Chapter 3

Harmonically Excited Vibration

3.1 (a) $\delta = \frac{W}{k} = \frac{50}{4000} = 0.0125$ m

(b) $\delta_{st} = \frac{F_0}{k} = \frac{60}{4000} = 0.015$ m

(c) $\omega_n = \sqrt{\frac{k}{m}} = \left(\frac{4000 \times 9.81}{50}\right)^{1/2} = 28.0143$ rad/sec

$\omega = 6$ Hz $= 37.6992$ rad/sec

$$X = \delta_{st} \left| \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2} \right| = 0.015 \left| \frac{1}{1-\left(\frac{37.6992}{28.0143}\right)^2} \right| = 0.0185 \text{ m}$$

3.2 $\tau_b = \frac{2\pi}{\omega_n - \omega} = \frac{2\pi}{2\pi\,(40.0-39.8)} = 5$ sec

3.3 $k = 4000$ N/m, $m = 10$ kg, $F(t) = 400 \cos 10t$ N

$F_0 = 400$ N, $\omega = 10$ rad/s

$\omega_n = \sqrt{\frac{k}{m}} = 20$ rad/s, $\quad \frac{\omega}{\omega_n} = \frac{10}{20} = 0.5 < 1$

Response is given by Eq. (3.9):

$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2}\right) \cos \omega_n t + \left(\frac{\dot{x}_0}{\omega_n}\right) \sin \omega_n t + \frac{F_0}{k - m\omega^2} \cos \omega t \qquad (E.1)$$

(a) $x_0 = 0.1$, $\dot{x}_0 = 0$:

Eq. (E.1) becomes

$$x(t) = \left\{0.1 - \frac{400}{4000 - 10(100)}\right\} \cos 20t + \frac{400}{4000 - 10(100)} \cos 10t$$

$$= -0.033333 \cos 20t + 0.133333 \cos 10t \qquad (E.2)$$

(b) $x_0 = 0$, $\dot{x}_0 = 10$:

Eq. (E.1) becomes

$$x(t) = \left\{0 - \frac{400}{4000 - 10(100)}\right\} \cos 20t + \frac{10}{20} \sin 20t + \left\{\frac{400}{4000 - 10(100)}\right\} \cos 10t$$

$$= -0.133333 \cos 20t + 0.5 \sin 20t + 0.133333 \cos 10t \qquad (E.3)$$

(c) $x_0 = 0.1$, $\dot{x}_0 = 10$:

Eq. (E.1) becomes

$$x(t) = \left\{ 0.1 - \frac{400}{4000 - 10\,(100)} \right\} \cos 20t + \frac{10}{20} \sin 20t$$

$$+ \left\{ \frac{400}{4000 - 10\,(100)} \right\} \cos 10t$$

$$= -0.033333 \cos 20t + 0.5 \sin 20t + 0.133333 \cos 10t \qquad (E.4)$$

3.4 $k = 4000$ N/m, $m = 10$ kg, $F(t) = 400 \cos 20t$ N,

$F_0 = 400$ N, $\omega = 20$ rad/s

$\omega_n = \sqrt{\frac{k}{m}} = 20$ rad/s , $\frac{\omega}{\omega_n} = \frac{20}{20} = 1$

Response is given by Eq. (3.15):

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \frac{\delta_{st}\, \omega_n t}{2} \sin \omega_n t \qquad (E.1)$$

where $\delta_{st} = F_0/k = 400/4000 = 0.1$

(a) $x_0 = 0.1$, $\dot{x}_0 = 0$:

Eq. (E.1) gives

$$x(t) = 0.1 \cos 20t + \frac{(0.1)(20)\,t}{2} \sin 20t$$

$$= 0.1 \cos 20t + t \sin 20t \qquad (E.2)$$

(b) $x_0 = 0$, $\dot{x}_0 = 10$:

Eq. (E.1) gives

$$x(t) = \frac{10}{20} \sin 20t + \frac{(0.1)(20)\,t}{2} \sin 20t$$

$$= 0.5 \sin 20t + t \sin 20t \qquad (E.3)$$

(c) $x_0 = 0.1$, $\dot{x}_0 = 10$:

Eq. (E.1) gives

$$x(t) = 0.1 \cos 20t + \frac{10}{20} \sin 20t + \frac{(0.1)(20)\,t}{2} \sin 20t$$

$$= 0.1 \cos 20t + 0.5 \sin 20t + t \sin 20t \qquad (E.4)$$

(3.5) $k = 4000 \text{ N/m}, \quad m = 10 \text{ kg}, \quad F(t) = 400 \cos 20.1 t \text{ N}$

$F_0 = 400 \text{ N}, \quad \omega = 20.1 \text{ rad/s}, \quad \omega^2 = 404.01 \text{ (rad/s)}^2$

$\omega_n = \sqrt{\frac{k}{m}} = 20 \text{ rad/s}$

Solution is given by Eq. (3.9):

$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2} \right) \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \left(\frac{F_0}{k - m\omega^2} \right) \cos \omega t \qquad (E.1)$$

(a) $x_0 = 0.1, \quad \dot{x}_0 = 0$:

Eq. (E.1) reduces to

$$x(t) = \left\{ 0.1 - \frac{400}{4000 - 10(404.01)} \right\} \cos 20 t + \left\{ \frac{400}{4000 - 10(404.01)} \right\} \cos 20.1 t$$

$$= 10.075062 \cos 20 t - 9.975062 \cos 20.1 t \qquad (E.2)$$

(b) $x_0 = 0, \quad \dot{x}_0 = 10$:

Eq. (E.1) reduces to

$$x(t) = -\left\{ \frac{400}{4000 - 10(404.01)} \right\} \cos 20 t + \frac{10}{20} \sin 20 t + \left\{ \frac{400}{4000 - 10(404.01)} \right\} \cos 20.1 t$$

$$= 9.975062 \cos 20 t + 0.5 \sin 20 t - 9.975062 \cos 20.1 t \qquad (E.3)$$

(c) $x_0 = 0.1, \quad \dot{x}_0 = 10$:

Eq. (E.1) gives

$$x(t) = \left\{ 0.1 - \frac{400}{4000 - 10(404.01)} \right\} \cos 20 t + \frac{10}{20} \sin 20 t + \left\{ \frac{400}{4000 - 10(404.01)} \right\} \cos 20.1 t$$

$$= 10.075062 \cos 20t + 0.5 \sin 20t$$
$$- 9.975062 \cos 20.1\, t \qquad (E.4)$$

(3.6) $k = 4000$ N/m, $m = 10$ kg, $F(t) = 400 \cos 30t$ N

$F_0 = 400$ N, $\omega = 30$ rad/s, $\omega^2 = 900$ (rad/s)^2

$\omega_n = \sqrt{\frac{k}{m}} = 20$ rad/s, $\frac{\omega}{\omega_n} = 1.5 > 1$

Solution is given by Eq. (3.9):

$$x(t) = \left(x_0 - \frac{F_0}{k - m\,\omega^2}\right) \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t$$
$$+ \left(\frac{F_0}{k - m\,\omega^2}\right) \cos \omega t \qquad (E.1)$$

(a) $x_0 = 0.1$, $\dot{x}_0 = 0$:

Eq. (E.1) yields

$$x(t) = \left\{0.1 - \frac{400}{4000 - 10(900)}\right\} \cos 20t + \frac{400}{4000 - 10(900)} \cos 30t$$

$$= 0.18 \cos 20t - 0.08 \cos 30t \qquad (E.2)$$

(b) $x_0 = 0$, $\dot{x}_0 = 10$:

Eq. (E.1) yields:

$$x(t) = -\left(\frac{400}{4000 - 10(900)}\right) \cos 20t + \frac{10}{20} \sin 20t$$
$$+ \left(\frac{400}{4000 - 10(900)}\right) \cos 30t$$

$$= 0.08 \cos 20t + 0.5 \sin 20t - 0.08 \cos 30t \qquad (E.3)$$

(c) $x_0 = 0.1$, $\dot{x}_0 = 10$:

Eq. (E.1) yields

$$x(t) = \left\{0.1 - \frac{400}{4000 - 10(900)}\right\} \cos 20t + \frac{10}{20} \sin 20t$$
$$+ \left\{\frac{400}{4000 - 10(900)}\right\} \cos 30t$$

$$= 0.18 \cos 20t + 0.5 \sin 20t - 0.08 \cos 30t \qquad (E.4)$$

3.7 $\delta_{st} = \frac{F_0}{k} = \frac{25}{2000} = 0.0125$ m

steady state solution at resonance $= x(t) = \frac{\delta_{st} \cdot \omega_n t}{2} \sin \omega_n t$

$= 0.00625\ \omega_n t\ \sin \omega_n t$ m

(a) At end of $\frac{1}{4}$ cycle, $\omega_n t = \frac{\pi}{2}$ and $x(t) = 0.00625\left(\frac{\pi}{2}\right) \sin \frac{\pi}{2} = 0.009817$ m

(b) At end of $2\frac{1}{2}$ cycles, $\omega_n t = 5\pi$ and $x(t) = 0.00625(5\pi) \sin 5\pi = 0$

(c) At end of $5\frac{3}{4}$ cycles, $\omega_n t = 11\frac{1}{2}\pi$ and

$$x(t) = 0.00625\left(\frac{23}{2}\pi\right) \sin \frac{23}{2}\pi = -0.2258 \text{ m}$$

3.8 $\delta_{st} = \frac{F_0}{k} = \frac{100}{4000} = 0.025$ m

$$X = \delta_{st} \cdot \frac{1}{\left|1 - \left(\frac{\omega}{\omega_n}\right)^2\right|}, \quad \left|1 - \left(\frac{\omega}{\omega_n}\right)^2\right| = \frac{\delta_{st}}{X} = \frac{0.025}{20 \times 10^{-3}} = 1.25$$

$$\frac{\omega}{\omega_n} = \sqrt{1.25 + 1} = 1.5$$

$$\omega_n = \omega/1.5 = 5(2\pi)/1.5 = 20.944 \text{ rad/sec}$$

$$m = k/\omega_n^2 = 4000/(20.944)^2 = 9.1189 \text{ kg}$$

3.9 $\omega_n = \sqrt{k/m} = \sqrt{5000/10} = 22.3607$ rad/sec

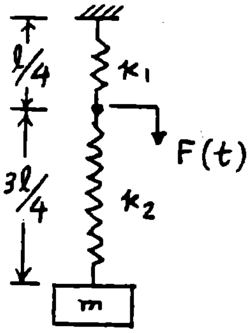
$\delta_{st} = F_0/k = 250/5000 = 0.05$ m

$$X = \delta_{st} \left\{ \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \right\}$$

i.e., $\omega = \omega_n \left(1 - \frac{\delta_{st}}{X}\right)^{\frac{1}{2}} = 22.3607\left[1 - \frac{0.05}{0.10}\right]^{\frac{1}{2}}$

$= 15.8114$ rad/sec

3.10



$k_1 = 4k$; $\frac{1}{4k} + \frac{1}{k_2} = \frac{1}{k}$, $k_2 = \frac{4}{3} k$

Force transmitted to the mass through k_2:

$$\underset{\sim}{F}(t) = \frac{k_2}{k_1 + k_2} F(t) = \frac{k_1 k_2}{k_1 + k_2} \left(\frac{F_0}{k_1} \right) \cos \omega t$$

$$= k \, \delta_{st} \cos \omega t \quad \text{where} \quad \delta_{st} = \frac{F_0}{k_1}$$

Steady state response of m:

$$x(t) = \frac{\underset{\sim}{F_0}}{k \left\{ 1 - \left(\frac{\omega}{\omega_n} \right)^2 \right\}} \cos \omega t$$

$$= \left\{ \frac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n} \right)^2} \right\} \cos \omega t \quad \text{with} \quad \underset{\sim}{F_0} = k \cdot \delta_{st}.$$

3.11 Equivalent stiffness of wing (beam) at location of engine:

$$k = \frac{\text{force}}{\text{deflection}} = \frac{3 E I}{\ell^3} = \frac{3 E \left(\frac{1}{12} b a^3 \right)}{\ell^3} = \frac{E b a^3}{4 \ell^3}$$

Magnitude of unbalanced force: $= m r \omega^2 = m r \left(\frac{2 \pi N}{60} \right)^2 = \frac{m r \pi^2 N^2}{900}$

Equivalent mass of wing at location of engine: $M = \frac{33}{140} m_w = \frac{33}{140} (a b \ell \rho)$

Equation of motion: $M \ddot{x} + k x = m r \omega^2 \sin \omega t$

Maximum steady state displacement of wing at location of engine:

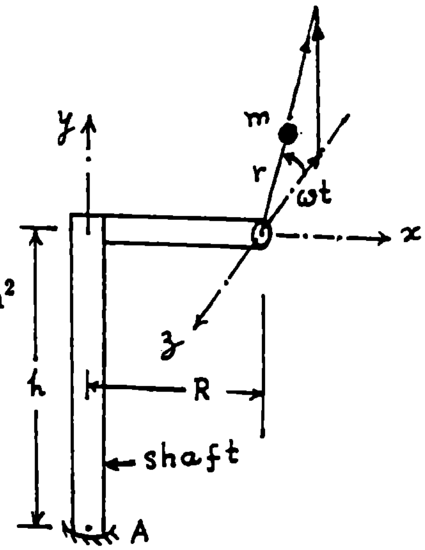
$$X = \left| \frac{m r \omega^2}{k - M \omega^2} \right| = \left| \frac{\left(\frac{m r \pi^2 N^2}{900} \right)}{\left\{ \frac{E b a^3}{4 \ell^3} - \frac{33}{140} a b \ell \rho \left(\frac{2 \pi N}{60} \right)^2 \right\}} \right|$$

$$= \left| \frac{m r \ell^3 N^2}{22.7973 \, E b a^3 - 0.2357 \, \rho a b \ell^4 N^2} \right|$$

(3.12) **Rotating unbalanced force, $m\,r\,\omega^2$, can be resolved into two components as:**

$$F_y = m\,r\,\omega^2 \sin \omega t \quad \text{(parallel y–axis)}$$
$$F_z = m\,r\,\omega^2 \cos \omega t \quad \text{(parallel z–axis)}$$

Maximum bending stress at A:



$$\sigma_b = \frac{1}{I_z}\,|M_z|\,\frac{d_0}{2} = \frac{m\,r\,\omega^2 R\left(\frac{d_0}{2}\right)}{\frac{\pi}{64}(d_o^4 - d_i^4)}$$

$$= \frac{(0.1)(0.1)(31.416^2)(0.5)\left(\frac{0.1}{2}\right)}{\frac{\pi}{64}(0.1^4 - 0.08^4)} = 8.5124\,(10^4)\ \text{N/m}^2$$

Maximum torsional stress at A:

$$\sigma_t = \frac{1}{J_y}\,|M_y|\left(\frac{d_o}{2}\right) = \frac{m\,r\,\omega^2 R\left(\frac{d_o}{2}\right)}{\frac{\pi}{32}(d_o^4 - d_i^4)}$$

$$= 4.2562\,(10^4)\ \text{N/m}^2$$

(3.13) Total stiffness with steel specimen:

$$k_{eq} = k_1 + k_2 = 1790 \times 10^3 + 130 \times 10^6 = 131.79 \times 10^6\ \text{N/m}$$

Force in specimen due to magnets (static) due to elongation $X = k_2 X$.
Force in specimen due to a.c. current in magnets (dynamic) due to elongation $X = k_{eq} X - m\,\omega^2 X$.

$$\text{Ratio of stresses} = \left|\frac{k_2 X}{k_{eq}X - m\,\omega^2 X}\right| = \frac{1}{2} \quad \text{i.e.,} \quad \left|\frac{k_2}{k_{eq} - m\,\omega^2}\right| = \frac{1}{2}.\text{sp}$$

$$\text{i.e} \left|\frac{130 \times 10^6}{131.79 \times 10^6 - 18\,\omega^2}\right| = \frac{1}{2}$$

Squaring both sides of this equation and rearranging gives:

$$1.69 \times 10^{16} = \frac{1}{4}(324\,\omega^4 - 4.74444 \times 10^9\,\omega^2 + 1.73686 \times 10^{16})$$

$$81\,\omega^4 - 1.118611 \times 10^9\,\omega^2 - 1.255785 \times 10^{16} = 0$$

or $\omega^2 = 21.14278 \times 10^6$ (positive value)

$\omega = 4598.128$ rad/sec $= 731.815$ Hz

3.14 Equation of motion: $m_{eq}\ \ddot{x} + k_{eq}\ x = F(t)$
where m_{eq} = mass of the valve and valve rod plus mass of spring at end

$$= \left(10 + \frac{7.5}{3}\right) = 12.5 \text{ kg}$$

$k_{eq} = 70 \times 10^3$ N/m, $F(t) = A\ p(t) = A\ p_0 \sin \omega t = (0.065)\ (70 \times 10^3) \sin \omega t$
$= 4550 \sin 8t$ N
Response of valve (steady state) $= x_p\ (t) = X \sin 8t$ in where

$$X = \frac{4550}{70 \times 10^3 - 12.5\ (8)^2} = 0.0575144 \text{ m} = 5.75144 \text{ cm}$$

3.15 (a) Equation of motion:

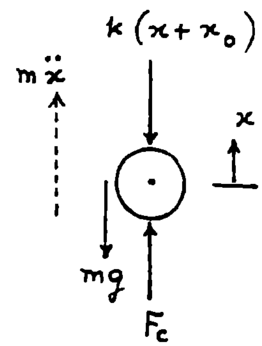
$$m_0\ \ddot{x} + k\ (x + x_0) + m_0\ g = F_c;\ \ \ddot{x} > 0 \qquad (1)$$

where F_c = force exerted on the follower by the cam, m_0 = mass of follower plus one third the mass of the spring, and x_0 = initial displacement of the spring.

(b) Force exerted on the follower by the cam:

$$F_c = m_0\ \ddot{x} + k\ (x + x_0) + m_0\ g \qquad (2)$$

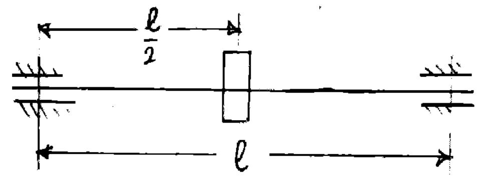
with $x = e \cos \omega t$.



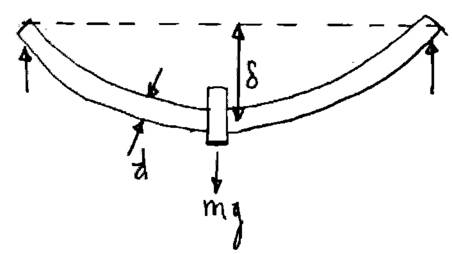
(c) Condition under which follower loses contact with the cam is when F_c is zero and $\ddot{x}$ is negative. Equation (1) can be used to state this condition as:

$$k\ (x + x_0) + m_0\ g \geq |m_0\ \ddot{x}| \qquad (3)$$

3.16 δ_{st} = static radial displacement of shaft under weight of turbine
δ = radial deflection of shaft during rotation



$k = \dfrac{48\,E\,I}{\ell^3}$ = stiffness of centrally loaded simply supported beam



$m\,\delta\,\omega^2 = k\,(\delta - \delta_{st})$ or $m\,\omega^2 = k - k \cdot \dfrac{\delta_{st}}{\delta}$

or $\dfrac{\delta}{\delta_{st}} = \dfrac{k}{k - m\,\omega^2}$ (E_1)

Critical speed is $\omega_{cri} = \sqrt{\dfrac{k}{m}}$ (E_2)

If critical speed = $\dfrac{1}{5}$th of operating speed,

$$\sqrt{\frac{k}{m}} = \frac{1}{5}\,\omega \qquad (E_3)$$

Here $m = 225$ kg

and $\omega = 3000 \times 2\pi/60 = 314.16$ rad/sec

For solid shaft (steel) of diameter d and length ℓ, Eq. (E_3) gives

$$\frac{48\,E\,I}{m\,\ell^3} = \frac{\omega^2}{25} \quad \text{with} \quad E = 200 \text{ GPa} \quad \text{and} \quad I = \frac{\pi\,d^4}{64}$$

i.e., $\dfrac{\ell^3}{d^4} = 530514.0$ (E_4)

Let $\ell = 30$ d in Eq. (E_4): $d = \dfrac{27000}{530514.0} = 0.050894$ m = 50.894 cm

and hence, $\ell = 1.52682$ m

3.17 $I = \frac{\pi}{64}(0.1^4 - 0.09^4) = 1.68812 \times 10^{-6}\ m^4$

$$k = \frac{48\,E\,I}{\ell^3} = \frac{48\,(200 \times 10^9)(1.68812 \times 10^{-6})}{(250 \times 10^{-2})^3} = 1.037181 \times 10^6\ N/m$$

$$m = 200\ kg;\ \omega_n = \sqrt{\frac{k}{m}} = 72.01323\ rad/sec$$

Eccentricity = r = 5 cm, eccentric mass = m_0 = 0.2 kg
Radial force due to eccentric mass at resonance = $F_0 = m_0\, r\, \omega^2$
$= (0.2)\ (5 \times 10^{-2})\ (72.01323)^2 = 51.85905$ N
Let x (t) = radial displacement of turbine
At resonance, Eq. (3.15) gives, for $x_0 = \dot{x}_0 = 0$

$$x\,(t) = \frac{1}{2}\,\delta_{st}\,\omega_n\,t\,\sin\omega_n\,t$$

where $$\delta_{st} = \frac{F_0}{k} = \frac{51.85905}{1.037181 \times 10^{16}} = 50 \times 10^{-6}\ m$$

