

Chapter 8

Continuous Systems

$$8.1 \quad c = \left(\frac{P}{\rho}\right)^{1/2} = \left(\frac{4000}{5}\right)^{1/2} = 28.2843 \text{ m/s}$$

$$8.2 \quad \rho = \text{mass density} \times \text{area} = 7830 \left\{ \frac{\pi}{4} (2 \times 10^{-3})^2 \right\} = 24.5987 \times 10^{-3} \text{ kg/m}$$

$$P = 250 \text{ N}, \quad l = 2 \text{ m}$$

$$(b) \quad c = \left(\frac{P}{\rho}\right)^{1/2} = \left(\frac{250}{24.5987 \times 10^{-3}}\right)^{1/2} = 100.8124 \text{ m/s}$$

$$(a) \quad \omega_1 = \frac{c\pi}{l} = \frac{100.8124 \pi}{2} = 158.3561 \text{ rad/s} = 25.2031 \text{ Hz}$$

$$8.3 \quad \omega_1 = 3000 (2\pi) \text{ rad/sec} = \frac{\pi c}{l} = \frac{\pi c}{2}$$

$$c = (6000 \pi \times 2 / \pi) = 12000 \text{ m/s}$$

$$\omega_3 = 3\pi c / l = 3\omega_1 = 9000 \text{ Hz}$$

$$c_{\text{original}} = (P/\rho)^{1/2} = 12000 \text{ m/s}$$

$$c_{\text{new}} = (1.2 P/\rho)^{1/2} = 1.0954 (P/\rho)^{1/2} = 1.0954 c_{\text{original}}$$

$\therefore \omega_1$ and ω_3 are increased by 9.54%

$$8.4 \quad P = 30000 \text{ N}, \quad \rho = 2 \text{ kg/m}$$

$$c = \left(\frac{P}{\rho}\right)^{1/2} = (30000/2)^{1/2} = 122.4745 \text{ m/s}$$

$$\text{Time taken} = \frac{300}{122.4745} = 2.4495 \text{ s}$$

$$8.5 \quad \text{At } x=0: \quad P \frac{\partial w}{\partial x} = m \frac{\partial^2 w}{\partial t^2} \quad (E_1)$$

$$\text{At } x=l: \quad P \frac{\partial w}{\partial x} = -k w \quad (E_2)$$

General solution is

$$w(x,t) = W(x) \cdot T(t) = \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c} \right) (C \cos \omega t + D \sin \omega t)$$

Equations (E₁) and (E₂) give: (E₃)

$$\left\{ P \left(-A \frac{\omega}{c} \sin \frac{\omega x}{c} + B \frac{\omega}{c} \cos \frac{\omega x}{c} \right) (C \cos \omega t + D \sin \omega t) \right. \\ \left. = -m \omega^2 \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c} \right) (C \cos \omega t + D \sin \omega t) \right\}_{x=0}$$

$$\text{i.e.,} \quad A (m \omega^2) + B \left(\frac{P \omega}{c} \right) = 0 \quad (E_4)$$

Eqs. (E₂) and (E₃) yield

$$P \left(-\frac{\omega}{c} A \sin \frac{\omega l}{c} + B \frac{\omega}{c} \cos \frac{\omega l}{c} \right) = -k \left(A \cos \frac{\omega l}{c} + B \sin \frac{\omega l}{c} \right)$$

i.e.,

$$A \left(-\frac{P\omega}{c} \sin \frac{\omega l}{c} + k \cos \frac{\omega l}{c} \right) + B \left(\frac{P\omega}{c} \cos \frac{\omega l}{c} + k \sin \frac{\omega l}{c} \right) = 0 \quad (E_5)$$

Eqs. (E₄) and (E₅) give the frequency equation:

$$\begin{vmatrix} (m\omega^2) & (P\omega/c) \\ \left(-\frac{P\omega}{c} \sin \frac{\omega l}{c} + k \cos \frac{\omega l}{c}\right) & \left(\frac{P\omega}{c} \cos \frac{\omega l}{c} + k \sin \frac{\omega l}{c}\right) \end{vmatrix} = 0$$

which, upon simplification, becomes

$$\tan \alpha = \left\{ \frac{Pk - \left(\frac{Pm c^2}{l^2}\right) \alpha^2}{\left(\frac{c^2 m k}{l}\right) \alpha + \left(\frac{P^2 c}{l}\right) \alpha} \right\} \quad (E_6)$$

8.6

$$l = 2\text{ m}, \quad d = 0.5\text{ mm}, \quad \rho = 7800\text{ kg/m}^3, \quad \omega_n = \frac{n c \pi}{l}, \quad c = \sqrt{\frac{P}{\rho}}$$

$$(a) \quad \omega_1 = 1(2\pi)\text{ rad/sec} = \frac{\pi c}{l} = \frac{\pi}{2} \sqrt{\frac{P}{7800}}$$

$$\text{i.e., } (2\pi)^2 = \frac{\pi^2}{4} \left(\frac{P}{7800}\right)$$

$$\text{i.e., } P = 124,800\text{ N}$$

$$(b) \quad \omega_1 = 5(2\pi)\text{ rad/sec} = \frac{\pi c}{l} = \frac{\pi}{2} \sqrt{\frac{P}{7800}}$$

$$\text{i.e., } 100\pi^2 = \frac{\pi^2}{4(7800)} P$$

$$\text{i.e., } P = 3,120,000\text{ N}$$

8.7

Let $x=0$ be fixed and $x=l$ be connected to the pin which can move in a frictionless slot.

$$W(x) = A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c}$$

$$w(0,t) = 0 \Rightarrow W(0) = 0 \Rightarrow A = 0$$

$$\frac{\partial w}{\partial x}(l,t) = 0 \Rightarrow \frac{dW}{dx}(l) = 0 \Rightarrow B \cos \frac{\omega l}{c} = 0 \Rightarrow \cos \frac{\omega l}{c} = 0$$

$$\therefore \frac{\omega l}{c} = (2n+1) \frac{\pi}{2}; \quad n = 0, 1, 2, \dots$$

$$\text{or } \omega_n = \frac{(2n+1) \pi c}{2l}; \quad n = 0, 1, 2, \dots$$

8.8

$$w(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \left[C_n \cos \frac{n\pi t}{l} + D_n \sin \frac{n\pi t}{l} \right]$$

where

$$C_n = \frac{2}{l} \int_0^l w_0(x) \sin \frac{n\pi x}{l} dx, \quad D_n = \frac{2}{\pi c n} \int_0^l \dot{w}_0(x) \sin \frac{n\pi x}{l} dx$$

Since $w_0(x) = w(x, 0) = 0$, $C_n = 0$

$$D_n = \frac{2}{\pi c n} \left[\int_0^{l/2} \frac{2ax}{l} \sin \frac{n\pi x}{l} dx + \int_{l/2}^l 2a \left(1 - \frac{x}{l}\right) \sin \frac{n\pi x}{l} dx \right]$$

$$= \frac{8al}{\pi^3 c n^3} \sin \frac{n\pi}{2} = \begin{cases} 0 & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}} \frac{8al}{\pi^3 n^3 c} & \text{if } n \text{ is odd} \end{cases}$$

$$w(x, t) = \frac{8al}{\pi^3 c} \sum_{n=1,3,5,\dots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^3} \sin \frac{n\pi x}{l} \sin \frac{n\pi c t}{l}$$

8.9

Eq. (8.18) is $c^2 \frac{d^2 W}{dx^2} = a W(x)$

Multiply this equation by $W(x)$ and integrate from 0 to l :

$$c^2 \int_0^l W(x) \frac{d^2 W(x)}{dx^2} dx = a \int_0^l [W(x)]^2 dx$$

This shows that the sign of a will be same as that of the integral on the left side. Integration by parts gives

$$I = \int_0^l W(x) \cdot \frac{d^2 W(x)}{dx^2} dx = W(l) \cdot \frac{dW}{dx}(l) - W(0) \frac{dW}{dx}(0) - \int_0^l \left[\frac{dW(x)}{dx} \right]^2 dx$$

Since the first two terms on the right side of this equation are zero for common boundary conditions, I and hence a will be negative.

Common boundary conditions (examples):

Fixed at ends: $W(0) = W(l) = 0$

Free at ends: $\frac{dW}{dx}(0) = \frac{dW}{dx}(l) = 0$

One end fixed and other end free: $W(0) = \frac{dW}{dx}(l) = 0$

One end fixed and other end connected to a spring:

$$W(0) = 0; \frac{dW}{dx}(l) = -k W(l)$$

8.10

$l = 2000 \text{ m}$, $\rho = 8890 \text{ kg/m}^3$, $0 \leq \omega_1 \text{ to } \omega_4 \leq 20 \text{ Hz}$

$$\omega_n = \frac{n c \pi}{l} = \frac{n \pi}{l} \sqrt{\frac{P}{\rho}}$$

For $\omega_4 \leq 20 (2\pi) \text{ rad/sec}$, $\frac{4\pi}{l} \sqrt{\frac{P}{\rho}} \leq 40\pi$

$$\text{i.e., } \sqrt{\frac{P}{8890}} \leq \frac{40\pi(2000)}{4\pi}$$

$$\text{i.e., } P \leq 35560 \times 10^8 \text{ N}$$

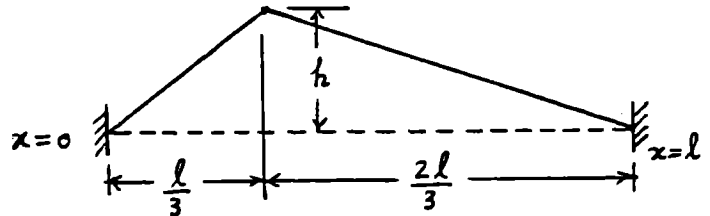
Let the permissible (yield) stress of the material be $300 \text{ MPa} = 3 \times 10^8 \text{ N/m}^2$.

