)^2 + k_3}$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k_1 k_2 l_1^2 + k_2 k_3 l_2^2}{m (k_1 l_1^2 + k_2 l_2^2 + k_3 l_3^2)}}$$

2.8  $m = 2000 \text{ kg}$ ,  $\delta_{st} = 0.02 \text{ m}$ .

$$\omega_n = (g / \delta_{st})^{1/2} = \left( \frac{9.81}{0.02} \right)^{1/2} = 22.1472 \text{ rad/sec}$$

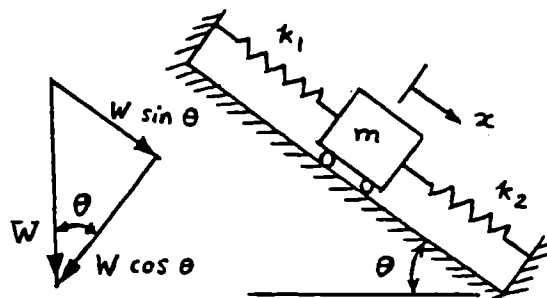
2.9 Let  $x$  be measured from the position of mass at which the springs are unstretched.

Equation of motion is

$$m \ddot{x} = -k_1(x + \delta_{st}) - k_2(x + \delta_{st}) + W \sin \theta \quad \text{--- (E}_1\text{)}$$

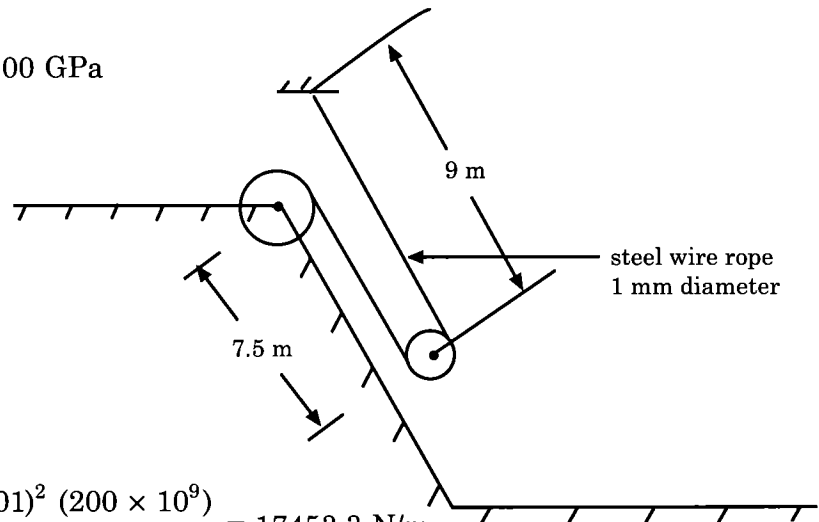
where  $\delta_{st} (k_1 + k_2) = W \sin \theta$ .

Thus Eq. (E<sub>1</sub>) becomes  $m \ddot{x} + (k_1 + k_2) x = 0 \Rightarrow \omega_n = \sqrt{\frac{k_1 + k_2}{m}}$ .



**2.10**

Cart weight: 20 kN  
 Steel Young's modulus  $E = 200$  GPa



Solution:

$$k_1 = \frac{A_1 E_1}{\ell_1} = \frac{\frac{\pi}{4} (0.001)^2 (200 \times 10^9)}{9} = 17453.3 \text{ N/m}$$

$$k_2 = \frac{A_2 E_2}{\ell_2} = \frac{17453.3 (7.5)}{9} = 14544.4 \text{ N/m}$$

$$k_{eq} = k_1 + k_2 = 31997.7 \text{ N/m}$$

Let  $x$  be measured from the unstretched length of the springs. The equation of motion is:

$$m \ddot{x} = -(k_1 + k_2) (x + \delta_{st}) + W \sin \theta$$

where  $(k_1 + k_2) \delta_{st} = W \sin \theta$

i.e.  $m \ddot{x} + (k_1 + k_2)x = 0$

Thus the natural frequency of vibration of the cart is given by

$$\omega_n = \sqrt{\frac{k_1 + k_2}{m}} = \sqrt{\frac{31997.7 \times 9.8}{20 \times 10^3}} = 3.9597 \text{ rad/s}$$

**2.11**

Weight of electronic chassis = 500 N. To be able to use the unit in a vibratory environment with a frequency range of 0 - 5 Hz, its natural frequency must be away from the frequency of the environment. Let the natural frequency be  $\omega_n = 10 \text{ Hz} = 62.832 \text{ rad/sec}$ . Since

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = 62.832$$

we have

$$k_{eq} = m \omega_n^2 = \left( \frac{500}{9.81} \right) (62.832)^2 = 20.1857 (10^4) \text{ N/m} \equiv 4 \text{ k}$$

so that  $k$  = spring constant of each spring = 50,464.25 N/m. For a helical spring,

$$k = \frac{G d^4}{8 n D^3}$$

Assuming the material of springs as steel with  $G = 80 (10^9) \text{ Pa}$ ,  $n = 5$  and  $d = 0.005 \text{ m}$ , we find

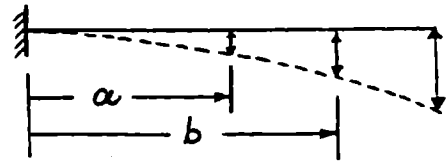
$$k = 50,464.25 = \frac{80 (10^9) (0.005)^4}{8 (5) D^3}$$

This gives

$$D^3 = \frac{1250 (10^{-3})}{50464.25} = 24,770.0 (10^{-9}) \text{ or } D = 0.0291492 \text{ m} = 2.91492 \text{ cm}$$

2.12 (i) with springs  $k_1$  and  $k_2$ :

Let  $y_a, y_b, y_l$  be deflections of beam at distances  $a, b, l$  from fixed end.



$$\frac{1}{2} (k_{12})_{eq} y_l^2 = \frac{1}{2} k_1 y_a^2 + \frac{1}{2} k_2 y_b^2$$

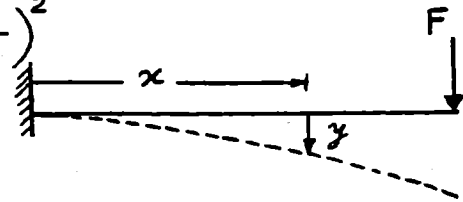
$$\text{i.e., } (k_{12})_{eq} = k_1 \left( \frac{y_a}{y_l} \right)^2 + k_2 \left( \frac{y_b}{y_l} \right)^2$$

$$y = \frac{F x^2}{6EI} (3l - x)$$

$$\text{@ } x = a, \quad y_a = \frac{F a^2}{6EI} (3l - a)$$

$$\text{@ } x = b, \quad y_b = \frac{F b^2}{6EI} (3l - b)$$

$$\text{@ } x = l, \quad y_l = \frac{F l^3}{3EI}$$



$$\omega_n = \left[ \frac{k_1 k_3 \left( \frac{y_a}{y_l} \right)^2 + k_2 k_3 \left( \frac{y_b}{y_l} \right)^2}{m \left\{ k_1 \left( \frac{y_a}{y_l} \right)^2 + k_2 \left( \frac{y_b}{y_l} \right)^2 + k_{beam} \right\}} \right]^{\frac{1}{2}} \quad \text{where } k_{beam} = \frac{3EI}{l^3}$$

$$= \left[ \frac{k_1 (3EI) a^4 (3l - a)^2 + k_2 (3EI) b^4 (3l - b)^2}{m l^3 \left\{ k_1 a^4 (3l - a)^2 + k_2 b^4 (3l - b)^2 + 12 EI l^3 \right\}} \right]^{\frac{1}{2}}$$

(ii) without springs  $k_1$  and  $k_2$ :

$$\omega_n = \sqrt{\frac{k_{beam}}{m}} = \sqrt{\frac{3EI}{m l^3}}$$

2.13 Let  $x_1, x_2 =$  displacements of pulleys 1, 2

$$x = 2x_1 + 2x_2 \quad \text{--- (E}_1\text{)}$$

Let  $P =$  tension in rope.

For equilibrium of pulley 1,

$$2P = k_1 x_1 \quad \text{--- (E}_2\text{)}$$

For equilibrium of pulley 2,

$$2P = k_2 x_2 \quad \text{--- (E}_3\text{)}$$

where  $\frac{1}{k_1} = \frac{1}{4k} + \frac{1}{4k} = \frac{1}{2k}$  ;  $k_1 = 2k$

and  $k_2 = k + k = 2k$

Combining Eqs. (E<sub>1</sub>) to (E<sub>3</sub>):

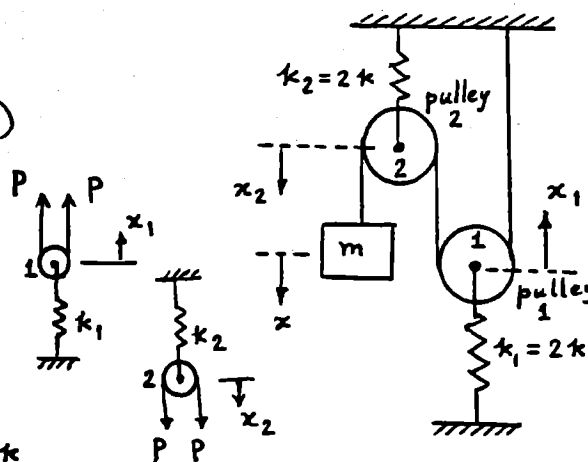
$$x = 2x_1 + 2x_2 = 2\left(\frac{2P}{k_1}\right) + 2\left(\frac{2P}{k_2}\right) = 4P\left(\frac{1}{2k} + \frac{1}{2k}\right) = \frac{4P}{k}$$

Let  $k_{eq} =$  equivalent spring constant of the system:

$$k_{eq} = \frac{P}{x} = \frac{k}{4}$$

Equation of motion of mass  $m$ :  $m\ddot{x} + k_{eq}x = 0$

$$\therefore \omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k}{4m}}$$



2.14 For a displacement of  $x$  of mass  $m$ , pulleys 1, 2 and 3 undergo displacements of  $2x$ ,  $4x$  and  $8x$ , respectively. The equation of motion of mass  $m$  can be written as

$$m\ddot{x} + F_0 = 0 \quad (1)$$

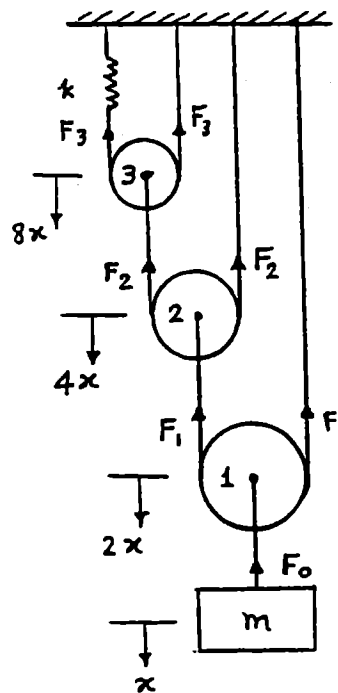
where  $F_0 = 2F_1 = 4F_2 = 8F_3$  as shown in figure.

Since  $F_3 = (8x)k$ , Eq. (1) can be rewritten as

$$m\ddot{x} + 8F_3 = 8(8k) = 0 \quad (2)$$

from which we can find

$$\omega_n = \sqrt{\frac{64k}{m}} = 8\sqrt{\frac{k}{m}} \quad (3)$$



2.15 (a)  $\omega_n = \sqrt{4k/M}$   
 (b)  $\omega_n = \sqrt{4k/(M+m)}$

Initial conditions:

velocity of falling mass  $m = v = \sqrt{2gl}$  ( $\because v^2 - \dot{u}^2 = 2gl$ )  
 $x=0$  at static equilibrium position.

$$x_0 = x(t=0) = -\frac{\text{weight}}{k_{eq}} = -\frac{mg}{4k}$$

Conservation of momentum:  $(M+m)\dot{x}_0 = m v = m\sqrt{2gl}$   
 $\dot{x}_0 = \dot{x}(t=0) = \frac{m}{M+m}\sqrt{2gl}$

Complete solution:  $x(t) = A_0 \sin(\omega_n t + \phi_0)$

$$\text{where } A_0 = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2} = \sqrt{\frac{m^2 g^2}{16 k^2} + \frac{m^2 g l}{2k(M+m)}}$$

$$\text{and } \phi_0 = \tan^{-1}\left(\frac{x_0 \omega_n}{\dot{x}_0}\right) = \tan^{-1}\left(\frac{-\sqrt{g}}{\sqrt{gl k (M+m)}}\right)$$

2.16

Velocity of hammer 20 m/s

Weight of hammer 50N

Weight of anvil 500N

Stiffness  $k = 20$  kN/m

Solution:

(a) Velocity of hammer  $= v = 20$  m/s,  $x = 0$  at static equilibrium position.

$$x_0 = x(t=0) = -\frac{\text{weight}}{k_{eq}} = -\frac{mg}{4k}$$

$$\text{Conservation of momentum: } (M+m)\dot{x}_0 = mv, \text{ or } \dot{x}_0 = \dot{x}(t=0) = \frac{mv}{M+m}$$

$$\text{Natural frequency: } \omega_n = \sqrt{\frac{4k}{M+m}}$$

Complete solution:  $x(t) = A_0 \sin(\omega_n t + \phi_0)$

where

$$A_0 = \left[ x_0^2 + \left( \frac{\dot{x}_0}{\omega_n} \right)^2 \right]^{\frac{1}{2}} = \left[ \frac{m^2 g^2}{16 k^2} + \frac{m^2 v^2}{(M + m) 4 k} \right]^{\frac{1}{2}}$$

and

$$\phi_0 = \tan^{-1} \left( -\frac{g \sqrt{M + m}}{v \sqrt{4 k}} \right)$$

since  $v = 20$  m/s,  $M = \frac{500}{9.8}$  kg,  $m = \frac{500}{9.8}$  kg,  $k = 20$  k/m,  $m = \frac{50}{9.8}$  kg,  $k = 20$  kN/m

$$A_0 = \left[ \left( \frac{50 (9.8)}{4 (20000) (9.8)} \right)^2 + \left( \frac{50 (20)}{9.8} \right)^2 \frac{9.8}{550 (4) (20000)} \right]^{\frac{1}{2}} = 0.04816 \text{ m}$$

$$\phi_0 = \tan^{-1} \left[ \frac{9.8 \sqrt{550}}{\sqrt{9.8 (20)} \sqrt{4(20000)}} \right] = \tan^{-1}(-0.01298) = -0.7436 \text{ deg}$$

(b)  $x = 0$  at static equilibrium position:  $x_0 = x(t = 0) = 0$ .

Conservation of momentum gives:

$$M \dot{x}_0 = mv \text{ or } x_0 = x(t = 0) = \frac{mv}{M}, \quad \dot{x}_0 = \dot{x}(t = 0) = 0$$

Complete solution:  $x(t) = A_0 \sin(\omega_n t + \phi_0)$

where

$$A_0 = \left[ x_0^2 + \left( \frac{\dot{x}_0}{\omega_n} \right)^2 \right]^{\frac{1}{2}} = \left[ \frac{m^2 v^2 (M)}{m^2 4 k} \right]^{\frac{1}{2}} = \frac{m v}{\sqrt{4 k M}} = \frac{50 (20) \sqrt{9.8}}{9.8 \sqrt{4 (20000) (500)}}$$

$$= 0.0505(\text{m})$$

and

$$\phi_0 = \tan^{-1} \left[ \frac{x_0 \omega_n}{\dot{x}_0} \right] = \tan^{-1}(0) = 0$$

$$(2.17) \quad k_1 = \frac{3E_1 I_1}{l_1^3} \quad (\text{at tip}) ; \quad k_2 = \frac{48E_2 I_2}{l_2^3} \quad (\text{at middle})$$

$$k_{eq} = k_1 + k_2$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\left(\frac{3E_1 I_1}{l_1^3} + \frac{48E_2 I_2}{l_2^3}\right) \frac{g}{W}}$$


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$$(2.18) \quad k = \frac{AE}{l} = \frac{\left\{\frac{\pi}{4}(0.01)^2\right\} \{2.07 \times 10^{11}\}}{20} = 0.8129 \times 10^6 \text{ N/m}$$

$$m = 1000 \text{ kg}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \left(\frac{0.8129 \times 10^6}{1000}\right)^{1/2} = 28.5114 \text{ rad/sec}$$

$$\dot{x}_0 = 2 \text{ m/s}, \quad x_0 = 0 \quad (\text{suddenly stopped while it has velocity})$$

$$\text{Period of ensuing vibration} = \tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{28.5114} = 0.2204 \text{ sec}$$

$$\text{Amplitude} = A = \dot{x}_0 / \omega_n = 2 / 28.5114 = 0.07015 \text{ m}$$


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$$(2.19) \quad \omega_n = 2 \text{ Hz} = 12.5664 \text{ rad/sec} = \sqrt{\frac{k}{m}}$$

$$\sqrt{k} = 12.5664 \sqrt{m}$$

$$\omega'_n = \sqrt{\frac{k'}{m'}} = \sqrt{\frac{k}{m+1}} = 6.2832 \text{ rad/sec}$$

$$\sqrt{k} = 6.2832 \sqrt{m+1}$$

$$= 12.5664 \sqrt{m}$$

$$\sqrt{m+1} = 2 \sqrt{m}, \quad m = \frac{1}{3} \text{ kg}$$

$$k = (12.5664)^2 m = 52.6381 \text{ N/m}$$


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$$(2.20) \quad \text{Cable stiffness} = k = \frac{AE}{l} = \frac{1}{4} \left[ \frac{\pi}{4} (0.01)^2 \right] 2.07 (10^{11}) = 4.0644 (10^6) \text{ N/m}$$

$$\tau_n = 0.1 = \frac{1}{f_n} = \frac{2\pi}{\omega_n}$$

$$\omega_n = \frac{2\pi}{0.1} = 20\pi = \sqrt{\frac{k}{m}}$$

Hence

$$m = \frac{k}{\omega_n^2} = \frac{4.0644 (10^6)}{(20\pi)^2} = 1029.53 \text{ kg}$$


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2.21

$$b = 2l \sin \theta$$

Neglect masses of links.

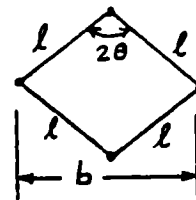
$$(a) \quad k_{eq} = k \left( \frac{4l^2 - b^2}{b^2} \right) = k \left( \frac{4l^2 - 4l^2 \sin^2 \theta}{4l^2 \sin^2 \theta} \right)$$

$$= k \left( \frac{\cos^2 \theta}{\sin^2 \theta} \right)$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k g \operatorname{cosec}^2 \theta}{W}}$$

(from solution of problem 1.8)

$$(b) \quad \omega_n = \sqrt{\frac{k g}{W}} \quad \text{since } k_{eq} = k.$$



2.22

$$y = \sqrt{\ell^2 - (\ell \sin \theta - x)^2} - \ell \cos \theta = \sqrt{\ell^2 \cos^2 \theta - x^2 + 2\ell x \sin \theta} - \ell \cos \theta$$

$$= \ell \cos \theta \sqrt{1 - \frac{x^2}{\ell^2 \cos^2 \theta} + \frac{2\ell x \sin \theta}{\ell^2 \cos^2 \theta}} - \ell \cos \theta$$

$$\frac{1}{2} k_{eq} x^2 = \frac{1}{2} k_1 y^2 + \frac{1}{2} k_2 y^2$$

where

$$y \approx \ell \cos \theta \left( 1 - \frac{1}{2} \frac{x^2}{\ell^2 \cos^2 \theta} + \frac{1}{2} \frac{2\ell x \sin \theta}{\ell^2 \cos^2 \theta} \right) - \ell \cos \theta$$

$$\approx \frac{x \sin \theta}{\cos \theta} = x \tan \theta$$

Thus  $k_{eq}$  can be expressed as

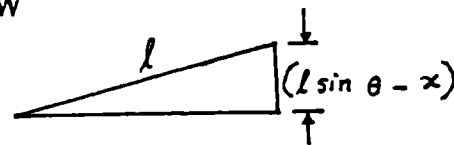
$$k_{eq} = (k_1 + k_2) \tan^2 \theta$$

Equation of motion:

$$m \ddot{x} + k_{eq} x = 0$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{(k_1 + k_2) g}{W}} \tan \theta$$



- 2.23 (a) Neglect masses of rigid links. Let  $x =$  displacement of W. Springs are in series.

$$k_{eq} = \frac{k}{2}$$

Equation of motion:

$$m \ddot{x} + k_{eq} x = 0$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k}{2m}}$$

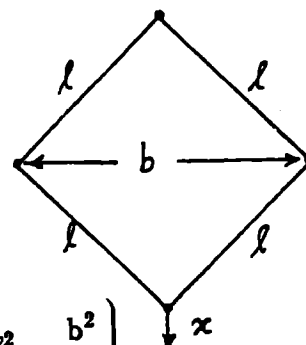
- (b) Under a displacement of  $x$  of mass, each spring will be compressed by an amount:

$$x_s = x \frac{2}{b} \sqrt{\ell^2 - \frac{b^2}{4}}$$

Equivalent spring constant:

$$\frac{1}{2} k_{eq} x^2 = 2 \left( \frac{1}{2} k x_s^2 \right)$$

or  $k_{eq} = 2 k \left( \frac{x_s}{x} \right)^2 = 2 k \left( \frac{4}{b^2} \right) \left( \ell^2 - \frac{b^2}{4} \right) = \frac{8 k}{b^2} \left( \ell^2 - \frac{b^2}{4} \right)$



Equation of motion:

$$m \ddot{x} + k_{eq} x = 0$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{8 k}{b^2 m} \left( \ell^2 - \frac{b^2}{4} \right)}$$

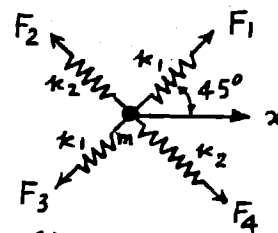
2.24  $F_1 = F_3 = k_1 x \cos 45^\circ$

$F_2 = F_4 = k_2 x \cos 135^\circ$

$F =$  force along  $x = F_1 \cos 45^\circ + F_2 \cos 135^\circ$   
 $+ F_3 \cos 45^\circ + F_4 \cos 135^\circ$   
 $= 2 x (k_1 \cos^2 45^\circ + k_2 \cos^2 135^\circ)$

$k_{eq} = \frac{F}{x} = 2 \left( \frac{k_1}{2} + \frac{k_2}{2} \right) = k_1 + k_2$

Equation of motion:  $m \ddot{x} + (k_1 + k_2) x = 0$



2.25 Let  $\alpha_i$  denote the angle made by  $i^{\text{th}}$  spring with respect to X axis.

Let  $x =$  displacement of mass along the direction defined by  $\theta$ .

If  $k_{eq} =$  equivalent spring constant, the equivalence of potential energies gives

$$\frac{1}{2} k_{eq} x^2 = \frac{1}{2} \sum_{i=1}^6 k_i \{x \cos(\theta - \alpha_i)\}^2$$

$$\begin{aligned} k_{eq} &= \sum_{i=1}^6 k_i \cos^2(\theta - \alpha_i) = \sum_{i=1}^6 k_i (\cos \theta \cos \alpha_i + \sin \theta \sin \alpha_i)^2 \\ &= \sum_{i=1}^6 k_i (\cos^2 \alpha_i \cos^2 \theta + \sin^2 \alpha_i \sin^2 \theta) \\ &\quad + 2 \sum_{i=1}^6 (\cos \alpha_i \sin \alpha_i \cos \theta \sin \theta) \end{aligned}$$

Natural frequency  $= \omega_n = \sqrt{\frac{k_{eq}}{m}}$

For  $\omega_n$  to be independent of  $\theta$ ,

$$\sum_{i=1}^6 k_i \cos^2 \alpha_i = \sum_{i=1}^6 k_i \sin^2 \alpha_i \dots (E_1)$$

and

$$\sum_{i=1}^6 k_i \cos \alpha_i \sin \alpha_i = 0 \dots (E_2)$$

(E<sub>1</sub>) and (E<sub>2</sub>) can be rewritten as

$$\sum_{i=1}^6 k_i \left( \frac{1}{2} + \frac{1}{2} \cos 2\alpha_i \right) = \sum_{i=1}^6 k_i \left( \frac{1}{2} - \frac{1}{2} \cos 2\alpha_i \right)$$

and

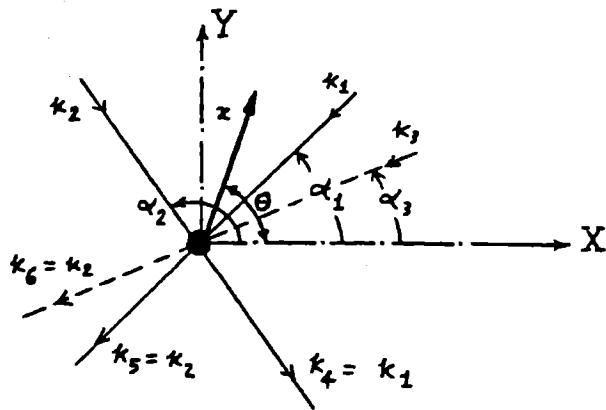
$$\frac{1}{2} \sum_{i=1}^6 k_i \sin 2\alpha_i = 0$$

i.e.

$$\sum_{i=1}^6 k_i \cos 2\alpha_i = 0 \dots (E_3)$$

and

$$\sum_{i=1}^6 k_i \sin 2\alpha_i = 0 \dots (E_4)$$



In the present example,  $(E_3)$  and  $(E_4)$  become

$$k_1 \cos 60^\circ + k_2 \cos 240^\circ + k_3 \cos 2\alpha_3 + k_1 \cos 420^\circ + k_2 \cos 600^\circ + k_3 \cos (360^\circ + 2\alpha_3) = 0$$

$$k_1 \sin 60^\circ + k_2 \sin 240^\circ + k_3 \sin 2\alpha_3 + k_1 \sin 420^\circ + k_2 \sin 600^\circ + k_3 \sin (360^\circ + 2\alpha_3) = 0$$

$$\text{i.e., } \left. \begin{aligned} k_1 - k_2 + 2k_3 \cos 2\alpha_3 &= 0 \\ \sqrt{3} k_1 - \sqrt{3} k_2 + 2k_3 \sin 2\alpha_3 &= 0 \end{aligned} \right\} ; \quad \begin{aligned} 2k_3 \cos 2\alpha_3 &= k_2 - k_1 \dots (E_5) \\ 2k_3 \sin 2\alpha_3 &= \sqrt{3}(k_2 - k_1) \dots (E_6) \end{aligned}$$

Squaring  $(E_5)$  and  $(E_6)$  and adding,

$$4k_3^2 = (k_2 - k_1)^2 (1 + 3)$$

$$\therefore k_3 = \pm (k_2 - k_1) \Rightarrow k_3 = |k_2 - k_1|$$

Dividing  $(E_6)$  by  $(E_5)$ ,

$$\tan 2\alpha_3 = \sqrt{3}$$

$$\therefore \alpha_3 = \frac{1}{2} \tan^{-1}(\sqrt{3}) = 30^\circ$$

2.26

$$T_1 = \frac{x}{a} T, \quad T_2 = \frac{x}{b} T$$

$$(a) \quad m \ddot{x} + (T_1 + T_2) = 0$$

$$m \ddot{x} + \left( \frac{T}{a} + \frac{T}{b} \right) x = 0$$

$$(b) \quad \omega_n = \sqrt{\frac{\frac{T}{a} + \frac{T}{b}}{m}} = \sqrt{\frac{T}{mab} (a+b)}$$



2.27

Jumper weight 720 N

Rope length 60 m

Rope stiffness 2 kN/m

Solution:

$$m = \frac{720}{9.8} \text{ kg}, k = 2 \text{ kN/m},$$

Velocity of jumper as he falls through 60 m:

$$mgh = \frac{1}{2} m v^2 \text{ or } v = \sqrt{2 g h} = \sqrt{2 (9.8) (60)} = 34.2928 \text{ m/s}$$

About static equilibrium position:

$$x_0 = x(t=0) = 0, \dot{x}_0 = \dot{x}(t=0) = 34.2928 \text{ m/s}$$

Response of jumper:

$$x(t) = A_0 \sin(\omega_n t + \phi_0)$$

where

$$A_0 = \left[ x_0^2 + \left( \frac{\dot{x}_0}{\omega_n} \right)^2 \right]^{\frac{1}{2}} = \frac{\dot{x}_0 \sqrt{m}}{\omega_n} = \frac{34.2928}{\sqrt{2000}} \sqrt{\frac{720}{9.8}} = 6.5726 \text{ m}$$

and

$$\phi_0 = \tan^{-1} \left[ \frac{x_0 \omega_n}{\dot{x}_0} \right] = 0$$

2.28

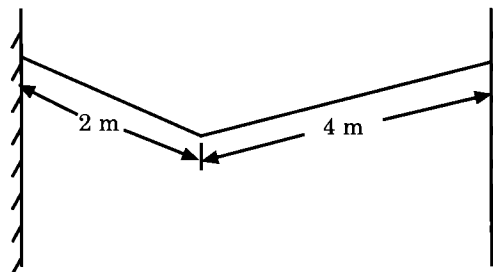
Aerobat weight 540 N

Natural frequency of vibration 10 rad/sec

The natural frequency of a vibrating rope is given by (see Problem 2.26):

Solution:

$$\omega_n = \sqrt{\frac{T(a+b)}{m a b}}$$



where  $T$  = tension in rope,  $m$  = mass, and  $a$  and  $b$  are lengths of the rope on both sides of the mass. For the given data:

$$10 = \left[ \frac{T(2+4)}{\left[ \frac{540}{9.8} \right] (2)(4)} \right]^{\frac{1}{2}} = \sqrt{T(0.013611)}$$

which yields

$$T = \frac{100}{0.013611} = 7346.9388 \text{ N}$$

2.29 When  $\omega = 0$ , total vertical height  $= 2l + h$

When  $\omega \neq 0$ , total vertical height  $= (2l \cos \theta + h)$

$$\text{spring force} = k[2l + h - (2l \cos \theta + h)] \\ = 2kl(1 - \cos \theta)$$

For vertical equilibrium of mass  $m$ ,

$$mg + T_2 \cos \theta = T_1 \cos \theta \quad \dots (E_1)$$

For horizontal equilibrium,  $F_c = (T_1 + T_2) \sin \theta$

$$T_2 = (F_c - T_1 \sin \theta) / \sin \theta \quad \dots (E_2)$$

From (E<sub>2</sub>), (E<sub>1</sub>) can be expressed as

$$mg + \left( \frac{F_c - T_1 \sin \theta}{\sin \theta} \right) \cos \theta = T_1 \cos \theta$$

$$\text{i.e. } T_1 = \frac{mg + F_c \cot \theta}{2 \cos \theta} = \frac{mg + m\omega^2 l \cos \theta}{2 \cos \theta}$$

$$T_2 = \frac{F_c - T_1 \sin \theta}{\sin \theta} = \frac{m\omega^2 l - \frac{mg}{2} \tan \theta - \frac{m\omega^2 l}{2} \sin \theta}{\sin \theta} \\ = \frac{ml\omega^2}{2} - \frac{mg}{2 \cos \theta}$$

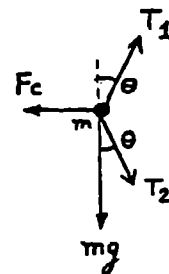
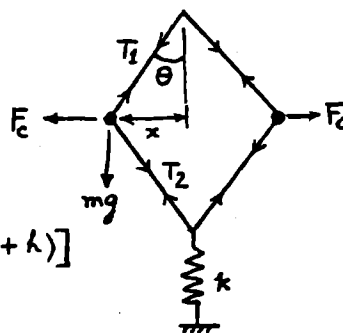
$$\text{spring force} = 2kl(1 - \cos \theta) = 2T_2 \cos \theta \\ = ml\omega^2 \cos \theta - mg$$

$$\cos \theta = \left( \frac{2kl + mg}{2kl + ml\omega^2} \right) \quad \dots (E_3)$$

This equation defines the equilibrium position of mass  $m$ .