To activate limit switch, x (t) = 1 cm and hence

$$10^{-2} = \frac{1}{2}\,(50 \times 10^{-6})\ (72.01323)\ t\,\sin 72.01323t$$

i.e., $t\,\sin 72.01323t = 5.554535 \quad (E_1)$

Eq. (E_1) is solved by trial and error (assuming values of t = 10, 9, 8, 7, 6, etc.)

$$t \approx 6.9058\ sec$$

3.18 Tip mass = 0.05 kg = m_0

$$I = \frac{1}{12}\,(0.5 \times 10^{-2})(0.125 \times 10^{-2})^3 = 8.138021 \times 10^{-13}\ m^4$$

$$k = \frac{3\,E\,I}{\ell^3} = \frac{3\,(200 \times 10^9)\,(8.138021 \times 10^{-13})}{(0.25)^3} = 31.25\ N/m$$

m = mass of beam = $7800 \times (0.25 \times (0.5 \times 10^{-2}) \times (0.125 \times 10^{-2})) = 12.1875 \times 10^{-3}$ kg

$$\omega_n = \left(\frac{k}{m_0 + m}\right)^{\frac{1}{2}} = \left(\frac{31.25}{0.05 + 0.0121875}\right)^{\frac{1}{2}} = 22.4168\ rad/sec$$

$$\frac{X}{Y} = \left\{\frac{1 + (2\gamma r)^2}{(1 - r^2) + (2\gamma r)^2}\right\}^{\frac{1}{2}}$$

i.e. $$\frac{6.25}{0.125} = 50 = \left\{\frac{1 + (2 \times 0.01r)^2}{(1 - r^2)^2 + (2 \times 0.01r)^2}\right\}^{\frac{1}{2}}$$

or $r^4 - 1.9996r^2 + 0.9996 = 0$

or $r = \dfrac{\omega}{\omega_n} = 0.9999$

$\therefore \quad \omega = 0.9999\, \omega_n = 22.41456$ rad/sec

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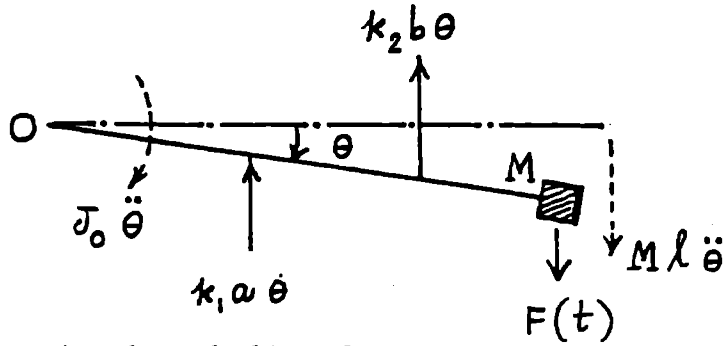
3.19



Equation of motion for rotational motion about the hinge O:

$$(J_0 + M\ell^2)\,\ddot{\theta} + (k_1 a^2 + k_2 b^2)\,\theta = F(t)\,\ell = F_0\,\ell \sin \omega t \tag{1}$$

Steady state response (using Eqs. (3.3) and (3.6)):

$$\theta_p(t) = \Theta \sin \omega t \tag{2}$$

$$\text{where} \quad \Theta = \frac{F_0\,\ell}{(k_1 a^2 + k_2 b^2) - (J_0 + M\ell^2)\,\omega^2} \tag{3}$$

$$\text{and} \quad J_0 = \frac{m\ell^2}{12} + m\left(\frac{\ell}{2}\right)^2 = \frac{1}{3}\, m\ell^2 \tag{4}$$

For given data, $J_0 = \frac{1}{3}(10)(1^2) = 3.3333$ kg–m^2, $\omega = \dfrac{1000\,(2\pi)}{60} = 104.72$ rad/sec, **and**

$$\Theta = \frac{500\,(1)}{5000\,(0.25^2 + 0.5^2) - (3.3333 + 50\,(1^2))\,(104.72^2)} = -8.5718\,(10^{-4}) \text{ rad}$$

(3.20) Equation of motion for rotation about O:

$$J_0 \ddot{\theta} = -k \frac{\theta \ell}{4} \frac{\ell}{4} - k \frac{\theta 3 \ell}{4} \frac{3 \ell}{4} + M_0 \cos \omega t$$

$$\text{i.e.,} \quad J_0 \ddot{\theta} + \left(\frac{5}{8} k \ell^2 \right) \theta = M_0 \cos \omega t$$

$$\text{where} \quad J_0 = \frac{1}{12} m \ell^2 + m \left(\frac{\ell}{4}\right)^2 = \frac{7}{48} m \ell^2 = \frac{7}{48} (10)(1^2) = 1.4583 \text{ kg–m}^2$$

and $\omega = 1000$ rpm $= 104.72$ rad/sec. Steady state solution is:

$$\theta_p(t) = \Theta \cos \omega t$$

where

$$\Theta = \frac{M_0}{\frac{5}{8} k \ell^2 - J_0 \omega^2} = \frac{100}{5000 \left(\frac{5}{8}\right)(1^2) - 1.4583 (104.72^2)} = -0.007772 \text{ rad}$$

(3.21) $k = 4000$ N/m, $m = 10$ kg, $c = 40$ N-s/m, $F(t) = 200 \cos 10t$,
$F_0 = 200$ N, $\omega = 10$ rad/s, $x_0 = 0.1$ m, $\dot{x}_0 = 0$

$$\omega_n = \sqrt{\frac{k}{m}} = 20 \text{ rad/s}, \quad \delta_{st} = \frac{F_0}{k} = \frac{200}{4000} = 0.05 \text{ m}$$

$$\zeta = c/c_c = \left(c/2\sqrt{km}\right) = \left(40/2\sqrt{4000(10)}\right) = 0.1$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = \sqrt{1-(0.1)^2}\,(20) = 19.899749 \text{ rad/s}$$

$$r = \frac{\omega}{\omega_n} = \frac{10}{20} = 0.5$$

$$X = \delta_{st}/\sqrt{(1-r^2)^2 + (2\zeta r)^2} = \frac{0.05}{\left\{(1-0.5^2)^2 + (2(0.1)(0.5))^2\right\}^{\frac{1}{2}}}$$

$$= 0.066082 \text{ m}$$

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{2 * 0.1 * 0.5}{1-0.5^2}\right) = 0.132552 \text{ rad}$$

steady state response, Eq. (3.25):

$$x_p(t) = X \cos(\omega t - \phi)$$

$$= 0.066082 \cos(10t - 0.132552) \text{ m}$$

Total response, Eq. (3.35):

$$x(t) = X_0 e^{-\zeta \omega_n t} \cos(\omega_d t - \phi_0) + X \cos(\omega t - \phi) \quad (E.1)$$

Using the initial conditions x_0 and $\dot{x}_0$, Eq.(E.1) gives

$$x_0 = X_0 \cos \phi_0 + X \cos \phi \quad (E.2)$$

or $$X_0 \cos \phi_0 = x_0 - X \cos \phi \quad (E.3)$$

$$\dot{x}_0 = -\zeta \omega_n X_0 \cos \phi_0 + \omega_d X_0 \sin \phi_0 + \omega X \sin \phi \quad (E.4)$$

or $$X_0 \sin \phi_0 = \frac{1}{\omega_d} \left\{ \dot{x}_0 + \zeta \omega_n X_0 \cos \phi_0 - \omega X \sin \phi \right\} \quad (E.5)$$

For known values, Eqs.(E.3) and (E.5) yield

$$X_0 \cos \phi_0 = 0.034498 , \quad X_0 \sin \phi_0 = -0.000922$$

Hence

$$X_0 = \left\{ (X_0 \cos \phi_0)^2 + (X_0 \sin \phi_0)^2 \right\}^{\frac{1}{2}} = 0.034510$$

$$\phi_0 = \tan^{-1} \left(\frac{X_0 \sin \phi_0}{X_0 \cos \phi_0} \right) = -0.026710$$

Thus the total response, Eq. (E.1), will be

$$x(t) = 0.034510 \, e^{-2t} \cos(19.899749 \, t + 0.026710) + 0.066082 \cos(10t - 0.132552) \text{ m} \quad (E.6)$$

3.22 $k = 4000$ N/m, $m = 10$ kg, $c = 40$ N-s/m, $F(t) = 200 \cos 10t$, $F_0 = 200$ N, $\omega = 10$ rad/s, $x_0 = 0$, $\dot{x}_0 = 10$ m/s

From solution of Problem 3.21,

$\zeta = 0.1$, $\omega_d = 19.899749$ rad/s, $r = 0.5$, $X = 0.066082$ m, $\phi = 0.132552$ rad

$$x_p(t) = 0.066082 \cos(10t - 0.132552) \text{ m}$$

$$X_0 \cos \phi_0 = x_0 - X \cos \phi = -0.065502$$

$$X_0 \sin \phi_0 = \frac{1}{\omega_d} \left\{ \dot{x}_0 + \zeta \omega_n X_0 \cos \phi_0 - \omega X \sin \phi \right\} = 0.491547$$

$$X_0 = \left\{ (X_0 \cos \phi_0)^2 + (X_0 \sin \phi_0)^2 \right\}^{\frac{1}{2}} = 0.495892$$

$$\phi_0 = \tan^{-1} \left(\frac{X_0 \sin \phi_0}{X_0 \cos \phi_0} \right) = -1.438320$$

Thus the total response, Eq. (3.35), is given by

$$x(t) = 0.495892\ e^{-2t}\ \cos(19.899749\ t + 1.438320)$$
$$+\ 0.066082\ \cos(10\ t - 0.132552)\quad m$$

(3.23) $k = 4000$ N/m, $m = 10$ kg, $c = 40$ N-s/m, $F(t) = 200\ \cos 20t$

$F_0 = 200$ N, $\omega = 20$ rad/s, $x_0 = 0.1$ m, $\dot{x}_0 = 0$

$$\omega_n = \sqrt{\frac{k}{m}} = 20 \text{ rad/s},\quad \delta_{st} = \frac{F_0}{k} = \frac{200}{4000} = 0.05$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{k m}} = \frac{40}{2\sqrt{4000\ (10)}} = 0.1$$

$$\omega_d = \sqrt{1-\zeta^2}\ \omega_n = \sqrt{1-0.1^2}\ (20) = 19.899749 \text{ rad/s}$$

$$r = \frac{\omega}{\omega_n} = 1$$

$$X = \frac{\delta_{st}}{\left\{(1-r^2)^2 + (2\zeta r)^2\right\}^{1/2}} = \frac{0.05}{\left\{(1-1^2)^2 + (2*0.1*1)^2\right\}^{1/2}} = 0.25$$

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$$

Steady state response, Eq. (3.25):

$$x_p(t) = X\cos(\omega t - \phi) = 0.25\ \cos\left(20t - \frac{\pi}{2}\right)\quad m$$

Total response, Eq. (3.35):

$$x(t) = X_0\ e^{-\zeta\omega_n t}\ \cos(\omega_d t - \phi_0) + X\cos(\omega t - \phi) \qquad (E.1)$$

Using the initial conditions x_0 and $\dot{x}_0$, Eq. (E.1) gives

$$X_0\cos\phi_0 = x_0 - X\cos\phi = 0.1 - 0.25\ \cos\frac{\pi}{2} = 0.1$$

$$X_0\sin\phi_0 = \frac{1}{\omega_d}\left\{\dot{x}_0 + \zeta\omega_n X_0\cos\phi_0 - \omega X\sin\phi\right\}$$
$$= \left(0 + 0.1*20*0.1 - 20*0.25*\sin\frac{\pi}{2}\right)/19.899749$$
$$= -0.241209$$

Hence $X_0 = \left\{(X_0\cos\phi_0)^2 + (X_0\sin\phi_0)^2\right\}^{\frac{1}{2}} = 0.261117$

$$\phi_0 = \tan^{-1}\left(\frac{X_0\sin\phi_0}{X_0\cos\phi_0}\right) = \tan^{-1}\left(\frac{-0.241209}{0.1}\right) = -1.177783$$

Total response :

$$x(t) = 0.261117\ e^{-2t} \cos(19.899749\ t + 1.777828)$$
$$+ 0.25 \cos(20\,t - \tfrac{\pi}{2})\ \ m$$

3.24 $k = 4000$ N/m, $m = 10$ kg, $c = 40$ N-s/m,

$F(t) = 200 \cos 20t$ N, $F_0 = 200$ N, $\omega = 20$ rad/s

$x_0 = 0$, $\dot{x}_0 = 10$ m/s

From solution of Problem 3.23,

$\zeta = 0.1$, $\omega_n = 20$ rad/s, $\omega_d = 19.899749$ rad/s, $r = 1$

$X = 0.25$, $\phi = \frac{\pi}{2}$

$x_p(t) = 0.25 \cos(20t - \frac{\pi}{2})$ m

$X_0 \cos\phi_0 = x_0 - X\cos\phi = 0 - 0 = 0$

$$X_0 \sin\phi_0 = \frac{1}{\omega_d}\left\{\dot{x}_0 + \zeta\omega_n X_0 \cos\phi_0 - \omega X \sin\phi\right\}$$
$$= \frac{1}{19.899749}\left\{10 + 0.1 * 20 * 0 - 20 * 0.25 * \sin\frac{\pi}{2}\right\}$$
$$= 0.251260$$

Hence $X_0 = \left\{(X_0\cos\phi_0)^2 + (X_0\sin\phi_0)^2\right\}^{\frac{1}{2}} = 0.251260$

$$\phi_0 = \tan^{-1}\left(\frac{X_0 \sin\phi_0}{X_0 \cos\phi_0}\right) = 1.570793$$

Total response:

$$x(t) = 0.251260\ e^{-2t} \cos(19.899749\ t - 1.570793)$$
$$+ 0.25 \cos(20\,t - \tfrac{\pi}{2})\ \ m$$

3.25 $m = 225$ kg, $F(t) = 900 \sin 100 \pi t$ N
Let $X_{max} = 0.125$ cm < 0.25 cm (maximum permissible value).

From Eq. (3.33), $X_{max} = \delta_{st} \dfrac{1}{2\zeta\sqrt{1-\zeta^2}} = 0.125 \times 10^{-2}$

Let $\zeta = 0.01$. Then $\delta_{st} = \dfrac{F_0}{k_{eq}} = \dfrac{900}{k_{eq}}$ and Eq. (1) gives

$$k_{eq} = \frac{900}{2(0.01)\sqrt{1-0.0001}\,(0.125 \times 10^{-2})} = 36.0018 \times 10^6 \text{ N/m}$$

Since shock mounts are in parallel, stiffness of each mounts = $k = 12.0006 \times 10^6$ N/m

$$\zeta = \frac{c_{eq}}{c_c} = \frac{c_{eq}}{\sqrt{2\,k_{eq}\,m}}$$

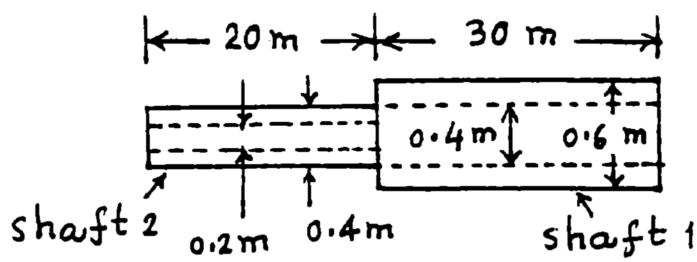
or $\quad c_{eq} = \zeta\sqrt{2\,k_{eq}\,m} = 0.01\sqrt{2\,(36.0018 \times 10^6)\,(225)} = 1272.824 \text{ N}\cdot\text{s/m}$

and $\quad$ hence $c = \dfrac{c_{eq}}{3} = 636.412 \text{ N}\cdot\text{s/m}$

3.26 **Equation of motion for torsional system:**

$$J_0\,\ddot{\theta} + c_t\,(\dot{\theta} - \dot{\alpha}) + k_t\,(\theta - \alpha) = 0 \qquad (1)$$

where θ = angular displacement of shaft and α = angular displacement of base of shaft = $\alpha_0 \sin \omega t$. Steady state response of propeller (Eq. (3.67)):



$$\theta_p(t) = \Theta \sin(\omega t - \phi) \tag{2}$$

$$\text{where } \Theta = \alpha_0 \left\{ \frac{k_t^2 + (c_t\,\omega)^2}{(k_t - J_0\,\omega^2)^2 - (c_t\,\omega)^2} \right\}^{\frac{1}{2}} \tag{3}$$

$$\text{and } \phi = \tan^{-1} \left\{ \frac{J_0\,c_t\,\omega^3}{k_t\,(k_t - J_0\,\omega^2) + (c_t\,\omega)^2} \right\} \tag{4}$$

Here $J_0 = 10^4$ kg–m^2, $\varsigma_t = 0.1$, and $\omega = 314.16$ rad/sec. Torsional stiffnesses of shafts:

$$(k_t)_1 = \frac{G_1\,J_1}{\ell_1} = \frac{(80\,(10^9))\left(\frac{\pi}{32}\,(0.6^4 - 0.4^4)\right)}{30} = 27.2272\,(10^6)\text{ N–m/rad}$$

$$(k_t)_2 = \frac{G_2\,J_2}{\ell_2} = \frac{(80\,(10^9))\left(\frac{\pi}{32}\,(0.4^4 - 0.2^4)\right)}{20} = 9.4248\,(10^6)\text{ N–m/rad}$$

Series springs give:

$$k_t = \frac{(k_t)_1\,(k_t)_2}{(k_t)_1 + (k_t)_2} = \frac{(27.2272\,(10^6))\,(9.4248\,(10^6))}{27.2272\,(10^6) + 9.4248\,(10^6)} = 7.0013\,(10^6)\text{ N–m/rad}$$

$$c_t = \varsigma\,(2\sqrt{J_0\,k_t}) = 0.1\,(2)\sqrt{(10^4)\,(7.0013\,(10^6))} = 52{,}919.8624\text{ N–m–s/rad}$$

From Eq. (3),

$$\Theta = 0.05 \left[\frac{(7.0013\,(10^6))^2 + \left\{5.2920\,(10^4)\,(314.16^2)\right\}^2}{\left\{7.0013\,(10^6) - (10^4)\,(314.16^2)\right\}^2 + \left\{5.2920\,(10^4)\,(314.16)\right\}^2} \right]^{\frac{1}{2}}$$

$$= 9.2028\,(10^{-4})\text{ rad}$$

$$\phi = \tan^{-1} \left\{ \frac{(10^4)\,(5.2920\,(10^4))\,(314.16^3)}{7.0013\,(10^6)\left[7.0013\,(10^6) - (10^4)\,(314.16^2)\right] + (5.2920\,(10^4)\,(314.16))^2} \right\}$$

$$= \tan^{-1}(59.3664) = 89.0350^\circ = 1.5540\text{ rad}$$

3.27 $X = \dfrac{\delta_{st}}{\{(1-r^2)^2 + (2\zeta r)^2\}^{1/2}}$

For maximum X, $\dfrac{dX}{dr} = -\delta_{st} \cdot \dfrac{1}{2} \dfrac{1}{\{(1-r^2)^2 + (2\zeta r)^2\}^{3/2}} \cdot \{2(1-r^2)(-2r) + 2(2\zeta r)(2\zeta)\} = 0$

i.e., $-4r(1-r^2) + 8r\zeta^2 = 0$

i.e., $r = \sqrt{1-2\zeta^2}$

$$X\Big|_{at\ r=\sqrt{1-2\zeta^2}} = \frac{\delta_{st}}{\left[\{1-(1-2\zeta^2)\}^2 + \left(2\zeta\sqrt{1-2\zeta^2}\right)^2\right]^{1/2}} = \frac{\delta_{st}}{2\zeta\sqrt{1-\zeta^2}}$$

$$\therefore \left(\frac{X}{\delta_{st}}\right)_{max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

3.28 Under a d.c. current (I) through the coil, core rotates by angle θ. Torque developed due to I balances the restoring torque of spring: $a\,I = k_t\,\theta$ where a is a constant and k_t is the torsional spring constant. Under an a.c. current I(t), torque developed is $T(t) = a\,I(t)$ and the equation of motion is:

$$J_0\,\ddot{\theta} + c_t\,\dot{\theta} + k_t\,\theta = T(t) = a\,I(t) = a\,I_0 \cos\omega t \qquad (1)$$

Steady state angular displacement of core:

$$\theta_p(t) = \Theta \cos(\omega t - \phi).sp \qquad (2)$$

$$\text{where}\quad \Theta = \frac{a\,I_0}{\left\{(k_t - J_0\,\omega^2)^2 + (c_t\,\omega)^2\right\}^{\frac{1}{2}}} = \frac{\left(\frac{a\,I_0}{k_t}\right)}{\left[\left\{1-(\frac{\omega}{\omega_n})^2\right\}^2 + \left(2\,\zeta\,\frac{\omega}{\omega_n}\right)^2\right]^{\frac{1}{2}}} \qquad (3)$$

When $\omega = 0$ (d.c. current) and $I_0 = 1$ ampere, Eq. (1) gives

$$\Theta_{dc} = \left(\frac{a}{k_t}\right) = 1 \text{ (reading corresponding } \Theta_{dc})$$

and hence $a = k_t = 62.5$.

When $\omega = 50$ Hz $= 314.16$ rad/sec and $I_0 = 5$ amperes, Eq. (3) gives:

$$\Theta_{ac} = \frac{\left(\frac{a\,(5)}{k_t}\right)}{\left[\left\{1-\left(\frac{314.16}{250}\right)^2\right\}^2 + \left\{2\,(1)\left(\frac{314.16}{250}\right)\right\}^2\right]^{\frac{1}{2}}} = 1.9386 \text{ amperes}$$

where $J_0 = 0.001$ N–m^2, $k_t = 62.5$ N-m/rad, $c_t = 0.5$ N-m-s/rad, and $\omega_n = \sqrt{\frac{k_t}{J_0}} = \sqrt{\frac{62.5}{0.001}} = 250$ rad/s. The steady state value of current indicated by ammeter = 1.9386 amperes (this shows that the ammeter is not accurate).