Assuming the diameter of cable as 0.1 m, area of cross-section is $\frac{\pi}{4} (0.1)^2 = 0.007854 \text{ m}^2$, and the permissible tension (with factor of safety of one) is

$$(0.007854)(3 \times 10^8) = 2,356,200 \text{ N}$$

$$\therefore \text{Initial tension} = 2.3562 \times 10^6 \text{ N}$$

8.11



Solution is given by
Eq. (8.30):

$$w(x,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} \cos \frac{n c \pi t}{l} \quad (E_1)$$

where

$$C_n = \frac{2}{l} \int_0^l w_0(x) \sin \frac{n\pi x}{l} dx \quad (E_2)$$

The initial deflection $w_0(x)$ is given by

$$w_0(x) = \begin{cases} 3hx/l & \text{for } 0 \leq x \leq l/3 \\ 3h(l-x)/2l & \text{for } l/3 \leq x \leq l \end{cases} \quad (E_3)$$

Substitution of Eq. (E3) into (E2) gives

$$\begin{aligned} C_n &= \frac{2}{l} \left\{ \int_0^{l/3} \frac{3hx}{l} \sin \frac{n\pi x}{l} dx + \int_{l/3}^l \frac{3h(l-x)}{2l} \sin \frac{n\pi x}{l} dx \right\} \\ &= \frac{6h}{l^2} \left\{ \frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l} - \frac{x l}{n\pi} \cos \frac{n\pi x}{l} \right\}_0^{l/3} - \frac{3h}{l} \left(\frac{l}{n\pi} \right) \left(\cos \frac{n\pi x}{l} \right)_0^{l/3} \\ &\quad - \frac{3h}{l^2} \left\{ \frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l} - \frac{x l}{n\pi} \cos \frac{n\pi x}{l} \right\}_{l/3}^l \\ &= \frac{9h}{n^2\pi^2} \sin \frac{n\pi}{3} \end{aligned}$$

$$= \frac{9h\sqrt{3}}{2\pi^2} \text{ for } n=1, \quad \frac{9h\sqrt{3}}{8\pi^2} \text{ for } n=2, \quad 0 \text{ for } n=3,$$

$$- \frac{9h\sqrt{3}}{32\pi^2} \text{ for } n=4, \quad - \frac{9h\sqrt{3}}{50\pi^2} \text{ for } n=5, \quad 0 \text{ for } n=6, \dots$$

$$\begin{aligned} w(x,t) &= \frac{9\sqrt{3}}{2\pi^2} h \sin \frac{\pi x}{l} \cos \frac{c\pi t}{l} + \frac{9\sqrt{3}}{8\pi^2} h \sin \frac{2\pi x}{l} \cos \frac{2c\pi t}{l} \\ &\quad - \frac{9\sqrt{3}}{32\pi^2} h \sin \frac{4\pi x}{l} \cos \frac{4c\pi t}{l} - \dots \end{aligned}$$

At $t=0$:

$$w(x,0) = \left\{ \frac{9\sqrt{3}}{2\pi^2} h \sin \frac{\pi x}{l} + \frac{9\sqrt{3}}{8\pi^2} h \sin \frac{2\pi x}{l} - \frac{9\sqrt{3}}{32\pi^2} h \sin \frac{4\pi x}{l} - \frac{9\sqrt{3}}{50\pi^2} h \sin \frac{5\pi x}{l} \right\}$$

At $t = \frac{l}{4c}$:

$$w(x, \frac{l}{4c}) = \frac{9\sqrt{3}}{2\pi^2} \frac{h}{\sqrt{2}} \sin \frac{\pi x}{l} - \frac{9\sqrt{3}}{32\pi^2} h \sin \frac{4\pi x}{l} + \frac{9\sqrt{3}}{50\pi^2} \frac{h}{\sqrt{2}} \sin \frac{5\pi x}{l}$$

At $t = \frac{l}{3c}$:

$$w(x, \frac{l}{3c}) = \frac{9\sqrt{3}}{4\pi^2} h \left\{ \sin \frac{\pi x}{l} - \frac{1}{4} \sin \frac{2\pi x}{l} + \frac{1}{16} \sin \frac{4\pi x}{l} - \frac{1}{25} \sin \frac{5\pi x}{l} \right\}$$

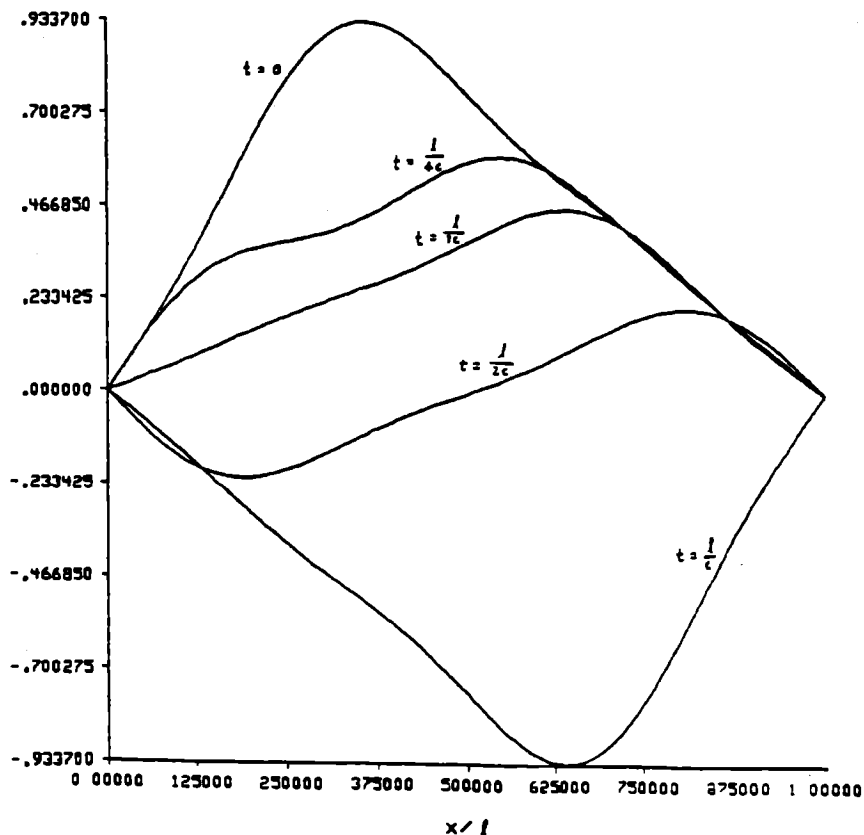
At $t = \frac{l}{2c}$:

$$w(x, \frac{l}{2c}) = -\frac{9\sqrt{3}}{8\pi^2} h \left\{ \sin \frac{2\pi x}{l} + \frac{1}{4} \sin \frac{4\pi x}{l} \right\}$$

At $t = \frac{l}{c}$:

$$w(x, \frac{l}{c}) = \frac{9\sqrt{3}}{2\pi^2} h \left\{ -\sin \frac{\pi x}{l} + \frac{1}{4} \sin \frac{2\pi x}{l} - \frac{1}{16} \sin \frac{4\pi x}{l} + \frac{1}{25} \sin \frac{5\pi x}{l} \right\}$$

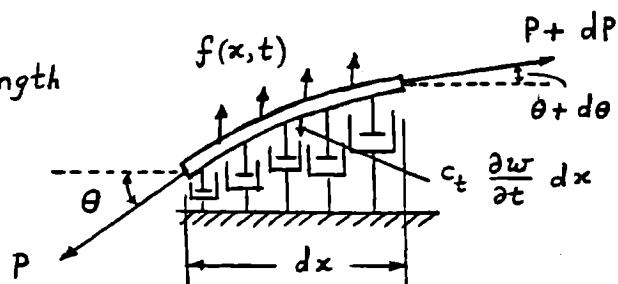
These deflection shapes are shown below:



8.12

c_t = viscous damping coefficient
= force/unit velocity/unit length

$$\begin{aligned} & (P + dP) \sin(\theta + d\theta) + f(x, t) dx \\ & - c_t \frac{\partial w}{\partial t} dx - P \sin \theta \\ & = \rho dx \frac{\partial^2 w}{\partial t^2} \quad \dots (E_1) \end{aligned}$$



with $dP = \frac{\partial P}{\partial x} dx$ and $\sin(\theta + d\theta) \approx \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} dx$

Eg. (E₁) can be rewritten as

$$\frac{\partial}{\partial x} \left(P \frac{\partial w}{\partial x} \right) + f(x, t) = \rho(x) \frac{\partial^2 w(x, t)}{\partial t^2} + c_t \frac{\partial w(x, t)}{\partial t}$$

8.13

Free vibration solution is given by

$$w(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \left\{ C_n \cos \frac{n\pi t}{l} + D_n \sin \frac{n\pi t}{l} \right\}$$

with $C_n = \frac{2}{l} \int_0^l w_0(x) \sin \frac{n\pi x}{l} dx$

and $D_n = \frac{2}{n\pi} \int_0^l \dot{w}_0(x) \sin \frac{n\pi x}{l} dx$

Since $\dot{w}_0(x) = \frac{\partial w}{\partial t}(x, t=0) = 0$, $D_n = 0$.

For $w_0(x) = w_0 \sin \frac{\pi x}{l}$, $C_n = \frac{2w_0}{l} \int_0^l \sin \frac{\pi x}{l} \sin \frac{n\pi x}{l} dx$

Using the relations

$$\int \sin m_1 x \sin n_1 x dx = \frac{\sin(m_1 - n_1)x}{2(m_1 - n_1)} - \frac{\sin(m_1 + n_1)x}{2(m_1 + n_1)}; \quad m_1 \neq n_1$$

$$\int \sin^2 m_1 x dx = \frac{x}{2} - \frac{1}{4m_1} \sin 2m_1 x,$$

we get for $n=1$,

$$C_1 = \frac{2w_0}{l} \int_0^l \sin^2 \frac{\pi x}{l} dx = w_0$$

and for $n=2, 3, \dots$,

$$C_n = \frac{2w_0}{l} \left\{ \frac{\sin \frac{\pi}{l}(n-1)x}{2 \frac{\pi}{l}(n-1)} - \frac{\sin \frac{\pi}{l}(n+1)x}{2 \frac{\pi}{l}(n+1)} \right\}_0^l = 0$$

$$\therefore w(x, t) = w_0 \sin \frac{\pi x}{l} \cos \frac{c\pi t}{l}$$

8.14

$$u(x,t) = \left(\tilde{A} \cos \frac{\omega x}{c} + \tilde{B} \sin \frac{\omega x}{c} \right) (C \cos \omega t + D \sin \omega t)$$

$$\frac{\partial u}{\partial x} = \frac{\omega}{c} \left(-\tilde{A} \sin \frac{\omega x}{c} + \tilde{B} \cos \frac{\omega x}{c} \right) (C \cos \omega t + D \sin \omega t)$$

$$\frac{\partial u}{\partial x}(0,t) = 0 \Rightarrow \tilde{B} = 0$$

$$\frac{\partial u}{\partial x}(l,t) = 0 \Rightarrow -\frac{\omega}{c} \tilde{A} \sin \frac{\omega l}{c} (C \cos \omega t + D \sin \omega t) = 0$$

$$\Rightarrow \sin \frac{\omega l}{c} = 0 ; \quad \frac{\omega_n l}{c} = n\pi$$

$$\omega_n = \frac{n\pi c}{l} = \frac{n\pi}{l} \sqrt{\frac{E}{\rho}}$$

$$u(x,t) = \sum_{n=1}^{\infty} \cos \frac{n\pi x}{l} \left[C_n \cos \frac{n\pi c t}{l} + D_n \sin \frac{n\pi c t}{l} \right]$$

$$\text{where } C_n = \frac{2}{l} \int_0^l u_0(x) \cos \frac{n\pi x}{l} dx$$

$$\text{and } D_n = \frac{2}{n\pi c} \int_0^l \dot{u}_0(x) \cos \frac{n\pi x}{l} dx$$