For small oscillations about the equilibrium position, Newton's second law gives

$$2m\ddot{y} + ky = 0, \quad \omega_n = \sqrt{\frac{2k}{m}}$$



$$F_c = m\omega^2 x \\ x = l \sin \theta$$

2.30

- (a) Let  $P =$  total spring force,  $F =$  centrifugal force acting on each ball. Equilibrium of moments about the pivot of bell crank lever (O) gives:

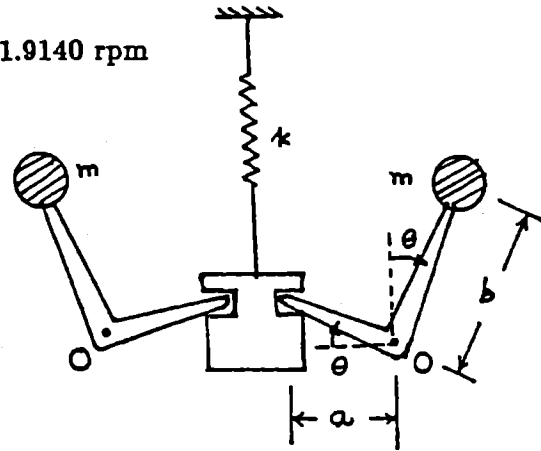
$$F \left( \frac{20}{100} \right) = \frac{P}{2} \left( \frac{12}{100} \right) \quad (1)$$

When  $P = 10^4 \left( \frac{1}{100} \right) = 100 \text{ N}$ , and

$$F = m r \omega^2 = m r \left( \frac{2 \pi N}{60} \right)^2 = \frac{25}{9.81} \left( \frac{16}{100} \right) \left( \frac{2 \pi N}{60} \right)^2 = 0.004471 \text{ N}^2$$

where  $N =$  speed of the governor in rpm. Equation (1) gives:

$$0.004471 \text{ N}^2 (0.2) = \frac{100}{2} (0.12) \quad \text{or} \quad N = 81.9140 \text{ rpm}$$



- (b) Consider a small displacement of the ball arm about the vertical position. Equilibrium about point O gives:

$$(m b^2) \ddot{\theta} + (k a \sin \theta) a \cos \theta = 0 \quad (2)$$

For small values of  $\theta$ ,  $\sin \theta \approx \theta$  and  $\cos \theta \approx 1$ , and hence Eq. (2) gives

$$m b^2 \ddot{\theta} + k a^2 \theta = 0$$

from which the natural frequency can be determined as

$$\omega_n = \left\{ \frac{k a^2}{m b^2} \right\}^{\frac{1}{2}} = \left\{ (10)^4 \left( \frac{0.12}{0.20} \right)^2 \frac{9.81}{25} \right\}^{\frac{1}{2}} = 37.5851 \text{ rad/sec}$$

2.31

$$SO' = \frac{a}{\sqrt{2}}, \quad OO' = h, \quad OS = \sqrt{h^2 + \frac{a^2}{2}}$$

When each wire stretches by  $x_s$ , let the resulting vertical displacement of the platform be  $x$ .

$$OS + x_s = \sqrt{(h+x)^2 + \frac{a^2}{2}}$$

$$x_s = \sqrt{h^2 + \frac{a^2}{2}} \left\{ \sqrt{\frac{(h+x)^2 + \frac{a^2}{2}}{h^2 + \frac{a^2}{2}}} - 1 \right\}$$

$$= \sqrt{h^2 + \frac{a^2}{2}} \left[ \sqrt{1 + \left\{ \frac{2hx + x^2}{(h^2 + \frac{a^2}{2})} \right\}} - 1 \right]$$

For small  $x$ ,  $x^2$  is negligible compared to  $2hx$  and  $\sqrt{1+\theta} \approx 1 + \frac{\theta}{2}$  and hence

$$x_s = \sqrt{h^2 + \frac{a^2}{2}} \left[ 1 + \frac{hx}{(h^2 + \frac{a^2}{2})} - 1 \right] = \frac{h}{\sqrt{h^2 + \frac{a^2}{2}}} x$$

Potential energy equivalence gives

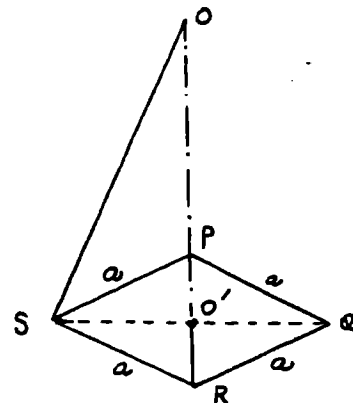
$$\frac{1}{2} k_{eq} x^2 = 4 \left( \frac{1}{2} k x_s^2 \right)$$

$$k_{eq} = 4k \left( \frac{x_s}{x} \right)^2 = \frac{4kh^2}{(h^2 + \frac{a^2}{2})}$$

Equation of motion of  $M$ :

$$M \ddot{x} + k_{eq} x = 0$$

$$\omega_n = \frac{2\pi}{\Omega_n} = \frac{2\pi}{(k_{eq}/M)^{1/2}} = \frac{\pi \sqrt{M}}{h} \left( \frac{2h^2 + a^2}{2k} \right)^{1/2}$$



2.32

Equation of motion:

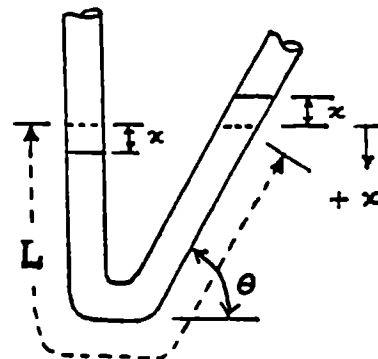
$$m \ddot{x} = \sum F_x$$

$$\text{i.e., } (LA\rho) \ddot{x} = -2(Ax\rho g)$$

$$\text{i.e., } \ddot{x} + \frac{2g}{L} x = 0$$

where  $A$  = cross-sectional area of the tube and  $\rho$  = density of mercury. Thus the natural frequency is given by:

$$\omega_n = \sqrt{\frac{2g}{L}}$$



2.33

Assume same area of cross section for all segments of the cable. Speed of blades = 300 rpm = 5 Hz = 31.416 rad/sec.

$$\omega_n^2 = \frac{k_{eq}}{m} = (2 (31.416))^2 = (62.832)^2$$

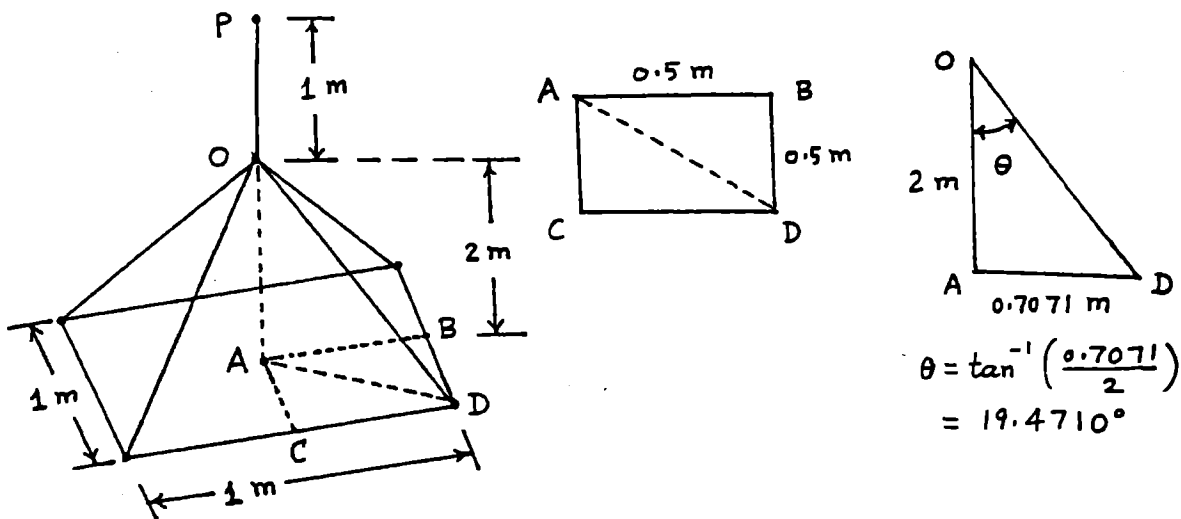
$$k_{eq} = m \omega_n^2 = 250 (62.832)^2 = 98.6965 (10^4) \text{ N/m} \quad (1)$$

$$AD = \sqrt{0.5^2 + 0.5^2} = 0.7071 \text{ m}, \quad OD = \sqrt{2^2 + 0.7071^2} = 2.1213 \text{ m}$$

Stiffness of cable segments:

$$k_{PO} = \frac{A E}{\ell_{PO}} = \frac{A (207) (10^9)}{1} = 207 (10^9) \text{ A N/m}$$

$$K_{OD} = \frac{A E}{\ell_{OD}} = \frac{A (207) (10^9)}{2.1213} = 97.5817 (10^9) \text{ A N/m}$$



The total stiffness of the four inclined cables ( $k_{ic}$ ) is given by:

$$k_{ic} = 4 k_{OD} \cos^2 \theta$$

$$= 4 (97.5817) (10^9) \text{ A} \cos^2 19.4710^\circ = 346.9581 (10^9) \text{ A N/m}$$

Equivalent stiffness of vertical and inclined cables is given by:

$$\frac{1}{k_{eq}} = \frac{1}{k_{PO}} + \frac{1}{k_{ic}}$$

$$\text{i.e., } k_{eq} = \frac{k_{PO} k_{ic}}{k_{PO} + k_{ic}}$$

$$= \frac{(207 (10^9) \text{ A}) (346.9581 (10^9) \text{ A})}{(207 (10^9) \text{ A}) + (346.9581 (10^9) \text{ A})} = 129.6494 (10^9) \text{ A N/m} \quad (2)$$

Equating  $k_{eq}$  given by Eqs. (1) and (2), we obtain the area of cross section of cables as:

$$A = \frac{98.6965 (10^4)}{129.6494 (10^9)} = 7.6126 (10^{-6}) \text{ m}^2$$

2.34

$$\frac{1}{2\pi} \left\{ \frac{k_1}{m} \right\}^{\frac{1}{2}} = 5 ; \quad \frac{k_1}{m} = 4 (\pi)^2 (25) = 986.9651$$

$$\frac{1}{2\pi} \left\{ \frac{k_1}{m + 5000} \right\}^{\frac{1}{2}} = 4.0825 ; \quad \frac{k_1}{m + 5000} = 4 (\pi)^2 (16.6668) = 657.9822$$

Using  $k_1 = \frac{A E}{\ell_1}$  we obtain

$$\begin{aligned} \frac{k_1}{m} &= \frac{A E}{\ell_1 m} = \frac{A (207) (10^9)}{2 m} = 986.9651 \\ \text{i.e., } A &= 9.5359 (10^{-9}) m \end{aligned} \quad (1)$$

Also

$$\begin{aligned} \frac{k_1}{m + 5000} &= \frac{A E}{\ell_1 (m + 5000)} = 657.9822 \\ \text{i.e., } \frac{A}{m + 5000} &= 6.3573 (10^{-9}) \end{aligned} \quad (2)$$

Using Eqs. (1) and (2), we obtain

$$\begin{aligned} A &= 9.5359 (10^{-9}) m = 6.3573 (10^{-9}) m + 31.7865 (10^{-6}) \\ \text{i.e., } 3.1786 (10^{-9}) m &= 31.7865 (10^{-6}) \\ \text{i.e., } m &= 10000.1573 \text{ kg} \end{aligned} \quad (3)$$

Equations (1) and (3) yield

$$A = 9.5359 (10^{-9}) m = 9.5359 (10^{-9}) (10000.1573) = 0.9536 (10^{-4}) \text{ m}^2$$


---

2.35

## Longitudinal Vibration:

Let  $W_1$  = part of weight  $w$  carried by length  $a$  of shaft $W_2 = W - W_1$  = weight carried by length  $b$ 

$$x = \text{Elongation of length } a = \frac{W_1 a}{AE}$$

$$y = \text{shortening of length } b = \frac{(W - W_1)(l - a)}{AE}$$

E = Young's modulus

A = area of cross-section  
=  $\pi d^2/4$ 

$$\text{Since } x = y, \quad W_1 = \frac{W(l-a)}{l}$$

$$x = \text{elongation or static deflection of length } a = \frac{W a (l-a)}{AE l}$$

Considering the shaft of length  $a$  with end mass  $W_1/g$  as a spring-mass system,

$$\omega_n = \sqrt{\frac{g}{x}} = \left( \frac{g l A E}{W a (l-a)} \right)^{1/2}$$

## Transverse vibration:

spring constant of a fixed-fixed beam with off-center load

$$= k = \frac{3EI l^3}{a^3 b^3} = \frac{3EI l^3}{a^3 (l-a)^3}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \left\{ \frac{3EI l^3 g}{W a^3 (l-a)^3} \right\}^{1/2} \quad \text{with } I = \left( \frac{\pi d^4}{64} \right)$$

= moment of inertia

## Torsional vibration:

If flywheel is given an angular deflection  $\theta$ , resisting torques offered by lengths  $a$  and  $b$  are  $\frac{GJ\theta}{a}$  and  $\frac{GJ\theta}{b}$ .

$$\text{Total resisting torque} = M_t = GJ \left( \frac{1}{a} + \frac{1}{b} \right) \theta$$

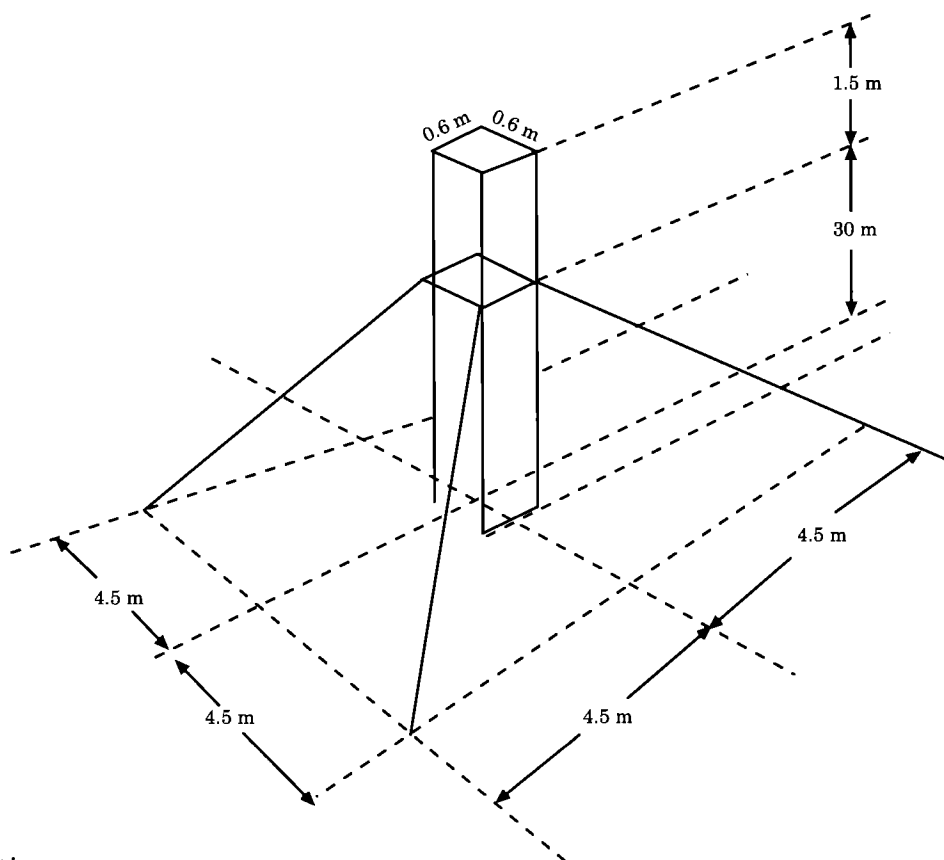
$$k_t = \frac{M_t}{\theta} = GJ \left( \frac{1}{a} + \frac{1}{b} \right) \quad \text{where } J = \frac{\pi d^4}{32} = \text{polar moment of inertia}$$

$$\omega_n = \sqrt{\frac{k_t}{J_o}} = \left\{ \frac{GJ}{J_o} \left( \frac{1}{a} + \frac{1}{b} \right) \right\}^{1/2}$$

where  $J_o$  = mass polar moment of inertia of the flywheel.

**2.36**Cable cross-sectional area  $3 \times 10^{-4} \text{m}^2$ 

Antenna tower can be modeled as a steel beam of square section of side 25 mm



Solution:

$m_{\text{eq}_{\text{end}}} = 3 (200 \times 109)$  equivalent mass of a uniform beam at the free end  
(see Problem 2.38)

$$M_{\text{eq}} = \frac{33}{140} m = \frac{33}{140} [0.025 (0.025) (45) (7.75 \times 10^3)] = 51.378 \text{ kg}$$

Stiffness of tower at free end:

$$k_b = \frac{3 E I}{L^3} = \frac{3 (200 \times 10^9) \left( \frac{1}{12} (0.025) (0.025)^3 \right)}{(45)^3} = 0.2143 \text{ N/m}$$

Length of each cable:

$$OA = 0.6 \sqrt{2} = 0.8485 \text{ m}, OB = 4.5 \sqrt{2} = 6.3640 \text{ m},$$

$$AB = OB - OA = 5.5155 \text{ m}$$

$$TB = \sqrt{TA^2 + AB^2} = \sqrt{30^2 + 5.5155^2} = 30.5028 \text{ m}$$

$$\tan \theta = \frac{AT}{AB} = \frac{3.0}{5.5155} = 5.4392, \theta = 79.5825^\circ$$

Axial stiffness of each cable:

$$k = \frac{A E}{\ell} = \frac{3 \times 10^{-4} (200 \times 10^9)}{30.5028} = 1967033 \text{ N/m}$$

Axial extension of each cable ( $y_c$ ) due to a horizontal displacement of  $x$  of tower:

$$\frac{1}{2} k y_c^2 = \frac{1}{2} k_{eq_c} x^2 \text{ or } k_{eq_c} = k \left( \frac{y_c}{x} \right)^2 = k \cos^2 \theta$$

This gives

$$k_{eq_c} = (1967033) \cos^2 79.5825^\circ = 64313 \text{ N/m}$$

In order to use the relation  $k_{eq_{end}} = k_b + 4 k_{eq_{end}} \left( \frac{y_{L1}}{y_L} \right)^2$

we find

$$\frac{y_{L1}}{y_L} = \frac{L_1^2 (3L - L_1)}{2 L^3} = \frac{30^2 [3(45 - 30)]}{2 (45)^3} = 0.5185$$

Thus

$$k_{eq_{end}} = k_b + 4 k_{eq_{end}} [0.5185]^2 = 0.2143 + 4(64313)(0.5185)^2 \\ = 69165 \text{ N/m}$$

Natural frequency:

$$\omega_n = \left( \frac{k_{eq_{end}}}{m_{eq_{end}}} \right)^{\frac{1}{2}} = \left( \frac{69165}{51.378} \right)^{\frac{1}{2}} = 36.6905 \text{ rad/sec}$$

2.37

Traffic sign thickness = 0.003 m

Post height: 1.8 m

Post cross section: 0.05 m × 0.006 m

Wind frequency: 1.1  $H_z$

Solution:

$$1.1 H_z = 6.9 \text{ rad/s}$$

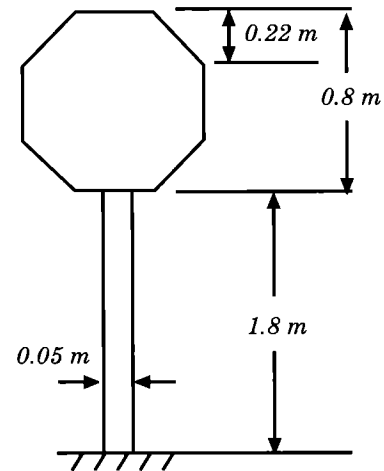
$$AB = \sqrt{0.22^2 + 0.22^2} = 0.3111 \text{ m}$$

$$BC = 0.8 - 0.22 - 0.22 = 0.36 \text{ m}$$

$$\text{Area} = 0.8(0.8) - 4 \left( \frac{1}{2} (0.22) (0.22) \right)$$

$$= 0.5432 \text{ m}^2$$

$$\text{Thickness} = 0.006 \text{ m,}$$



Weight of density of steel =  $75950 \text{ N/m}^3$

Weight of sign =  $75950 (0.003) (0.5432) = 123.8 \text{ N}$

Weight of post =  $1.8 (0.05) (0.006) (75950) = 41.01 \text{ N}$

Stiffness of sign post (cantilever beam):  $k = \frac{3 E I}{\ell^3}$

Area moments of inertia of the cross section of the sign post:

$$I_{xx} = \frac{1}{12} (0.05) (0.006)^3 = 9 \times 10^{-10} (\text{m}^4)$$

$$I_{yy} = \frac{1}{12} (0.006) (0.05)^3 = 6.25 \times 10^{-8} (\text{m}^4)$$

Bending stiffness of the sign post:

$$k_{xz} = \frac{3 E I_{yy}}{\ell^3} = \frac{3 \times (200 \times 10^9) (6.25 \times 10^{-8})}{1.8^3} = 6430 \text{ N/m}$$

$$k_{yz} = \frac{3 E I_{xx}}{\ell^3} = \frac{3 \times (200 \times 10^9) (9 \times 10^{-10})}{1.8^3} = 92.59 \text{ N/m}$$

$$k_t = 5.33 \frac{a b^3}{\ell} G \left\{ 1 - 0.63 \frac{b}{a} \left( 1 - \frac{b^4}{12 a^4} \right) \right\}$$

(see Ref: N. H. Cook, Mechanics of Materials for Design, McGraw-Hill, New York, 1984, p. 342).

Thus,

$$\begin{aligned} k_t &= 5.33 \left[ \frac{(0.025) (0.003)^3}{1.8} \right] (83 \times 10^9) \left\{ 1 - (0.63) \left( \frac{0.003}{0.025} \right) \left( 1 - \frac{0.003^4}{12 (0.025)^4} \right) \right\} \\ &= 153.3547 \text{ N} \cdot \text{m/rad} \end{aligned}$$

Natural frequency for bending in xz plane:

$$\omega_{xz} = \left\{ \frac{k_{xz}}{m} \right\}^{\frac{1}{2}} = \left\{ \frac{6430}{\left( \frac{123.8}{9.8} \right)} \right\}^{\frac{1}{2}} = 22.5610 \text{ rad/sec}$$

Natural frequency for bending in yz plane:

$$\omega_{yz} = \left\{ \frac{k_{yz}}{m} \right\}^{\frac{1}{2}} = k_{yz} = 2.7073 \text{ rad/sec}$$

By approximating the shape of the sign as a square of side 0.8 m, we can find its mass moment of inertia as:

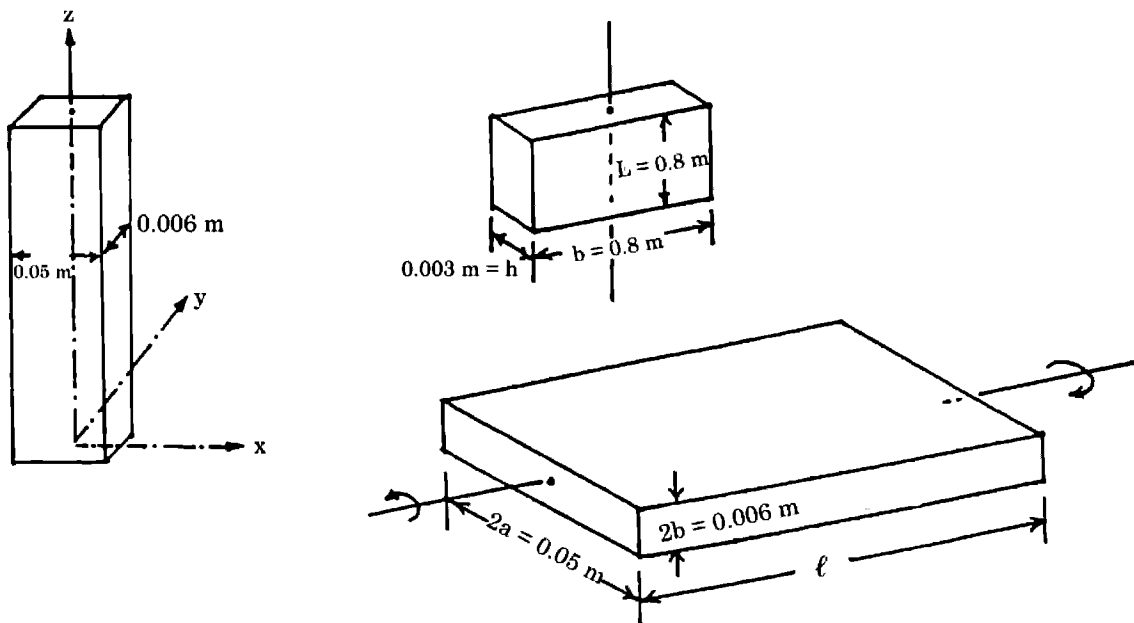
$$I_{oo} = \frac{\gamma L}{3} (b^3 h + h^3 b) = \left( \frac{75950}{9.8} \right) \left( \frac{0.8}{3} \right) [0.8^3(0.003) + (0.003)^3(0.8)]$$

$$= 3.1744 \text{ kg.m}^2$$

Natural torsional frequency:

$$\omega_t = \left\{ \frac{k_t}{I_{oo}} \right\}^{\frac{1}{2}} = \left\{ \frac{153.3547}{3.1744} \right\}^{\frac{1}{2}} = 6.9505 \text{ rad/sec}$$

Thus the mode of vibration (close to resonance) is torsion in xy plane.



2.38 (a) Pivoted:

Let  $l = h$ .

$$k_{eq} = 4 k_{column} = 4 \left( \frac{3EI}{l^3} \right) = \frac{12EI}{l^3}$$

Let  $m_{eff1}$  = effective mass due to self weight of columns

$$\text{Equation of motion: } \left( \frac{W}{g} + m_{eff1} \right) \ddot{x} + k_{eq} x = 0$$

$$\text{Natural frequency of horizontal vibration} = \omega_n = \sqrt{\frac{12EI}{l^3 \left( \frac{W}{g} + m_{eff1} \right)}}$$

(b) Fixed:

since the joint between column and floor does not permit rotation, each column will bend with inflection point at middle.

When force  $F$  is applied at ends,

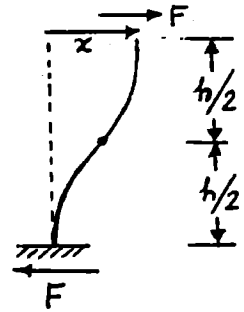
$$x = 2 \frac{F \left(\frac{l}{2}\right)^3}{3EI} = \frac{Fl^3}{12EI}$$

$$k_{\text{column}} = \frac{12EI}{l^3} \quad ; \quad k_{\text{eq}} = 4 k_{\text{column}} = \frac{48EI}{l^3}$$

Let  $m_{\text{eff}2}$  = effective mass of each column at top end

$$\text{Equation of motion: } \left(\frac{W}{g} + m_{\text{eff}2}\right) \ddot{x} + k_{\text{eq}} x = 0$$

$$\text{Natural frequency of horizontal vibration} = \omega_n = \sqrt{\frac{48EI}{l^3 \left(\frac{W}{g} + m_{\text{eff}2}\right)}}$$



Effective mass (due to self weight):

(a) Let  $m_{\text{eff}1}$  = effective part of mass of beam ( $m$ ) at end.

Thus vibrating inertia force at end is due to  $(M + m_{\text{eff}1})$ .