3.29 Eq. (3.34): $\frac{X_{res}}{\delta_{st}} = \frac{X}{\delta_{st}}\Big|_{\omega=\omega_n} = \frac{1}{2\zeta}$

i.e., $\delta_{st} = 2\zeta\left(\frac{20}{1000}\right) = 0.04\,\zeta$ $\quad (E_1)$

Eq. (3.30): $\frac{X}{\delta_{st}} = \frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$; $r = 0.75 = \frac{\omega}{\omega_n}$

i.e., $\frac{0.01}{\delta_{st}} = \frac{1}{\sqrt{(1-0.75^2)^2+(2\zeta \times 0.75)^2}}$ $\quad (E_2)$

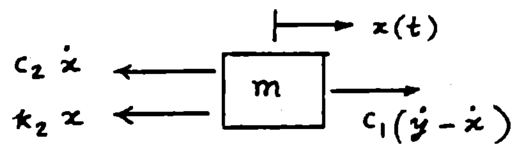
Eqs. (E_1) and (E_2) give

$$\frac{0.01}{0.04\,\zeta} = \frac{1}{\sqrt{0.1914+2.25\,\zeta^2}}$$

i.e., $0.1914 + 2.25\,\zeta^2 = 16\,\zeta^2$

i.e., $\zeta = 0.1180$

3.30 (a) Equation of motion of mass:



$m\ddot{x} = c_1(\dot{y}-\dot{x}) - c_2\dot{x} - k_2 x$

i.e., $m\ddot{x} + (c_1+c_2)\dot{x} + k_2 x = c_1\dot{y} = -c_1\omega Y \sin\omega t$

(b) $$x_p(t) = \frac{-(c_1\omega Y/k_2)}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\sin(\omega t-\phi)$$

where $r = \omega/\omega_n$, $\zeta = (c_1+c_2)\omega/(2 r k)$ and $\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$.

(c) steady-state force transmitted to point P:

$$= k_2 x_p + c_2 \dot{x}_p$$

$$= \frac{-(c_1\omega Y)}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\left\{\sin(\omega t-\phi) + \frac{c_2\omega}{k_2}\cos(\omega t-\phi)\right\}$$

3.31 Eq. (3.34) gives $\left(\frac{X}{\delta_{st}}\right)_{\omega=\omega_n} = \frac{1}{2\zeta}$

If $X = \frac{1}{\sqrt{2}} X_{max} = \frac{1}{\sqrt{2}} X\Big|_{\omega=\omega_n}$, Eq. (3.30) gives

$$\frac{1}{\sqrt{2}} \cdot \frac{1}{2\zeta} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

Squaring and rearranging

$$8\zeta^2 = (1-r^2)^2 + 4\zeta^2 r^2 = 1 - 2r^2 + r^4 + 4r^2\zeta^2$$

$$r^4 + r^2(4\zeta^2 - 2) + (1 - 8\zeta^2) = 0$$

$$r^2 = 1 - 2\zeta^2 \pm 2\zeta\sqrt{1+\zeta^2}$$

Neglecting terms involving ζ^2,

$$r^2 = \frac{\omega^2}{\omega_n^2} = 1 \pm 2\zeta$$

Let $\omega = \omega_1$ when $r^2 = 1 - 2\zeta$ and $\omega = \omega_2$ when $r^2 = 1 + 2\zeta$

$$\frac{\omega_2^2 - \omega_1^2}{\omega_n^2} = \frac{(\omega_2 + \omega_1)(\omega_2 - \omega_1)}{\left(\frac{\omega_2 + \omega_1}{2}\right)^2} = 4\zeta$$

$$\therefore \quad \frac{\omega_2 - \omega_1}{\omega_2 + \omega_1} = \zeta$$

3.32 $k_t = \frac{\pi G}{32\, l} d^4 = \frac{\pi\,(79.3 \times 10^9)}{32\,(1)} \left(\frac{4}{100}\right)^4 = 19930.31$ N-m/rad

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \sqrt{\frac{19930.31}{10}} = 44.6434 \text{ rad/sec}$$

$$\theta_{st} = M_{t0}/k_t = 1000/19930.31 = 0.0502 \text{ rad}$$

$$\zeta_t = \frac{c_t}{2 J_0\, \omega_n} = \frac{300}{2(10)(44.6434)} = 0.336$$

(a) Eq. (3.30), when written for a torsional system, gives

$$\frac{\Theta}{\theta_{st}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

i.e., $$\frac{(2/57.2956)}{0.0502} = \frac{1}{\sqrt{(1-r^2)^2 + (2 \times 0.336\, r)^2}}$$

i.e., $$r^4 - 1.5484\, r^2 - 1.0679 = 0$$

i.e., $$r^2 = 2.0655,\ -0.5171$$

$$\therefore \quad \omega = r\,\omega_n = \sqrt{2.0655}\,(44.6434) = 64.16 \text{ rad/sec}$$

(b) Maximum torque transmitted to the support:

$$M_t(t) = k_t\,\theta(t) + c_t\,\dot{\theta}(t)$$

$$= k_t\,\Theta\cos(\omega t-\phi) - c_t\,\Theta\,\omega\,\sin(\omega t-\phi)$$

$$(M_t)_{max} = \sqrt{(k_t\,\Theta)^2 + (c_t\,\Theta\,\omega)^2}$$

$$= \sqrt{\left\{19930.31\left(\frac{2}{57.2956}\right)\right\}^2 + \left\{300\left(\frac{2}{57.2956}\right)(64.16)\right\}^2}$$

$$= 967.2 \text{ N-m}$$

3.33 Complete solution is $x(t) = X_0\, e^{-\zeta\omega_n t}\cos(\omega_d t+\phi_0) + X\cos(\omega t-\phi)$

$\omega = 2\pi(3.5) = 21.9912$ rad/sec, $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2500}{10}} = 15.8114$ rad/sec

$\delta_{st} = \frac{F_0}{k} = \frac{180}{2500} = 0.072$ m

$\zeta = \frac{c}{2m\omega_n} = \frac{45}{2(10)(15.8114)} = 0.1423$, $r = \frac{\omega}{\omega_n} = \frac{21.9912}{15.8114} = 1.3908$

$\zeta\omega_n = 2.25$, $\omega_d = \sqrt{1-\zeta^2}\,\omega_n = 15.6505$

$$X = \frac{\delta_{st}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = \frac{0.072}{\left[(1-1.3908^2)^2 + (2\times 0.1423\times 1.3908)^2\right]^{1/2}}$$

$= 0.07095$ m

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{0.3958}{-0.9343}\right) = -22.9591^\circ$$

$$x(t) = X_0\, e^{-2.25t}\cos(15.6505t+\phi_0) + 0.07095\cos(21.9912t + 22.9591^\circ)$$

$$\dot{x}(t) = -2.25\,X_0\,e^{-2.25t}\cos(15.6505t+\phi_0) - 15.6505\,X_0\,e^{-2.25t}\sin(15.6505t+\phi_0)$$
$$- 21.9912\,(0.07095)\sin(21.9912\,t + 22.9591^\circ)$$

$x(0) = 0.015 = X\cos\phi_0 + 0.07095\cos 22.9591^\circ$

$X\cos\phi_0 = -0.05033$ ---- (E_1)

$\dot{x}(0) = 5 = -2.25\,X_0\cos\phi_0 - 15.6505\,X_0\sin\phi_0 - 1.5603\sin 22.9591^\circ$

$$X\sin\phi_0 = \frac{-0.6086 - 2.25\,X_0\cos\phi_0 - 5}{15.6505} = -0.3511 \quad \text{----}(E_2)$$

Eqs. (E_1) and (E_2) give

$$X_0 = \left\{(-0.05033)^2 + (-0.3511)^2\right\}^{1/2} = 0.3547$$

$$\phi_0 = \tan^{-1}\left(\frac{0.3511}{0.05033}\right) = \tan^{-1}(6.9760) = 81.8423^\circ$$

3.34 (a) Eq. (3.38) gives $\frac{1}{2\zeta} \approx \left(\frac{X}{\delta_{st}}\right)_{max} = \frac{0.5}{0.25} = 2$

$\zeta = 0.25$

(b) Eqs. (3.42) yield

$$\left(\frac{\omega_1}{\omega_n}\right)^2 \approx 1 - 2\,\zeta = 0.5, \quad \omega_1 = \omega_n \sqrt{0.5} = (5 \times 2\,\pi)\sqrt{0.5} = 22.2145 \text{ rad/sec}$$

$$\left(\frac{\omega_2}{\omega_n}\right)^2 \approx 1 + 2\,\zeta = 1.5, \quad \omega_2 = \omega_n \sqrt{1.5} = 38.4766 \text{ rad/sec}$$

3.35 Amplitude of vibration under base excitation:

$$X = Y \left\{ \frac{\sqrt{k^2 + (c\,\omega)^2}}{\left[\left(k - m\,\omega^2\right)^2 + (c\,\omega)^2\right]^{\frac{1}{2}}} \right\}$$

$$= \frac{(0.2)\sqrt{k^2 + c^2\,(157.08)^2}}{\left[\left\{k - 2000\,(157.08)^2\right\}^2 + c^2\,(157.08)^2\right]^{\frac{1}{2}}} = 0.1 \text{ m} \qquad (1)$$

Let $k = 5\,(10^6)$ N/m. Then Eq. (1) gives:

$$\frac{\sqrt{25\,(10^{12}) + 2.4674\,(10^4)\,c^2}}{\sqrt{1966.7717\,(10^{12}) + 2.4674\,(10^4)\,c^2}} = 0.5$$

i.e., $1.85055\,(10^4)\,c^2 = 466.6929\,(10^{12})$ i.e., $c = 158805.0$ N–s/m

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3.36

$$\frac{X}{Y} = \left[\frac{k^2 + c^2\,\omega^2}{\left(k - m\,\omega^2\right)^2 + c^2\,\omega^2}\right]^{\frac{1}{2}} \text{.sp}$$

$$\text{or} \quad \frac{10^{-6}}{Y} = \left[\frac{(10^6) + (10^3\,(200\,\pi))^2}{\left\{10^6 - \left(\frac{5000}{9.81}\right)(200\,\pi)^2\right\}^2 + \left\{(10^3)\,(200\,\pi)\right\}^2}\right]^{\frac{1}{2}}$$

or $Y = 169.5294\,(10^{-6})$ m

(3.37) Equation of motion:

$$I_0 \ddot{\theta} + \left(k \frac{\ell}{4} \theta\right) \frac{\ell}{4} + \left(c \frac{\ell}{4} \dot{\theta}\right) \frac{\ell}{4} + \left(k \frac{3\ell}{4} \theta\right) \frac{3\ell}{4} = M_0 \cos \omega t$$

$$\text{or} \quad I_0 \ddot{\theta} + c \frac{\ell^2}{16} \dot{\theta} + \frac{5}{8} k \ell^2 \theta = M_0 \cos \omega t$$

$$\text{where } I_0 = \frac{m \ell^2}{12} + m \left(\frac{\ell}{4}\right)^2 = \frac{7}{48} m \ell^2 = \frac{7}{48} (10) (1^2) = 1.4583 \text{ kg-m}^2$$

$$\frac{c \ell^2}{16} = \frac{(1000)(1^2)}{16} = 62.5 \text{ N-m-s/rad}$$

$$\frac{5}{8} k \ell^2 = \frac{5}{8} (5000) (1^2) = 3125.0 \text{ N-m/rad}$$

$$\omega = \frac{1000 (2\pi)}{60} = 104.72 \text{ rad/sec}$$

Equation of motion becomes:

$$1.4583 \ddot{\theta} + 62.5 \dot{\theta} + 3125.0 \theta = 100 \cos 104.72 t$$

Steady state response is given by Eq. (3.28):

$$\theta_p(t) = \Theta \cos(\omega t - \phi) = \Theta \cos(104.72 t - \phi).sp$$

$$\text{where} \quad \Theta = \frac{100}{\left[\left\{3125.0 - 1.4583 (104.72^2)\right\}^2 + \left\{62.5 (104.72)\right\}^2\right]^{\frac{1}{2}}} = 0.006927 \text{ rad}$$

$$\text{and} \quad \phi = \tan^{-1}\left(\frac{62.5 (104.72)}{3125.0 - 1.4583 (104.72^2)}\right) = -0.4705 \text{ rad} = -26.9606^\circ$$

3.38 $m = 100$ kg, $F_0 = 100$ N, $X_{max} = 0.005$ m at $\omega = 300$ rpm $= 31.416$ rad/sec. Equations (3.33) and (3.34) yield:

$$\omega = \omega_n \sqrt{1 - 2\,\varsigma^2} = \sqrt{\frac{k}{m}}\,\sqrt{1 - 2\,\varsigma^2} = 31.416$$

$$\text{or} \quad k\,(1 - 2\,\varsigma^2) = (100)\,(31.416^2) = 98{,}696.5056 \qquad (1)$$

$$\text{and} \quad X_{max} = \delta_{st}\,\frac{1}{2\,\varsigma\sqrt{1 - \varsigma^2}} = \frac{F_0}{k}\,\frac{1}{2\,\varsigma\sqrt{1 - \varsigma^2}} = 0.005$$

$$\text{or} \quad k\,\varsigma\sqrt{1 - \varsigma^2} = \frac{F_0}{2\,(0.005)} = 10{,}000.0 \qquad (2)$$

Divide Eq. (1) by (2):

$$\frac{1 - 2\,\varsigma^2}{\varsigma\sqrt{1 - \varsigma^2}} = 9.8696 \qquad (3)$$

Squaring Eq. (3) and rearranging leads to:

$$101.4090\,\varsigma^4 - 101.4090\,\varsigma^2 + 1 = 0 \quad \text{or} \quad \varsigma = 0.0998,\ 0.9950$$

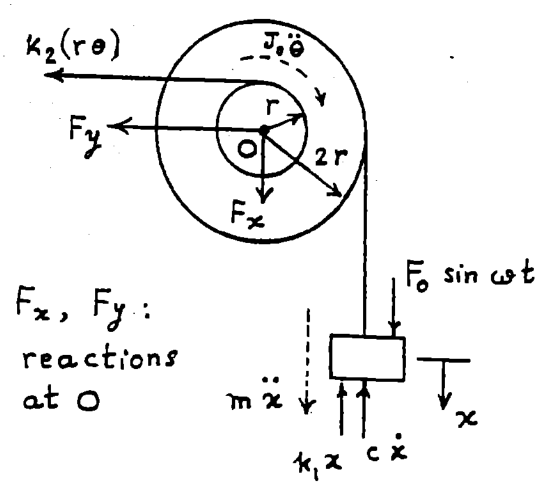
Using $\varsigma = 0.0998$ in Eq. (1), we obtain

$$k = \frac{98696.5056}{1 - 2\,(0.0998^2)} = 100{,}702.4994 \text{ N/m}$$

Since $\varsigma = \dfrac{c}{2\,m\,\omega_n}$, we find

$$c = 2\,m\,\omega_n\,\varsigma = 2\,(100)\sqrt{\frac{100702.4944}{1000}}\,(0.0998) = 633.4038 \text{ N–s/m}$$

3.39



Equation of motion for rotation of pulley about O:

$$-k_2\,(\theta\,r)\,r - J_0\,\ddot{\theta} - k_1\,x\,(2\,r) - c\,\dot{x}\,(2\,r) + F_0\sin\omega\,t\,(2\,r) - m\,\ddot{x}\,(2\,r) = 0 \qquad (1)$$

where $\theta = x/(2r)$. Equation (1) can be rearranged as:

$$\left(\frac{J_0}{2\,r} + 2\,m\,r\right)\ddot{x} + 2\,c\,r\,\dot{x} + \left(2\,k_1\,r + \frac{1}{2}\,k_2\,r\right)x = 2\,r\,F_0\sin\omega\,t \qquad (2)$$

For given data, Eq. (2) becomes

$$11\,\ddot{x} + 50\,\dot{x} + 112.5\,x = 5 \sin 20\,t \tag{3}$$

Steady state response is given by Eq. (3.25):

$$x_p(t) = X \cos(\omega t - \phi)$$

where $X = \dfrac{5}{\left[\left\{112.5 - 11\,(20^2)\right\}^2 + \left\{50\,(20)\right\}^2\right]^{\frac{1}{2}}} = 0.001136$ m

and $\phi = \tan^{-1}\left(\dfrac{50\,(20)}{112.5 - 11\,(20^2)}\right) = -0.2291$ rad $= -13.1287°$

(3.40) (a)

$$\sum M_0 = 0 \text{ (about hinge):}$$

$$I_0\,\ddot{\theta} + \left(k\,\theta\,\frac{3\,\ell}{4}\right)\frac{3\,\ell}{4} + (c\,\ell\,\dot{\theta})\,\ell = \frac{\ell}{2}\,F_0 \sin\omega t$$

$$\text{or}\quad I_0\,\ddot{\theta} + c\,\ell^2\,\dot{\theta} + \frac{9}{16}\,k\,\ell^2\,\theta = \frac{F_0\,\ell}{2}\sin\omega t$$

Magnitude of steady state response:

$$\Theta_a = \left(\frac{F_0\,\ell}{2}\right) \Big/ \left[\left\{\frac{9}{16}\,k\,\ell^2 - I_0\,\omega^2\right\}^2 + (c\,\ell^2\,\omega)^2\right]^{\frac{1}{2}} \tag{1}$$

(b)

$$\sum M_0 = 0 \text{ (about hinge):}$$

$$I_0\,\ddot{\theta} + (k\,\ell\,\theta)\,\ell + \left(c\,\frac{3\,\ell}{4}\,\dot{\theta}\right)\frac{3\,\ell}{4} = \frac{\ell}{2}\,F_0 \sin\omega t$$

$$\text{or}\quad I_0\,\ddot{\theta} + \frac{9}{16}\,c\,\ell^2\,\dot{\theta} + k\,\ell^2\,\theta = \frac{F_0\,\ell}{2}\sin\omega t$$

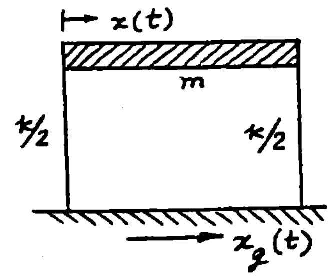
Magnitude of steady state response:

$$\Theta_b = \left(\frac{F_0\,\ell}{2}\right) \Big/ \left[\left\{k\,\ell^2 - I_0\,\omega^2\right\}^2 + \left\{\frac{9}{16}\,c\,\ell^2\,\omega\right\}^2\right]^{\frac{1}{2}} \tag{2}$$

Usually, c is small compared to k. If the term containing c is negligible, Θ_a will be smaller than Θ_b. Hence arrangement (a) is desirable.

3.41 $\ddot{y}(t) = \ddot{x}_g(t) = A \cos \omega t$; $\dot{y}(t) = \frac{A}{\omega} \sin \omega t + B_1$

$$y(t) = -\frac{A}{\omega^2} \cos \omega t + B_1 t + B_2$$



Assuming $y(0) = \dot{y}(0) = 0$, we get

$$y(t) = -\frac{A}{\omega^2} \cos \omega t$$

Equation of motion:

$$m\ddot{x} + k(x - y) = 0$$

i.e.,
$$m\ddot{z} + kz = -m\ddot{y} = -m\ddot{x}_g(t) = -mA \cos \omega t$$

where $z = x - y$

Solution is :
$$z(t) = \frac{-mA \cos \omega t}{k - m\omega^2}$$

$$\therefore \quad x(t) = z(t) + y(t) = -\left(\frac{m}{k - m\omega^2} + \frac{1}{\omega^2}\right) A \cos \omega t$$

3.42 From solution of problem 3.41,

$$x(t) = \left|\frac{-mA}{k - m\omega^2}\right| \sin \omega t - \frac{A}{\omega^2} \sin \omega t$$

$$= \left|\frac{-2000\left(\frac{100}{1000}\right)}{0.1 * 10^6 - 2000\,(25)^2}\right| \sin 25t - \left(\frac{100}{1000}\right)\frac{1}{(25)^2} \sin 25t$$

For maximum $x(t)$,

$$x(t) = \left(\frac{-200}{1.15 \times 10^6} - \frac{1}{6250}\right) \sin 25t = -3.3391 \times 10^{-4} \sin 25t \ \text{m}$$

$\therefore$ *Maximum horizontal displacement of floor* $= 0.3339$ mm

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3.43 $m(\ddot{x}-\ddot{y}) + k(x-y) = -m\ddot{y} = -m\ddot{x}_g$ (E$_1$)

Here $y(t) = x_g(t) = X_g \cos \omega t$, and Eq. (E$_1$) becomes

$$m\ddot{z} + kz = m\omega^2 X_g \cos \omega t \quad \text{with} \quad z = x - y$$

Solution is:

$$z(t) = \frac{m\omega^2 X_g \cos \omega t}{k - m\omega^2} = \frac{X_g r^2 \cos \omega t}{1 - r^2}$$

with $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{0.5 \times 10^6}{2000}} = 15.8114$ rad/sec

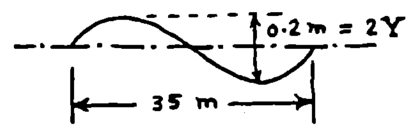
and $r = \omega/\omega_n = 200/15.8114 = 12.6491$

$$z(t) = \left(\frac{15}{1000}\right)\left\{\frac{12.6491^2}{1 - 12.6491^2}\right\} \cos 200t = -0.01509 \cos 200t \ \text{m}$$

$$x(t) = y(t) + z(t) = \{0.015 \cos 200t - |0.01509| \cos 200t\} \ \text{m}$$

∴ Amplitude of vibration of floor = 0.03009 m = 30.09 mm.