8.15

$$(a) u(x,t) = \left(\tilde{A} \cos \frac{\omega x}{c} + \tilde{B} \sin \frac{\omega x}{c} \right) (C \cos \omega t + D \sin \omega t)$$

$$\text{At } x=0: M_1 \frac{\partial^2 u}{\partial t^2} = AE \frac{\partial u}{\partial x}$$

$$-\omega^2 M_1 \tilde{A} = AE \frac{\omega}{c} \tilde{B} \Rightarrow \tilde{B} = -\left(\frac{\omega M_1 c}{AE} \right) \tilde{A}$$

$$\text{At } x=l: M_2 \frac{\partial^2 u}{\partial t^2} = -AE \frac{\partial u}{\partial x}$$

$$-\omega^2 M_2 \left(\tilde{A} \cos \frac{\omega l}{c} + \tilde{B} \sin \frac{\omega l}{c} \right) = -AE \frac{\omega}{c} \left(-\tilde{A} \sin \frac{\omega l}{c} + \tilde{B} \cos \frac{\omega l}{c} \right)$$

$$\text{i.e. } \tilde{A} \left(-\omega^2 M_2 \cos \frac{\omega l}{c} - AE \frac{\omega}{c} \sin \frac{\omega l}{c} \right) = \tilde{B} \left(-\frac{AE\omega}{c} \cos \frac{\omega l}{c} + \omega^2 M_2 \sin \frac{\omega l}{c} \right)$$

$$\text{i.e. } \omega^2 M_2 \cos \frac{\omega l}{c} + \frac{AE\omega}{c} \sin \frac{\omega l}{c} + \frac{\omega M_1 c}{AE} \left(-\omega^2 M_2 \sin \frac{\omega l}{c} + \frac{AE\omega}{c} \cos \frac{\omega l}{c} \right) = 0$$

This is the frequency equation.

$$(b) u(x,t) = \left(\tilde{A} \cos \frac{\omega x}{c} + \tilde{B} \sin \frac{\omega x}{c} \right) (C \cos \omega t + D \sin \omega t) \dots (E_1)$$

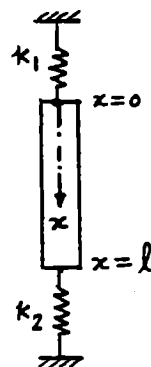
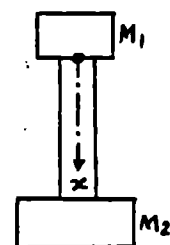
$$\text{At } x=0, \quad k_1 u = AE \frac{\partial u}{\partial x} \Rightarrow \tilde{B} = \frac{k_1 c}{AE \omega} \tilde{A} \dots (E_2)$$

$$\text{At } x=l, \quad k_2 u = -AE \frac{\partial u}{\partial x}$$

$$\Rightarrow k_2 \left(\tilde{A} \cos \frac{\omega l}{c} + \tilde{B} \sin \frac{\omega l}{c} \right) = -AE \frac{\omega}{c} \left\{ -\tilde{A} \sin \frac{\omega l}{c} + \tilde{B} \cos \frac{\omega l}{c} \right\}$$

Substituting (E₂) into (E₃), we get

$$\tilde{A} \left[\left(k_2 - \frac{k_1 c}{AE \omega} \right) \cos \frac{\omega l}{c} + \left(\frac{k_1 k_2 c}{AE \omega} - \frac{AE \omega}{c} \right) \sin \frac{\omega l}{c} \right] = 0$$



Hence the frequency equation is given by

$$\left(k_2 - \frac{k_1 c}{AE\omega}\right) \cos \frac{\omega l}{c} + \left(\frac{k_1 k_2 c}{AE\omega} - \frac{AE\omega}{c}\right) \sin \frac{\omega l}{c} = 0$$

$$\text{i.e., } \tan \frac{\omega l}{c} = \left(\frac{k_1 c^2 - k_2 AE\omega c}{k_1 k_2 c^2 - A^2 E^2 \omega^2}\right)$$

$$(c) \quad u(x,t) = \left(\tilde{A} \cos \frac{\omega x}{c} + \tilde{B} \sin \frac{\omega x}{c}\right) (C \cos \omega t + D \sin \omega t) \quad \dots (E_1)$$

$$\text{At } x=0, \quad k u = AE \frac{\partial u}{\partial x} \Rightarrow \tilde{B} = \frac{k c}{AE\omega} \tilde{A} \quad \dots (E_2)$$

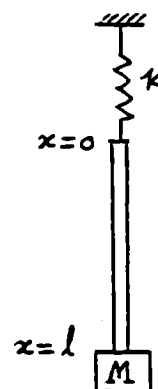
$$\begin{aligned} \text{At } x=l, \quad AE \frac{\partial u}{\partial x} &= -M \frac{\partial^2 u}{\partial t^2} \\ \Rightarrow AE \left(-\tilde{A} \frac{\omega}{c} \sin \frac{\omega l}{c} + \tilde{B} \frac{\omega}{c} \cos \frac{\omega l}{c}\right) &= M \left(\tilde{A} \cos \frac{\omega l}{c} + \tilde{B} \sin \frac{\omega l}{c}\right) \omega^2 \quad \dots (E_3) \end{aligned}$$

Substituting (E₂) into (E₃), we get

$$\frac{AE\omega}{c} \tilde{A} \left(-\sin \frac{\omega l}{c} + \frac{k c}{AE\omega} \cos \frac{\omega l}{c}\right) = M \omega^2 \tilde{A} \left(\cos \frac{\omega l}{c} + \frac{k c}{AE\omega} \sin \frac{\omega l}{c}\right)$$

This gives the frequency equation

$$\tan \frac{\omega l}{c} = \left\{ \frac{AE\omega c (k - M \omega^2)}{A^2 E^2 \omega^2 - M \omega^2 k c^2} \right\}$$



8.16

For a clamped-free bar in longitudinal vibration, fundamental frequency = $\frac{\pi c}{2l}$

If the bar is fixed at $x=0$ and attached to a mass M at $x=l$, its fundamental frequency = $\frac{\alpha_1 c}{l}$.

$$\text{Here } \frac{\alpha_1 c}{l} = \frac{1}{2} \left(\frac{\pi c}{2l}\right) \Rightarrow \alpha_1 = \frac{\pi}{4} = 0.7854$$

As an approximation, use linear relationship for the values in Table 8.1. When $\beta = 0.1$, $\alpha_1 = 0.3113$

$$\text{When } \beta = 1.0, \alpha_1 = 0.8602$$

$$\text{This gives } \alpha_1(\beta) = a + b\beta \approx 0.6099\beta + 0.2503$$

$$\text{For } \alpha_1 = 0.7854, \text{ we get } \beta = \frac{0.7854 - 0.2503}{0.6099} = 0.8774 = \frac{m}{M}$$

$$\therefore \text{Mass to be attached} = M \approx \frac{m}{0.8774} = 1.1397 m.$$

8.17

Equation of motion for free vibration $c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad (E_1)$

Assume separation of variables in Eq. (E₁): $u(x,t) = U(x) \cdot T(t)$

$$\text{so that } c^2 \frac{d^2 U}{dx^2} T = U \frac{d^2 T}{dt^2} \quad \text{or} \quad c^2 \frac{1}{U} \frac{d^2 U}{dx^2} = \frac{1}{T} \frac{d^2 T}{dt^2} = \omega = -\omega^2 \quad \dots (E_2)$$

$$\text{i.e.,} \quad c^2 \frac{d^2 U(x)}{dx^2} + \omega^2 U(x) = 0 \quad (E_3)$$

$$\text{and} \quad \frac{d^2 T(t)}{dt^2} + \omega^2 T(t) = 0 \quad (E_4)$$

To show that the constant a on the right hand side of Eq. (E₂) is a negative quantity, multiply Eq. (E₃) by $U(x)$ and integrate w.r.t. x from 0 to l :

$$c^2 \int_0^l U(x) \frac{d^2 U}{dx^2} dx = a \int_0^l \underbrace{\{U(x)\}^2}_{\text{positive always}} dx \quad (E_5)$$

Eq. (E₅) shows that a will have the same sign as the integral on the left side:

$$I = \int_0^l U(x) \frac{d^2 U(x)}{dx^2} dx \quad (E_6)$$

Integrate (E₆) by parts to get

$$I = U(l) \cdot \frac{dU(l)}{dx} - U(0) \cdot \frac{dU(0)}{dx} - \int_0^l \left\{ \frac{dU(x)}{dx} \right\}^2 dx \quad (E_7)$$

since $U(0) = 0$, the second term on the r.h.s. of (E₇) will be zero. Also, $AE \frac{dU(l)}{dx} = -\kappa U(l)$ (E₈)

at $x=l$. Multiplication of (E₈) by $U(l)$ gives

$$U(l) \cdot \frac{dU}{dx}(l) = -\kappa \{U(l)\}^2 < 0$$

This shows that $I < 0$ and hence $a < 0$.

In Eqs. (E₃) and (E₄), there exists an eigen (normal) function for each frequency (constant) ω . Let $U_m(x)$ and $U_n(x)$ denote the normal functions corresponding to the frequencies ω_m and ω_n , respectively. Eq. (E₃) gives

$$c^2 \frac{d^2 U_m}{dx^2} + \omega_m^2 U_m = 0 \quad \dots (E_9); \quad c^2 \frac{d^2 U_n}{dx^2} + \omega_n^2 U_n = 0 \quad \dots (E_{10})$$

Multiply (E₉) by U_n , (E₁₀) by U_m and subtract the resulting equations one from the other to get

$$c^2 U_n \frac{d^2 U_m}{dx^2} - c^2 U_m \frac{d^2 U_n}{dx^2} + (\omega_m^2 - \omega_n^2) U_m U_n = 0 \quad (E_{11})$$

Integrate (E₁₁) with respect to x from 0 to l to get, after integration by parts, the desired orthogonality relation:

$$\int_0^l U_m(x) U_n(x) dx = 0 \quad , \quad \omega_m \neq \omega_n \quad (E_{12})$$

8.18

Set up two coordinates x_1 and x_2 as shown.

$$u_1(x_1, t) = \left(\tilde{A}_1 \cos \frac{\omega x_1}{c_1} + \tilde{B}_1 \sin \frac{\omega x_1}{c_1} \right) (C \cos \omega t + D \sin \omega t)$$

$$u_2(x_2, t) = \left(\tilde{A}_2 \cos \frac{\omega x_2}{c_2} + \tilde{B}_2 \sin \frac{\omega x_2}{c_2} \right) (C \cos \omega t + D \sin \omega t)$$

$$u_1(0, t) = 0 \Rightarrow \tilde{A}_1 = 0$$

$$u_1(l_1, t) = u_2(0, t) \Rightarrow \tilde{B}_1 \sin \frac{\omega l_1}{c_1} = \tilde{A}_2$$

$$A_1 E_1 \frac{\partial u_1}{\partial x_1}(l_1, t) = A_2 E_2 \frac{\partial u_2}{\partial x_2}(0, t) = \text{tensile force same in both areas}$$

