Chapter 5

Two Degree of Freedom Systems

5.1 Equations of motion

$$\left.\begin{aligned} m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 &= 0 \\ m_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 &= 0 \end{aligned}\right\} \quad (E_1)$$

With $x_i(t) = X_i \cos(\omega t + \phi)$; $i = 1,2$, Eqs. (E_1) give the frequency equation

$$\begin{vmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 \end{vmatrix} = 0$$

or $$\omega^4 - \left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \right) \omega^2 + \frac{k_1 k_2}{m_1 m_2} = 0 \quad (E_2)$$

Roots of Eq. (E_2) are

$$\omega_1^2 , \omega_2^2 = \frac{k_1 + k_2}{2 m_1} + \frac{k_2}{2 m_2} \mp \sqrt{\frac{1}{4}\left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right)^2 - \frac{k_1 k_2}{m_1 m_2}} \quad (E_3)$$

If $\vec{X}^{(1)} = \begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} = r_1 X_1^{(1)} \end{Bmatrix}$ and $\vec{X}^{(2)} = \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} = r_2 X_1^{(2)} \end{Bmatrix}$,

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m_1 \omega_1^2 + k_1 + k_2}{k_2} = \frac{k_2}{-m_2 \omega_1^2 + k_2} \quad (E_4)$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m_1 \omega_2^2 + k_1 + k_2}{k_2} = \frac{k_2}{-m_2 \omega_2^2 + k_2} \quad (E_5)$$

General solution of (E_1) is

$$\left.\begin{aligned} x_1(t) &= X_1^{(1)} \cos(\omega_1 t + \phi_1) + X_1^{(2)} \cos(\omega_2 t + \phi_2) \\ x_2(t) &= r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) + r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2) \end{aligned}\right\} \quad (E_6)$$

where $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1 and ϕ_2 can be found using Eqs. (5.18).

For $m_1 = m$, $m_2 = 2m$, $k_1 = k$ and $k_2 = 2k$, (E_3) gives

$$\omega_1^2 = (2 - \sqrt{3}) \frac{k}{m} \quad , \quad \omega_2^2 = (2 + \sqrt{3}) \frac{k}{m} \quad (E_7)$$

when $k = 1000$ N/m and $m = 20$ kg,

$\omega_1 = 3.6603$ rad/sec and $\omega_2 = 13.6603$ rad/sec

$$r_1 = \frac{k_2}{-m_2\omega_1^2 + k_2} = 1.36604, \quad r_2 = \frac{k_2}{-m_2\omega_2^2 + k_2} = -0.36602$$

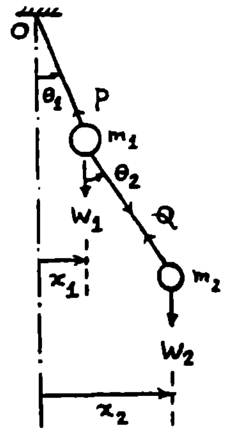
With $x_1(0) = 1$, $\dot{x}_1(0) = 0$, $x_2(0) = -1$ and $\dot{x}_2(0) = 0$, Eqs. (5.18) give $X_1^{(1)} = -0.36602$, $X_1^{(2)} = -1.36603$, $\phi_1 = 0$, $\phi_2 = 0$

Response of the system is

$$x_1(t) = -0.36602 \cos 3.6603\,t - 1.36603 \cos 13.6603\,t$$

$$x_2(t) = -0.5 \cos 3.6603\,t + 0.5 \cos 13.6603\,t$$

5.2 Taking moments about O and mass m_1,



$$m_1 \ell_1^2 \ddot{\theta}_1 = -W_1(\ell_1 \sin\theta_1) + Q \sin\theta_2 (\ell_1 \cos\theta_1) - Q\cos\theta_2(\ell_1 \sin\theta_1)$$

$$= -W_1 \ell_1 \theta_1 + W_2 \ell_1 (\theta_2 - \theta_1) \qquad (E_1)$$

assuming $Q \simeq W_2$.

$$m_2 \ell_2^2 \ddot{\theta}_2 + m_2 \ell_2 (\ell_1 \ddot{\theta}_1) = -W_2(\ell_2 \sin\theta_2)$$

$$= -W_2 \ell_2 \theta_2 \qquad (E_2)$$

Using the relations $\theta_1 = \frac{x_1}{\ell_1}$ and $\theta_2 = \frac{x_2 - x_1}{\ell_2}$, Eqs. (E_1) and (E_2) become

$$m_1 \ell_1 \ddot{x}_1 + \left[W_1 + W_2\left(\frac{\ell_1+\ell_2}{\ell_2}\right)\right] x_1 - \frac{W_2 \ell_1}{\ell_2} x_2 = 0 \qquad (E_3)$$

$$m_2 \ell_2 \ddot{x}_2 - W_2 x_1 + W_2 x_2 = 0 \qquad (E_4)$$

When $m_1 = m_2 = m$, $\ell_1 = \ell_2 = \ell$ and $W_1 = W_2 = mg$, Eqs. (E_3) and (E_4) give

$$\left.\begin{aligned} m\ell\ddot{x}_1 + 3mg\,x_1 - mg\,x_2 &= 0 \\ m\ell\ddot{x}_2 - mg\,x_1 + mg\,x_2 &= 0 \end{aligned}\right\} \qquad (E_5)$$

For harmonic motion $x_i(t) = X_i \cos\omega t$; $i = 1,2$, Eqs. (E_5) become

$$\left.\begin{aligned} -\omega^2 m\ell\, X_1 + 3mg\, X_1 - mg\, X_2 &= 0 \\ -\omega^2 m\ell\, X_2 - mg\, X_1 + mg\, X_2 &= 0 \end{aligned}\right\} \qquad (E_6)$$

from which the frequency equation can be obtained as

$$\omega^4 m^2 \ell^2 - (4 m^2 \ell g)\,\omega^2 + 2 m^2 g^2 = 0$$

i.e. $\quad \omega_1^2, \omega_2^2 = (2 \mp \sqrt{2})\,\frac{g}{\ell}$

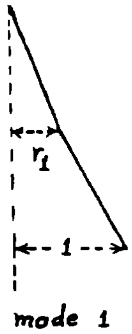
$\therefore \quad \omega_1 = 0.7654 \sqrt{\frac{g}{\ell}}, \quad \omega_2 = 1.8478 \sqrt{\frac{g}{\ell}}$

Ratio of amplitudes is given by Eq. (E_6) as

$$\frac{X_1}{X_2} = \frac{mg}{-\omega^2 m l + 3 mg} = \frac{1}{\left(-\omega^2 \frac{l}{g} + 3\right)}$$

In mode 1, $\omega_1 = 0.7654 \sqrt{\frac{g}{l}}$, $r_1 = \left(\frac{X_1}{X_2}\right)^{(1)} = 0.4142$

No node.

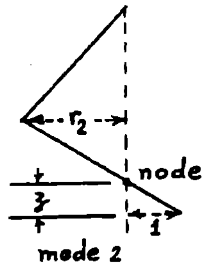


In mode 2, $\omega_2 = 1.8478 \sqrt{\frac{g}{l}}$,

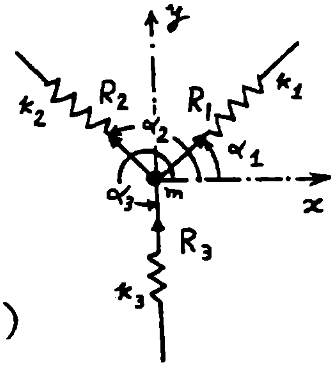
$$r_2 = \left(\frac{X_1}{X_2}\right)^{(2)} = -2.4133$$

one node located at z:

$$\frac{z}{1} = \frac{1-z}{2.4133} \quad \text{or} \quad z = 0.2930$$



5.3 Let R_1, R_2 and R_3 be the restoring forces in springs. Equations of motion of mass m in x and y directions are



$$m\ddot{x} = \sum_{i=1}^{3} R_i \cos\alpha_i \qquad (E_1)$$

$$m\ddot{y} = \sum_{i=1}^{3} R_i \sin\alpha_i \qquad (E_2)$$

where $R_i = -k_i\,(x\cos\alpha_i + y\sin\alpha_i)$ $\quad (E_3)$

Eqs. (E_1) to (E_3) give

$$m\ddot{x} + \sum_{i=1}^{3} k_i\,(x\cos^2\alpha_i + y\sin\alpha_i\cos\alpha_i) = 0 \qquad (E_4)$$

$$m\ddot{y} + \sum_{i=1}^{3} k_i\,(x\sin\alpha_i\cos\alpha_i + y\sin^2\alpha_i) = 0 \qquad (E_5)$$

For $\alpha_1 = 45°$, $\alpha_2 = 135°$, $\alpha_3 = 270°$ and $k_1 = k_2 = k_3 = k$, Eqs. (E_4) and (E_5) reduce to

$$m\ddot{x} + kx = 0 \qquad (E_6)$$

$$m\ddot{y} + 2ky = 0 \qquad (E_7)$$

These equations are uncoupled. For harmonic motion, $x(t) = X\cos(\omega t + \phi)$, $y(t) = Y\cos(\omega t + \phi)$, and hence

$\omega_1 = \sqrt{\frac{k}{m}}$ for motion in x direction

$\omega_2 = \sqrt{\frac{2k}{m}}$ for motion in y direction

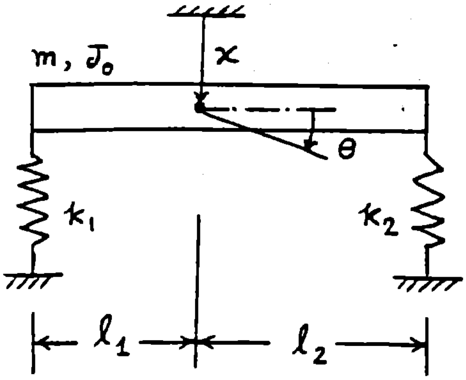
Natural modes are given by

$$x(t) = X\cos\left(\sqrt{\frac{k}{m}}\,t + \phi_1\right)$$

$$y(t) = Y\cos\left(\sqrt{\frac{2k}{m}}\,t + \phi_2\right)$$

where X, ϕ_1, Y and ϕ_2 can be determined from initial conditions.

5.4 Equations of motion in terms of x and θ:



$$m\ddot{x} + k_1(x - l_1\theta) + k_2(x + l_2\theta) = 0 \quad (E_1)$$

$$J_0\ddot{\theta} - k_1 l_1(x - l_1\theta) + k_2 l_2(x + l_2\theta) = 0 \quad (E_2)$$

For free vibration,

$$x(t) = X \cos(\omega t + \phi) \quad (E_3)$$

$$\theta(t) = \Theta \cos(\omega t + \phi) \quad (E_4)$$

and Eqs. (E_1) and (E_2) become

$$\begin{bmatrix} -m\omega^2 + k_1 + k_2 & -(k_1 l_1 - k_2 l_2) \\ -(k_1 l_1 - k_2 l_2) & -J_0\omega^2 + k_1 l_1^2 + k_2 l_2^2 \end{bmatrix} \begin{Bmatrix} X \\ \Theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (E_5)$$

Frequency equation is

$$\begin{vmatrix} -m\omega^2 + k_1 + k_2 & -(k_1 l_1 - k_2 l_2) \\ -(k_1 l_1 - k_2 l_2) & -J_0\omega^2 + k_1 l_1^2 + k_2 l_2^2 \end{vmatrix} = 0 \quad (E_6)$$

i.e.,

$$\begin{vmatrix} -\omega^2 + 5000 & 100 \\ 100 & -0.3\,\omega^2 + 2030 \end{vmatrix} = 0$$

i.e.,

$$0.3\,\omega^4 - 3530\,\omega^2 + 10.14 \times 10^6 = 0$$

i.e., $\omega^2 = 6785.3373\,,\quad 4981.3293$

$\therefore \quad \omega_1 = 70.5785$ rad/sec , $\quad \omega_2 = 82.3732$ rad/sec

Mode shapes:

$$(-1000\,\omega_1^2 + 5 \times 10^6)\,X + 0.1 \times 10^6\,\Theta = 0$$

or $$\left.\frac{X}{\Theta}\right|_{\omega_1} = \frac{-0.1 \times 10^6}{-1000\,\omega_1^2 + 5 \times 10^6} = -5.3476$$

and

$$\left.\frac{X}{\Theta}\right|_{\omega_2} = \frac{-0.1 \times 10^6}{-1000\,\omega_2^2 + 5 \times 10^6} = 0.05601$$

5.5 k_g = stiffness of girder $= \dfrac{48\,E\,I}{\ell^3} = \dfrac{48\,(17 \times 10^9)}{9^3} = 1.11934 \times 10^9$ N/m

k = stiffness of rope $= \dfrac{A\,E}{\ell} = \dfrac{A\,(200 \times 10^9)}{6} = 3.3333 \times 10^{10}\,A$ N/m

m_1 = mass of trolley = 3600 kg
m_2 = mass of load = 900 kg

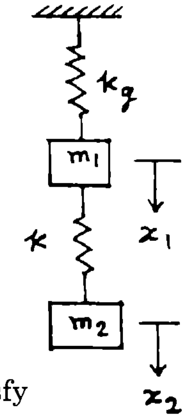
Desired frequency value: $\omega_1 > 20$ Hz. Let $\omega_1 \approx 25$ Hz = 157.08 rad/sec
or $\omega_1^2 \approx 24674.1264$ (rad/sec)^2

Fundamental natural frequency is given by (see Eq. (E_3) in solution of Problem 5.1):

$$\omega_1^2 = \frac{k_g + k}{2\,m_1} + \frac{k}{2\,m_2} - \sqrt{\left\{\frac{1}{4}\left(\frac{k_g + k}{m_1} + \frac{k}{m_2}\right)^2 - \frac{k_g\,k}{m_1\,m_2}\right\}} \tag{1}$$

Using the known values of kg, m_1 and m_2, a series of trial values of A are given and Eq. (1) is evaluated to find the corresponding values of ω_1^2. The results are given in the table below.

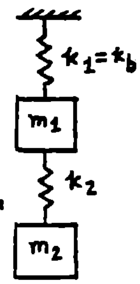
A (10^{-3} m^2)	k (N/m)	ω_1^2(rad/sec)^2	ω_1 (rad/sec)
0.4	13.3333 (10^6)	14,631.88	120.962
0.5	16.6667 (10^6)	18,230.206	135.02
0.6	20.0 (10^6)	21,803.27	147.66
0.7	23.3333 (10^6)	25,350.53	159.22
0.8	26.6667 (10^6)	28,871.44	169.916
0.9	30.0 (10^6)	32,365.116	179.90
1.0	33.3333 (10^6)	35,830.995	189.29



It can be seen that A = 0.7 × 10^{-3} m^2 yield a frequency that best satisfy $\omega \approx 157.08$ rad/sec

---

5.6 $k_1 = \frac{48\,EI}{l^3} = \frac{48\,(2.06 \times 10^{11})\,(0.02)}{(40)^3} = 3.09 \times 10^6$ N/m



$k_2 = 0.3 \times 10^6$ N/m, $m_1 = 1000$ kg, $m_2 = 5000$ kg

Eq. (E_3) in the solution of problem 5.1 gives

$$\omega_1^2, \omega_2^2 = \frac{3.39 \times 10^6}{2000} + \frac{0.3 \times 10^6}{10\,000} \mp \sqrt{\frac{1}{4}\left(\frac{3.39 \times 10^6}{1000} + \frac{0.3 \times 10^6}{5000}\right)^2 - \frac{3.09 \times 0.3 \times 10^{12}}{5 \times 10^6}}$$

$$= (1.725 \mp 1.6704)\,10^3$$

$\omega_1 = 7.3892$ rad/s, $\omega_2 = 58.2701$ rad/s

From Eqs. (E_4) and (E_5) of solution of problem 5.1,

$$r_1 = \frac{k_2}{-m_2\,\omega_1^2 + k_2} = \frac{0.3 \times 10^6}{-5000\,(54.6003) + 0.3 \times 10^6} = 11.1117$$

$$r_2 = \frac{k_2}{-m_2\,\omega_2^2 + k_2} = \frac{0.3 \times 10^6}{-5000\,(3395.4046) + 0.3 \times 10^6} = -0.01799$$

Mode shapes are $\vec{X}^{(1)} = \begin{Bmatrix} 1.0 \\ 11.1117 \end{Bmatrix} X_1^{(1)}$, $\vec{X}^{(2)} = \begin{Bmatrix} 1.0 \\ -0.01799 \end{Bmatrix} X_1^{(2)}$

5.7 Frequency equation:

$$\left| \left[-\omega^2 [m] + [k] \right] \right| = 0$$

or

$$\begin{vmatrix} (k_{11} - \omega^2 m_1) & k_{12} \\ k_{21} & (k_{22} - \omega^2 m_2) \end{vmatrix} = 0 \tag{1}$$

Expansion of the determinantal equation (1) gives:

$$(m_1 m_2)\, \omega^4 - (m_1 k_{22} + m_2 k_{11})\, \omega^2 + (k_{11} k_{22} - k_{12}^2) = 0 \tag{2}$$

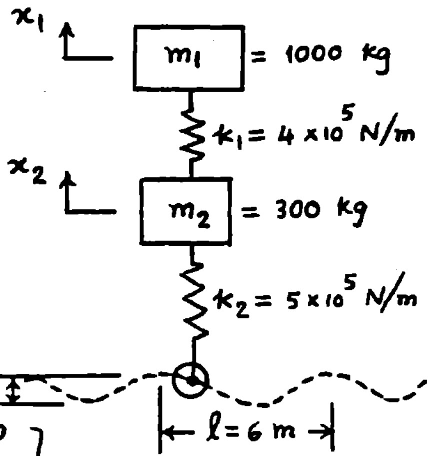
Roots of Eq. (2):

$$\omega_2^2,\ \omega_1^2 = \frac{(m_1 k_{22} + m_2 k_{11}) \pm \sqrt{\left\{(m_1 k_{22} - m_2 k_{11})^2 + 4\, m_1 m_2 k_{12}^2\right\}}}{2\, m_1 m_2} \tag{3}$$

Substitution of known expressions for k_{11}, k_{12}, and k_{22} into Eq. (3) yields:

$$\omega_2^2,\ \omega_1^2 = \frac{48}{7} \frac{E I}{m_1 m_2} \left[(m_1 + 8 m_2) \pm \sqrt{(m_1 - 8 m_2)^2 + 25\, m_1 m_2} \right] \tag{4}$$

5.8 Equations of motion :



$$\left. \begin{aligned} m_1 \ddot{x}_1 + k_1 x_1 - k_1 x_2 &= 0 \\ m_2 \ddot{x}_2 + (k_1 + k_2) x_2 - k_1 x_1 &= 0 \end{aligned} \right\} \quad (E_1)$$

Let $\quad x_i(t) = X_i \cos(\omega t + \phi); \quad i = 1, 2 \qquad (E_2)$

Eq. (E_1) becomes

$$\begin{bmatrix} -m_1 \omega^2 + k_1 & -k_1 \\ -k_1 & -m_2 \omega^2 + k_1 + k_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Frequency equation is

$$\begin{vmatrix} -m_1 \omega^2 + k_1 & -k_1 \\ -k_1 & -m_2 \omega^2 + k_1 + k_2 \end{vmatrix} = 0$$

i.e.,

$$m_1 m_2 \omega^4 - (m_1 k_1 + m_1 k_2)\, \omega^2 - k_1 m_2 \omega^2 + k_1 k_2 = 0$$

i.e.,

$$\omega^2 = \Big[(m_1 k_1 + m_1 k_2 + m_2 k_1) \pm \big(m_1^2 k_1^2 + m_1^2 k_2^2 + m_2^2 k_1^2 + 2 m_1^2 k_1 k_2 - 2 m_1 m_2 k_1 k_2 + 2 m_1 m_2 k_1^2 \big)^{\frac{1}{2}} \Big] \Big/ 2 m_1 m_2 \qquad (E_3)$$

Since $m_1 = 1000$ kg, $m_2 = 300$ kg, $k_1 = 4 \times 10^5$ N/m and $k_2 = 5 \times 10^5$ N/m, Eq. (E_3) gives

$$\omega_1 = 14.4539 \text{ rad/sec}, \qquad \omega_2 = 56.4897 \text{ rad/sec}$$

$$f_1 = \frac{14.4539}{2\pi} \text{ Hz} = \frac{s_1 (1000)}{3600} \left(\frac{1}{l}\right) = \frac{s_1}{21.6}$$

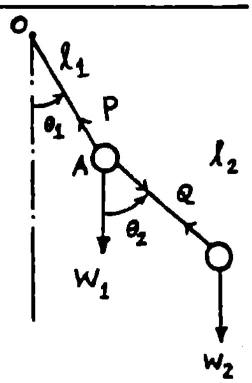
where $l = 6$ m and s_1 is in km/hr.

$$\therefore \; s_1 = \text{critical velocity \# 1} = \frac{14.4539}{2\pi}(21.6) = 49.6887 \text{ km/hr}$$

$$f_2 = \frac{56.4897}{2\pi} \text{ Hz} = s_2 \left(\frac{1000}{3600}\right)\left(\frac{1}{l}\right) = \frac{s_2}{21.6}$$

$$\therefore \; s_2 = \text{critical velocity \# 2} = \frac{56.4897}{2\pi}(21.6) = 194.1968 \text{ km/hr.}$$

5.9 Equations of motion for rotation about O and A give



$$m_1 l_1^2 \ddot{\theta}_1 = - W_1 l_1 \sin\theta_1 + Q \sin\theta_2 (l_1 \cos\theta_1) - Q \cos\theta_2 (l_1 \sin\theta_1)$$

$$= - W_1 l_1 \theta_1 + Q l_1 (\theta_2 - \theta_1)$$

$$= - W_1 l_1 \theta_1 + W_2 l_1 (\theta_2 - \theta_1) \quad ---- \quad (E_1)$$

since $Q \simeq W_2$.

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_2 (l_1 \ddot{\theta}_1) = - W_2 l_2 \sin\theta_2$$

$$= - W_2 l_2 \theta_2 \quad ---- \quad (E_2)$$

For $m_1 = m_2 = m$ and $l_1 = l_2 = l$, Eqs. (E_1) and (E_2) become

$$m l \ddot{\theta}_1 + 2mg\,\theta_1 - mg\,\theta_2 = 0$$

$$m l \ddot{\theta}_1 + m l \ddot{\theta}_2 + mg\,\theta_2 = 0$$

Assuming $\theta_i(t) = \Theta_i \cos(\omega t + \phi)$; $i = 1, 2$, we get

$$\begin{bmatrix} -\omega^2 m l + 2mg & -mg \\ -\omega^2 m l & -\omega^2 m l + mg \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad ---- \quad (E_3)$$

Defining $\lambda = \dfrac{\omega^2 m l}{mg} = \dfrac{\omega^2 l}{g}$, frequency equation can be obtained as

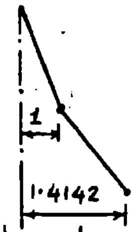
$$\begin{vmatrix} -\lambda + 2 & -1 \\ -\lambda & -\lambda + 1 \end{vmatrix} = \lambda^2 - 4\lambda + 2 = 0$$

$\lambda_1 = 2-\sqrt{2} = 0.5858$, $\lambda_2 = 2+\sqrt{2} = 3.4142$

$\omega_1 = 0.7654 \sqrt{\frac{g}{l}}$, $\omega_2 = 1.8478 \sqrt{\frac{g}{l}}$

For ω_1, first equation in (E_3) gives for $\Theta_1 = 1$,

$$\Theta_2 = -\lambda_1 + 2 = \sqrt{2} = 1.4142$$

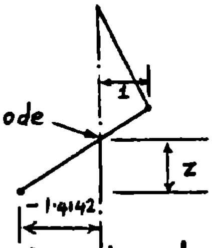


For ω_2, first equation in (E_3) gives for $\Theta_1 = 1$,

$$\Theta_2 = -\lambda_2 + 2 = -\sqrt{2} = -1.4142$$

Location of node :

$$\frac{z}{1.4142} = \frac{1-z}{1} \quad ; \quad z = 0.4142$$



5.10 Eq. (E_3) in the solution of problem 5.1 gives

$$\omega_1^2, \omega_2^2 = \frac{k_1+k_2}{2m_1} + \frac{k_2}{2m_2} \mp \sqrt{\frac{1}{4}\left(\frac{k_1+k_2}{m_1} + \frac{k_2}{m_2}\right)^2 - \frac{k_1 k_2}{m_1 m_2}}$$

For $m_1 = m_2 = m$ and $k_1 = k_2 = k$, we get

$$\omega_1^2, \omega_2^2 = \frac{k}{m} + \frac{k}{2m} \mp \sqrt{\frac{1}{4}\left(\frac{3k}{m}\right)^2 - \frac{k^2}{m^2}} = \frac{3k}{2m} \mp \frac{k}{m}\sqrt{\frac{5}{4}}$$

$$\omega_1^2 = 0.382 \frac{k}{m} , \quad \omega_2^2 = 2.618 \frac{k}{m}$$

$$\omega_1 = 0.6181 \sqrt{\frac{k}{m}} , \quad \omega_2 = 1.6180 \sqrt{\frac{k}{m}}$$

Mode shapes are given by (see Eqs. (E_4) and (E_5) of problem 5.1)

$$r_1 = \frac{k_2}{-m_2 \omega_1^2 + k_2} = 1.6181 \quad ; \quad \vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1.6181 \end{Bmatrix} X_1^{(1)}$$

$$r_2 = \frac{k_2}{-m_2 \omega_2^2 + k_2} = -0.6180 \quad ; \quad \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ -0.6180 \end{Bmatrix} X_1^{(2)}$$

5.11 For the system of Fig. 5.3 (a),

$$\vec{X}^{(1)T} [m] \vec{X}^{(2)} = \{ X_1^{(1)} \quad X_2^{(1)} \} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{Bmatrix} = X_1^{(1)} X_1^{(2)} \{1 \quad r_1\} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} 1 \\ r_2 \end{Bmatrix}$$

$$= X_1^{(1)} X_1^{(2)} (m_1 + m_2 r_1 r_2)$$

But

$$m_1 + m_2 r_1 r_2 = m_1 + m_2 \left(\frac{k_2}{-m_2\omega_1^2 + k_2 + k_3}\right)\left(\frac{k_2}{-m_2\omega_2^2 + k_2 + k_3}\right) \equiv \frac{N}{D}$$

where

$$N = m_1 (-m_2 \omega_1^2 + k_2 + k_3)(-m_2 \omega_2^2 + k_2 + k_3) + m_2 k_2^2 \text{ , and}$$

$$D = (-m_2 \omega_1^2 + k_2 + k_3)(-m_2 \omega_2^2 + k_2 + k_3)$$

By substituting

$$\omega_1^2, \omega_2^2 = \frac{(k_1+k_2)m_2 + (k_2+k_3)m_1}{2 m_1 m_2} \mp \frac{1}{2}\left\{\left[\frac{(k_1+k_2)m_2 + (k_2+k_3)m_1}{m_1 m_2}\right]^2 - 4\,\frac{(k_1+k_2)(k_2+k_3) - k_2^2}{m_1 m_2}\right\}^{1/2}, \text{ we get}$$

$$N = m_1 m_2^2\, \omega_1^2 \omega_2^2 - m_1 m_2\, k_2\, \omega_2^2 - m_1 m_2\, k_3\, \omega_2^2 - m_1 m_2\, k_2\, \omega_1^2 + m_1 k_2^2 + m_1 k_2 k_3 - m_1 m_2 k_3 \omega_1^2 + m_1 k_2 k_3 + m_1 k_3^2 + m_2 k_2^2$$

$$= m_1 m_2^2 \left\{ \left[\frac{(k_1+k_2)m_2 + (k_2+k_3)m_1}{2 m_1 m_2}\right]^2 - \frac{1}{4}\left[\left(\frac{(k_1+k_2)m_2 + (k_2+k_3)m_1}{m_1 m_2}\right)^2 - 4\left(\frac{(k_1+k_2)(k_2+k_3) - k_2^2}{m_1 m_2}\right)\right]\right\} - m_1 m_2 k_2 \left\{\frac{(k_1+k_2)m_2 + (k_2+k_3)m_1}{m_1 m_2}\right\}$$

$$- m_1 m_2 k_3 \left\{\frac{(k_1+k_2)m_2 + (k_2+k_3)m_1}{m_1 m_2}\right\} + m_1 k_2^2 + m_2 k_2^2 + 2 m_1 k_2 k_3 + m_1 k_3^2$$

$$= 0$$

5.12 Eq. (5.10) gives

$$\omega_1^2, \omega_2^2 = \frac{1}{2}\left\{\frac{(800)1 + (700)2}{2}\right\} \mp \frac{1}{2}\left[\left\{\frac{(800)1 + (700)2}{2}\right\}^2 - 4\left\{\frac{(800)(700) - (500)^2}{2}\right\}\right]^{1/2}$$

$$= 550 \mp 384.0573 = 165.9427,\ 934.0573$$

$\omega_1 = 12.8819$ rad/s , $\omega_2 = 30.5624$ rad/s

5.13 Eq. (E_3) in the solution of problem 5.1 gives

$$\omega_1^2, \omega_2^2 = \frac{8000}{2} + \frac{6000}{2} \mp \sqrt{\frac{1}{4}\left(\frac{8000}{1} + \frac{6000}{1}\right)^2 - \frac{12 \times 10^6}{1}} = 917.2,\ 13082.8$$