3.44 Time taken by car to travel one cycle (35 m) is



$$\tau = \frac{35 \times 3600}{60 \times 1000} = 2.1 \text{ sec}$$

Excitation frequency $= \omega = \frac{2\pi}{\tau} = 2.992$ rad/sec

$\omega_n = 2\pi(2) = 12.5664$ rad/sec , $r = \frac{\omega}{\omega_n} = 0.2381$, $\zeta = 0.15$

Amplitude of vibration of car is given by Eq. (3.68):

$$\frac{X}{Y} = \left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}\right]^{1/2} \qquad (E_1)$$

$$X = 0.1 \left\{\frac{1 + (2 \times 0.15 \times 0.2381)^2}{(1 - 0.2381^2)^2 + (2 \times 0.15 \times 0.2381)^2}\right\}^{1/2}$$

$$= 0.105977 \text{ m}$$

The most unfavorable speed corresponds to the maximum of $\frac{X}{Y}$ in Eq. (E$_1$). For maximum of $\frac{X}{Y}$ with respect to r,

$$\frac{d}{dr}\left[\frac{1 + 4\zeta^2 r^2}{1 + r^4 - 2r^2 + 4\zeta^2 r^2}\right] = 0$$

i.e., $$\frac{(1 + r^4 - 2r^2 + 4\zeta^2 r^2)(8\zeta^2 r) - (1 + 4\zeta^2 r^2)(4r^3 - 4r + 8\zeta^2 r)}{(1 + r^4 - 2r^2 + 4\zeta^2 r^2)^2} = 0$$

i.e., $$-4r(2\zeta^2 r^4 + r^2 - 1) = 0$$

i.e., $$r = 0 \quad \text{or} \quad r^2 = \frac{-1 \pm \sqrt{1 + 8\zeta^2}}{4\zeta^2}$$

Feasible value of $r^2 = \dfrac{-1 + \sqrt{1+8(0.15)^2}}{4(0.15)^2} = 0.9586$

$r = \dfrac{\omega}{\omega_n} = 0.9791$

$\omega = 0.9791\,(12.5664) = 12.3035$ rad/sec $= \dfrac{2\pi}{\tau}$

where $\tau = \dfrac{35 \times 3600}{s \times 1000}$ and s = speed of car in km/hr.

$\therefore \quad s = \dfrac{12.3035 \times 35 \times 3.6}{2\pi} = 246.7279$ km/hr.

3.45 Equations (3.73) and (3.68) give

$$F_T = m\omega^2 X = m\omega^2 Y \left[\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right]^{1/2}$$

$$\frac{F_T}{kY} = \frac{m\omega^2}{k}\left[\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right]^{1/2}$$

$$= r^2\left[\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right]^{1/2}$$

3.46 Eq.(3.75): $\quad m\ddot{z} + c\dot{z} + kz = -m\ddot{y} = m\omega^2 Y\cos\omega t$

steady-state solution is:

$$z(t) = \frac{m\omega^2 Y\cos(\omega t-\phi_1)}{(k-m\omega^2)^2+(c\omega)^2} = Z\cos(\omega t-\phi_1)$$

where $\phi_1 = \tan^{-1}\left(\dfrac{c\omega}{k-m\omega^2}\right)$

Damping force $= c\,\dfrac{dz}{dt} = -c\omega Z\sin(\omega t-\phi_1)$

Energy absorbed per cycle by the damper (E):

$$E = \int_{t=0}^{2\pi/\omega} c\,\frac{dz}{dt}\cdot dz = \int_0^{2\pi/\omega}\{-c\omega Z\sin(\omega t-\phi_1)\}\{-\omega Z\sin(\omega t-\phi_1)\}\,dt$$

$$= c\,\omega^2 Z^2\int_0^{2\pi/\omega}\sin^2(\omega t-\phi_1)\,dt = \pi c\omega Z^2$$

Since $Z = m\omega^2 Y\Big/\sqrt{(k-m\omega^2)^2+(c\omega)^2}$,

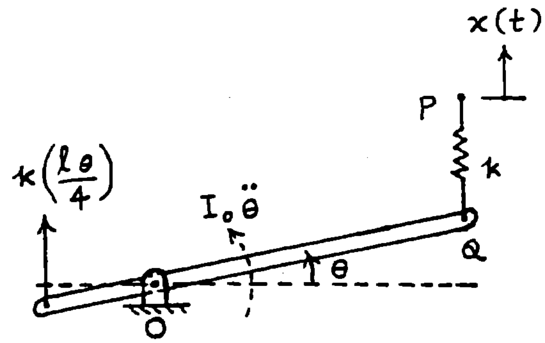
$$E = \left\{\frac{\pi c\omega\,(m^2\omega^4 Y)}{(k-m\omega^2)^2+c^2\omega^2}\right\}$$

For maximum power, $\dfrac{dE}{dc} = 0$

i.e., $$\frac{\{(k-m\omega^2)^2+c^2\omega^2\}\,(\pi\omega^5 m^2 Y) - \pi c\,\omega^5 m^2 Y\,(2c\,\omega^2)}{\{(k-m\omega^2)^2+c^2\omega^2\}^2} = 0$$

i.e., $$c = \left(\frac{k-m\omega^2}{\omega}\right).$$

(3.47)



Linear displacement of point Q due to $\theta = \frac{3\,\ell}{4}\,\theta$ and net compression of spring PQ = $\frac{3}{4}\,\ell\,\theta - x(t)$. Equation of motion:

$$I_0\,\ddot{\theta} = -\frac{k\,\ell\,\theta}{4}\,\frac{\ell}{4} - k\left(\frac{3\,\ell\,\theta}{4} - x(t)\right)\frac{3\,\ell}{4} \tag{1}$$

where $I_0 = \frac{1}{12}\,m\,\ell^2 + m\,(\frac{\ell}{4})^2 = \frac{7}{48}\,m\,\ell^2 = \frac{7}{48}\,(10)\,(1^2) = 1.4583$ kg–m^2

Hence Eq. (1) can be rewritten as

$$I_0\,\ddot{\theta} + \left(\frac{5}{8}\,k\,\ell^2\right)\theta = \left(\frac{3}{4}\,k\,\ell\,x_0\right)\sin\omega\,t \tag{2}$$

Steady state angular displacement of the bar is given by Eq. (3.6):

$$\Theta = \left(\frac{3}{4}\,k\,\ell\,x_0\right)\Big/\left(\frac{5}{8}\,k\,\ell^2 - I_0\,\omega^2\right) \tag{3}$$

$$= \left(\frac{3}{4}\,(1000)\,(1)\,(0.01)\right)\Big/\left(\frac{5}{8}\,(1000)\,(1^2) - 1.4583\,(10^2)\right) = 0.01565 \text{ rad}$$

and hence $\theta(t) = \Theta\sin\omega\,t = 0.01565\sin 10\,t$ rad

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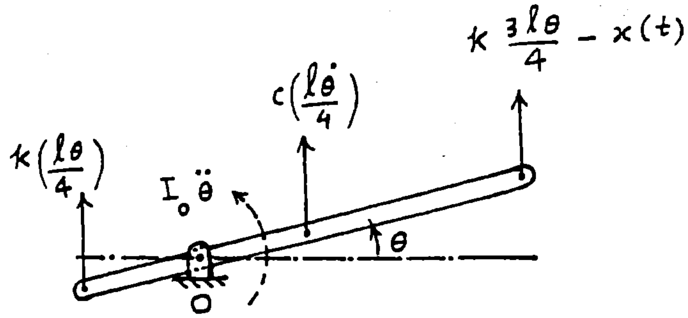
(3.48) Equation of motion:

$$I_0\,\ddot{\theta} = -k\,\frac{\ell\,\theta}{4}\,(\frac{\ell}{4}) - c\,\frac{\ell}{4}\,\dot{\theta}\,(\frac{\ell}{4}) - k\left(\frac{3\,\ell}{4}\,\theta - x(t)\right)\frac{3\,\ell}{4}$$

i.e., $$I_0\,\ddot{\theta} + \frac{1}{16}\,c\,\ell^2\,\dot{\theta} + \frac{5}{8}\,k\,\ell^2 = \frac{3}{4}\,k\,\ell\,x(t) = \frac{3}{4}\,k\,\ell\,x_0\sin\omega\,t \tag{1}$$

where $$I_0 = \frac{1}{12}\,m\,\ell^2 + m\left(\frac{\ell}{4}\right)^2 = \frac{7}{48}\,m\,\ell^2 = \frac{7}{48}\,(10)\,(1^2) = 1.4583 \text{ kg–m}^2 \tag{2}$$



Using given data, Eq. (1) can be expressed as

$$1.4583\,\ddot{\theta} + \frac{1}{16}\,(500)\,(1^2)\,\dot{\theta} + \frac{5}{8}\,(1000)\,(1^2)\,\theta = \frac{3}{4}\,(1000)\,(1)\,(0.01)\sin 10\,t$$

$$\text{i.e.,}\quad 1.4583\,\ddot{\theta} + 31.25\,\dot{\theta} + 625.0\,\theta = 7.5\sin 10\,t \tag{3}$$

Steady state angular displacement of the bar is given by Eq. (3.28) with:

$$\Theta = \frac{7.5}{\left\{\left[625.0 - 1.4583\,(10^2)\right]^2 + 31.25^2\,(10^2)\right\}^{\frac{1}{2}}} = 0.01311 \text{ rad}$$

$$\phi = \tan^{-1}\left(\frac{31.25\,(10)}{625.0 - 1.4583\,(10^2)}\right) = 0.5779 \text{ rad}$$

$$\therefore\ \theta(t) = \Theta\sin(\omega\,t - \phi) = 0.01311\sin(10\,t - 0.5779) \text{ rad}$$

(3.49) Displacement transmissibility (T):

$$T = \frac{X}{Y} = \left[\frac{1 + (2\,\zeta\,r)^2}{(1 - r^2)^2 + (2\,\zeta\,r)^2}\right]^{\frac{1}{2}}$$

For maximum of T,

$$\frac{dT}{dr} = \frac{1}{2}\left[\frac{1 + 4\,\zeta^2\,r^2}{(1 + r^4 - 2\,r^2) + 4\,\zeta^2\,r^2}\right]^{-\frac{1}{2}}$$
$$\frac{\left[(1 - r^2)^2 + (2\,\zeta\,r)^2\right](8\,\zeta^2\,r) - (1 + 4\,\zeta^2\,r^2)\left[4\,r^3 - 4\,r + 8\,\zeta^2\,r\right]}{\left[(1 - r^2)^2 + (2\,\zeta\,r)^2\right]^2} = 0$$

This equation can be simplified to obtain:

$$(2\,\zeta^2)\,r^4 + r^2 - 1 = 0$$

$$\text{Solution:}\quad r^2 = \frac{-1 \pm \sqrt{1 + 8\,\zeta^2}}{4\,\zeta^2}$$

$$\text{or}\quad r = r_m = \frac{1}{2\,\zeta}\sqrt{\sqrt{1 + 8\,\zeta^2} - 1}$$

3.50

Empty

m = 500 kg

at speed = 90 km/hr = 25 m/s

and wavelength = 3.7 m

$$\omega = 2\,\pi\,f = 2\,\pi\left(\frac{25}{3.7}\right)$$

$$= 42.454 \text{ rad/sec}$$

$k = 450 \times 10^3$ N/m

$\zeta = 0.2$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{450 \times 10^3}{500}}$$

$$= 30 \text{ rad/sec}$$

$$r = \frac{\omega}{\omega_n} = \frac{42.454}{30} = 1.41513$$

$$(2\,\zeta\,r)^2 = (2 \times 0.2 \times 1.41513)^2$$

$$= 0.320415$$

$$(1 - r^2)^2 = 1.005193$$

$$\frac{X}{Y} = \left\{\frac{1 + (2\,\zeta\,r)^2}{(1 - r^2)^2 + (2\,\zeta\,r)^2}\right\}^{\frac{1}{2}}$$

$$= \left\{\frac{1 + 0.320416}{1.005193 + 0.320416}\right\}^{\frac{1}{2}}$$

$$= 0.99804$$

Amplitude of vibration of automobile is diminished by a factor of 0.99804

Fully loaded

m = 1500 kg

ω = 42.454 rad/sec

$k = 450 \times 10^3$ N/m

$\zeta = 0.2$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{450 \times 10^3}{1500}}$$

$$= 17.32051 \text{ rad/sec}$$

$$r = \frac{\omega}{\omega_n} = 2.45108$$

$$(2\,\zeta\,r)^2 = 0.961249$$

$$(1 - r^2)^2 = 25.078$$

$$\frac{X}{Y} = \left\{\frac{1 + (2\,\zeta\,r)^2}{(1 - r^2)^2 + (2\,\zeta\,r)^2}\right\}^{\frac{1}{2}}$$

$$= \left\{\frac{1 + 0.961249}{25.078 + 0.961249}\right\}^{\frac{1}{2}}$$

$$= 0.274443$$

Amplitude of vibration of automobile is diminished by a factor of 0.274443

(3.51) Equation of motion: $M\ddot{x} + c\dot{x} + kx = m e \omega^2 \sin \omega t$

where $\omega = \dfrac{3000\,(2\pi)}{60} = 314.16$ rad/sec, M = 100 kg, c = 2000 N-s/m, k = 10^6 N/m, m = 0.1 kg and e = r = 0.1 m. Steady state response is:

$$x_p(t) = X \sin(\omega t - \phi)$$

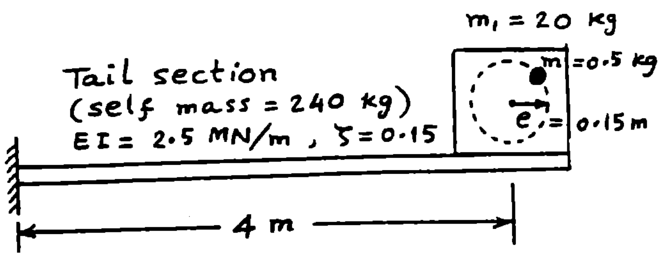
$$\text{where}\quad X = \frac{m e \omega^2}{\left[(k - M\omega^2)^2 + (c\omega)^2\right]^{\frac{1}{2}}}$$

$$= \frac{0.1\,(0.1)\,(314.16^2)}{\left[\left\{10^6 - 100\,(314.16^2)\right\}^2 + (2000\,(314.16))^2\right]^{\frac{1}{2}}} = 110.9960\,(10^{-6})\text{ m}$$

$$\text{and}\quad \phi = \tan^{-1}\left(\frac{c\omega}{k - M\omega^2}\right) = \tan^{-1}\left(\frac{2000\,(314.16)}{10^6 - 100\,(314.16^2)}\right)$$
$$= -0.07072\text{ rad} = -4.0520^\circ$$

(3.52) k = spring constant of cantilever beam

$$= \frac{3EI}{l^3} = \frac{3\,(2.5\times10^6)}{4^3}$$
$$= 0.1172\times10^6\ \text{N/m}$$



$$\omega_n = \sqrt{\frac{k}{m_1 + 0.25\, m_b}} = \sqrt{\frac{0.1172\times10^6}{20 + 0.25\,(240)}} = 38.2753\ \text{rad/sec}$$

$\omega = 2\pi\,(1500)/60 = 157.08$ rad/sec

$r = \omega/\omega_n = 157.08/38.2753 = 4.1040$, $\quad r^2 = 16.8428$

Forced response is given by Eq. (3.79):

$$x_p(t) = X \sin(\omega t - \phi)$$

where
$$X = \frac{m e}{m_1}\cdot\frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$$= \frac{(0.5)(0.15)}{20}\ \frac{16.8428}{\sqrt{(1 - 16.8428)^2 + (2\times0.15\times4.1040)^2}}$$

$$= 3.9747\times10^{-3}\ \text{m} = 3.9747\ \text{mm}$$

3.53 $\delta_{st} = \frac{45}{1000}\ m = \frac{Mg}{k} = \frac{380 \times 9.81}{k}$

i.e., $k = 82,840\ N/m$

$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{82,840}{380}} = 14.7648\ rad/sec$; $\omega = \frac{2\pi(1750)}{60} = 183.26\ rad/sec$

$r = \frac{\omega}{\omega_n} = \frac{183.26}{14.7648} = 12.412$; $r^2 = 154.0566$

(i) Amplitude of vibration

$$X = \frac{me}{M} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{0.15}{380} \frac{154.0566}{\sqrt{(153.0566)^2 + 0}}$$

$$= 3.9732 \times 10^{-4}\ m$$

(ii) Force transmitted to ground

$$= kX = (82840)(3.9732 \times 10^{-4}) = 32.9140\ N$$

3.54 $I = \frac{1}{12}(0.5)(0.1)^3 = 0.4167 \times 10^{-4}\ m^4$

$$k = \frac{192\,EI}{l^3} = \frac{192\ (2.07 \times 10^{11})(0.4167 \times 10^{-4})}{(5)^3} = 1.3248 \times 10^7\ N/m$$

(a) $\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{13.248 \times 10^6}{75}} = 420.2856\ rad/sec$

$\omega = 2\pi(1200)/60 = 125.664\ rad/sec$

$r = \omega/\omega_n = 125.664/420.2856 = 0.299$, $r^2 = 0.0894$

Amplitude of steady-state vibration is given by Eq. (3.30) with $\zeta = 0$:

$$X = \frac{\delta_{st}}{|r^2 - 1|} = \frac{F_0}{k|(r^2-1)|} = \frac{5000}{(1.3248 \times 10^7)(0.9106)}$$

$$= 0.4145 \times 10^{-3}\ m$$

(b) Using the effective mass due to self weight of beam (for a cantilever) to be valid here also,

$$\omega_n = \sqrt{\frac{k}{M + 0.2357\ m}}$$

where M = mass of motor = 75 kg, and

m = mass of beam = $(5 \times 0.5 \times 0.1)\left(\frac{76.5 \times 10^3}{9.81}\right) = 1949.5313\ kg$

$$\omega_n = \sqrt{\frac{13.248 \times 10^6}{75 + (1949.5313)(0.2357)}} = 157.4339\ rad/sec$$

$r = \omega/\omega_n = 125.664/157.4339 = 0.7982$, $r^2 = 0.6371$

$$X = \frac{\delta_{st}}{|r^2 - 1|} = \frac{F_0}{k\,|r^2-1|} = \frac{5000}{(1.3248 \times 10^7)(0.3629)}$$

$$= 1.0400 \times 10^{-3}\ m$$

3.55 Let width $= 0.5$ m and thickness $= t$ m.

$$I = \frac{1}{12}(0.5)\,t^3 = \frac{t^3}{24} \quad m^4$$

$$k = \frac{3EI}{l^3} = \frac{3(2.07\times10^{11})(t^3/24)}{(5)^3} = 2.07\times10^{8}\,t^3 \quad N/m$$

$$\omega_n = \sqrt{\frac{k}{M + 0.2357\,m}}$$

where m = mass of beam $= (5\times0.5\times t)\left(\frac{76.5\times10^{3}}{9.81}\right) = 19495.41\,t$ kg

$$\omega_n = \sqrt{\frac{2.07\times10^{8}\;t^3}{75 + 0.2357\,(19495.41\,t)}}$$

$$r = \frac{\omega}{\omega_n} = 125.664\sqrt{\frac{75 + 4595.0688\,t}{2.07\times10^{8}\,t^3}}$$

$$X = \frac{\delta_{st}}{|r^2-1|} = \frac{F_0}{k\,|r^2-1|}$$

i.e.,

$$0.5 = \frac{5000}{(2.07\times10^{8}\,t^3)\left\{(125.664)^2\left[\frac{75+4595.0688\,t}{2.07\times10^{8}\,t^3}\right]-1\right\}}$$

i.e., $1.3108\times10^{4}\,t^3 - 4595.069\,t - 74.367 = 0$

By trial and error, the value of t is found as

$t \approx 0.6$ m.

Since this is too large, we can start with a new width such as 1.0 m.

3.56 $m = (600/9.81)$ N , $\omega = 2\pi(1000)/60 = 104.72$ rad/sec

$k = 6\,(6000) = 36{,}000$ N/m

$$\omega_n = \sqrt{k/m} = \sqrt{36000\Big/\left(\frac{600}{9.81}\right)} = 24.2611 \text{ rad/sec}$$

$r = \omega/\omega_n = 104.72/24.2611 = 4.3164$, $r^2 = 18.6311$

$$X = \frac{F_0}{k\,|r^2-1|} = \frac{m_0\,e\,\omega^2}{k\,|r^2-1|}$$ where m_0 = unbalanced mass and e = eccentricity

i.e.,

$$2.5\times10^{-3} = \frac{m_0\,e\,(104.72)^2}{36000\,|17.6311|}$$

i.e., $m_0\,e = 0.1447$ kg-m

$\therefore$ Unbalance $= W_0\,e = m_0 g\,e = 0.1447\,(9.81) = 1.4195$ N-m

---

3.57 Compressor weight 5000 N operating at 1500 rpm
A helical spring with stiffness 8×10^6 N/m
another with a stiffness of 2.5×10^6 N/m

Solution: $m = \frac{5000}{9.8} = 510.2$ kg, $\omega = \frac{2\pi(1500)}{60} = 157.08$ rad/sec

Possible isolators are: (with only one spring)

(i) $k = 8 \times 10^6$ N/m, $\zeta = 0$
(ii) $k = 2.5 \times 10^6$ N/m, $\zeta = 0$
(iii) $k = 8 \times 10^6$ N/m, $\zeta = 0.15$
(iv) $k = 2.5 \times 10^6$ N/m, $\zeta = 0.15$

We will compare the force transmissibilities of these isolators.