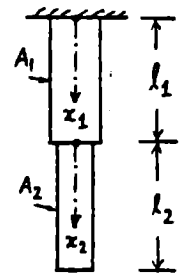
$$\text{i.e. } A_1 E_1 \tilde{B}_1 \frac{\omega}{c_1} \cos \frac{\omega l_1}{c_1} = A_2 E_2 \frac{\omega}{c_2} \tilde{B}_2 \Rightarrow \tilde{B}_2 = \frac{A_1 E_1 c_2}{A_2 E_2 c_1} \cos \frac{\omega l_1}{c_1} \cdot \tilde{B}_1$$

$$u_2(x_2, t) = \tilde{B}_1 \left(\sin \frac{\omega l_1}{c_1} \cos \frac{\omega x_2}{c_2} + \frac{A_1 E_1 c_2}{A_2 E_2 c_1} \cos \frac{\omega l_1}{c_1} \sin \frac{\omega x_2}{c_2} \right) (C \cos \omega t + D \sin \omega t)$$

$$\frac{\partial u_2}{\partial x_2}(l_2, t) = 0 \Rightarrow \tilde{B}_1 \frac{\omega}{c_2} \left\{ -\sin \frac{\omega l_1}{c_1} \cdot \sin \frac{\omega l_2}{c_2} + \frac{A_1 E_1 c_2}{A_2 E_2 c_1} \cos \frac{\omega l_1}{c_1} \cdot \cos \frac{\omega l_2}{c_2} \right\} = 0$$

\therefore Frequency equation is

$$\tan \frac{\omega l_1}{c_1} \cdot \tan \frac{\omega l_2}{c_2} = \frac{A_1 E_1 c_2}{A_2 E_2 c_1}$$



8.19

(a) Axial vibration:

$$\beta = \frac{m_0}{m} = \frac{\text{mass of rod}}{\text{end mass}}$$

Using $\rho = 76.5 \text{ kN/m}^3$ for steel, we find

$$m_0 = \frac{\pi d^2 \ell \rho}{4} = \frac{\pi}{4} \left(\frac{5}{100} \right)^2 (1) \left(\frac{76.5 (10^3)}{9.81} \right) = 15.3117 \text{ kg}$$

$$\frac{m_0}{m} = \frac{15.3117}{100} = 0.1531$$

From Table 8.1, the value of α_1 for $\beta = 0.1531$ (using linear interpolation between values of $\beta = 0.1$ and $\beta = 1.0$) is:

$$\alpha_1 = 0.6099 (0.1531) + 0.2503 = 0.3437$$

$$\omega_1 = \frac{\alpha_1 c}{\ell} = \frac{\alpha_1}{\ell} \sqrt{\frac{E}{\rho}} = \frac{0.3437}{1} \sqrt{\frac{207 (10^9) (9.81)}{76500}} = 1770.7958 \text{ rad/sec}$$

(b) Torsional vibration:

In this case, we use the result of Example 8.6.

$$\beta = \frac{\bar{J}_{\text{rod}}}{I_0} = \frac{J \rho \ell}{I_0}$$

where J = polar moment of inertia of the shaft:

$$J = \frac{\pi}{32} d^4 = \frac{\pi}{32} (0.05)^4 = 2.4544 (10^{-8}) \text{ m}^4$$

$$\rho = 76500/9.81 \text{ kg/m}^3 ; \ell = 1 \text{ m}$$

$$\bar{J}_{\text{rod}} = (2.4544 (10^{-8})) \frac{76500}{9.81} (1) = 1.9140 (10^{-4}) \text{ kg-m}^2$$

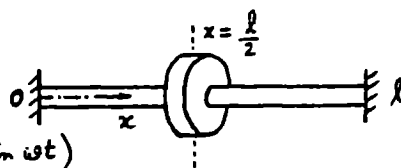
$$\beta = \frac{1.9140 (10^{-4})}{10} = 1.9140 (10^{-5})$$

Since β is close to zero, we have from Example 8.6,

$$\omega_1 \approx \frac{c}{\ell} \sqrt{\beta} = \sqrt{\frac{G\beta}{\rho}} = \sqrt{\frac{(80 (10^9)) (9.81)}{76500}} = 14.0126 \text{ rad/sec}$$

8.20

Consider half the shaft and half the disc.



$$\theta(x, t) = \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c} \right) (C \cos \omega t + D \sin \omega t)$$

$$\theta(0, t) = 0 \Rightarrow A = 0$$

$$GJ \frac{\partial \theta}{\partial x} \left(\frac{l}{2}, t \right) = - \frac{J_0}{2} \frac{\partial^2 \theta}{\partial t^2} \left(\frac{l}{2}, t \right)$$

$$\Rightarrow GJ \frac{\omega}{c} \cdot B \cos \frac{\omega l}{2c} = \frac{J_0 \omega^2}{2} \cdot B \sin \frac{\omega l}{2c}$$

$$\Rightarrow \tan \frac{\omega l}{2c} = \frac{2GJ}{J_0 \omega c} = \frac{2c}{\omega l} \cdot \frac{GJ l}{J_0 c^2}$$

Frequency equation: $\alpha \tan \alpha = \beta$ where $\alpha = \frac{\omega l}{2c}$ and $\beta = \frac{GJ l}{J_0 c^2}$

If roots of this equation are given by α_n ,

$$\omega_n = \frac{2\alpha_n c}{l} \text{ and}$$

$$= \left(\frac{J_0 l}{J_0} \right)$$

$$\theta(x, t) = \sum_{n=1}^{\infty} \sin \frac{\omega_n x}{c} \left(C_n \cos \omega_n t + D_n \sin \omega_n t \right) \quad \text{---- (E}_1\text{)}$$

$$\theta(x, 0) = \sum_{n=1}^{\infty} \sin \frac{\omega_n x}{c} (C_n) = 0 \Rightarrow C_n = 0 \quad \text{---- (E}_2\text{)}$$

$$\dot{\theta}(x, 0) = \sum_{n=1}^{\infty} \sin \frac{\omega_n x}{c} (\omega_n D_n) = \frac{2 \dot{\theta}_0 x}{l} \quad \text{---- (E}_3\text{)}$$

as $\dot{\theta} \Big|_{\frac{l}{2}, t=0} = \dot{\theta}_0$ and hence $\dot{\theta}(x, 0) = \frac{2 \dot{\theta}_0 x}{l}$

Multiply Eq. (E₃) by $\sin \frac{\omega_n x}{c}$ and integrate from 0 to $\frac{\pi c}{2\omega_n}$:

$$\omega_n D_n \int_0^{\left(\frac{\pi c}{2\omega_n}\right)} \sin^2 \frac{\omega_n x}{c} dx = \frac{2 \dot{\theta}_0}{l} \int_0^{\left(\frac{\pi c}{2\omega_n}\right)} x \sin \frac{\omega_n x}{c} dx \quad \text{(E}_4\text{)}$$

i.e.,
$$D_n = \frac{8 c \dot{\theta}_0}{\pi l \omega_n^2} \quad \text{(E}_5\text{)}$$

$$\therefore \theta(x, t) = \sum_{n=1}^{\infty} \left(\frac{8c \dot{\theta}_0}{\pi l \omega_n^2} \right) \sin \frac{\omega_n x}{c} \sin \omega_n t \quad (E_6)$$

8.21 $\theta(x, t) = \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c} \right) (C \cos \omega t + D \sin \omega t)$

$\theta(0, t) = 0 \Rightarrow A = 0$

$\theta(l, t) = 0 \Rightarrow \sin \frac{\omega l}{c} = 0 \quad ; \quad \frac{\omega_n l}{c} = n\pi$

$\therefore \omega_n = \frac{n\pi c}{l} = \frac{n\pi}{l} \sqrt{\frac{G}{J}} \quad ; \quad n = 1, 2, 3, \dots$

8.22 Boundary conditions:

At $x=0$, $GJ \frac{\partial \theta}{\partial x}(0, t) = k_{t1} \theta(0, t) + c_{t1} \frac{\partial \theta}{\partial t}(0, t) + J_1 \frac{\partial^2 \theta}{\partial t^2}(0, t)$

At $x=l$, $GJ \frac{\partial \theta}{\partial x}(l, t) = -k_{t2} \theta(l, t) - c_{t2} \frac{\partial \theta}{\partial t}(l, t) - J_2 \frac{\partial^2 \theta}{\partial t^2}(l, t)$

8.23 $\theta(x, t) = \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c} \right) (C \cos \omega t + D \sin \omega t)$

$\theta(0, t) = 0 \Rightarrow A = 0$

$\frac{\partial \theta}{\partial x}(l, t) = 0 \Rightarrow \cos \frac{\omega l}{c} = 0 \quad ; \quad \frac{\omega_n l}{c} = (n - \frac{1}{2})\pi$

$\therefore \omega_n = (n - \frac{1}{2})\pi \sqrt{\frac{G}{Jl^2}} \quad ; \quad n = 1, 2, \dots$

8.24 $\theta(x, t) = \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c} \right) (C \cos \omega t + D \sin \omega t) \quad \text{----} (E_1)$

at $x=0$: $J_1 \frac{\partial^2 \theta}{\partial t^2} = GJ \frac{\partial \theta}{\partial x} \quad \text{----} (E_2)$

at $x=l$: $-J_2 \frac{\partial^2 \theta}{\partial t^2} = GJ \frac{\partial \theta}{\partial x} \quad \text{----} (E_3)$

Eqs. (E₁) and (E₂) give $(J_1 \omega^2) A + \left(\frac{GJ\omega}{c} \right) B = 0 \quad \text{----} (E_4)$

(E₁) and (E₃) give $\left(J_2 \omega^2 \cos \frac{\omega l}{c} + \frac{GJ\omega}{c} \sin \frac{\omega l}{c} \right) A + \left(J_2 \omega^2 \sin \frac{\omega l}{c} - \frac{GJ\omega}{c} \cos \frac{\omega l}{c} \right) B = 0 \quad \text{----} (E_5)$

Setting the determinant of the coefficients of A and B in (E₄) and (E₅) equal to zero, we get the frequency equation

$$\begin{vmatrix} J_1 \omega^2 & \frac{GJ\omega}{c} \\ \left(J_2 \omega^2 \cos \frac{\omega l}{c} + \frac{GJ\omega}{c} \sin \frac{\omega l}{c} \right) & \left(J_2 \omega^2 \sin \frac{\omega l}{c} - \frac{GJ\omega}{c} \cos \frac{\omega l}{c} \right) \end{vmatrix} = 0$$

i.e. $J_1 J_2 \omega^2 \sin \frac{\omega l}{c} - (J_1 + J_2) \frac{GJ\omega}{c} \cos \frac{\omega l}{c} - \frac{G^2 J^2}{c^2} \sin \frac{\omega l}{c} = 0$

8.25 Equation of motion: $c^2 \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 \theta}{\partial t^2}$ ----- (E₁)

Let steady state vibration be $\theta(x, t) = X(x) \cdot \cos \omega t$ ----- (E₂)