$\omega_1 = 30.2853$ rad/sec , $\omega_2 = 114.3801$ rad/s

Eqs. (E_4) and (E_5) of solution of problem 5.1 give

$$r_1 = \frac{k_2}{-m_2 \omega_1^2 + k_2} = \frac{6000}{-917.2 + 6000} = 1.1805\ ; \quad \vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1.1805 \end{Bmatrix} X_1^{(1)}$$

$$r_2 = \frac{k_2}{-m_2 \omega_2^2 + k_2} = \frac{6000}{-13082.8 + 6000} = -0.8471\ ; \quad \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ -0.8471 \end{Bmatrix} X_1^{(2)}$$

5.14 Same as example 5.1 with $m_1 = m_2 = 4400$ kg and $k_1 = k_2 = k_3 = 9 \times 10^6$ N/m

$$\omega_1 = \sqrt{\frac{k}{m}} = \sqrt{\frac{9 \times 10^6}{4400}} = 45.226 \text{ rad/sec}$$

$$\omega_2 = \sqrt{\frac{3\,k}{m}} = 78.335 \text{ rad/sec}$$

General motion is given by Eq. (E_{10}) of Example 5.1:

$$x_1(t) = x_1^{(1)} \cos(45.226\,t + \phi_1) + x_1^{(2)} \cos(78.335\,t + \phi_2)$$
$$x_2(t) = x_1^{(1)} \cos(45.226\,t + \phi_1) - x_1^{(2)} \cos(78.335\,t + \phi_2)$$

5.15 Eq.(5.10) gives

$$\omega_1^2, \omega_2^2 = \frac{1}{2}\left\{\frac{3000(2)+4000(1)}{2}\right\} \mp \frac{1}{2}\left[\left\{\frac{3000(2)+4000(1)}{2}\right\}^2 - 4\left\{\frac{3000(4000)-1000^2}{2}\right\}\right]^{\frac{1}{2}}$$

$$= 1633.9746,\ 3366.0254$$

$\omega_1 = 40.4225$ rad/s, $\omega_2 = 58.0175$ rad/s

$$r_1 = \frac{k_2}{-m_2\omega_1^2 + k_2 + k_3} = \frac{1000}{-2(1633.9746)+4000} = 1.3660$$

$$r_2 = \frac{k_2}{-m_2\omega_2^2 + k_2 + k_3} = \frac{1000}{-2(3366.0254)+4000} = -0.3660$$

When $x_1(0) = x_2(0) = \dot{x}_2(0) = 0$ and $\dot{x}_1(0) = 20$ m/s,

$$x_1^{(1)} = \frac{1}{(-0.366-1.366)}\left[\frac{+0.366(20)}{40.4225}\right] = -0.1046$$

$$x_1^{(2)} = \frac{1}{-1.732}\left[\frac{1.366(20)}{58.0175}\right] = -0.2719$$

$$\phi_1 = \phi_2 = \tan^{-1}(\infty) = \frac{\pi}{2}$$

Motion of the two masses is given by Eq. (5.15):

$$x_1(t) = 0.1046 \sin 40.4225t + 0.2719 \sin 58.0175t$$

$$x_2(t) = +(1.3660)(0.1046) \sin 40.4225t - (-0.3660)(-0.2719) \sin 58.0175t$$
$$= 0.1429 \sin 40.4225t - 0.09952 \sin 58.0175t$$

5.16 (a) $\omega_1^2 = 917.2$, $\omega_2^2 = 13082.8$, $r_1 = 1.1805$, $r_2 = -0.8471$

$x_1(0) = 0.2$, $x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$

Eq. (5.18) gives

$$x_1^{(1)} = \frac{1}{(-0.8471-1.1805)}\left[(-0.8471)(0.2)\right] = 0.08356$$

$$x_1^{(2)} = \frac{1}{(-0.8471-1.1805)}\left[(-1.1805)(0.2)\right] = 0.11644$$

$$\phi_1 = \phi_2 = \tan^{-1}(0) = 0$$

$$x_1(t) = 0.08356 \cos 30.2853t + 0.11644 \cos 114.3801t$$

$$x_2(t) = (1.1805)(0.08356) \cos 30.2853t + (-0.8471)(0.11644) \cos 114.3801t$$
$$= 0.09864 \cos 30.2853t - 0.09864 \cos 114.3801t$$

(b) $x_1(0) = 0.2$, $x_2(0) = \dot{x}_1(0) = 0$, $\dot{x}_2(0) = 5.0$

Eq. (5.18) gives

$$x_1^{(1)} = \frac{1}{-2.0276}\left[\{-0.8471(0.2)\}^2 + \frac{(5)^2}{917.2}\right]^{1/2} = -0.1167$$

$$X_1^{(2)} = \frac{1}{-2.0276}\left[\{-1.1805\,(0.2)\}^2 + \frac{(-5)^2}{13082.8}\right]^{1/2} = -0.1184$$

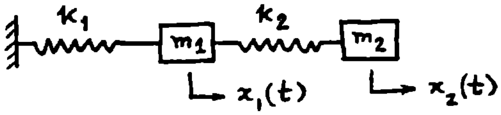
$$\phi_1 = \tan^{-1}\left\{\frac{5.0}{30.2853\quad(-0.8471)(0.2)}\right\} = \tan^{-1}(-0.9745) = 135.7399^\circ$$

$$\phi_2 = \tan^{-1}\left\{\frac{-5.0}{114.3801\,(-1.1805)(0.2)}\right\} = \tan^{-1}(0.1851) = 10.4895^\circ$$

$$x_1(t) = -0.1167\cos(30.2853\,t + 135.7399^\circ) - 0.1184\cos(114.3801\,t + 10.4895^\circ)$$

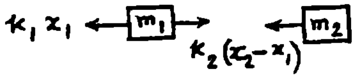
$$x_2(t) = (1.1805)(-0.1167)\cos(30.2853\,t + 135.7399^\circ) - 0.8471\,(-0.1184)\cos(114.3801\,t + 10.4895^\circ)$$
$$= -0.1378\cos(30.2853\,t + 135.7399^\circ) + 0.1003\cos(114.3801\,t + 10.4895^\circ)$$

5.17 Equivalent system is shown in figure:



$$k_i = 2\left(\frac{24\,E\,I_i}{h_i^3}\right)\quad;\quad i = 1, 2$$

$$k_1 = k_2 = k = \frac{48\,E\,I}{h^3}\;;\; m_1 = 2m,\; m_2 = m$$



Equations of motion:
$$m_1\ddot{x}_1 + (k_1 + k_2)\,x_1 - k_2 x_2 = 0$$
$$m_2\ddot{x}_2 - k_2 x_1 + k_2 x_2 = 0$$

For harmonic motion $x_i(t) = X_i\cos(\omega t + \phi)$; $i = 1, 2$, we get

$$\begin{bmatrix} (-\omega^2 m_1 + k_1 + k_2) & -k_2 \\ -k_2 & (-\omega^2 m_2 + k_2) \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{----} (E_1)$$

Frequency equation is

$$\begin{vmatrix} (-\omega^2 m_1 + k_1 + k_2) & -k_2 \\ -k_2 & (-\omega^2 m_2 + k_2) \end{vmatrix} = 0$$

or
$$\omega^4 m_1 m_2 - \omega^2(m_2 k_1 + m_2 k_2 + m_1 k_2) + k_1 k_2 = 0$$

$$\omega^2 = \frac{(m_2 k_1 + m_2 k_2 + m_1 k_2) \pm \sqrt{(m_2 k_1 + m_2 k_2 + m_1 k_2)^2 - 4 m_1 m_2 k_1 k_2}}{2 m_1 m_2} \quad \text{---} (E_2)$$

For given data,
$$\omega^2 = \frac{(mk + mk + 2mk) \pm \sqrt{(mk + mk + 2mk)^2 - 8m^2k^2}}{4m^2} = \frac{k}{m}\left(1 \pm \frac{1}{\sqrt{2}}\right)$$

$$\omega_1 = 0.5412\sqrt{\frac{k}{m}} = 3.7495\sqrt{\frac{EI}{mh^3}}\;;\quad \omega_2 = 1.3066\sqrt{\frac{k}{m}} = 9.0524\sqrt{\frac{EI}{mh^3}}$$

From Eq. (E_1), we get

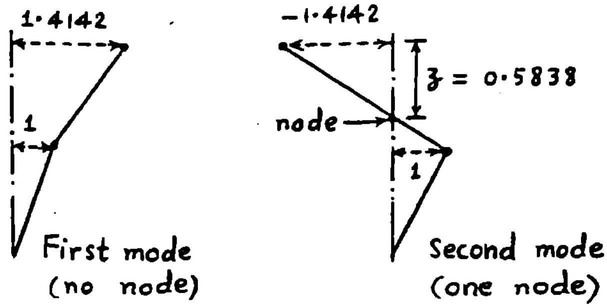
$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-\omega_1^2 m_1 + k_1 + k_2}{k_2} = \frac{-2m\,\omega_1^2 + 2k}{k} = \frac{-2(0.2929\,k) + 2k}{k}$$
$$= 1.4142$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-\omega_2^2 m_1 + k_1 + k_2}{k_2} = \frac{-2m\,\omega_2^2 + 2k}{k} = \frac{-2(1.7071\,k) + 2k}{k}$$
$$= -1.4142$$

Mode shapes are:

$$\vec{X}^{(1)} = \begin{Bmatrix} 1.0 \\ 1.4142 \end{Bmatrix} X_1^{(1)}$$

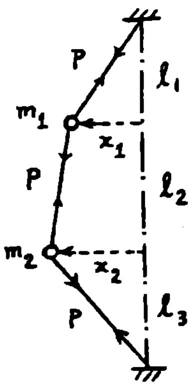
$$\vec{X}^{(2)} = \begin{Bmatrix} 1.0 \\ -1.4142 \end{Bmatrix} X_1^{(2)}$$



Location of node:

$$\frac{z}{1.4142} = \frac{1-z}{1} \; ; \; z = 0.5838$$

5.18 Let P be the tension in the string. Horizontal components of tension (along x_1 direction) in the string lying above and below m_1 are $-\frac{x_1 P}{l_1}$ and $-\frac{(x_1 - x_2) P}{l_2}$, respectively.



Newton's second law gives

$$m_1 \ddot{x}_1 = -\frac{x_1 P}{l_1} - \frac{(x_1 - x_2) P}{l_2} \quad \text{or} \quad m_1 \ddot{x}_1 + \frac{x_1}{l_1} P + \left(\frac{x_1 - x_2}{l_2}\right) P = 0$$

Similarly

$$m_2 \ddot{x}_2 + \frac{x_2}{l_3} P - \left(\frac{x_1 - x_2}{l_2}\right) P = 0$$

With $x_i(t) = X_i \cos(\omega t + \phi)$; $i = 1,2$, and $l_1 = l_2 = l_3 = l$, $m_1 = m_2 = m$,

$$\begin{bmatrix} \left(-m\,\omega^2 + \frac{2P}{l}\right) & -\frac{P}{l} \\ -\frac{P}{l} & \left(-m\,\omega^2 + \frac{2P}{l}\right) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{----} \; (E_1)$$

This gives the frequency equation

$$\left(-m\,\omega^2 + \frac{2P}{l}\right)^2 - \left(\frac{P}{l}\right)^2 = \left(-m\,\omega^2 + \frac{3P}{l}\right)\left(-m\,\omega^2 + \frac{P}{l}\right) = 0$$

$$\therefore \quad \omega_1 = \sqrt{\frac{P}{ml}} \; , \quad \omega_2 = \sqrt{\frac{3P}{ml}}$$

From first of Eqs. (E_1),

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m\,\omega_1^2 + \frac{2P}{l}}{\left(\frac{P}{l}\right)} = 1 \; ; \quad r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m\,\omega_2^2 + \frac{2P}{l}}{\left(\frac{P}{l}\right)} = -1$$

mode shapes are: $\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} X_1^{(1)}$, $\vec{X}^{(2)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} X_1^{(2)}$

No node — one node at middle of the two masses

(5.19) For $m_1 = 3m$, $m_2 = m$, $k_1 = 3k$ and $k_2 = k$, Eq. (E_2) in solution of problem 5.17 gives

$$\omega^2 = \frac{(3mk + mk + 3mk) \pm \sqrt{(3mk+mk+3mk)^2 - 36k^2m^2}}{6m^2} = \frac{k}{6m}(7 \pm \sqrt{13})$$

$$\omega^2 = 0.5657\,\frac{k}{m}, \quad 1.7676\,\frac{k}{m}$$

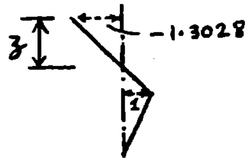
$$\omega_1 = 0.7521\sqrt{\frac{k}{m}}, \quad \omega_2 = 1.3295\sqrt{\frac{k}{m}}$$

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-\omega_1^2 m_1 + k_1 + k_2}{k_2} = \frac{-0.5657(3) + 3 + 1}{1} = 2.3029$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-\omega_2^2 m_1 + k_1 + k_2}{k_2} = \frac{-1.7676(3) + 3 + 1}{1} = -1.3028$$

$$\vec{X}^{(1)} = \begin{Bmatrix} 1.0 \\ 2.3029 \end{Bmatrix} X_1^{(1)}, \quad \vec{X}^{(2)} = \begin{Bmatrix} 1.0 \\ -1.3028 \end{Bmatrix} X_1^{(2)}$$

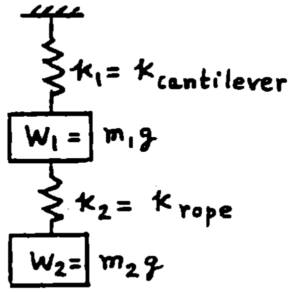
No node — one node



$$\frac{z}{1.3028} = \frac{1-z}{1}; \quad z = 0.5657$$

(5.20) $$k_1 = \frac{3EI}{b^3} = \frac{3E\left(\frac{1}{12}at^3\right)}{b^3} = \frac{Eat^3}{4b^3}$$

$$k_2 = \frac{AE}{l} = \frac{\pi d^2 E}{4l}$$



From problem 5.1,

$$\omega_{1,2}^2 = \frac{k_1 + k_2}{2m_1} + \frac{k_2}{2m_2} \mp \sqrt{\frac{1}{4}\left(\frac{k_1+k_2}{m_1} + \frac{k_2}{m_2}\right)^2 - \frac{k_1 k_2}{m_1 m_2}}$$

$$= \left(\frac{Eat^3}{4b^3} + \frac{\pi d^2 E}{4l}\right)\frac{g}{2W_1} + \frac{\pi d^2 E g}{8 l W_2}$$

$$\mp \sqrt{\left[\frac{1}{4}\left\{\left(\frac{Eat^3}{4b^3} + \frac{\pi d^2 E}{4l}\right)\frac{g}{W_1} + \frac{\pi d^2 E g}{4 l W_2}\right\}^2 - \frac{E^2 a t^3 \pi d^2 g^2}{16 l b^3 W_1 W_2}\right]}$$

(5.21) From solution of Problem 5.1, we find that for $m_1 = m$, $m_2 = 2m$, $k_1 = k$ and $k_2 = 2k$,

$$\omega_1^2 = (2 - \sqrt{3})\,\frac{k}{m}\ ; \ \omega_2^2 = (2 + \sqrt{3})\,\frac{k}{m}$$

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{k_2}{-m_2\,\omega_1^2 + k_2} = \frac{1}{-1 + \sqrt{3}}$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{k_2}{-m_2\,\omega_2^2 + k_2} = \frac{1}{-1-\sqrt{3}}$$

First mode shape:

$$\begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \cos(\omega_1 t + \phi_1) \\ r_1\, X_1^{(1)} \cos(\omega_1 t + \phi_1) \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \cos(\omega_1 t + \phi_1) \\ \left(\dfrac{X_1^{(1)}}{\sqrt{3}-1}\right) \cos(\omega_1 t + \phi_1) \end{Bmatrix}$$

For the motion to be identical with the first normal mode, we need to have $X_1^{(2)} = 0$. This requires that (from Eq. (5.18)):

$$\frac{1}{r_2 - r_1}\left[\left\{-r_1\, x_1(0) + x_2(0)\right\}^2 + \frac{1}{\omega_2^2}\left\{r_1\, \dot{x}_1(0) - \dot{x}_2(0)\right\}^2\right]^{\frac{1}{2}} = 0$$

or

$$x_2(0) = r_1\, x_1(0) = \frac{x_1(0)}{\sqrt{3}-1}$$

$$\dot{x}_2(0) = r_1\, \dot{x}_1(0) = \frac{\dot{x}_1(0)}{\sqrt{3}-1}$$

5.22 Let $m_1 = m$, $m_2 = 2\,m$, $k_1 = k$, $k_2 = 2\,k$.
Initial conditions: $x_1(0) = 0$, $x_2(0) = 0.1$ m, $\dot{x}_1(0) = 0$, $\dot{x}_2(0) = 0$
Eqs. (5.18) yield:

$$X_1^{(1)} = \frac{1}{r_2 - r_1}\left[(0-0.1)^2\right]^{\frac{1}{2}} = \frac{0.1}{r_2 - r_1} = \frac{0.1}{\left(\dfrac{-1}{\sqrt{3}+1} - \dfrac{1}{\sqrt{3}-1}\right)} = -\frac{0.1}{\sqrt{3}}$$

$$X_1^{(2)} = \frac{1}{r_2 - r_1}\left[(0.1)^2\right]^{\frac{1}{2}} = \frac{0.1}{r_2 - r_1} = -\frac{0.1}{\sqrt{3}}$$

$$\phi_1 = \tan^{-1}(0) = 0$$

$$\phi_2 = \tan^{-1}(0) = 0$$

where ω_1 and ω_2 are given by Eq. (E3) of solution of Problem 5.1.

Resulting motion:

$$x_1(t) = X_1^{(1)} \cos(\omega_1 t + \phi_1) + X_1^{(2)} \cos(\omega_2 t + \phi_2) = -\frac{0.1}{\sqrt{3}}\left\{\cos \omega_1 t + \cos \omega_2 t\right\}$$

$$x_2(t) = r_1\, X_1^{(1)} \cos(\omega_1 t + \phi_1) + r_2\, X_1^{(2)} \cos(\omega_2 t + \phi_2)$$

$$= \left(\frac{1}{\sqrt{3}-1}\right)\left(-\frac{0.1}{\sqrt{3}}\right) \cos \omega_1 t + \left(-\frac{1}{\sqrt{3}+1}\right)\left(-\frac{0.1}{\sqrt{3}}\right) \cos \omega_2 t$$

$$= -\frac{0.1}{\sqrt{3}}\left[\left(\frac{1}{\sqrt{3}-1}\right)\cos\omega_1 t - \left(\frac{1}{\sqrt{3}+1}\right)\cos\omega_2 t\right]$$

(5.23) $\omega_1^2 \geq (2\,\pi \times 10)^2 = 3947.8602\ (\text{rad/sec})^2$

From solution of problem 5.2, this inequality can be expressed as:

$$\omega_1^2 = \left(\frac{E\,a\,t^3}{4\,b^3} + \frac{\pi\,d^2\,E}{4\,\ell}\right)\frac{1}{2m_1} + \frac{\pi\,d^2\,E}{8\,\ell\,m_2}$$

$$- \sqrt{\left\{\frac{1}{4}\left\{\left(\frac{E\,a\,t^3}{4\,b^3} + \frac{\pi\,d^2\,E}{4\,\ell}\right)\frac{1}{m_1} + \frac{\pi\,d^2\,E}{4\,\ell\,m_2}\right\}^2 - \frac{E^2\,a\,t^3\,\pi\,d^2}{16\,\ell\,b^3\,m_1\,m_2}\right\}}$$

$$\geq 3947.8602 \qquad (E_1)$$

Data: $E = 200 \times 10^9$ Pa, $m_1 = 450$ kg, $m_2 = 225$ kg
$b = 0.75$ m, $\ell = 1.5$ m

Unknowns: a, t, d

Let $a = 10\,t$ and $d = t$

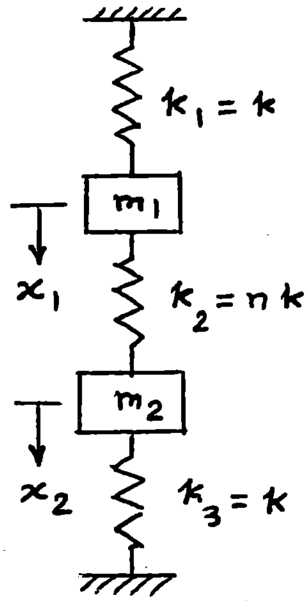
For this data, t is incremented from 2.5 mm in increment of 0.25 mm and the left hand side of the inequality (E_1) is evaluated. This gives a value of t = 38.75 mm for satisfying (E_1).

$\therefore$ Design is t = 38.75 mm, d = t = 38.75 mm, a = 10 t = 387.5 mm

---

(5.24) Equations of motion:

$$\left.\begin{aligned} m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2 x_2 &= 0 \\ m_2\ddot{x}_2 - k_2 x_1 + (k_2+k_3)x_2 &= 0 \end{aligned}\right\} \qquad (1)$$



Eigenvalue problem:

$$\begin{bmatrix} (-m_1\omega^2 + k_1 + k_2) & -k_2 \\ -k_2 & (-m_2\omega^2 + k_2 + k_3) \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (2)$$

For the data $k_1 = k_3 = 8$, $k_2 = n\,k = 8$, $m_1 = m_2 = m = 2$, Eq. (2) becomes

$$\begin{bmatrix} -2\omega^2+16 & -8 \\ -8 & -2\omega^2+16 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3}$$

Frequency equation:

$$(-2\omega^2+16)^2 - 8^2 = 0$$

or $\qquad \omega^2 = 4,\ 12$

or $\qquad \omega = 2,\ 3.4641 \qquad (4)$

For $\omega_1^2 = 4$; Eq. (3) gives

$$[-2(4)+16]\,X_1 = 8\,X_2 \qquad \text{or} \qquad X_1 = X_2$$

$$\therefore \quad \vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} X_1^{(1)} \tag{5}$$

For $\omega_2^2 = 12$; Eq. (3) gives

$$[-2(12)+16]\,X_1 = 8\,X_2 \qquad \text{or} \qquad X_2 = -X_1$$

$$\therefore \quad \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} X_1^{(2)} \tag{6}$$

Free vibration responses of masses m_1 and m_2 are given by

$$x_1(t) = X_1^{(1)} \cos(2t+\phi_1) + X_1^{(2)} \cos(3.4641\,t+\phi_2) \tag{7}$$

$$x_2(t) = X_1^{(1)} \cos(2t+\phi_1) - X_1^{(2)} \cos(3.4641\,t+\phi_2) \tag{8}$$

where $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1 and ϕ_2 are constants to be determined from the initial conditions. Using the given initial conditions, Eqs. (7) and (8) yield

$$x_1(t=0) = 1 = X_1^{(1)} \cos\phi_1 + X_1^{(2)} \cos\phi_2 \tag{9}$$

$$x_2(t=0) = 0 = X_1^{(1)} \cos\phi_1 - X_1^{(2)} \cos\phi_2 \tag{10}$$

$$\dot{x}_1(t=0) = 0 = -2\,X_1^{(1)} \sin\phi_1 - 3.4641\,X_1^{(2)} \sin\phi_2 \tag{11}$$

$$\dot{x}_2(t=0) = 1 = -2\,X_1^{(1)} \sin\phi_1 + 3.4641\,X_1^{(2)} \sin\phi_2 \tag{12}$$

Eqs. (9) and (10) give:

$$X_1^{(1)} \cos\phi_1 = X_1^{(2)} \cos\phi_2 = \frac{1}{2} \tag{13}$$

Eqs. (11) and (12) yield:

$$X_1^{(1)} \sin\phi_1 = -0.25, \qquad X_1^{(2)} \sin\phi_2 = 0.1443 \tag{14}$$

From Eqs. (13) and (14), we can find

$$X_1^{(1)} = \left\{ \left(\tfrac{1}{2}\right)^2 + \left(-\tfrac{1}{4}\right)^2 \right\}^{\frac{1}{2}} = 0.5590$$

$$\phi_1 = \tan^{-1}\left(\frac{X_1^{(1)} \sin \phi_1}{X_1^{(1)} \cos \phi_1} \right) = \tan^{-1}(-0.5) = -0.4636 \text{ rad}$$

$$X_1^{(2)} = \left\{ \left(\tfrac{1}{2}\right)^2 + (0.1443)^2 \right\}^{\frac{1}{2}} = 0.5204$$

$$\phi_2 = \tan^{-1}\left(\frac{X_1^{(2)} \sin \phi_2}{X_1^{(2)} \cos \phi_2} \right) = \tan^{-1}(0.2886) = 0.2810 \text{ rad}$$

Free vibration responses of m_1 and m_2:

$$x_1(t) = 0.5590 \cos(2t - 0.4636) + 0.5204 \cos(3.4641\, t + 0.2810)$$

$$x_2(t) = 0.5590 \cos(2t - 0.4636) - 0.5204 \cos(3.4641\, t + 0.2810)$$

5.25 From solution of problem 5.24, the free vibration responses of masses m_1 and m_2 are given by

$$x_1(t) = X_1^{(1)} \cos(2t + \phi_1) + X_1^{(2)} \cos(3.4641\, t + \phi_2) \qquad (1)$$

$$x_2(t) = X_1^{(1)} \cos(2t + \phi_1) - X_1^{(2)} \cos(3.4641\, t + \phi_2) \qquad (2)$$

where $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1 and ϕ_2 are constants to be determined from the initial conditions. Using the given initial conditions, Eqs. (1) and (2) lead to

$$x_1(0) = 1 = X_1^{(1)} \cos \phi_1 + X_1^{(2)} \cos \phi_2 \qquad (3)$$

$$x_2(0) = 0 = X_1^{(1)} \cos \phi_1 - X_1^{(2)} \cos \phi_2 \qquad (4)$$

$$\dot{x}_1(0) = 0 = -2\, X_1^{(1)} \sin \phi_1 - 3.4641\, X_1^{(2)} \sin \phi_2 \qquad (5)$$

$$\dot{x}_2(0) = 0 = -2\, X_1^{(1)} \sin \phi_1 + 3.4641\, X_1^{(2)} \sin \phi_2 \qquad (6)$$

Eqs. (3) and (4) give:

$$X_1^{(1)} \cos \phi_1 = \tfrac{1}{2} \qquad (7)$$

$$X_1^{(2)} \cos \phi_2 = \tfrac{1}{2} \qquad (8)$$

Eqs. (5) and (6) give:

$$X_1^{(1)} \sin \phi_1 = 0 \qquad (9)$$

$$X_1^{(2)} \sin \phi_2 = 0 \qquad (10)$$

Eqs. (7) and (9) give:

$$x_1^{(1)} = \left\{ \left(\tfrac{1}{2}\right)^2 + (0)^2 \right\}^{\frac{1}{2}} = \tfrac{1}{2} \quad ; \quad \phi_1 = \tan^{-1}(0) = 0$$