Force transmissibility $= T_r = \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}}$

(i) $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{8 \times 10^6}{510.2}} = 125.2203$ rad/sec

$$r = \frac{\omega}{\omega_n} = \frac{157.08}{125.2203} = 1.2544, \quad r^2 = 1.5736$$

$$T_r = \frac{1}{|1 - r^2|} = \frac{1}{0.5736} = 1.7434$$

(ii) $\omega_n = \sqrt{\frac{2.5 \times 10^6}{510.2}} = 70$ rad/sec

$$r = \frac{\omega}{\omega_n} = \frac{157.08}{70} = 2.244, \quad r^2 = 5.0355$$

$$T_r = \frac{1}{|1 - r^2|} = \frac{1}{4.0355} = 0.2478$$

(iii) $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{8 \times 10^6}{510.2}} = 125.2203$ rad/sec

$r = 1.2544$, $r^2 = 1.5736$, $\zeta = 0.15$

$$T_r = \sqrt{\frac{1 + (2 \times 1.2544 \times 0.15)^2}{(1 - 1.5736)^2 + (2 \times 1.2544 \times 0.15)^2}} = 1.5575$$

(iv) $\omega_n = 70$ rad/sec, $r = 2.244$, $r^2 = 5.0355$, $\zeta = 0.15$

$$T_r = \sqrt{\frac{1 + (2 \times 2.244 \times 0.15)^2}{(1 - 5.0355)^2 + (2 \times 2.244 \times 0.15)^2}} = 0.2946$$

$\therefore$ Isolation (ii) is best.

3.58 Amplitude of vibration have been observed to be 0.015 m at resonance and 0.004 m beyond resonance.

Solution: Eq. (3.82)

$$\frac{MX}{me} = \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\,\zeta\, r)^2}}$$

when $r = 1$

$$\frac{MX}{me} = \frac{1}{2\,\zeta} \quad \text{or} \quad \frac{M}{me} = \frac{1}{2\,\zeta\, X} = \frac{1}{2\,\zeta\,(0.015)} = \frac{1}{0.03\,\zeta} \qquad (E_1)$$

when r = large

$$\frac{MX}{me} \approx 1 \quad \text{or} \quad \frac{M}{me} \approx \frac{1}{X} = \frac{1}{0.004} \qquad (E_2)$$

Combining (E_1) and (E_2),

$$\frac{M}{me} = \frac{1}{0.004} = \frac{1}{0.03\,\zeta}$$

$\therefore\ \zeta = 0.1333$

3.59 Motor weight 3500 N, running at 1800 rpm.
Helical springs wire diameter 0.0065 m, coil diameter 0.08 m,
Rotor weight 500 N, center of mass offset by 0.2 mm

Solution:
For each spring:

$$\underset{\sim}{k} = \frac{G\, d^4}{64\, n\, R^3} = \frac{(83 \times 10^9)\,(0.0065)^4}{64\,(8)\,(0.04)^3} = 4521.5 \text{ N/m}$$

Total k = 4 (4521.5) = 18085 N/m

$$\omega = \frac{2\,\pi\,(1800)}{60} = 188.496 \text{ rad/sec}$$

$$m = \frac{500}{9.8} \text{ kg}, \quad M = \frac{3500}{9.8} \text{ kg}, \quad \zeta = 0$$

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{18085}{(3500/9.8)}} = 7.1162 \text{ rad/sec}$$

$$r = \frac{188.496}{7.1162} = 26.4882,$$

$$r^2 = 701.6244$$

$$X = \frac{me}{M}\,\frac{r^2}{\sqrt{(1 - r^2)^2 + (2\,\zeta\, r)^2}} = \frac{500\,(0.0002)}{3500}\left(\frac{701.6244}{700.6244}\right)$$

$$= 2.8612 \times 10^{-5} \text{ m}$$

3.60 shaft diameter 5 mm,
rotor weight 100 N,
eccentricity e 0.0002 m
find: (b) the power needed to drive the shaft.

Solution:

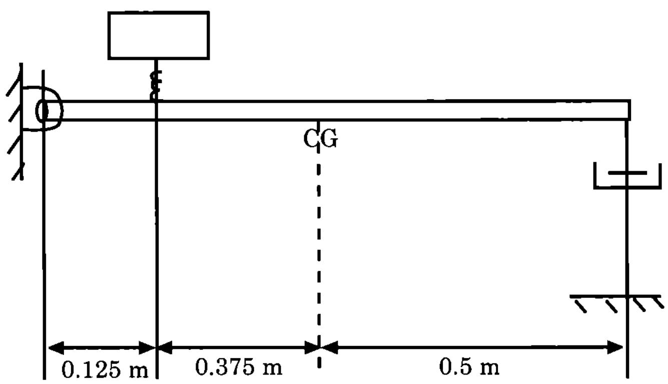
$$\omega = \frac{2\,\pi\,(1500)}{60} = 157.08 \text{ rad/sec}$$

(a) Force due to eccentricity of rotor

$$= me\,\omega^2 = \left(\frac{100}{9.8}\right)(0.0002)\,(157.08)^2 = 50.3554 \text{ N}$$

(b) P = (Force) (eccentricity) (angular velocity)
= (50.3554) (0.0002) (157.08)
= 1.5820 (W)

3.61 Plate weight 500 N,
Dashpot C = 200 N-sec/m
Fan weight 200 N, rotating at 750 rpm,
Spring k = 40 kN/m
Center of gravity of fan is 3 mm offset from axis of rotation.



Solution:

$$\omega_{n,\,fan} = \sqrt{\frac{k}{m_{fan}}} = \sqrt{\frac{40 \times 10^3}{200/9.8}} = 44.2719 \text{ rad/sec}$$

$$\omega = \frac{2\,\pi\,(750)}{60} = 78.54 \text{ rad/sec}$$

$$(J_p)_{plate + fan} = \frac{1}{3}\left(\frac{500}{9.8}\right)(1)^2 + \left(\frac{200}{9.8}\right)(0.125)^2 = 17.3257 \text{ kg}\cdot\text{m}^2$$

$$F_0 = me\,\omega^2 = \left(\frac{200}{9.8}\right)(0.003)\,(78.54)^2 = 377.6652 \text{ N}$$

Point R is subjected to the force F (t) = $F_0 \cos \omega t$ = 377.6652 cos 78.54t (N)
Assume S is not moving.
Then R is displaced by

$$x(t) = \frac{F_0 \cos \omega t}{\left|k - m\omega^2\right|} = \frac{F_0 \cos \omega t}{k\left|1 - \left(\frac{\omega}{\omega_n}\right)^2\right|} = \frac{377.6652 \cos \omega t}{(40 \times 10^3)\left|1 - \left(\frac{78.54}{44.2719}\right)^2\right|}$$

$$= 0.004397 \cos 78.54t \text{ (m)}$$

θ = angular displacement of plate PQ.
Displacement of S = 0.125 θ m
Extension of spring RS = (0.125 θ – 0.004397 cos 78.54t) m
Restoring moment of spring force about
P = (40×10^3) (0.125 θ – 0.004397 cos 78.54t) (0.125) N · m
Velocity of Q = (1) $\dot{\theta} = \dot{\theta}$ (m/s)
Damping force at Q = $\dot{\theta}$ (200) = 200 $\dot{\theta}$ (N)
Moment of damping force about P = (200 $\dot{\theta}$)(1) = 200 $\dot{\theta}$ (N-m)
Equation of motion of plate PQ:

$$J_p \ddot{\theta} + 200\dot{\theta} + 5000(0.125\theta - 0.004397 \cos 78.54t) = 0$$

i.e. $17.3257\ddot{\theta} + 200\dot{\theta} + 625\theta = 21.985 \cos 78.54t$ $\quad (E_1)$

Comparing (E_1) with Eq. (3.24)

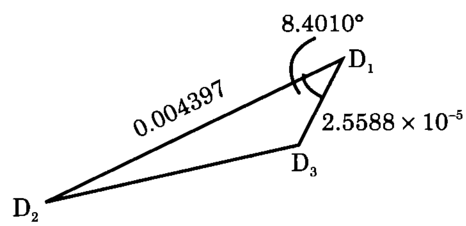
$$\theta_p(t) = \Theta \cos(\omega t - \phi)$$

where, from Eqs. (3.30) and (3.31), we get

$$\Theta = \frac{(21.985/625)}{\sqrt{(1 - 170.9986)^2 + (2 \times 0.9610 \times 13.0766)^2}}$$

$$= 2.0470 \times 10^{-4} \text{ (rad)}$$

$$\phi = \tan^{-1}\left(\frac{25.1333}{-169.9986}\right) = -8.4010°$$



Steady state motion of $\theta = \theta_p$ (1)
$= (2.0470 \times 10^{-4}) \cos (78.54t + 8.4010°)$ (m)

Displacement of S = Θ (0.125) = 2.5588×10^{-5} (m)

D_2D_3 = maximum deformation of spring ≈ 0.004397 m

Max. force transmitted to point S = k (D_2D_3) = (40×10^3) (0.004397) = 175.88 N

3.62

$$I = \int_0^{2\pi/\omega} \sin \omega t \cdot \cos(\omega t - \phi)\, dt$$

$$= \int_0^{2\pi/\omega} \sin \omega t \left[\cos \omega t \cdot \cos\phi + \sin \omega t \cdot \sin\phi \right] dt$$

$$= \int_0^{2\pi/\omega} \left\{ \cos\phi\,(\sin \omega t \cdot \cos \omega t) + \sin\phi\,(\sin^2 \omega t) \right\} dt$$

$$= \int_0^{2\pi/\omega} \left\{ \cos\phi \left(\frac{\sin 2\omega t}{2}\right) + \sin\phi \left(\frac{1 - \cos 2\omega t}{2}\right) \right\} dt$$

$$= \frac{\cos\phi}{2}\left(-\frac{\cos 2\omega t}{2\omega}\right)\Bigg|_0^{2\pi/\omega} + \frac{\sin\phi}{2}\left(t - \frac{\sin 2\omega t}{2}\right)\Bigg|_0^{2\pi/\omega}$$

$$= \frac{\pi}{\omega} \sin\phi$$

$$\Delta W' = \omega F_0 X \cdot I = \omega F_0 X \sin\phi$$

3.63

Let $x(t)$ = displacement of mass m

New length of each spring, $k_1 = (l^2 + x^2)^{1/2}$

New tension in each spring $k_1 = T = \left(\sqrt{l^2 + x^2} - l\right) k_1 + T_0$

Horizontal component of new tension in each spring k_1

$$= T x / \sqrt{l^2 + x^2}$$

Vertical component of new tension in each spring $k_1 = \dfrac{T\, l}{\sqrt{l^2 + x^2}}$

Total friction force $= \mu\, mg + \dfrac{2 T l}{\sqrt{l^2 + x^2}}$

when mass moves to right:

Equation of motion of mass m:

$$m\ddot{x} + k_2 x + \frac{2 T x}{\sqrt{l^2 + x^2}} - \mu\left[mg + \frac{2 T l}{\sqrt{l^2 + x^2}}\right] = p_0 A \sin\omega t$$

where A = area of piston.

i.e., $m\ddot{x} + x\left(k_2 + 2\dfrac{T_0}{l}\right) = \mu m g + 2\mu T_0 + p_0 A \sin\omega t$

Similarly, when the mass moves to left:

$$m\ddot{x} + x\left(k_2 + 2\frac{T_0}{l}\right) = -\mu m g - 2\mu T_0 + p_0 A \sin\omega t$$

3.64 $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2100}{2}} = 32.403703$ rad/sec

N = vertical force = $mg = 2\,(9.81) = 19.62$ N

$$\frac{\omega}{\omega_n} = \frac{2.5173268 \times 2\pi}{32.403703} = 0.4881191$$

$$X = \frac{F_0}{k}\left[\frac{1 - \left(\frac{4\mu N}{\pi F_0}\right)^2}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2}\right]^{1/2}$$

i.e., $$0.075 = \frac{120}{2100}\left[\frac{1 - \left\{\frac{4\mu\,(19.62)}{\pi\,(120)}\right\}^2}{(1 - 0.4881191^2)}\right]^{1/2}$$

i.e., $$1.3125 = \left(\frac{1 - 0.04334\,\mu^2}{0.5802473}\right)^{1/2}$$

i.e., $$0.9995666 = 1 - 0.04334\,\mu^2$$

i.e., $$\mu = 0.1$$

3.65 (a) $k = \frac{W}{\delta_{st}} = \frac{5000}{0.05} = 10^5$ N/m

When $\omega = \omega_n$, Eq. (3.102) gives

$$X = \frac{F_0}{k\beta} \Rightarrow 0.1 = \frac{1000}{(10^5)\,\beta} \Rightarrow \beta = 0.1$$

(b) $\Delta W = \pi\, c_{eq}\, \omega\, X^2 = \pi \beta k X^2$ where $c_{eq} = \frac{\beta k}{\omega}$ from Eq. (3.100)

$\Delta W = \pi\,(0.1)\,(10^5)\,(0.1)^2 = 314.16$ Joules/cycle

(c) Steady state amplitude at one-quarter of resonant frequency:

$$\frac{\omega}{\omega_n} = 0.25$$

$$X = \frac{F_0}{k\left[\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \beta^2\right]^{1/2}} = \frac{1000}{10^5\left[\left\{1 - 0.25^2\right\}^2 + (0.1)^2\right]^{1/2}}$$

$= 0.01061$ m

(d) Steady state amplitude at thrice the resonant frequency:

$$\frac{\omega}{\omega_n} = 3$$

$$X = \frac{1000}{10^5\left[\left(1 - 3^2\right)^2 + (0.1)^2\right]^{1/2}} = 0.00125 \text{ m}$$

3.66 $\Delta W = \pi \beta k X^{\gamma}$

$$\left.\begin{aligned} 3.8 &= \pi \beta (60\,000)(0.04)^{\gamma} \\ 9.5 &= \pi \beta (60000)(0.06)^{\gamma} \end{aligned}\right\} \Rightarrow \quad \begin{aligned} \beta (0.04)^{\gamma} &= 0.0000202 \\ \beta (0.06)^{\gamma} &= 0.0000504 \end{aligned}$$

Taking logarithms,

$$\ln \beta + \gamma \ln (0.04) = \ln (0.0000202)$$
$$\ln \beta + \gamma \ln (0.06) = \ln (0.0000504)$$

i.e.,
$$\ln \beta - 3.218876\, \gamma = -10.809828 \quad ---- (E_1)$$
$$\ln \beta - 2.813411\, \gamma = -9.895511 \quad ---- (E_2)$$

subtracting (E_1) from (E_2), $\quad 0.405465\, \gamma = 0.914309$

$$\gamma = 2.254964$$

From (E_1), $\ln \beta = -10.809828 + 3.218876\,(2.254964) = -3.551378$

$$\beta = 0.028685$$

3.67 Harmonic force F (t) = 20 cos 3 π t N

Displacement $x(t) = 0.01 \cos\left(3\pi t - \frac{\pi}{3}\right)$ m

Solution:

Work done $= W = \int F\, dx = \int F\, \dot{x}\, dt$

If $F(t) = F_0 \cos \omega t$ and $x(t) = X \cos \omega t - \phi)$, work done in one cycle

$$= W = -\int_0^{2\pi/\omega} F_0 \cos \omega t .\, \omega X \sin (\omega t - \phi)\, dt$$

$$= -\frac{F_0\, \omega X \cos\phi}{2}\left(-\frac{1}{2\omega}\cos 2\omega t\right)_0^{\frac{2\pi}{\omega}} + \frac{F_0\, \omega X \sin\phi}{2}\left(t + \frac{1}{2\omega}\sin 2\omega t\right)_0^{\frac{2\pi}{\omega}}$$

$$= F_0\, \pi X \sin\phi$$

Given: $F_0 = 20$ N, $\omega = 3\pi$ rad/sec, $\tau = \frac{2}{3}$ sec, $\phi = \frac{\pi}{3}$, $X = 0.01$ m

$$W = F_0\, \pi X \sin\phi = (20)\, \pi\, (0.01) \sin\frac{\pi}{3} = 0.5441 \text{ (J)}$$

(i) In one second, it will complete $1\frac{1}{2}$ cycles

$$W\big|_{1\,\text{sec}} = 1.5\, W = 0.8162 \text{ (J)}$$

(ii) In four seconds, it will complete 6 cycles

$$W\big|_{6\,\text{sec}} = 6\, W = 3.2646 \text{ (J)}$$

3.68 Damping force $= F = c(\dot{x})^n$

Energy dissipated per quarter cycle during harmonic motion $x(t) = X \sin \omega t$ is

$$\frac{\Delta W}{4} = \int_0^{\pi/2\omega} c(\dot{x})^n\, dx = \int_0^{\pi/2\omega} c\, (\omega X \cos \omega t)^n\, dx$$

But $dx = \dot{x}\, dt = \omega X \cos \omega t \cdot dt$

$$\Delta W = 4c\,\omega^{n+1}\,X^{n+1}\int_0^{\pi/2\omega}\cos^{n+1}\omega t\cdot dt$$

$$= 4c\,\omega^{n+1}\,X^{n+1}\left\{\frac{1}{(n+1)\omega}\cos^n\omega t\cdot\sin\omega t\,\Big|_0^{\pi/2\omega}+\frac{n}{n+1}\int_0^{\pi/2\omega}\cos^{n-1}\omega t\cdot dt\right\}$$

$$= 4c\,\omega^{n+1}\,X^{n+1}\left(\frac{n}{n+1}\right)\int_0^{\pi/2\omega}\cos^{n-1}\omega t\cdot dt$$

Equating this expression to $\pi\, c_{eq}\,\omega\, X^2$, we obtain

$$c_{eq} = \frac{4c\,\omega^n\,X^{n-1}}{\pi}\left(\frac{n}{n+1}\right)\int_0^{\pi/2\omega}\cos^{n-1}\omega t\cdot dt \equiv c\,\omega^n\,X^{n-1}\,\alpha_n$$

where $\alpha_n = \frac{4}{\pi}\left(\frac{n}{n+1}\right)\int_0^{\pi/2\omega}\cos^{n-1}\omega t\cdot dt$ ---- (E_1)

For example, for $n=2$, (E_1) becomes

$$\alpha_n = \frac{4}{\pi}\left(\frac{2}{3}\right)\int_0^{\pi/2\omega}\cos\omega t\cdot dt = \frac{8}{3\pi}\left(\frac{\sin\omega t}{\omega}\right)_0^{\pi/2\omega} = \frac{8}{3\pi\omega}$$

and hence $c_{eq} = \frac{8c\omega X}{3\pi}$

which can be seen to be same as the expression found in Example 3.7.
For few other values of n, α_n can be found as follows:

n	1	2	3	4
α_n	$\frac{1}{\omega}$	$\frac{8}{3\pi\omega}$	$\frac{3}{4\omega}$	$\frac{32}{15\pi\omega}$

The amplitude can be found as

$$X = \frac{F_0}{\sqrt{(k-m\omega^2)^2+c_{eq}^2\,\omega^2}} = \frac{F_0}{\sqrt{k^2(1-r^2)^2+c_{eq}^2\,\omega^2}}$$

$$= \frac{F_0}{\sqrt{k^2(1-r^2)^2+c^2\,\omega^{2(n+1)}\,X^{2(n-1)}\,\alpha_n^2}}$$

(3.69) Energy dissipated per cycle for viscous damping $= \pi c\,\omega X^2$
Energy dissipated per cycle for Coulomb damping $= 4\mu N X$

Equivalent viscous damping constant (c_{eq}) is given by

$$\pi\, c_{eq}\,\omega X^2 = \pi c\,\omega X^2 + 4\mu N X$$

$$c_{eq} = \left(c + \frac{4\mu N}{\pi\omega X}\right)$$

Amplitude X is given by

$$X = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c_{eq}^2 \omega^2}} = \frac{F_0}{\sqrt{k^2(1-r^2)^2 + c_{eq}^2 \omega^2}}$$

substituting for c_{eq}, squaring and rearranging,

$$X^2 \left\{ k^2 (1-r^2)^2 + c^2 \omega^2 \right\} + X \left(\frac{8 \mu N c \omega}{\pi} \right) + \left(\frac{16 \mu^2 N^2}{\pi^2} - F_0^2 \right) = 0$$

3.70 (a) Equation of motion $m\ddot{x} \pm \mu N + c(\dot{x})^3 + k x = F_0 \cos \omega t$

Thus the system has combined Coulomb and velocity-cubed damping.