(E₁) and (E₂) give $\frac{d^2 X}{dx^2} + \frac{\omega^2}{c^2} X = 0$

Solution is $X(x) = A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c}$

$\theta(x=0, t) = 0 \Rightarrow X(x=0) = 0 \Rightarrow A = 0$

At free end, $GJ \frac{\partial \theta}{\partial x} (x=l, t) = M_{t0} \cos \omega t$

$B GJ \frac{\omega}{c} \cos \frac{\omega l}{c} \cos \omega t = M_{t0} \cos \omega t$

$B = \frac{M_{t0} c}{GJ \omega} \sec \frac{\omega l}{c}$

$\therefore \theta(x, t) = \frac{M_{t0} c}{GJ \omega} \sec \frac{\omega l}{c} \sin \frac{\omega x}{c} \cdot \cos \omega t$

8.26 From solution of problem 8.21, $\omega_n = \frac{n \pi}{l} \sqrt{\frac{G}{J}}$

$\therefore \omega_1 = \frac{\pi}{l} \sqrt{\frac{G}{J}} = \frac{\pi}{2} \sqrt{\frac{0.8 \times 10^{11}}{7800}} = 5030.5861 \text{ rad/sec}$

8.27 Let the angular displacement of the shaft be measured from the position occupied at the instant the shaft is stopped. Then the initial conditions are

$\theta(x, 0) = 0, \quad \frac{\partial \theta}{\partial t} (x, 0) = \underline{\omega}$ ----- (E₁)

Angular displacement of shaft is

$\theta(x, t) = \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c} \right) (C \cos \omega t + D \sin \omega t)$ --- (E₂)

As the shaft is supported at $x=0$ and free at $x=l$,

$\theta(0, t) = 0, \quad \frac{\partial \theta}{\partial x} (l, t) = 0$ ----- (E₃)

Eqs. (E₂) and (E₃) give

$A = 0$
 $B \frac{\omega}{c} \cos \frac{\omega l}{c} = 0 \quad \text{or} \quad \frac{\omega_n l}{c} = \frac{n \pi}{2}; n=1, 3, \dots$

$\theta_n(x, t) = \sin \frac{n \pi x}{2l} (C_n \cos \omega_n t + D_n \sin \omega_n t)$ ----- (E₄)

$\theta(x, t) = \sum_{n=1, 3, \dots}^{\infty} \sin \frac{n \pi x}{2l} (C_n \cos \omega_n t + D_n \sin \omega_n t)$ ----- (E₅)

At $t=0$, $\theta(x, 0) = \sum_{n=1, 3, \dots}^{\infty} \sin \frac{n \pi x}{2l} \cdot C_n$ ----- (E₆)

$\frac{\partial \theta}{\partial t} (x, 0) = \sum_{n=1, 3, \dots}^{\infty} \sin \frac{n \pi x}{2l} \omega_n D_n = \underline{\omega}$ ----- (E₇)

Since $\theta(x, 0) = 0$, $C_n = 0$. By multiplying both sides of (E₇) by $\sin \frac{n \pi x}{2l}$, integrating from 0 to l , and noting that

$$\int_0^l \sin^2 \frac{n\pi x}{2l} dx = \frac{l}{2} \quad \text{and} \quad \int_0^l \omega \sin \frac{n\pi x}{2l} dx = \omega \frac{2l}{n\pi}$$

we get $\omega \frac{2l}{n\pi} = \frac{\omega_n l}{2} D_n = \frac{n\pi c}{4} D_n \Rightarrow D_n = \frac{8l\omega}{n^2\pi^2 c}; n=1,3,\dots$

$$\theta(x,t) = \sum_{n=1,3,\dots}^{\infty} \sin \frac{n\pi x}{2l} \left(\frac{8l\omega}{n^2\pi^2 c} \right) \sin \omega_n t$$

8.28

$$I = \frac{1}{12} (0.1)(0.3)^3 = 2.25 \times 10^{-4} \text{ m}^4$$

$$A = 0.03 \text{ m}^2, \quad l = 2 \text{ m}, \quad E = 20.5 \times 10^{10} \text{ N/m}^2, \quad \rho = 7.83 \times 10^3 \text{ kg/m}^3$$

Fig. 8.15 gives the values of $\beta_n l$:

$$\omega_n = (\beta_n l)^2 \sqrt{\frac{EI}{\rho A l^4}}$$

$$\text{Here } \sqrt{\frac{EI}{\rho A l^4}} = \left\{ \frac{(20.5 \times 10^{10})(2.25 \times 10^{-4})}{7830 \times 0.03 \times 16} \right\}^{\frac{1}{2}} = 110.7814$$

(a) For pinned-pinned beam:

$$\beta_1 l = \pi, \quad \omega_1 = \pi^2 (110.7814) = 1093.3737 \text{ rad/sec}$$

$$\beta_2 l = 2\pi, \quad \omega_2 = 4\omega_1 = 4373.4948 \text{ rad/sec}$$

$$\beta_3 l = 3\pi, \quad \omega_3 = 9\omega_1 = 9840.3634 \text{ rad/sec}$$

(b) For fixed-fixed beam:

$$\beta_1 l = 4.730841, \quad \omega_1 = 2479.3826 \text{ rad/sec}$$

$$\beta_2 l = 7.853205, \quad \omega_2 = 6832.2023 \text{ rad/sec}$$

$$\beta_3 l = 10.995608, \quad \omega_3 = 13393.8474 \text{ rad/sec}$$

(c) For fixed-free beam:

$$\beta_1 l = 1.875104, \quad \omega_1 = 389.5091 \text{ rad/sec}$$

$$\beta_2 l = 4.694091, \quad \omega_2 = 2441.0117 \text{ rad/sec}$$

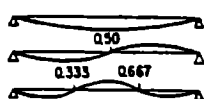
$$\beta_3 l = 7.854757, \quad \omega_3 = 6834.9030 \text{ rad/sec}$$

(d) For free-free beam:

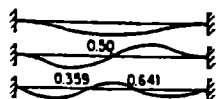
$$\beta_1 l = 0, \quad \omega_1 = 0; \quad \beta_2 l = 4.730841, \quad \omega_2 = 2479.3826 \text{ rad/sec}$$

$$\beta_3 l = 7.853205, \quad \omega_3 = 6832.2023 \text{ rad/sec.}$$

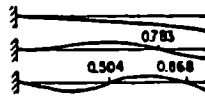
Mode shapes: The mode shapes are given in Fig. 8.15 (equations only). They result in the following.



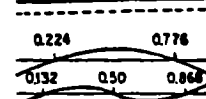
Pinned-pinned



Fixed-fixed



Fixed-free



Free-free

8.29

Boundary conditions are:

$$\text{At } x=0, \text{ fixed end } \Rightarrow W(0) = 0 \dots (E_1), \quad \frac{dW}{dx}(0) = 0 \dots (E_2)$$

$$\text{At } x=l, \text{ free end } \Rightarrow \frac{d^2W}{dx^2}(l) = 0 \dots (E_3), \quad \frac{d^3W}{dx^3}(l) = 0 \dots (E_4)$$

The deflection (normal) function is given by

$$W(x) = C_1 \cos \beta x + C_2 \sin \beta x + C_3 \cosh \beta x + C_4 \sinh \beta x \quad (E_5)$$

from which

$$\frac{dW}{dx}(x) = \beta [-C_1 \sin \beta x + C_2 \cos \beta x + C_3 \sinh \beta x + C_4 \cosh \beta x] \quad (E_6)$$

$$\text{Eqs. (E}_1\text{) and (E}_5\text{) give } C_1 + C_3 = 0 \quad (E_7)$$

$$\text{Eqs. (E}_2\text{) and (E}_6\text{) yield } \beta (C_2 + C_4) = 0 \quad (E_8)$$

Use of (E₇) and (E₈) in (E₅) leads to

$$W(x) = C_1 (\cos \beta x - \cosh \beta x) + C_2 (\sin \beta x - \sinh \beta x) \quad (E_9)$$

Use of Eqs. (E₃), (E₄) and (E₉) yields

$$C_1 (\cos \beta l + \cosh \beta l) + C_2 (\sin \beta l + \sinh \beta l) = 0 \quad (E_{10})$$

$$C_1 (\sin \beta l - \sinh \beta l) - C_2 (\cos \beta l + \cosh \beta l) = 0 \quad (E_{11})$$

The frequency equation can be obtained by setting the coefficient matrix in (E₁₀) and (E₁₁) to zero as

$$\begin{vmatrix} \cos \beta l + \cosh \beta l & \sin \beta l + \sinh \beta l \\ \sin \beta l - \sinh \beta l & -\cos \beta l - \cosh \beta l \end{vmatrix} = 0 \quad (E_{12})$$

Upon simplification, Eq. (E₁₂) yields the frequency equation

$$\cos \beta l \cdot \cosh \beta l = -1 \quad (E_{13})$$

First four roots of (E₁₃) are given by

$$\beta_1 l = 1.875104, \quad \beta_2 l = 4.694091, \quad \beta_3 l = 7.854757, \quad \beta_4 l = 10.995541.$$

8.30

Boundary conditions are:

$$\text{At } x=0; \quad EI \frac{d^3 W}{dx^3} = 0 \quad \dots (E_1), \quad EI \frac{d^2 W}{dx^2} = -k_t \frac{dW}{dx} \quad \dots (E_2)$$

$$\text{At } x=l; \quad EI \frac{d^3 W}{dx^3} = k W \quad \dots (E_3), \quad EI \frac{d^2 W}{dx^2} = 0 \quad \dots (E_4)$$

Eq. (8.105) gives

$$\int_0^l W_i W_j dx = - \frac{c^2}{\omega_i^2 - \omega_j^2} \left[W_i W_j''' - W_j W_i''' + W_j' W_i'' - W_i' W_j'' \right]_0^l \quad \dots (E_5)$$

At $x=l$, Eq. (E₄) gives

$$W_i'' = 0 \quad \dots (E_6), \quad W_j'' = 0 \quad \dots (E_7)$$

Eq. (E₃) gives

$$EI W_i''' = k W_i \quad \dots (E_8), \quad EI W_j''' = k W_j \quad \dots (E_9)$$

Multiplying (E₈) by W_j , (E₉) by W_i , and subtracting one from the other, we get

$$EI (W_j''' W_i - W_i''' W_j) = k (W_j W_i - W_i W_j) = 0 \quad \dots (E_{10})$$

This shows that the r.h.s. of (E₅) is zero at $x=l$.

$$\text{At } x=0, \text{ Eq. (E}_1\text{) gives } W_i''' = 0 \quad \dots (E_{11}), \quad W_j''' = 0 \quad \dots (E_{12})$$

Eq. (E₂) gives

$$EI W_i'' = -k_t W_i' \quad \dots (E_{13}), \quad EI W_j'' = -k_t W_j' \quad \dots (E_{14})$$

Multiplying (E₁₃) by W_j' , (E₁₄) by W_i' , and subtracting one from the other, we get

$$EI (W_i'' W_j' - W_j'' W_i') = -k_t (W_i' W_j' - W_j' W_i') = 0 \quad \dots (E_{15})$$

This shows that the r.h.s. of Eq. (E₅) is zero at $x=0$.