Assume deflection shape during vibration same as the static deflection shape with a tip load:

$$y(x,t) = Y(x) \cos(\omega_n t - \phi) \quad \text{where} \quad Y(x) = \frac{F x^2 (3l - x)}{6EI}$$

$$Y(x) = \frac{Y_0}{2l^3} x^2 (3l - x) \quad \text{where} \quad Y_0 = \frac{Fl^3}{3EI} = \text{max. tip deflection}$$

$$y(x,t) = \frac{Y_0}{2l^3} (3x^2 l - x^3) \cos(\omega_n t - \phi) \quad (E_1)$$

Max. strain energy of beam = Max. work by force  $F$

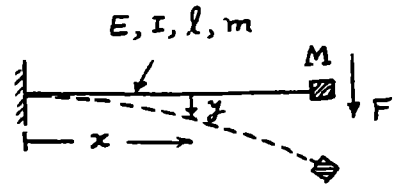
$$= \frac{1}{2} F Y_0 = \frac{3}{2} \frac{EI}{l^3} Y_0^2 \quad (E_2)$$

Max. kinetic energy due to distributed mass of beam

$$= \frac{1}{2} \frac{m}{l} \int_0^l \dot{y}^2(x,t) \Big|_{\text{max}} dx + \frac{1}{2} (\dot{y}_{\text{max}})^2 M$$

$$= \frac{1}{2} \omega_n^2 Y_0^2 \left(\frac{33}{140} m\right) + \frac{1}{2} \omega_n^2 Y_0^2 M \quad (E_3)$$

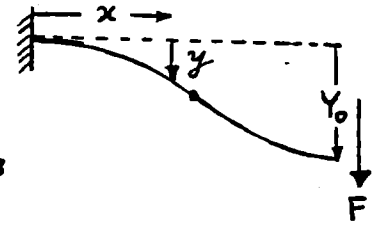
$$\therefore m_{\text{eff}1} = \frac{33}{140} m = 0.2357 m$$



(b) Let  $Y(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$   
 $Y(0) = 0, \frac{dY}{dx}(0) = 0, Y(l) = Y_0, \frac{dY}{dx}(l) = 0$

This leads to  $Y(x) = \frac{3Y_0}{l^2} x^2 - \frac{2Y_0}{l^3} x^3$

$y(x,t) = Y_0 \left( 3 \frac{x^2}{l^2} - 2 \frac{x^3}{l^3} \right) \cos(\omega_n t - \phi)$  (E4)



Maximum strain energy  $= \frac{1}{2} EI \int_0^l \left( \frac{\partial^2 y}{\partial x^2} \right)^2 dx \Big|_{\max}$   
 $= \frac{6EI Y_0^2}{l^3}$  (E5)

Max. kinetic energy  $= \frac{1}{2} M \omega_n^2 Y_0^2 + \frac{1}{2} \left( \frac{m}{l} \right) Y_0^2 \omega_n^2 \int_0^l \left( \frac{3x^2}{l^2} - \frac{2x^3}{l^3} \right)^2 dx$   
 $= \frac{1}{2} \omega_n^2 Y_0^2 \left( M + \frac{13}{35} m \right)$  (E6)

$\therefore m_{eff 2} = \frac{13}{35} m = 0.3714 m$

2.39

Object weight 50 N

$$l_1 = 0.30 \text{ m}, \quad D_1 = 0.05 \text{ m}, \quad d_1 = 0.045 \text{ m}$$

$$l_2 = 0.25 \text{ m}, \quad D_2 = 0.04 \text{ m}, \quad d_2 = 0.035 \text{ m}$$

$$l_3 = 0.20 \text{ m}, \quad D_3 = 0.03 \text{ m}, \quad d_3 = 0.025 \text{ m}$$

$$E_1 = E_2 = E_3 = 70 \text{ GPa}$$

Solution: Stiffness of segments:

$$A_1 = \frac{\pi}{4} (D_1^2 - d_1^2) = \frac{\pi}{4} (0.05^2 - 0.045^2) = 3.7306 \times 10^{-4} \text{ m}^2$$

$$k_1 = \frac{A_1 E_1}{L_1} = \frac{(3.7306 \times 10^{-4}) (70 \times 10^9)}{0.30} = 87.0483(10^6) \text{ N/m}$$

$$A_2 = \frac{\pi}{4} (D_2^2 - d_2^2) = \frac{\pi}{4} (0.04^2 - 0.035^2) = 2.9452 \times 10^{-4} \text{ m}^2$$

$$k_2 = \frac{A_2 E_2}{L_2} = \frac{(2.9425 \times 10^{-4}) (70 \times 10^9)}{0.30} = 68.7223(10^6) \text{ N/m}$$

$$A_3 = \frac{\pi}{4} (D_3^2 - d_3^2) = \frac{\pi}{4} (0.03^2 - 0.025^2) = 2.1598 \times 10^{-4} \text{ m}^2$$

$$k_3 = \frac{A_3 E_3}{L_3} = \frac{(2.1598 \times 10^{-4}) (70 \times 10^9)}{0.30} = 50.3964(10^6) \text{ N/m}$$

Equivalent stiffness (springs in series):

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = 1.14879(10^{-8}) + 1.45513(10^{-8}) + 1.98427(10^{-8})$$

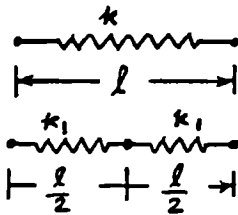
$$= 4.58819(10^{-8})$$

or  $K_{eq} = 21.7951(10^6) \text{ N/m}$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k_{eq} g}{w}} = \sqrt{\frac{(21.7951 \times 10^6) (9.8)}{50}} = 2066.8429 \text{ rad/sec}$$

2.40



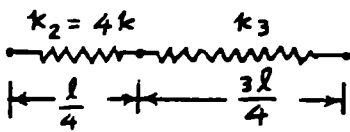
$$\frac{1}{k_{total}} = \frac{1}{k_1} + \frac{1}{k_1}$$

$$k_{total} = \frac{k_1}{2} \equiv k; \quad k_1 = 2k$$

$$\tau_n = 2\pi \sqrt{\frac{m}{k_{eq}}}$$

$$0.5 = 2\pi \sqrt{\frac{m}{4k}}$$

$$\sqrt{\frac{m}{k}} = \frac{1}{2\pi}$$



$$\frac{1}{k_{total}} = \frac{1}{k_2} + \frac{1}{k_3} = \frac{1}{4k} + \frac{1}{k_3} = \frac{1}{k}$$

$$k_3 = \frac{4}{3}k$$

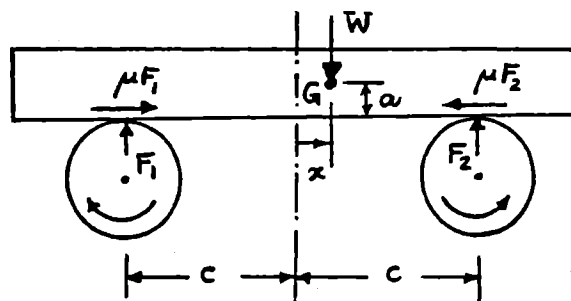
$$\tau_n = 2\pi \sqrt{\frac{m}{k_{eq}}} \quad \text{where } k_{eq} = 4k + \frac{4}{3}k = \frac{16}{3}k$$

$$\therefore \tau_n = 2\pi \sqrt{\frac{3m}{16k}} = \frac{2\pi\sqrt{3}}{4} \sqrt{\frac{m}{k}} = \frac{2\pi\sqrt{3}}{4} \left(\frac{1}{2\pi}\right) = 0.4330 \text{ sec}$$

2.41 Let  $\mu =$  coefficient of friction

$x =$  displacement of c.g. of block

$F_1, F_2 =$  net reactions between roller and block on left and right sides.



Reactions at left and right due to static load  $W$  are  $W(c-x)/2c$  and  $W(c+x)/2c$ , respectively.

$M =$  moment about  $G$  due to motion of block  $= (\mu F_2 - \mu F_1)a$

Reactions at left and right to balance  $M = \frac{M}{2c} = \frac{\mu a}{2c}(F_2 - F_1)$

$$F_1 = \frac{W(c-x)}{2c} - \frac{\mu a}{2c}(F_2 - F_1); \quad F_2 = \frac{W(c+x)}{2c} + \frac{\mu a}{2c}(F_2 - F_1)$$

Subtraction gives  $F_2 - F_1 = \frac{Wx}{c} + \frac{\mu a}{c}(F_2 - F_1)$

$$\text{i.e., } F_2 - F_1 = \frac{Wx}{c} \left( \frac{c}{c - \mu a} \right) = \frac{Wx}{c - \mu a}$$

Restoring force  $= \mu(F_2 - F_1) = \left( \frac{\mu W x}{c - \mu a} \right)$

Equation of motion:  $\frac{W}{g} \ddot{x} + \frac{\mu W}{(c - \mu a)} x = 0$

$$\omega_n = \omega = \sqrt{\frac{\mu W g}{W(c - \mu a)}} = \sqrt{\frac{\mu g}{c - \mu a}}$$

Solving this, we get  $\omega = [c\omega^2 / (g + a\omega^2)]$

2.42 From problem 2.41,

$$\text{Restoring force without springs} = \mu (F_2 - F_1) = \frac{\mu W x}{c - \mu a}$$

$$\text{spring restoring force} = 2 k x$$

$$\text{Total restoring force} = \frac{\mu W x}{c - \mu a} + 2 k x$$

$$\text{Equation of motion: } \frac{W}{g} \ddot{x} + \left( \frac{\mu W}{c - \mu a} + 2 k \right) x = 0$$

$$\omega_n = \omega = \left\{ \frac{[\mu W + 2 k (c - \mu a)] g}{(c - \mu a) W} \right\}^{1/2}$$

Solution of this equation gives

$$\mu = \left( \frac{\omega^2 W c - 2 k g c}{W g + W \omega^2 a - 2 k g a} \right)$$

2.43

(a) Natural frequency of vibration of electromagnet (without the automobile):

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{(1.8 \times 10^6)}{1,400}} = 35.8569 \text{ rad/sec}$$

(b) When the automobile is dropped, the electromagnet moves up by a distance ( $x_0$ ) from its static equilibrium position.

$$\begin{aligned} x_0 &= \text{static deflection (elongation of cable) under the weight of automobile} \\ &= \frac{W_{\text{auto}}}{k} = \frac{(900)(9.81)}{1.8 \times 10^6} = 0.004905 \text{ m} \end{aligned}$$

Resultant motion of electromagnet (tx upwards):

$$x(t) = A_0 \sin(\omega_n t + \phi_0)$$

where

$$A_0 = \left\{ x_0^2 + \left( \frac{\dot{x}_0}{\omega_n} \right)^2 \right\}^{1/2} = x_0 = 4.905 \text{ mm}$$

$$\text{and } \phi_0 = \tan^{-1} \left( \frac{x_0 \omega_n}{\dot{x}_0} \right) = \tan^{-1}(\infty) = \frac{\pi}{2} \text{ rad}$$

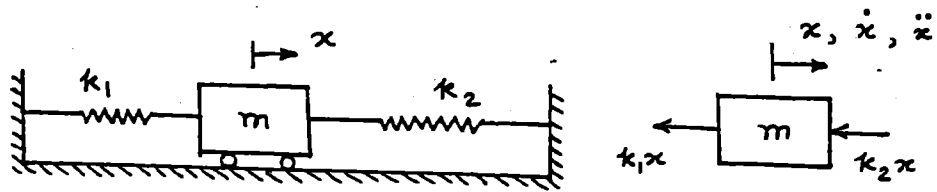
$$\text{Hence } x(t) = 4.905 \sin \left( 35.8569t + \frac{\pi}{2} \right) = 4.905 \cos(35.8569)t \text{ mm}$$

(c) Maximum  $x(t)$

$$x(t)|_{\text{max}} = A_0 = 4.905 \text{ mm}$$

$$\begin{aligned} \text{Maximum tension in cable during motion} &= kx(t)|_{\text{max}} + \text{Weight of electromagnet} \\ &= (1.8 \times 10^6)(0.004905) + (1,400)(9.81) = 22563 \text{ N} \end{aligned}$$

2.44



- (a) Newton's second law of motion:

$$F(t) = -k_1 x - k_2 x = m \ddot{x} \text{ or } m \ddot{x} + (k_1 + k_2) x = 0$$

- (b) D'Alembert's principle:

$$F(t) - m \ddot{x} = 0 \text{ or } -k_1 x - k_2 x - m \ddot{x} = 0$$

$$\text{Thus } m \ddot{x} + (k_1 + k_2) x = 0$$

- (c) Principle of virtual work:

When mass  $m$  is given a virtual displacement  $\delta x$ ,

Virtual work done by the spring forces =  $-(k_1 + k_2) x \delta x$

Virtual work done by the inertia force =  $-(m \ddot{x}) \delta x$

According to the principle of virtual work, the total virtual work done by all forces must be equal to zero:

$$-m \ddot{x} \delta x - (k_1 + k_2) x \delta x = 0 \text{ or } m \ddot{x} + (k_1 + k_2) x = 0$$

- (d) Principle of conservation of energy:

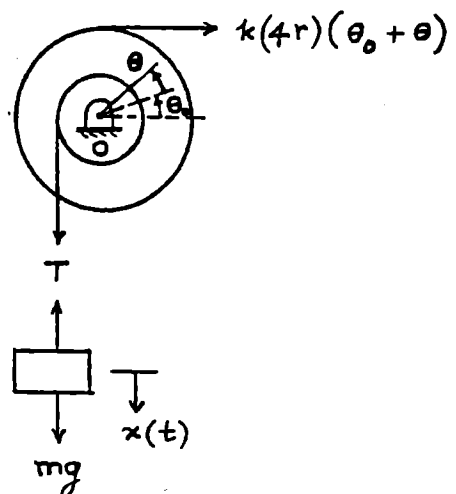
$$T = \text{kinetic energy} = \frac{1}{2} m \dot{x}^2$$

$$U = \text{strain energy} = \text{potential energy} = \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 x^2$$

$$T + U = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} (k_1 + k_2) x^2 = c = \text{constant}$$

$$\frac{d}{dt} (T + U) = 0 \text{ or } m \ddot{x} + (k_1 + k_2) x = 0$$

2.45



Equation of motion:

$$\text{Mass } m: m g - T = m \ddot{x} \quad (1)$$

$$\text{Pulley } J_0: J_0 \ddot{\theta} = T r - k 4 r (\theta + \theta_0) 4 r \quad (2)$$

where  $\theta_0 =$  angular deflection of the pulley under the weight,  $mg$ , given by:

$$m g r = k (4 r \theta_0) 4 r \quad \text{or} \quad \theta_0 = \frac{m g}{16 r k} \quad (3)$$

Substituting Eqs. (1) and (3) into (2), we obtain

$$J_0 \ddot{\theta} = (m g - m \ddot{x}) r - k 16 r^2 \left( \theta + \frac{m g}{16 r k} \right) \quad (4)$$

Using  $x = r \theta$  and  $\ddot{x} = r \ddot{\theta}$ , Eq. (4) becomes

$$(J_0 + m r^2) \ddot{\theta} + (16 r^2 k) \theta = 0$$

- 2.46 Consider the springs connected to the pulleys (by rope) to be in series. Then

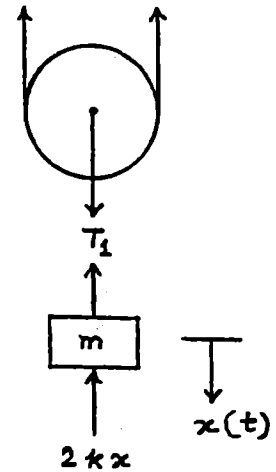
$$\frac{1}{k_{eq}} = \frac{1}{k} + \frac{1}{5k} \quad \text{or} \quad k_{eq} = \frac{5}{6} k$$

Let the displacement of mass  $m$  be  $x$ .

Then the extension of the rope (springs connected to the pulleys) =  $2x$ . From the free body diagram, the equation of motion of mass  $m$ :

$$m \ddot{x} + 2 k x + k_{eq} (2 x) = 0$$

$$\text{or} \quad m \ddot{x} + \frac{11}{3} k x = 0$$



- 2.47  $T =$  kinetic energy =  $T_{mass} + T_{pulley}$

$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2 = \frac{1}{2} (m r^2 + J_0) \dot{\theta}^2$$

$$U = \text{potential energy} = \frac{1}{2} k x_s^2 = \frac{1}{2} k (4 r \theta)^2 = \frac{1}{2} k (16 r^2) \theta^2$$

Using  $\frac{d}{dt} (T + U) = 0$  gives

$$(m r^2 + J_0) \ddot{\theta} + (16 r^2 k) \theta = 0$$

- 2.48  $T =$  kinetic energy =  $\frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2$

$$U = \text{potential energy} = \frac{1}{2} k x_s^2$$

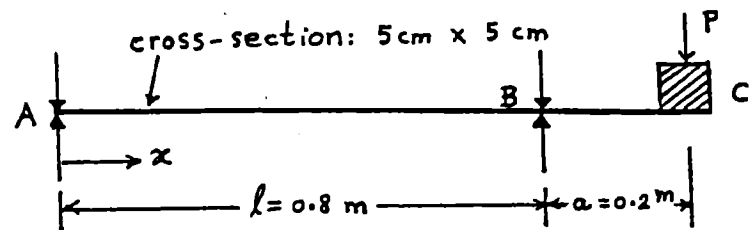
where  $\theta = \frac{x}{r}$ ,  $x_s = \text{extension of spring} = 4 r \theta = 4 x$ . Hence

$$T = \frac{1}{2} \left( m + \frac{J_0}{r^2} \right) \dot{x}^2 ; U = \frac{1}{2} (16 k) x^2$$

Using the relation  $\frac{d}{dt} (T + U) = 0$ , we obtain the equation of motion of the system as:

$$\left( m + \frac{J_0}{r^2} \right) \ddot{x} + 16 k x = 0$$

2.49



Due to a load  $P$  at  $C$ , deflection at point  $C$  is given by (from Appendix B):

$$y(x) = \frac{P(x-\ell)}{6EI\ell} \left[ a(3x-\ell) - (x-\ell)^2 \right] ; \ell \leq x \leq \ell + a$$

$$y_C = y(x = \ell + a) = \frac{P a^2}{3EI\ell} (\ell + a)$$

Moment of inertia of cross section of beam:

$$I = \frac{1}{12} (0.05) (0.05)^3 = 52.0833 (10^{-8}) \text{ m}^4$$

Equivalent stiffness:

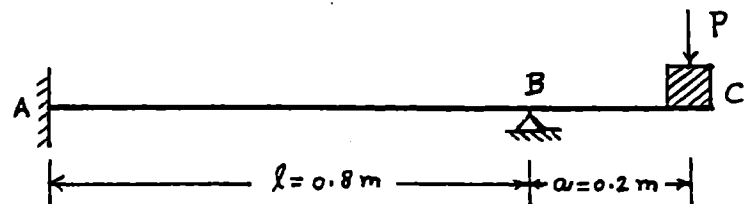
$$k_{eq} = \frac{P}{y_C} = \frac{3EI\ell}{a^2(\ell + a)} = \frac{3(207(10^9))(52.0833(10^{-8}))(0.8)}{(0.2)^2(0.8 + 0.2)}$$

$$= 6.4687 (10^6) \text{ N/m}$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{6.4687(10^6)}{50}} = 359.6872 \text{ rad/sec}$$

2.50



From Appendix B, the deflection of fixed-pinned beam with an overhang, due to load  $P$  at the free end, is given by:

$$y(x) = \frac{P a}{4 E I \ell} \left[ x^2 - \ell x^2 - \left( \frac{2 \ell}{3 a} + 1 \right) (x - \ell)^3 \right]; \ell \leq x \leq \ell + a$$

Using  $a = 0.2$ ,  $\ell = 0.8$ ,  $x = a + \ell = 1.0$ , and

$$I = \frac{1}{12} (0.05) (0.05)^3 = 52.0833 (10^{-8}) \text{ m}^4$$

we obtain

$$y_C = \frac{P (0.2)}{4 (207 (10^9)) (52.0833 (10^{-8})) (0.8)} \left[ 1^2 - 0.8 (1)^2 - \left( \frac{1.6}{0.6} + 1 \right) (0.2)^3 \right]$$

$$= P (9.895652 (10^{-8}))$$

$$k_{eq} = \frac{P}{y_C} = 1010.5448 (10^4) \text{ N/m}$$

$$\omega_n = \left\{ \frac{k_{eq}}{m} \right\}^{\frac{1}{2}} = \left\{ \frac{10.1054 (10^8)}{50} \right\}^{\frac{1}{2}} = 449.5642 \text{ rad/sec}$$

2.51 Data:  $k = 500 \text{ N/m}$ ,  $m = 2 \text{ kg}$ ,  $x_0 = 0.1 \text{ m}$ ,  $\dot{x}_0 = 5 \text{ m/s}$

$$\omega_n = \sqrt{k/m} = (500/2)^{\frac{1}{2}} = 15.8114 \text{ rad/s}$$

$$\text{Eq. (2.18): } x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t$$

$$= 0.1 \cos 15.8114 t + \left( \frac{5}{15.8114} \right) \sin 15.8114 t$$

$$\therefore x(t) = 0.1 \cos 15.8114 t + 0.3162 \sin 15.8114 t \text{ m}$$

$$\dot{x}(t) = -1.5811 \sin 15.8114 t + 5 \cos 15.8114 t \text{ m/s}$$

$$\ddot{x}(t) = -24.9994 \cos 15.8114 t - 79.0570 \sin 15.8114 t \frac{\text{m}}{\text{s}^2}$$

2.52 Data:  $\omega_n = 10 \text{ rad/s}$ ,  $x_0 = 0.05 \text{ m}$ ,  $\dot{x}_0 = 1 \text{ m/s}$

Response of undamped system:

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t$$

$$= 0.05 \cos 10 t + \left( \frac{1}{10} \right) \sin 10 t$$

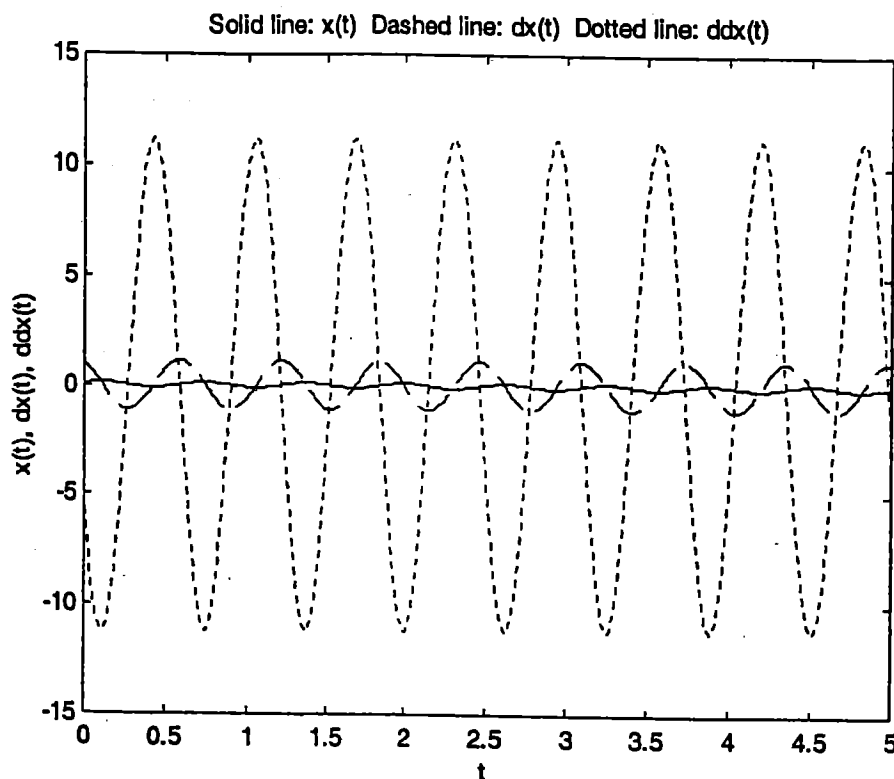
$$\therefore x(t) = 0.05 \cos 10t + 0.1 \sin 10t \quad \text{m} \quad (\text{E.1})$$

$$\dot{x}(t) = -0.5 \sin 10t + \cos 10t \quad \text{m/s} \quad (\text{E.2})$$

$$\ddot{x}(t) = -5 \cos 10t - 10 \sin 10t \quad \text{m/s}^2 \quad (\text{E.3})$$

Plotting of Eqs. (E.1) to (E.3):

```
% Ex2_52.m
for i = 1: 1001
    t(i) = (i-1)*5/1000;
    x(i) = 0.05 * cos(10*t(i)) + 0.1*sin(10*t(i));
    dx(i) = -0.5*sin(10*t(i)) + cos(10*t(i));
    ddx(i) = -5*cos(10*t(i)) - 10*sin(10*t(i));
end
plot(t, x);
hold on;
plot(t, dx, '--');
plot(t, ddx, ':');
xlabel('t');
ylabel('x(t), dx(t), ddx(t)');
title('Solid line: x(t) Dashed line: dx(t) Dotted line: ddx(t)');
```



2.53 Data:  $\omega_d = 2 \text{ rad/s}$ ,  $\zeta = 0.1$ ,  $X_0 = 0.01 \text{ m}$ ,  $\phi_0 = 1 \text{ rad}$   
Initial conditions ?

$$\omega_d = \sqrt{1 - \zeta^2} \omega_n, \quad \omega_n = \omega_d / \sqrt{1 - \zeta^2} = 2 / \sqrt{1 - 0.01} = 2.0101 \text{ rad/s} \quad (\text{E.1})$$

Eqs. (2.73), (2.75):  $\rightarrow$

$$X_0 = \left\{ x_0^2 + \left( \frac{\dot{x}_0 + \zeta \omega_n x_0}{\omega_d} \right)^2 \right\}^{\frac{1}{2}} = 0.01 \quad (\text{E.2})$$

$$\phi_0 = \tan^{-1} \left( - \frac{\dot{x}_0 + \zeta \omega_n x_0}{\omega_d x_0} \right) = 1 \quad (\text{E.3})$$

Eqs. (E.2) and (E.3) lead to:

$$x_0^2 + \left( \frac{\dot{x}_0 + 0.20101 x_0}{2} \right)^2 = 0.0001 \quad (\text{E.4})$$

$$- \left( \frac{\dot{x}_0 + 0.20101 x_0}{2 x_0} \right) = \tan 1 = 0.7854$$

$$\text{or } - (\dot{x}_0 + 0.20101 x_0) = 1.5708 x_0 \quad (\text{E.5})$$

Substitution of Eq. (E.5) into (E.4) yields

$$x_0 = 0.007864 \text{ m} \quad (\text{E.6})$$

Eqs. (E.6) and (E.5) give

$$\dot{x}_0 = -0.013933 \text{ m/s} \quad (\text{E.7})$$

2.54 Without passengers,

$$(\omega_n)_1 = \sqrt{\frac{k}{m}} = 20 \text{ rad/s} \Rightarrow k = 400 \text{ m} \quad (\text{E.1})$$

With passengers,

$$(\omega_n)_2 = \sqrt{\frac{k}{m+500}} = 17.32 \text{ rad/s} \quad (\text{E.2})$$

squaring Eq. (E.2), we get

$$\frac{k}{m+500} = (17.32)^2 = 299.9824 \quad (\text{E.3})$$

Using  $k = 400 \text{ m}$  in (E.3) gives

$$m = 1499.6481 \text{ kg}$$

$$(2.55) \quad \omega_n = \sqrt{k/m} = \sqrt{3200/2} = 40 \text{ rad/s}$$

$$x_0 = 0$$

$$X_0 = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2} = 0.1$$

$$\text{i.e.} \quad \frac{\dot{x}_0}{\omega_n} = 0.1 \quad \text{or} \quad \dot{x}_0 = 0.1 \omega_n = 4 \text{ m/s}$$

(2.56)

$$D = 1.5 \text{ cm}, \quad G = 79.3 \text{ GPa}, \quad \rho = 7800 \text{ kg/m}^3$$

$$f = 193 \text{ Hz}, \quad k = 4.62 \text{ kN/m}$$

$$k = \text{Spring rate} = \frac{d^4 G}{8 D^3 N} \Rightarrow \frac{d^4 (79.3 \times 10^9)}{8 (1.5 \times 10^{-2})^3 N} = 4.62 \times 10^3$$

$$\text{or} \quad \frac{d^4}{N} = 1.5730 \times 10^{-12} \text{ (E.1)}$$

$$f = \frac{1}{2} \sqrt{\frac{k}{W}}$$

$$\text{where } W = \left(\frac{\pi d^2}{4}\right) \pi D N \rho = \frac{\pi^2}{4} (1.5 \times 10^{-2}) (7800) N d^2 = 288.686 N d^2$$

$$\text{Hence } f = \frac{1}{2} \sqrt{\frac{(4.62 \times 10^3)}{288.686 N d^2}} = 193$$

$$\text{or } N d^2 = 107.409 \times 10^{-6} \text{ (E.2)}$$

Eqs. (E.1) and (E.2) yield

$$N = \frac{d^4}{1.5730 \times 10^{-12}} = \frac{107.409 \times 10^{-6}}{d^2}$$

$$d^6 = 1.68954 \times 10^{-16}$$

$$\text{or} \quad d = 0.00235123 \text{ m} = 2.35123 \text{ mm}$$

$$\text{Hence } N = 19.429$$

2.57

Data:  $D = 1.5 \text{ cm}$ ,  $G = 30 \text{ GPa}$ ,  $\rho = 3000 \text{ kg/m}^3$   
 $f = 193 \text{ Hz}$ ,  $k = 4.62 \text{ kN/m}$

$$k = \text{Spring rate} = \frac{d^4 G}{8 D^3 N} \Rightarrow \frac{d^4 (30 \times 10^9)}{8 (1.5 \times 10^{-2})^3 N} = 4.62 \times 10^3$$

$$\text{or } \frac{d^4}{N} = 4.158 \times 10^{-12} \quad (\text{E.1})$$

$$f = \frac{1}{2} \sqrt{\frac{k}{W}}$$

$$\text{where } W = \left( \frac{\pi d^2}{4} \right) \pi D N \rho = \frac{\pi^2}{4} (1.5 \times 10^{-2}) (3000) N d^2 = 111.033 N d^2$$

$$\text{Hence } f = \frac{1}{2} \sqrt{\frac{4.62 \times 10^3}{111.033 N d^2}} = 193$$

$$\text{or } N d^2 = 279.264 \times 10^{-6} \quad (\text{E.2})$$

Eqs. (E.1) and (E.2) yield

$$d^6 = 1.16118 \times 10^{-15} \quad \text{or } d = 0.00324203 \text{ m}$$

Hence  $N = 26.569397$

2.58

By neglecting the effect of self weight of the beam, and using a single degree of free-

dom model, the natural frequency of the system can be expressed as  $\omega_n = \sqrt{\frac{k}{m}}$

where  $m$  = mass of the machine, and  $k$  = stiffness of the cantiliver beam:

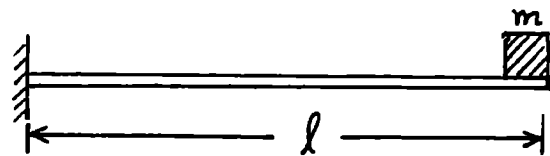
$$k = \frac{3 E I}{\ell^3}$$

where  $\ell$  = length,  $E$  = Young's Modulus, and  $I$  = area moment of inertia of the beam section.