For Coulomb damping, $c_{eq1} = \dfrac{4 \mu N}{\pi \omega X}$ (E_1)

For velocity-cubed damping, the equivalent viscous damping coefficient can be obtained from the solution of problem 3.68:

$$c_{eq2} = c\, \omega^3 X^2 \alpha_3 \qquad (E_2)$$

where

$$\alpha_3 = \frac{4}{\pi} \left(\frac{3}{4} \right) \int_0^{\pi/2\omega} \cos^2 \omega t \; dt = \frac{3}{4\omega} \qquad (E_3)$$

$$\therefore \quad c_{eq2} = \frac{3}{4} c\, \omega^2 X^2 \qquad (E_4)$$

and

$$c_{eq} = c_{eq1} + c_{eq2} = \frac{4 \mu N}{\pi \omega X} + \frac{3}{4} c \omega^2 X^2 \qquad (E_5)$$

(b) Steady state amplitude under harmonic force:

$$X = \frac{F_0}{\sqrt{k^2(1-r^2)^2 + c_{eq}^2 \omega^2}} = \frac{F_0}{\sqrt{k^2 (1-r^2)^2 + \left\{ \frac{4 \mu N}{\pi \omega X} + \frac{3}{4} c \omega^2 X^2 \right\}^2 \omega^2}} \qquad (E_6)$$

(c) Amplitude ratio:

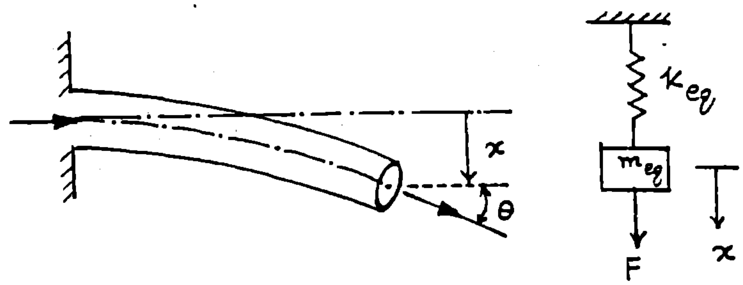
$$\frac{X}{\delta_{st}} = \frac{X}{(F_0/k)} = \frac{1}{\sqrt{(1-r^2)^2 + \left(\frac{c_{eq}^2 \omega^2}{k^2} \right)}}$$

$$= \frac{1}{\sqrt{(1-r^2)^2 + \left\{ \frac{4 \mu N}{\pi X k} + \frac{3}{4} \frac{c \omega^3 X^2}{k} \right\}^2}} \qquad (E_7)$$

At resonance, $r = 1$ and Eq. (E_7) reduces to

$$\left. \frac{X}{\delta_{st}} \right|_{resonance} = \frac{1}{\left\{ \frac{4 \mu N}{\pi X k} + \frac{3}{4k} c \omega^3 X^2 \right\}} \qquad (E_8)$$

(3.71) Model the pipe as a single degree of freedom system with m_{eq} = equivalent mass at end $= \frac{33}{140}$ m (m = mass of pipe; see Problem 2.46) and $k_{eq} = \frac{3\,E\,I}{\ell^3}$. Slope of pipe at end:



$$\theta = \frac{F\,\ell^2}{2\,E\,I} = \frac{F\,\ell^3}{3\,E\,I}\left(\frac{3}{2\,\ell}\right) = \frac{3\,x}{2\,\ell}$$

where x = end deflection of the cantilever pipe under a transverse load F. Force induced due to fluid velocity v is $\rho\,A\,v^2$. Force acting on the single degree of freedom system (in vertical direction):

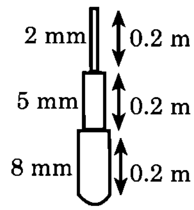
$$F = \rho\,A\,v^2 \sin\theta \approx \rho\,A\,v^2\,\theta = \rho\,A\,v^2\,\frac{3\,x}{2\,\ell}$$

Equation of motion: $m_{eq}\,\ddot{x} + k_{eq}\,x = F$

or $$\frac{33}{140}\,m\,\ddot{x} + \left(\frac{3\,E\,I}{\ell^3} - \frac{3\,\rho\,A\,v^2}{2\,\ell}\right)x = 0$$

Instability occurs when $\frac{3\,E\,I}{\ell^3} - \frac{3\,\rho\,A\,v^2}{2\,\ell} < 0$ or $v > \sqrt{\frac{2\,E\,I}{\rho\,A\,\ell^2}}$

3.72 Speed range 80 ~ 120 kph



Solution:

Assume R_e greater than 1000, Strouhal number (St) for shedding is taken as:

$St = \frac{fd}{V} = 0.21$ where f = frequency of vortex shedding, d = diameter of cylinder and V = velocity of fluid (air).

At 80 kph,

$$V = \frac{80 \times 10^3}{3600} = 22.22 \text{ m/s}$$

$$f = \frac{0.21\text{ V}}{d} = \frac{4.667}{d} \text{ Hz (d in meters)}$$

For the three sections of the antenna, the vortex frequencies are:

$$f_1 = \frac{4.667}{0.008} = 583.4 \text{ Hz}$$

$$f_2 = \frac{4.667}{0.005} = 933.4 \text{ Hz}$$

$$f_3 = \frac{4.667}{0.002} = 2\,333.5 \text{ Hz}$$

at 120 kph,

$$V = \frac{120 \times 10^3}{3600} = 33.33 \text{ m/s}$$

$$f = \frac{0.21\text{ V}}{d} = \frac{7}{d} \text{ Hz (d in meters)}$$

for the three sections of the antenna, the vortex frequencies are:

$$f_1 = \frac{7}{0.008} = 875 \text{ Hz}$$

$$f_2 = \frac{7}{0.005} = 1400 \text{ Hz}$$

$$f_3 = \frac{7}{0.002} = 3500 \text{ Hz}$$

natural frequency is much smaller, so no instability occurs.

3.73 (a) Equivalent mass of single d.o.f. system $= m_{eq} = M + \frac{33}{140}$ m where m = mass of cylindrical part of the sign post:

$$m = \frac{\pi}{4} (D^2 - d^2)\, h\, \rho = \frac{\pi}{4} (0.25^2 - 0.2^2)\, (10)\, (\frac{76500}{9.81}) = 1378.0527 \text{ kg}$$

$$\therefore m_{eq} = 200 + \frac{33}{140} (1378.0527) = 524.8267 \text{ kg}$$

$$I = \frac{\pi}{64} (D^4 - d^4) = \frac{\pi}{4} (0.25^4 - 0.2^4) = 113.208\, (10^{-6})\, m^4$$

Equivalent stiffness of the system:

$$k_{eq} = \frac{3\, E\, I}{h^3} = \frac{3\, (207\, (10^9))\, (113.208\, (10^{-6}))}{10^3} = 70{,}302.168 \text{ N/m}$$

Natural frequency of transverse vibration of sign post:

$$\omega_1 = \left(\frac{k_{eq}}{m_{eq}}\right)^{\frac{1}{2}} = \left(\frac{70302.168}{524.8267}\right)^{\frac{1}{2}} = 11.5738 \text{ rad/sec} = 1.8420 \text{ Hz}$$

(b) Wind velocity corresponding to maximum vibration of sign post (V) is given by:

$$St = 0.21 = \frac{f_1\, D}{V} \quad \text{or} \quad V = \frac{f_1\, D}{0.21} = \frac{(1.8420)\, (0.25)}{0.21} = 2.1929 \text{ m/s}$$

(c) Maximum force acting on the system due to wind velocity:

$$F(t) = F_0 \sin \omega t = \frac{1}{2}\, c\, \rho\, V^2\, A \sin \omega t = \frac{1}{2}\, (1)\, (1.2215)\, (2.19299^2)\, (8.0) \sin \omega t \text{ N}$$
$$= 23.4958 \sin \omega t \text{ N}$$

where c = 1 for a cylinder, ρ = density of air = 1.2215 kg/m^3, A = projected area of cylindrical part = (0.8)(10) = 8.0 m^2, and ω = frequency of wind force. Equation of motion:

$$m_{eq}\, \ddot{x} + c_{eq}\, \dot{x} + k_{eq}\, x = F(t)$$

and the maximum steady state displacement of the sign post occurs when $\omega = \omega_1$ and is given by Eq. (3.34):

$$X = \frac{\delta_{st}}{2\, \varsigma} = \frac{F_0}{k_{eq}\, (2)\, \varsigma} = \frac{23.4958}{2\, (0.1)\, (70302.168)} = 0.001671 \text{ m}$$

3.74 (a) Equation of motion $m\ddot{x} + c\dot{x} + kx = F_0 x$

or $m\ddot{x} + c\dot{x} + (k - F_0)x = 0$ (E_1)

Assuming the solution $x(t) = C e^{st}$ (E_2)

where C is a constant, Eq. (E_1) gives the auxiliary equation

$$s^2 + \frac{c}{m}s + \left(\frac{k-F_0}{m}\right) = 0 \qquad (E_3)$$

Roots of (E_3) are

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k-F_0}{m}\right)} \qquad (E_4)$$

First consider the case of positive stiffness $(k > F_0)$. For this case, following possibilities exist.

1. If $\left(\frac{c}{2m}\right)^2 > \left(\frac{k-F_0}{m}\right)$:

   Both s_1 and s_2 will be real and negative and hence

   $$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} \qquad (E_5)$$

   will be stable.

2. If $\left(\frac{c}{2m}\right)^2 = \left(\frac{k-F_0}{m}\right)$:

   Both s_1 and s_2 will be identical, real and negative.

   Solution $$x(t) = (C_1 + C_2 t)\, e^{s_1 t} \qquad (E_6)$$

   will be stable since $e^{s_1 t} \to 0$ as $t \to \infty$.

3. If $\left(\frac{c}{2m}\right)^2 < \left(\frac{k-F_0}{m}\right)$:

   Here s_1 and s_2 will be complex conjugates and solution will be

   $$x(t) = C\, e^{-\left(\frac{c}{2m}\right)t} \sin\left(\sqrt{\left\{-\left(\frac{c}{2m}\right)^2 + \left(\frac{k-F_0}{m}\right)\right\}}\, t + \phi\right) \qquad (E_7)$$

   This represents a converging oscillatory motion and hence the system will be stable.

Next consider the case of negative stiffness $(k < F_0)$. Here

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 + \left(\frac{F_0 - k}{m}\right)} \qquad (E_8)$$

Thus s_1 will be positive and s_2 will be negative, and the solution becomes

$$x(t) = C_1 e^{+s_1 t} + C_2 e^{-|s_2| t} \qquad (E_9)$$

This solution can be seen to diverge as $t \to \infty$.

(b) Equation of motion $m\ddot{x} + c\dot{x} + kx = F_0 \dot{x}$

or $$\ddot{x} + \left(\frac{c - F_0}{m}\right)\dot{x} + \frac{k}{m}x = 0 \qquad (E_{10})$$

Assuming $x(t) = C\,e^{st}$ the auxiliary equation becomes

$$s^2 + \left(\frac{c - F_0}{m}\right) s + \frac{k}{m} = 0 \qquad (E_{11})$$

and hence

$$s_{1,2} = -\left(\frac{c - F_0}{2m}\right) \pm \sqrt{\left(\frac{c - F_0}{2m}\right)^2 - \frac{k}{m}} \qquad (E_{12})$$

First consider the case of positive damping ($c > F_0$) in (E_{10}). For this case, it can be seen that the system will be stable for all possible values of $\left\{ \left(\frac{c - F_0}{2m}\right)^2 - \frac{k}{m} \right\}$.

Next, consider the case of negative damping ($c < F_0$). Depending on the sign of the quantity under the radical in Eq. (E_{12}), we will have three types of solution.

1. $\left(\frac{c - F_0}{2m}\right)^2 > \frac{k}{m}$. Here both s_1 and s_2 are real and positive and hence $x(t) = C_1\,e^{s_1 t} + C_2\,e^{s_2 t}$ (E_{13})
   This denotes a diverging nonoscillatory motion; so the system is unstable.

2. $\left(\frac{c - F_0}{2m}\right)^2 = \frac{k}{m}$. Here s_1 and s_2 are identical and are real and positive. Hence $x(t) = (C_1 + C_2\,t)\,e^{s_1 t}$ (E_{14})
   This represents a diverging nonoscillatory solution; so the system will be unstable.

3. $\left(\frac{c - F_0}{2m}\right)^2 < \frac{k}{m}$. Here s_1 and s_2 are complex conjugates and hence

$$s_{1,2} = \left(\frac{F_0 - c}{2m}\right) \pm i\,\sqrt{\frac{k}{m} - \left(\frac{c - F_0}{2m}\right)^2} \qquad (E_{15})$$

   The solution becomes

$$x(t) = X\,e^{\left(\frac{F_0 - c}{2m}\right)t} \sin\left(\sqrt{\frac{k}{m} - \left(\frac{c - F_0}{2m}\right)^2}\; t + \phi\right) \qquad (E_{16})$$

   Since the exponent is positive, Eq. (E_{16}) denotes a diverging oscillatory motion and hence the system is unstable.

Thus the condition for dynamic stability of the system can be stated as

$$F_0 \le c \qquad (E_{17})$$

(c) Equation of motion $m\ddot{x} + c\dot{x} + k x = F_0 \ddot{x}$

or $(m - F_0)\ddot{x} + c\dot{x} + k x = 0$ (E_{18})

With the solution $x(t) = C\, e^{st}$ (E_{19})

the auxiliary equation will be

$$s^2 + \left(\frac{c}{m-F_0}\right) s + \left(\frac{k}{m-F_0}\right) = 0 \qquad (E_{20})$$

The roots are

$$s_{1,2} = -\frac{c}{2(m-F_0)} \pm \sqrt{\left\{\frac{c}{2(m-F_0)}\right\}^2 - \left(\frac{k}{m-F_0}\right)} \qquad (E_{21})$$

First consider the case of positive mass $(m > F_0)$ in (E_{18}). In this case, the system will be stable for all values of

$$\left[\left\{\frac{c}{2(m-F_0)}\right\}^2 - \left(\frac{k}{m-F_0}\right) \right].$$

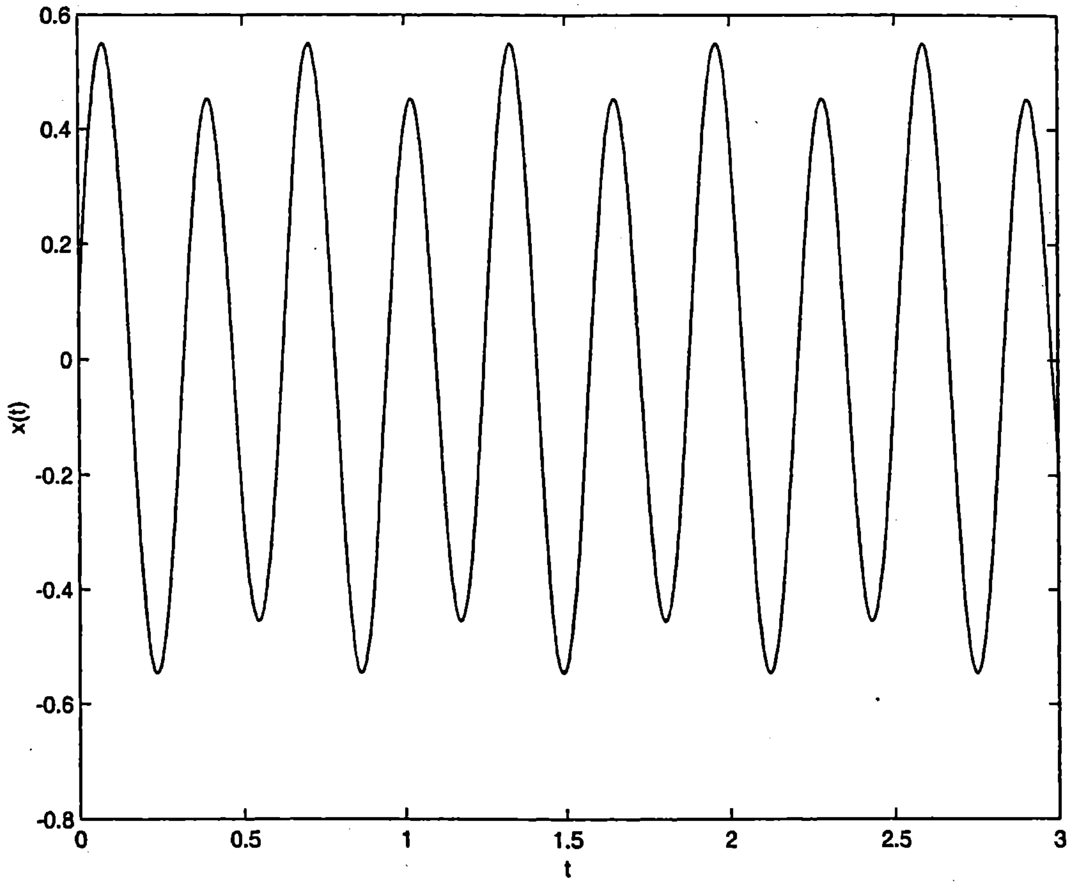
Next consider the case of negative mass $(m < F_0)$ in (E_{18}). For this case s_1 and s_2 can be expressed as

$$s_{1,2} = \frac{c}{2(F_0 - c)} \pm \sqrt{\left\{\frac{c}{2(F_0-m)}\right\}^2 + \left(\frac{k}{F_0-m}\right)} \qquad (E_{22})$$

This shows that s_1 will be positive and s_2 will be negative ; thus the solution will be divergent.

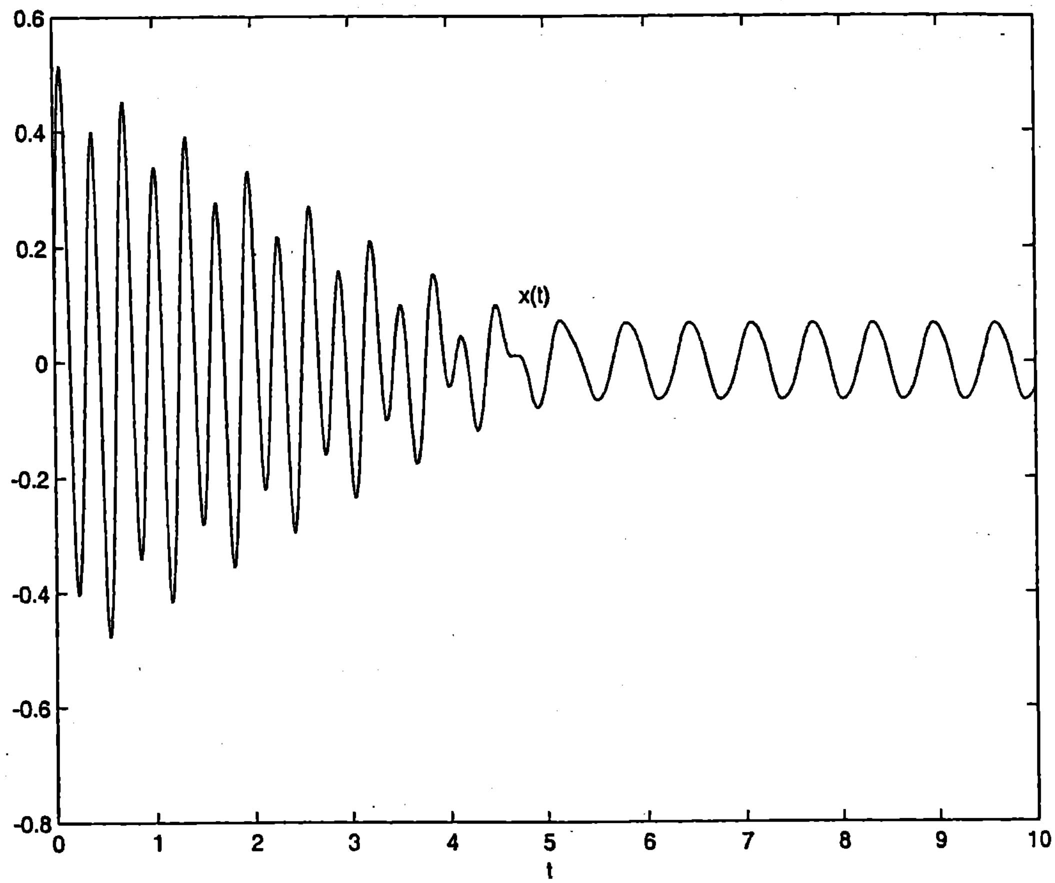
3.75

```
% Ex3_75.m
k = 4000;
m = 10;
w = 10;
F0 = 200;
wn = sqrt(k/m);
x0 = 0.1;
x0_dot = 10;
f_0 = F0/m;
for i = 1: 501
    t(i) = 3 * (i-1)/500;
    x(i) = x0_dot*sin(wn*t(i))/wn + (x0 - f_0/(wn^2-w^2))*cos(wn*t(i))...
        + f_0/(wn^2-w^2)*cos(w*t(i));
end
plot(t,x);
xlabel('t');
ylabel('x(t)');
```



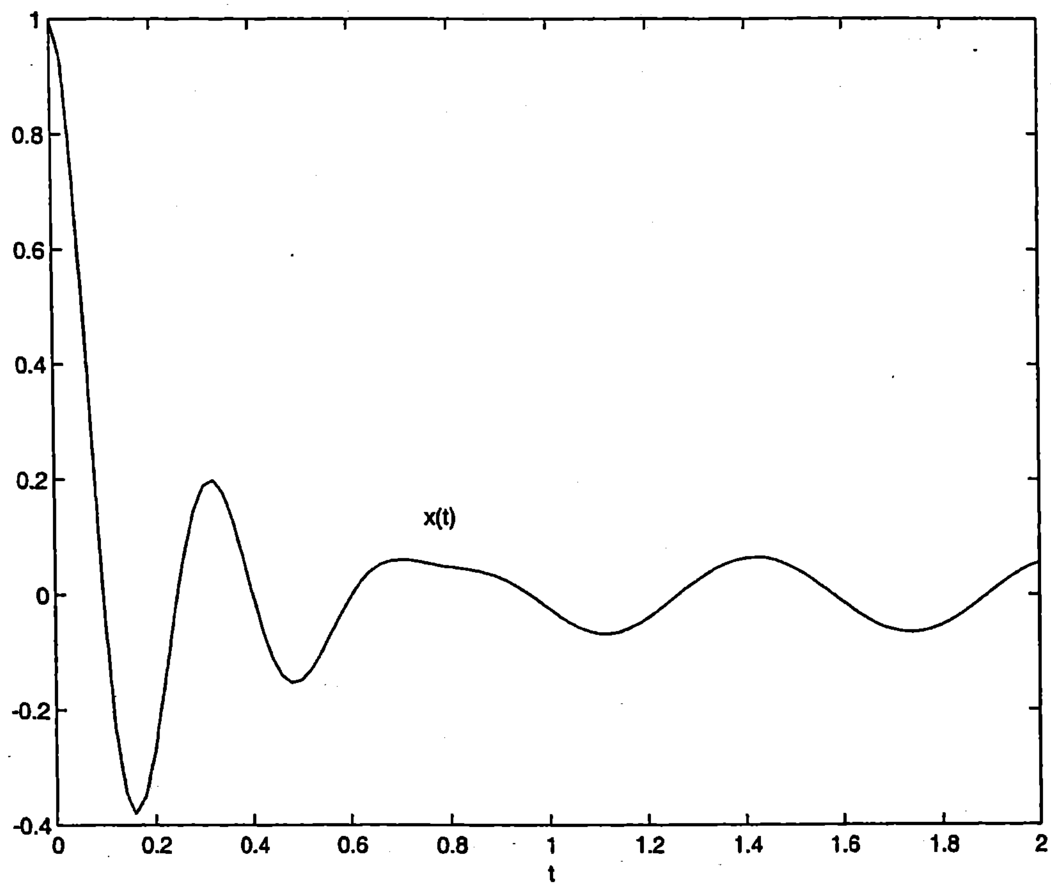
3.76

```
% Ex3_76.m
% This program will use the function dfunc3_76.m, they should
% be in the same folder
tspan = [0: 0.01: 10];
x0 = [0.1; 10];
[t,x] = ode23('dfunc3_76', tspan, x0);
disp('      t          x(t)     xd(t)');
disp([t x]);
plot(t,x(:,1));
xlabel('t');
gtext('x(t)');

% dfunc3_76.m
function f = dfunc3_76(t,x)
F0 = 200;
w = 10;
u = 0.3;
m = 10;
k = 4000;
f = zeros(2,1);
f(1) = x(2);
f(2) = (F0/m)*sin(w*t) - 9.81*u*sign(x(2)) - (k/m)*x(1);
```