Eqs. (8) and (10) yield:

$$x_1^{(2)} = \left\{ \left(\tfrac{1}{2}\right)^2 + (0)^2 \right\}^{\frac{1}{2}} = \tfrac{1}{2} \quad ; \quad \phi_2 = \tan^{-1}(0) = 0$$

Hence, the free vibration responses of m_1 and m_2 are:

$$x_1(t) = \tfrac{1}{2} \cos 2t + \tfrac{1}{2} \cos 3.4641\, t$$

$$x_2(t) = \tfrac{1}{2} \cos 2t - \tfrac{1}{2} \cos 3.4641\, t$$

(5.26) Results of Example 5.1:

$$\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} X_1^{(1)} , \quad \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} X_1^{(2)} , \quad \omega_1^2 = \frac{k}{m}, \quad \omega_2^2 = \frac{3k}{m}$$

$$[m] = m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \quad [k] = k \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\vec{X}^{(1)T} \vec{X}^{(2)} = \{1 \quad 1\} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} X_1^{(1)} X_1^{(2)} = 0$$

$$\vec{X}^{(1)T} [m] \vec{X}^{(2)} = X_1^{(1)} X_1^{(2)} m \{1 \quad 1\} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

$$= m X_1^{(1)} X_1^{(2)} \{1 \quad 1\} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = 0$$

$$\vec{X}^{(1)T} [m] \vec{X}^{(1)} = m \left(X_1^{(1)}\right)^2 \{1 \quad 1\} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$= 2m \left(X_1^{(1)}\right)^2 = c_1 = \text{constant}$$

$$\vec{X}_1^{(1)T} [k] \vec{X}^{(1)} = k \left(X_1^{(1)}\right)^2 \{1 \quad 1\} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$= k \left(X_1^{(1)}\right)^2 \{1 \quad 1\} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = 2k \left(X_1^{(1)}\right)^2 = 2k \left(\frac{c_1}{2m}\right)$$

$$= c_1 \frac{k}{m} = c_1 \omega_1^2$$

$$\vec{X}^{(2)T} [m] \vec{X}^{(2)} = m \left(X_1^{(2)}\right)^2 \{1 \quad -1\} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

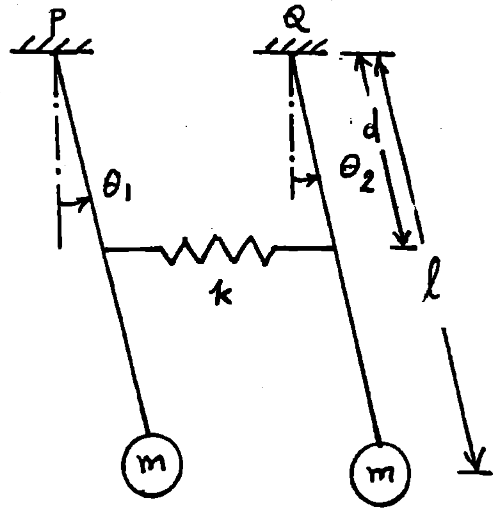
$$= 2m \left(X_1^{(2)}\right)^2 = c_2 = \text{constant}$$

$$\vec{X}^{(2)T} [k] \vec{X}^{(2)} = k \left(X_1^{(2)}\right)^2 \{1 \quad -1\} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

$$= k \left(X_1^{(2)}\right)^2 \{1 \quad -1\} \begin{Bmatrix} 3 \\ -3 \end{Bmatrix} = 6k \left(X_1^{(2)}\right)^2$$

$$= 6k \left(\frac{c_2}{2m}\right) = c_2 \left(\frac{3k}{m}\right) = c_2 \omega_2^2$$

5.27 (a) Equations of motion:



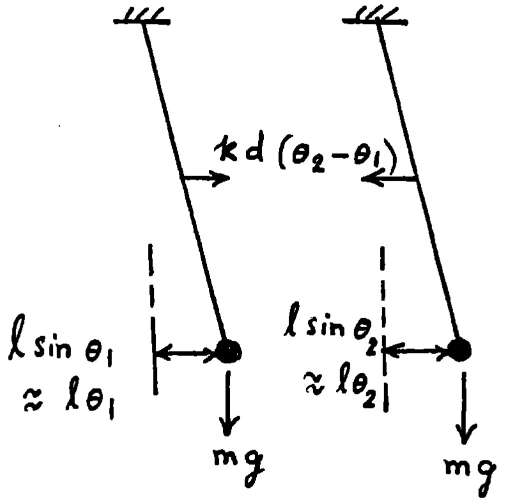
Assume: θ_1, θ_2 are small.

Moment equilibrium equations of the two masses about P and Q:

$$ml^2\ddot{\theta}_1 + mgl\theta_1 + kd^2(\theta_1-\theta_2)=0 \quad (1)$$

$$ml^2\ddot{\theta}_2 + mgl\theta_2 - kd^2(\theta_1-\theta_2)=0 \quad (2)$$

(b) Natural frequencies and mode shapes:



Free body diagram

Assume: Harmonic motion with

$$\theta_i(t) = \Theta_i \cos(\omega t-\phi)\,;\ i=1,2 \quad (3)$$

where Θ_1 and Θ_2 are amplitudes of θ_1 and θ_2, respectively, ω is the natural frequency, and ϕ is the phase angle.

Using Eq. (3), Eqs. (1) and (2) can be expressed in matrix form as

$$-\omega^2 ml^2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}\Theta_1\\ \Theta_2\end{Bmatrix} + \begin{bmatrix}mgl+kd^2 & -kd^2\\ -kd^2 & mgl+kd^2\end{bmatrix}\begin{Bmatrix}\Theta_1\\ \Theta_2\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix} \quad (4)$$

Frequency equation:

$$\begin{vmatrix}-\omega^2 ml^2+mgl+kd^2 & -kd^2\\ -kd^2 & -\omega^2 ml^2+mgl+kd^2\end{vmatrix}=0$$

or

$$\omega^4-\omega^2\left(\frac{2g}{l}+\frac{2kd^2}{ml^2}\right)+\left(\frac{g^2}{l^2}+\frac{2gkd^2}{ml^3}\right)=0 \quad (5)$$

Solution of Eq. (5) gives

$$\omega_1^2=\frac{g}{l}\,,\qquad \omega_2^2=\frac{g}{l}+\frac{2kd^2}{ml^2} \quad (6)$$

By substituting for ω_1^2 and ω_2^2 into Eq. (4), we obtain

$$\left(\frac{\Theta_2}{\Theta_1}\right)^{(1)} = 1 \qquad \text{or} \qquad \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \Theta_1^{(1)}$$

and

$$\left(\frac{\Theta_2}{\Theta_1}\right)^{(2)} = -1 \qquad \text{or} \qquad \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \Theta_1^{(2)}$$

Thus the motion of the masses in the two modes is given by

$$\vec{\theta}^{(1)}(t) = \begin{Bmatrix} \theta_1^{(1)}(t) \\ \theta_2^{(1)}(t) \end{Bmatrix} = \Theta_1^{(1)} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \cos(\omega_1 t + \phi_1) \tag{7}$$

$$\vec{\theta}^{(2)}(t) = \begin{Bmatrix} \theta_1^{(2)}(t) \\ \theta_2^{(2)}(t) \end{Bmatrix} = \Theta_1^{(2)} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \cos(\omega_2 t + \phi_2) \tag{8}$$

(c) Free vibration response:

Using linear superposition of natural modes, the free vibration response of the system is given by

$$\vec{\theta}(t) = c_1\, \vec{\theta}^{(1)}(t) + c_2\, \vec{\theta}^{(2)}(t) \tag{9}$$

By choosing $c_1 = c_2 = 1$, with no loss of generality, Eqs. (7) to (9) lead to

$$\theta_1(t) = \Theta_1^{(1)} \cos(\omega_1 t + \phi_1) + \Theta_1^{(2)} \cos(\omega_2 t + \phi_2) \tag{10}$$

$$\theta_2(t) = \Theta_1^{(1)} \cos(\omega_1 t + \phi_1) - \Theta_1^{(2)} \cos(\omega_2 t + \phi_2) \tag{11}$$

where $\Theta_1^{(1)}$, ϕ_1, $\Theta_1^{(2)}$ and ϕ_2 are constants to be determined from the initial conditions. When $\theta_1(0) = a$, $\theta_2(0) = 0$, $\dot{\theta}_1(0) = 0$ and $\dot{\theta}_2(0) = 0$, Eqs. (10) and (11) yield

$$\left.\begin{aligned} a &= \Theta_1^{(1)} \cos\phi_1 + \Theta_1^{(2)} \cos\phi_2 \\ 0 &= \Theta_1^{(1)} \cos\phi_1 - \Theta_1^{(2)} \cos\phi_2 \\ 0 &= -\omega_1 \Theta_1^{(1)} \sin\phi_1 - \omega_2 \Theta_1^{(2)} \sin\phi_2 \\ 0 &= -\omega_1 \Theta_1^{(1)} \sin\phi_1 + \omega_2 \Theta_1^{(2)} \sin\phi_2 \end{aligned}\right\} \tag{12}$$

Eqs. (12) can be solved for $\Theta_1^{(1)}$, ϕ_1, $\Theta_1^{(2)}$ and ϕ_2 to obtain

$$\left.\begin{aligned} \theta_1(t) &= a \cos\frac{\omega_2 - \omega_1}{2} t \cdot \cos\frac{\omega_2 + \omega_1}{2} t \\ \theta_2(t) &= a \sin\frac{\omega_2 - \omega_1}{2} t \cdot \sin\frac{\omega_2 + \omega_1}{2} t \end{aligned}\right\} \tag{13}$$

(d) Conditions for beating:

When $\frac{2kd^2}{ml^2} << \frac{g}{l}$ or $k << \frac{mgl}{2d^2}$, (14)

the two frequency components in Eqs. (13), namely, $\frac{\omega_2-\omega_1}{2}$ and $\frac{\omega_2+\omega_1}{2}$, can be approximated as

$$\Omega_1 = \frac{\omega_2-\omega_1}{2} \simeq \frac{k}{2m}\frac{d^2}{\sqrt{g l^3}} \tag{15}$$

and

$$\Omega_2 = \frac{\omega_2+\omega_1}{2} \simeq \sqrt{\frac{g}{l}} + \frac{k}{2m}\frac{d^2}{\sqrt{g l^3}} \tag{16}$$

This implies that the motions of the pendulums are given by

$$\left.\begin{aligned} \theta_1(t) &\simeq a \cos\Omega_1 t \cdot \cos\Omega_2 t \\ \theta_2(t) &\simeq a \sin\Omega_1 t \cdot \sin\Omega_2 t \end{aligned}\right\} \tag{17}$$

This motion, Eqs. (17), denotes beating phenomenon.

5.28 With $k_{t1} = k_t$, $k_{t2} = 2k_t$, $J_1 = J_0$, $J_2 = 2J_0$, $k_{t3} = 0$ and $M_{t1} = M_{t2} = 0$, Eqs. (5.20) give

$$J_0\ddot{\theta}_1 + 3k_t\theta_1 - 2k_t\theta_2 = 0$$

$$2J_0\ddot{\theta}_2 - 2k_t\theta_1 + 2k_t\theta_2 = 0$$

For harmonic solution, $\theta_i(t) = \Theta_i \cos(\omega t + \phi)$; $i = 1, 2$,

$$\begin{bmatrix} (-\omega^2 J_0 + 3k_t) & -2k_t \\ -2k_t & (-2\omega^2 J_0 + 2k_t) \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

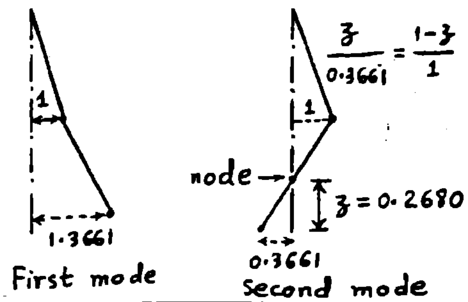
Frequency equation is

$$\begin{vmatrix} -\omega^2 J_0 + 3k_t & -2k_t \\ -2k_t & -2\omega^2 J_0 + 2k_t \end{vmatrix} = 2J_0^2\omega^4 - 8J_0k_t\omega^2 + 2k_t^2 = 0$$

$$\omega^2 = (2 \mp \sqrt{3})\frac{k_t}{J_0} \quad ; \quad \omega_1 = 0.5176\sqrt{\frac{k_t}{J_0}}, \quad \omega_2 = 1.9319\sqrt{\frac{k_t}{J_0}}$$

$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = \frac{-J_0\,\omega_1^2 + 3\,k_t}{2\,k_t} = 1.3661$$

$$r_2 = \frac{\Theta_2^{(2)}}{\Theta_1^{(2)}} = \frac{-J_0\,\omega_2^2 + 3\,k_t}{2\,k_t} = -0.3661$$



5.29 Equation of motion of mass m: $\quad m\ddot{x} = -k_2(x - r\theta) \quad ---(E_1)$

Equation of motion of cylinder of mass m_0 and mass moment of inertia $J_0 = \frac{1}{2} m_0 r^2$: $\quad J_0\ddot{\theta} = -k_1 r^2\theta - k_2(r\theta - x)r \quad ---(E_2)$

For $x(t) = X\cos(\omega t + \phi)$ and $\theta(t) = \Theta\cos(\omega t + \phi)$, Eqs. (E_1) and (E_2) give the frequency equation

$$\begin{vmatrix} -m\omega^2 + k_2 & -k_2 r \\ -k_2 r & -\frac{1}{2} m_0 r^2\omega^2 + k_1 r^2 + k_2 r^2 \end{vmatrix} = 0$$

i.e. $$\omega^4 - \omega^2\left(\frac{k_2}{m} + \frac{2\{k_1 + k_2\}}{m_0}\right) + \frac{2\,k_1 k_2}{m_0\,m} = 0$$

$$\omega_1^2,\ \omega_2^2 = \frac{k_2}{2m} + \frac{(k_1 + k_2)}{m_0} \mp \sqrt{\frac{1}{4}\left(\frac{k_2}{m} + \frac{2\,k_1}{m_0} + \frac{2\,k_2}{m_0}\right)^2 - \frac{2\,k_1 k_2}{m\,m_0}}$$

5.30 For $J_1 = J_0$, $J_2 = 2\,J_0$, $k_{t_1} = k_{t_2} = k_{t_3} = k_t$, and $M_{t1} = M_{t2} = 0$, Eqs. (5.20) give

$$J_0\,\ddot{\theta}_1 + 2\,k_t\,\theta_1 - k_t\,\theta_2 = 0$$

$$2\,J_0\,\ddot{\theta}_2 - k_t\,\theta_1 + 2k_t\,\theta_2 = 0$$

For harmonic motion, these equations give

$$\begin{bmatrix} -\omega^2 J_0 + 2\,k_t & -k_t \\ -k_t & -2\omega^2 J_0 + 2\,k_t \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

from which the frequency equation can be obtained as

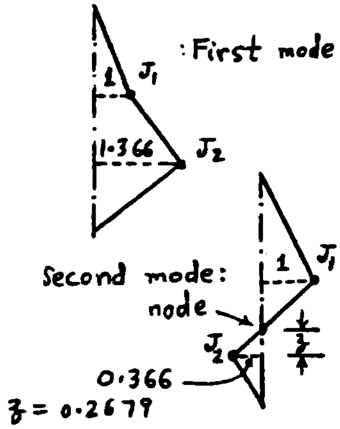
$$2\,J_0^2\,\omega^4 - 6\,J_0\,k_t\,\omega^2 + 3\,k_t^2 = 0$$

$$\omega_1^2,\ \omega_2^2 = \frac{1}{2}(3 \mp \sqrt{3})\,\frac{k_t}{J_0}$$

$$\therefore\ \omega_1 = 0.79623\sqrt{\frac{k_t}{J_0}}\ ; \qquad \omega_2 = 1.53819\sqrt{\frac{k_t}{J_0}}$$

$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = \frac{-\omega_1^2 J_0 + 2k_t}{k_t} = 1.36603$$

$$r_2 = \frac{\Theta_2^{(2)}}{\Theta_1^{(2)}} = \frac{-\omega_2^2 J_0 + 2\,k_t}{k_t} = -0.36603$$



5.31 Eqs. (5.20) give

$$J_0 \ddot{\theta}_1 + 6 k_t \theta_1 - 5 k_t \theta_2 = 0$$

$$5 J_0 \ddot{\theta}_2 - 5 k_t \theta_1 + 5 k_t \theta_2 = 0$$

These equations can be expressed as, for harmonic motion,

$$\begin{bmatrix} -\omega^2 J_0 + 6 k_t & -5 k_t \\ -5 k_t & -5\omega^2 J_0 + 5 k_t \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Frequency equation is

$$J_0^2 \omega^4 - 7 k_t J_0 \omega^2 + k_t^2 = 0$$

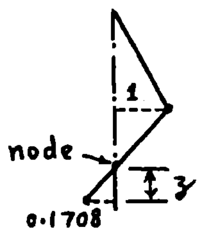
$$\omega_1^2,\ \omega_2^2 = \frac{k_t}{J_0}\left(\frac{7}{2} \mp \frac{1}{2}\sqrt{45}\right) = 0.1459 \frac{k_t}{J_0},\ 6.8541 \frac{k_t}{J_0}$$

$$\omega_1 = 0.38197 \sqrt{\frac{k_t}{J_0}}, \quad \omega_2 = 2.61803 \sqrt{\frac{k_t}{J_0}}$$

$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = \frac{-\omega_1^2 J_0 + 6 k_t}{5 k_t} = 1.1708$$

$$r_2 = \frac{\Theta_2^{(2)}}{\Theta_1^{(2)}} = \frac{-\omega_2^2 J_0 + 6 k_t}{5 k_t} = -0.1708$$

First mode

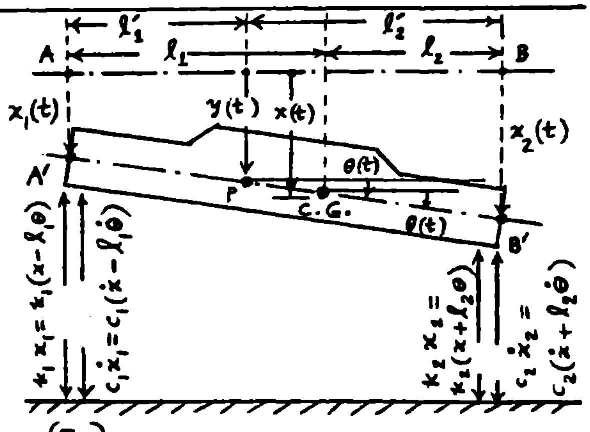


Second mode

$$\frac{z}{0.1708} = \frac{1-z}{1}$$

$$z = 0.1459$$

5.32 (i) Using $x(t)$ and $\theta(t)$:



For translatory motion:

$$m\ddot{x} = -k_1(x - l_1\theta) - c_1(\dot{x} - l_1\dot{\theta}) - k_2(x + l_2\theta) - c_2(\dot{x} + l_2\dot{\theta}) \quad \text{---} (E_1)$$

For rotational motion about C.G.:

$$J_0 \ddot{\theta} = k_1(x - l_1\theta) l_1 + c_1(\dot{x} - l_1\dot{\theta}) l_1 - k_2(x + l_2\theta) l_2 - c_2(\dot{x} + l_2\dot{\theta}) l_2 \quad \text{---} (E_2)$$

Eqs. (E_1) and (E_2) can be rewritten as

$$\begin{bmatrix} m & 0 \\ 0 & J_0 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_1 l_1 + c_2 l_2 \\ -c_1 l_1 + c_2 l_2 & c_1 l_1^2 + c_2 l_2^2 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_1 l_1 + k_2 l_2 \\ -k_1 l_1 + k_2 l_2 & k_1 l_1^2 + k_2 l_2^2 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

(ii) Using $y(t)$ and $\theta(t)$:

For translatory motion:

$$m\ddot{y} = -k_1(y - l_1'\theta) - c_1(\dot{y} - l_1'\dot{\theta}) - k_2(y + l_2'\theta) - c_2(\dot{y} + l_2'\dot{\theta}) - me\ddot{\theta} \qquad \text{---- } (E_3)$$

For rotational motion:

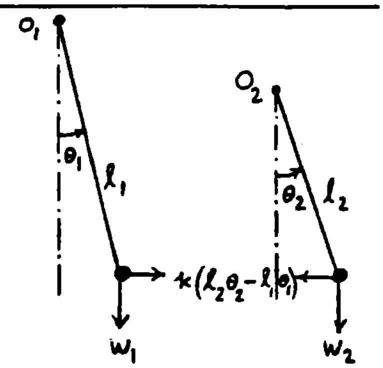
$$J_p\ddot{\theta} = k_1(y - l_1'\theta)l_1' + c_1(\dot{y} - l_1'\dot{\theta})l_1' - k_2(y + l_2'\theta)l_2' - c_2(\dot{y} + l_2'\dot{\theta})l_2' - me\ddot{y} \qquad \text{---- } (E_4)$$

Eqs. (E_3) and (E_4) can be rewritten as

$$\begin{bmatrix} m & me \\ me & J_p \end{bmatrix} \begin{Bmatrix} \ddot{y} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_1 l_1' + c_2 l_2' \\ -c_1 l_1' + c_2 l_2' & c_1 l_1'^2 + c_2 l_2'^2 \end{bmatrix} \begin{Bmatrix} \dot{y} \\ \dot{\theta} \end{Bmatrix}$$

$$+ \begin{bmatrix} k_1 + k_2 & -k_1 l_1' + k_2 l_2' \\ -k_1 l_1' + k_2 l_2' & k_1 l_1'^2 + k_2 l_2'^2 \end{bmatrix} \begin{Bmatrix} y \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

5.33 For small angular deflections, equations of motion are



$$m_1 l_1^2 \ddot{\theta}_1 = -W_1 l_1 \sin\theta_1 + k(l_2\theta_2 - l_1\theta_1) l_1 \cos\theta_1$$

$$m_2 l_2^2 \ddot{\theta}_2 = -W_2 l_2 \sin\theta_2 - k(l_2\theta_2 - l_1\theta_1) l_2 \cos\theta_2$$

or

$$m_1 l_1^2 \ddot{\theta}_1 + \theta_1(W_1 l_1 + k l_1^2) - k l_1 l_2 \theta_2 = 0$$

$$m_2 l_2^2 \ddot{\theta}_2 + \theta_2(W_2 l_2 + k l_2^2) - k l_1 l_2 \theta_1 = 0$$

For harmonic motion, $\theta_i(t) = \Theta_i \cos(\omega t + \phi)$; $i = 1,2$, we get

$$\begin{bmatrix} -\omega^2 m_1 l_1^2 + W_1 l_1 + k l_1^2 & -k l_1 l_2 \\ -k l_1 l_2 & -\omega^2 m_2 l_2^2 + W_2 l_2 + k l_2^2 \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Frequency equation is

$$\omega^4(m_1 m_2 l_1^2 l_2^2) - \omega^2\left[m_2 l_2^2 (W_1 l_1 + k l_1^2) + m_1 l_1^2 (W_2 l_2 + k l_2^2)\right]$$

$$+ \left[W_1 l_1 W_2 l_2 + W_2 l_2 \, k l_1^2 + W_1 l_1 \, k l_2^2\right] = 0 \qquad \text{----} (E_1)$$

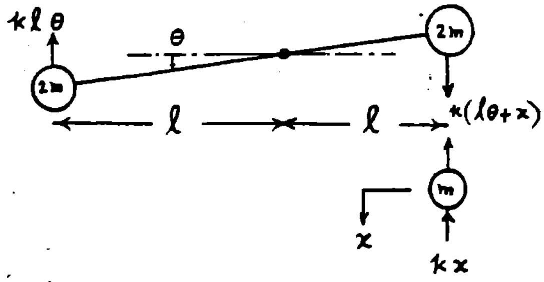
Roots of this equation give the natural frequencies ω_1 and ω_2.

Amplitude ratios are given by

$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = \frac{-\omega_1^2 m_1 l_1^2 + W_1 l_1 + k l_1^2}{k l_1 l_2}$$

$$r_2 = \frac{\Theta_2^{(2)}}{\Theta_1^{(2)}} = \frac{-\omega_2^2 m_1 l_1^2 + W_1 l_1 + k l_1^2}{k l_1 l_2}$$ where $W_1 = m_1 g$, $W_2 = m_2 g$.

5.34 Equations of motion:



$$4m\ell^2\ddot{\theta} = -k\ell\theta\cdot\ell - k(\ell\theta+x)\ell$$

$$m\ddot{x} = -kx - k(\ell\theta+x)$$

i.e.
$$4m\ell^2\ddot{\theta} + 2k\ell^2\theta + k\ell x = 0$$
$$m\ddot{x} + 2kx + k\ell\theta = 0$$

For harmonic motion, these equations give

$$\begin{bmatrix} -4m\ell^2\omega^2 + 2k\ell^2 & k\ell \\ k\ell & -m\omega^2+2k \end{bmatrix} \begin{Bmatrix} \Theta \\ X \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Frequency equation is $4m^2\omega^4 - 10km\,\omega^2 + 3k^2 = 0$

$$\omega^2 = \frac{k}{m}\left(\frac{5}{4} \mp \frac{\sqrt{13}}{4}\right) = 0.3486\,\frac{k}{m},\quad 2.1514\,\frac{k}{m}$$

$$\omega_1 = 0.5904\sqrt{\frac{k}{m}},\qquad \omega_2 = 1.4668\sqrt{\frac{k}{m}}$$

Amplitude ratios are

$$r_1 = \frac{X^{(1)}}{\Theta^{(1)}} = \frac{-4m\ell^2\omega_1^2 + 2k\ell^2}{-k\ell} = -0.6056\,\ell$$

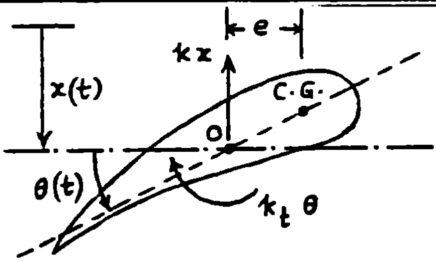
$$r_2 = \frac{X^{(2)}}{\Theta^{(2)}} = \frac{-4m\ell^2\omega_2^2 + 2k\ell^2}{-k\ell} = 6.6056\,\ell$$

Mode shapes are

$$\vec{X}^{(1)} = \begin{Bmatrix} \Theta^{(1)} \\ X^{(1)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -0.6056\ell \end{Bmatrix}\Theta^{(1)}$$

$$\vec{X}^{(2)} = \begin{Bmatrix} \Theta^{(2)} \\ X^{(2)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 6.6056\,\ell \end{Bmatrix}\Theta^{(2)}$$

5.35 Equations of motion:



$$m(\ddot{x} - e\ddot{\theta}) = -kx$$

$$J_{CG}\,\ddot{\theta} = -k_t\theta - kxe$$

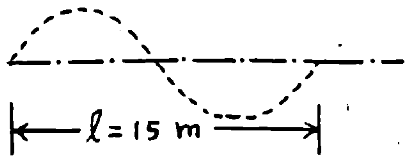
i.e.
$$m\ddot{x} + kx - me\ddot{\theta} = 0$$
$$(J_0 - me^2)\ddot{\theta} + k_t\theta + kex = 0$$

For harmonic motion, we get the frequency equation as

$$\begin{vmatrix} -m\omega^2 + k & me\omega^2 \\ ke & -(J_0 - me^2)\omega^2 + k_t \end{vmatrix} = 0$$

or
$$(J_0 - me^2)m\,\omega^4 - (J_0 k + m k_t)\omega^2 + k k_t = 0$$

Roots of this equation give the natural frequencies of the system.