Assuming  $E = 200 \text{ GPa}$  for steel and  $72 \text{ GPa}$  for aluminium, we have:

$$(\omega_n)_{\text{steel}} = \left\{ \frac{3 (200 \times 10^9) I}{m \ell^3} \right\}^{\frac{1}{2}}$$

$$(\omega_n)_{\text{Al}} = \left\{ \frac{3 (72 \times 10^6) I}{m \ell^3} \right\}^{\frac{1}{2}}$$



$$\text{Ratio of natural freq.} = \frac{(\omega_n)_{\text{steel}}}{(\omega_n)_{\text{Al}}} = \left( \frac{200}{72} \right)^{\frac{1}{2}} = 1.6667 = \frac{1}{0.6}$$

Thus the natural frequency is reduced to 60% of its value if aluminium is used instead of steel.

2.59) For free vibration, apply Newton's second law of motion:

$$ml\ddot{\theta} + mg \sin \theta = 0 \quad (\text{E.1})$$

For small angular displacements, Eq. (E.1) reduces to

$$ml\ddot{\theta} + mg\theta = 0 \quad (\text{E.2})$$

$$\text{or } \ddot{\theta} + \omega_n^2 \theta = 0 \quad (\text{E.3})$$

$$\text{where } \omega_n = \sqrt{\frac{g}{l}} \quad (\text{E.4})$$

Solution of Eq. (E.3) is:

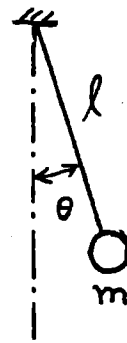
$$\theta(t) = \theta_0 \cos \omega_n t + \frac{\dot{\theta}_0}{\omega_n} \sin \omega_n t \quad (\text{E.5})$$

where  $\theta_0$  and  $\dot{\theta}_0$  denote the angular displacement and angular velocity at  $t=0$ . The amplitude of motion is given by

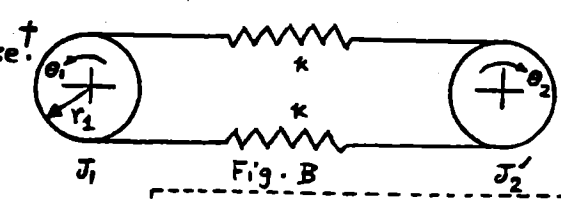
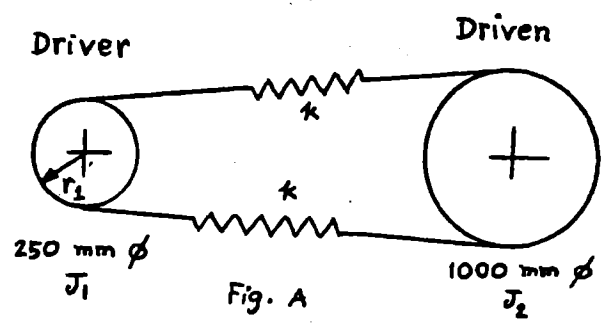
$$\Theta = \left\{ \theta_0^2 + \left( \frac{\dot{\theta}_0}{\omega_n} \right)^2 \right\}^{\frac{1}{2}} \quad (\text{E.6})$$

Using  $\Theta = 0.5 \text{ rad}$ ,  $\theta_0 = 0$  and  $\dot{\theta}_0 = 1 \text{ rad/s}$ , Eq. (E.6) gives

$$0.5 = \frac{\dot{\theta}_0}{\omega_n} = \frac{1}{\omega_n} \quad \text{or } \omega_n = 2 \text{ rad/s}$$



2.60 The system of Fig. (A) can be drawn in equivalent form as shown in Fig. (B) where both pulleys have the same radius  $r_1$ . We notice in Fig. (B) that vibration can take place in only one way with one pulley moving clockwise and the other moving counter clockwise.



When pulleys rotate in opposite directions,  $\frac{\theta_1}{\theta_2} = \frac{J_2}{J_1}$ .

The spring force, which has the same value on either pulley is  $-k_t(\theta_1 + \theta_2)$  where  $k_t =$  torsional spring constant of the system. Equation of motion is

$$J_1 \ddot{\theta}_1 + k_t(\theta_1 + \theta_2) = 0 \quad \& \quad J_2 \ddot{\theta}_2 + k_t(\theta_1 + \theta_2) = 0$$

i.e.  $J_1 \ddot{\theta}_1 + k_t(1 + \frac{J_1}{J_2})\theta_1 = 0 \quad \& \quad J_2 \ddot{\theta}_2 + k_t(\frac{J_2}{J_1} + 1)\theta_2 = 0$

$$k_t = \frac{\Delta M_t}{\Delta \theta} = \left( \frac{\text{force in springs}}{\text{due to } \Delta \theta} \right) \frac{r_1}{\Delta \theta}$$

$$= (2k r_1 \Delta \theta) \frac{r_1}{\Delta \theta} = 2k r_1^2$$

$$= 2k \left( \frac{125}{1000} \right)^2 = k/32 \text{ N-m/rad}$$

$\therefore E_2 \cdot (E_1)$  gives, for  $\omega = 12\pi \text{ rad}$ ,  $k = 454.7935 \text{ N/m}$ .

Either of these equations gives  $\omega = \left\{ k_t \left( \frac{J_1 + J_2}{J_1 J_2} \right) \right\}^{1/2} \dots (E_1)$

Here  $J_1 = 0.2/4 = 0.05 \text{ kg-m}^2$ ,  
 $J_2' = J_2 (\text{speed ratio})^2 = 0.2 \left( \frac{1}{4} \right)^2 = 0.0125 \text{ kg-m}^2$

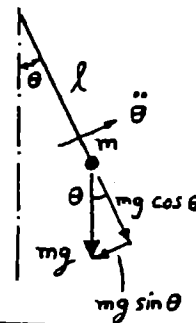
† The other possible motion is rotation of the two pulleys as a whole (as rigid body) in same direction. This will have a natural frequency of zero. See section 5.7.

$$\textcircled{2.61} \quad ml \ddot{\theta} + mg \sin \theta = 0$$

For small  $\theta$ ,  $ml \ddot{\theta} + mg \theta = 0$

$$\omega_n = \sqrt{\frac{g}{l}}$$

$$T_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{\frac{9.81}{0.5}}} = 1.4185 \text{ sec}$$



$$\textcircled{2.62} \quad (a) \quad \omega_n = \sqrt{\frac{g}{l}}$$

$$(b) \quad ml^2 \ddot{\theta} + \kappa a^2 \sin \theta + mgl \sin \theta = 0 ; \quad ml^2 \ddot{\theta} + (\kappa a^2 + mgl) \theta = 0$$

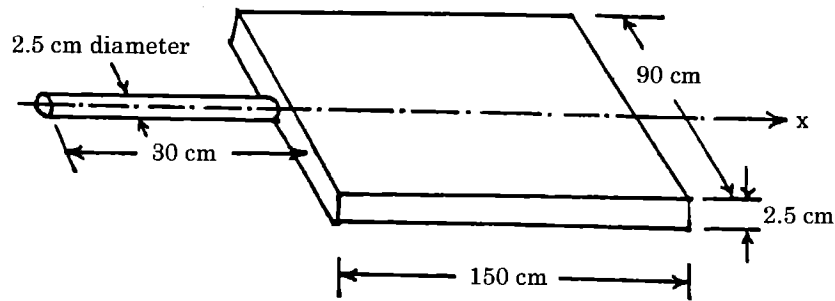
$$\omega_n = \sqrt{\frac{\kappa a^2 + mgl}{ml^2}}$$

$$(c) \quad ml^2 \ddot{\theta} + \kappa a^2 \sin \theta - mgl \sin \theta = 0 ; \quad ml^2 \ddot{\theta} + (\kappa a^2 - mgl) \theta = 0$$

$$\omega_n = \sqrt{\frac{\kappa a^2 - mgl}{ml^2}}$$

configuration (b) has the highest natural frequency.

2.63



$$m = \text{mass of a panel} = (150 \times 10^{-2}) (90 \times 10^{-2}) (2.5 \times 10^{-2}) (3000) = 101.25 \text{ kg}$$

$$J_0 = \text{mass moment inertia of panel about x-axis} = \frac{m}{12} (a^2 + b^2)$$

$$= \frac{101.25}{12} ((2.5 \times 10^{-2})^2 + (90 \times 10^{-2})^2) = 6.83965 \text{ kg m}^2$$

$$I_0 = \text{polar moment of inertia of rod} = \frac{\pi}{32} d^4 = \frac{\pi}{32} (2.5 \times 10^{-2})^4 = 38.3495 \times 10^{-9} \text{ m}^4$$

$$k_t = \frac{G I_0}{\ell} = \frac{(26 \times 10^9) (38.3495 \times 10^{-9})}{(30 \times 10^{-2})} = 3323.62 \text{ N-m/rad}$$

$$\omega_n = \left( \frac{k_t}{J_0} \right)^{\frac{1}{2}} = \left( \frac{3323.64}{6.83965} \right)^{\frac{1}{2}} = 22.044 \text{ rad/sec}$$

2.64

$I_0 = \text{polar moment of inertia of cross section of shaft AB}$

$$= \frac{\pi}{32} d^4 = \frac{\pi}{32} (2.5 \times 10^{-2})^4 = 38.3495 \times 10^{-9} \text{ m}^4$$

$$k_t = \text{torsional stiffness of shaft AB} = \frac{G I_0}{\ell}$$

$$= \frac{(83 \times 10^9) (38.3495 \times 10^{-9})}{15 \times 10^{-2}} = 21220.06 \text{ N-m/rad}$$

$J_0 = \text{mass moment of inertia of the three blades about y-axis}$

$$= 3 J_0 |PQ = 3 \left( \frac{1}{3} m \ell^2 \right) = m \ell^2 = 1(30 \times 10^{-2})^2 = 0.09 \text{ kg m}^2$$

$$\omega_n = \left( \frac{k_t}{J_0} \right)^{\frac{1}{2}} = \left( \frac{21220.06}{0.09} \right)^{\frac{1}{2}} = 485.57 \text{ rad/sec}$$

2.65  $J_0 =$  mass moment of inertia of the ring  $= 1.0 \text{ kg-m}^2$ .  
 $I_{os} =$  polar moment of inertia of the cross section of steel shaft

$$= \frac{\pi}{32} (d_{os}^4 - d_{is}^4) = \frac{\pi}{4} (0.05^4 - 0.04^4) = 36.2266 (10^{-8}) \text{ m}^4$$

$I_{ob} =$  polar moment of inertia of cross section of brass shaft

$$= \frac{\pi}{32} (d_{ob}^4 - d_{ib}^4) = \frac{\pi}{32} (0.04^4 - 0.03^4) = 17.1806 (10^{-8}) \text{ m}^4$$

$k_{ts} =$  torsional stiffness of steel shaft

$$= \frac{G_s I_{os}}{\ell} = \frac{(80 (10^9)) (36.2266 (10^{-8}))}{2} = 14490.64 \text{ N-m/rad}$$

$k_{tb} =$  torsional stiffness of brass shaft

$$= \frac{G_b I_{ob}}{\ell} = \frac{(40 (10^9)) (17.1806 (10^{-8}))}{2} = 3436.12 \text{ N-m/rad}$$

$k_{teq} = k_{ts} + k_{tb} = 17,926.76 \text{ N-m/rad}$

Torsional natural frequency:

$$\omega_n = \sqrt{\frac{k_{teq}}{J_0}} = \sqrt{\frac{17926.76}{1}} = 133.8908 \text{ rad/sec}$$

Natural time period:

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{133.8908} = 0.04693 \text{ sec}$$

2.66 Kinetic energy of system is

$$T = T_{rod} + T_{bob} = \frac{1}{2} \left( \frac{1}{3} m l^2 \right) \dot{\theta}^2 + \frac{1}{2} M l^2 \dot{\theta}^2$$

Potential energy of system is

(since mass of the rod acts through its center)

$$U = U_{rod} + U_{bob} = \frac{1}{2} m g l (1 - \cos \theta) + \frac{1}{2} M g l (1 - \cos \theta)$$

Equation of motion:

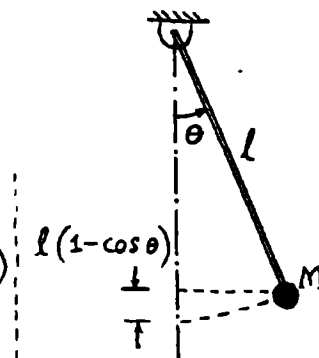
$$\frac{d}{dt} (T + U) = 0$$

$$\text{i.e. } \left( M + \frac{m}{3} \right) l^2 \ddot{\theta} + \left( M + \frac{m}{2} \right) g l \sin \theta = 0$$

For small angles,

$$\ddot{\theta} + \frac{\left( M + \frac{m}{2} \right) g}{\left( M + \frac{m}{3} \right) l} \theta = 0$$

$$\omega_n = \sqrt{\frac{\left( M + \frac{m}{2} \right) g}{\left( M + \frac{m}{3} \right) l}}$$



(2.67) For the shaft, 
$$J = \frac{\pi d^4}{32} = \frac{\pi (0.05)^4}{32} = 61.3594 \times 10^{-8} \text{ m}^4$$

$$k_t = \frac{GJ}{l} = \frac{(0.793 \times 10^{11}) (61.3594 \times 10^{-8})}{2} = 24329.002 \text{ N-m/rad}$$

For the disc,

$$J_o = \frac{MD^2}{8} = \left( \rho \frac{\pi D^2}{4} h \right) \frac{D^2}{8} = \frac{\rho \pi D^4 h}{32}$$

$$= \frac{(7.83 \times 10^3) \pi (1)^4 (0.1)}{32} = 76.8710 \text{ kg-m}^2$$

$$\omega_n = \sqrt{\frac{k_t}{J_o}} = \left( \frac{24329.002}{76.8710} \right)^{1/2} = 17.7902 \text{ rad/sec}$$

(2.68) Equation of motion

$$J_A \ddot{\theta} = -W d \theta - 2k \left( \frac{l}{3} \theta \right) \frac{l}{3}$$

$$- 2k \left( \frac{2l}{3} \theta \right) \frac{2l}{3} - k_t \theta$$

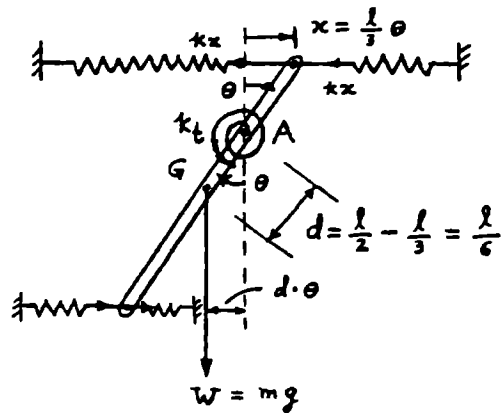
where

$$J_A = J_G + m d^2 = \frac{1}{12} m l^2 + m \frac{l^2}{36}$$

$$= \frac{1}{9} m l^2$$

$$\therefore \frac{m l^2}{9} \ddot{\theta} + (mgd + 2k \frac{l^2}{9} + \frac{8k l^2}{9} + k_t) \theta = 0$$

$$\omega_n = \sqrt{\frac{(mgd + \frac{2}{9} k l^2 + \frac{8}{9} k l^2 + k_t) 9}{m l^2}} = \sqrt{\frac{9mgd + 10k l^2 + 9k_t}{m l^2}}$$



For given data,

$$\omega_n = \sqrt{\frac{9(10)(9.81)(5/6) + 10(2000)(5)^2 + 9(1000)}{10(5)^2}} = 45.1547 \frac{\text{rad}}{\text{sec}}$$

(2.69)  $J_O = \frac{1}{2} m R^2$ ,  $J_C = \frac{1}{2} m R^2 + m R^2$

Let angular displacement =  $\theta$

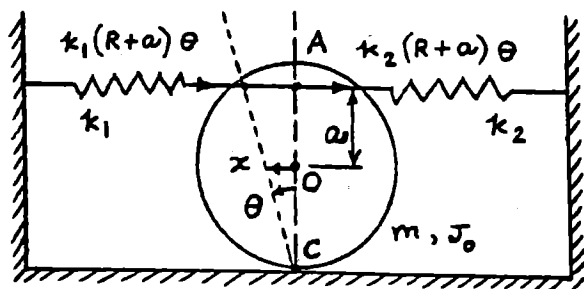
Equation of motion:

$$J_C \ddot{\theta} + k_1(R+a)^2 \theta + k_2(R+a)^2 \theta = 0$$

$$\omega_n = \sqrt{\frac{(k_1 + k_2)(R+a)^2}{J_C}} = \sqrt{\frac{(k_1 + k_2)(R+a)^2}{1.5 m R^2}} \quad (E_1)$$

Equation (E<sub>1</sub>) shows that  $\omega_n$  increases with the value of  $a$ .

$\therefore \omega_n$  will be maximum when  $a = R$ .



(2.70) Net  $g$  acting on the pendulum =  $9.81 - 5 = 4.81 \text{ m/sec}^2 = g_n$

$$\omega_n = \sqrt{\frac{g_n}{l}} = \sqrt{\frac{4.81}{5}} = 3.1016 \text{ rad/sec}$$

$$T_n = 2\pi/\omega_n = 2.0258 \text{ sec}$$

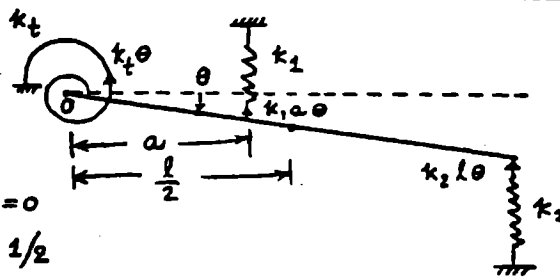
(2.71) Equation of motion:

$$J_O \ddot{\theta} = -k_t \theta - (k_1 a \theta) a - (k_2 l \theta) l$$

where  $J_O = \frac{1}{12} m l^2 + m \left(\frac{l}{2}\right)^2 = \frac{1}{3} m l^2$

$$\therefore \frac{1}{3} m l^2 \ddot{\theta} + (k_t + k_1 a^2 + k_2 l^2) \theta = 0$$

$$\omega_n = \left\{ \frac{3(k_t + k_1 a^2 + k_2 l^2)}{m l^2} \right\}^{1/2}$$



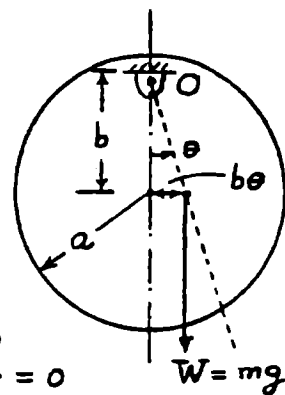
(2.72)  $J_O = J_G + m b^2 = \frac{1}{2} m a^2 + m b^2$

Equation of motion:

$$J_O \ddot{\theta} + m g b \theta = 0$$

$$\omega_n = \sqrt{\frac{m g b}{J_O}} = \sqrt{\frac{2 g b}{a^2 + 2 b^2}}$$

$$\frac{\partial \omega_n}{\partial b} = \frac{1}{2} \left( \frac{2 g b}{a^2 + 2 b^2} \right)^{-1/2} \left\{ \frac{(a^2 + 2 b^2)(2g) - 2 g b (4b)}{(a^2 + 2 b^2)^2} \right\} = 0$$



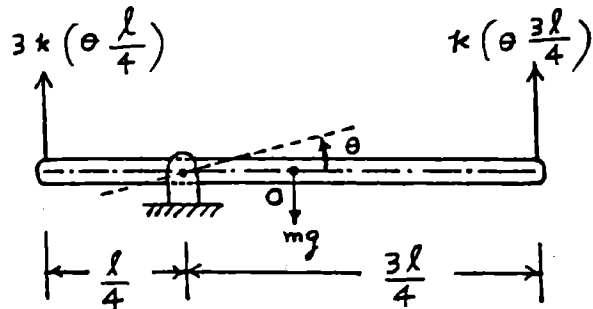
$$\text{i.e., } b = \pm \frac{a}{\sqrt{2}}$$

$$\omega_n \Big|_{b = + a/\sqrt{2}} = \sqrt{\frac{2g \frac{a}{\sqrt{2}}}{a^2 + 2(a^2/2)}} = \sqrt{\frac{g}{\sqrt{2} a}}$$

$b = - a/\sqrt{2}$  gives imaginary value for  $\omega_n$ .

Since  $\omega_n = 0$  when  $b = 0$ , we have  $\omega_n / \text{max}$  at  $b = \frac{a}{\sqrt{2}}$ .

2.73



Let  $\theta$  be measured from static equilibrium position so that gravity force need not be considered.

(a) Newton's second law of motion:

$$J_0 \ddot{\theta} = -3k \left(\theta \frac{\ell}{4}\right) \frac{\ell}{4} - k \left(\theta \frac{3\ell}{4}\right) \left(\frac{3\ell}{4}\right) \quad \text{or} \quad J_0 \ddot{\theta} + \frac{3}{4} k \ell^2 \theta = 0$$

(b) D'Alembert's principle:

$$M(t) - J_0 \ddot{\theta} = 0 \quad \text{or} \quad -3k \left(\theta \frac{\ell}{4}\right) \left(\frac{\ell}{4}\right) - k \left(\theta \frac{3\ell}{4}\right) \left(\frac{3\ell}{4}\right) - J_0 \ddot{\theta} = 0$$

$$\text{or} \quad J_0 \ddot{\theta} + \frac{3}{4} k \ell^2 \theta = 0$$

(c) Principle of virtual work:

Virtual work done by spring force:

$$\delta W_s = -3k \left(\theta \frac{\ell}{4}\right) \left(\frac{\ell}{4} \delta\theta\right) - k \left(\theta \frac{3\ell}{4}\right) \left(\frac{3\ell}{4} \delta\theta\right)$$

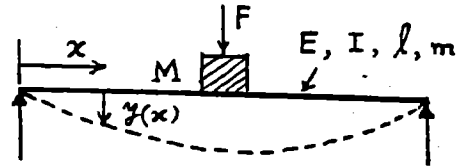
Virtual work done by inertia moment =  $-(J_0 \ddot{\theta}) \delta\theta$

Setting total virtual work done by all forces/moments equal to zero, we obtain

$$J_0 \ddot{\theta} + \frac{3}{4} k \ell^2 \theta = 0$$

2.74

Let  $m_{\text{eff}}$  = effective part of mass of beam ( $m$ ) at middle. Thus vibratory inertia force at middle is due to  $(M + m_{\text{eff}})$ . Assume a deflection shape:  $y(x, t) = Y(x) \cos(\omega_n t - \phi)$  where  $Y(x)$  = static deflection shape due to load at middle given by:



$$Y(x) = Y_0 \left( 3 \frac{x}{\ell} - 4 \frac{x^3}{\ell^3} \right); \quad 0 \leq x \leq \frac{\ell}{2}$$

where  $Y_0 =$  maximum deflection of the beam at middle  $= \frac{F \ell^3}{48 E I}$

Maximum strain energy of beam = maximum work done by force  $F = \frac{1}{2} F Y_0$ .

Maximum kinetic energy due to distributed mass of beam:

$$\begin{aligned} &= 2 \left\{ \frac{1}{2} \frac{m}{\ell} \int_0^{\ell/2} \dot{y}^2(x,t) |_{\max} dx \right\} + \frac{1}{2} (\dot{y}_{\max})^2 M \\ &= \frac{m \omega_n^2}{\ell} \int_0^{\ell/2} Y^2(x) dx + \frac{1}{2} \omega_n^2 Y_{\max}^2 M \\ &= \frac{m \omega_n^2}{\ell} \int_0^{\ell/2} Y_0^2 \left( \frac{9x^2}{\ell^2} + 16 \frac{x^6}{\ell^6} - 24 \frac{x^4}{\ell^4} \right) dx + \frac{1}{2} Y_0^2 M \omega_n^2 \\ &= \frac{m \omega_n^2 Y_0^2}{\ell} \left[ \frac{9}{\ell^2} \frac{x^3}{3} + \frac{16}{\ell^6} \frac{x^7}{7} - \frac{24}{\ell^4} \frac{x^5}{5} \right] \Big|_0^{\ell/2} + \frac{1}{2} Y_0^2 M \omega_n^2 \\ &= \frac{1}{2} Y_0^2 \omega_n^2 \left[ \frac{17}{35} m + M \right] \end{aligned}$$

This shows that  $m_{\text{eff}} = \frac{17}{35} m = 0.4857 m$

**2.75** For small angular rotation of bar PQ about P,

$$\frac{1}{2} (k_{12})_{eq} (\theta l_3)^2 = \frac{1}{2} k_1 (\theta l_1)^2 + \frac{1}{2} k_2 (\theta l_2)^2$$

$$(k_{12})_{eq} = \frac{k_1 l_1^2 + k_2 l_2^2}{l_3^2}$$

Since  $(k_{12})_{eq}$  and  $k_3$  are in series,

$$k_{eq} = \frac{(k_{12})_{eq} k_3}{(k_{12})_{eq} + k_3} = \frac{k_1 k_3 l_1^2 + k_2 k_3 l_2^2}{k_1 l_1^2 + k_2 l_2^2 + k_3 l_3^2}$$

$T =$  kinetic energy  $= \frac{1}{2} m \dot{x}^2$ ,  $U =$  potential energy  $= \frac{1}{2} k_{eq} x^2$

If  $x = X \cos \omega_n t$ ,

$$T_{\max} = \frac{1}{2} m \omega_n^2 X^2, \quad U_{\max} = \frac{1}{2} k_{eq} X^2$$

$$T_{\max} = U_{\max} \quad \text{gives} \quad \omega_n = \sqrt{\frac{k_1 k_3 l_1^2 + k_2 k_3 l_2^2}{m(k_1 l_1^2 + k_2 l_2^2 + k_3 l_3^2)}}$$

2.76 When mass  $m$  moves by  $x$ , spring  $k_1$  deflects by  $x/4$ .

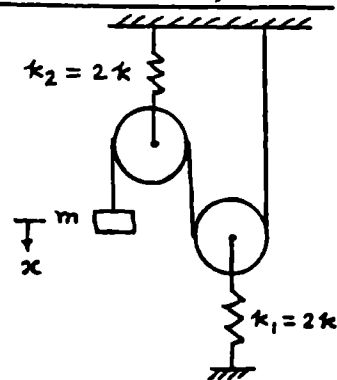
$$T = \text{kinetic energy} = \frac{1}{2} m (\dot{x})^2$$

$$U = \text{potential energy} = 2 \left\{ \frac{1}{2} (2k) \left( \frac{x}{4} \right)^2 \right\} = \frac{1}{8} k x^2$$

For harmonic motion,

$$T_{\max} = \frac{1}{2} m \omega_n^2 X^2, \quad U_{\max} = \frac{1}{8} k X^2$$

$$T_{\max} = U_{\max} \quad \text{gives} \quad \omega_n = \sqrt{\frac{k}{4m}}$$



2.77 Refer to the figure of solution of problem 2.24.

$$T = \frac{1}{2} m \dot{x}^2, \quad U = \frac{1}{2} \left[ 2k_1 (x \cos 45^\circ)^2 + 2k_2 (x \cos 135^\circ)^2 \right] = \frac{1}{2} (k_1 + k_2) x^2$$

For harmonic motion,

$$T_{\max} = \frac{1}{2} m \omega_n^2 X^2, \quad U_{\max} = \frac{1}{2} (k_1 + k_2) X^2$$

$$T_{\max} = U_{\max} \quad \text{gives} \quad \omega_n = \sqrt{\frac{k_1 + k_2}{m}}$$

2.78 Kinetic energy (K.E.) =  $\frac{1}{2} m \dot{x}^2$

Potential energy (P.E.) =  $\frac{1}{2} T_1 x + \frac{1}{2} T_2 x$  = work done in displacing mass  $m$  by distance  $x$  against the total force (tension) of  $T_1 + T_2$ .

$$T_1 = \frac{x}{a} T, \quad T_2 = \frac{x}{b} T \quad \text{from solution of problem 2.26}$$

$$\text{Max. K.E.} = \frac{1}{2} m \omega_n^2 X^2, \quad \text{Max. P.E.} = \frac{1}{2} T \left( \frac{1}{a} + \frac{1}{b} \right) X^2$$

$$\text{Max. K.E.} = \text{Max. P.E.} \quad \text{gives} \quad \omega_n = \sqrt{\frac{T(a+b)}{mab}} = \sqrt{\frac{Tl}{ma(l-a)}}$$

$$2.79 \quad T = \text{K.E.} = \frac{1}{2} \mathcal{J}_A \dot{\theta}^2 = \frac{1}{2} (\mathcal{J}_G + m d^2) \dot{\theta}^2 = \frac{1}{2} \left( \frac{1}{12} m l^2 + m \frac{l^2}{36} \right) \dot{\theta}^2 = \frac{1}{2} \left( \frac{m l^2}{9} \right) \dot{\theta}^2$$

$$U = \text{P.E.} = mgd(1 - \cos \theta) + 2 \left( \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 \right) + \frac{1}{2} k_t \theta^2$$

$$\text{with } \cos \theta \approx 1 - \frac{1}{2} \theta^2, \quad x_1 = \frac{l}{3} \theta \quad \text{and} \quad x_2 = \frac{2l}{3} \theta$$

$$U = mg \frac{l}{6} \frac{\theta^2}{2} + \kappa \frac{l^2}{9} \theta^2 + \kappa \frac{4l^2}{9} \theta^2 + \frac{1}{2} \kappa_t \theta^2$$

$$T_{\max} = \frac{1}{2} \left( \frac{ml^2}{9} \right) \Theta^2, \quad U_{\max} = \frac{1}{2} \frac{mg l}{6} \Theta^2 + \frac{1}{2} \left( \frac{10 \kappa l^2}{9} \right) \Theta^2 + \frac{1}{2} \kappa_t \Theta^2$$

$T_{\max} = U_{\max}$  gives

$$\omega_n = \sqrt{\frac{mg \frac{l}{6} + \frac{10 \kappa l^2}{9} + \kappa_t}{\frac{ml^2}{9}}} = 45.1547 \frac{\text{rad}}{\text{sec}} \quad \text{for given data}$$

2.80 Refer to the figure in the solution of problem 2.71.

$$T = \frac{1}{2} J_0 \dot{\theta}^2$$

$$U = \frac{1}{2} \kappa_t \theta^2 + \frac{1}{2} \kappa_1 (\theta a)^2 + \frac{1}{2} \kappa_2 (\theta l)^2$$

For  $\theta(t) = \Theta \cos \omega_n t$ ,

$$T_{\max} = \frac{1}{2} J_0 \omega_n^2 \Theta^2, \quad U_{\max} = \frac{1}{2} (\kappa_t + \kappa_1 a^2 + \kappa_2 l^2) \Theta^2$$

$T_{\max} = U_{\max}$  gives

$$\omega_n = \sqrt{\frac{\kappa_t + \kappa_1 a^2 + \kappa_2 l^2}{J_0}} = \sqrt{\frac{3(\kappa_t + \kappa_1 a^2 + \kappa_2 l^2)}{m l^2}}$$

since  $J_0 = ml^2/3$ .