3.77

```
% Ex3_77.m
% This program will use the function dfunc3_77.m, they should
% be in the same folder
tspan = [0: 0.02: 2];
x0 = [1; 0];
[t,x] = ode23('dfunc3_77', tspan, x0);
disp('        t           x(t)      xd(t)');
disp([t x]);
plot(t,x(:,1));
xlabel('t');
gtext('x(t)');

% dfunc3_77.m
function f = dfunc3_77(t,x)
m = 100;
k = 40000;
zeta = 0.25;
Y = 0.05;
w = 10;
c = 2 * zeta * sqrt(k*m);
f = zeros(2,1);
f(1) = x(2);
f(2) = k*Y*sin(w*t)/m + c*w*Y*cos(w*t)/m - c*x(2)/m - (k/m)*x(1);
```




```
>> Ex3_77
       t          x(t)       xd(t)
         0       1.0000           0
    0.0200       0.9272     -6.9328
    0.0400       0.7381    -11.5448
    0.0600       0.4823    -13.6094
    0.0800       0.2096    -13.3072
    0.1000      -0.0372    -11.1218
    0.1200      -0.2270     -7.7195
      .
      .
      .
    1.8000      -0.0523      0.3904
    1.8200      -0.0434      0.4869
    1.8400      -0.0329      0.5637
    1.8600      -0.0210      0.6179
    1.8800      -0.0083      0.6473
    1.9000       0.0047      0.6510
    1.9200       0.0176      0.6286
    1.9400       0.0297      0.5811
    1.9600       0.0406      0.5103
    1.9800       0.0499      0.4194
    2.0000       0.0572      0.3120
```

3.78

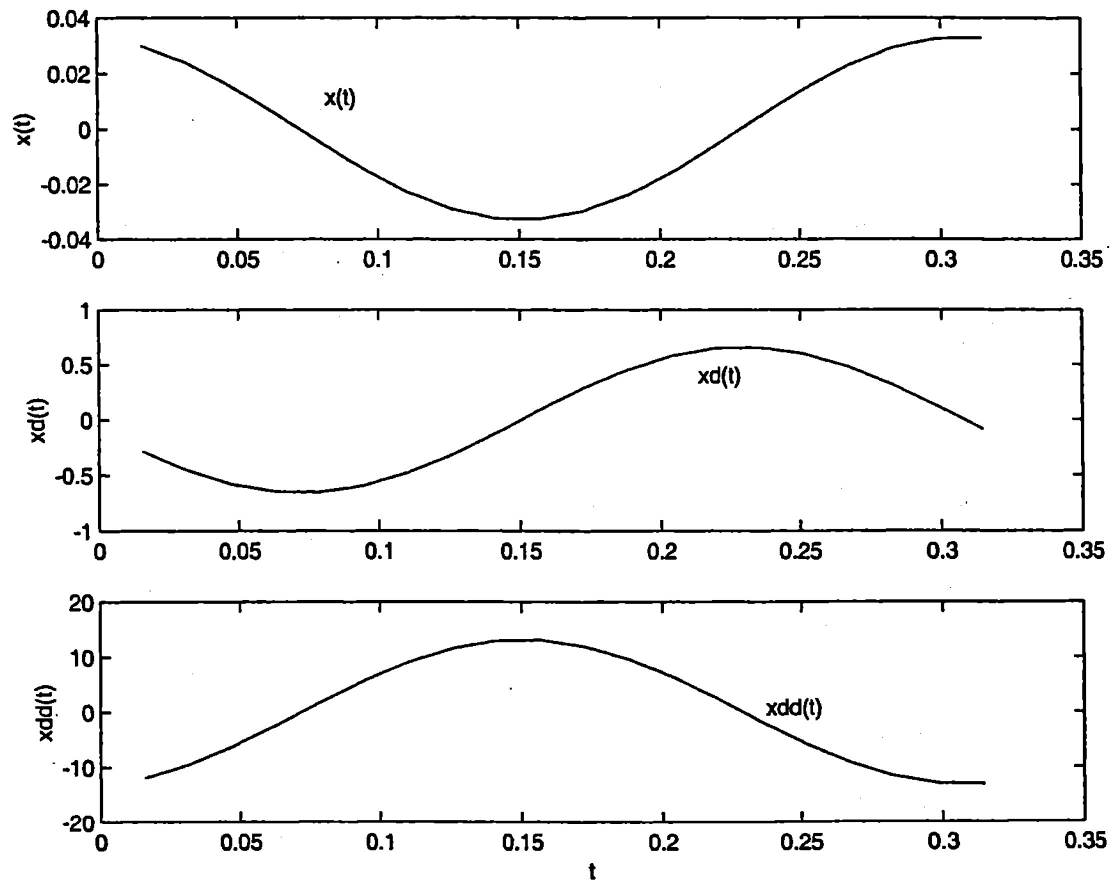
```
%=========================================================================
%
% Ex3_78.m (Program3.m)
% Main program which calls HARESP
%
%=========================================================================
% Run "Ex3_78.m" in MATLAB command window. Ex3_78.m and haresp.m should
% be in the same folder,and set the path to this folder
% following seven lines contain problem-dependent data
xm=10.0;
xk=1000;
zeta=0.1;
xc=2*zeta*sqrt(xk*xm);
f0=100.0;
om=20.0;
n=20;
ic=1;
% end of problem-dependent data
[t,x,xd,xdd,xamp,xphi]=haresp(xm,xc,xk,f0,om,ic,n);
% following lines output the results
fprintf('Steady state response of an undamped\n');
fprintf('Single degree of freedom system under harmonic force\n\n');
fprintf('Given data\n');
fprintf('xm = %10.8e\n',xm);
fprintf('xc = %10.8e\n',xc);
fprintf('xk = %10.8e\n',xk);
fprintf('f0 = %10.8e\n',f0);
fprintf('om = %10.8e\n',om);
fprintf('ic = %1.0f\n',ic);
fprintf('n  = %2.0f\n\n\n',n);
fprintf('Response: \n\n');
fprintf('   i          x(i)                    xd(i)                xdd(i)');
fprintf('\n\n');
for i=1:n
   fprintf('  %2.0f    %10.8e          %10.8e     %10.8e\n',i,x(i),...
      xd(i),xdd(i));
end
subplot(311);
plot(t,x);
ylabel('x(t)');
gtext('x(t)');
subplot(312);
plot(t,xd);
ylabel('xd(t)');
gtext('xd(t)');
subplot(313);
plot(t,xdd);
ylabel('xdd(t)');
gtext('xdd(t)');
xlabel('t');
```

```
%=====================================================================
%.
%function haresp.m
%
%=====================================================================
function [t,x,xd,xdd,xamp,xphi]=haresp(xm,xc,xk,f0,om,ic,n);
omn=sqrt(xk/xm);
xai=xc/(2.0*xm*omn);
dst=f0/xk;
r=om/omn;
xamp=dst/sqrt((1.0-r^2)^2+(2.0*xai*r)^2);
xphi=atan(2.0*xai*r/(1.0-r^2));
delt=2.0*3.1416/(om*n);
time=0.0;
if ic~=0
   for i=1:n
      time=time+delt;
      t(i) = time;
      x(i)=xamp*cos(om*time-xphi);
      xd(i)=-xamp*om*sin(om*time-xphi);
      xdd(i)=-xamp*om^2*cos(om*time-xphi);
   end
else
   for i=1:n
      time=time+delt;
      t(i) = time;
      x(i)=xamp*sin(om*time-xphi);
      xd(i)=xamp*om*cos(om*time-xphi);
      xdd(i)=-xamp*om^2*sin(om*time-xphi);
   end
end
```

```
>> Ex3_78
Steady state response of an undamped
Single degree of freedom system under harmonic force

Given data
xm = 1.00000000e+001
xc = 2.00000000e+001
xk = 1.00000000e+003
f0 = 1.00000000e+002
om = 2.00000000e+001
ic = 1
n  = 20


Response:
```

i	x(i)	xd(i)	xdd(i)
1	2.97987095e-002	-2.85475021e-001	-1.19194838e+001
2	2.39294085e-002	-4.55669383e-001	-9.57176339e+000
3	1.57177193e-002	-5.81259445e-001	-6.28708774e+000
4	5.96746320e-003	-6.49951518e-001	-2.38698528e+000
5	-4.36693253e-003	-6.55021513e-001	1.74677301e+000
6	-1.42738605e-002	-5.95973141e-001	5.70954420e+000

7	-2.27835571e-002	-4.78586496e-001	9.11342282e+000
8	-2.90630300e-002	-3.14352252e-001	1.16252120e+001
9	-3.24975978e-002	-1.19346877e-001	1.29990391e+001
10	-3.27510596e-002	8.73410566e-002	1.31004238e+001
11	-2.97986046e-002	2.85479399e-001	1.19194419e+001
12	-2.39292411e-002	4.55672899e-001	9.57169644e+000
13	-1.57175058e-002	5.81261754e-001	6.28700233e+000
14	-5.96722446e-003	6.49952395e-001	2.38688978e+000
15	4.36717313e-003	6.55020872e-001	-1.74686925e+000
16	1.42740794e-002	5.95971044e-001	-5.70963176e+000
17	2.27837329e-002	4.78583148e-001	-9.11349314e+000
18	2.90631454e-002	3.14347982e-001	-1.16252582e+001
19	3.24976417e-002	1.19342102e-001	-1.29990567e+001
20	3.27510275e-002	-8.73458687e-002	-1.31004110e+001



3.79

```
%Ex3_79.m
Y= 0.05;
zeta = 0.1;
wn = 8.164966;
w = 2.90889;
r = w/wn;
x = Y * sqrt( (1 + (2 * zeta * r)^2)/((1 - r^2)^2 + (2 * zeta * r)^2) );
disp('   w                    wn                    x');
disp([w wn x]);
w = 14.54445;
r = w/wn;
x = Y * sqrt( (1 + (2 * zeta * r)^2)/((1 - r^2)^2 + (2 * zeta * r)^2) );
disp([w wn x]);
w = 29.08890;
r = w/wn;
x = Y * sqrt( (1 + (2 * zeta * r)^2)/((1 - r^2)^2 + (2 * zeta * r)^2) );
disp([w wn x]);
wn = 6.324555;
w = 2.90889;
r = w/wn;
x = Y * sqrt( (1 + (2 * zeta * r)^2)/((1 - r^2)^2 + (2 * zeta * r)^2) );
disp([w wn x]);
w = 14.54445;
r = w/wn;
x = Y * sqrt( (1 + (2 * zeta * r)^2)/((1 - r^2)^2 + (2 * zeta * r)^2) );
disp([w wn x]);
w = 29.08890;
r = w/wn;
x = Y * sqrt( (1 + (2 * zeta * r)^2)/((1 - r^2)^2 + (2 * zeta * r)^2) );
disp([w wn x]);
```

```
>> Ex3_79
   w                    wn                    x
   2.90889000000000     8.16496600000000     0.05722376420338

  14.54445000000000     8.16496600000000     0.02410324256879

  29.08890000000000     8.16496600000000     0.00524102723160

   2.90889000000000     6.32455500000000     0.06325355032007

  14.54445000000000     6.32455500000000     0.01275990975243

  29.08890000000000     6.32455500000000     0.00336736169683
```

(3.80)

```
Results of Ex3_80
********************

Please input n and ic:
20 0

Please input xm, xc, xk, f0, om:
6 210 14000 450 10


STEADY STATE RESPONSE OF AN UNDERDAMPED
SINGLE DEGREE OF FREEDOM SYSTEM UNDER HARMONIC FORCE


GIVEN DATA:
XM = 6
XC = 210
F0 = 450
OM = 10
IC = 0
N = 20

RESONSE:
```

I	X(I)	XD(I)	XDD(I)
0	0.00524340	0.32760185	-0.52434011
1	0.01511025	0.29536476	-1.51102462
2	0.02349799	0.23421518	-2.34979883
3	0.02958557	0.15013888	-2.95855729
4	0.03277710	0.05136587	-3.27771021
5	0.03276017	-0.05243521	-3.27601652
6	0.02953642	-0.15110355	-2.95364203
7	0.02342143	-0.23498074	-2.34214313
8	0.01501378	-0.29585628	-1.50137789
9	0.00513647	-0.32777121	-0.51364663
10	-0.00524364	-0.32760146	0.52436417
11	-0.01511046	-0.29536365	1.51104632
12	-0.02349816	-0.23421345	2.34981604
13	-0.02958568	-0.15013670	2.95856832
14	-0.03277714	-0.05136346	3.27771398
15	-0.03276013	0.05243762	3.27601267
16	-0.02953631	0.15110572	2.95363093
17	-0.02342126	0.23498246	2.34212587
18	-0.01501356	0.29585738	1.50135615
19	-0.00513623	0.32777159	0.51362255

(3.81)

```
Results of Ex3_81
**********************

Please input n and ic:
20 1

Please input xm, xc, xk, f0, om:
10 45 2500 180 20


STEADY STATE RESPONSE OF AN UNDERDAMPED
SINGLE DEGREE OF FREEDOM SYSTEM UNDER HARMONIC FORCE


GIVEN DATA:
XM = 10
XC = 45
F0 = 180
OM = 20
IC = 1
N = 20

RESONSE:
```

I	X(I)	XD(I)	XDD(I)
0	0.06755697	-1.55232612	-27.02278847
1	0.04026567	-1.89387550	-16.10626822
2	0.00903287	-2.05003830	-3.61314690
3	-0.02308414	-2.00552814	9.23365606
4	-0.05294150	-1.76470199	21.17660023
5	-0.07761655	-1.35113372	31.04662161
6	-0.09469392	-0.80530645	37.87756926
7	-0.10250195	-0.18064981	41.00077932
8	-0.10027632	0.46169017	40.11052882
9	-0.08823491	1.05883649	35.29396208
10	-0.06755640	1.55233604	27.02256039
11	-0.04026497	1.89388142	16.10598995
12	-0.00903211	2.05003963	3.61284569
13	0.02308488	2.00552474	-9.23395073
14	0.05294215	1.76469421	-21.17685951
15	0.07761705	1.35112232	-31.04682013
16	0.09469422	0.80529254	-37.87768758
17	0.10250201	0.18063475	-41.00080586
18	0.10027615	-0.46170490	-40.11046098
19	0.08823452	-1.05884946	-35.29380650

3.82 The main program which calls HARESP and the output are given below.

```
C ===================================================================
C
C MAIN PROGRAM WHICH CALLS HARESP
C
C ===================================================================
C FOLLOWING 2 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION X(20),XD(20),XDD(20)
      DATA XM,XC,XK,F0,OM,N,IC/6.0,210.0,14000.0,450.0,10.0,20,0/
C END OF PROBLEM-DEPENDENT DATA
      CALL HARESP (XM,XC,XK,F0,OM,IC,N,X,XD,XDD,XAMP,XPHI)
      WRITE (21,100)
  100 FORMAT (//,40H STEADY STATE RESPONSE OF AN UNDERDAMPED,/,
     2 53H SINGLE DEGREE OF FREEDOM SYSTEM UNDER HARMONIC FORCE)
      WRITE (21,200) XM,XC,XK,F0,OM,IC,N
  200 FORMAT (//,12H GIVEN DATA:,/,5H XM =,E15.8,/,5H XC =,E15.8,/,
     2 5H XK =,E15.8,/,5H F0 =,E15.8,/,5H OM =,E15.8,/,5H IC =,I2,/,
     3 5H N  =,I2)
      WRITE (21,300)
  300 FORMAT (//,10H RESPONSE:,//,5H   I  ,3X,5H X(I),12X,6H XD(I),
     2 11X,7H XDD(I),/)
      DO 400 I=1,N
  400 WRITE (21,500) I,X(I),XD(I),XDD(I)
  500 FORMAT (I4,2X,E15.8,2X,E15.8,2X,E15.8)
      STOP
      END
```

```
STEADY STATE RESPONSE OF AN UNDERDAMPED
SINGLE DEGREE OF FREEDOM SYSTEM UNDER HARMONIC FORCE


GIVEN DATA:
XM = 0.60000000E+01
XC = 0.21000000E+03
XK = 0.14000000E+05
F0 = 0.45000000E+03
OM = 0.10000000E+02
IC = 0
N  =20


RESPONSE:

   I    X(I)              XD(I)             XDD(I)

   1  0.52434015E-02    0.32760188E+00   -0.52434015E+00
   2  0.15110249E-01    0.29536480E+00   -0.15110248E+01
   3  0.23497989E-01    0.23421521E+00   -0.23497992E+01
   4  0.29585576E-01    0.15013890E+00   -0.29585576E+01
   5  0.32777105E-01    0.51365890E-01   -0.32777107E+01
   6  0.32760169E-01   -0.52435201E-01   -0.32760170E+01
   7  0.29536424E-01   -0.15110357E+00   -0.29536424E+01
   8  0.23421437E-01   -0.23498075E+00   -0.23421438E+01
```

9	0.15013780E-01	-0.29585633E+00	-0.15013779E+01
10	0.51364703E-02	-0.32777125E+00	-0.51364702E+00
11	-0.52436423E-02	-0.32760152E+00	0.52436423E+00
12	-0.15110461E-01	-0.29536372E+00	0.15110462E+01
13	-0.23498155E-01	-0.23421355E+00	0.23498156E+01
14	-0.29585691E-01	-0.15013662E+00	0.29585693E+01
15	-0.32777146E-01	-0.51363401E-01	0.32777145E+01
16	-0.32760132E-01	0.52437652E-01	0.32760131E+01
17	-0.29536312E-01	0.15110572E+00	0.29536314E+01
18	-0.23421254E-01	0.23498255E+00	0.23421254E+01
19	-0.15013559E-01	0.29585743E+00	0.15013559E+01
20	-0.51362254E-02	0.32777163E+00	0.51362258E+00

3.83 Eq. (3.35) gives the complete solution

$$x(t) = X_0\, e^{-\zeta\omega_n t} \cos(\omega_d t + \phi_0) + X \cos(\omega t - \phi) \quad \text{(E.1)}$$

Differentiation gives

$$\dot{x}(t) = -\zeta\omega_n X_0\, e^{-\zeta\omega_n t} \cos(\omega_d t + \phi_0) - \omega_d X_0\, e^{-\zeta\omega_n t} \sin(\omega_d t + \phi_0) - \omega X \sin(\omega t - \phi) \quad \text{(E.2)}$$

$$\ddot{x}(t) = X_0\, e^{-\zeta\omega_n t} (\zeta^2\omega_n^2 - \omega_d^2) \cos(\omega_d t + \phi_0) + 2\zeta\omega_n\omega_d X_0\, e^{-\zeta\omega_n t} \sin(\omega_d t + \phi_0) - \omega^2 X \cos(\omega t - \phi) \quad \text{(E.3)}$$

Let $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$ be known. Then

$$x_0 = X_0 \cos\phi_0 + X \cos\phi \quad \text{(E.4)}$$

$$\dot{x}_0 = -\zeta\omega_n X_0 \cos\phi_0 - \omega_d X_0 \sin\phi_0 + \omega X \sin\phi \quad \text{(E.5)}$$

Eqs. (E.4) and (E.5) give

$$X_0 = \left[(X_0\cos\phi_0)^2 + (X_0 \sin\phi_0)^2\right]^{1/2} = \left[(x_0 - X\cos\phi)^2 + \left\{\frac{-\dot{x}_0 - \zeta\omega_n x_0 + \zeta\omega_n X \cos\phi + \omega X \sin\phi}{\omega_d}\right\}^2\right]^{1/2} \quad \text{(E.6)}$$

$$\phi_0 = \tan^{-1}\left\{\frac{-\dot{x}_0 - \zeta\omega_n x_0 + \zeta\omega_n X\cos\phi + \omega X \sin\phi}{\omega_d (x_0 - X\cos\phi)}\right\} \quad \text{(E.7)}$$

Now the computer program TOTALR can be written. In addition to the arguments used in section 3.12, the following arguments are also used:

XO = value of $x(t)$ at $t=0$. Input

XDO = value of $\frac{dx}{dt}(t)$ at $t=0$. Input.