5.36 Speed becomes unfavorable when it is related to l as

$v \tau_n = l$

i.e., $v = \dfrac{l}{\tau_n} = l f_n = \dfrac{l \omega_n}{2\pi}$

Example 5.7 gives

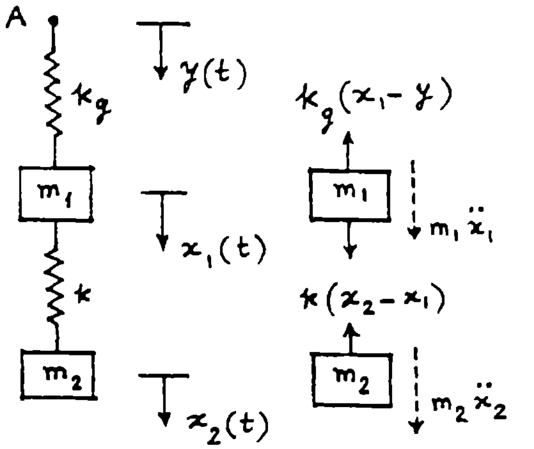
$\omega_1 = 5.8593$ rad/sec (bouncing)

$\omega_2 = 9.4341$ rad/sec (pitching)

$\therefore \; v_1 = \dfrac{l \omega_1}{2\pi} = \dfrac{15\,(5.8593)}{2\pi} = 13.9880$ m/s (bouncing)

$v_2 = \dfrac{l \omega_2}{2\pi} = \dfrac{15\,(9.4341)}{2\pi} = 22.5222$ m/s (pitching)

5.37



Free body diagrams of masses

Equations of motion:

$m_1 \ddot{x}_1 + (k_g + k)\, x_1 - k\, x_2 = k_g\, y$

$m_2 \ddot{x}_2 - k\, x_1 + k\, x_2 = 0$

Since velocity of crane in z-direction = 9m/min = 0.15 m/s, τ = time to complete one cycle = $\dfrac{3}{0.15}$ = 20 sec, $\omega = \dfrac{2\pi}{\tau} = \dfrac{2\pi}{20} = 0.31416$ rad/s

Base motion for m_1 (girder motion due to unevenness of rails):

$$y(t) = Y \sin \omega t$$

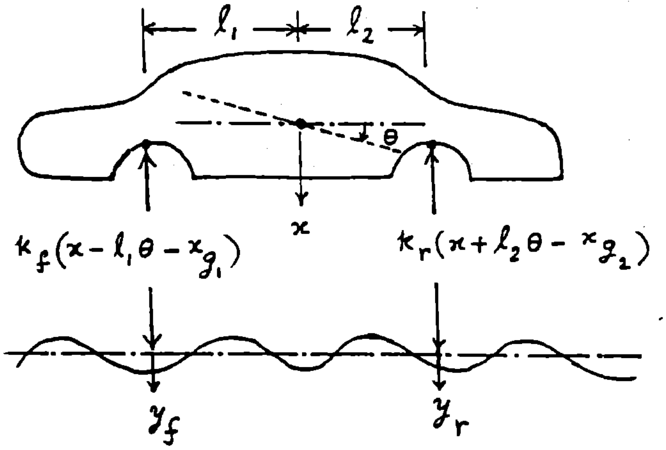
where Y = 0.05 m and ω = 0.31416 rad/sec.

5.38 **Road surface varies sinusoidally with amplitude, $Y = 0.05$ m and wavelength, $d = 10$ m. If v = velocity of automobile (m/sec), time to travel one wave length $= \tau = d/v$ sec. $\tau = 10/v$ sec, $\omega = \dfrac{2\pi}{\tau} = \dfrac{2\pi v}{10}$ rad/sec.**

$v = 50$ km/hr $= (50\,(10^3))/(60\,(60)) = 13.8889$ m/sec,

$J_0 = m\, r_g^2 = 1000\,(0.9)^2 = 810$ kg–m^2.

Equations of motion:



y_f (y_r) = ground or base displacement of front (rear) wheels, downwards

For motion along x:

$$m\,\ddot{x} + x\,(k_f + k_r) + \theta\,(k_r\,\ell_2 - k_f\,\ell_1) = k_f\,y_f + k_r\,y_r \tag{1}$$

For motion along θ:

$$J_0\,\ddot{\theta} + x\,(\ell_2\,k_r - \ell_1\,k_f) + \theta\,(k_r\,\ell_2^2 + k_f\,\ell_1^2) = k_r\,\ell_2\,y_r - k_f\,\ell_1\,y_f \tag{2}$$

where the ground (base) motions can be expressed as

$$y_f(t) = Y \sin \omega\, t = 0.05 \sin \frac{2\,\pi\,v}{10}\, t \ \ m \tag{3}$$

$$y_r(t) = Y \sin(\omega\, t - \phi) = 0.05 \sin \left(\frac{2\,\pi\,v}{10}\, t - \frac{2\,\pi\,(\ell_1 + \ell_2)}{d} \right) \ m \tag{4}$$

For given data, Eqs. (1) and (2) take the form:

$$1000\,\ddot{x} + 40\,(10^3)\,x + 15000\,\theta = 900 \sin 8.7267\,t + 1100 \sin(8.7267\,t - 1.5708) \tag{5}$$

$$810\,\ddot{\theta} + 15000\,x + 67500\,\theta = 1650 \sin(8.7267\,t - 1.5708) - 900 \sin 8.7267\,t \tag{6}$$

5.39 Natural frequencies are given by:

$$\left[-\omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \right] \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{1}$$

where m_1 = mass of pulley = 90 kg; m_2 = mass of motor = 225 kg

X_1 = amplitude of pulley, X_2 = amplitude of motor

$$\frac{E\,I}{\ell^3} = \frac{(200 \times 10^9)\left(\dfrac{\pi}{64} \times 0.05^4\right)}{(2.25)^3} = 5386.82 \text{ N/m}$$

Frequency equation becomes:

$$\begin{vmatrix} (-\omega^2 m_1 + k_{11}) & k_{12} \\ k_{12} & (-\omega^2 m_2 + k_{22}) \end{vmatrix} = 0$$

or $$(m_1 m_2)\,\omega^4 - (k_{11} m_2 + k_{22} m_1)\,\omega^2 + (k_{11} k_{22} - k_{12}^2) = 0 \tag{2}$$

From known data, Eq. (2) can be expressed as:

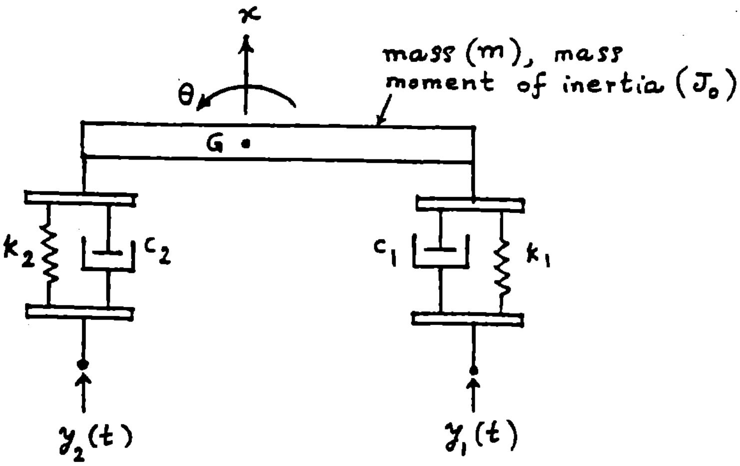
$$20250\,\omega^4 - 335.1033 \times 10^6\,\omega^2 + 2.030784 \times 10^{11} = 0 \tag{3}$$

Roots of Eq. (3):

$$\omega^2 = 15918.309,\ 630.00177$$

or $$\omega_1 = 25.01 \text{ rad/sec},\ \omega_2 = 126.1678 \text{ rad/sec}$$

5.40



1. **Model the bicycle and the rider as a two d.o.f system as shown in figure.**
2. **Find the equivalent stiffness (k_1) and damping coefficient (c_1) of the front wheel in the vertical direction.**
3. **Find the equivalent stiffness (k_2) and damping coefficient (c_2), if applicable, of the rear wheel in the vertical direction.**
4. **Describe the road roughness under the wheels as $y_1(t)$ and $y_2(t)$.**
5. **Derive the equations of motion of the system subjected to base excitation.**
6. **Solve the resulting system of equations to find the steady state response.**

5.41 (a)

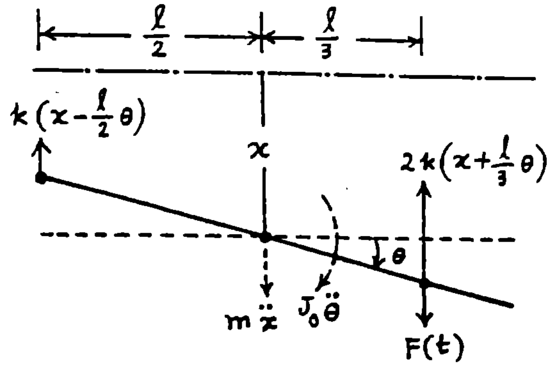
Choose unknown coordinates as $x(t)$ and $\theta(t)$. Equations of motion:

$$m\,\ddot{x} = -k\,(x - \ell\,\theta/2) - 2\,k\,(x + \ell\,\theta/3) + F(t)$$
$$J_0\,\ddot{\theta} = k\,(x - \ell\,\theta/2)\,(\ell/2) - 2\,k\,(x + \ell\,\theta/3)\,(\ell/3) + F(t)\,(\ell/3)$$

or

$$\begin{bmatrix} m & 0 \\ 0 & J_0 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} 3\,k & k\,\ell/6 \\ k\,\ell/6 & 17\,k\,\ell^2/36 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} F(t) \\ \ell\,F(t)/3 \end{Bmatrix}$$

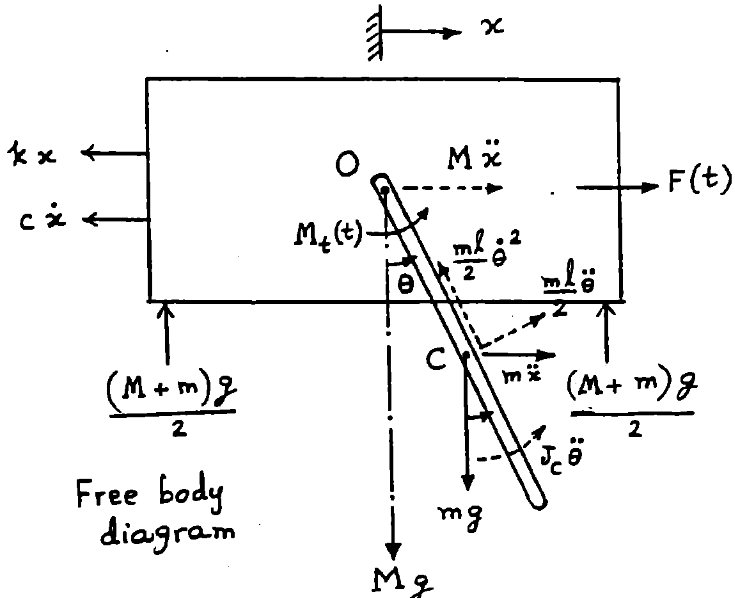
where $J_0 = \dfrac{m\,\ell^2}{12}$ and $F(t) = F_0 \sin \omega\, t$.



(b)

Elastic or static coupling.

(5.42)



Equations of motion with coordinates x(t) and θ(t):

For motion along x:

$$M\,\ddot{x} = -\,k\,x - c\,\dot{x} - (m\,\ell/2)\,\ddot{\theta}\cos\theta - m\,\ddot{x} + (m\,\ell/2)\,\dot{\theta}^2 \sin\theta + F(t) \qquad (1)$$

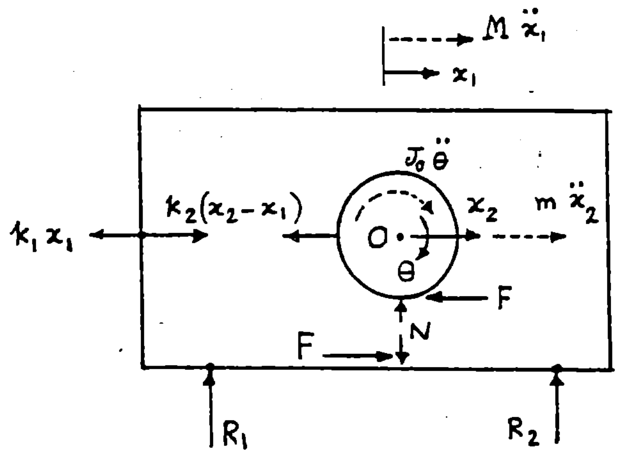
For rotation about O:

$$J_c\,\ddot{\theta} + (m\,\ell/2)\,\ddot{\theta}\,(\ell/2) + m\,\ddot{x}\,(\ell\cos\theta/2) = -\,m\,g\,(\ell/2)\sin\theta + M_t(t) \qquad (2)$$

Using $J_c = \dfrac{1}{12}\,m\,\ell^2$, $\cos\theta \approx 1$ and $\sin\theta \approx \theta$ and neglecting the nonlinear term involving $\dot{\theta}^2$ in Eq. (1), Eqs. (1) and (2) can be rewritten in matrix form as:

$$\begin{bmatrix} (M+m) & m\,\ell/2 \\ m\,\ell/2 & (J_c + m\,\ell^2/4) \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} k & 0 \\ 0 & m\,g\,\ell/2 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} F(t) \\ M_t(t) \end{Bmatrix}$$

5.43



Free body diagram

N = normal reaction between cylinder and trailer, F = friction force, R_1, R_2 = reactions between trailer and ground.

Equation of motion for linear motion of cylinder:

$$\sum F = m\ddot{x}_2 \quad \text{or} \quad m\ddot{x}_2 = -F - k_2(x_2 - x_1) \tag{1}$$

Equation of motion for rotational motion of cylinder:

$$\sum M_0 = J_0\ddot{\theta} \quad \text{or} \quad J_0\ddot{\theta} = F\,r \tag{2}$$

where $J_0 = \frac{1}{2} m r^2$ and $\theta = \frac{x_2 - x_1}{r}$.

Equation of motion for linear motion of trailer:

$$\sum F = M\ddot{x}_1 \quad \text{or} \quad M\ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1) + F \tag{3}$$

Eq. (2) gives

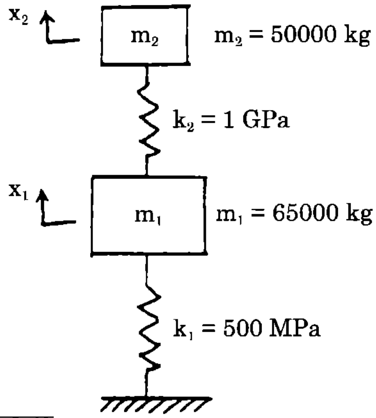
$$F = \frac{J_0\ddot{\theta}}{r} = \frac{1}{r}\left(\frac{1}{2} m r^2\right)\left(\frac{\ddot{x}_2 - \ddot{x}_1}{r}\right) = \frac{m}{2}(\ddot{x}_2 - \ddot{x}_1) \tag{4}$$

Substitution of Eq. (4) into Eqs. (1) and (3) yields the equations of motion as:

$$\frac{3m}{2}\ddot{x}_2 - \frac{1}{2} m\ddot{x}_1 - k_2 x_1 + k_2 x_2 = 0 \tag{5}$$

$$\left(M + \frac{m}{2}\right)\ddot{x}_1 - \frac{m}{2}\ddot{x}_2 + x_1(k_1 + k_2) - k_2 x_2 = 0 \tag{6}$$

5.44 mass of tup: 2000 kg,
mass of frame: 20000 kg.
mass of anvil: 30000 kg,
mass of foundation block: 65000 kg,
stiffness of elastic pad: 1 GPa,
stiffness of isolation: 500 MPa,
striking velocity of tup: 4.5 m/s.



Solution:

(a) Natural frequencies of the system:
From problem 5.1,

$$\omega_{1,2}^2 = \frac{k_1 + k_2}{2\,m_1} + \frac{k_2}{2\,m_2} \mp \sqrt{\frac{1}{4}\left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right)^2 - \frac{k_1\,k_2}{m_1\,m_2}}$$

$$\omega_{1,2}^2 = \frac{1.5 \times 10^9}{130000} + \frac{10^9}{100000} \mp \sqrt{\frac{1}{4}\left(\frac{1.5 \times 10^9}{65000} + \frac{10^9}{50000}\right)^2 - \frac{500 \times 10^{15}}{3.25 \times 10^9}}$$

or $\omega_1 = 62.6894$ rad/sec.
$\omega_2 = 197.8559$ rad/sec.

(b) Initial conditions of the system:

Let ϑ_2 = initial velocity of anvil and frame just after the impact of tup.

From conservation of momentum principle, momentum of tup plus momentum of m_2 just before impact = momentum of tup plus momentum of m_2 just after impact.

i.e.
$$m_{tup}\,\vartheta_{tup} + m_2\,(0) = m_{tup}\,\vartheta_0 + m_2\,\vartheta_2 \tag{E$_1$}$$

where ϑ_0 = velocity of rebound of tup after impact
Also,

$$\text{Coefficient of restitution (e)} = \left(\frac{\text{relative velocity after impact}}{\text{relative velocity before impact}}\right)$$

i.e.
$$e = \frac{\vartheta_2 - \vartheta_0}{\vartheta_{tup}} \quad \text{or} \quad \vartheta_0 = \vartheta_2 - e\,\vartheta_{tup} \tag{E$_2$}$$

From Eqs. (E_1) and (E_2),

$$\vartheta_2 = \frac{m_{tup}\,\vartheta_{tup}\,(1 + e)}{m_{tup} + m_2}$$

For given data,

$$\vartheta_2 = \frac{2000\,(-4.5)\,(1 + 0.5)}{2000 + 50000} = -0.2596 \text{ m/s}$$

∴ Initial conditions are:

$x_1(0) = 0, \quad \dot{x}_1(0) = 0, \quad x_2(0) = 0, \quad \dot{x}_2(0) = -0.2596$ m/s.

(c) Displacements of anvil and foundation block:
we can use results of section 5.3 with $k_3 = 0$

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m_1\,\omega_1^2 + k_1 + k_2}{k_2} = \frac{-65000\,(62.6894)^2 + 10^9 + 0.5 \times 10^9}{10^9}$$
$$= 1.2445$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m_1\,\omega_2^2 + k_1 + k_2}{k_2} = \frac{-65000\,(197.8559)^2 + 1.5 \times 10^9}{10^9}$$
$$= -1.0446$$

Response of the system can be computed using Eq. (5.18):

$$X_1^{(1)} = \frac{1}{r_2 - r_1}\left(\frac{\dot{x}_2}{\omega_1}\right) = \frac{1}{-2.2891}\left(\frac{+0.2596}{62.6894}\right) = -1.809 \times 10^{-3}\ (\text{m})$$

$$X_1^{(2)} = \frac{1}{r_2 - r_1}\left(\frac{\dot{x}_2}{\omega_2}\right) = \frac{1}{-2.2891}\left(\frac{0.2596}{197.8559}\right) = -0.5732 \times 10^{-3}\ (\text{m})$$

$$\phi_1 = \tan^{-1}\left\{\frac{\dot{x}_2\,(0)}{0}\right\} = \frac{-\pi}{2}$$

$$\phi_2 = \tan^{-1}\left\{-\frac{\dot{x}_2\,(0)}{0}\right\} = \frac{\pi}{2}$$

Response is given by Eq. (5.15):

$$X_1\,(t) = X_1^{(1)}\cos(\omega_1 t + \phi_1) + X_1^{(2)}\cos(\omega_2 t + \phi_2)$$
$$= -1.809 \times 10^{-3}\cos\left(62.6894\,t - \frac{\pi}{2}\right) - 0.5732 \times 10^{-3}\cos\left(197.8559\,t + \frac{\pi}{2}\right)\ \text{m}$$

$$X_2\,(t) = r_1 X_1^{(1)}\cos(\omega_1 t + \phi_1) + r_2\,X_1^{(2)}\cos(\omega_2 t + \phi_2)$$
$$= -2.2513 \times 10^{-3}\cos\left(62.6894\,t - \frac{\pi}{2}\right) + 0.5988 \times 10^{-3}\cos\left(197.8559\,t + \frac{\pi}{2}\right)\ \text{m}$$

5.45 (a) Natural frequencies:

Equations of motion:
$$m_1\ddot{x}_1 + k_1 x_1 - k_1 x_2 = F_1(t) \qquad (E_1)$$
$$m_2\ddot{x}_2 + (k_1 + k_2)\,x_2 - k_1 x_1 = 0 \qquad (E_2)$$

Frequency equation:
$$\begin{vmatrix} -\omega^2 m_1 + k_1 & -k_1 \\ -k_1 & -m_2\omega^2 + k_1 + k_2 \end{vmatrix} = 0$$

or
$$\omega^4 - \left(\frac{k_1}{m_1} + \frac{k_1 + k_2}{m_2}\right)\omega^2 + \frac{k_1 k_2}{m_1 m_2} = 0$$

$$\therefore \omega_{1,2}^2 = \frac{k}{2m_1} + \frac{k_1 + k_2}{2m_2} \mp \sqrt{\frac{1}{4}\left(\frac{k_1}{m_1} + \frac{k_1 + k_2}{m_2}\right)^2 - \frac{k_1 k_2}{m_1 m_2}}$$

Here $m_1 = 2 \times 10^5$ kg, $m_2 = 2.5 \times 10^5$ kg, $k_1 = 150 \times 10^6$ N/m and $k_2 = 75 \times 10^6$ N/m.

$$\omega_{1,2}^2 = \frac{150 \times 10^6}{4 \times 10^5} + \frac{225 \times 10^6}{5 \times 10^5} \mp \sqrt{\frac{1}{4}\left(\frac{150 \times 10^6}{2 \times 10^5} + \frac{225 \times 10^6}{2.5 \times 10^5}\right)^2 - \frac{150 \times 75 \times 10^{12}}{5 \times 10^{10}}}$$

$$= 150,\ 1500 \quad (\text{rad/sec})^2$$

$$\therefore \omega_1 = 12.2474 \text{ rad/sec}, \quad \omega_2 = 38.7298 \text{ rad/sec}$$

(b) Response:

Assuming zero initial conditions, the Laplace transforms of (E_1) and (E_2) can be written as

$$m_1 s^2 \bar{x}_1(s) + k_1 \bar{x}_1(s) - k_1 \bar{x}_2(s) = \bar{F}_1(s)$$
$$m_2 s^2 \bar{x}_2(s) + (k_1 + k_2)\bar{x}_2(s) - k_1 \bar{x}_1(s) = 0$$

i.e.
$$(m_1 s^2 + k_1)\bar{x}_1(s) - k_1 \bar{x}_2(s) = \bar{F}_1(s)$$
$$-k_1 \bar{x}_1(s) + (m_2 s^2 + k_1 + k_2)\bar{x}_2(s) = 0$$

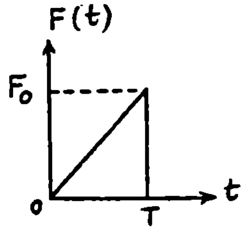
Solution of these equations gives

$$\bar{x}_1(s) = \left\{\frac{(m_2 s^2 + k_1 + k_2)}{m_1 m_2 s^4 + s^2(m_1 k_1 + m_1 k_2 + m_2 k_1) + k_1 k_2}\right\}\bar{F}_1(s) \quad \text{----}(E_3)$$

$$\bar{x}_2(s) = \left\{\frac{k_1}{m_1 m_2 s^4 + s^2(m_1 k_1 + m_1 k_2 + m_2 k_1) + k_1 k_2}\right\}\bar{F}_1(s) \quad \text{----}(E_4)$$

For the forcing function given,

$$\bar{F}_1(s) = \bar{F}(s) = \frac{F_0}{T}\left\{\frac{1}{s^2} - e^{-Ts}\left(\frac{T}{s} - \frac{1}{s^2}\right)\right\} \quad \text{----}(E_5)$$



For the given data, Eqs. (E_3) to (E_5) become

$$\bar{x}_1(s) = \frac{2.5 \times 10^5 s^2 + 225 \times 10^6}{(5 \times 10^{10} s^4 + 825 \times 10^{11} s^2 + 11250 \times 10^{12})}\bar{F}_1(s)$$

$$= \frac{s^2 + 900}{2 \times 10^5 s^4 + 330 \times 10^6 s^2 + 45 \times 10^9}\bar{F}_1(s) \quad \text{----}(E_6)$$

$$\bar{x}_2(s) = \frac{150 \times 10^6}{5\times 10^{10}\, s^4 + 825 \times 10^{11}\, s^2 + 11250 \times 10^{12}}\ \bar{F}_1(s)$$

$$= \frac{30}{10^4\, s^4 + 165 \times 10^5\, s^2 + 225 \times 10^7}\ \bar{F}_1(s) \qquad \text{---- } (E_7)$$

where $\bar{F}_1(s) = 2\times 10^5 \left[\frac{1}{s^2} - e^{-0.5 s}\left(\frac{1}{2s} - \frac{1}{s^2}\right)\right]$ ---- (E_8)

The inverse transforms of (E_6) and (E_7) yield $x_1(t)$ and $x_2(t)$.

5.46 Equations of motion for free vibration are (from Eqs. (5.1) and (5.2)):

$$\left.\begin{aligned} m_1\ddot{x}_1 + (c_1+c_2)\dot{x}_1 + (k_1+k_2)x_1 - c_2\dot{x}_2 - k_2 x_2 = 0 \\ m_2\ddot{x}_2 + (c_2+c_3)\dot{x}_2 + (k_2+k_3)x_2 - c_2\dot{x}_1 - k_2 x_1 = 0 \end{aligned}\right\} \qquad (E_1)$$

Assuming the solution as

$$x_i(t) = \tau_i\, e^{s_i t} \quad ; \quad i = 1,2$$

Eqs. (E_1) can be rewritten as

$$\begin{bmatrix} m_1 s^2 + (c_1+c_2)s + (k_1+k_2) & -(c_2 s + k_2) \\ -(c_2 s + k_2) & m_2 s^2 + (c_2+c_3)s + (k_2+k_3)\end{bmatrix} \begin{Bmatrix}\tau_1\\ \tau_2\end{Bmatrix} = \begin{Bmatrix}0\\0\end{Bmatrix} \qquad (E_2)$$

For a non-trivial solution of Eqs. (E_2),

$$\begin{vmatrix} m_1 s^2 + (c_1+c_2)s + k_1+k_2 & -(c_2 s + k_2) \\ -(c_2 s + k_2) & m_2 s^2 + (c_2+c_3)s + k_2+k_3\end{vmatrix} = 0$$

i.e.,

$$s^4(m_1 m_2) + s^3\left[m_1(c_2+c_3) + m_2(c_1+c_2)\right] + s^2\left[m_1(k_2+k_3) + (c_1+c_2)(c_2+c_3) + m_2(k_1+k_2) - c_2^2\right] + s\left[(c_1+c_2)(k_2+k_3) + (c_2+c_3)(k_1+k_2) - 2c_2k_2\right] + \left[(k_1+k_2)(k_2+k_3) - k_2^2\right] = 0 \qquad (E_3)$$

This equation can be expressed as

$$a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0 \qquad (E_4)$$

where $a_0, a_1, \ldots, a_4$ can be identified by comparing Eqs. (E_4) and (E_3).

Nature of possible solutions:

If s_1, s_2, s_3 and s_4 are the roots of Eq. (E_4), the general solution of the system can be expressed as

$$\left.\begin{aligned} x_1(t) &= \mathcal{C}_1^{(1)} e^{s_1 t} + \mathcal{C}_1^{(2)} e^{s_2 t} + \mathcal{C}_1^{(3)} e^{s_3 t} + \mathcal{C}_1^{(4)} e^{s_4 t} \\ x_2(t) &= \mathcal{C}_2^{(1)} e^{s_1 t} + \mathcal{C}_2^{(2)} e^{s_2 t} + \mathcal{C}_2^{(3)} e^{s_3 t} + \mathcal{C}_2^{(4)} e^{s_4 t} \end{aligned}\right\} \qquad (E_5)$$

where the constants $\mathcal{C}_1^{(i)}$, $i = 1$ to 4, can be found from the four initial conditions of the system, namely, $x_1(0)$, $x_2(0)$, $\dot{x}_1(0)$ and $\dot{x}_2(0)$. The ratios of amplitudes $\mathcal{C}_1^{(i)}/\mathcal{C}_2^{(i)}$ can be determined from Eqs. (E_2) as

$$\frac{\mathcal{C}_1^{(i)}}{\mathcal{C}_2^{(i)}} = \frac{c_2 s_i + k_2}{m_1 s_i^2 + (c_1+c_2) s_i + k_1 + k_2} = \frac{m_2 s_i^2 + (c_2+c_3) s_i + k_2 + k_3}{c_2 s_i + k_2};$$

$$i = 1, 2, 3, 4 \qquad (E_6)$$

If any root s_i has a positive real part, $x_1(t)$ and $x_2(t)$ will increase with time. If all s_i have negative real parts as

$$s_j = -r_j + i\, w_j$$

then the solution, $x_1(t)$, can be expressed as

$$x_1(t) = \sum_{j=1}^{4} \mathcal{C}_1^{(j)} e^{-r_j t} e^{i w_j t} = \sum_{j=1}^{4} \mathcal{C}_1^{(j)} e^{-r_j t} \left(\cos w_j t + i \sin w_j t\right)$$

If two roots s_1 and s_2 are complex conjugates as

$$s_1 = -(r_1 + i\, w_1) \quad \text{and} \quad s_2 = -(r_1 - i\, w_1),$$

$x_1(t)$ can be expressed as

$$\begin{aligned} x_1(t) = {} & e^{-r_1 t}\left\{\mathcal{C}_1^{(1)} \cos w_1 t - i\, \mathcal{C}_1^{(1)} \sin w_1 t\right\} \\ & + e^{-r_1 t}\left\{\mathcal{C}_1^{(2)} \cos w_1 t + i\, \mathcal{C}_1^{(2)} \sin w_1 t\right\} \\ & + \mathcal{C}_1^{(3)} e^{s_3 t} + \mathcal{C}_1^{(4)} e^{s_4 t} \end{aligned}$$

Similar expressions can be derived for $x_2(t)$.