2.81 When prism is displaced by  $x$  from equilibrium position, the weight of oil displaced

$$= \rho_o g abx = \text{restoring force}$$

$$\text{Mass of prism} = m = \rho_w abh$$

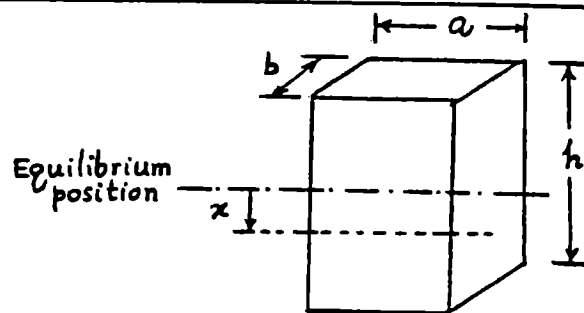
Equation of motion:

$$m \ddot{x} + \text{restoring force} = 0$$

$$\rho_w abh \ddot{x} + \rho_o g abx = 0$$

$$\omega_n = \sqrt{\frac{\rho_o g ab}{\rho_w abh}} = \sqrt{\frac{\rho_o g}{\rho_w h}} \quad (E1)$$

Since  $\omega_n$  is independent of cross-section of the prism,  $\omega_n$  remains same even for a circular wooden prism.



2.82 
$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2 = \frac{1}{2} \left( m R^2 + \frac{1}{2} m R^2 \right) \dot{\theta}^2$$

since  $x = R \theta$  and  $J_0 = \frac{1}{2} m R^2$ .

$$U = \frac{1}{2} \kappa_1 x_1^2 + \frac{1}{2} \kappa_2 x_2^2 = \frac{1}{2} (\kappa_1 + \kappa_2) (R + a)^2 \theta^2$$

where  $x_1 = (R + a) \theta$ . Using  $\frac{d}{dt} (T + U) = 0$ , we obtain

$$\left(\frac{3}{2} m R^2\right) \ddot{\theta} + (k_1 + k_2) (R + a)^2 \theta = 0$$

(2.83) Let  $x(t)$  be measured from static equilibrium position of mass.  $T =$  kinetic energy of the system:

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2 = \frac{1}{2} \left( m + \frac{J_0}{r^2} \right) \dot{x}^2$$

since  $\dot{\theta} = \frac{\dot{x}}{r} =$  angular velocity of pulley.  $U =$  potential energy of the system:

$$U = \frac{1}{2} k y^2 = \frac{1}{2} k (16 x^2)$$

since  $y = \theta (4 r) = 4 x =$  deflection of spring.  $\frac{d}{dt} (T + U) = 0$  leads to:

$$m \ddot{x} + \frac{J_0}{r^2} \ddot{x} + 16 k x = 0$$

This gives the natural frequency:

$$\omega_n = \sqrt{\frac{16 k r^2}{m r^2 + J_0}}$$

(2.84) For pendulum,  $\omega_n = \sqrt{g/l}$  in vacuum = 0.5 Hz =  $\pi$  rad/sec

$$l = g/\pi^2 = 9.81/\pi^2 = 0.9940 \text{ m}$$

$\omega_d = \omega_n \sqrt{1 - \zeta^2}$  in viscous medium = 0.45 Hz =  $0.9 \pi$  rad/sec

$$\zeta^2 = \frac{\omega_n^2 - \omega_d^2}{\omega_n^2} = \pi^2 \left( \frac{1 - 0.81}{1} \right) = 1.8752$$

$$\zeta = 1.3694$$

Equation of motion:  $m l^2 \ddot{\theta} + c_t \dot{\theta} + m g l \theta = 0$

$$c_{ct} = 2(m l^2) \omega_n = 2(1)(0.994)^2(\pi) = 6.2080$$

Since  $\zeta = \frac{c_t}{c_{ct}} = 1.3694$ ,  $c_t = 8.5013 \text{ N-m-sec/rad}$ .

(2.85) From Eq. (2.85),

$$\ln \left( \frac{x_j}{x_{j+1}} \right) = \ln(18) \Rightarrow \frac{2 \pi \zeta}{\sqrt{1 - \zeta^2}} = 2.8904$$

$$\zeta = \left\{ \frac{(2.8904)^2}{(2.8904)^2 + 4 \pi^2} \right\}^{\frac{1}{2}} = 0.4179$$

(a) If damping is doubled,

$$\zeta_{\text{new}} = 0.8358$$

$$\ln \left( \frac{x_j}{x_{j+1}} \right) = \frac{2\pi \zeta_{\text{new}}}{\sqrt{1 - \zeta_{\text{new}}^2}} = \frac{2\pi (0.8358)}{\sqrt{1 - (0.8358)^2}} = 9.5656$$

$$\therefore \frac{x_j}{x_{j+1}} = 14265.362$$

(b) If damping is halved,

$$\zeta = 0.2090$$

$$\ln \left( \frac{x_j}{x_{j+1}} \right) = \frac{2\pi \zeta}{\sqrt{1 - \zeta^2}} = \frac{2\pi (0.2090)}{\sqrt{1 - (0.2090)^2}} = 1.3428$$

$$\therefore \frac{x_j}{x_{j+1}} = 3.8296$$

2.86

$$x(t) = X e^{-\zeta \omega_n t} \sin \omega_d t \quad \text{where} \quad \omega_d = \sqrt{1 - \zeta^2} \omega_n$$

For maximum or minimum of  $x(t)$ ,

$$\frac{dx}{dt} = X e^{-\zeta \omega_n t} (-\zeta \omega_n \sin \omega_d t + \omega_d \cos \omega_d t) = 0$$

As  $e^{-\zeta \omega_n t} \neq 0$  for finite  $t$ ,

$$-\zeta \omega_n \sin \omega_d t + \omega_d \cos \omega_d t = 0$$

$$\text{i.e.} \quad \tan \omega_d t = \frac{\sqrt{1 - \zeta^2}}{\zeta}$$

Using the relation

$$\sin \omega_d t = \pm \frac{\tan \omega_d t}{\sqrt{1 + \tan^2 \omega_d t}} = \pm \frac{(\sqrt{1 - \zeta^2}/\zeta)}{\sqrt{1 + \left(\frac{\sqrt{1 - \zeta^2}}{\zeta}\right)^2}} = \pm \sqrt{1 - \zeta^2}$$

we obtain

$$\sin \omega_d t = \sqrt{1 - \zeta^2}, \quad \cos \omega_d t = \zeta$$

and

$$\sin \omega_d t = -\sqrt{1 - \zeta^2}, \quad \cos \omega_d t = -\zeta$$

$$\frac{d^2 x}{dt^2} = X e^{-\zeta \omega_n t} \left[ \zeta^2 \omega_n^2 \sin \omega_d t - 2\zeta \omega_n \omega_d \cos \omega_d t - \omega_d^2 \sin \omega_d t \right]$$

When  $\sin \omega_d t = \sqrt{1 - \zeta^2}$  and  $\cos \omega_d t = \zeta$ ,

$$\frac{d^2 x}{dt^2} = -X e^{-\zeta \omega_n t} \omega_n^2 \sqrt{1 - \zeta^2} < 0$$

$\therefore \sin \omega_d t = \sqrt{1 - \zeta^2}$  corresponds to maximum of  $x(t)$ .

When  $\sin \omega_d t = -\sqrt{1 - \zeta^2}$  and  $\cos \omega_d t = -\zeta$ ,

$$\frac{d^2x}{dt^2} = X e^{-\gamma \omega_n t} \omega_n^2 \sqrt{1-\gamma^2} > 0$$

$\therefore \sin \omega_d t = -\sqrt{1-\gamma^2}$  corresponds to minimum of  $x(t)$ .

Enveloping curves:

Let the curve passing through the maximum (or minimum) points be

$$x(t) = C e^{-\gamma \omega_n t}$$

For maximum points,  $t_{\max} = \frac{\sin^{-1}(\sqrt{1-\gamma^2})}{\omega_d}$

and

$$C e^{-\gamma \omega_n t_{\max}} = X e^{-\gamma \omega_n t_{\max}} \sin \omega_d t_{\max}$$

i.e.

$$C = X \sqrt{1-\gamma^2}$$

$$\therefore x_1(t) = X \sqrt{1-\gamma^2} e^{-\gamma \omega_n t}$$

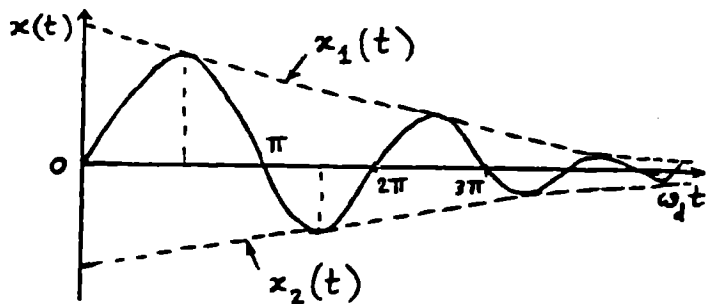
Similarly for minimum points,  $t_{\min} = \frac{\sin^{-1}(-\sqrt{1-\gamma^2})}{\omega_d}$

and

$$C e^{-\gamma \omega_n t_{\min}} = X e^{-\gamma \omega_n t_{\min}} \sin \omega_d t_{\min}$$

$$\text{i.e. } C = -X \sqrt{1-\gamma^2}$$

$$\therefore x_2(t) = -X \sqrt{1-\gamma^2} e^{-\gamma \omega_n t}$$



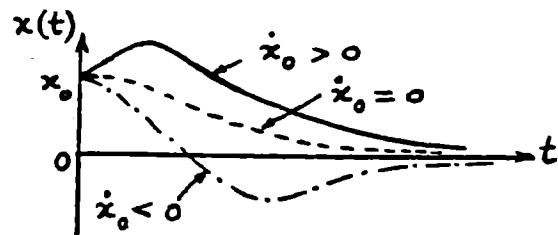
$$(2.87) \quad x(t) = [x_0 + (\dot{x}_0 + \omega_n x_0)t] e^{-\omega_n t} \quad \text{--- (E}_1\text{)}$$

For  $x_0 > 0$ , graph of  $E_1$  (E<sub>1</sub>) is shown for different  $\dot{x}_0$ . We assume  $\dot{x}_0 > 0$  as it is the only case that gives a maximum.

For maximum of  $x(t)$ ,

$$\frac{dx}{dt} = e^{-\omega_n t} \{ -(\dot{x}_0 + \omega_n x_0) \omega_n t + \dot{x}_0 \} = 0$$

$$t_m = \frac{\dot{x}_0}{\omega_n (\dot{x}_0 + \omega_n x_0)} \quad \text{--- (E}_2\text{)}$$



$$\frac{d^2x}{dt^2} = -e^{-\omega_n t} \left\{ 2\omega_n \dot{x}_0 + \omega_n^2 x_0 - \omega_n^2 (\dot{x}_0 + \omega_n x_0)t \right\} \dots (E_3)$$

(E<sub>2</sub>) and (E<sub>3</sub>) give

$$\begin{aligned} \left. \frac{d^2x}{dt^2} \right|_{t=t_m} &= -e^{-\omega_n t_m} \left\{ 2\omega_n \dot{x}_0 + \omega_n^2 x_0 - \omega_n^2 (\dot{x}_0 + \omega_n x_0)t_m \right\} \\ &= -e^{-\omega_n t_m} \left( \frac{\dot{x}_0}{\omega_n (\dot{x}_0 + \omega_n x_0)} \right) \left\{ \omega_n \dot{x}_0 + \omega_n^2 x_0 \right\} \dots (E_4) \end{aligned}$$

For  $x_0 > 0$  and  $\dot{x}_0 > 0$ ,  $\left. \frac{d^2x}{dt^2} \right|_{t=t_m} < 0$

Hence  $t_m$  given by Eq. (E<sub>2</sub>) corresponds to a maximum of  $x(t)$ .

$$\begin{aligned} x \Big|_{t=t_m} &= \left\{ x_0 + (\dot{x}_0 + \omega_n x_0) \frac{\dot{x}_0}{\omega_n (\dot{x}_0 + \omega_n x_0)} \right\} e^{-\omega_n t_m} \\ &= \left( x_0 + \frac{\dot{x}_0}{\omega_n} \right) e^{-\left( \frac{\dot{x}_0}{\dot{x}_0 + \omega_n x_0} \right)} \dots (E_5) \end{aligned}$$

(2.88) Equation (2.92) can be expressed as  $\delta = \frac{1}{m} \ln \left( \frac{x_0}{x_m} \right)$   
 For half cycle,  $m = \frac{1}{2}$  and hence  $\delta = 2 \ln \left( \frac{x_0}{x_{\frac{1}{2}}} \right) = 2 \ln \left( \frac{1}{0.15} \right)$   
 $= 3.7942$

Necessary damping ratio  $\zeta_0$  is

$$\begin{aligned} \zeta_0 &= \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = \frac{3.7942^2}{\sqrt{4\pi^2 + 3.7942^2}} \\ &= 0.5169 \end{aligned}$$

(a)

If  $\zeta = \frac{3}{4} \zeta_0 = 0.3877$ , the overshoot can be determined by finding  $\delta$  from Eq. (2.85):

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi(0.3877)}{\sqrt{1-0.3877^2}} = 2.6427 = 2 \ln \left( \frac{x_0}{x_{\frac{1}{2}}} \right)$$

$$\ln \left( \frac{x_0}{x_{\frac{1}{2}}} \right) = 1.32135$$

$$x_{\frac{1}{2}} = x_0 / e^{1.32135} = 0.266775 x_0$$

$\therefore$  overshoot is 26.6775%

(b)

If  $\zeta = \frac{5}{4} \zeta_0 = 0.6461$ ,  $\delta$  is given by

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi(0.6461)}{\sqrt{1-(0.6461)^2}} = 5.3189 = 2 \ln \left( \frac{x_0}{x_{\frac{1}{2}}} \right)$$

$$\frac{x_0}{x_{\frac{1}{2}}} = 14.2888, \quad x_{\frac{1}{2}} = 0.0700 x_0$$

$$\therefore \text{overshoot} = 7\%$$

(2.89) (i) (a) Viscous damping, (b) Coulomb damping.

(iii) (a)  $\tau_d = 0.2 \text{ sec}$ ,  $f_d = 5 \text{ Hz}$ ,  $\omega_d = 31.416 \text{ rad/sec}$ .  
 (b)  $\tau_n = 0.2 \text{ sec}$ ,  $f_n = 5 \text{ Hz}$ ,  $\omega_n = 31.416 \text{ rad/sec}$ .

(ii) (a)  $\frac{x_i}{x_{i+1}} = e^{\zeta \omega_n \tau_d}$

$$\ln \left( \frac{x_i}{x_{i+1}} \right) = \ln 2 = 0.6931 = \frac{2 \pi \zeta}{\sqrt{1 - \zeta^2}}$$

$$\text{or } 39.9590 \zeta^2 = 0.4804 \quad \text{or } \zeta = 0.1096$$

Since  $\omega_d = \omega_n \sqrt{1 - \zeta^2}$ , we find

$$\omega_n = \frac{\omega_d}{\sqrt{1 - \zeta^2}} = \frac{31.416}{\sqrt{0.98798}} = 31.6065 \text{ rad/sec}$$

$$k = m \omega_n^2 = \left( \frac{500}{9.81} \right) (31.6065)^2 = 5.0916 (10^4) \text{ N/m}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2 m \omega_n}$$

$$\text{Hence } c = 2 m \omega_n \zeta = 2 \left( \frac{500}{9.81} \right) (31.6065) (0.1096) = 353.1164 \text{ N-s/m}$$

(b) From Eq. (2.116):

$$k = m \omega_n^2 = \frac{500}{9.81} (31.416)^2 = 5.0304 (10^4) \text{ N/m}$$

Using  $N = W = 500 \text{ N}$ ,

$$\mu = \frac{0.002 k}{4 W} = \frac{(0.002) (5.0304 (10^4))}{4 (500)} = 0.0503$$

(2.90) (a)  $c_c = 2 \sqrt{k m} = 2 \sqrt{5000 \times 50} = 1000 \text{ N-s/m}$

(b)  $c = c_c/2 = 500 \text{ N-s/m}$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = \sqrt{\frac{k}{m}} \sqrt{1 - \left( \frac{c}{c_c} \right)^2} = \sqrt{\frac{5000}{50}} \sqrt{1 - \left( \frac{1}{2} \right)^2}$$

$$= 8.6603 \text{ rad/sec}$$

(c) From Eq. (2.85),  $\delta = \frac{2\pi}{\omega_d} \left( \frac{c}{2m} \right) = \frac{2\pi}{8.6603} \left( \frac{500}{2 \times 50} \right)$   
 $= 3.6276$

2.91  $m = 2000 \text{ kg}$ ,  $v = \dot{x}_0 = 10 \text{ m/sec}$ ,  $k = 40,000 \text{ N/m}$   
 $c = 20,000 \text{ N-sec/m}$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{40000}{2000}} = 4.4721 \text{ rad/sec}$$

$$c_c = 2\sqrt{km} = 25,298.221 \text{ N-sec/m}$$

$$\gamma = c/c_c = 0.7906$$

$$\omega_d = \omega_n \sqrt{1-\gamma^2} = 4.4721 \sqrt{1-(0.7906)^2} = 2.7384 \text{ rad/sec}$$

$$\tau_d = 2\pi/\omega_d = 2.2945 \text{ sec}$$

(a) For  $x_0 = 0$  and  $\dot{x}_0 = 10 \text{ m/sec}$ , Eq. (2.72) gives

$$x(t) = e^{-\gamma\omega_n t} \frac{\dot{x}_0}{\omega_n \sqrt{1-\gamma^2}} \sin \omega_n \sqrt{1-\gamma^2} t$$

At  $x_{\max}$ ,  $\omega_n t \approx \frac{\pi}{2}$  and  $\sin \omega_n \sqrt{1-\gamma^2} t \approx 1$

$$\therefore x_{\max} \approx e^{-0.7906(\pi/2)} \cdot \left(\frac{10}{2.7384}\right) \cdot (1) = 1.0548 \text{ m}$$

(b)  $t = \tau_d/4 = 2.2945/4 = 0.5736 \text{ sec}$ .

2.92  $\omega_n = 200 \text{ cycles/min} = 20.944 \text{ rad/sec}$ ,  $\omega_d = 180 \text{ cycles/min} = 18.8496 \frac{\text{rad}}{\text{sec}}$   
 $J_0 = 0.2 \text{ kg-m}^2$

Since  $\omega_d = \sqrt{1-\gamma^2} \omega_n$ ,  $\gamma = \sqrt{1 - \left(\frac{\omega_d}{\omega_n}\right)^2} = \sqrt{1 - \left(\frac{18.8496}{20.944}\right)^2} = 0.4359$

$$= \frac{c_t}{(c_t)_{\text{cri}}} = \frac{c_t}{2J_0\omega_n}$$

$$c_t = 2J_0\omega_n\gamma = 2(0.2)(20.944)(0.4359)$$

$$= 3.6518 \text{ N-m-s/rad}$$

Eq. (2.72) can be used to obtain  $\theta(t)$  for  $\dot{\theta}_0 = 0$ ,  $\theta_0 = 2^\circ = 0.03491 \text{ rad}$  and  $t = \tau_d = \frac{2\pi}{\omega_d} = 0.3333 \text{ sec}$ ,

$$\theta(t) = e^{-\gamma\omega_n t} \theta_0 \left\{ \cos \omega_d t + \frac{\gamma\omega_n}{\omega_d} \sin \omega_d t \right\}$$

$$= e^{-(0.4359)(20.944)(0.3333)} (0.03491) \left\{ \cos 18.8496 \times 0.3333 \right.$$

$$\left. + \frac{0.4359 \times 20.944}{18.8496} \sin 18.8496 \times 0.3333 \right\}$$

$$= 0.001665 \text{ rad} = 0.09541^\circ$$

- 2.93** Assume that the bicycle and the boy fall as a rigid body by 5 cm at point A. Thus the mass ( $m_{eq}$ ) will be subjected to an initial downward displacement of 5 cm ( $t = 0$  assumed at point A):

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{(50000)(9.81)}{800}} = 24.7614 \text{ rad/sec}$$

$$c_c = 2 m \omega_n = 2 \left[ \frac{800}{9.81} \right] (24.7614) = 4038.5566 \text{ N-s/m}$$

$$\zeta = \frac{c}{c_c} = \frac{1000.0}{4038.5566} = 0.2476 \text{ (underdamping)}$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 24.7614 \sqrt{1 - 0.2476^2} = 23.9905 \text{ rad/sec}$$

Response of the system:

$$x(t) = X e^{-\zeta \omega_n t} \sin(\omega_d t + \phi)$$

$$\text{where } X = \left\{ x_0^2 + \left( \frac{\dot{x}_0 + \zeta \omega_n x_0}{\omega_d} \right)^2 \right\}^{\frac{1}{2}}$$

$$= \left\{ (0.05)^2 + \left( \frac{(0.2476)(24.7614)(0.05)}{23.9905} \right)^2 \right\}^{\frac{1}{2}} = 0.051607 \text{ m}$$

$$\text{and } \phi = \tan^{-1} \left( \frac{x_0 \omega_d}{\dot{x}_0 + \zeta \omega_n x_0} \right) = \tan^{-1} \left( \frac{0.05 (23.9905)}{0.2476 (24.7614) (0.05)} \right) = 75.6645^\circ$$

Thus the displacement of the boy (positive downward) in vertical direction is given by

$$x(t) = 0.051607 e^{-6.1309 t} \sin(23.9905 t + 75.6645^\circ) \text{ m}$$

**2.94**

Reduction in amplitude of viscously damped free vibration in one cycle = 1 cm

$$\frac{x_1}{x_2} = \frac{15}{14} = 1.07143; \ln \frac{x_1}{x_2} = 0.0689942 = \frac{2 \pi \zeta}{\sqrt{1 - \zeta^2}}$$

$$\text{i.e. } 0.0047602 = \frac{4 \pi^2 \zeta^2}{1 - \zeta^2}$$

$$\text{or } 0.0047602 = 39.48318 \zeta^2 \text{ or } \zeta = 0.0109801$$

(2.95)  $\tau_d = 0.2 \text{ sec} = \frac{2\pi}{\omega_d}$  ,  $\omega_d = 31.416 \text{ rad/sec}$   
 From Eq. (2.92)  $\delta = \frac{1}{50} \ln 10 = 0.04605$   

$$\gamma = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = \frac{0.04605}{\sqrt{(2\pi)^2 + 0.04605^2}} = 0.007329$$

When damping is neglected,

$$\omega_n = \omega_d / \sqrt{1 - \gamma^2} = 31.417 \text{ rad/sec} ; \zeta_n = \frac{2\pi}{\omega_n} = 0.19999 \text{ sec}$$

$$\text{Proportional decrease in period} = \left( \frac{0.2 - 0.19999}{0.2} \right) = 0.00005$$

(2.96) For critically damped system, Eq. (2.80) gives

$$x(t) = \{x_0 + (\dot{x}_0 + \omega_n x_0)t\} e^{-\omega_n t} \quad (E_1)$$

$$\dot{x}(t) = e^{-\omega_n t} \{ \dot{x}_0 - \dot{x}_0 \omega_n t - \omega_n^2 x_0 t \} \quad (E_2)$$

Let  $t_m =$  time at which  $x = x_{\max}$  and  $\dot{x} = 0$  occur.

Here  $x_0 = 0$  and  $\dot{x}_0 =$  initial recoil velocity. By setting

$\dot{x}(t) = 0$ , Eq. (E2) gives

$$t_m = \frac{\dot{x}_0}{\omega_n (\dot{x}_0 + \omega_n x_0)} = \frac{\dot{x}_0}{\omega_n \dot{x}_0} = \frac{1}{\omega_n} \quad (E_3)$$

With Eq. (E3) for  $t_m$  and  $x_0 = 0$ , (E1) gives

$$x_{\max} = \dot{x}_0 t_m e^{-\omega_n t_m} = \frac{\dot{x}_0 e^{-1}}{\omega_n}$$

$$\text{i.e. } \dot{x}_0 = \omega_n x_{\max} e = \omega_n (0.5) (2.7183) \quad (E_4)$$

$$\text{Using } \dot{x}_0 = 10 \text{ m/sec, } \omega_n = 10 / (0.5 * 2.7183) = 7.3575 \frac{\text{rad}}{\text{sec}}$$

When mass of gun is 500 kg,  
 the stiffness of the spring is

$$k = \omega_n^2 m = (7.3575)^2 (500) = 27,066.403 \text{ N/m}$$

2.97  $k = 5000 \text{ N/m}$ ,  $c_c = 0.2 \text{ N-s/mm} = 200 \text{ N-s/m}$   
 $= 2\sqrt{km} = 2\sqrt{5000m}$   
 $m = 2 \text{ kg}$   
 $\omega_n = \sqrt{k/m} = \sqrt{5000/2} = 50 \text{ rad/sec}$

Logarithmic decrement  $= \delta = \frac{2\pi\gamma}{\sqrt{1-\gamma^2}} = 2.0$

i.e.,  $\gamma = \frac{c}{c_c} = 0.3033$  and  $c = 0.3033(0.2) = 60.66 \text{ N-s/m}$

Assuming  $x_0 = 0$  and  $\dot{x}_0 = 1 \text{ m/s}$ ,

$$x(t) = e^{-\gamma\omega_n t} \frac{\dot{x}_0}{\omega_n \sqrt{1-\gamma^2}} \sin \sqrt{1-\gamma^2} \omega_n t$$

For  $x_{\max}$ ,  $\omega_n t \approx \pi/2$  and  $\sin \sqrt{1-\gamma^2} \omega_n t \approx 1$

$$\therefore x_{\max} \approx e^{-0.3033(\pi/2)} \frac{1}{50 \sqrt{1-0.3033^2}} (1) = 0.01303 \text{ m}$$

2.98 For an overdamped system, Eq. (2.81) gives

$$x(t) = e^{-\gamma\omega_n t} (C_1 e^{\omega_d t} + C_2 e^{-\omega_d t}) \quad (E_1)$$

Using the relations  $e^{\pm x} = \cosh x \pm \sinh x$  (E2)

Eq. (E1) can be rewritten as

$$x(t) = e^{-\gamma\omega_n t} (C_3 \cosh \omega_d t + C_4 \sinh \omega_d t) \quad (E_3)$$

where  $C_3 = C_1 + C_2$  and  $C_4 = C_1 - C_2$ .

Differentiating (E3),

$$\dot{x}(t) = e^{-\gamma\omega_n t} [C_3 \omega_d \sinh \omega_d t + C_4 \omega_d \cosh \omega_d t] - \gamma\omega_n e^{-\gamma\omega_n t} [C_3 \cosh \omega_d t + C_4 \sinh \omega_d t] \quad (E_4)$$

Initial conditions  $x(t=0) = x_0$  and  $\dot{x}(t=0) = \dot{x}_0$  with (E3) and (E4) give

$$C_3 = x_0, \quad C_4 = (\dot{x}_0 + \gamma\omega_n x_0) / \omega_d \quad (E_5)$$

Thus (E3) becomes

$$x(t) = x_0 e^{-\gamma\omega_n t} \left( \cosh \omega_d t + \frac{\gamma\omega_n}{\omega_d} \sinh \omega_d t \right) + \frac{\dot{x}_0}{\omega_d} e^{-\gamma\omega_n t} \sinh \omega_d t \quad (E_6)$$

(i) When  $\dot{x}_0 = 0$ , Eq. (E6) gives

$$x(t) = x_0 e^{-\gamma \omega_n t} \left( \cosh \omega_d t + \frac{\gamma \omega_n}{\omega_d} \sinh \omega_d t \right) \quad (E7)$$

since  $e^{-\gamma \omega_n t}$ ,  $\cosh \omega_d t$ ,  $\frac{\gamma \omega_n}{\omega_d}$  and  $\sinh \omega_d t$  do not change sign (always positive) and  $e^{-\gamma \omega_n t}$  approaches zero with increasing  $t$ ,  $x(t)$  will not change sign.