The program and the output are given below.

```
C =======================================================================
C
C SOLUTION OF PROBLEM 3.83
C MAIN PROGRAM WHICH CALLS TOTALR
C HARMONIC FORCE IS ASSUMED TO BE F(T)=F0*COS(OM*T)
C DATA OF PROBLEM 3.33 IS USED IN MAIN PROGRAM
C
C =======================================================================
C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION X(20),XD(20),XDD(20)
      DATA XM,XC,XK,F0,OM,N/10.0,45.0,2500.0,180.0,21.9912,20/
      DATA X0,XD0/0.015,5.0/
C END OF PROBLEM-DEPENDENT DATA
      CALL TOTALR (XM,XC,XK,F0,OM,N,X,XD,XDD,XAMP,XPHI,X0,XD0)
      WRITE (21,100)
  100 FORMAT (//,33H TOTAL RESPONSE OF AN UNDERDAMPED,/,
     2 53H SINGLE DEGREE OF FREEDOM SYSTEM UNDER HARMONIC FORCE)
      WRITE (21,200) XM,XC,XK,F0,OM,N
  200 FORMAT (//,12H GIVEN DATA:,/,5H XM =,E15.8,/,5H XC =,E15.8,/,
     2 5H XK =,E15.8,/,5H F0 =,E15.8,/,5H OM =,E15.8,/,
     3 5H N  =,I2)
      WRITE (21,250) X0,XD0
  250 FORMAT (/,20H INITIAL CONDITIONS:,/,6H X0  =,E15.8,/,
     2 6H XD0 =,E15.8)
      WRITE (21,300)
  300 FORMAT (//,10H RESPONSE:,//,5H   I ,3X,5H X(I),12X,6H XD(I),
     2 11X,7H XDD(I),/)
      DO 400 I=1,N
  400 WRITE (21,500) I,X(I),XD(I),XDD(I)
  500 FORMAT (I4,2X,E15.8,2X,E15.8,2X,E15.8)
      STOP
      END
C =======================================================================
C
C SUBROUTINE TOTALR
C
C =======================================================================
      SUBROUTINE TOTALR (XM,XC,XK,F0,OM,N,X,XD,XDD,XAMP,XPHI,X0,XD0)
      DIMENSION X(N),XD(N),XDD(N)
      OMN=SQRT(XK/XM)
      XAI=XC/(2.0*XM*OMN)
      OMD=OMN*SQRT(1.0-XAI**2)
      DST=F0/XK
      R=OM/OMN
      XAMP=DST/SQRT((1.0-R**2)**2+(2.0*XAI*R)**2)
      XPHI=ATAN(2.0*XAI*R/(1.0-R**2))
      DELT=2.0*3.1416/(OMD*REAL(N))
      XCOS=X0-XAMP*COS(XPHI)
      XSIN=(-XD0-XAI*OMN*X0+XAI*OMN*XAMP*COS(XPHI)+OM*XAMP*SIN
     2 (XPHI))/OMD
      XZ=SQRT(XCOS**2+XSIN**2)
      PZ=ATAN(XSIN/XCOS)
      TIME=0.0
      DO 10 I=1,N
      TIME=TIME+DELT
```

```
      EX1=EXP(-XAI*OMN*TIME)
      EC1=COS(OMD*TIME+PZ)
      EC2=COS(OM*TIME-XPHI)
      ES1=SIN(OMD*TIME+PZ)
      ES2=SIN(OM*TIME-XPHI)
      X(I)=XZ*EX1*EC1+XAMP*EC2
      XD(I)=-XAI*OMN*XZ*EX1*EC1-OMD*XZ*EX1*ES1-OM*XAMP*ES2
10    XDD(I)=XZ*EX1*EC1*((XAI*OMN)**2-OMD**2)+2.0*XAI*OMN*OMD*XZ*
     2 EX1*ES1-(OM**2)*XAMP*EC2
      RETURN
      END
```

```
TOTAL RESPONSE OF AN UNDERDAMPED
SINGLE DEGREE OF FREEDOM SYSTEM UNDER HARMONIC FORCE


GIVEN DATA:
XM = 0.10000000E+02
XC = 0.45000000E+02
XK = 0.25000000E+04
F0 = 0.18000000E+03
OM = 0.21991199E+02
N  =20

INITIAL CONDITIONS:
X0  = 0.15000000E-01
XD0 = 0.50000000E+01


RESPONSE:

  I      X(I)              XD(I)              XDD(I)

  1  -0.10720424E-01   -0.62618327E+01    0.14583885E+02
  2  -0.13126461E+00   -0.56405468E+01    0.46769913E+02
  3  -0.23313771E+00   -0.44189043E+01    0.73777733E+02
  4  -0.30557248E+00   -0.27362981E+01    0.92193657E+02
  5  -0.34123138E+00   -0.79179698E+00    0.99568497E+02
  6  -0.33717862E+00    0.11797444E+01    0.94842743E+02
  7  -0.29529962E+00    0.29382858E+01    0.78578743E+02
  8  -0.22207454E+00    0.42714534E+01    0.52945923E+02
  9  -0.12771350E+00    0.50244703E+01    0.21447788E+02
 10  -0.24761723E-01    0.51222720E+01   -0.11575108E+02
 11   0.73627755E-01    0.45800018E+01   -0.41590317E+02
 12   0.15549700E+00    0.34999285E+01   -0.64561966E+02
 13   0.21168864E+00    0.20550721E+01   -0.77567314E+02
 14   0.23700854E+00    0.46208435E+00   -0.79236084E+02
 15   0.23077171E+00   -0.10522565E+01   -0.69936806E+02
 16   0.19665717E+00   -0.22854314E+01   -0.51678036E+02
 17   0.14191106E+00   -0.30888264E+01   -0.27741650E+02
 18   0.76043457E-01   -0.33878446E+01   -0.21128597E+01
 19   0.92454441E-02   -0.31893401E+01    0.21192802E+02
 20  -0.49206819E-01   -0.25754893E+01    0.38788284E+02
```

(3.84) The main program and output are given.

```
C ==========================================================================
C
C MAIN PROGRAM WHICH CALLS HARESP
C
C ==========================================================================
C FOLLOWING 2 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION X(20),XD(20),XDD(20)
      DATA XM,XC,XK,F0,OM,N,IC/10.0,45.0,2500.0,180.0,20.0,20,1/
C END OF PROBLEM-DEPENDENT DATA
      CALL HARESP (XM,XC,XK,F0,OM,IC,N,X,XD,XDD,XAMP,XPHI)
      WRITE (21,100)
 100  FORMAT (//,40H STEADY STATE RESPONSE OF AN UNDERDAMPED,/,
     2 53H SINGLE DEGREE OF FREEDOM SYSTEM UNDER HARMONIC FORCE)
      WRITE (21,200) XM,XC,XK,F0,OM,IC,N
 200  FORMAT (//,12H GIVEN DATA:,/,5H XM =,E15.8,/,5H XC =,E15.8,/,
     2 5H XK =,E15.8,/,5H F0 =,E15.8,/,5H OM =,E15.8,/,5H IC =,I2,/,
     3 5H N  =,I2)
      WRITE (21,300)
 300  FORMAT (//,10H RESPONSE:,//,5H    I ,3X,5H X(I),12X,6H XD(I),
     2 11X,7H XDD(I),/)
      DO 400 I=1,N
 400  WRITE (21,500) I,X(I),XD(I),XDD(I)
 500  FORMAT (I4,2X,E15.8,2X,E15.8,2X,E15.8)
      STOP
      END
```

```
STEADY STATE RESPONSE OF AN UNDERDAMPED
SINGLE DEGREE OF FREEDOM SYSTEM UNDER HARMONIC FORCE


GIVEN DATA:
XM = 0.10000000E+02
XC = 0.45000000E+02
XK = 0.25000000E+04
F0 = 0.18000000E+03
OM = 0.20000000E+02
IC = 1
N  =20


RESPONSE:

   I   X(I)              XD(I)              XDD(I)

   1   0.67556962E-01  -0.15523261E+01  -0.27022785E+02
   2   0.40265664E-01  -0.18938755E+01  -0.16106266E+02
   3   0.90328613E-02  -0.20500383E+01  -0.36131446E+01
   4  -0.23084141E-01  -0.20055280E+01   0.92336569E+01
   5  -0.52941486E-01  -0.17647020E+01   0.21176596E+02
   6  -0.77616550E-01  -0.13511336E+01   0.31046618E+02
   7  -0.94693922E-01  -0.80530620E+00   0.37877567E+02
   8  -0.10250194E+00  -0.18064982E+00   0.41000774E+02
```

9	-0.10027631E+00	0.46169037E+00	0.40110523E+02
10	-0.88234901E-01	0.10588365E+01	0.35293961E+02
11	-0.67556389E-01	0.15523360E+01	0.27022554E+02
12	-0.40264975E-01	0.18936812E+01	0.16105991E+02
13	-0.90321312E-02	0.20500395E+01	0.36128526E+01
14	0.23084890E-01	0.20055246E+01	-0.92339563E+01
15	0.52942146E-01	0.17646941E+01	-0.21176857E+02
16	0.77617034E-01	0.13511224E+01	-0.31046814E+02
17	0.94694205E-01	0.80529296E+00	-0.37877682E+02
18	0.10250200E+00	0.18063448E+00	-0.41000801E+02
19	0.10027614E+00	-0.46170485E+00	-0.40110458E+02
20	0.88234514E-01	-0.10588492E+01	-0.35293804E+02

3.85 Equation of motion is $m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$

when $y(t) = Y \sin \omega t$, $x_p(t) = X \cos(\omega t - \phi_1 - \phi_2)$

Complete solution can be expressed as

$$x(t) = X_0 \, e^{-\zeta\omega_n t} \cos(\omega_d t + \phi_0) + X \cos(\omega t - \phi_1 - \phi_2) \qquad (E.1)$$

with $X = Y \left[\dfrac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2} \right]^{1/2}$,

$$\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right), \quad \phi_2 = \tan^{-1}\left(\frac{1}{2\zeta r}\right), \quad r = \frac{\omega}{\omega_n}.$$

If the initial conditions are known $x(t=0) = x_0$ and $\dot{x}(t=0) = \dot{x}_0$,

$$x_0 = X_0 \cos\phi_0 + X \cos(\phi_1 + \phi_2)$$

and $$\dot{x}_0 = -\zeta\omega_n X_0 \cos\phi_0 - \omega_d X_0 \sin\phi_0 - \omega X \sin(-\phi_1 - \phi_2)$$

Hence

$$X_0 = \left[\left\{ x_0 - X\cos(\phi_1+\phi_2) \right\}^2 + \left\{ \frac{-\dot{x}_0 - \zeta\omega_n x_0 + \zeta\omega_n X \cos(\phi_1+\phi_2) + \omega X \sin(\phi_1+\phi_2)}{\omega_d} \right\}^2 \right]^{1/2}$$

$$\phi_0 = \tan^{-1}\left[\frac{-\dot{x}_0 - \zeta\omega_n x_0 + \zeta\omega_n X\cos(\phi_1+\phi_2) + \omega X \sin(\phi_1+\phi_2)}{\omega_d \left\{ x_0 - X\cos(\phi_1+\phi_2) \right\}} \right]$$

If necessary, the velocity $\dot{x}(t)$ and acceleration $\ddot{x}(t)$ can be found from Eq. (E.1). The computer program and output are given below.

```
C ==================================================================
C
C SOLUTION OF PROBLEM 3.85
C MAIN PROGRAM WHICH CALLS BASEX
C RESPONSE OF A SINGLE D.O.F. SYSTEM SUBJECTED TO BASE EXCITATION,
C Y(T)=Y*SIN(OM*T)
C
C ==================================================================
```

```
C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION X(20),XD(20),XDD(20)
      DATA XM,XC,XK,Y,OM,N/2.0,10.0,100.0,0.1,25.0,20/
      DATA X0,XD0/0.01,5.0/
C END OF PROBLEM-DEPENDENT DATA
      CALL BASEX (XM,XC,XK,Y,OM,N,X,XD,XDD,X0,XD0)
      PRINT 100
 100  FORMAT (//,33H TOTAL RESPONSE OF AN UNDERDAMPED,/,
     2 52H SINGLE D.O.F. SYSTEM UNDER HARMONIC BASE EXCITATION)
      PRINT 200, XM,XC,XK,Y,OM,N
 200  FORMAT (//,12H GIVEN DATA:,/,5H XM =,E15.8,/,5H XC =,E15.8,/,
     2 5H XK =,E15.8,/,5H Y  =,E15.8,/,5H OM =,E15.8,/,5H N  =,I2)
      PRINT 300,X0,XD0
 300  FORMAT (/,20H INITIAL CONDITIONS:,/,6H X0  =,E15.8,/,6H XD0 =,
     2 E15.8)
      PRINT 400
 400  FORMAT (//,10H RESPONSE:,//,5H   I ,3X,5H X(I),12X,6H XD(I),
     2 11X,7H XDD(I),/)
      DO 500 I=1,N
 500  PRINT 600, I,X(I),XD(I),XDD(I)
 600  FORMAT (I4,2X,E15.8,2X,E15.8,2X,E15.8)
      STOP
      END
C ==================================================================
C
C SUBROUTINE BASEX
C
C ==================================================================
      SUBROUTINE BASEX (XM,XC,XK,Y,OM,N,X,XD,XDD,X0,XD0)
      DIMENSION X(N),XD(N),XDD(N)
      OMN=SQRT(XK/XM)
      XAI=XC/(2.0*XM*OMN)
      OMD=OMN*SQRT(1.0-XAI**2)
      R=OM/OMN
      DELT=2.0*3.1416/(OMD*REAL(N))
      XAMP=Y*SQRT(1.0+(2.0*XAI*R)**2/((1.0-R**2)**2+(2.0*XAI*R)**2))
      PHI1=ATAN(2.0*XAI*R/(1.0-R*R))
      PHI2=ATAN(1.0/(2.0*XAI*R))
      XCC=XC
      TIME=0.0
      DO 10 I=1,N
      TIME=TIME+DELT
      XC=X0-XAMP*COS(PHI1+PHI2)
      XS=(-XD0-XAI*OMN*XC+OM*XAMP*SIN(PHI1+PHI2))/OMD
      XZ=SQRT(XC**2+XS**2)
      PZ=ATAN(XS/XC)
      EX=EXP(-XAI*OMN*TIME)
      CS=COS(OMD*TIME+PZ)
      SI=SIN(OMD*TIME+PZ)
      CS12=COS(OM*TIME-PHI1-PHI2)
      SI12=SIN(OM*TIME-PHI1-PHI2)
      X(I)=XZ+EX*CS+XAMP*CS12
      XD(I)=-XAI*OMN*XZ*EX*CS-OMD*XZ*EX*SI-OM*XAMP*SI12
 10   XDD(I)=XZ*EX*CS*((XAI*OMN)**2-OMD**2)+2.0*XAI*OMN*OMD*XZ*EX*SI
     2 -(OM**2)*XAMP*CS12
```

```
      XC=XCC
      RETURN
      END
```

TOTAL RESPONSE OF AN UNDERDAMPED
SINGLE D.O.F. SYSTEM UNDER HARMONIC BASE EXCITATION

GIVEN DATA:
XM = 0.20000000E+01
XC = 0.10000000E+02
XK = 0.10000000E+03
Y = 0.10000000E+00
OM = 0.25000000E+02
N =20

INITIAL CONDITIONS:
XO = 0.99999998E-02
XDO = 0.50000000E+01

RESPONSE:

I	X(I)	XD(I)	XDD(I)
1	-0.50330855E-01	-0.57580528E+01	-0.10305269E+02
2	-0.30772516E+00	-0.44899216E+01	0.62558521E+02
3	-0.43412885E+00	-0.65008521E+00	0.85064461E+02
4	-0.38465756E+00	0.22720275E+01	0.28117821E+02
5	-0.27407524E+00	0.18295681E+01	-0.40405914E+02
6	-0.24066399E+00	-0.41657627E+00	-0.39765869E+02
7	-0.28432682E+00	-0.90879965E+00	0.23411983E+02
8	-0.27970907E+00	0.14345939E+01	0.64174423E+02
9	-0.14624044E+00	0.38781688E+01	0.26183165E+02
10	0.39102390E-01	0.33305802E+01	-0.47345394E+02
11	0.12840362E+00	0.26650119E+00	-0.67513680E+02
12	0.80883794E-01	-0.18088942E+01	-0.10970811E+02
13	0.96553117E-02	-0.68976790E+00	0.50778744E+02
14	0.38462169E-01	0.18209940E+01	0.40712753E+02
15	0.14735566E+00	0.22107751E+01	-0.27525837E+02
16	0.19996722E+00	-0.31552225E+00	-0.66972618E+02
17	0.11764607E+00	-0.28227291E+01	-0.26468327E+02
18	-0.18114097E-01	-0.23140993E+01	0.45068512E+02
19	-0.63696094E-01	0.51316518E+00	0.59701672E+02
20	0.10478826E-01	0.21295214E+01	0.43720150E+00

3.86 Requirement:
(a) power output at least 700 W
(b) amplitude between 2 mm ~ 5 mm

Solution:
Unbalanced force in vertical direction = $m e \omega^2 \sin \omega t$ (E_1)
Unbalanced force in horizontal direction = 0
Let M = total mass of the shaker
Equation of motion is $M \ddot{x} + c \dot{x} + k x = m e \omega^2 \sin \omega t$ (E_2)
Steady state solution of (E_2) is

$$x(t) = X \sin(\omega t - \phi) \quad (E_3)$$

where $$X = \frac{m e r^2}{M [(1 \pm r^2)^2 + (2 \zeta r)^2]^{\frac{1}{2}}} \quad (E_4)$$

and $$\phi = \tan^{-1}\left(\frac{2 \zeta r}{1 - r^2}\right) \quad (E_5)$$

where $$r = \frac{\omega}{\omega_n} = \omega \sqrt{\left(\frac{M}{k}\right)} \quad (E_6)$$

Frequency range:

20 Hz to 30 Hz

i.e., 125.664 rad/sec to 188.496 rad/sec

$\therefore$ 125.664 rad/sec [<] ω [<] 188.496 rad/sec (E_7)

2 mm $\le X \le$ 5 mm where X is given by (E_4).

Mean power output over a time period [r] is given by

$$P = \frac{1}{\tau} \int_0^{\tau} F(\tau) \frac{dx}{dt}(\tau)\, d\tau \quad (E_9)$$

where $\tau = \frac{2\pi}{\omega}$,

$F(\tau) = m e \omega^2 \sin \omega t$, and

$\frac{dx}{dt} = \omega X \cos(\omega t - \phi)$

$$P \ge 700 \text{ W} \quad (E_{10})$$

$$\frac{M}{m} \ge 50 \quad (E_{11})$$

Procedure:

Find ω, e, M, m, k and c so as to satisfy the requirements stated in (E_7), (E_8), (E_{10}) and (E_{11}).

3.87 W is 500 KN,
height is 20 m,
yield strength 200 GPa,
ground acceleration a 0.5 g.

Solution:

$$m = \frac{500 \times 10^3}{9.8} = 51020.4 \text{ kg},$$

$\ell = 20$ m,

E = 200 GPa,

$$k = \frac{3\,E\,I}{t^3} = \frac{3 \times (200 \times 10^9)}{20^3} \frac{\pi}{64} (D^4 - d^4) = 3681554\,(D^4 - d^4) \text{ N/m}$$

$\omega = 2\,\pi\,(15) = 94.248$ rad/s, $\tau = 0.15$

Ground acceleration $\ddot{y}\,(t) = 4.9 \sin 94.248t\ (m/s^2)$ (E_1)

Equation of motion:

$$m\,\ddot{z} + c\,\dot{z} + k\,z = -\,m\,\ddot{y} = -\,250000 \sin 94.248t \qquad (E_2)$$

where $z = x - y$ = relative displacement

Let $z\,(t) = z \sin (94.248t - \phi)$ (E_3)

with $$z = \frac{-\,250000}{\sqrt{(k - m\,\omega^2) + c^2\,\omega^2}} \qquad (E_4)$$

and $$\phi = \tan^{-1}\left(\frac{c\,\omega}{k - m\,\omega^2}\right) \qquad (E_5)$$

max. bending moment = M = FL

$$\text{max. bending stress} = \frac{M\left(\frac{D}{2}\right)}{E\,I} = \frac{F\,L}{E\,I} \cdot \frac{D}{2} = \frac{3\,D}{2\,L^2}\,y_{max}$$

if maximum relative displacement, $y_{max} = Z$ is known, max. bending stress ([@]$_1$) is given by

$\sigma_1 = \frac{3\,D}{2\,t^2} \cdot Z$, direct compresssive stress (σ_2) due to weight of water is

$$\sigma_2 = \frac{mg}{\frac{\pi}{4}(D^2 - d^2)} = \frac{2 \times 10^6}{\pi\,(D^2 - d^2)}$$

Total stress = $\sigma_1 + \sigma_2 \le 200 \times 10^9$ Pa

i.e. $$\frac{3\,D}{2\,L^2}\,Z + \frac{2 \times 10^6}{\pi\,(D^2 - d^2)} \le 200 \times 10^9 \qquad (E_6)$$

Natural frequency

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3681554\,(D^4 - d^4)}{51020.4}} \ge 30\,\pi \text{ rad/sec}$$

i.e. $$D^4 - d^4 \ge 123.1 \qquad (E_7)$$

Weight of steel column

$$W_s = \frac{\pi}{4}\,(D^2 - d^2)\,\ell\,\rho_{weight} = \frac{\pi}{4}\,(D^2 - d^2)\,(20)\,(7.75 \times 10^3 \times 9.8)$$
$$= 1193020\,(D^2 - d^2)\text{ N}$$

Problem:

Find $\begin{Bmatrix} D \\ d \end{Bmatrix}$ to minimize W_s subject to restrictions of (E_6) and (E_7). Also use the conditions:

$D \ge d$, $D \ge 0$ and $d \ge 0$.

Procedure:
Plot the inequalities (E_6) and (E_7) in the D – d space and draw contours of W_s and identify the minimum weight design.