5.47 Known data: $m_1 = m_2 = 10$ kg, $k_1 = k_2 = 2000$ N/m, $k_3 = 2$ N/m
$c_1 = 100$ N-s/m, $c_2 = c_3 = 1$ N-s/m
$x_1(0) = 0.2$ m, $x_2(0) = 0.1$ m, $\dot{x}_1(0) = \dot{x}_2(0) = 0$

Eqs. (E_3) and E_4 of the solution of Problem 5.46 give

$a_0 = m_1 m_2 = 100$

$a_1 = 10(2) + 10(101) = 1030$

$a_2 = 10(2002) + 101(2) + 10(4000) - 1 = 60221$

$a_3 = 101(2002) + 2(4000) - 2(1)(2000) = 206202$

$a_4 = 4000(2002) - 4 \times 10^6 = 4008000$

and

$$100 s^4 + 1030 s^3 + 60221 s^2 + 206202 s + 4008000 = 0 \qquad (E_1)$$

Using PROGRAM 10, the roots of Eq. (E_1) can be found as

$$s_{1,2} = -1.4714 \pm i\ 8.8272 \qquad (E_2)$$

$$s_{3,4} = -3.6786 \pm i\ 22.0668 \qquad (E_3)$$

Thus the solution is given by

$$x_1(t) = \sum_{j=1}^{4} C_1^{(j)} e^{s_j t}, \quad x_2(t) = \sum_{j=1}^{4} C_2^{(j)} e^{s_j t} \qquad (E_4, E_5)$$

where $C_1^{(j)}$, $j = 1,2,3,4$, can be found from the initial conditions, and the ratios of amplitudes $\left\{ \frac{C_1^{(j)}}{C_2^{(j)}} \right\}$ can be obtained from Eq. (E_6) in problem 5.46:

$$\frac{C_1^{(j)}}{C_2^{(j)}} = \frac{c_2 s_j + k_2}{m_1 s_j^2 + (c_1 + c_2) s_j + k_1 + k_2} = \frac{s_j + 2000}{10 s_j^2 + 101 s_j + 4000}; \quad j = 1,2,3,4 \qquad (E_6)$$

For $j = 1$, $s_1 = -1.4714 + i\ 8.8272$ and (E_6) gives[†]

$$\alpha_1 = C_1^{(1)} / C_2^{(1)} = 0.6207 - i\ 0.1239 \qquad (E_7)$$

For $j = 2$, $s_2 = -1.4714 - i\ 8.8272$ and (E_6) gives

$$\alpha_2 = C_1^{(2)} / C_2^{(2)} = 0.6207 + i\ 0.1239 \qquad (E_8)$$

For $j = 3$, $s_3 = -3.6786 + i\ 22.0668$ and (E_6) yields

$$\alpha_3 = C_1^{(3)} / C_2^{(3)} = -1.3808 - i\ 0.7758 \qquad (E_9)$$

For $j=4$, $s_4 = -3.6786 - i\,22.0668$ and (E_6) yields

$$\alpha_4 = C_1^{(4)}/C_2^{(4)} = -1.3808 + i\,0.7758 \qquad (E_{10})$$

Thus the solution of Eqs. (E_4) and (E_5) can be rewritten as

$$x_1(t) = \sum_{j=1}^{4} \alpha_j\, C_2^{(j)}\, e^{s_j t} \quad , \quad x_2(t) = \sum_{j=1}^{4} \alpha_j\, e^{s_j t} \qquad (E_{11}, E_{12})$$

Since the pairs (α_1, α_2), (α_3, α_4), (s_1, s_2) and (s_3, s_4) are complex conjugates, we can express them as

$$\left.\begin{aligned} \alpha_1, \alpha_2 &= p_1 \pm i\, q_1 \quad ; \quad \alpha_3, \alpha_4 = p_2 \pm i\, q_2 \\ s_1, s_2 &= u_1 \pm i\, v_1 \quad ; \quad s_3, s_4 = u_2 \pm i\, v_2 \end{aligned}\right\} \qquad (E_{13})$$

and (E_{11}) and (E_{12}) can be simplified further. However, we proceed directly with (E_{11}) and (E_{12}) and use the initial conditions to evaluate the constants $C_2^{(j)}$; $j = 1,2,3,4$:

$$\left.\begin{aligned} x_1(0) &= \alpha_1 C_2^{(1)} + \alpha_2 C_2^{(2)} + \alpha_3 C_2^{(3)} + \alpha_4 C_2^{(4)} = 0.2 \\ x_2(0) &= C_2^{(1)} + C_2^{(2)} + C_2^{(3)} + C_2^{(4)} = 0.1 \\ \dot{x}_1(0) &= s_1\alpha_1 C_2^{(1)} + s_2\alpha_2 C_2^{(2)} + s_3\alpha_3 C_2^{(3)} + s_4 s_4 C_2^{(4)} = 0 \\ \dot{x}_2(0) &= s_1 C_2^{(1)} + s_2 C_2^{(2)} + s_3 C_2^{(3)} + s_4 C_2^{(4)} = 0 \end{aligned}\right\} \qquad (E_{14})$$

Once $C_2^{(j)}$, $j = 1,2,3,4$ are determined from Eqs. (E_{14}), the displacements of masses $x_1(t)$ and $x_2(t)$ can be obtained using Eqs. (E_{11}) and (E_{12}).

---

† If $s = a + ib$, $s^2 = (a^2 - b^2) + i\,(2ab)$

If $x = \dfrac{a+bi}{c+di}$, it can be rewritten as

$$x = \frac{(a+bi)(c-di)}{(c^2+d^2)} = \left(\frac{ac+bd}{c^2+d^2}\right) + i\left(\frac{bc-ad}{c^2+d^2}\right)$$

5.48 $m = 0.2$ kg, $e = 0.15$ m, $m_1 = 400$ kg, $k_1 = 400$ kN/m, $m_2 = 900$ kg, $k_2 = 200$ kN/m, $c_2 = 40$ N · s/m

Solution:

$$\omega = \frac{2\,\pi\,(1200)}{60} = 125.664 \text{ rad/sec}$$

$$F_1\,(t) = m\,e\,\omega^2 \cos\omega\,t = (0.2)\,(0.15)\,(125.664)^2 \cos\,(125.664\,t)$$
$$= 473.7432 \cos\,(125.664\,t)\ (N)$$

$$F_{10} = 473.7432 \text{ N}, \quad F_{20} = 0$$

Equations of motion are [substitute $k_1 = k_2$, $c_1 = c_2$, $m_1 = m_2$, $F_1 = 0$, $k_2 = k_1$, $c_2 = 0$, $m_2 = m_1$, $F_2 = F_1$, $k_3 = 0$, $c_3 = 0$, in Eq. (5.3)]:

$$\begin{bmatrix} m_2 & 0 \\ 0 & m_1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_2 \\ \ddot{x}_1 \end{Bmatrix} + \begin{bmatrix} c_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_2 \\ \dot{x}_1 \end{bmatrix} + \begin{bmatrix} k_2 + k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} x_2 \\ x_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_1\,(t) \end{Bmatrix} \qquad (E_1)$$

or

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1\,(t) \\ 0 \end{Bmatrix} \qquad (E_2)$$

Comparing (E_2) with Eq. (5.27), we find that $m_{11} = m_1$, $m_{12} = 0$, $m_{22} = m_2$, $c_{11} = 0$, $c_{12} = 0$, $c_{22} = c_2$, $k_{11} = k_1$, $k_{12} = 0$, and $k_{22} = k_1 + k_2$.

Application of Eq. (5.31) leads to

$$Z_{11}\,(i\,\omega) = -\omega^2\,m_{11} + i\,\omega\,c_{11} + k_{11} = -m_1\,\omega^2 + k_1$$
$$Z_{12}\,(i\,\omega) = -\omega^2\,m_{12} + i\,\omega\,c_{12} + k_{12} = -k_1$$
$$Z_{22}\,(i\,\omega) = -\omega^2\,m_{22} + i\,\omega\,c_{22} + k_{22} = -m_2\,\omega^2 + i\,\omega\,c_2 + k_1 + k_2$$

Response of the system can be expressed as

$$X_j\,(t) = X_j\,e^{i\,\omega\,t} = X_j \cos\omega\,t \quad \text{(real part)}$$

with X_j given by Eq. (5.35):

$$X_1\,(i\,\omega) = \frac{Z_{22}\,(i\,\omega)\cdot F_{10} - Z_{12}\,(i\,\omega)\cdot F_{20}}{Z_{11}\,(i\,\omega)\cdot Z_{22}\,(i\,\omega) - Z_{12}^2\,(i\,\omega)}$$

$$= \frac{(-m_2\,\omega^2 + i\,\omega\,c_2 + k_1 + k_2)\,F_{10}}{(-m_1\,\omega^2 + k_1)\,(-m_2\,\omega^2 + i\,\omega\,c_2 + k_1 + k_2) - k_1^2}$$

$$= \frac{\left\{-900\,(125.664)^2 + i\,(125.664)\,(40) + 600\times 10^3\right\}(473.7432)}{\left\{\left[-400\,(125.664)^2 + 400\times 10^3\right]\cdot \left[-900\,(125.664)^2 + i\,(125.664)\,(40) + 600\times 10^3\right] - 400^2\times 10^6\right\}}$$

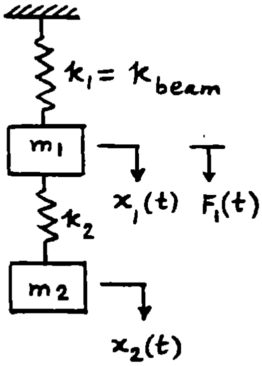
$$= (-8.0230 - i\,5.8964\times 10^{-6})\times 10^{-5}$$

$$X_2(i\,\omega) = \frac{-Z_{12}(i\,\omega)\,F_{10} + Z_{11}(i\,\omega)\,F_{20}}{Z_{11}(i\,\omega)\,Z_{22}(i\,\omega) - Z_{12}^2(i\,\omega)} = \frac{k_1\,F_{10}}{Z_{11}(i\,\omega)\,Z_{22}(i\,\omega) - Z_{12}^2(i\,\omega)}$$

$$= \frac{400 \times 10^3\,(473.7432)}{(8.0539 \times 10^{13} - i\,2.0133 \times 10^{13})}$$

$$= (2.2145 + i\,0.5536) \times 10^{-6}$$

5.49 $k_1 = k_{beam} = \dfrac{192\,E\left(\frac{1}{12}\,a\,t^3\right)}{l^3} = \dfrac{16\,E\,a\,t^3}{l^3}$



Equations of motion:

$$\left.\begin{aligned} m_1\ddot{x}_1 + k_1 x_1 + k_2(x_1 - x_2) &= F_1(t) = F_0 \cos\omega t \\ m_2\ddot{x}_2 + k_2(x_2 - x_1) &= 0 \end{aligned}\right\} \ldots (E_1)$$

Assuming harmonic response

$$x_j(t) = X_j \cos\omega t \quad ; \quad j = 1, 2$$

Eqs. (E_1) yield

$$X_1 = \frac{(k_2 - m_2\omega^2)\,F_0}{(k_1 + k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2^2}$$

$$X_2 = \frac{k_2\,F_0}{(k_1 + k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2^2}$$

For no steady state vibration of the beam, $X_1 = 0$ and hence the condition to be satisfied is

$$\frac{k_2}{m_2} = \omega^2$$

5.50 Equations of motion:

$$m_1\ddot{x}_1 + (k_1 + k_2)\,x_1 - k_2 x_2 = F_1(t) = F_0 \sin\omega t \qquad \text{----} \ (E_1)$$

$$m_2\ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0 \qquad \text{----} \ (E_2)$$

We use $F_0\, e^{i\omega t}$ (with $i = \sqrt{-1}$) for $F_1(t)$ and consider only the imaginary part at the end.

Let $\quad x_j(t) = X_j\, e^{i\omega t} \quad ; \quad j = 1, 2$

Eqs. (E_1) and (E_2) become

$$-m_1 \omega^2 X_1 e^{i\omega t} + (k_1 + k_2) X_1 e^{i\omega t} - k_2 X_2 e^{i\omega t} = F_0 e^{i\omega t}$$

$$-m_2 \omega^2 X_2 e^{i\omega t} + k_2 X_2 e^{i\omega t} - k_2 X_1 e^{i\omega t} = 0$$

i.e. $$[Z(i\omega)]\, \vec{X} = \vec{\underset{\sim}{F_0}} \qquad \text{----}\ (E_3)$$

where $\vec{X} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$, $\quad \vec{\underset{\sim}{F_0}} = \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}$,

$$[Z(i\omega)] = \begin{bmatrix} Z_{11}(i\omega) & Z_{12}(i\omega) \\ Z_{21}(i\omega) & Z_{22}(i\omega) \end{bmatrix},$$

$Z_{11}(i\omega) = -m_1\omega^2 + k_1 + k_2, \quad Z_{12}(i\omega) = Z_{21}(i\omega) = -k_2,$

$Z_{22}(i\omega) = -m_2\omega^2 + k_2.$

Eqs. (5.35) give

$$X_1(i\omega) = \frac{(-m_2\omega^2 + k_2)\, F_0}{(-m_1\omega^2 + k_1 + k_2)(-m_2\omega^2 + k_2) - k_2^2}$$

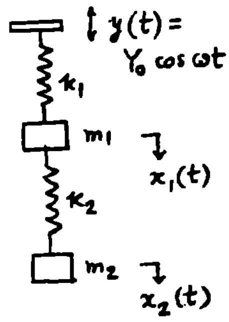
$$X_2(i\omega) = \frac{k_2\, F_0}{(-m_1\omega^2 + k_1 + k_2)(-m_2\omega^2 + k_2) - k_2^2}$$

Since $F_0 \sin\omega t = \mathrm{Im}\,(F_0\, e^{i\omega t})$, $\quad x_j(t) = \mathrm{Im}\,(X_j\, e^{i\omega t}) = X_j \sin\omega t$

$$\therefore\ x_1(t) = \frac{(-m_2\omega^2 + k_2)\, F_0}{(-m_1\omega^2 + k_1 + k_2)(-m_2\omega^2 + k_2) - k_2^2} \sin\omega t$$

$$x_2(t) = \frac{k_2\, F_0}{(-m_1\omega^2 + k_1 + k_2)(-m_2\omega^2 + k_2) - k_2^2} \sin\omega t$$

5.51 Equations of motion:



$$m_1 \ddot{x}_1 = -k_1(x_1 - y) - k_2(x_1 - x_2)$$

$$m_2 \ddot{x}_2 = -k_2(x_2 - x_1)$$

or $$m_1 \ddot{x}_1 + (k_1 + k_2)\, x_1 - k_2 x_2 = k_1 Y_0 \cos\omega t \quad \text{---}\ (E_1)$$

$$m_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0 \quad \text{---}\ (E_2)$$

As there is no damping, the masses vibrate either in phase or 180° out of phase with respect to the base motion. Hence the response can be taken as

$$x_j(t) = X_j \cos \omega t \quad ; \quad j = 1,2 \qquad \text{--- } (E_3)$$

Eqs. (E_1) and (E_2) reduce to

$$(-\omega^2 m_1 + k_1 + k_2)\, X_1 - k_2 X_2 = k_1 Y_0$$

$$-k_2 X_1 + (-\omega^2 m_2 + k_2)\, X_2 = 0$$

i.e. $[Z(i\omega)]\, \vec{X} = \vec{F}$ where $\vec{X} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$, $\vec{F} = \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix} = \begin{Bmatrix} k_1 Y_0 \\ 0 \end{Bmatrix}$,

$$Z_{11}(i\omega) = -\omega^2 m_1 + k_1 + k_2 , \qquad Z_{12}(i\omega) = Z_{21}(i\omega) = -k_2 ,$$

$$Z_{22}(i\omega) = -\omega^2 m_2 + k_2 .$$

Eqs. (5.35) and (E_3) give

$$x_1(t) = \frac{(-\omega^2 m_2 + k_2)\, k_1 Y_0}{(-m_1\omega^2 + k_1 + k_2)(-m_2\omega^2 + k_2) - k_2^2} \cos \omega t$$

$$x_2(t) = \frac{k_1 k_2 Y_0}{(-m_1\omega^2 + k_1 + k_2)(-m_2\omega^2 + k_2) - k_2^2} \cos \omega t$$

5.52 Equations of motion:

$$m_1 \ddot{x}_1 + (c_1 + c_2)\dot{x}_1 + (k_1 + k_2)\, x_1 - c_2 \dot{x}_2 - k_2 x_2 = F_1(t) = F_0\, e^{i\omega t}$$

$$m_2 \ddot{x}_2 + c_2 \dot{x}_2 + k_2 x_2 - c_2 \dot{x}_1 - k_2 x_1 = 0$$

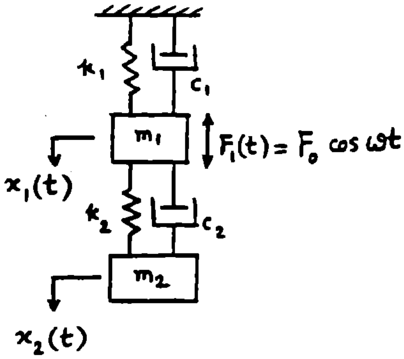
Let $x_j(t) = X_j\, e^{i\omega t}$; $j = 1,2$

$k_1 = k_2 = 500$ N/m

$m_1 = m_2 = 1$ kg

$c_1 = c_2 = c = 200$ N·s/m

$\omega = 1$ rad/s



Equations of motion become

$$[-\omega^2 m_1 + i\omega(c_1 + c_2) + k_1 + k_2]\, X_1 - (i\omega c_2 + k_2)\, X_2 = F_0 \qquad \text{--- } (E_1)$$

$$-(i\omega c_2 + k_2)\, X_1 + [-\omega^2 m_2 + i\omega c_2 + k_2]\, X_2 = 0 \qquad \text{--- } (E_2)$$

For given data, (E_1) and (E_2) become

$$[Z(i\omega)]\, \vec{X} = \vec{F}_0 \qquad \text{--- } (E_3)$$

where $Z_{11}(i\omega) = 400\, i + 999$

$$Z_{12}(i\omega) = Z_{21}(i\omega) = -200\, i - 500$$

$$Z_{22}(i\omega) = 200\, i + 499$$

$$\vec{X} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} , \quad \vec{F}_0 = \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}$$

Solution of (E_3) is, using Eqs. (5.35),

$$X_1 = \frac{(200\,i + 499)\,F_0}{(400i+999)(200\,i+499) - (-200i-500)^2} = \frac{(200i+499)\,F_0}{(199400\,i + 208501)}$$

$$= \frac{(200\,i + 499)(-199400\,i + 208501)\,F_0}{(199400\,i + 208501)(-199400\,i + 208501)}$$

$$= (17.2915 \times 10^{-4} - 6.9444 \times 10^{-4}\,i)\,F_0 \qquad \text{----} \; (E_4)$$

$$X_2 = \frac{(200\,i + 500)\,F_0}{(400\,i+999)(200\,i+499) - (-200i-500)^2} = \frac{(200\,i + 500)\,F_0}{(199400i + 208501)}$$

$$= \frac{(200\,i + 500)(-199400\,i + 208501)\,F_0}{(199400\,i + 208501)(-199400\,i + 208501)}$$

$$= (17.3165 \times 10^{-4} - 6.9684 \times 10^{-4}\,i)\,F_0 \qquad \text{----} \; (E_5)$$

Final solution is given by the real parts as

$$x_1(t) = \mathrm{Re}\left(X_1\, e^{i\omega t}\right) = \mathrm{Re}\left(X_1 \cos\omega t + i\,X_1 \sin\omega t\right)$$

$$= \mathrm{Re}\left[(17.2915\times10^{-4} - 6.9444\times10^{-4}\,i)\,F_0 \cos\omega t + (17.2915\times10^{-4}\,i + 6.9444\times10^{-4})\,F_0 \sin\omega t\right]$$

$$= 17.2915\times10^{-4}\,F_0 \cos\omega t + 6.9444\times10^{-4}\,F_0 \sin\omega t \qquad \text{---} \; (E_6)$$

$$x_2(t) = \mathrm{Re}\left(X_2\, e^{i\omega t}\right) = \mathrm{Re}\left(X_2 \cos\omega t + i\,X_2 \sin\omega t\right)$$

$$= \mathrm{Re}\left[(17.3165\times10^{-4} - 6.9684\times10^{-4}\,i)\,F_0 \cos\omega t + (17.3165\times10^{-4}\,i + 6.9684\times10^{-4})\,F_0 \sin\omega t\right]$$

$$= 17.3165\times10^{-4}\,F_0 \cos\omega t + 6.9684\times10^{-4}\,F_0 \sin\omega t \qquad \text{---} \; (E_7)$$

5.53 Equations of motion:

$$m_1 \ddot{x}_1 + (k_1+k_2)\,x_1 - k_2 x_2 = F_{10} \cos\omega t = \mathrm{Re}\left(F_{10}\, e^{i\omega t}\right)$$

$$m_2 \ddot{x}_2 + (k_2+k_3)\,x_2 - k_2 x_1 = F_{20} \cos\omega t = \mathrm{Re}\left(F_{20}\, e^{i\omega t}\right)$$

Assuming $x_j(t) = X_j\, e^{i\omega t}$; $j = 1,2$ along with $F_j(t) = F_{j0}\, e^{i\omega t}$; $j=1,2$, the equations of motion can be expressed as

$$(-\omega^2 m_1 + k_1 + k_2)\,X_1 - k_2 X_2 = F_{10}$$

$$-k_2 X_1 + (-\omega^2 m_2 + k_2 + k_3)\,X_2 = F_{20}$$

i.e. $$[Z(i\omega)]\,\vec{X} = \vec{F_0} \qquad \text{----} \; (E_1)$$

where $Z_{11}(i\omega) = -\omega^2 m_1 + k_1 + k_2$, $Z_{12}(i\omega) = Z_{21}(i\omega) = -k_2$,

$Z_{22}(i\omega) = -\omega^2 m_2 + k_2 + k_3$,

$$\vec{X} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}, \quad \vec{F_0} = \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix}$$

Solution of (E_1) can be expressed, using Eqs. (5.35), as

$$X_1 = \frac{(-\omega^2 m_2 + k_2 + k_3) F_{10} + k_2 F_{20}}{(-\omega^2 m_1 + k_1 + k_2)(-\omega^2 m_2 + k_2 + k_3) - k_2^2} \quad \text{----} (E_2)$$

$$X_2 = \frac{k_2 F_{10} + (-\omega^2 m_1 + k_1 + k_2) F_{20}}{(-\omega^2 m_1 + k_1 + k_2)(-\omega^2 m_2 + k_2 + k_3) - k_2^2} \quad \text{----} (E_3)$$

Since X_1 and X_2 are real (since there is no damping), the final solution is given by

$$x_1(t) = X_1 \cos \omega t$$

$$x_2(t) = X_2 \cos \omega t$$

where X_1 and X_2 are given by (E_2) and (E_3).

5.54 From the solution of problem 5.50, we have

$$x_1(t) = \frac{(-m_2\omega^2 + k_2) F_0}{(-m_1\omega^2 + k_1 + k_2)(-m_2\omega^2 + k_2) - k_2^2} \sin \omega t$$

$$x_2(t) = \frac{k_2 F_0}{(-m_1\omega^2 + k_1 + k_2)(-m_2\omega^2 + k_2) - k_2^2} \sin \omega t$$

For the data $F_1(t) = 50 \sin 4\pi t$, $F_0 = 50$ N, $\omega = 4\pi$ rad/s, $m_1 = 10$ kg, $m_2 = 5$ kg, $k_1 = 8000$ N/m and $k_2 = 2000$ N/m,

$$x_1(t) = \frac{(-5 \times 16\pi^2 + 2000)\, 50}{(-10 \times 16\pi^2 + 8000 + 2000)(-5 \times 16\pi^2 + 2000) - (2000)^2} \sin 4\pi t$$

$$= 0.009773 \sin 4\pi t \quad \text{meters}$$

$$x_2(t) = \frac{2000(50)}{(-10 \times 16\pi^2 + 8000 + 2000)(-5 \times 16\pi^2 + 2000) - (2000)^2} \sin 4\pi t$$

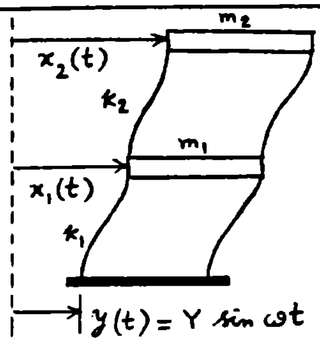
$$= 0.016148 \sin 4\pi t \quad \text{meters}$$

5.55 k_1 = total stiffness = 800 N/m

k_2 = total stiffness = 600 N/m

$m_1 = 50$ kg, $m_2 = 50$ kg

$Y = 0.2$ m, $\omega = \pi$ rad/s



Equations of motion:

$$m_1 \ddot{x}_1 = -k_1 (x_1 - y) - k_2 (x_1 - x_2)$$

$$m_2 \ddot{x}_2 = -k_2 (x_2 - x_1)$$

i.e. $$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = k_1 y = k_1 Y \sin \omega t$$

$$m_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0$$

Assuming $x_i(t) = X_i \sin \omega t$; $i = 1, 2$, we get

$$(-m_1\omega^2 + k_1 + k_2)\, X_1 - k_2 X_2 = k_1 Y$$

$$-k_2 X_1 + (-m_2\omega^2 + k_2)\, X_2 = 0$$

For given data, these equations take the form

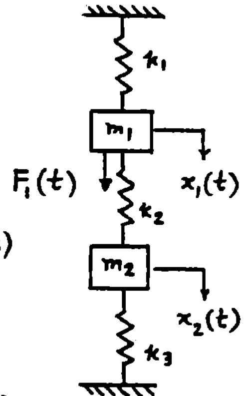
$$(-50\pi^2 + 1400)\, X_1 - 600\, X_2 = (800)(0.2)$$

$$-600\, X_1 + (-50\pi^2 + 600)\, X_2 = 0$$

Solution of these equations gives $X_1 = -0.06469$ m, $X_2 = -0.36439$ m

$\therefore\ x_1(t) = -0.06469 \sin \pi t$ m ; $x_2(t) = -0.36439 \sin \pi t$ m.