(ii) When  $x_0 = 0$ , Eq. (E6) gives

$$x(t) = \frac{\dot{x}_0}{\omega_d} e^{-\gamma \omega_n t} \sinh \omega_d t \quad (E8)$$

Here also,  $\omega_d$ ,  $e^{-\gamma \omega_n t}$  and  $\sinh \omega_d t$  do not change sign (always positive) and  $e^{-\gamma \omega_n t}$  approaches zero with increasing  $t$ ,  $x(t)$  will not change sign.

2.99

Newton's second law of motion:

$$\sum F = m \ddot{x} = -kx - c\dot{x} + F_f \quad (1)$$

$$\sum M = J_0 \ddot{\theta} = -F_f R \quad (2)$$

where  $F_f$  = friction force.

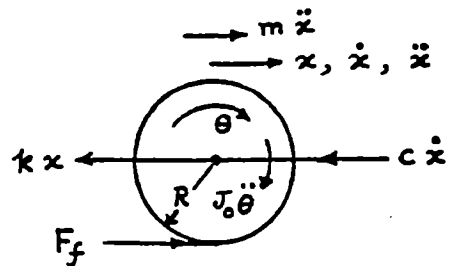
Using  $J_0 = \frac{m R^2}{2}$  and  $\ddot{\theta} = \frac{\ddot{x}}{R}$ , Eq. (2) gives

$$F_f = -\frac{1}{2R} (m R^2) \frac{\ddot{x}}{R} = -\frac{1}{2} m \ddot{x} \quad (3)$$

Substitution of Eq. (3) into (1) yields:

$$\frac{3}{2} m \ddot{x} + c \dot{x} + kx = 0 \quad (4)$$

The undamped natural frequency is:  $\omega_n = \sqrt{\frac{2k}{3m}} \quad (5)$



2.100 Newton's second law of motion: (measuring  $x$  from static equilibrium position of cylinder)

$$\sum F = m \ddot{x} = -kx - c\dot{x} - kx + F_f \quad (1)$$

$$\sum M = J_0 \ddot{\theta} = -F_f R \quad (2)$$

where  $F_f$  = friction force. Using  $J_0 = \frac{1}{2} m R^2$  and  $\ddot{\theta} = \frac{\ddot{x}}{R}$ , Eq. (2) gives

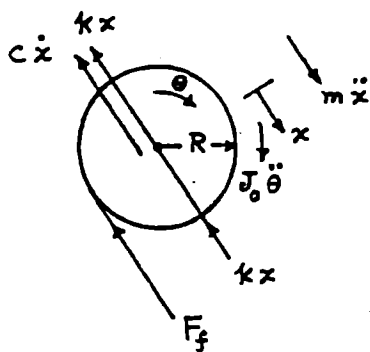
$$F_f = -\frac{1}{2} m \ddot{x} \quad (3)$$

Substitution of Eq. (3) into (1) gives

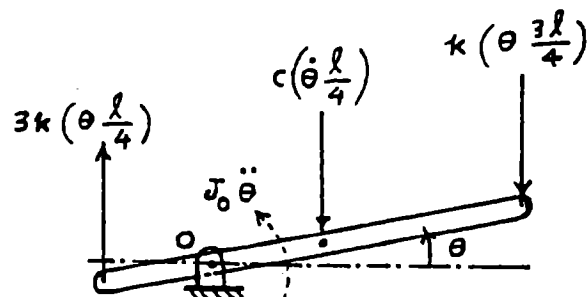
$$\frac{3}{2} m \ddot{x} + c\dot{x} + 2kx = 0 \quad (4)$$

Undamped natural frequency of the system:

$$\omega_n = \sqrt{\frac{4k}{3m}} \quad (4)$$



2.101



Consider a small angular displacement of the bar  $\theta$  about its static equilibrium position. Newton's second law gives:

$$\sum M = J_0 \ddot{\theta} = -k \left( \theta \frac{3\ell}{4} \right) \left( \frac{3\ell}{4} \right) - c \left( \dot{\theta} \frac{\ell}{4} \right) \left( \frac{\ell}{4} \right) - 3k \left( \theta \frac{\ell}{4} \right) \left( \frac{\ell}{4} \right)$$

$$\text{i.e., } J_0 \ddot{\theta} + \frac{c\ell^2}{16} \dot{\theta} + \frac{3}{4} k \ell^2 \theta = 0$$

where  $J_0 = \frac{7}{48} m \ell^2$ . The undamped natural frequency of torsional vibration is given by:

$$\omega_n = \sqrt{\frac{3 k \ell^2}{4 J_0}} = \sqrt{\frac{36 k}{7 m}}$$

**2.102** Let  $\delta x =$  virtual displacement given to cylinder. Virtual work done by various forces:

Inertia forces:  $\delta W_i = -(J_0 \ddot{\theta}) (\delta\theta) - (m \ddot{x}) \delta x = -(J_0 \ddot{\theta}) \left(\frac{\delta x}{R}\right) - (m \ddot{x}) \delta x$

Spring force:  $\delta W_s = -(k x) \delta x$

Damping force:  $\delta W_d = -(c \dot{x}) \delta x$

By setting the sum of virtual works equal to zero, we obtain:

$$-\frac{J_0}{R} \left(\frac{\ddot{x}}{R}\right) - m \ddot{x} - k x - c \dot{x} = 0 \quad \text{or} \quad \frac{3}{2} m \ddot{x} + c \dot{x} + k x = 0$$

**2.103** Let  $\delta x =$  virtual displacement given to cylinder from its static equilibrium position. Virtual works done by various forces:

Inertia forces:  $\delta W_i = -(J_0 \ddot{\theta}) \delta\theta - (m \ddot{x}) \delta x = -(J_0 \frac{\ddot{x}}{R}) \left(\frac{\delta x}{R}\right) - (m \ddot{x}) \delta x$

Spring force:  $\delta W_s = -(k x) \delta x - (k x) \delta x = -2 k x \delta x$

Damping force:  $\delta W_d = -(c \dot{x}) \delta x$

By setting the sum of virtual works equal to zero, we find

$$-\frac{J_0}{R} \frac{\ddot{x}}{R} - m \ddot{x} - 2 k x - c \dot{x} = 0 \quad (1)$$

Using  $J_0 = \frac{1}{2} m R^2$ , Eq. (1) can be rewritten as

$$\frac{3}{2} m \ddot{x} + c \dot{x} + 2 k x = 0 \quad (2)$$

**2.104** See figure given in the solution of Problem 2.101. Let  $\delta\theta$  be virtual angular displacement given to the bar about its static equilibrium position. Virtual works done by various forces:

Inertia force:  $\delta W_i = -(J_0 \ddot{\theta}) \delta\theta$

Spring forces:

$$\delta W_s = -\left(k \theta \frac{3 \ell}{4}\right) \left(\frac{3 \ell}{4} \delta\theta\right) - \left(3 k \theta \frac{\ell}{4}\right) \left(\frac{\ell}{4} \delta\theta\right) = -\left(\frac{3}{4} k \ell^2 \theta\right) \delta\theta$$

Damping force:  $\delta W_d = -\left(c \dot{\theta} \frac{\ell}{4}\right) \left(\frac{\ell}{4} \delta\theta\right)$

By setting the sum of virtual works equal to zero, we get the equation of motion as:

$$J_0 \ddot{\theta} + c \frac{\ell^2}{16} \dot{\theta} + \frac{3}{4} k \ell^2 \theta = 0$$

- 2.105 See solution of Problem 2.81. When wooden prism is given a displacement  $x$ , equation of motion becomes:  $m \ddot{x} + \text{restoring force} = 0$   
 where  $m = \text{mass of prism} = 40 \text{ kg}$  and  $\text{restoring force} = \text{weight of fluid displaced} = \rho_0 g a b x = \rho_0 (9.81) (0.4) (0.6) x = 2.3544 \rho_0 x$  where  $\rho_0$  is the density of the fluid. Thus the equation of motion becomes:

$$40 \ddot{x} + 2.3544 \rho_0 x = 0$$

$$\text{Natural frequency} = \omega_n = \sqrt{\frac{2.3544 \rho_0}{40}}$$

$$\text{Since } \tau_n = \frac{2\pi}{\omega_n} = 0.5, \text{ we find}$$

$$\omega_n = \frac{2\pi}{0.5} = 4\pi = \sqrt{\frac{2.3544 \rho_0}{40}}$$

$$\text{Hence } \rho_0 = 2682.8816 \text{ kg/m}^3.$$

- 2.106 Let  $x = \text{displacement of mass}$  and  $P = \text{tension in rope on the left of mass}$ .  
 Equations of motion:

$$\sum F = m \ddot{x} = -kx - P \text{ or } P = -m \ddot{x} - kx \quad (1)$$

$$\sum M = J_0 \ddot{\theta} = P r_2 - c (\dot{\theta} r_1) \quad (2)$$

Using Eq. (1) in (2), we obtain

$$-(m \ddot{x} + kx) r_2 - c \dot{\theta} r_1 = J_0 \ddot{\theta} \quad (3)$$

With  $x = \theta r_2$ , Eq. (3) can be written as:

$$(J_0 + m r_2^2) \ddot{\theta} + c r_1 \dot{\theta} + k r_2^2 \theta = 0 \quad (4)$$

For given data, Eq. (4) becomes

$$[5 + 10 (0.25)^2] \ddot{\theta} + c (0.1) \dot{\theta} + k (0.25)^2 \theta = 0$$

$$\text{or } 5.625 \ddot{\theta} + 0.1 c \dot{\theta} + 0.0625 k \theta = 0 \quad (5)$$

Since amplitude is reduced by 80% in 10 cycles,

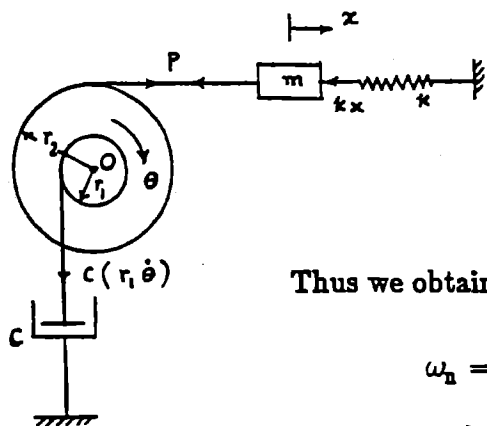
$$\frac{x_1}{x_{11}} = \frac{1.0}{0.2} = 5 = e^{10 \zeta \omega_n \tau_d}$$

$$\ln \frac{x_1}{x_{11}} = \ln 5 = 1.6094 = 10 \zeta \omega_n \tau_d \quad (6)$$

Since the natural frequency (assumed to be undamped torsional vibration frequency) is 5 Hz,  $\omega_n = 2\pi(5) = 31.416 \text{ rad/sec}$ . Also

$$\tau_d = \frac{1}{f_d} = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{0.2}{\sqrt{1-\zeta^2}} \quad (7)$$

Eq. (6) gives



$$1.6094 = 10 \zeta (31.416) \left( \frac{0.2}{\sqrt{1-\zeta^2}} \right) = \frac{62.832 \zeta}{\sqrt{1-\zeta^2}}$$

$$\text{i.e., } \sqrt{1-\zeta^2} = \frac{62.832}{1.6094} \zeta = 39.0406 \zeta$$

$$\text{i.e., } \zeta = 0.02561$$

Thus we obtain:

$$\omega_n = \sqrt{\frac{0.0625 k}{5.625}} = 31.416 \text{ or } k = 8.8827 (10^4) \text{ N/m}$$

$$\zeta = 0.02561 = \frac{c}{c_c} = \frac{c}{2 m_{eq} \omega_n} = \frac{0.10 c}{2 (5.625) (31.416)}$$

$$\text{or } c = 90.5134 \text{ N-s/m}$$

2.107

$$\text{Torque} = 2 \times 10^{-3} \text{ N-m}$$

$$\text{angle} = 50^\circ = 80 \text{ divisions}$$

For a torsional system, Eq. (2.84) gives

$$\frac{\theta_1}{\theta_2} = e^{\zeta \omega_n \tau_d} \quad (E_1)$$

(b) For one cycle,  $\tau_d = 2 \text{ sec}$  and (E<sub>1</sub>) gives

$$\frac{80}{5} = e^{2 \zeta \omega_n} \quad \text{or} \quad \zeta \omega_n = \frac{1}{2} \ln(16) = 1.3863 \quad (E_2)$$

$$\text{Since } \tau_d = \frac{2\pi}{\sqrt{\omega_n^2 - \zeta^2 \omega_n^2}},$$

$$\omega_n^2 = \frac{(2\pi)^2}{\tau_d^2} + \zeta^2 \omega_n^2 = \frac{4\pi^2}{4} + 1.3863^2 = 11.7915 \quad (E_3)$$

$$\text{i.e., } \omega_n = 3.4339 \text{ rad/sec}$$

(d) Since angular displacement of rotor under applied torque

$$= 50^\circ = 0.8727 \text{ rad,}$$

$$k_t = \text{torque/angular displacement} = 2 \times 10^{-3} / 0.8727$$

$$= 2.2917 \times 10^{-3} \text{ N-m/rad} \quad (E_4)$$

(a) Mass moment of inertia of rotor is

$$J_o = \frac{k_t}{\omega_n^2} = 2.2917 \times 10^{-3} / 11.7915 = 1.9436 \times 10^{-4} \text{ N-m-s}^2 \quad (E_5)$$

$$(c) c_t = 2 J_o \zeta \omega_n \quad (E_6)$$

$$\text{Eqs. (E}_2\text{) and (E}_3\text{) give } \zeta = \frac{\zeta \omega_n}{\omega_n} = \frac{1.3863}{3.4339} = 0.4037$$

$$\text{Eq. (E}_6\text{) gives } c_t = 5.3887 \times 10^{-4} \text{ N-m-s/rad.}$$

2.108

(a)  $m = 10 \text{ kg}$   
 $c = 150 \text{ N-s/m}$   
 $k = 1000 \text{ N/m}$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}}$$

$$= 10 \text{ rad/s}$$

$$\zeta = \frac{c}{2m\omega_n}$$

$$= \frac{150}{2(10)(10)} = 0.75$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

$$= 10 \sqrt{1 - 0.75^2}$$

$$= 6.61438 \text{ rad/s}$$

(under-damped)

(b)  $m = 10 \text{ kg}$   
 $c = 200 \text{ N-s/m}$   
 $k = 1000 \text{ N/m}$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$= 10 \text{ rad/s}$$

$$\zeta = \frac{c}{2m\omega_n}$$

$$= \frac{200}{2(10)(10)} = 1.0$$

$$\omega_d = 10 \sqrt{1 - 1.0^2}$$

$$= 0$$

(critically-damped)

(c)  $m = 10 \text{ kg}$   
 $c = 250 \text{ N-s/m}$   
 $k = 1000 \text{ N/m}$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$= 10 \text{ rad/s}$$

$$\zeta = \frac{c}{2m\omega_n}$$

$$= \frac{250}{2(10)(10)} = 1.25$$

$$\omega_d = \text{not applicable}$$

(over-damped)

2.109

(a) Underdamped system: Response: Eq. (2.70)

$$X_0 = \left\{ x_0^2 + \left( \frac{\dot{x}_0 + \zeta \omega_n x_0}{\omega_d} \right)^2 \right\}^{1/2} \quad (\text{E.1})$$

Using  $x_0 = 0.1$ ,  $\dot{x}_0 = 10$ ,  $\zeta = 0.75$ ,  $\omega_n = 10$ ,  $\omega_d = 6.61438$ ,  
 Eq. (E.1) gives  $X_0 = 1.62832 \text{ m}$ .

$$\phi_0 = \tan^{-1} \left( - \frac{\dot{x}_0 + \zeta \omega_n x_0}{\omega_d x_0} \right)$$

$$= \tan^{-1} \left( - \frac{10 + 0.75(10)(0.1)}{6.61438(0.1)} \right) = -86.47908^\circ$$

$$= -1.50935 \text{ rad}$$

Eq. (2.70) gives:

$$x(t) = 1.62832 e^{-7.5t} \cos(6.61438t + 1.50935) \text{ m}$$

(b) Critically damped system: Response: Eq. (2.80)

$$x(t) = \left\{ x_0 + (\dot{x}_0 + \omega_n x_0) t \right\} e^{-\omega_n t}$$

$$= \{0.1 + (10 + 10 * 0.1) t\} e^{-10 t}$$

$$= (0.1 + 11 t) e^{-10 t} \text{ m}$$

(c) overdamped system: Response: Eq. (2.81)

Using  $\sqrt{\zeta^2 - 1} = \sqrt{1.25^2 - 1} = 0.75$ , we obtain

$$C_1 = \frac{x_0 \omega_n \{\zeta + \sqrt{\zeta^2 - 1}\} + \dot{x}_0}{2 \omega_n \sqrt{\zeta^2 - 1}} \quad \text{Eq. (2.82)}$$

$$= \frac{0.1 (10) \{1.25 + 0.75\} + 10}{2 (10) (0.75)} = 0.8$$

$$C_2 = \frac{-x_0 \omega_n \{\zeta - \sqrt{\zeta^2 - 1}\} - \dot{x}_0}{2 \omega_n \sqrt{\zeta^2 - 1}} \quad \text{Eq. (2.82)}$$

$$= \frac{-0.1 (10) \{1.25 - 0.75\} - 10}{2 (10) (0.75)} = -0.7$$

Eq. (2.81) gives

$$x(t) = C_1 e^{(-\zeta + \sqrt{\zeta^2 - 1}) \omega_n t} + C_2 e^{(-\zeta - \sqrt{\zeta^2 - 1}) \omega_n t}$$

$$= 0.8 e^{(-1.25 + 0.75)(10)t} - 0.7 e^{(-1.25 - 0.75)(10)t}$$

$$= 0.8 e^{-5t} - 0.7 e^{-20t} \text{ m}$$

2.110 Energy dissipated in a cycle of motion,  $x(t) = X \sin \omega_d t$ , is given by

$$\Delta W = \pi c \omega_d X^2 \quad \text{(E.1)}$$

$$(a) \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 10 \text{ rad/s}$$

$$\zeta = \frac{c}{2 m \omega_n} = \frac{50}{2 (10) (10)} = 0.25$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 10 \sqrt{1 - 0.25^2} = 9.682458 \text{ rad/s}$$

For  $X = 0.2 \text{ m}$ , Eq. (E.1) gives

$$\Delta W = \pi (50) (9.682458) (0.2^2) = 60.83682 \text{ Joules}$$

$$(b) \omega_n = \sqrt{\frac{k}{m}} = 10 \text{ rad/s}$$

$$\zeta = \frac{c}{2m\omega_n} = \frac{150}{2(10)(10)} = 0.75$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 10 \sqrt{1 - 0.75^2} = 6.614378 \text{ rad/s}$$

For  $X = 0.2 \text{ m}$ , Eq. (E.1) gives

$$\Delta W = \pi (150) (6.614378) (0.2^2) = 124.678385 \text{ Joules}$$

$$(2.111) \quad m = 20 \text{ kg}, \quad k = 4000 \text{ N/m}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{20}} = 14.1421 \text{ rad/sec}$$

Amplitudes of successive cycles : 50, 45, 40, 35 mm

Amplitudes of successive cycles diminish by 5 mm =  $5 \times 10^{-3} \text{ m}$

System has Coulomb damping.

$$\frac{4\mu N}{k} = 5 \times 10^{-3} \Rightarrow \mu N = \left\{ \frac{(5 \times 10^{-3})(4000)}{4} \right\} = 5 \text{ N}$$

= damping force

Frequency of damped vibration = 14.1421 rad/sec.

$$(2.112) \quad m = 20 \text{ kg}, \quad k = 10000 \text{ N/m}, \quad \frac{4\mu N}{k} = \frac{150 - 100}{4} \text{ mm} = 12.5 \times 10^{-3} \text{ m}$$

$$\mu = \frac{(12.5 \times 10^{-3})(10000)}{4(20 \times 9.81)} = 0.1593$$

$$\text{Time elapsed} = 4\tau_n = 4 \times \frac{2\pi}{\omega_n} = 8\pi \sqrt{\frac{m}{k}} = 1.124 \text{ sec}$$

$$(2.113) \quad m = 10 \text{ kg}, \quad k = 3000 \text{ N/m}, \quad \mu = 0.12, \quad X = 100 \text{ mm}$$

$$\frac{4\mu N}{k} = \frac{4(0.12)(10 \times 9.81)}{3000} = 0.0157 \text{ m} = 15.7 \text{ mm}$$

As  $6\left(\frac{4\mu N}{k}\right) = 94.2 \text{ mm}$ , mass comes to rest at  $(100 - 94.2) = 5.8 \text{ mm}$

$$(2.114) \quad mg = 25 \text{ N}, \quad k = 1000 \text{ N/m}, \quad \text{damping force} = \text{constant}$$

mass released with  $x_0 = 10 \text{ cm}$  and  $\dot{x}_0 = 0$ .

$$\text{Static deflection of spring due to self weight of mass} = \frac{25}{1000}$$

$$= 0.025 \text{ m}$$

$$\text{at } t = 0: \quad x = 0.1 \text{ m}, \quad \dot{x} = 0$$

$$x_0 = 0.1$$

$$x_1 = x_0 - 2 \frac{\mu N}{k}, \quad x_2 = x_0 - \frac{4 \mu N}{k}$$

$$x_3 = x_0 - \frac{6 \mu N}{k}, \quad x_4 = x_0 - \frac{8 \mu N}{k} = 0$$

i.e.,  $x_0 = \frac{8 \mu N}{k} = 0.1$

$$\text{Magnitude of damping force} = \mu N = \frac{x_0 k}{8} = \frac{(0.1)(1000)}{8}$$

$$= 12.5 \text{ N}$$

2.115  $m = 20 \text{ kg}, \quad k = 10,000 \text{ N/m}, \quad \mu N = 50 \text{ N}, \quad x_0 = 0.05 \text{ m}$

(a) Number of half cycles elapsed before mass comes to rest ( $r$ ) is given by:

$$r \geq \left\{ \frac{x_0 - \frac{\mu N}{k}}{2 \frac{\mu N}{k}} \right\} = \frac{0.05 - \left( \frac{50}{10000} \right)}{2 \left( \frac{50}{10000} \right)} = 4.5$$

$$\therefore r = 5$$

(b) Time elapsed before mass comes to rest:

$$t_p = 2\pi \sqrt{\frac{m}{k}} = 2\pi \sqrt{\frac{20}{10000}} = 0.2810 \text{ sec}$$

$$\text{Time taken} = (2.5 \text{ cycles}) t_p = 0.7025 \text{ sec}$$

(c) Final extension of spring after 5 half-cycles:

$$x_5 = 0.05 - 5 \left( \frac{2 \mu N}{k} \right) = 0.05 - 5 \left( 2 * \frac{50}{10000} \right) = 0$$

(displacement from static equilibrium position = 0)

$$\text{But static deflection} = \frac{mg}{k} = \frac{20 * 9.81}{10000} = 0.01962 \text{ m}$$

$$\therefore \text{Final extension of spring} = 1.9620 \text{ cm.}$$

2.116 (a) Equation of motion for angular oscillations of pendulum:

$$J_0 \ddot{\theta} + mgl \sin \theta \pm mg \mu \frac{d}{2} \cos \theta = 0$$

$$\text{For small angles, } \ddot{\theta} + \frac{mgl}{J_0} \left( \theta \pm \frac{\mu d}{2l} \right) = 0$$

This shows that the angle of swing decreases by  $\left( \frac{\mu d}{2l} \right)$  in each quarter cycle.

(b) For motion from right to left:

$$\theta(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t + \frac{\mu d}{2l}$$

$$\text{where } \omega_n = \sqrt{\frac{mgl}{J_0}}$$

$$\text{Let } \theta(t=0) = \theta_0 \text{ and } \dot{\theta}(t=0) = 0. \text{ Then } A_1 = \theta_0 - \frac{\mu d}{2l}, \quad A_2 = 0$$

$$\theta(t) = \left(\theta_0 - \frac{\mu d}{2l}\right) \cos \omega_n t + \frac{\mu d}{2l}$$

For motion from left to right:

$$\theta(t) = A_3 \cos \omega_n t + A_4 \sin \omega_n t - \frac{\mu d}{2l}$$

At  $\omega_n t = \pi$ ,  $\theta = -\theta_0 + \frac{2\mu d}{2l}$ ,  $\dot{\theta} = 0$  from previous solution.

$$A_3 = \theta_0 - \frac{3\mu d}{2l}, \quad A_4 = 0$$

$$\theta(t) = \left(\theta_0 - \frac{3\mu d}{2l}\right) \cos \omega_n t - \frac{\mu d}{2l}$$

(c) The motion ceases when  $\left(\theta_0 - n \frac{4\mu d}{2l}\right) < \frac{\mu d}{2l}$   
where  $n$  denotes the number of cycles.

2.117  $x(t) = X \sin \omega t$  (under sinusoidal force  $F_0 \sin \omega t$ )

Damping force =  $\mu N$

Total displacement per cycle =  $4X$

$$\text{Energy dissipated per cycle} = \Delta W = 4\mu N X \quad (E_1)$$

If  $c_{eq}$  = equivalent viscous damping constant, energy dissipated per cycle is given by  $E_2$ . (2.98):

$$\Delta W = \pi c_{eq} \omega X^2 \quad (E_2)$$

Equating  $(E_1)$  and  $(E_2)$  gives

$$c_{eq} = \frac{4\mu N X}{\pi \omega X^2} = \frac{4\mu N}{\pi \omega X} \quad (E_3)$$

2.118 Due to viscous damping:

$$\delta = \ln \left( \frac{X_m}{X_{m+1}} \right) = 2\pi \zeta$$

$$\begin{aligned} \zeta_1 &= \text{percent decrease in amplitude per cycle at } X_m \\ &= 100 \left( \frac{X_m - X_{m+1}}{X_m} \right) = 100 \left( 1 - \frac{X_{m+1}}{X_m} \right) = 100 \left( 1 - e^{-2\pi \zeta} \right) \end{aligned}$$

Due to Coulomb damping:

$$\begin{aligned} \zeta_2 &= \text{percent decrease in amplitude per cycle at } X_m \\ &= 100 \left( \frac{X_m - X_{m+1}}{X_m} \right) = 100 \left( \frac{4\mu N}{\pi X_m} \right) \end{aligned}$$

When both types of damping are present:

$$\zeta_1 + \zeta_2 \Big|_{X_m = 20 \text{ mm}} = 2 \quad ; \quad \zeta_1 + \zeta_2 \Big|_{X_m = 10 \text{ mm}} = 3$$

i.e.,

$$100 (1 - e^{-2\pi\gamma}) + \frac{400}{0.02} \left( \frac{\mu N}{k} \right) = 2$$

$$100 (1 - e^{-2\pi\gamma}) + \frac{400}{0.01} \left( \frac{\mu N}{k} \right) = 3$$

The solution of these equations gives

$$50 (1 - e^{-2\pi\gamma}) = 0.5 \quad \text{and} \quad \frac{\mu N}{k} = 0.5 \times 10^{-6} \text{ m}$$

2.119

Coulomb damping.

- (a) Natural frequency =  $\omega_n = \frac{2\pi}{T_n} = \frac{2\pi}{1} = 6.2832 \text{ rad/sec}$ . Reduction in amplitude in each cycle:

$$\begin{aligned} &= \frac{4\mu N}{k} = 4\mu g \frac{m}{k} = \frac{4\mu g}{\omega_n^2} = 4\mu \left( \frac{9.81}{6.2832^2} \right) \\ &= 0.9940 \mu = \frac{0.5}{100} = 0.005 \text{ m} \end{aligned}$$

Kinetic coefficient of friction =  $\mu = 0.00503$ 

- (b) Number of half-cycles executed ( $r$ ) is:

$$r \geq \frac{(x_0 - \frac{\mu N}{k})}{(\frac{2\mu N}{k})} = \frac{(x_0 - \frac{\mu g}{\omega_n^2})}{(\frac{2\mu g}{\omega_n^2})}$$

$$\geq \frac{\left( 0.1 - \frac{0.00503 (9.81)}{6.2832^2} \right)}{\left( \frac{2 (0.00503) (9.81)}{6.2832^2} \right)}$$

$$\geq 39.5032$$

$$\geq 40$$

Thus the block stops oscillating after 20 cycles.

$$2.120 \quad \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10,000}{5}} = 44.721359 \text{ rad/s}$$

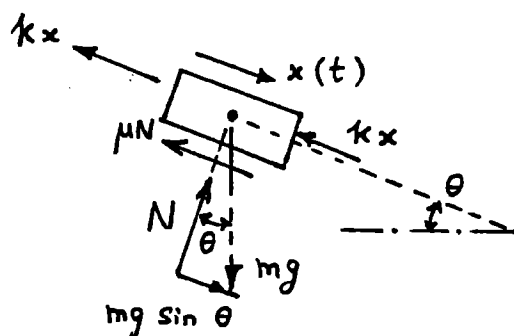
$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{44.721359} = 0.140497 \text{ s}$$

$$\begin{aligned} \text{Time taken to complete 10 cycles} &= 10 \tau_n \\ &= 1.40497 \text{ s} \end{aligned}$$

2.121

$$(a) \quad \theta = 30^\circ$$

$$N = mg \cos \theta$$



Case 1: When  $x = +$  and  $\dot{x} = +$  or  $x = -$  and  $\dot{x} = +$  :

$$m\ddot{x} = -2kx - \mu N + mg \sin \theta$$

$$\text{or} \quad m\ddot{x} + 2kx = -\mu mg \cos \theta + mg \sin \theta \quad (E.1)$$

Case 2: When  $x = +$  and  $\dot{x} = -$  or  $x = -$  and  $\dot{x} = -$  :

$$m\ddot{x} = -2kx + \mu N + mg \sin \theta$$

$$\text{or} \quad m\ddot{x} + 2kx = \mu mg \cos \theta + mg \sin \theta \quad (E.2)$$

Eqs. (E.1) and (E.2) can be written as a single equation as:

$$m\ddot{x} + \mu mg \cos \theta \operatorname{sgn}(\dot{x}) + 2kx + mg \sin \theta = 0 \quad (E.3)$$

$$(b) \quad x_0 = 0.1 \text{ m}, \quad \dot{x}_0 = 5 \text{ m/s}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{20}} = 7.071068 \text{ rad/s}$$

Solution of Eq. (E.1):

$$\begin{aligned} x(t) &= A_1 \cos \omega_n t + A_2 \sin \omega_n t - \frac{\mu mg \cos \theta}{k} \\ &\quad + \frac{mg \sin \theta}{k} \end{aligned} \quad (E.4)$$

Solution of Eq. (E.2):

$$x(t) = A_3 \cos \omega_n t + A_4 \sin \omega_n t + \frac{\mu mg \cos \theta}{k} + \frac{mg \sin \theta}{k} \quad (E.5)$$

Using the initial conditions in each half cycle, the constants  $A_1$  and  $A_2$  or  $A_3$  and  $A_4$  are to be found. For example, in the first half cycle, the motion starts from left toward right with  $x_0 = 0.1$  and  $\dot{x}_0 = 5$ . These values can be used in Eq. (E.4) to find  $A_1$  and  $A_2$ .