8.31

$$w(0,t) = \frac{\partial^2 w}{\partial x^2}(0,t) = w(l,t) = \frac{\partial^2 w}{\partial x^2}(l,t) = 0; \quad t \geq 0$$

$$W(x) = C_1 (\cos \beta x + \cosh \beta x) + C_2 (\cos \beta x - \cosh \beta x)$$

$$+ C_3 (\sin \beta x + \sinh \beta x) + C_4 (\sin \beta x - \sinh \beta x)$$

$$\frac{d^2 W}{dx^2}(x) = C_1 \beta^2 (-\cos \beta x + \cosh \beta x) + C_2 \beta^2 (-\cos \beta x - \cosh \beta x) \\ + C_3 \beta^2 (-\sin \beta x + \sinh \beta x) + C_4 \beta^2 (-\sin \beta x - \sinh \beta x)$$

$$W(0) = 0 \Rightarrow C_1 = 0$$

$$\frac{d^2 W}{dx^2}(0) = 0 \Rightarrow C_2 = 0$$

$$W(l) = 0 \Rightarrow C_3 (\sin \beta l + \sinh \beta l) + C_4 (\sin \beta l - \sinh \beta l) = 0 \quad \dots (E_1)$$

$$\frac{d^2 W}{dx^2}(l) = 0 \Rightarrow C_3 (-\sin \beta l + \sinh \beta l) - C_4 (\sin \beta l + \sinh \beta l) = 0 \quad \dots (E_2)$$

Setting the determinant of the coefficient matrix of C_3 and C_4 in (E₁) and (E₂) to zero, we get the frequency equation

$$-(\sin \beta l + \sinh \beta l)^2 + (\sin \beta l - \sinh \beta l)^2 = 0$$

$$\text{or } \sin \beta l \sinh \beta l = 0$$

Since $\sinh \beta l \neq 0$, the frequency equation becomes

$$\sin \beta l = 0$$

$$\therefore \beta_n l = n\pi \quad ; \quad \omega_n = \frac{n^2 \pi^2}{l^2} \sqrt{\frac{EI}{\rho A}} \quad ; \quad n = 1, 2, \dots$$

8.32

$$w(0, t) = \frac{\partial^2 w}{\partial x^2}(0, t) = \frac{\partial^2 w}{\partial x^2}(l, t) = \frac{\partial^3 w}{\partial x^3}(l, t) = 0 \quad ; \quad t \geq 0$$

$$W(x) = C_1(\cos \beta x + \cosh \beta x) + C_2(\cos \beta x - \cosh \beta x) \\ + C_3(\sin \beta x + \sinh \beta x) + C_4(\sin \beta x - \sinh \beta x)$$

$$\frac{d^2 W}{dx^2}(x) = C_1 \beta^2 (-\cos \beta x + \cosh \beta x) + C_2 \beta^2 (-\cos \beta x - \cosh \beta x) \\ + C_3 \beta^2 (-\sin \beta x + \sinh \beta x) + C_4 \beta^2 (-\sin \beta x - \sinh \beta x)$$

$$\frac{d^3 W}{dx^3}(x) = C_1 \beta^3 (\sin \beta x + \sinh \beta x) + C_2 \beta^3 (\sin \beta x - \sinh \beta x) \\ + C_3 \beta^3 (-\cos \beta x + \cosh \beta x) + C_4 \beta^3 (-\cos \beta x - \cosh \beta x)$$

$$W(0) = 0 \Rightarrow C_1 = 0$$

$$\frac{d^2 W}{dx^2}(0) = 0 \Rightarrow C_2 = 0$$

$$\frac{d^2 W}{dx^2}(l) = 0 \Rightarrow C_3(-\sin \beta l + \sinh \beta l) - C_4(\sin \beta l + \sinh \beta l) = 0 \quad \text{--- (E}_1\text{)}$$

$$\frac{d^3 W}{dx^3}(l) = 0 \Rightarrow C_3(-\cos \beta l + \cosh \beta l) - C_4(\cos \beta l + \cosh \beta l) = 0 \quad \text{--- (E}_2\text{)}$$

Setting the determinant of coefficients of C_3 and C_4 in Eqs.

(E₁) and (E₂) to zero, the frequency equation can be obtained:

$$-(-\sin \beta l + \sinh \beta l)(\cos \beta l + \cosh \beta l) + (\sin \beta l + \sinh \beta l)(-\cos \beta l \\ + \cosh \beta l) = 0$$

$$\text{or } \tan \beta l - \tanh \beta l = 0$$

8.33

For a simply supported beam, the first three natural frequencies are given by

$$\omega_1 = (\beta_1 l)^2 (EI/\rho A l^4)^{\frac{1}{2}} = \pi^2 (EI/\rho A l^4)^{\frac{1}{2}}$$

$$\omega_2 = (\beta_2 l)^2 (EI/\rho A l^4)^{\frac{1}{2}} = 4\pi^2 (EI/\rho A l^4)^{\frac{1}{2}}$$

$$\omega_3 = (\beta_3 l)^2 (EI/\rho A l^4)^{\frac{1}{2}} = 9\pi^2 (EI/\rho A l^4)^{\frac{1}{2}}$$

For steel, $E = 2.07 \times 10^{11} \text{ N/m}^2$, $\rho = 7880 \text{ kg/m}^3$, $l = 1.0 \text{ m}$.

Setting $\omega_1 \geq 1500 \text{ Hz} = 9424.8 \text{ rad/sec} = \pi^2 \left\{ \frac{(2.07 \times 10^{11}) I}{(7880) A (1)} \right\}^{1/2}$

we get

$$(9424.8)^2 \geq \pi^4 \left(\frac{2.07 \times 10^{11}}{7880} \right) \left(\frac{I}{A} \right)$$

or $\frac{I}{A} \leq 0.03471$ (E₁)

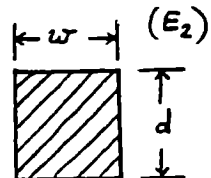
Setting $\omega_3 \leq 5000 \text{ Hz} = 31416.0 \text{ rad/sec} = 9\pi^2 \left\{ \frac{(2.07 \times 10^{11}) I}{7880 A (1)} \right\}^{1/2}$

we get

$$\frac{I}{A} \geq 0.0047617$$

Let the beam have a rectangular section

$$A = wd, \quad I = \frac{1}{12} wd^3, \quad \frac{I}{A} = \frac{d^2}{12}$$



Let $\frac{I}{A} = 0.005$ to satisfy inequalities (E₁) and (E₂).

Then $d^2 = 0.06$ or $d = 0.2449 \text{ m}$

Taking $w = 0.1 \text{ m}$ (w can have any value), we get

$$A = wd = 0.02449 \text{ m}^2 \quad \text{and} \quad I = 1.2240 \times 10^{-4} \text{ m}^4.$$

8.34

From the solution of problem 8.31, we find for $\sin \beta l = 0$,

$$C_3 = C_4$$

Hence $w_n(x) = C_n \sin \beta_n x = C_n \sin \frac{n\pi x}{l}$

General free vibration solution is

$$w(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} (A_n \cos \omega_n t + B_n \sin \omega_n t)$$

When beam vibrates in the first mode,

$$w(x,t) = \sin \frac{\pi x}{l} (A_1 \cos \omega_1 t + B_1 \sin \omega_1 t)$$

Bending moment in the beam is

$$M(x,t) = EI \frac{\partial^2 w}{\partial x^2}(x,t) = -\frac{\pi^2 EI}{l^2} \sin \frac{\pi x}{l} (A_1 \cos \omega_1 t + B_1 \sin \omega_1 t)$$

$$= -\frac{\pi^2}{l^2} EI w(x,t)$$

If the amplitude of vibration is Y , the maximum bending moment is

$$|M_{\max}| = \frac{\pi^2}{l^2} EI Y$$

For the given data,

$$|M_{\max}| = \frac{\pi^2}{(1)^2} (20.5 \times 10^{10}) \left(\frac{10^3}{10^{12}} \right) \left(\frac{10}{1000} \right) = 20.2328 \text{ N-m}$$

8.35

$$W(x) = C_1 \cos \beta x + C_2 \sin \beta x + C_3 \cosh \beta x + C_4 \sinh \beta x$$

$$\frac{d^2 W}{dx^2}(x) = -C_1 \beta^2 \cos \beta x - C_2 \beta^2 \sin \beta x + C_3 \beta^2 \cosh \beta x + C_4 \beta^2 \sinh \beta x$$

$$\frac{d^3 W}{dx^3}(x) = C_1 \beta^3 \sin \beta x - C_2 \beta^3 \cos \beta x + C_3 \beta^3 \sinh \beta x + C_4 \beta^3 \cosh \beta x$$

$$EI \frac{d^2 W}{dx^2}(0) = 0 \Rightarrow C_1 = C_3 \quad \text{--- (E}_1\text{)}$$

$$EI \frac{d^3 W}{dx^3}(0) = -\tilde{k}_1 W(0) \Rightarrow EI \beta^3 (-C_2 + C_4) + \tilde{k}_1 (C_1 + C_3) = 0$$

$$C_2 (-EI \beta^3) + C_3 (+2 \tilde{k}_1) + C_4 (EI \beta^3) = 0 \quad \text{--- (E}_2\text{)}$$

$$EI \frac{d^2 W}{dx^2}(l) = 0 \Rightarrow C_2 (-\sin \beta l) + C_3 (\cosh \beta l - \cos \beta l) + C_4 (\sinh \beta l) = 0 \quad \text{--- (E}_3\text{)}$$

$$EI \frac{d^3 W}{dx^3}(l) = +\tilde{k}_2 W(l) \Rightarrow$$

$$C_2 (-EI \beta^3 \cos \beta l - \tilde{k}_2 \sin \beta l) + C_3 (EI \beta^3 \sin \beta l + EI \beta^3 \sinh \beta l - \tilde{k}_2 \cos \beta l - \tilde{k}_2 \cosh \beta l) + C_4 (EI \beta^3 \cosh \beta l - \tilde{k}_2 \sinh \beta l) = 0 \quad \text{--- (E}_4\text{)}$$

Setting the determinant of the coefficients of C_2 , C_3 and C_4 in (E₂) to (E₄) to zero, we get the frequency equation:

$$\begin{vmatrix} -EI \beta^3 & +2 \tilde{k}_1 & EI \beta^3 \\ -\sin \beta l & (\cosh \beta l - \cos \beta l) & \sinh \beta l \\ (-EI \beta^3 \cos \beta l - \tilde{k}_2 \sin \beta l) & (EI \beta^3 \{\sin \beta l + \sinh \beta l\} - \tilde{k}_2 \{\cos \beta l + \cosh \beta l\}) & (EI \beta^3 \cosh \beta l - \tilde{k}_2 \sinh \beta l) \end{vmatrix} = 0$$