5.56 Equations of motion:



$$m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2 x_2 = F_1(t) \quad \text{---} \ (E_1)$$

$$m_2\ddot{x}_2 + (k_2+k_3)x_2 - k_2 x_1 = 0 \quad \text{---} \ (E_2)$$

Laplace transforms of (E_1) and (E_2) are

$$m_1[s^2\bar{x}_1(s) - s\,x_1(0) - \dot{x}_1(0)] + (k_1+k_2)\bar{x}_1(s) - k_2\bar{x}_2(s) = \bar{F}_1(s)$$

$$m_2[s^2\bar{x}_2(s) - s\,x_2(0) - \dot{x}_2(0)] + (k_2+k_3)\bar{x}_2(s) - k_2\bar{x}_1(s) = 0$$

Rearranging these equations, we get

$$(m_1 s^2 + k_1 + k_2)\bar{x}_1(s) - k_2\bar{x}_2(s) = \bar{F}_1(s) + s\,m_1 x_1(0) + m_1\dot{x}_1(0) \ \text{----} \ (E_3)$$

$$-k_2\bar{x}_1(s) + (m_2 s^2 + k_2 + k_3)\bar{x}_2(s) = s\,m_2 x_2(0) + m_2\dot{x}_2(0) \ \text{---} \ (E_4)$$

When $k_1 = k_2 = k_3 = k$ and $m_1 = m_2 = m$, (E_3) and (E_4) give

$$(m s^2 + 2k)\,\bar{x}_1(s) - k\,\bar{x}_2(s) = \bar{F}_1(s) + s\,m\,x_1(0) + m\,\dot{x}_1(0) \quad \text{---} \ (E_5)$$

$$-k\,\bar{x}_1(s) + (m s^2 + 2k)\,\bar{x}_2(s) = s\,m\,x_2(0) + m\,\dot{x}_2(0) \quad \text{---} \ (E_6)$$

Solution of Eqs. (E_5) and (E_6) gives

$$\bar{x}_1(s) = \frac{(ms^2+2k)\{\bar{F}_1(s) + m\,x_1(0)\cdot s + m\,\dot{x}_1(0)\} + k\{m\,x_2(0)\cdot s + m\,\dot{x}_2(0)\}}{(ms^2+2k)(ms^2+2k) - k^2}$$

$$\bar{x}_2(s) = \frac{k\{\bar{F}_1(s) + m\,x_1(0)\cdot s + \dot{x}_1(0)\,m\} + (ms^2+2k)\{m\,x_2(0)\cdot s + m\,\dot{x}_2(0)\}}{(ms^2+2k)(ms^2+2k) - k^2}$$

These equations become, for $x_1(0) = x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$, $F_1(t) = 5\,u(t)$, $\bar{F}_1(s) = \frac{5}{s}$, $m = 1$ and $k = 100$,

$$\bar{x}_1(s) = \frac{5(s^2+200)}{s\,[(s^2+200)^2 - 10000]} = \frac{5(s^2+200)}{s(s^2+300)(s^2+100)} \quad \text{---} \ (E_7)$$

$$\bar{x}_2(s) = 100\left(\frac{5}{s}\right)\frac{1}{[(s^2+200)^2 - 10000]} = \frac{500}{s(s^2+300)(s^2+100)} \ \text{---} \ (E_8)$$

By expressing

$$\bar{x}_1(s) = \frac{5(s^2+200)}{s(s^2+300)(s^2+100)} = \frac{A_1}{s} + \frac{A_2 s + A_3}{s^2+100} + \frac{A_4 s + A_5}{s^2+300}$$

$$\bar{x}_2(s) = \frac{500}{s(s^2+300)(s^2+100)} = \frac{B_1}{s} + \frac{B_2 s + B_3}{s^2+100} + \frac{B_4 s + B_5}{s^2+300}$$

we can find $A_1, A_2, \ldots, B_1, B_2, \ldots$ (partial fractions method).

This leads to

$$\bar{x}_1(s) = \frac{1}{30 s} - \frac{s}{40(s^2+100)} - \frac{s}{120(s^2+300)} \quad \text{---}(E_9)$$

$$\bar{x}_2(s) = \frac{1}{60 s} - \frac{s}{40(s^2+100)} + \frac{s}{120(s^2+300)} \quad \text{---}(E_{10})$$

Inverse Laplace transforms of (E_9) and (E_{10}) give

$$x_1(t) = \left(\frac{1}{30} - \frac{1}{40}\cos 10t - \frac{1}{120}\cos 10\sqrt{3}\, t\right) u(t)$$

$$x_2(t) = \left(\frac{1}{60} - \frac{1}{40}\cos 10t + \frac{1}{120}\cos 10\sqrt{3}\, t\right) u(t)$$

It is to be noted that $x_1(t) = \frac{1}{30}$ meter and $x_2(t) = \frac{1}{60}$ meter are the static deflections associated with 5 N static force applied to mass m_1. $\omega_1 = 10$ rad/s and $\omega_2 = 10\sqrt{3}$ rad/s are the two resonant frequencies associated with the two degree of freedom system.

5.57 Equivalent mass of cylinder with respect to $x_2 = (m_2)_{eq} = m_2 + \frac{J_0}{r^2}$

Equations of motion: $m_1 \ddot{x}_1 = -k(x_1 - x_2)$

$$(m_2)_{eq}\, \ddot{x}_2 = -k(x_2 - x_1)$$

i.e.

$$m_1 \ddot{x}_1 + k x_1 - k x_2 = 0 \quad \text{----}(E_1)$$

$$\left(m_2 + \frac{J_0}{r^2}\right)\ddot{x}_2 + k x_2 - k x_1 = 0 \quad \text{----}(E_2)$$

Assuming $x_i(t) = X_i \cos(\omega t + \phi)$; $i = 1,2$, Eqs. (E_1) and (E_2) can be written as

$$(-m_1\omega^2 + k) X_1 - k X_2 = 0$$

$$-k X_1 + \left(-\omega^2\left\{m_2 + \frac{J_0}{r^2}\right\} + k\right) X_2 = 0$$

Frequency equation is:

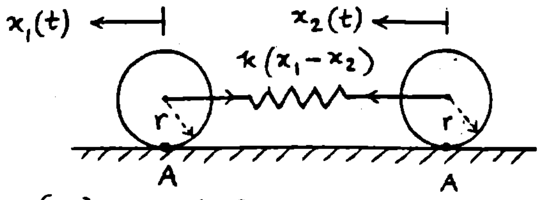
$$(-m_1\omega^2 + k)\left(-\omega^2 m_2 - \omega^2 \frac{J_0}{r^2} + k\right) - k^2 = 0$$

or

$$\omega^4\left(m_1 m_2 + \frac{m_1 J_0}{r^2}\right) - \omega^2\left(m_1 k + m_2 k + \frac{k J_0}{r^2}\right) = 0$$

$$\omega_1 = 0, \qquad \omega_2 = \left\{\left(m_1 k + m_2 k + \frac{k J_0}{r^2}\right)\Big/\left(m_1 m_2 + \frac{m_1 J_0}{r^2}\right)\right\}^{1/2}$$

5.58 Equivalent mass of each cylinder for translatory motion, referred to point A, is



$$m_{eq} = \frac{J_A}{r^2} = \frac{\frac{mr^2}{2} + mr^2}{r^2} = \frac{3m}{2} = (m_1)_{eq} = (m_2)_{eq}$$

Equations of motion:

$$(m_1)_{eq}\,\ddot{x}_1 + k(x_1 - x_2) = 0$$

$$(m_2)_{eq}\,\ddot{x}_2 - k(x_1 - x_2) = 0$$

Frequency equation:

$$\begin{vmatrix} -(m_1)_{eq}\,\omega^2 + k & -k \\ -k & -(m_2)_{eq} + k \end{vmatrix} = 0$$

or $\quad (m_1)_{eq}(m_2)_{eq}\,\omega^4 - \omega^2\left[(m_1)_{eq}\,k + (m_2)_{eq}\,k\right] = 0$

$$\omega_1 = 0, \qquad \omega_2 = \sqrt{\frac{(m_1)_{eq}\,k + (m_2)_{eq}\,k}{(m_1)_{eq}\,(m_2)_{eq}}} = \sqrt{\frac{4k}{3m}}$$

5.59 For harmonic motion $x_i(t) = X_i \cos(\omega t + \phi)$; $i = 1,2$ given equations lead to

$$(-\omega^2 a_1 + b_1)\,X_1 + c_1\,X_2 = 0$$

$$b_2\,X_1 + (-\omega^2 a_2 + c_2)\,X_2 = 0$$

Frequency equation is

$$\begin{vmatrix} -\omega^2 a_1 + b_1 & c_1 \\ b_2 & -\omega^2 a_2 + c_2 \end{vmatrix} = 0$$

or $\quad \omega^4(a_1 a_2) - \omega^2(a_1 c_2 + b_1 a_2) + (b_1 c_2 - c_1 b_2) = 0$

Condition for degeneracy is: $\quad b_1 c_2 - c_1 b_2 = 0$

5.60 Equations of motion:

$$J_1\,\ddot{\theta}_1 + k_t\,\theta_1 - k_t\,\theta_2 = 0$$

$$J_2\,\ddot{\theta}_2 + k_t\,\theta_2 - k_t\,\theta_1 = 0$$

For $\theta_i(t) = \Theta_i \cos(\omega t + \phi)$; $i = 1, 2$,

$$\begin{bmatrix} -\omega^2 J_1 + k_t & -k_t \\ -k_t & -\omega^2 J_2 + k_t \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Frequency equation is $\quad \omega^4(J_1 J_2) - \omega^2(J_1 k_t + J_2 k_t) = 0$

$$\omega_1 = 0, \qquad \omega_2 = \sqrt{\frac{k_t(J_1 + J_2)}{J_1 J_2}}$$

Amplitude ratios:

$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = \frac{-\omega_1^2 J_1 + k_t}{k_t} = 1$$

$$r_2 = \frac{\Theta_2^{(2)}}{\Theta_1^{(2)}} = \frac{-\omega_2^2 J_1 + k_t}{k_t} = -\frac{J_1}{J_2}$$

General solution is given by equations similar to Eqs. (5.15). With $\dot{\theta}_1(0) = \dot{\theta}_2(0) = 0$, we obtain from equations similar to Eqs. (5.18):

$$\Theta_1^{(1)} = \frac{1}{r_2 - r_1}\left\{r_2\,\theta_1(0) - \theta_2(0)\right\} = -\left(\frac{J_2}{J_1+J_2}\right)\left[-\frac{J_1}{J_2}\,\theta_1(0) - \theta_2(0)\right]$$

$$= \left\{\frac{J_1\,\theta_1(0) + J_2\,\theta_2(0)}{J_1 + J_2}\right\}$$

$$\Theta_1^{(2)} = \frac{1}{r_2 - r_1}\left\{-r_1\,\theta_1(0) + \theta_2(0)\right\} = -\left(\frac{J_2}{J_1+J_2}\right)\left[\frac{J_1}{J_2}\,\theta_1(0) + \theta_2(0)\right]$$

$$= -\left\{\frac{J_1\,\theta_1(0) + J_2\,\theta_2(0)}{J_1 + J_2}\right\}$$

$$\phi_1 = \phi_2 = 0$$

$$\theta_1(t) = \Theta_1^{(1)}\cos 0t + \Theta_1^{(2)}\cos\omega_2 t = \Theta_1^{(1)} + \Theta_1^{(2)}\cos\omega_2 t$$

$$\theta_2(t) = \Theta_1^{(1)} - \frac{J_1}{J_2}\,\Theta_1^{(2)}\cos\omega_2 t$$

5.61 When $k_{t1} = 0$, the system becomes identical to the system of problem 5.60 with $k_t = k_{t1}$. Normal modes are given by

$$\vec{\Theta}^{(1)} = \begin{Bmatrix}\Theta_1^{(1)}\\ \Theta_2^{(1)}\end{Bmatrix} = \begin{Bmatrix}1\\1\end{Bmatrix}\Theta_1^{(1)}\quad;\quad \vec{\Theta}^{(2)} = \begin{Bmatrix}\Theta_1^{(2)}\\ \Theta_2^{(2)}\end{Bmatrix} = \begin{Bmatrix}1\\ -\left(\frac{J_1}{J_2}\right)\end{Bmatrix}\Theta_1^{(2)}$$

Equations of motion can be rewritten as

$$\ddot{\theta}_1 + \frac{k_t}{J_1}(\theta_1 - \theta_2) = 0 \qquad \text{---- } (E_1)$$

$$\ddot{\theta}_2 - \frac{k_t}{J_2}(\theta_1 - \theta_2) = 0 \qquad \text{---- } (E_2)$$

Subtracting (E_2) from (E_1) gives

$$(\ddot{\theta}_1 - \ddot{\theta}_2) + (\theta_1 - \theta_2)\left(\frac{k_t}{J_1} + \frac{k_t}{J_2}\right) = 0 \quad \text{---- } (E_3)$$

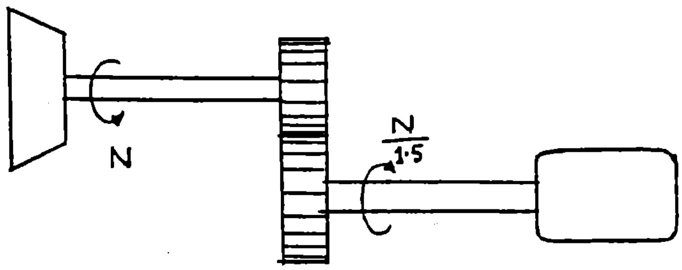
Defining $\alpha = \theta_1 - \theta_2$, (E_3) can be written as

$$\ddot{\alpha} + \left(\frac{k_t}{J_1} + \frac{k_t}{J_2}\right)\alpha = 0 \qquad \text{---- } (E_4)$$

This is a single equation for which the natural frequency is

$$\omega = \sqrt{\left(\frac{k_t}{J_1} + \frac{k_t}{J_2}\right)} = \sqrt{\frac{k_t(J_1 + J_2)}{J_1 J_2}} \equiv \omega_2 \text{ of problem 5.45.}$$

5.62



Since the length of shaft 1 is small and its diameter large, it will be very rigid and hence the turbine and gear 1 are assumed to be rigidly connected. This helps in modeling the system as a two d.o.f. system.

$$J_{01} = J_{turbine} + J_{gear1} + \frac{J_{gear2}}{1.5^2} = 3000 + 500 + (1000/2.25) = 3944.4444 \text{ kg-m}^2$$

$$k_{t2} = \left(\frac{GJ}{\ell}\right)_{shaft2} = \frac{(80\,(10^9))\left(\frac{\pi}{32}\,(0.1^4)\right)}{1} = 7.854\,(10^5) \text{ N/m}$$

$$J_{02} = J_{generator} = 2000 \text{ kg-m}^2$$

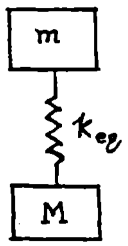
System is a semi-definite system. Its natural frequencies are given by (see Eq. (5.40)):

$$\omega_1 = 0$$

$$\omega_2 = \sqrt{\frac{k_{t2}\,(J_{01} + J_{02})}{J_{01}\,J_{02}}} = \sqrt{\frac{(78.54\,(10^4))\,(5944.4444)}{(3944.4444)\,(2000)}} = 24.3273 \text{ rad/sec}$$

5.63 Natural frequencies are given by Eq. (5.40):

$$\omega_1 = 0 \quad ; \quad \omega_2 = \sqrt{\frac{k\,(m_1 + m_2)}{m_1\,m_2}} = \sqrt{\frac{6\,k\,(m + M)}{m\,M}}$$



Assumption: Balloon is a point mass.

$k_{eq} = 12\,k\,\cos^2 45°$
$= 6\,k$

5.64 Turbine mass moment of inertia 0.5 kg · m^2, generator mass moment of inertia 0.25 kg · m^2, shaft inner diameter 0.02 m, shaft outer diameter 0.04 m, shaft length 0.4 m, turbine power 70 kW.

Solution:

$$\text{Speed of shaft} = 6000 \text{ rpm} = \frac{6000\,(2\,\pi)}{60} = 628.32 \text{ rad/sec}$$

Torsional stiffness (spring constant) of the hollow steel shaft:

$$k_t = \frac{\pi\,G\,(d_0^4 - d_i^4)}{32\,\ell} = \frac{\pi\,(83 \times 10^9)\,(0.04^4 - 0.02^4)}{32\,(0.4)} = 48.8910 \times 10^3 \text{ N} \cdot \text{m/s}$$

Natural frequencies of the system, given by an equation similar to Eq. (5.40):

$$\omega_1 = 0,\ \omega_2 = \left\{\frac{k_t\,(J_1 + J_2)}{J_1\,J_2}\right\}^{\frac{1}{2}} = \left\{\frac{48.8910 \times 10^3\,(0.5 + 0.25)}{0.5\,(0.25)}\right\}^{\frac{1}{2}} = 541.6143 \text{ rad/sec.}$$

Second mode shape is given by

$$\alpha = \frac{\Theta_2}{\Theta_1} = \frac{k_{tz} - J_1 \omega_2^2}{k_{tz}} = \frac{48.8910 \times 10^3 - 0.5\,(541.6143)^2}{48.8910 \times 10^3}$$

$$\alpha = -2.0 \tag{E$_1$}$$

Torque transmitted before turbine is stopped (T):

$$T = \frac{700 \times 10^3}{628.32} = 111.4\ \text{N} \cdot \text{m}$$

In view of the fact that $\omega_1 = 0$ corresponds to the same angular motions of the two mass moments of inertia, we have constant angular displacements and constant angular velocities so that the free vibration can be expressed as:

$$\theta_1 = c_1 + c_2\, t + c_3 \cos \omega_2\, t + c_4 \sin \omega_2\, t$$
$$\theta_2 = c_1 + c_2\, t + c_3\, \alpha \cos \omega_2\, t + c_4\, \alpha \sin \omega_2\, t$$

where c_1, c_2, c_3 and c_4 are determined by the initial conditions.

The total initial angular displacement between the turbine and generator is given by

$$\phi_0 = \frac{T}{k_t} = \frac{111.4}{48.8910 \times 10^3} = 0.00228\ \text{rad} = \Theta_1 - \Theta_2 \tag{E$_2$}$$

The initial angular displacements at the instant when turbine is suddenly stopped can be found by solving Eqs. (E$_1$) and (E$_2$):

$$\Theta_2 = -2\,\Theta_1$$
$$3\,\Theta_1 = 0.00228 \quad \text{or} \quad \Theta_1 = 0.00076\ \text{rad}$$

The initial angular velocities (speeds) at the instant when turbine is stopped are given by

$$\dot{\Theta}_1 = \dot{\Theta}_2 = 628.32\ \text{rad/sec}$$

Using the conditions

$$\theta_1\,(t = 0) = \Theta_1 = 0.00076, \qquad \theta_2\,(t = 0) = \Theta_2 = -0.00152$$
$$\dot{\theta}_1\,(t = 0) = \dot{\Theta}_1 = 628.32\ \text{rad/sec}, \quad \dot{\theta}_2\,(t = 0) = \dot{\Theta}_2 = 628.32\ \text{rad/sec}$$

the constants c_1, c_2, c_3 and c_4 can be determined.

5.65 Equations of motion :

$$J_1 \ddot{\theta}_1 + c_{t1}\dot{\theta}_1 - c_{t2}(\dot{\theta}_2 - \dot{\theta}_1) + k_{t1}\theta_1 - k_{t2}(\theta_2 - \theta_1) = 0$$

$$J_2 \ddot{\theta}_2 + c_{t2}(\dot{\theta}_2 - \dot{\theta}_1) + k_{t2}(\theta_2 - \theta_1) = 0$$

i.e.,

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} c_{t1}+c_{t2} & -c_{t2} \\ -c_{t2} & c_{t2} \end{bmatrix} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} k_{t1}+k_{t2} & -k_{t2} \\ -k_{t2} & k_{t2} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \text{--- } (E_1)$$

Let $\theta_j(t) = X_j \, e^{st}$ ----- (E_2)

Eqs. (E_1) yield :

$$\left(\begin{bmatrix} J_1 s^2 & 0 \\ 0 & J_2 s^2 \end{bmatrix} + \begin{bmatrix} (c_{t1}+c_{t2})s & -c_{t2}s \\ -c_{t2}s & c_{t2}s \end{bmatrix} + \begin{bmatrix} (k_{t1}+k_{t2}) & -k_{t2} \\ -k_{t2} & k_{t2} \end{bmatrix} \right) \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} e^{st} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \text{---- } (E_3)$$

The characteristic equation becomes:

$$\begin{vmatrix} J_1 s^2 + (c_{t1}+c_{t2})s + k_{t1}+k_{t2} & -(c_{t2}s + k_{t2}) \\ -(c_{t2}s + k_{t2}) & J_2 s^2 + c_{t2}s + k_{t2} \end{vmatrix} = 0$$

i.e.,

$$a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0 \qquad \text{--- } (E_4)$$

where

$$a_0 = J_1 J_2$$

$$a_1 = J_1 c_{t2} + J_2 (c_{t1} + c_{t2})$$

$$a_2 = J_1 k_{t2} + c_{t2}(c_{t1} + c_{t2}) + J_2(k_{t1} + k_{t2}) - c_{t2}^2$$

$$a_3 = k_{t2}(c_{t1} + c_{t2}) + c_{t2}(k_{t1} + k_{t2}) - 2\, c_{t2} k_{t2}$$

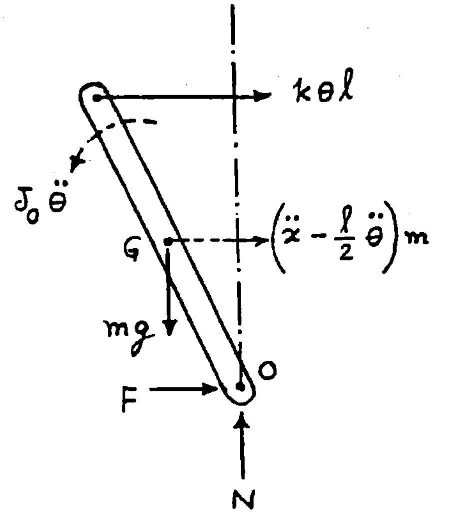
$$a_4 = k_{t2}(k_{t1} + k_{t2}) - k_{t2}^2$$

For the stability of the system, the conditions derived in section 5.8 are applicable:

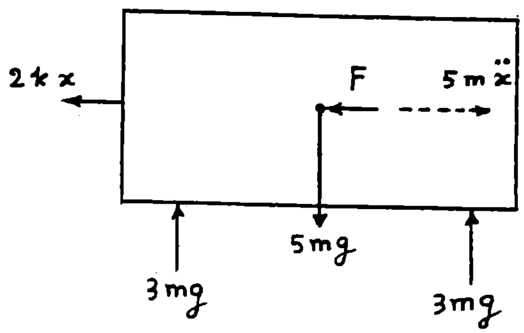
$$a_i > 0 \quad ; \quad i = 0, 1, 2, 3, 4$$

$$a_1 a_2 a_3 - a_1^2 a_4 - a_0 a_3^2 > 0$$

5.66



Free body diagram of bar



Free body diagram of trailer

Equations of motion of bar:

$$m\left(\ddot{x}-\frac{\ell}{2}\ddot{\theta}\right)=k\,\theta\,\ell+F \tag{1}$$

$$J_0\,\ddot{\theta}-m\left(\ddot{x}-\frac{\ell}{2}\ddot{\theta}\right)\frac{\ell}{2}=-k\,\theta\,\ell\,(\ell)+m\,g\,\frac{\ell}{2}\sin\theta \tag{2}$$

Equation of motion of trailer:

$$5\,m\,\ddot{x}=-F-2\,k\,x \quad \text{or} \quad F=-2\,k\,x-5\,m\,\ddot{x} \tag{3}$$

Equations (1) and (2) can be rewritten as:

$$m\left(\ddot{x}-\frac{\ell}{2}\ddot{\theta}\right)-k\,\theta\,\ell+2\,k\,x+5\,m\,\ddot{x}=0 \tag{5}$$

$$J_0\,\ddot{\theta}-\frac{m\,\ell}{2}\,(\ddot{x}-\frac{\ell}{2}\,\ddot{\theta})+k\,\theta\,\ell^2-\frac{m\,g\,\ell\,\theta}{2}=0 \tag{6}$$

$$\text{where}\quad J_0=\frac{1}{3}\,m\,\ell^2 \tag{7}$$

Equations (5) and (6) can be expressed in matrix form as:

$$\begin{bmatrix}6\,m & -\dfrac{m\,\ell}{2}\\ -\dfrac{m\,\ell}{2} & \dfrac{7}{12}\,m\,\ell^2\end{bmatrix}\begin{Bmatrix}\ddot{x}\\ \ddot{\theta}\end{Bmatrix}+\begin{bmatrix}2\,k & -k\,\ell\\ 0 & (k\,\ell^2-\dfrac{1}{2}\,m\,g\,\ell)\end{bmatrix}\begin{Bmatrix}x\\ \theta\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix} \tag{8}$$

Assuming a solution of the form:

$$x(t)=X\,e^{s\,t}\quad\text{and}\quad \theta(t)=\Theta\,e^{s\,t} \tag{9}$$

Eq. (8) can be expressed as:

$$\left[s^2\begin{bmatrix}6\,m & -\dfrac{m\,\ell}{2}\\ -\dfrac{m\,\ell}{2} & \dfrac{m\,\ell^2}{3}\end{bmatrix}+\begin{bmatrix}2\,k & -k\,\ell\\ 0 & -\left(\dfrac{m\,g\,\ell}{2}-k\,\ell^2\right)\end{bmatrix}\right]\begin{Bmatrix}X\\ \Theta\end{Bmatrix}e^{s\,t}=\begin{Bmatrix}0\\ 0\end{Bmatrix} \tag{10}$$

By setting the determinant of the coefficient matrix in Eq. (10) equal to zero, we obtain:

$$\begin{vmatrix}(6\,m\,s^2+2\,k) & -\left(\dfrac{m\,\ell\,s^2}{2}+k\,\ell\right)\\ -\left(\dfrac{m\,\ell\,s^2}{2}\right) & \left(\dfrac{m\,\ell^2\,s^2}{3}-\dfrac{m\,g\,\ell}{2}+k\,\ell^2\right)\end{vmatrix}=0 \tag{11}$$

which, upon expansion, gives:

$$\left(\frac{7}{4}\,m^2\,\ell^2\right)s^4+\left(\frac{37}{6}\,m\,k\,\ell^2-3\,m^2\,g\,\ell\right)s^2+\left(-\,m\,k\,g\,\ell+2\,k^2\,\ell^2\right)=0 \tag{12}$$

A comparison of Eq. (12) with Eq. (5.43) gives:

$$a_0=\frac{7}{4}\,m^2\,\ell^2$$
$$a_1=0$$
$$a_2=\frac{37}{6}\,m\,k\,\ell^2-3\,m^2\,g\,\ell$$
$$a_3=0$$
$$a_4=2\,k^2\,\ell^2-m\,k\,g\,\ell$$

Conditions for the stability of the system:

1. All coefficients a_i must be positive:

$$a_2 \geq 0 \text{ or } k \geq \frac{18}{37}\frac{m\,g}{\ell}$$
$$a_4 \geq 0 \text{ or } k \geq \frac{1}{2}\frac{m\,g}{\ell}$$

2.

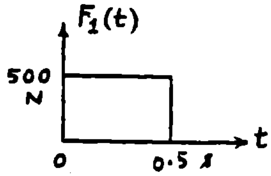
$$a_1\, a_2\, a_3 > a_0\, a_3^2 + a_4\, a_1^2$$

This is not applicable since both sides of the inequality are zero.

Thus the condition for stability is: $k \geq \frac{1}{2}\frac{m\,g}{\ell}$.

5.67 Equations of motion are (Eqs. (5.1) and (5.2))

$$2m\,\ddot{x}_1 + 3k\,x_1 - 2k\,x_2 = F_1(t)$$
$$m\,\ddot{x}_2 - 2k\,x_1 + 3k\,x_2 = 0$$



Hence $[m] = \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} = \begin{bmatrix} 20 & 0 \\ 0 & 10 \end{bmatrix}$, $[k] = \begin{bmatrix} 3k & -2k \\ -2k & 3k \end{bmatrix} = \begin{bmatrix} 6000 & -4000 \\ -4000 & 6000 \end{bmatrix}$,

$\vec{F} = \begin{Bmatrix} F_1(t) \\ 0 \end{Bmatrix}$

Frequency equation is $\left| -\omega^2 [m] + [k] \right| = \omega^4 - 900\,\omega^2 + 100\,000 = 0$

Hence $\omega_1 = 11.3949$ rad/s, $\omega_2 = 27.7517$ rad/s

and $\tau_1 = 0.5514$ s, $\tau_2 = 0.2264$ s.

We select $\Delta t = 0.02$ s and use central difference method for numerical solution (see chapter 11 for details).