- 2.122 Friction force  $= \mu N = 0.2 (5) = 1 \text{ N}$ .  $k = \frac{25}{0.10} = 250 \text{ N/m}$ . Reduction in amplitude in each cycle  $= \frac{4 \mu N}{k} = \frac{4 (1)}{250} = 0.016 \text{ m}$ . Number of half-cycles executed before the motion ceases ( $r$ ):

$$r \geq \left( \frac{x_0 - \frac{\mu N}{k}}{\frac{2 \mu N}{k}} \right) = \frac{0.1 - 0.004}{0.008} \geq 12$$

Thus after 6 cycles, the mass stops at a distance of  $0.1 - 6 (0.016) = 0.004 \text{ m}$  from the unstressed position of the spring.

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{250 (9.81)}{5}} = 22.1472 \text{ rad/sec}$$

$$\tau_n = \frac{2 \pi}{\omega_n} = 0.2837 \text{ sec}$$

Thus total time of vibration  $= 6 \tau_n = 1.7022 \text{ sec}$ .

- 2.123 Energy dissipated in each full load cycle is given by the area enclosed by the hysteresis loop.

The area can be found by counting the squares enclosed by the hysteresis loop. In Fig. 2.100, the number of squares is  $\approx 33$ . Since each square  $= \frac{100 \times 1}{1000} = 0.1 \text{ N-m}$ , the energy dissipated in a cycle is

$$\Delta W = 33 \times 0.1 = 3.3 \text{ N-m} = \pi k \beta X^2$$

Since the maximum deflection  $= X = 4.3 \text{ mm}$ , and the slope of the force-deflection curve is

$$k = \frac{1800 \text{ N}}{11 \text{ mm}} = 1.6364 \times 10^5 \text{ N/m},$$

the hysteresis damping constant  $\beta$  is given by

$$\beta = \frac{\Delta W}{\pi k X^2} = \frac{3.3}{\pi (1.6364 \times 10^5) (0.0043)^2} = 0.3472$$

$$\delta = \pi \beta = \text{logarithmic decrement} = \pi (0.3472) = 1.0908$$

$$\text{Equivalent viscous damping ratio} = \zeta_{eq} = \beta/2 = 0.1736.$$


---

$$(2.124) \quad \frac{X_j}{X_{j+1}} = \frac{2 + \pi \beta}{2 - \pi \beta} = 1.1 \quad , \quad \beta = 0.03032$$

$$c_{eq} = \beta \sqrt{m k} = 0.03032 \sqrt{1 \times 2} = 0.04288 \text{ N-s/m}$$

$$\Delta W = \pi k \beta X^2 = \pi (2) (0.03032) \left(\frac{10}{1000}\right)^2 = 19.05 \times 10^{-6} \text{ N-m}$$


---

$$(2.125) \quad \text{Logarithmic decrement} = \delta = \ln \left( \frac{X_j}{X_{j+1}} \right) \approx \pi \beta$$

$$\text{For } n \text{ cycles,} \quad \delta = \frac{1}{n} \ln \left( \frac{X_0}{X_n} \right) \approx \pi \beta$$

$$\frac{1}{100} \ln \left( \frac{30}{20} \right) = 0.004055 = \pi \beta$$

$$\beta = 0.001291$$


---

$$(2.126) \quad \delta = \frac{1}{n} \ln \frac{X_0}{X_n}$$

$$= \frac{1}{100} \ln \frac{25}{10} = \frac{1}{100} \ln 2.5 = 0.0091629$$

$$\delta = \pi \frac{h}{k}$$

$$\text{or } h = \frac{\delta k}{\pi} = \frac{(0.0091629) (200)}{\pi} = 0.583327 \text{ N/m}$$


---

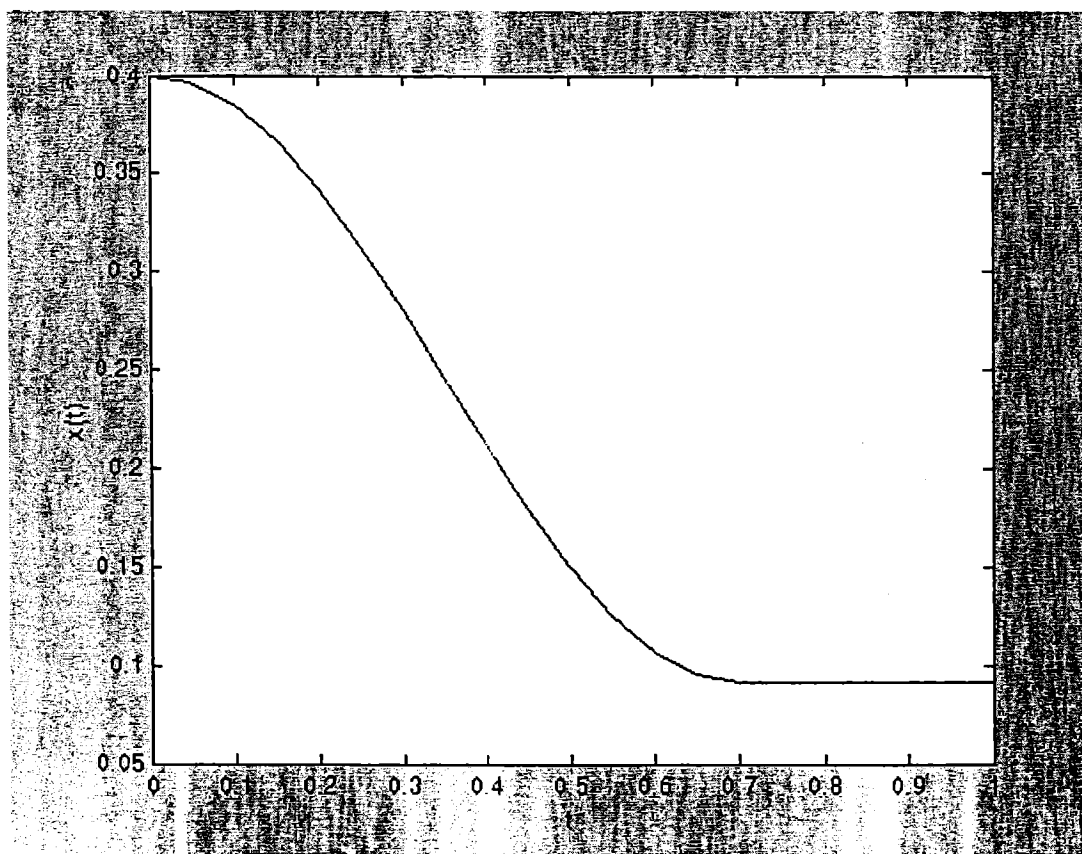
2.127

```

% Ex2_127.m
% This program will use dfunc1.m
tspan = [0: 0.05: 8];
x0 = [0.4; 0.0];
[t, x] = ode23('dfunc1', tspan, x0);
plot(t, x(:, 1));
xlabel('t');
ylabel('x(t)');

% dfunc1.m
function f = dfunc1(t, x)
u = 0.5;
k = 100;
m = 5;
f = zeros(2,1);
f(1) = x(2);
f(2) = -u * 9.81 * sign(x(2)) - k * x(1) / m;

```



2.128

```

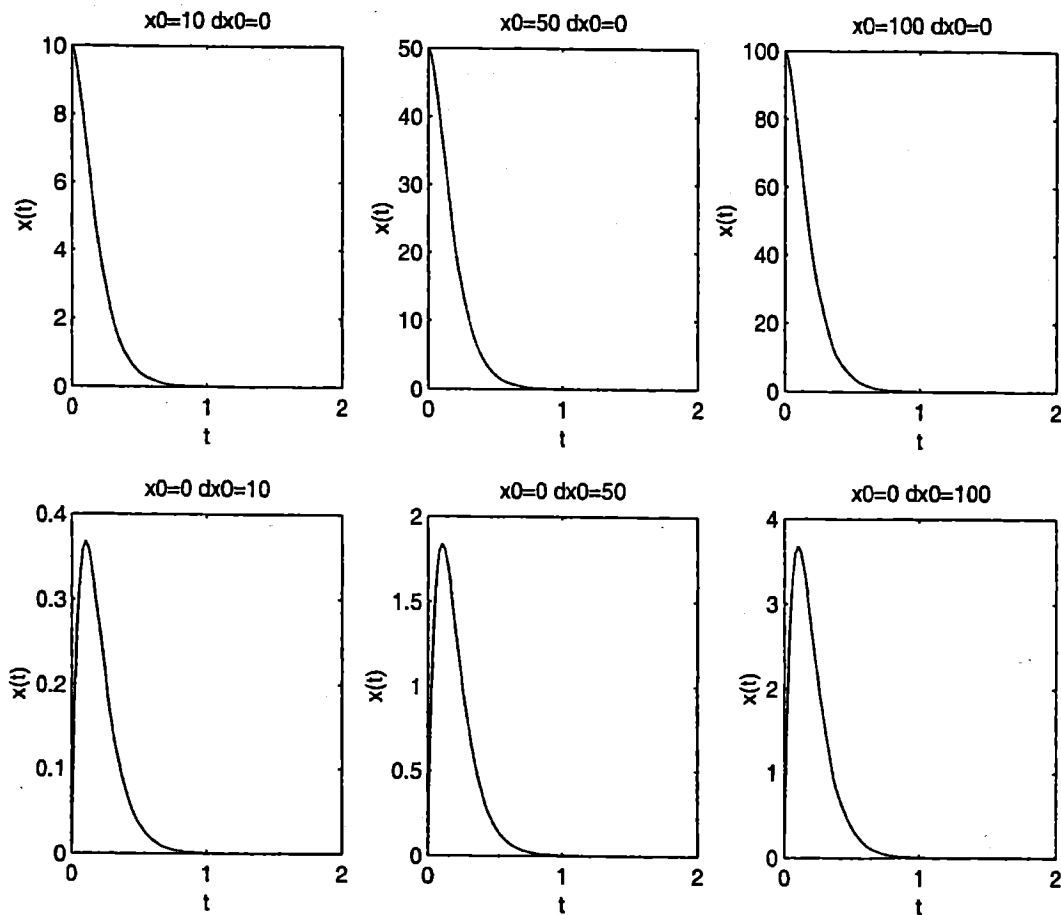
%Ex2_128.m
wn = 10;
dx0 = 0;
x0 = 10;
for i = 1:101
    t(i) = 2*(i-1)/100;
    x1(i) = (x0 + ( dx0 + wn*x0)*t(i) ) * exp(-wn*t(i));
end

```

```

x0 = 50;
for i = 1:101
    t(i) = 2*(i-1)/100;
    x2(i) = (x0 + ( dx0 + wn*x0)*t(i) )*exp(-wn*t(i));
end
x0 = 100;
for i = 1:101
    t(i) = 2*(i-1)/100;
    x3(i) = (x0 + ( dx0 + wn*x0)*t(i) )*exp(-wn*t(i));
end
x0 = 0;
dx0 = 10;
for i = 1:101
    t(i) = 2*(i-1)/100;
    x4(i) = (x0 + ( dx0 + wn*x0)*t(i) )*exp(-wn*t(i));
end
dx0 = 50;
for i = 1:101
    t(i) = 2*(i-1)/100;
    x5(i) = (x0 + ( dx0 + wn*x0)*t(i) )*exp(-wn*t(i));
end
dx0 = 100;
for i = 1:101
    t(i) = 2*(i-1)/100;
    x6(i) = (x0 + ( dx0 + wn*x0)*t(i) )*exp(-wn*t(i));
end
subplot(231);
plot(t,x1);
title('x0=10 dx0=0');
xlabel('t');
ylabel('x(t)');
subplot(232);
plot(t,x2);
title('x0=50 dx0=0');
xlabel('t');
ylabel('x(t)');
subplot(233);
plot(t,x3);
title('x0=100 dx0=0');
xlabel('t');
ylabel('x(t)');
subplot(234);
plot(t,x4);
title('x0=0 dx0=10');
xlabel('t');
ylabel('x(t)');
subplot(235);
plot(t,x5);
title('x0=0 dx0=50');
xlabel('t');
ylabel('x(t)');
subplot(236);
plot(t,x6);
title('x0=0 dx0=100');
xlabel('t');
ylabel('x(t)');

```

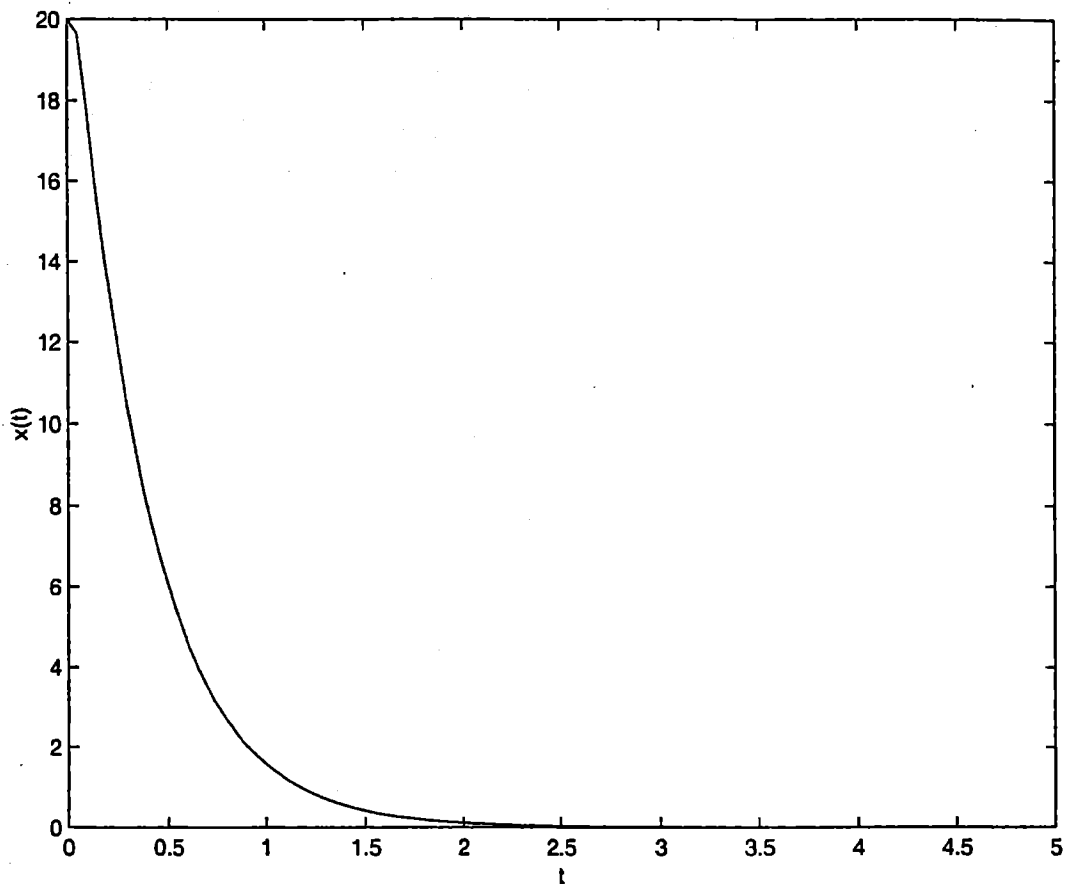


2.129

```

%Ex2_129.m
wn = 10;
zeta = 2.0;
dx0 = 50;
x0 = 20;
c1 = ( x0*wn*( zeta + sqrt(zeta^2-1) ) + dx0 )/( 2*wn*sqrt(zeta^2-1) );
c2 = ( -x0*wn*( zeta - sqrt(zeta^2-1) ) - dx0 )/( 2*wn*sqrt(zeta^2-1) );
for i = 1:101
    t(i) = 5*(i-1)/100;
    x(i) = c1*exp( (-zeta + sqrt(zeta^2-1)) *wn*t(i) ) ...
          + c2*exp( (-zeta - sqrt(zeta^2-1)) *wn*t(i) );
end
plot(t,x);
xlabel('t');
ylabel('x(t)');

```



2.130

```
Results of Ex2_130_a
*****
>> program2
Free vibration analysis
of a single degree of freedom analysis
```

Data:

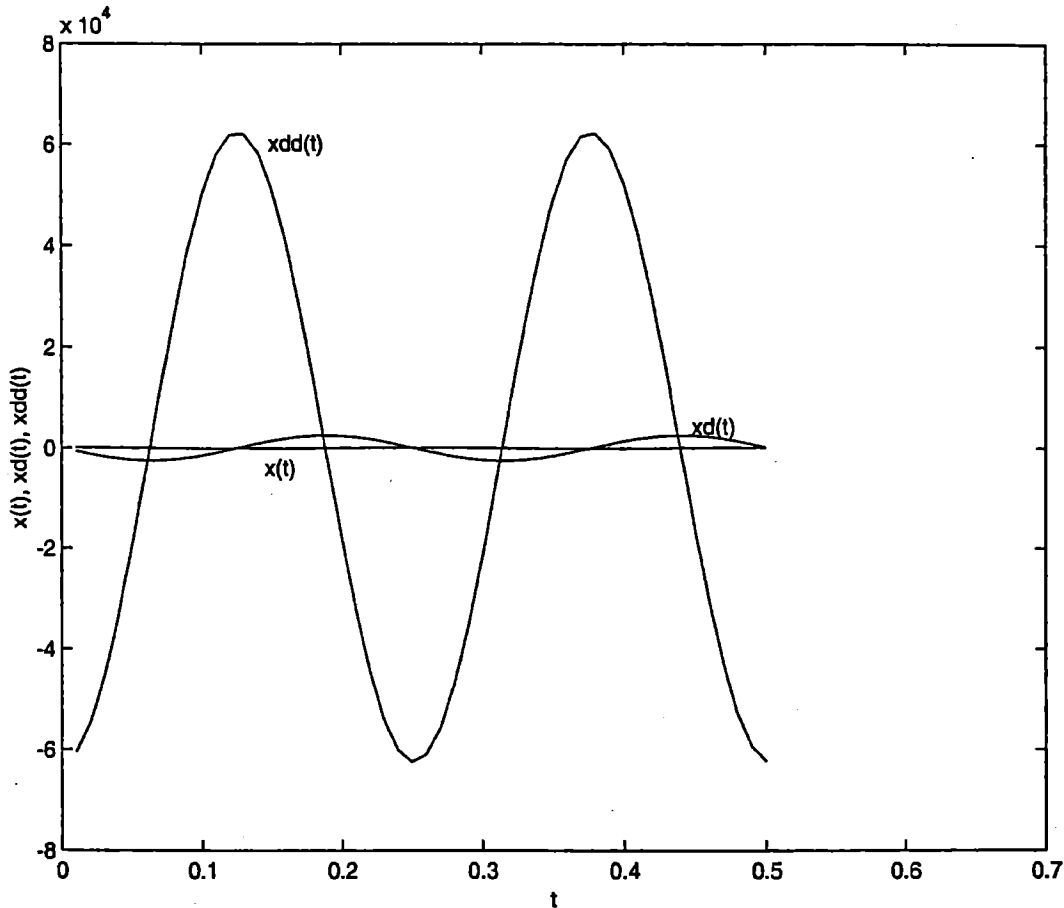
```
m=      4.00000000e+000
k=      2.50000000e+003
c=      0.00000000e+000
x0=      1.00000000e+002
xd0=     -1.00000000e+001
n=       50
delt=    1.00000000e-002
```

system is undamped

Results:

i	time(i)	x(i)	xd(i)	xdd(i)
1	1.000000e-002	9.679228e+001	-6.282079e+002	-6.049518e+004
2	2.000000e-002	8.756649e+001	-1.207348e+003	-5.472905e+004
3	3.000000e-002	7.289623e+001	-1.711420e+003	-4.556014e+004
4	4.000000e-002	5.369364e+001	-2.109085e+003	-3.355853e+004
5	5.000000e-002	3.115264e+001	-2.375618e+003	-1.947040e+004
6	6.000000e-002	6.674722e+000	-2.494445e+003	-4.171701e+003
⋮				

44	4.400000e-001	8.425659e-001	2.499931e+003	-5.266037e+002
45	4.500000e-001	2.555609e+001	2.417001e+003	-1.597256e+004
46	4.600000e-001	4.868066e+001	2.183793e+003	-3.042541e+004
47	4.700000e-001	6.877850e+001	1.814807e+003	-4.298656e+004
48	4.800000e-001	8.460003e+001	1.332986e+003	-5.287502e+004
49	4.900000e-001	9.516153e+001	7.682859e+002	-5.947596e+004
50	5.000000e-001	9.980636e+001	1.558176e+002	-6.237897e+004



2.131

Results of Ex2\_130\_b

\*\*\*\*\*

&gt;&gt; program2

Free vibration analysis

of a single degree of freedom analysis

Data:

```

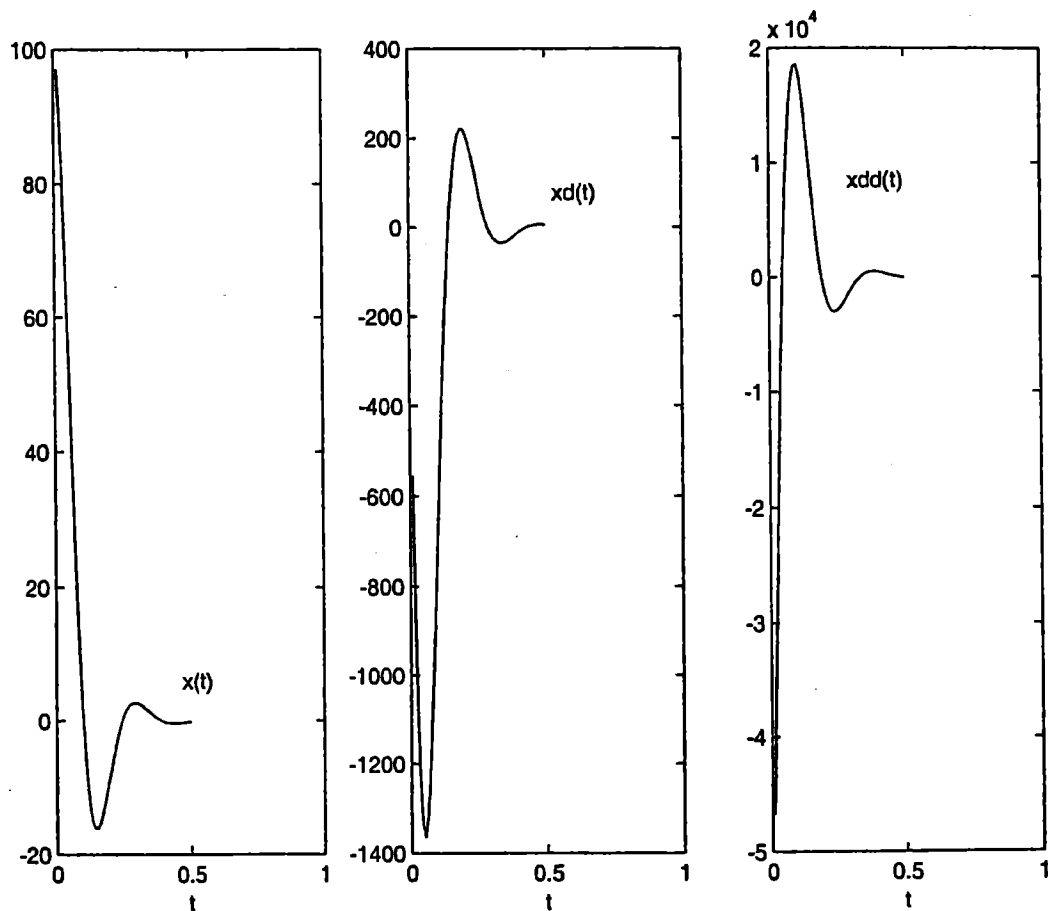
m=      4.00000000e+000
k=      2.50000000e+003
c=      1.00000000e+002
x0=      1.00000000e+002
xd0=     -1.00000000e+001
n=       50
delt=    1.00000000e-002

```

system is under damped

Results:

i	time(i)	x(i)	xd(i)	xdd(i)
1	1.000000e-002	9.704707e+001	-5.547860e+002	-4.678477e+004
2	2.000000e-002	8.940851e+001	-9.485455e+002	-3.216668e+004
3	3.000000e-002	7.854100e+001	-1.203024e+003	-1.901253e+004
4	4.000000e-002	6.575661e+001	-1.335030e+003	-7.722135e+003
5	5.000000e-002	5.218268e+001	-1.364393e+003	1.495649e+003
6	6.000000e-002	3.874058e+001	-1.312202e+003	8.592187e+003
⋮				
45	4.500000e-001	-4.071590e-001	3.283084e+000	1.723973e+002
46	4.600000e-001	-3.667451e-001	4.698554e+000	1.117518e+002
47	4.700000e-001	-3.150951e-001	5.542443e+000	5.837337e+001
48	4.800000e-001	-2.575358e-001	5.894760e+000	1.359090e+001
49	4.900000e-001	-1.985409e-001	5.844858e+000	-2.203340e+001
50	5.000000e-001	-1.416733e-001	5.484453e+000	-4.856551e+001



2.132

Results of Ex2\_130\_c

\*\*\*\*\*

&gt;&gt; program2

Free vibration analysis

of a single degree of freedom analysis

Data:

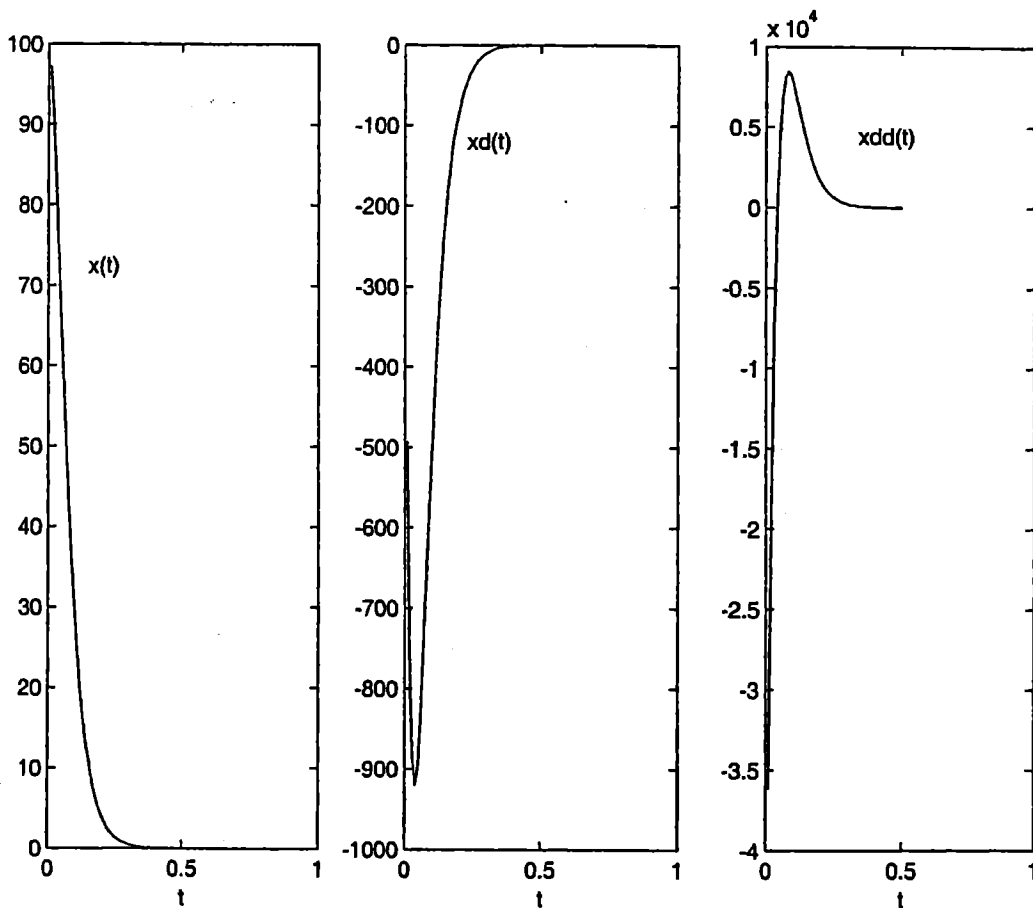
```

m=      4.00000000e+000
k=      2.50000000e+003
c=      2.00000000e+002
x0=     1.00000000e+002
xd0=    -1.00000000e+001
n=      50
delt=   1.00000000e-002