8.36

Because of symmetry, consider only half of beam to the left of M. Boundary conditions are:

$$w(0, t) = 0, \quad EI \frac{\partial^2 w}{\partial x^2}(0, t) = 0$$

$$\text{For symmetry, } \frac{\partial w}{\partial x} = 0 \text{ at } x = \frac{l}{2}.$$

$$EI \frac{\partial^3 w}{\partial x^3} = +\frac{M}{2} \frac{\partial^2 w}{\partial t^2} \text{ at } x = \frac{l}{2}$$

$$W(x) = C_1 (\cos \beta x + \cosh \beta x) + C_2 (\cos \beta x - \cosh \beta x) + C_3 (\sin \beta x + \sinh \beta x) + C_4 (\sin \beta x - \sinh \beta x)$$

$$W(0) = 0 \Rightarrow C_1 = 0$$

$$\frac{d^2 W}{dx^2}(0) = 0 \Rightarrow C_2 = 0$$

$$W(x) = C_3 (\sin \beta x + \sinh \beta x) + C_4 (\sin \beta x - \sinh \beta x)$$

$$\frac{dW}{dx}(x) = \beta C_3 (\cos \beta x + \cosh \beta x) + \beta C_4 (\cos \beta x - \cosh \beta x)$$

$$\frac{d^3 W}{dx^3}(x) = \beta^3 C_3 (-\cos \beta x + \cosh \beta x) - \beta^3 C_4 (\cos \beta x + \cosh \beta x)$$

$$\frac{dW}{dx}\left(\frac{l}{2}\right) = 0 \Rightarrow C_3 \left(\cos \frac{\beta l}{2} + \cosh \frac{\beta l}{2}\right) + C_4 \left(\cos \frac{\beta l}{2} - \cosh \frac{\beta l}{2}\right) = 0 \quad \text{--- (E}_1\text{)}$$

$$EI \frac{d^3 W}{dx^3}\left(\frac{l}{2}\right) = -\frac{M}{2} \omega^2 W\left(\frac{l}{2}\right) \Rightarrow$$

$$EI \beta^3 \left[C_3 \left(-\cos \frac{\beta l}{2} + \cosh \frac{\beta l}{2}\right) - C_4 \left(\cos \frac{\beta l}{2} + \cosh \frac{\beta l}{2}\right) \right] = -\frac{M}{2} \omega^2 \left[C_3 \left(\sin \frac{\beta l}{2} + \sinh \frac{\beta l}{2}\right) + C_4 \left(\sin \frac{\beta l}{2} - \sinh \frac{\beta l}{2}\right) \right] \quad \text{--- (E}_2\text{)}$$

i.e. $R_1 \cos \frac{\beta l}{2} + R_2 \cosh \frac{\beta l}{2} = 0$

$$R_1 \left(-\cos \frac{\beta l}{2} + \lambda \sin \frac{\beta l}{2}\right) + R_2 \left(\cosh \frac{\beta l}{2} + \lambda \sinh \frac{\beta l}{2}\right) = 0$$

where $R_1 = C_3 + C_4$, $R_2 = C_3 - C_4$, $\lambda = \frac{M \omega^2}{2EI \beta^3}$.

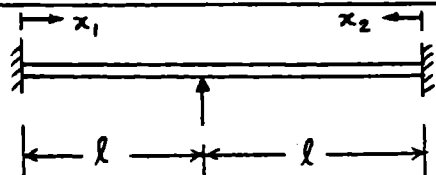
By setting the coefficient matrix of R_1 and R_2 equal to zero, we obtain the frequency equation:

$$\cos \frac{\beta l}{2} \left(\cosh \frac{\beta l}{2} + \lambda \sinh \frac{\beta l}{2}\right) + \cosh \frac{\beta l}{2} \left(\cos \frac{\beta l}{2} + \lambda \sin \frac{\beta l}{2}\right) = 0$$

or $\lambda \left(-\tanh \frac{\beta l}{2} + \tan \frac{\beta l}{2}\right) = 2$

8.37

For convenience, set up two coordinates x_1 and x_2 as shown.



$$W_1(x_1) = C_1 \cos \beta x_1 + C_2 \sin \beta x_1 + C_3 \cosh \beta x_1 + C_4 \sinh \beta x_1$$

$$\frac{dW_1}{dx_1}(x_1) = -C_1 \beta \sin \beta x_1 + C_2 \beta \cos \beta x_1 + C_3 \beta \sinh \beta x_1 + C_4 \beta \cosh \beta x_1$$

$$W_1(0) = 0 \Rightarrow C_1 + C_3 = 0 ; \quad C_1 = -C_3$$

$$\frac{dW_1}{dx_1}(0) = 0 \Rightarrow C_2 + C_4 = 0 ; \quad C_2 = -C_4$$

$$W_1(x_1) = C_3 (\cosh \beta x_1 - \cos \beta x_1) + C_4 (\sinh \beta x_1 - \sin \beta x_1)$$

$$W_1(l) = 0 \Rightarrow C_3 = -C_4 \left(\frac{\sin \beta l - \sinh \beta l}{\cos \beta l - \cosh \beta l} \right)$$

$$W_1(x_1) = C_4 \left[(\sinh \beta x_1 - \sin \beta x_1) - \left(\frac{\sin \beta l - \sinh \beta l}{\cos \beta l - \cosh \beta l} \right) (\cosh \beta x_1 - \cos \beta x_1) \right] \quad \text{--- (E}_1\text{)}$$

$$W_2(x_2) = C'_1 \cos \beta x_2 + C'_2 \sin \beta x_2 + C'_3 \cosh \beta x_2 + C'_4 \sinh \beta x_2$$

$$\frac{dW_2}{dx_2}(x_2) = -c'_1 \beta \sin \beta x_2 + c'_2 \beta \cos \beta x_2 + c'_3 \beta \sinh \beta x_2 + c'_4 \beta \cosh \beta x_2$$

$$W_2(0) = 0 \Rightarrow c'_1 = -c'_3$$

$$\frac{dW_2}{dx_2}(0) = 0 \Rightarrow c'_2 = -c'_4$$

$$W_2(l) = 0 \Rightarrow c'_3 = -c'_4 \left(\frac{\sin \beta l - \sinh \beta l}{\cos \beta l - \cosh \beta l} \right)$$

$$W_2(x_2) = c'_4 \left[(\sinh \beta x_2 - \sin \beta x_2) - \left(\frac{\sin \beta l - \sinh \beta l}{\cos \beta l - \cosh \beta l} \right) (\cosh \beta x_2 - \cos \beta x_2) \right] \quad \text{--- (E}_2\text{)}$$

$$\frac{dW_1}{dx_1}(x_1=l) = -\frac{dW_2}{dx_2}(x_2=l) \quad \text{gives}$$

$$c_4 \left[(\cosh \beta l - \cos \beta l) - \left(\frac{\sin \beta l - \sinh \beta l}{\cos \beta l - \cosh \beta l} \right) (\sinh \beta l + \sin \beta l) \right]$$

$$+ c'_4 \left[(\cosh \beta l - \cos \beta l) - \left(\frac{\sin \beta l - \sinh \beta l}{\cos \beta l - \cosh \beta l} \right) (\sinh \beta l + \sin \beta l) \right] = 0 \quad \text{--- (E}_3\text{)}$$

$$\frac{d^2 W_1}{dx_1^2}(x_1=l) = -\frac{d^2 W_2}{dx_2^2}(x_2=l) \quad \text{gives}$$

$$c_4 \left[(\sinh \beta l + \sin \beta l) - \left(\frac{\sin \beta l - \sinh \beta l}{\cos \beta l - \cosh \beta l} \right) (\cosh \beta l + \cos \beta l) \right]$$

$$+ c'_4 \left[(\sinh \beta l + \sin \beta l) - \left(\frac{\sin \beta l - \sinh \beta l}{\cos \beta l - \cosh \beta l} \right) (\cosh \beta l + \cos \beta l) \right] = 0 \quad \text{--- (E}_4\text{)}$$

Since the determinant of the coefficients of c_4 and c'_4 in (E₃) and (E₄) is identically zero, we set the coefficients of c_4 and c'_4 in (E₃) and (E₄) to zero. This leads to the frequency equations

$$\cos \beta l \cosh \beta l = 1$$

$$\text{and } \tan \beta l = \tanh \beta l$$

8.38

General free vibration solution of a simply supported beam is

$$w(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} (A_n \cos \omega_n t + B_n \sin \omega_n t)$$

$$\text{At } t=0, \quad \frac{\partial w}{\partial t}(x,0) = 0 \quad \text{and} \quad EI \frac{\partial^4 w}{\partial x^4}(x,0) = f_0$$

$$\text{Thus } \sum_{n=1}^{\infty} B_n \omega_n \sin \frac{n\pi x}{l} = 0 \quad \& \quad B_n = 0$$

$$\text{and } EI \sum_{n=1}^{\infty} A_n \left(\frac{n\pi}{l} \right)^4 \sin \frac{n\pi x}{l} = f_0 \quad \text{--- (E}_1\text{)}$$

Multiplying Eq. (E₁) by $\sin \frac{n\pi x}{l}$ and integrating from 0 to l , we obtain

$$A_n = \frac{2l^3}{EI n^4 \pi^4} \int_0^l f_0 \sin \frac{n\pi x}{l} dx = \frac{2l^4 f_0}{EI n^5 \pi^5} (1 - \cos n\pi)$$

$$= \frac{4f_0 l^4}{EI n^5 \pi^5} \quad ; \quad n = 1, 3, 5, \dots$$

\therefore Solution is $w(x, t) = \frac{4f_0 l^4}{\pi^5 EI} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^5} \sin \frac{n\pi x}{l} \cos \omega_n t$

8.39

Let $W(x) = (1 - \frac{x}{l})^2$; $\frac{dW}{dx} = -\frac{2}{l}(1 - \frac{x}{l})$; $\frac{d^2W}{dx^2} = \frac{2}{l^2}$

$$N = \int_0^l EI \left(\frac{d^2W}{dx^2} \right)^2 dx = \frac{EI_0}{l} \int_0^l x \left(\frac{4}{l^2} \right) dx = \frac{4EI_0}{l^3}$$

$$D = \int_0^l \rho A \{W(x)\}^2 dx = \frac{\rho A_0}{l} \int_0^l x \left(1 - \frac{x}{l}\right)^4 dx = \frac{\rho A_0 l}{30}$$

$$\omega^2 = \frac{N}{D} \approx \frac{4EI_0 (30)}{l^3 (\rho A_0 l)} = \frac{120 EI_0}{\rho A_0 l^4}$$

$$\omega \approx \sqrt{120} \left(\frac{EI_0}{\rho A_0 l^4} \right)^{1/2}$$

8.40

(a) Equation of motion for a uniform beam is

$$EI \frac{\partial^4 w}{\partial x^4} = f(x, t) - \rho A \frac{\partial^2 w}{\partial t^2} \quad \text{---- (E}_1\text{)}$$