The main program which calls CDIFF, the subroutine EXTFUN and the output are given below [CDIFF is in Program 15.F]:

```
C ======================================================================
C
C PROGRAM
C MAIN PROGRAM WHICH CALLS CDIFF
C
C ======================================================================
C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
      REAL M(2,2),K(2,2),MC(2,2),MK(2,2),MCI(2,2),MMC(2,2)
      DIMENSION C(2,2),XI(2),XDI(2),XDDI(2),XM1(2),F(2),R(2),RR(2),
     2 XMK(2),XMI(2),XM2(2),XP1(2),ZA(2),ZB(2),ZC(2),LA(2),LB(2,2),
     3 S(2),X(50,2),XD(50,2),XDD(50,2)
      DATA N,NSTEP,NSTEP1,DELT/2,49,50,0.02/
      DATA XI/0.0,0.0/
      DATA XDI/0.0,0.0/
      DATA M/20.0,0.0,0.0,10.0/
      DATA C/0.0,0.0,0.0,0.0/
      DATA K/6000.0,-4000.0,-4000.0,6000.0/
C END OF PROBLEM-DEPENDENT DATA
      CALL CDIFF (M,C,K,XI,XDI,XDDI,N,NSTEP,DELT,F,R,RR,XM1,XM2,XP1,
     2 MC,MK,MCI,XMK,MMC,XMI,ZA,ZB,ZC,LA,LB,S,X,XD,XDD,NSTEP1)
```

```
      WRITE (13,10)
10    FORMAT (//,38H SOLUTION BY CENTRAL DIFFERENCE METHOD,/)
      WRITE (13,20) N,NSTEP,DELT
20    FORMAT (12H GIVEN DATA:,/,3H N=,I5,4X,7H NSTEP=,I5,4X,6H DELT=,
     2 E15.8,/)
      WRITE (13,30)
30    FORMAT (10H SOLUTION:,//,5H STEP,3X,5H TIME,3X,7H X(I,1),3X,
     2 8H XD(I,1),2X,9H XDD(I,1),4X,7H X(I,2),3X,8H XD(I,2),2X,
     3 9H XDD(I,2),/)
      DO 40, I=1,NSTEP1
      TIME=REAL(I-1)*DELT
40    WRITE (13,50) I,TIME,X(I,1),XD(I,1),XDD(I,1),X(I,2),XD(I,2),XDD(
     2 I,2)
50    FORMAT (1X,I4,F8.4,6(1X,E10.4))
      STOP
      END
C =====================================================================
C
C SUBROUTINE EXTFUN
C THIS SUBROUTINE IS PROBLEM-DEPENDENT
C
C =====================================================================
      SUBROUTINE EXTFUN (F,TIME,N)
      DIMENSION F(N)
      F(1)=0.0
      F(2)=0.0
      IF (TIME .LE. 0.5) F(1)=500.0
      RETURN
      END
```

```
SOLUTION BY CENTRAL DIFFERENCE METHOD

GIVEN DATA:
N=     2     NSTEP=    49     DELT= 0.20000000E-01

SOLUTION:
```

STEP	TIME	X(I,1)	XD(I,1)	XDD(I,1)	X(I,2)	XD(I,2)	XDD(I,2)
1	0.0000	0.0000E+00	0.0000E+00	0.2500E+02	0.0000E+00	0.0000E+00	0.0000E+00
2	0.0200	0.5000E-02	0.0000E+00	0.2500E+02	0.0000E+00	0.0000E+00	0.0000E+00
3	0.0400	0.1940E-01	0.4850E+00	0.2350E+02	0.8000E-03	0.2000E-01	0.2000E+01
4	0.0600	0.4154E-01	0.9134E+00	0.1934E+02	0.4512E-02	0.1128E+00	0.7280E+01
5	0.0800	0.6905E-01	0.1241E+01	0.1344E+02	0.1379E-01	0.3247E+00	0.1391E+02
6	0.1000	0.9938E-01	0.1446E+01	0.7043E+01	0.3080E-01	0.6572E+00	0.1935E+02
7	0.1200	0.1302E+00	0.1530E+01	0.1347E+01	0.5632E-01	0.1063E+01	0.2127E+02
8	0.1400	0.1600E+00	0.1515E+01	-.2810E+01	0.8917E-01	0.1459E+01	0.1831E+02
9	0.1600	0.1877E+00	0.1436E+01	-.5164E+01	0.1262E+00	0.1747E+01	0.1050E+02
10	0.1800	0.2129E+00	0.1323E+01	-.6059E+01	0.1630E+00	0.1846E+01	-.6577E+00
	⋮						
46	0.9000	0.2345E+00	-.1023E+01	-.8737E+01	0.1991E+00	-.5925E+00	-.2714E+02
47	0.9200	0.2101E+00	-.1165E+01	-.5533E+01	0.1715E+00	-.1120E+01	-.2565E+02
48	0.9400	0.1842E+00	-.1258E+01	-.3717E+01	0.1365E+00	-.1566E+01	-.1889E+02
49	0.9600	0.1571E+00	-.1325E+01	-.2966E+01	0.9808E-01	-.1837E+01	-.8195E+01
50	0.9800	0.1290E+00	-.1379E+01	-.2515E+01	0.6131E-01	-.1879E+01	0.3993E+01

5.68 $m_1 = m_2 = 40$ kg, $k_1 = k_2 = 3000$ N/m,
$k_3 = 0$, $c_1 = c_2 = c_3 = 0$, $X_1(0) = X_2(0) = 0.05$ m, $\dot{X}_1(0) = \dot{X}_2(0) = 0$.

Solution:

(a) Frequency equation is:

$$\left|-\omega^2 [m] + [K]\right| = \left|-\omega^2 \begin{bmatrix} 40 & 0 \\ 0 & 40 \end{bmatrix} + \begin{bmatrix} 6000 & -3000 \\ -3000 & 3000 \end{bmatrix}\right| = 0$$

$$\omega^4 - 225\,\omega^2 + 5625 = 0$$

```
PROGRAM 6.F
MAIN PROGRAM FOR CALLING THE SUBROUTINE QUART

SOLUTION OF: A(1)*(X**4)+A(2)*(X**3)+A(3)*(X**2)+A(4)*X+A(5)=0
          DIMENSION A(5),RR(4),RI(4)
FOLLOWING LINE CONTAINS PROBLEM-DEPENDENT DATA
          DATA A/1.0,0.0,-225.0,0.0,5625.0/
END OF PROBLEM-DEPENDENT DATA
          WRITE (26,10) (A(I),T=1,5)
10        FORMAT (//,31H SOLUTION OF A QUARTIC EQUATION,//,6H
          DATA:,/,
   2      7H A(1) =,E15.6,/,7H A(2) =,E15.6,/,7H A(3) =,E15.6,/,
   3      7H A(4) =,E15.6,/,7H A(5) =,E15.6,/)
          CALL QUART (A,RR,RI)
          WRITE (26,20)
20        FORMAT (/,7H ROOTS:,//,9H ROOT NO.,3X,10H REAL PART,
          5X,
   2      15H IMAGINARY PART,/)
          DO 30 I=1,4
30        WRITE (26,40) I, RR(I),RI(I)
40        FORMAT (I5,3X,E15.6,3X,E15.6)
          STOP
          END
SOLUTION OF A QUARTIC EQUATION

DATA:
A(1) =    0.100000E+01
A(2) =    0.000000E+00
A(3) =    -0.225000E+03
A(4) =    0.000000E+00
A(5) =    0.562500E+04

ROOTS:

ROOT NO.      REAL PART     IMAGINARY PART
    1        -0.140126E+02    0.000000E+00
    2        -0.535230E+01    0.000000E+00
    3         0.535230E+01    0.000000E+00
    4         0.140126E+02    0.000000E+00
```

(b) $(-40\,\omega_j^2 + 6000)\,X_j^{(1)} - 3000\,X_j^{(2)} = 0;\ j = 1.2$

Writing $X_j^{(2)} = \left(\dfrac{-40\,\omega_j^2 + 6000}{3000}\right) X_j^{(1)} = r_j\, X_j^{(1)}$

$$r_1 = \frac{\{-40\,(5.3523)^2 + 6000\}}{3000} = 1.6180384$$

$$r_2 = \frac{\{-40\,(14.0126)^2 + 6000\}}{3000} = -\,0.6180395$$

Eqs. (5.18) give

$$X_1^{(1)} = -0.03618 \text{ m},$$

$$X_1^{(2)} = -0.01382 \text{ m},$$

$$\phi_1 = \phi_2 = 0$$

Displacements of masses m_1 and m_2 are given by Eqs. (5.15):

$$X_1\,(t) = -0.03618 \cos\,(5.3523\ t) - 0.01382 \cos\,(14.0126\ t)$$

$$X_2\,(t) = -0.05854 \cos\,(5.3523\ t) - 0.008541 \cos\,(14.0126\ t)$$

5.69

```
      DIMENSION C(2,2),FZ(2)
      REAL K(2,2),M(2,2)
      COMPLEX Z(2,2),X(2),AA,BB,DEN
C INPUT DATA
      DATA  M/20.0,0.0,0.0,20.0/
      DATA  C/200,0.0,0.0,0.0/
      DATA  K/7000.0,-35000.0,-35000.0,7000.0/
      DATA  FZ/5.0,10.0/
            OMF=5.0
C END OF INPUT DATA
      DO 10 I=1,2
      DO 10 J=1.2
      A=-(OMF**2)*M(I,J)+K(I,J)
      B=OMF*C(I,J)
10    Z(I,J)=CMPLX(A,B)
      DEN=Z(1,1)*Z(2,2)-Z(1,2)*Z(1,2)
      AA=Z(2,2)*CMPLX(FZ(1),0.0)
      BB=Z(1,2)*CMPLX(FZ(2),0.0)
      X(1)=(AA-BB)/DEN
      AA=Z(1,1)*CMPLX(FZ(2),0.0)
      BB=Z(1,2)*CMPLX(FZ(1),0.0)
      X(2)=(AA-BB)/DEN
      PRINT 20,X(1),X(2)
20    FORMAT (//,2X,25H SOLUTION OF PROBLEM 5.49,//,2X,
      27HX(1) =,2E15.8,/,2X,7H X(2) =,2E15.8,/)
      PRINT 30, OMF
30    FORMAT (2X,6H OMF=,E15.8)
      STOP
      END
SOLUTION OF PROBLEM 5.69
      X(1)=0.58883249E-02-0.24365482E-02
      X(2)=0.10203046E-01-0.28426396E-02
      OMF=0.50000000E+01
```

5.70 Free vibration response of the system shown in Fig. 5.16:

$k_1 = 1000,\quad k_2 = 500,\quad m_1 = 2,\quad m_2 = 1,$

$x_1(0) = 1,\quad x_2(0) = 0,\quad \dot{x}_1(0) = -1,\quad \dot{x}_2(0) = 0$

$$\omega_1^2, \omega_2^2 = \frac{k_1+k_2}{2m_1} + \frac{k_2}{2m_2} \mp \left\{\frac{1}{4}\left(\frac{k_1+k_2}{m_1} + \frac{k_2}{m_2}\right)^2 - \frac{k_1 k_2}{m_1 m_2}\right\}^{\frac{1}{2}}$$

$$= \frac{1500}{4} + \frac{500}{2} \mp \left\{\frac{1}{4}\left(\frac{1500}{2} + \frac{500}{1}\right)^2 - \frac{5\times10^5}{2}\right\}^{\frac{1}{2}}$$

$$= 250;\ 1000$$

$$\omega_1 = 15.8114 \text{ rad/s},\quad \omega_2 = 31.6228 \text{ rad/s} \tag{E1}$$

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{k_2}{-m_2\omega_1^2 + k_2} = \frac{500}{-1(250)+500} = 2$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{k_2}{-m_2\omega_2^2 + k_2} = \frac{500}{-1(1000)+500} = -1 \tag{E2}$$

$$x_1(t) = X_1^{(1)}\cos(\omega_1 t + \phi_1) + X_1^{(2)}\cos(\omega_2 t + \phi_2)$$

$$x_2(t) = r_1 X_1^{(1)}\cos(\omega_1 t + \phi_1) + r_2 X_1^{(2)}\cos(\omega_2 t + \phi_2) \tag{E3}$$

Initial conditions yield:

$$x_1(0) = 1 = X_1^{(1)}\cos(15.8114\,t + \phi_1) + X_1^{(2)}\cos(31.6228\,t + \phi_2)$$

$$x_2(0) = 0 = 2X_1^{(1)}\cos(15.8114\,t + \phi_1) - X_1^{(2)}\cos(31.6228\,t + \phi_2)$$

$$\dot{x}_1(0) = -1 = -\omega_1 X_1^{(1)}\sin(15.8114\,t + \phi_1) - \omega_2 X_1^{(2)}\sin(31.6228\,t + \phi_2)$$

$$\dot{x}_2(0) = 0 = -r_1\omega_1 X_1^{(1)}\sin(15.8114\,t + \phi_1) - r_2\omega_2 X_1^{(2)}\sin(31.6228\,t + \phi_2)$$

or (E4)

$$X_1^{(1)}\cos\phi_1 + X_1^{(2)}\cos\phi_2 = 1 \tag{E5}$$

$$2X_1^{(1)}\cos\phi_1 - X_1^{(2)}\cos\phi_2 = 0 \tag{E6}$$

$$-15.8114\,X_1^{(1)}\sin\phi_1 - 31.6228\,X_1^{(2)}\sin\phi_2 = -1 \tag{E7}$$

$$-31.6228\,X_1^{(1)}\sin\phi_1 + 31.6228\,X_1^{(2)}\sin\phi_2 = 0 \tag{E8}$$

Solution of Eqs. (E5) and (E6):

$$X_1^{(1)}\cos\phi_1 = \frac{1}{3} \tag{E9}$$

$$X_1^{(2)}\cos\phi_2 = \frac{2}{3} \tag{E10}$$

Solution of Eqs. (E_7) and (E_8):

$$X_1^{(1)} \sin \phi_1 = 0.02108 \qquad (E_{11})$$

$$X_1^{(2)} \sin \phi_2 = 0.02108 \qquad (E_{12})$$

Eqs. (E_9) and (E_{11}) yield: $X_1^{(1)} = 0.334$, $\phi_1 = 0.06316$ rad

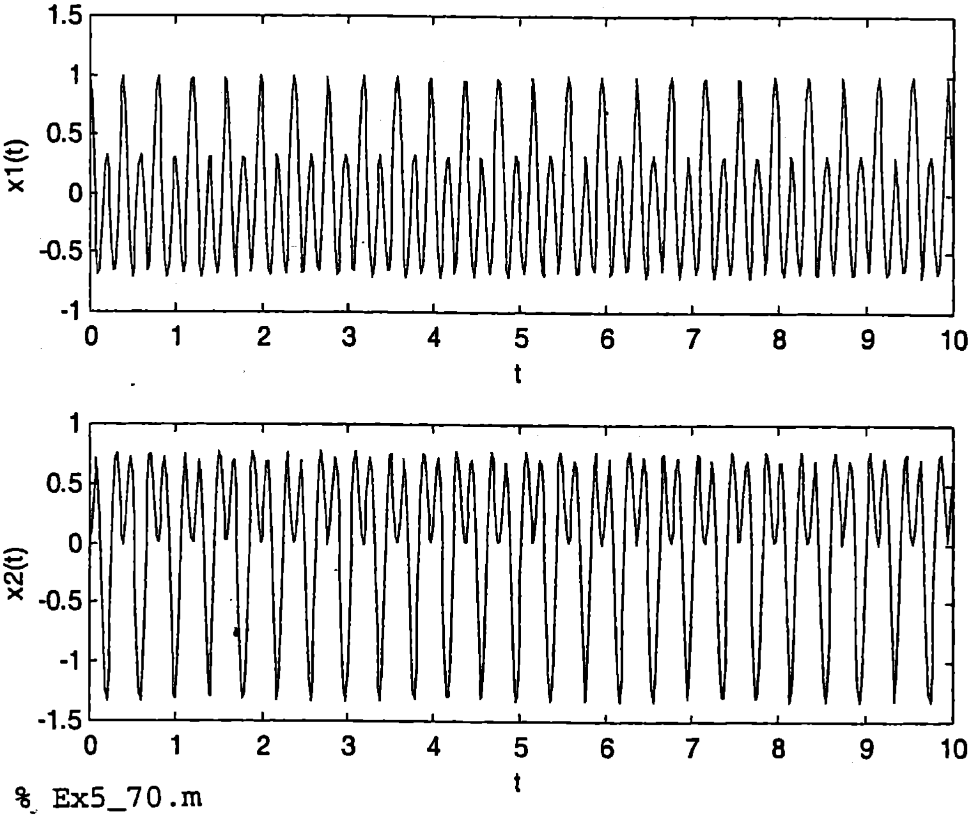
Eqs. (E_{10}) and (E_{12}) yield: $X_1^{(2)} = 0.667$, $\phi_2 = 0.03161$ rad

Response:

$$x_1(t) = 0.334 \cos(15.8114\,t + 0.06316) + 0.667 \cos(31.6228\,t + 0.03161) \qquad (E_{13})$$

$$x_2(t) = 0.668 \cos(15.8114\,t + 0.06316) - 0.667 \cos(31.6228\,t + 0.03161) \qquad (E_{14})$$

Plotting of Eqs. (E_{13}) and (E_{14}):




```
% Ex5_70.m
for i = 1: 501
   t(i) = 10 * (i-1)/500;
   x1(i) = 0.334 * cos(15.8114*t(i) + 0.06316)...
      + 0.667 * cos(31.6228*t(i) + 0.03161);
   x2(i) = 0.668 * cos(15.8114*t(i) + 0.06316)...
      - 0.667 * cos(31.6228*t(i) + 0.03161);
```

```
end
subplot(211);
plot(t,x1);
xlabel('t');
ylabel('x1(t)')
subplot(212);
plot(t,x2);
xlabel('t');
ylabel('x2(t)')
```

5.71 For the initial conditions

$x_1(0) = 1, \quad x_2(0) = 2, \quad \dot{x}_1(0) = 1$ and $\dot{x}_2(0) = -2$,

Eqs. (E_3) of Solution of Problem 5.70 yield

$$x_1(0) = 1 = X_1^{(1)} \cos\phi_1 + X_1^{(2)} \cos\phi_2 \qquad (E_1)$$

$$x_2(0) = 2 = 2\, X_1^{(1)} \cos\phi_1 - X_1^{(2)} \cos\phi_2 \qquad (E_2)$$

$$\dot{x}_1(0) = 1 = -15.8114\, X_1^{(1)} \sin\phi_1 - 31.6228\, X_1^{(2)} \sin\phi_2 \qquad (E_3)$$

$$\dot{x}_2(0) = -2 = -31.6228\, X_1^{(1)} \sin\phi_1 + 31.6228\, X_1^{(2)} \sin\phi_2 \qquad (E_4)$$

Eqs. (E_1) and (E_2) give:

$$X_1^{(1)} \cos\phi_1 = 1, \qquad X_1^{(2)} \cos\phi_2 = 0 \qquad (E_5)$$

Eqs. (E_3) and (E_4) yield

$$X_1^{(1)} \sin\phi_1 = 0.02108, \qquad X_1^{(2)} \sin\phi_2 = -0.04216 \qquad (E_6)$$

Equations (E_5) and (E_6) can be used to obtain

$$X_1^{(1)} = 1.000222, \quad \phi_1 = 0.02108 \text{ rad}$$

$$X_1^{(2)} = 0.04216, \quad \phi_2 = \frac{\pi}{2} \text{ rad}$$

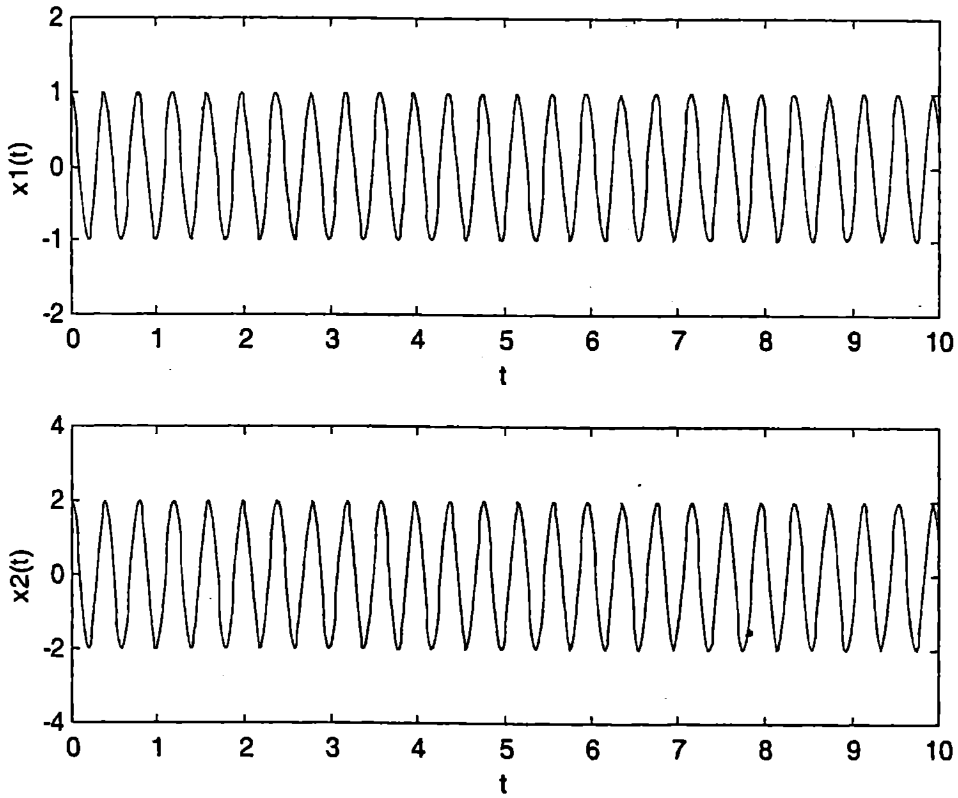
Response of the system:

$$x_1(t) = 1.000222 \cos(15.8114\, t + 0.02108) + 0.04216 \cos\left(31.6228\, t + \frac{\pi}{2}\right) \qquad (E_7)$$

$$x_2(t) = 2.000444 \cos(15.8114\, t + 0.02108) - 0.04216 \cos\left(31.6228\, t + \frac{\pi}{2}\right) \qquad (E_8)$$

Plotting of Eqs. (E_7) and (E_8):

```
% Ex5_71.m
for i = 1: 501
   t(i) = 10 * (i-1)/500;
   x1(i) = 1.000222 * cos(15.8114*t(i) + 0.02108)...
      + 0.04216 * cos(31.6228*t(i) + pi/2);
   x2(i) = 2.000444 * cos(15.8114*t(i) + 0.02108)...
      - 0.04216 * cos(31.6228*t(i) + pi/2);
end
subplot(211);
plot(t,x1);
xlabel('t');
ylabel('x1(t)')
subplot(212);
plot(t,x2);
xlabel('t');
ylabel('x2(t)')
```



5.72

```
% Ex5_72.m
>>A = 1e6*[25 -5; -5 5]
A =
    25000000    -5000000
    -5000000     5000000
>>B = [10000 0; 0 5000]
B =
       10000           0
           0        5000
>>[V, D] = eig(A, B)
V =
    0.8719    0.2703
   -0.4896    0.9628
D =
  1.0e+003 *
    2.7808         0
         0    0.7192
```

5.73 Differential equations:

$$2\ddot{x}_1 + 20\dot{x}_1 - 5\dot{x}_2 + 50x_1 - 10x_2 = 2\sin 3t \qquad (E_1)$$

$$10\ddot{x}_2 - 5\dot{x}_1 + 5\dot{x}_2 - 10x_1 + 10x_2 = 5\cos 5t \qquad (E_2)$$

Let
$$y_1 = x_1$$
$$\dot{y}_1 = y_2 = \dot{x}_1$$
$$y_3 = x_2$$
$$\dot{y}_3 = y_4 = \dot{x}_2$$

Equations (E_1) and (E_2) can be rewritten as

$$2\dot{y}_2 + 20y_2 - 5y_4 + 50y_1 - 10y_3 = 2\sin 3t$$

$$10\dot{y}_4 - 5y_2 + 5y_4 - 10y_1 + 10y_3 = 5\cos 5t$$

or

$$\frac{d}{dt}\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = \begin{Bmatrix} y_2 \\ -10y_2 + 2.5y_4 - 25y_1 + 5y_3 + \sin 3t \\ y_4 \\ 0.5y_2 - 0.5y_4 + y_1 - y_3 + 0.5\cos 5t \end{Bmatrix} \qquad (E_3)$$

or
$$\dot{\vec{y}} = \vec{f} \qquad (E_4)$$

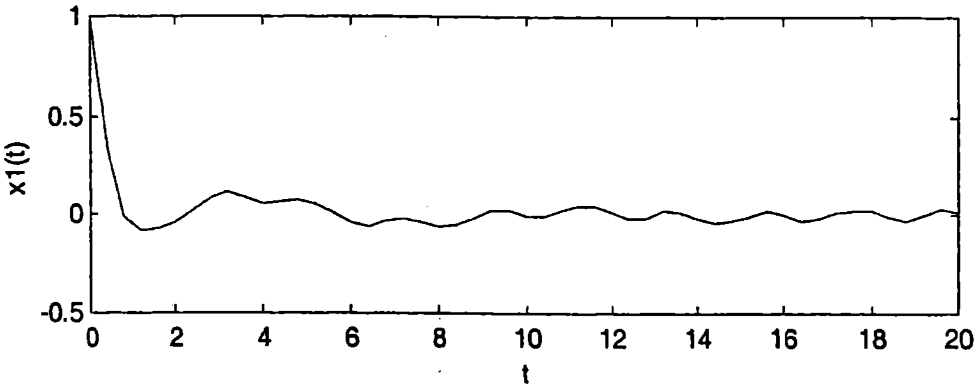
with
$$\vec{y} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}, \quad \vec{y}(0) = \begin{Bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{Bmatrix}$$

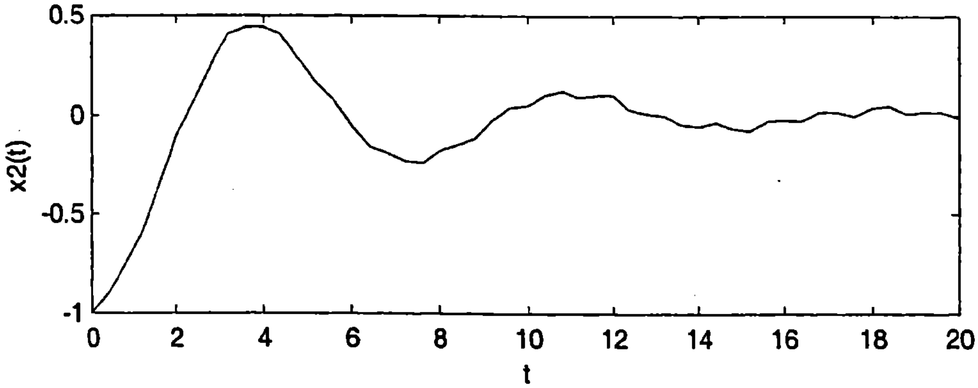
and $\vec{f}$ is given by the right hand side of Eq. (E_3).