```

system is critically damped

Results:



i	time(i)	x(i)	xd(i)	xdd(i)
1	1.000000e-002	9.727222e+001	-4.925915e+002	-3.616556e+004
2	2.000000e-002	9.085829e+001	-7.611960e+002	-1.872663e+004
3	3.000000e-002	8.252244e+001	-8.868682e+002	-7.233113e+003
4	4.000000e-002	7.342874e+001	-9.196986e+002	9.196986e+001
5	5.000000e-002	6.432033e+001	-8.946112e+002	4.530357e+003
⋮				

44	4.400000e-001	1.996855e-002	-4.576266e-001	1.040098e+001
45	4.500000e-001	1.587541e-002	-3.644970e-001	8.302721e+000
46	4.600000e-001	1.261602e-002	-2.901765e-001	6.623815e+000
47	4.700000e-001	1.002181e-002	-2.309008e-001	5.281410e+000
48	4.800000e-001	7.957984e-003	-1.836505e-001	4.208785e+000
49	4.900000e-001	6.316833e-003	-1.460059e-001	3.352274e+000
50	5.000000e-001	5.012349e-003	-1.160293e-001	2.668750e+000

---

Results of Ex2\_130\_d

2.133

\*\*\*\*\*

>> program2

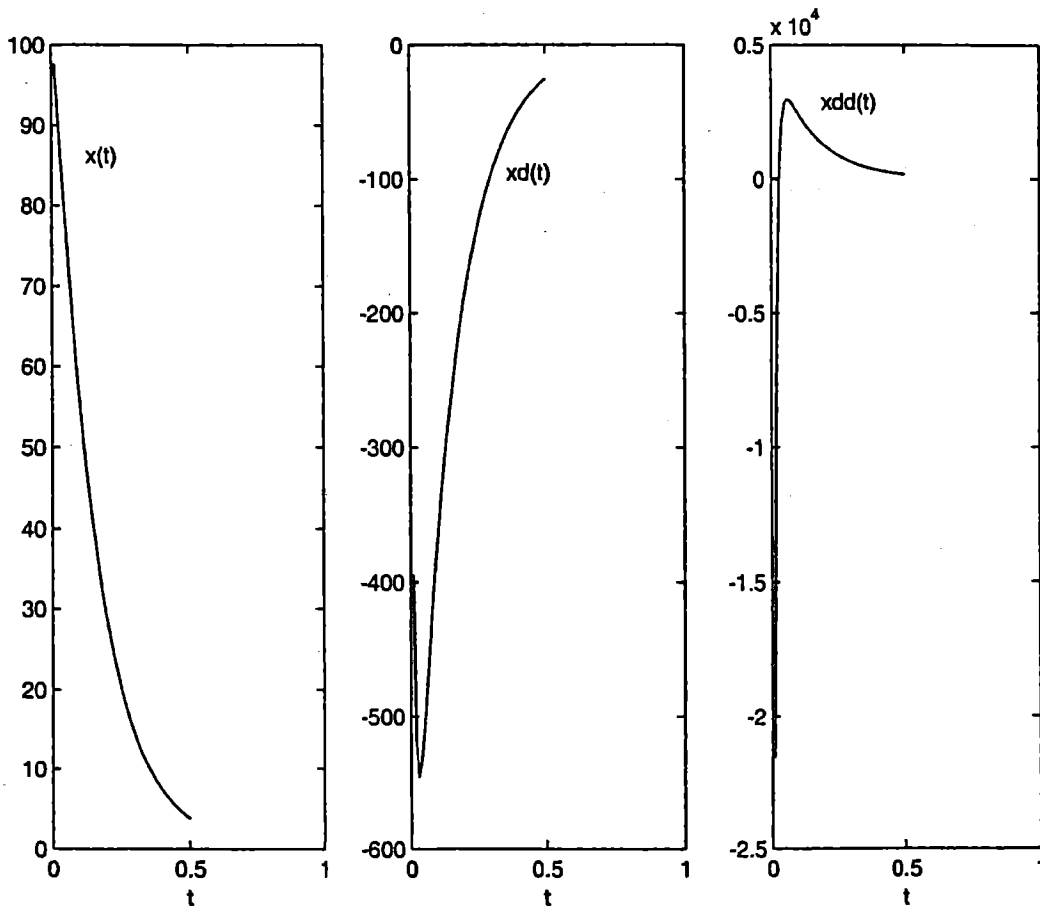
Free vibration analysis  
of a single degree of freedom analysis

Data:

m= 4.00000000e+000  
k= 2.50000000e+003  
c= 4.00000000e+002  
x0= 1.00000000e+002  
xd0= -1.00000000e+001  
n= 50  
delt= 1.00000000e-002

system is over damped

Results:



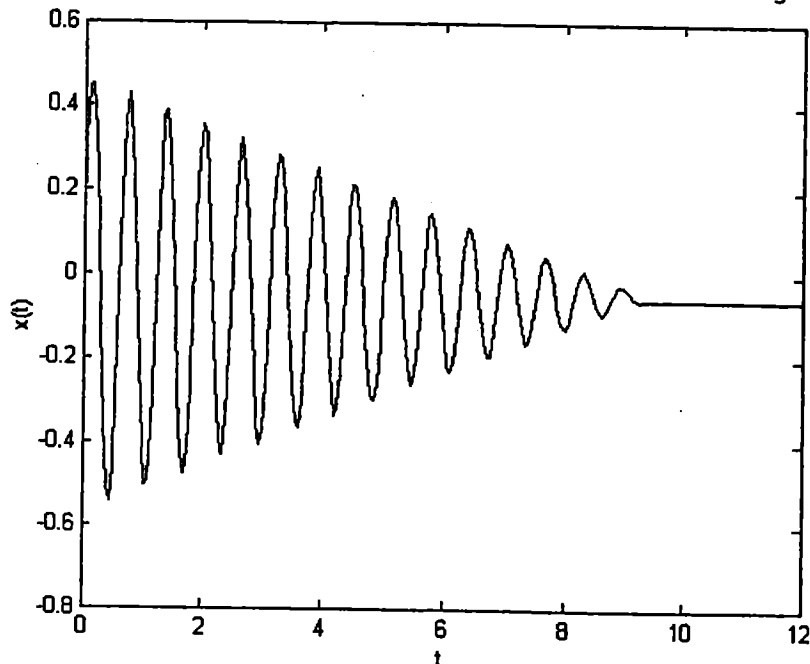
i	time(i)	x(i)	xd(i)	xdd(i)
1	1.000000e-002	9.764929e+001	-3.945541e+002	-2.157540e+004
2	2.000000e-002	9.294636e+001	-5.205155e+002	-6.039927e+003
3	3.000000e-002	8.756294e+001	-5.463949e+002	-8.734340e+001
4	4.000000e-002	8.214078e+001	-5.344391e+002	2.105923e+003
5	5.000000e-002	7.691749e+001	-5.090344e+002	2.830006e+003
⋮				
45	4.500000e-001	5.281309e+000	-3.537806e+001	2.369881e+002
46	4.600000e-001	4.939118e+000	-3.308581e+001	2.216329e+002
47	4.700000e-001	4.619098e+000	-3.094209e+001	2.072727e+002
48	4.800000e-001	4.319813e+000	-2.893726e+001	1.938429e+002
49	4.900000e-001	4.039920e+000	-2.706233e+001	1.812832e+002
50	5.000000e-001	3.778161e+000	-2.530888e+001	1.695374e+002

```

2.134 % Ex2_134.m
% This program will use dfunc2_134.m
tspan = [0: 0.05: 12];
x0 = [0.1; 5];
[t, x] = ode23('dfunc2_134', tspan, x0);
plot(t, x(:, 1));
xlabel('t');
ylabel('x(t)');

% dfunc2_134.m
function f = dfunc2_134(t, x)
u = 0.1;
k = 1000;
m = 20;
g = 9.81;
theta = 30 * pi/180;
f = zeros(2,1);
f(1) = x(2);
f(2) = -u*g*cos(theta)*sign(x(2)) - 2*k*x(1)/m - g*sin(theta);

```



2.135

Results of Ex2\_135

\*\*\*\*\*

Please input M, K, C, X0, XD0, DELT:

4 2500 0 100 -10 0.01

Please input n:

50

FREE VIBRATION ANALYSIS  
OF A SINGLE DEGREE OF FREEDOM SYSTEM

DATA

M = 4  
K = 2500  
C = 0  
X0 = 100  
XD0 = -10  
N = 50  
DELT = 0.01

SYSTEM IS UNDAMPED

RESULTS:

I	TIME(I)	X(I)	KD(I)	KDD(I)
0	0.010000	96.792281	-628.207911	-60495.175367
1	0.020000	87.566486	-1207.347713	-54729.053733
2	0.030000	72.896231	-1711.420483	-45560.144615
3	0.040000	53.693642	-2109.085416	-33558.526371
4	0.050000	31.152642	-2375.617633	-19470.401495
5	0.060000	6.674722	-2494.445451	-4171.701358
.	.	.	.	.
44	0.450000	25.556088	2417.000750	-15972.555129
45	0.460000	48.680657	2183.792919	-30425.410466
46	0.470000	68.778498	1814.807420	-42986.561143
47	0.480000	84.600025	1332.985987	-52875.015650
48	0.490000	95.161532	768.285940	-59475.957780
49	0.500000	99.806357	155.817595	-62378.972923

2.136

Results of Ex2\_136

\*\*\*\*\*

Please input M, K, C, X0, XD0, DELT:

4 2500 100 100 -10 0.01

Please input n:

50

FREE VIBRATION ANALYSIS  
OF A SINGLE DEGREE OF FREEDOM SYSTEM

DATA

M = 4  
K = 2500  
C = 100  
X0 = 100  
XD0 = -10  
N = 50  
DELT = 0.01

SYSTEM IS UNDER DAMPED

RESULTS:

I	TIME(I)	X(I)	XD(I)	XDD(I)
0	0.010000	97.047075	-554.786003	-46784.771735
1	0.020000	89.408515	-948.545502	-32166.684060
2	0.030000	78.540996	-1203.023676	-19012.530355
3	0.040000	65.756612	-1335.029917	-7722.134854
4	0.050000	52.182677	-1364.392905	1495.649356
5	0.060000	38.740577	-1312.201909	8592.187235
:	:	:	:	:
45	0.460000	-0.366745	4.698554	111.751841
46	0.470000	-0.315095	5.542443	58.373370
47	0.480000	-0.257536	5.894760	13.590899
48	0.490000	-0.198541	5.844858	-22.033396
49	0.500000	-0.141673	5.484453	-48.565509

Results of Ex2\_137

\*\*\*\*\*

2.137

Please input M, K, C, X0, XD0, DELT:

4 2500 200 100 -10 0.01

Please input n:

50

FREE VIBRATION ANALYSIS  
OF A SINGLE DEGREE OF FREEDOM SYSTEM

DATA

M = 4  
K = 2500  
C = 200  
X0 = 100  
XD0 = -10  
N = 50  
DELT = 0.01

SYSTEM IS CRITICALLY DAMPED

RESULTS:

I	TIME(I)	X(I)	XD(I)	XDD(I)
0	0.010000	97.272218	-492.591495	-36165.561364
1	0.020000	90.858293	-761.195978	-18726.634119
2	0.030000	82.522437	-886.868203	-7233.112839
3	0.040000	73.428736	-919.698603	91.969860
4	0.050000	64.320327	-894.611228	4530.357100
5	0.060000	55.648662	-835.622450	7000.708775
:	:	:	:	:
45	0.460000	0.012616	-0.290177	6.623815
46	0.470000	0.010022	-0.230901	5.281410
47	0.480000	0.007958	-0.183651	4.208785
48	0.490000	0.006317	-0.146006	3.352274
49	0.500000	0.005012	-0.116029	2.668750

2.138

Results of Ex2\_138

\*\*\*\*\*

Please input M, K, C, X0, XD0, DELT:

4 2500 400 100 -10 0.01

Please input n:

50

FREE VIBRATION ANALYSIS  
OF A SINGLE DEGREE OF FREEDOM SYSTEM

DATA

M = 4  
K = 2500  
C = 400  
X0 = 100  
XD0 = -10  
N = 50  
DELT = 0.01

SYSTEM IS OVER DAMPED

RESULTS:

I	TIME(I)	X(I)	XD(I)	XDD(I)
0	0.010000	97.649291	-394.554074	-21575.399270
1	0.020000	92.946355	-520.515453	-6039.926862
2	0.030000	87.562936	-546.394916	-87.343400
3	0.040000	82.140779	-534.439095	2105.922757
4	0.050000	76.917491	-509.034379	2830.005865
5	0.060000	71.972668	-479.680638	2985.146261
:	:	:	:	:
44	0.450000	5.281309	-35.378062	236.988076
45	0.460000	4.939118	-33.085814	221.632927
46	0.470000	4.619098	-30.942087	207.272682
47	0.480000	4.319813	-28.937259	193.842880
48	0.490000	4.039920	-27.062330	181.283234
49	0.500000	3.778161	-25.308882	169.537364

---

2.139 The input data for Program 2.F and results are given below.

```
C THE FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
  DIMENSION X(50),XD(50),XDD(50),T(50)
  DATA M,K,C,X0,XD0,N,DELTA/
  2 4.0,2500.0,0.0,0.1,-10.0,50,0.01/
C END OF PROBLEM-DEPENDENT DATA
```

```
-----
FREE VIBRATION ANALYSIS
OF A SINGLE DEGREE OF FREEDOM SYSTEM
```

DATA

```
M = 0.40000000E+01
K = 0.25000000E+04
C = 0.00000000E+00
X0 = 0.10000000E+00
XD0 = -0.10000000E+02
N = 50
DELTA = 0.99999998E-02
```

SYSTEM IS UNDAMPED

RESULTS:

I	TIME(I)	X(I)	XD(I)	XDD(I)
1	0.100000E-01	-0.207036E-02	-0.103076E+02	0.129397E+01
2	0.200000E-01	-0.104012E+00	-0.997438E+01	0.650075E+02
3	0.300000E-01	-0.199487E+00	-0.902097E+01	0.124679E+03
4	0.400000E-01	-0.282558E+00	-0.750667E+01	0.176599E+03
5	0.500000E-01	-0.348062E+00	-0.552565E+01	0.217539E+03
6	0.600000E-01	-0.391924E+00	-0.320107E+01	0.244953E+03
7	0.700000E-01	-0.411419E+00	-0.677464E+00	0.257137E+03
8	0.800000E-01	-0.405334E+00	0.188826E+01	0.253334E+03
9	0.900000E-01	-0.374047E+00	0.433659E+01	0.233779E+03
10	0.100000E+00	-0.319503E+00	0.651528E+01	0.199690E+03
:				
46	0.460000E+00	0.398512E+00	-0.264442E+01	-0.249070E+03
47	0.470000E+00	0.359954E+00	-0.502704E+01	-0.224971E+03
48	0.480000E+00	0.299016E+00	-0.709711E+01	-0.186885E+03
49	0.490000E+00	0.219486E+00	-0.872591E+01	-0.137179E+03
50	0.500000E+00	0.126310E+00	-0.981218E+01	-0.789439E+02

The problem-dependent data and results are given (given by Program 2.F).

2.140

```
C THE FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
  DIMENSION X(50),XD(50),XDD(50),T(50)
  DATA M,K,C,X0,XD0,N,DELTA/
  2 4.0,2500.0,100.0,0.1,-10.0,50,0.01/
C END OF PROBLEM-DEPENDENT DATA
```

```
-----
FREE VIBRATION ANALYSIS
OF A SINGLE DEGREE OF FREEDOM SYSTEM
```

## DATA

```

M = 0.40000000E+01
K = 0.25000000E+04
C = 0.10000000E+03
X0 = 0.10000000E+00
XDO = -0.10000000E+02
N = 50
DELTA = 0.99999998E-02

```

SYSTEM IS UNDER DAMPED

## RESULTS:

I	TIME(I)	X(I)	XD(I)	XDD(I)
1	0.100000E-01	-0.957279E-02	0.807168E+01	-0.195809E+03
2	0.200000E-01	0.613786E-01	0.612586E+01	-0.191508E+03
3	0.300000E-01	0.113259E+00	0.427345E+01	-0.177623E+03
4	0.400000E-01	0.147433E+00	0.259570E+01	-0.157038E+03
5	0.500000E-01	0.165937E+00	0.114627E+01	-0.132367E+03
6	0.600000E-01	0.171219E+00	-0.456091E-01	-0.105872E+03
7	0.700000E-01	0.165914E+00	-0.971334E+00	-0.794127E+02
8	0.800000E-01	0.152654E+00	-0.163885E+01	-0.544377E+02
9	0.900000E-01	0.133930E+00	-0.206856E+01	-0.319924E+02
10	0.100000E+00	0.111980E+00	-0.228942E+01	-0.127520E+02
:				
45	0.450000E+00	-0.113169E-03	-0.310021E-01	0.845783E+00
46	0.460000E+00	-0.381386E-03	-0.227080E-01	0.806065E+00
47	0.470000E+00	-0.569294E-03	-0.149992E-01	0.730789E+00
48	0.480000E+00	-0.684317E-03	-0.817052E-02	0.631961E+00
49	0.490000E+00	-0.736251E-03	-0.240284E-02	0.520228E+00
50	0.500000E+00	-0.736195E-03	0.222121E-02	0.404592E+00

2.141

The problem-dependent data (to be used in Program 2.F) and output are given.

```

C THE FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
  DIMENSION X(50),XD(50),XDD(50),T(50)
  DATA M,K,C,X0,XDO,N,DELTA/
  2 4.0,2500.0,200.0,0.1,-10.0,50,0.01/
C END OF PROBLEM-DEPENDENT DATA

```

```

-----
FREE VIBRATION ANALYSIS
OF A SINGLE DEGREE OF FREEDOM SYSTEM

```

## DATA

```

M = 0.40000000E+01
K = 0.25000000E+04
C = 0.20000000E+03
X0 = 0.10000000E+00
XDO = -0.10000000E+02
N = 50
DELTA = 0.99999998E-02

```

~~SYSTEM IS CRITICALLY DAMPED~~~~RESULTS:~~

I	TIME(I)	X(I)	XD(I)	XDD(I)
1	0.100000E-01	0.194700E-01	-0.632776E+01	0.304219E+03
2	0.200000E-01	-0.303265E-01	-0.379082E+01	0.208495E+03
<del>3</del>	<del>0.300000E-01</del>	<del>-0.590458E-01</del>	<del>-0.206660E+01</del>	<del>0.140234E+03</del>
4	0.400000E-01	-0.735759E-01	-0.919699E+00	0.919699E+02
5	0.500000E-01	-0.787888E-01	-0.179066E+00	0.581963E+02
<del>6</del>	<del>0.600000E-01</del>	<del>-0.780956E-01</del>	<del>0.278913E+00</del>	<del>0.348641E+02</del>
7	0.700000E-01	-0.738539E-01	0.543044E+00	0.190065E+02
8	0.800000E-01	-0.676676E-01	0.676676E+00	0.845846E+01
<del>9</del>	<del>0.900000E-01</del>	<del>-0.606045E-01</del>	<del>0.724619E+00</del>	<del>0.164687E+01</del>
10	0.100000E+00	-0.533553E-01	0.718244E+00	-0.256515E+01
:	:	:	:	:
<del>46</del>	<del>0.460000E+00</del>	<del>-0.339359E-04</del>	<del>0.772422E-03</del>	<del>-0.174112E-01</del>
47	0.470000E+00	-0.270210E-04	0.616356E-03	-0.139296E-01
48	0.480000E+00	-0.215048E-04	0.491539E-03	-0.111364E-01
<del>49</del>	<del>0.490000E+00</del>	<del>-0.171069E-04</del>	<del>0.391783E-03</del>	<del>-0.889736E-02</del>
50	0.500000E+00	-0.136023E-04	0.312109E-03	-0.710396E-02

2.142 The problem-dependent data (to be used in Program2.F) and results are given.

```

C THE FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
DIMENSION X(50),XD(50),XDD(50),I(50)
DATA M,K,C,X0,XD0,N,DELT/
2 4.0,2500.0,400.0,0.1,-10.0,50,0.01/
C END OF PROBLEM-DEPENDENT DATA

```

FREE VIBRATION ANALYSIS  
OF A SINGLE DEGREE OF FREEDOM SYSTEM

DATA

```

M = 0.40000000E+01
K = 0.25000000E+04
C = 0.40000000E+03
X0 = 0.10000000E+00
XD0 = -0.10000000E+02
N = 50
DELT = 0.99999998E-02

```

~~SYSTEM IS OVER DAMPED~~~~RESULTS:~~

I	TIME(I)	X(I)	XD(I)	XDD(I)
---	---------	------	-------	--------

1	0.100000E-01	0.351455E-01	-0.390559E+01	0.368593E+03
2	0.200000E-01	0.990550E-02	-0.151007E+01	0.144816E+03
3	0.300000E-01	0.230870E-03	-0.569458E+00	0.568015E+02
4	0.400000E-01	-0.333730E-02	-0.201042E+00	0.221900E+02
5	0.500000E-01	-0.451879E-02	-0.576071E-01	0.858495E+01
6	0.600000E-01	-0.477582E-02	-0.257606E-02	0.324249E+01
7	0.700000E-01	-0.468266E-02	0.177700E-01	0.114966E+01
8	0.800000E-01	-0.446433E-02	0.245564E-01	0.334569E+00
9	0.900000E-01	-0.420854E-02	0.260878E-01	0.215607E-01
10	0.100000E+00	-0.394902E-02	0.256257E-01	-0.944359E-01
⋮				
46	0.460000E+00	-0.354993E-03	0.237800E-02	-0.159296E-01
47	0.470000E+00	-0.331992E-03	0.222393E-02	-0.148975E-01
48	0.480000E+00	-0.310481E-03	0.207983E-02	-0.139322E-01
49	0.490000E+00	-0.290364E-03	0.194507E-02	-0.130295E-01
50	0.500000E+00	-0.271551E-03	0.181905E-02	-0.121853E-01

2.143 The equations for the natural frequencies of vibration were derived in Problem 2.35.

Operating speed of turbine is:

$$\omega_0 = (2400) \frac{2\pi}{60} = 251.328 \text{ rad/sec}$$

Thus we need to satisfy:

$$\omega_n|_{\text{axial}} = \left\{ \frac{g l A E}{W a (l-a)} \right\}^{1/2} \geq \omega_0 \quad (E_1)$$

$$\omega_n|_{\text{transverse}} = \left\{ \frac{3 E I l^3 g}{W a^3 (l-a)^3} \right\}^{1/2} \geq \omega_0 \quad (E_2)$$

$$\omega_n|_{\text{circumferential}} = \left\{ \frac{G J}{J_0} \left( \frac{1}{a} + \frac{1}{l-a} \right) \right\}^{1/2} \geq \omega_0 \quad (E_3)$$

where  $A = \frac{\pi d^2}{4}$ ,  $W = 1000 \times 9.81 = 9810 \text{ N}$ ,

$$I = \frac{\pi d^4}{64}, \quad J = \frac{\pi d^2}{32}, \quad J_0 = 500 \text{ kg-m}^2,$$

and  $E = 207 \times 10^9 \text{ N/m}^2$ ,  $G = 79.3 \times 10^9 \text{ N/m}^2$  (for steel).

The unknowns  $d$ ,  $l$  and  $a$  can be determined to satisfy the inequalities  $(E_1)$ ,  $(E_2)$  and  $(E_3)$  using a trial and error procedure.

2.144 From solution of problem 2.38, the requirements can be stated as:

$$\omega_n |_{\text{pivot ends}} = \sqrt{\frac{12 E I}{\ell^3 \left( \frac{W}{g} + m_{\text{eff}_2} \right)}} \geq \omega_0 \quad (\text{E.1})$$

where  $E = 30 \times 10^6$  psi and  $I = \frac{\pi}{64} [d^4 - (d - 2t)^4]$

$$\omega_n |_{\text{fixed ends}} = \sqrt{\frac{48 E I}{\ell^3 \left( \frac{W}{g} + m_{\text{eff}_2} \right)}} \geq \omega_0 \quad (\text{E.2})$$

with  $m_{\text{eff}_1} = (0.2357 m)$ ,  $m_{\text{eff}_2} = (0.3714 m)$ ,

$$m = \text{mass of each column} = \frac{\pi}{4} [d^2 - (d - 2t)^2] \frac{\ell \rho}{g},$$

$$\rho = 7,800 \text{ kg/m}^3, g = 9.81 \text{ m/sec}^2$$

$$\ell = \text{length of column} = 240 \text{ cm}$$

$$W = \text{mass of floor} = 1,800 \text{ kg}$$

$$\underline{W} = \text{mass of columns} = 4 \left\{ \frac{\pi}{4} [d^2 - (d - 2t)^2] \ell \rho \right\} \quad (\text{E.3})$$

Frequency limit =  $\omega_0 = 50 \times 2\pi = 314.16$  rad/sec.

Problem: Find  $d$  and  $t$  such that  $\underline{W}$  given by Eq. (E.3) is minimized while satisfying the inequalities (E.1) and (E.2).

This problem can be solved either by graphical optimization or by using a trial and error procedure.

2.145  $J_0 = \frac{ml^2}{12} + \frac{ml^2}{4} + Ml^2 = \frac{1}{3}ml^2 + Ml^2$  --- (E1)

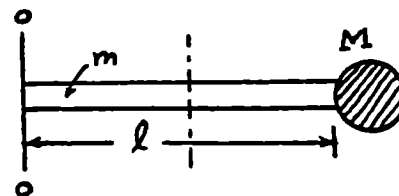
(i) Viscous damping:

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \left( \frac{k_t}{\frac{1}{3}ml^2 + Ml^2} \right)^{\frac{1}{2}} \quad \text{--- (E2)}$$

$$(c_t)_{\text{cri}} = 2 J_0 \omega_n = 2 \sqrt{J_0 k_t} \quad \text{--- (E3)}$$

For critical damping, Eq. (2.80) gives

$$\theta(t) = \left\{ \theta_0 + (\dot{\theta}_0 + \omega_n \theta_0) t \right\} e^{-\omega_n t} \quad \text{--- (E4)}$$



For  $\theta_0 = 75^\circ = 1.309 \text{ rad}$  and  $\dot{\theta}_0 = 0$ ,

$$\theta(t) = (1.309 + 1.309 \omega_n t) e^{-\omega_n t} \quad \text{--- (E5)}$$

For  $\theta = 5^\circ = 0.08727 \text{ rad}$ , Eq. (E5) becomes

$$0.08727 = 1.309 (1 + \omega_n t) e^{-\omega_n t} \quad \text{--- (E6)}$$

Let time to return = 2 sec. Then Eq. (E6) gives

$$0.08727 = 1.309 (1 + 2\omega_n) e^{-2\omega_n} \quad \text{--- (E7)}$$

Solve (E7) by trial and error to find  $\omega_n$ . Then choose the values of  $m$ ,  $M$  and  $k_t$  to get the desired value of  $\omega_n$ . Find the damping constant  $(c_t)_{\text{cri}}$  using Eq. (E3).

### (ii) Coulomb damping:

- (a) Follow the procedure of part(i) to find the value of  $\omega_n$ .
- (b) Derive expression for the equivalent torsional viscous damping constant  $(c_t)_{\text{eq}}$  for Coulomb damping. This expression, for small amounts of damping, is

$$(c_t)_{\text{eq}} = \left\{ 4 T_d / \pi \omega_n \Theta \right\} \quad \text{--- (E8)}$$

where  $T_d$  = friction (damping) torque, and  $\Theta$  = amplitude of angular oscillations.

- (c) If  $(c_t)_{\text{eq}}$  is to be equal to  $(c_t)_{\text{cri}} = 2\sqrt{J_0 k_t}$ , we find

$$T_d = \frac{\pi \omega \Theta}{4} (2\sqrt{J_0 k_t}) \quad \text{--- (E9)}$$


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2.146 Let  $x$  = vertical displacement of the mass (lunar excursion module),  $x_s$  = resulting deflection of each inclined leg (spring). From equivalence of potential energy, we find:

$$k_{eq} = \text{stiffness of each leg in vertical direction} = k \cos^2 \alpha$$

Hence for the four legs, the equivalent stiffness in vertical direction is:

$$k_{eq} = 4 k \cos^2 \alpha$$

Similarly, the equivalent damping coefficient of the four legs in vertical direction is:

$$c_{eq} = 4 c \cos^2 \alpha$$

where  $c$  = damping constant of each leg (in axial motion). Modeling the system as a single degree of freedom system, the equation of motion is:

$$m_{eq} \ddot{x} + c_{eq} \dot{x} + k_{eq} x = 0$$

and the damped period of vibration is:

$$\tau_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{2\pi}{\sqrt{\frac{k_{eq}}{m_{eq}}} \sqrt{1 - \left[ \frac{c_{eq}^2}{4 k_{eq} m_{eq}} \right]}}$$

Using  $m_{eq} = 2000$  kg,  $k_{eq} = 4 k \cos^2 \alpha$ ,  $c_{eq} = 4 c \cos^2 \alpha$ , and  $\alpha = 20^\circ$ , the values of  $k$  and  $c$  can be determined (by trial and error) so as to achieve a value of  $\tau_d$  between 1 s and 2 s. Once  $k$  and  $c$  are known, the spring (helical) and damper (viscous) can be designed suitably.

2.147 Assume no damping. Neglect masses of telescoping boom and strut. Find stiffness of telescoping boom in vertical direction (see Example 2.4). Find the equivalent stiffness of telescoping boom together with the strut in vertical direction. Model the system as a single degree of freedom system with natural time period:

$$\tau_n = \frac{2\pi}{\omega_n} = 2\pi \sqrt{\frac{m_{eq}}{k_{eq}}}$$

Using  $\tau_n = 1$  s and  $m = 100 + 50 = 150$  kg, determine the axial stiffness of the strut ( $k_s$ ). Once  $k_s$  is known, the cross section of the strut ( $A_s$ ) can be found from:

$$k_s = \frac{A_s E_s}{\ell_s}$$

with  $E_s = 200$  GPa and  $\ell_s =$  length of strut (known).