Solution of Eq. (E_4) using MATLAB:

```
% Ex5_73.m
% This program will use the function dfun5_73.m, they should
% be in the same folder
tspan = [0: 0.4: 20];
y0 = [1; 0; -1; 0];
[t,y] = ode23('dfun5_73', tspan, y0);
disp('        t        x1(t)     xd1(t)     x2(t)      xd2(t)');
disp([t y]);
subplot(211);
plot(t,y(:,1));
xlabel('t');
ylabel('x1(t)');
subplot(212);
plot(t,y(:,3));
xlabel('t');
ylabel('x2(t)');

% dfun5_73.m
function f = dfun5_73(t,y)
f = zeros(4,1);
f(1) = y(2);
f(2) = -10*y(2) + 2.5*y(4) - 25*y(1) + 5*y(3) + sin(3*t);
f(3) = y(4);
f(4) = 0.5*y(2) - 0.5*y(4) + y(1) - y(3) + 0.5*cos(5*t);
```





Results of Ex5_73
****

```
>>Ex5_73
        t       x1(t)      xd1(t)      x2(t)      xd2(t)
        0      1.0000           0    -1.0000           0
   0.4000      0.3177     -1.4828    -0.8995      0.3577
   0.8000     -0.0076     -0.3482    -0.7604      0.3375
   1.2000     -0.0763     -0.0594    -0.5974      0.5230
   1.6000     -0.0741      0.0612    -0.3445      0.6835
   2.0000     -0.0356      0.1222    -0.1033      0.4929
   2.4000      0.0214      0.1625     0.0716      0.4371
   .
   .
   .
  18.8000     -0.0268      0.0072     0.0066     -0.0481
  19.2000      0.0010      0.1087     0.0196      0.0720
  19.6000      0.0331      0.0247     0.0233     -0.0721
  20.0000      0.0178     -0.0827    -0.0117     -0.0459
```

5.74 Equations:

$$2m\,\ddot{x}_1 + 3k\,x_1 - 2k\,x_2 = F_1(t)$$
$$m\,\ddot{x}_2 - 2k\,x_1 + 3k\,x_2 = 0$$

i.e., $\ddot{x}_1 = -300\,x_1 + 200\,x_2 + \frac{1}{20}F_1(t)$

$\ddot{x}_2 = 400\,x_1 - 600\,x_2$

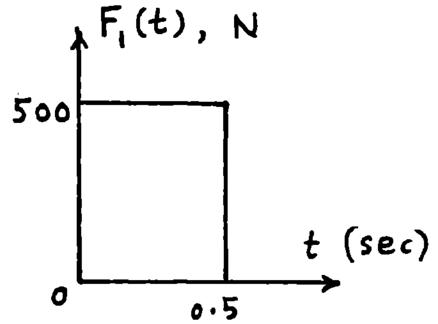
Let

$$\vec{y} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{Bmatrix} \quad \text{and} \quad \vec{y}(0) = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \text{zero initial conditions assumed}$$

Then equations to be solved are:

$$\frac{d\vec{y}}{dt} = \begin{Bmatrix} y_2 \\ -300\,y_1 + 200\,y_3 + \frac{1}{20}F_1(t) \\ y_4 \\ 400\,y_1 - 600\,y_3 \end{Bmatrix} \qquad (E_1)$$

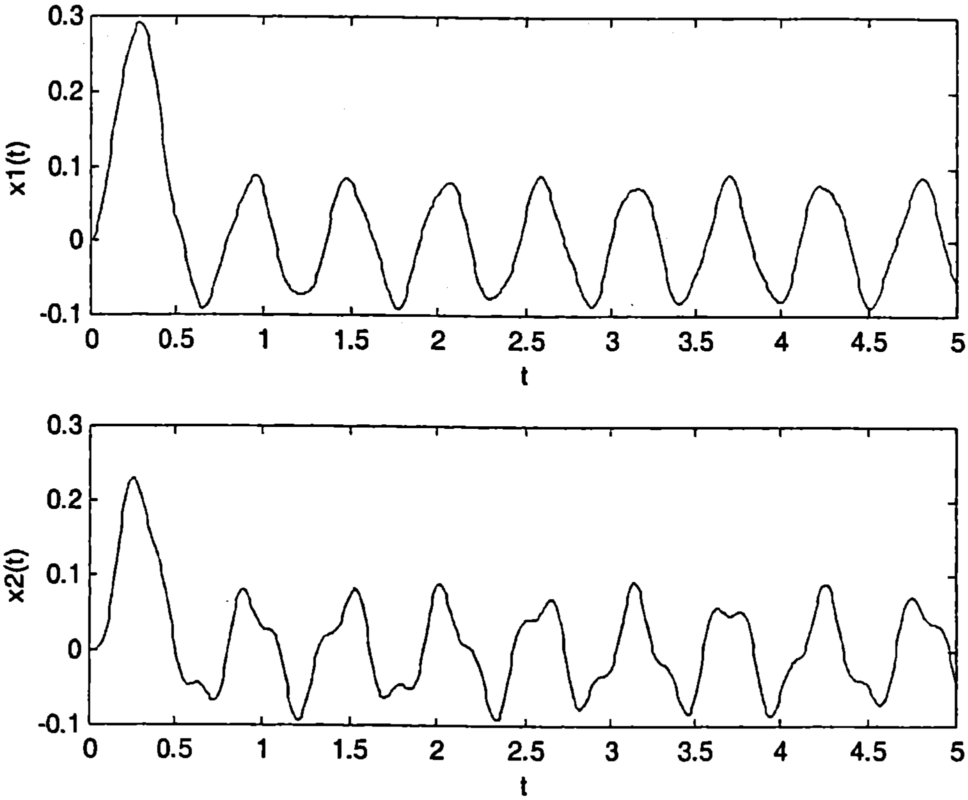
with $F_1(t)$ shown in the figure:



MATLAB Solution of Eq. (E_1):

```
% Ex5_74.m
% This program will use the function dfun5_74.m, they should
% be in the same folder
tspan = [0: 0.01: 5];
y0 = [0; 0; 0; 0];
[t,y] = ode23('dfun5_74', tspan, y0);
subplot(211);
plot(t,y(:,1));
xlabel('t');
ylabel('x1(t)');
subplot(212);
plot(t,y(:,3));
xlabel('t');
ylabel('x2(t)');

% dfun5_74.m
function f = dfun5_74(t,y)
F1 = 500 * stepfun(t, 0.0) - 500 * stepfun(t, 0.5);
f = zeros(4,1);
f(1) = y(2);
f(2) = -300*y(1) + 200*y(3) + F1/20;
f(3) = y(4);
f(4) = 400*y(1) - 600*y(3);
```



5.75 Frequency equation, Eq. (5.9):

$$m_1 m_2 \omega^4 - \{(k_1+k_2)m_2 + (k_2+k_3)m_1\}\omega^2 + \{(k_1+k_2)(k_2+k_3) - k_2^2\} = 0 \qquad (E_1)$$

With $m_1 = m_2 = 0.2$, $k_1 = k_2 = 18$ and $k_3 = 0$, Eq. (E_1) becomes

$$0.04\,\omega^4 - 10.8\,\omega^2 + 324 = 0$$

or $$\omega^4 - 270\,\omega^2 + 8100 = 0 \qquad (E_2)$$

Solution of Eq. (E_2) using MATLAB:

```
% Ex5_75.m
>>roots([1 0 -270 0 8100])
ans =
  15.35001820805078
 -15.35001820805078
   5.86318522754564
  -5.86318522754564
```

5.76
$$\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \qquad (E_1)$$

$$F_j(t) = F_{j0}\, e^{i\omega t} \; ; \; j = 1,2 \; ; \; i = \sqrt{-1} \qquad (E_2)$$

$$x_j(t) = X_j\, e^{i\omega t} \; ; \; j = 1,2 \qquad (E_3)$$

Eqs. (5.35):

$$X_1(i\omega) = \frac{Z_{22}(i\omega)\,F_{10} - Z_{12}(i\omega)\,F_{20}}{Z_{11}(i\omega)\,Z_{22}(i\omega) - Z_{12}^2(i\omega)} \qquad (E_4)$$

$$X_2(i\omega) = \frac{-Z_{12}(i\omega)\,F_{10} + Z_{11}(i\omega)\,F_{20}}{Z_{11}(i\omega)\,Z_{22}(i\omega) - Z_{12}^2(i\omega)} \qquad (E_5)$$

where
$$Z_{rs}(i\omega) = -\omega^2 m_{rs} + i\omega c_{rs} + k_{rs} \; ; \; r, s = 1,2 \qquad (E_6)$$

<u>Data:</u> $m_{11} = m_{22} = 0.1$, $m_{12} = 0$, $c_{11} = 1.0$, $c_{12} = c_{22} = 0$, $k_{11} = 40$, $k_{22} = 20$, $k_{12} = -20$, $F_{10} = 1$, $F_{20} = 2$, $\omega = 5$

Hence $\quad Z_{11}(i\omega) = 37.5 + 5i$, $Z_{12}(i\omega) = -20$

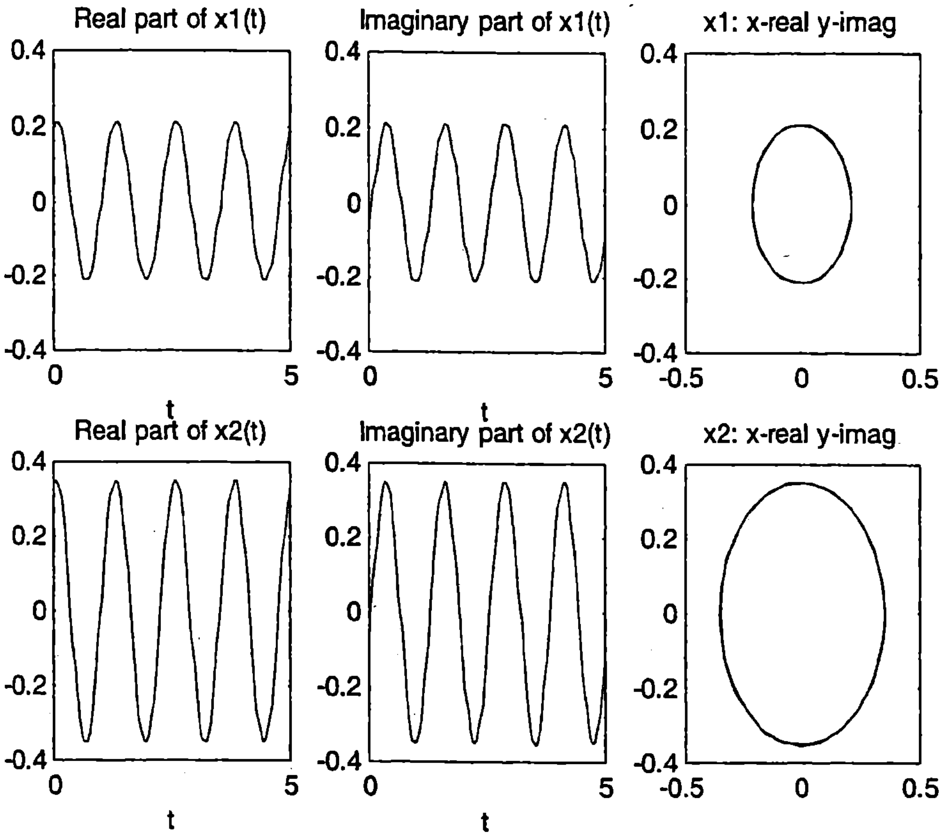
and $Z_{22}(i\omega) = 17.5$

Solution using MATLAB:

(Real and Imaginary parts of $x_1(t)$ and $x_2(t)$ given by Eq. (E_3))

```
% Ex5_76.m
m11 = 0.1;
m22 = 0.1;
m12 = 0;
c11 = 1.0;
c12 = 0;
c22 = 0;
k11 = 40;
k22 = 20;
k12 = -20;
F10 = 1;
F20 = 2;
w = 5;
z11 = complex((-w^2*m11 + k11), w*c11);
z12 = complex((-w^2*m12 + k12), w*c12);
z22 = complex((-w^2*m22 + k22), w*c22);
X1 = (z22*F10 - z12*F20)/(z11*z22 - z12*z12)
X2 = (-z12*F10 + z11*F20)/(z11*z22 - z12*z12)
for i = 1: 101
   t(i) = 5*(i-1)/100;
   x1(i) = X1 * exp(complex(0, w*t(i)));
   x2(i) = X2 * exp(complex(0, w*t(i)));
end
subplot(231);
plot(t, real(x1));
xlabel('t');
title('Real part of x1(t)');
subplot(232);
plot(t, imag(x1));
xlabel('t');
title('Imaginary part of x1(t)');
subplot(233);
plot(real(x1), imag(x1));
title('x1: x-real y-imag');
subplot(234);
plot(t, real(x2));
xlabel('t');
title('Real part of x2(t)');
```

```
subplot(235);
plot(t, imag(x2));
xlabel('t');
title('Imaginary part of x2(t)');
subplot(236);
plot(real(x2), imag(x2));
title('x2: x-real y-imag');
```



(5.77) Roots of the equation:

$$x^4 - 32x^3 + 244x^2 - 20x - 1200 = 0$$

Using MATLAB:

```
% Ex5_77.m
>>roots([1 -32 244 -20 -1200])
ans =
  20.00000000000001
  11.15980239097340
   2.77656274263302
  -1.93636513360642
```

5.78

```
Results of Ex5_78 using Program88.cpp
****************************************
Please input the coefficient array a
(a[0] is the coefficient of x^4 and nonzero):
1.0 0.0 -270 0.0 8100

SOLUTION OF QUARTIC EQUATION

DATA:
A[0] =     1.000000
A[1] =     0.000000
A[2] =  -270.000000
A[3] =     0.000000
A[4] =  8100.000000


ROOTS:

          ROOT NO.        REAL PART   IMAGINARY PART

                 1     -15.35001821       0.00000000
                 2      -5.86318523       0.00000000
                 3       5.86318523       0.00000000
                 4      15.35001821       0.00000000
```

5.79

```
Results of Ex5_79 using Program88.cpp
****************************************
Please input the coefficient array a
(a[0] is the coefficient of x^4 and nonzero):
1.0 -32 244 -20 -1200

SOLUTION OF QUARTIC EQUATION

DATA:
A[0] =     1.000000
A[1] =   -32.000000
A[2] =   244.000000
A[3] =   -20.000000
A[4] = -1200.000000


ROOTS:

          ROOT NO.        REAL PART   IMAGINARY PART

                 1      11.15980239       0.00000000
                 2      20.00000000       0.00000000
                 3      -1.93636513       0.00000000
                 4       2.77656274       0.00000000
```

5.80

```
C =================================================================
C
C PROGRAM 6.F
C MAIN PROGRAM FOR CALLING THE SUBROUTINE QUART
C
C =================================================================
C SOLUTION OF:  A(1)*(X**4)+A(2)*(X**3)+A(3)*(X**2)+A(4)*X+A(5)=0
      DIMENSION A(5),RR(4),RI(4)
C FOLLOWING LINE CONTAINS PROBLEM-DEPENDENT DATA
      DATA A/1.0,0.0,-270.0,0.0,8100.0/
C END OF PROBLEM-DEPENDENT DATA
      PRINT 10,(A(I),I=1,5)
 10   FORMAT (//,31H SOLUTION OF A QUARTIC EQUATION,//,6H DATA:,/,
     2 7H A(1) =,E15.6,/,7H A(2) =,E15.6,/,7H A(3) =,E15.6,/,
     3 7H A(4) =,E15.6,/,7H A(5) =,E15.6,/)
      CALL QUART (A,RR,RI)
      PRINT 20
 20   FORMAT (/,7H ROOTS:,//,9H ROOT NO.,3X,10H REAL PART,5X,
     2 15H IMAGINARY PART,/)
      DO 30 I=1,4
 30   PRINT 40,I,RR(I),RI(I)
 40   FORMAT (I5,3X,E15.6,3X,E15.6)
      STOP
      END
C =================================================================
C
C SUBROUTINE QUART
C
C =================================================================
      SUBROUTINE QUART (A,RR,RI)
      DIMENSION A(5),RR(4),RI(4),B(4),RRC(3),RIC(3)
      DO 10 I=2,5
 10   A(I)=A(I)/A(1)
      B(1)=1.0
      B(2)=-A(3)
      B(3)=A(4)*A(2)-4.0*A(5)
      B(4)=A(5)*(4.0*A(3)-A(2)**2)-A(4)**2
      CALL CUBIC (B,RRC,RIC)
      IF (RIC(2) .NE. 0.0) GO TO 20
      X=AMAX1(RRC(1),RRC(2),RRC(3))
      RRC(1)=X
 20   X=RRC(1)/2.0
      IF ((X**2-A(5)) .GT. 0.0) GO TO 30
      Y=0.0
      Z=SQRT((A(2)/2.0)**2+2.0*X-A(3))
C ADD TO ABOVE EQN.
      GO TO 40
 30   Y=SQRT(X**2-A(5))
      Z=-(A(4)-A(2)*X)/(2.0*Y)
 40   C1=1.0
      C2=A(2)/2.0+Z
      C3=X+Y
      CALL QUADRA (C1,C2,C3,QR1,QR2,QI1,QI2)
      RR(1)=QR1
      RR(2)=QR2
      RI(1)=QI1
      RI(2)=QI2
      C1=1.0
      C2=A(2)/2.0-Z
      C3=X-Y
      CALL QUADRA (C1,C2,C3,QR1,QR2,QI1,QI2)
      RR(3)=QR1
      RR(4)=QR2
      RI(3)=QI1
      RI(4)=QI2
      RETURN
      END
```

```
C ====================================================================
C
C SUBROUTINE QUADRA
C
C ====================================================================
C       SOLUTION OF QUADRATIC EQUATION  A1*(X**2)+A2*(X)+A3 = 0
C       A1,A2,A3 ARE INPUT, (RR1,RI1) AND (RR2,RI2) ARE ROOTS (OUTPUT)
C       A1 MUST NOT BE EQUAL TO ZERO
        SUBROUTINE QUADRA (A1,A2,A3,RR1,RR2,RI1,RI2)
        RAD=A2**2-4.0*A1*A3
        IF (RAD) 20,10,10
 10     SRAD=SQRT(RAD)
        RR1=(-A2-SRAD)/(2.0*A1)
        RR2=(-A2+SRAD)/(2.0*A1)
        RI1=0.0
        RI2=0.0
        RETURN
 20     SRAD=SQRT(-RAD)
        RR1=-A2/(2.0*A1)
        RR2=RR1
        RI1=SRAD/(2.0*A1)
        RI2=-RI1
        RETURN
        END
C ====================================================================
C
C SUBROUTINE CUBIC
C
C ====================================================================
C       ROOTS OF CUBIC EQUATION  A(1)*(X**3)+A(2)*(X**2)+A(3)*X+A(4)=0
        SUBROUTINE CUBIC (A,RR,RI)
        DIMENSION A(4),RR(3),RI(3)
        DO 10 I=1,3
        RR(I)=0.0
 10     RI(I)=0.0
        A0=A(1)
        A1=A(2)/3.0
        A2=A(3)/3.0
        A3=A(4)
        G=(A0**2)*A3-3.0*A0*A1*A2+2.0*(A1**3)
        H=A0*A2-A1**2
        Y1=G**2+4.0*(H**3)
        IF (Y1 .LT. 0.0) GO TO 100
        Y2=SQRT(Y1)
        Z1=(G+Y2)/2.0
        Z2=(G-Y2)/2.0
        IF(Z1 .LT. 0.0) GO TO 21
        Z3=Z1**(1.0/3.0)
        GO TO 22
 21     Z3=(-Z1)**(1.0/3.0)
        Z3=-Z3
 22     IF(Z2 .LT. 0.0) GO TO 23
        Z4=Z2**(1.0/3.0)
        GO TO 24
 23     Z4=(-Z2)**(1.0/3.0)
        Z4=-Z4
 24     CONTINUE
        RR(1)=-(A1+Z3+Z4)/A0
        RR(2)=(-2.0*A1+Z3+Z4)/(2.0*A0)
        RI(2)=SQRT(3.0)*(Z4-Z3)/(2.0*A0)
        RR(3)=RR(2)
        RI(3)=-RI(2)
        GO TO 200
 100    SH=SQRT(-H)
        XK=2.0*SH
        THETA=ACOS(G/(2.0*H*SH))/3.0
        XY1=2.0*SH*COS(THETA)
        PI=3.1416
        XY2=2.0*SH*COS(THETA+(2.0*PI/3.0))
        XY3=2.0*SH*COS(THETA+(4.0*PI/3.0))
```

```
      RR(1)=(XY1-A1)/A0
      RR(2)=(XY2-A1)/A0
      RR(3)=(XY3-A1)/A0
200   RETURN
      END
```

```
SOLUTION OF A QUARTIC EQUATION

DATA:
A(1) =    0.100000E+01
A(2) =    0.000000E+00
A(3) =   -0.270000E+03
A(4) =    0.000000E+00
A(5) =    0.810000E+04


ROOTS:

ROOT NO.     REAL PART       IMAGINARY PART

   1      -0.153500E+02      0.000000E+00
   2      -0.586319E+01      0.000000E+00
   3       0.586319E+01      0.000000E+00
   4       0.153500E+02      0.000000E+00
```

5.81

```
C ==================================================================
C
C PROGRAM 6.F
C MAIN PROGRAM FOR CALLING THE SUBROUTINE QUART
C
C ==================================================================
C SOLUTION OF:  A(1)*(X**4)+A(2)*(X**3)+A(3)*(X**2)+A(4)*X+A(5)=0
      DIMENSION A(5),RR(4),RI(4)
C FOLLOWING LINE CONTAINS PROBLEM-DEPENDENT DATA
      DATA A/1.0,-32.0,244.0,-20.0,-1200.0/
C END OF PROBLEM-DEPENDENT DATA
      PRINT 10,(A(I),I=1,5)
 10   FORMAT (//,31H SOLUTION OF A QUARTIC EQUATION,//,6H DATA:,/,
     2 7H A(1) =,E15.6,/,7H A(2) =,E15.6,/,7H A(3) =,E15.6,/,
     3 7H A(4) =,E15.6,/,7H A(5) =,E15.6,/)
      CALL QUART (A,RR,RI)
      PRINT 20
 20   FORMAT (/,7H ROOTS:,//,9H ROOT NO.,3X,10H REAL PART,5X,
     2 15H IMAGINARY PART,/)
      DO 30 I=1,4
 30   PRINT 40,I,RR(I),RI(I)
 40   FORMAT (I5,3X,E15.6,3X,E15.6)
      STOP
      END
```

```
SOLUTION OF A QUARTIC EQUATION

DATA:
A(1) =    0.100000E+01
A(2) =   -0.320000E+02
A(3) =    0.244000E+03
A(4) =   -0.200000E+02
A(5) =   -0.120000E+04


ROOTS:

ROOT NO.     REAL PART       IMAGINARY PART

   1       0.111598E+02      0.000000E+00
   2       0.200000E+02      0.000000E+00
   3      -0.193637E+01      0.000000E+00
   4       0.277656E+01      0.000000E+00
```

5.82 The system shown in Fig. A can be drawn in equivalent form as shown in Fig. B where both pulleys have the same radius, r_1.

The equivalent mass moment of inertia of pulley 2 can be computed in different speed ratios as:

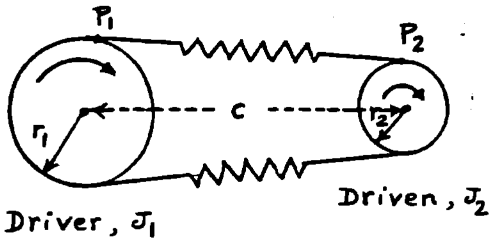


Fig. A

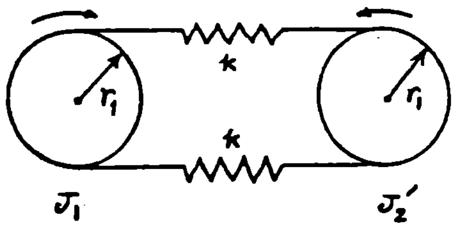


Fig. B

$$J_2' = J_2 \,(\text{speed ratio})^2$$

$$= J_2\left(\frac{150}{350}\right)^2 ; \; J_2\left(\frac{250}{350}\right)^2 ;$$

$$J_2\left(\frac{450}{350}\right)^2 ; \; J_2\left(\frac{750}{350}\right)^2$$

or $\quad J_2' = 0.1837\ J_2 \;;\; 0.5102\ J_2 \;;\; 1.6531\ J_2 \;;\; 4.5918\ J_2$

Stiffness of the belt (on each side) is given by

$$k = \frac{AE}{l}$$

where A = cross-sectional area of belt, E = Young's modulus and l = length of the belt. Length of the belt (distance P_1P_2 in Fig. A) is given by

$$l = \frac{1}{2}\left[4c^2 - (D-d)^2\right]^{\frac{1}{2}}$$

In this example, $c = 5$ m, $D = 1$ m, $d = 0.25$ m and hence

$$l = \frac{1}{2}\left[4(5)^2 - (1-0.25)^2\right]^{1/2} = 4.9859 \text{ m}$$

$$\therefore\ k = \frac{A\,(10^{10})}{4.9859} = 2.0057 \times 10^9\ A \quad N/m$$

Equation of motion:

$$J_1\ddot{\theta}_1 + k_t(\theta_1+\theta_2) = 0 \;\Rightarrow\; J_1\ddot{\theta}_1 + k_t\,\theta_1\left(1 + \frac{J_1}{J_2'}\right) = 0$$

$$J_2\ddot{\theta}_2 + k_t(\theta_1+\theta_2) = 0 \;\Rightarrow\; J_2\ddot{\theta}_2 + k_t\,\theta_2\left(\frac{J_2'}{J_1} + 1\right) = 0$$

$$\therefore \quad \omega_n = \sqrt{k_t \left(\frac{J_1 + J_2'}{J_1\, J_2'}\right)}$$

where $k_t = 2 k\, r_1^2$ (see solution of problem 5.60).

Here $J_1 = 0.1$ kg-m^2 and $J_2 = 0.2$ kg-m^2. In order for the natural frequency ω_n to be away from the speeds 150, 250, 350, 450 and 750 rpm {or, 15.708, 26.180, 36.652, 47.124 and 78.540 rad/sec},

$$\omega_n \leq 15.708 \text{ rad/sec}$$

$$\omega_n \geq 78.540 \text{ rad/sec}$$

Since ω_n involves A (through k_t), it can be determined from the above inequalities.

5.83 Velocity of tup before impact is given by:

$$\frac{1}{2}\, m_{tup}\, v^2 = m_{tup}\, g\, h \quad \text{or} \quad v = \sqrt{2\, g\, h} = \sqrt{2\,(9.81)\,(2)} = 6.2642 \text{ m/sec}$$

(a) Impact is inelastic:

Conservation of momentum leads to:

$$m_{tup}\, v_{tup} + m_{anvil}\,(0) = (m_{tup} + m_{anvil})\, v_0$$

$$\text{or} \quad v_0 = \frac{(1000)\,(6.2642)}{(1000 + 5000)} = 1.0440 \text{ m/sec}$$

(b) Natural frequencies:

$$\omega_{2,1}^2 = \frac{k_1 + k_2}{2\, m_1} + \frac{k_2}{2\, m_2} \pm \sqrt{\frac{1}{4}\left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right)^2 - \frac{k_1\, k_2}{m_1\, m_2}}$$

Thus the natural frequency requirement can be stated as:

$$f_1^2 = \frac{\omega_1^2}{(2\,\pi)^2}$$

$$= \frac{1}{(2\,\pi)^2}\left\{\frac{k_1 + k_2}{50000} + \frac{k_2}{10000} - \sqrt{\frac{1}{4}\left(\frac{k_1 + k_2}{25000} + \frac{k_2}{5000}\right)^2 - \frac{k_1\, k_2}{125\,(10^6)}}\right\} > (5^2) \qquad (1)$$

(c) Free vibration response:

Initial conditions:

$$x_1(0) = x_2(0) = \dot{x}_1(0) = 0, \quad \dot{x}_2(0) = v_0 = 1.0440 \text{ m/sec}$$

Maximum forces in the springs:

$$F_1 = k_1 \, x_1 \big|_{max} \tag{2}$$

$$F_2 = k_2 \, (x_2 - x_1) \big|_{max} \tag{3}$$

For a helical spring, the shear stress (τ) under an axial force F is given by:

$$\tau = k_s \, \frac{8 \, F \, D}{\pi \, d^3} \tag{4}$$

where k_s = shear stress correction factor = $\dfrac{2D + d}{2\,D}$, D = mean coil diameter, and d = wire diameter.

Ref: J. E. Shigley and C. R. Mischke, "Mechanical Engineering Design," 5th Ed., McGraw-Hill, New York, 1989.

Since stress is to be less than the yield stress witth a factor of safety of 1.5, we have

$$\tau_1 \le \frac{\tau_{yield}}{1.5} \tag{5}$$

$$\tau_2 \le \frac{\tau_{yield}}{1.5} \tag{6}$$

where τ_1 and τ_2 denote the shear stresses induced in the springs k_1 and k_2, respectively, and τ_{yield} is the shear stress corresponding to the yield stress of the material.