Let $w(x, t) = \sum_{n=1}^{\infty} W_n(x) T_n(t)$ where $W_n(x) = n^{\text{th}}$ normal mode

satisfying the boundary conditions and the equation

$$EI \frac{d^4 W}{dx^4} = \omega_n^2 \rho A W_n \quad ; \quad n = 1, 2, \dots \quad \text{---- (E}_2\text{)}$$

Eq. (E₁) becomes for the assumed solution,

$$EI \sum_{n=1}^{\infty} \frac{d^4 W}{dx^4} \cdot T_n(t) = f(x, t) - \rho A \sum_{n=1}^{\infty} W_n(x) \cdot \frac{d^2 T_n}{dt^2} \quad \text{---- (E}_3\text{)}$$

which, in view of (E₂), becomes

$$\sum_{n=1}^{\infty} \omega_n^2 W_n(x) T_n(t) = \frac{1}{\rho A} f(x, t) - \sum_{n=1}^{\infty} W_n(x) \frac{d^2 T_n}{dt^2} \quad \text{---- (E}_4\text{)}$$

Multiplying Eq. (E₄) by $W_m(x)$ and integrating from 0 to l , we get

$$\frac{d^2 T_n}{dt^2} + \omega_n^2 T_n = \frac{2}{\rho A l} F_n(t) \quad \text{---- (E}_5\text{)}$$

where

$$F_n(t) = \int_0^l f(x, t) W_n(x) dx \quad \text{---- (E}_6\text{)}$$

Note that the orthogonality of normal modes, namely,

$$\int_0^l W_m(x) W_n(x) dx = \begin{cases} 0 & \text{if } m = n \\ l/2 & \text{if } m \neq n \end{cases}$$

is used in deriving (E5). The solution of Eq. (E5) can be written as

$$T_n(t) = A_n \cos \omega_n t + B_n \sin \omega_n t + \frac{2}{\rho A l \omega_n} \int_0^l F_n(\tau) \sin \omega_n(t-\tau) d\tau$$

where A_n and B_n are constants to be determined from initial conditions. Thus the total solution can be expressed as

$$w(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \omega_n t + B_n \sin \omega_n t + \frac{2}{\rho A l \omega_n} \int_0^t F_n(\tau) \sin \omega_n(t-\tau) d\tau \right] W_n(x) \quad \text{---- (E8)}$$

(b) For a simply supported beam,

$$W_n(x) = \sin \frac{n\pi x}{l}$$

For the given loading, (E6) becomes $F_n(t) = F_0 \sin \frac{n\pi a}{l} \sin \omega t$

Hence (E7) becomes

$$T_n(t) = \frac{2 F_0 l^3}{EI \pi^4} \sin \frac{n\pi a}{l} \left[\frac{1}{n^4 - \left(\frac{\omega}{\omega_1}\right)^2} \sin \omega t - \frac{\left(\frac{\omega}{\omega_1}\right)}{n^2 \{n^4 - \left(\frac{\omega}{\omega_1}\right)^2\}} \sin n^2 \omega_1 t \right]$$

where $\omega_1 = \frac{\pi^2}{l^2} \sqrt{\frac{EI}{\rho A}}$ = fundamental frequency of the beam

Thus the forced response of the beam is given by

$$w(x,t) = \frac{2 F_0 l^3}{\pi^4 EI} \left[\sum_{n=1}^{\infty} \frac{\sin \left(\frac{n\pi a}{l}\right)}{n^4 - \left(\frac{\omega}{\omega_1}\right)^2} \sin \frac{n\pi x}{l} \right] \sin \omega t - \frac{2 \left(\frac{\omega}{\omega_1}\right) F_0 l^3}{\pi^4 EI} \left[\sum_{n=1}^{\infty} \frac{\sin \left(\frac{n\pi a}{l}\right)}{n^2 \{n^4 - \left(\frac{\omega}{\omega_1}\right)^2\}} \sin \frac{n\pi x}{l} \sin n^2 \omega_1 t \right]$$

8.41

Derivation of Eq. (E5):

If shear deformation is zero, $M = EI \frac{\partial^2 w}{\partial x^2}$

$$\text{Eq. (8.131)} \Rightarrow \frac{\partial V}{\partial x} = -\rho A \frac{\partial^2 w}{\partial t^2} + f$$

$$\text{Eq. (8.132)} \Rightarrow \frac{\partial M}{\partial x} - V = \rho I \frac{\partial^2 \phi}{\partial t^2}, \quad \frac{\partial^2 M}{\partial x^2} - \frac{\partial V}{\partial x} = \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2}$$

$$\text{or } \frac{\partial^2 M}{\partial x^2} + \rho A \frac{\partial^2 w}{\partial t^2} - f = \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2}$$

$$\text{or } EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} - \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2} - f = 0 \quad \text{---- (E.1)}$$

Derivation of Eq. (E6):

Eq. (E.1) becomes, for $f=0$,

$$\frac{EI}{\rho A} \frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} - \frac{I}{A} \frac{\partial^4 w}{\partial x^2 \partial t^2} = 0 \quad \text{--- (E.2)}$$

If $w(x,t) = C \sin \frac{n\pi x}{l} \cos \omega_n t$, Eq. (E.2) becomes

$$\alpha^2 \left(\frac{n\pi}{l} \right)^4 - \omega_n^2 - r^2 \left(\frac{n\pi}{l} \right)^2 \omega_n^2 = 0 \quad \left. \begin{array}{l} \alpha^2 = \frac{EI}{\rho A} \\ r^2 = \frac{I}{A} \end{array} \right\}$$

$$\text{i.e. } \omega_n^2 = \frac{\alpha^2 \left(\frac{n\pi}{l} \right)^4}{1 + r^2 \left(\frac{n\pi}{l} \right)^2} = \frac{\alpha^2 n^4 \pi^4}{l^4 \left(1 + \frac{n^2 \pi^2 r^2}{l^2} \right)}$$

8.42

Derivation of (E₇):

If rotary inertia is neglected,

$$M = EI \frac{\partial \phi}{\partial x} \quad \text{(E.1)}$$

$$V = \kappa AG \left(\phi - \frac{\partial w}{\partial x} \right) \quad \text{(E.2)}$$

Eq. (8.131) \Rightarrow

$$-\frac{\partial V}{\partial x} + f = \rho A \frac{\partial^2 w}{\partial t^2} \quad \text{(E.3)}$$

Eq. (8.132) \Rightarrow

$$\frac{\partial M}{\partial x} - V = 0 \quad \text{(E.4)}$$

Using (E.1) and (E.2), (E.3) and (E.4) can be written as

$$-\kappa AG \left(\frac{\partial \phi}{\partial x} - \frac{\partial^2 w}{\partial x^2} \right) + f = \rho A \frac{\partial^2 w}{\partial t^2}$$

$$\text{or } \frac{\partial \phi}{\partial x} = \frac{\partial^2 w}{\partial x^2} + \frac{f}{\kappa AG} - \frac{\rho A}{\kappa AG} \frac{\partial^2 w}{\partial t^2} \quad \text{(E.5)}$$

$$\text{and } EI \frac{\partial^2 \phi}{\partial x^2} - \kappa AG \left(\phi - \frac{\partial w}{\partial x} \right) = 0$$

$$\text{or } EI \frac{\partial^2}{\partial x^2} \left(\frac{\partial \phi}{\partial x} \right) - \kappa AG \frac{\partial \phi}{\partial x} + \kappa AG \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{(E.6)}$$

Substitution of (E.5) into (E.6) gives

$$EI \frac{\partial^4 w}{\partial x^4} + \frac{EI}{\kappa AG} \frac{\partial^2 f}{\partial x^2} - \frac{\rho AEI}{\kappa AG} \frac{\partial^4 w}{\partial x^2 \partial t^2} - f + \rho A \frac{\partial^2 w}{\partial t^2} = 0 \quad \text{(E.7)}$$

$$\text{If } f=0, \text{ (E.7) becomes } EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} - \frac{\rho EI}{\kappa G} \frac{\partial^4 w}{\partial x^2 \partial t^2} = 0 \quad \text{(E.8)}$$

Derivation of (E₈):

With $w(x,t) = C \sin \frac{n\pi x}{l} \cos \omega_n t$, (E.8) gives

$$\omega_n^2 = \frac{\alpha^2 n^4 \pi^4}{l^4 \left(1 + \frac{E}{\kappa G} \cdot \frac{r^2 n^2 \pi^2}{l^2} \right)}$$

8.43

Multiply Eq. (8.83) by $w(x)$ and integrate from 0 to l :

$$\int_0^l W(x) \frac{d^4 W}{dx^4} dx = \frac{\omega^2}{c^2} \int_0^l [W(x)]^2 dx$$

This shows that the sign of ω^2 will be same as that of the

left hand side term. Integrating the left hand side expression by parts, we get

$$\left. \frac{d^3 W}{dx^3}(x) \cdot W(x) \right|_0^l - \left. \frac{d^2 W}{dx^2}(x) \cdot \frac{dW}{dx}(x) \right|_0^l + \int_0^l \left[\frac{d^2 W}{dx^2}(x) \right]^2 dx \quad (E_1)$$

For common boundary conditions, the first two terms in (E₁) will be zero, and (E₁) will be positive. Hence ω^2 is positive.

Common boundary conditions:

Simply supported end: $W(x) = \frac{d^2 W}{dx^2} = 0$

Fixed end: $W(x) = \frac{dW}{dx} = 0$

Free end: $\frac{d^2 W}{dx^2}(x) = \frac{d^3 W}{dx^3} = 0$

8.44

Equation of motion:

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = F_0 \sin \omega t \quad (1)$$

For steady state response, we assume the particular solution as

$$w(x, t) = W(x) \sin \omega t \quad (2)$$

Substitution of Eq. (2) into (1) gives

$$\frac{d^4 W}{dx^4} - \frac{\omega^2}{c^2} W = \frac{F_0}{\rho A c^2} \quad (3)$$

$$\text{where } c^2 = \frac{EI}{\rho A} \quad (4)$$

The complete solution of Eq. (3) can be written as

$$W(x) = C_1 \cos \beta x + C_2 \sin \beta x + C_3 \cosh \beta x + C_4 \sinh \beta x - \frac{F_0}{\rho A c^2} \quad (5)$$

$$\text{where } \beta^2 = \frac{\omega^2}{c^2} \quad (6)$$

The boundary conditions for a simply supported beam are

$$W(x=0) = 0 \quad (7)$$

$$\frac{d^2 W}{dx^2}(x=0) = 0 \quad (8)$$

$$W(x=l) = 0 \quad (9)$$

$$\frac{d^2 W}{dx^2}(x=l) = 0 \quad (10)$$

Equations (7) and (5) give:

$$C_1 + C_3 - \frac{F_0}{\rho A c^2} = 0 \quad (11)$$

Equations (8) and (5) yield:

$$-C_1 \beta^2 + C_3 \beta^2 = 0 \quad \text{or} \quad C_1 = C_3 \quad (12)$$

Eqs. (11) and (12) provide:

$$C_1 = C_3 = \frac{F_0}{2 \rho A c^2} \quad (13)$$

Eqs. (9) and (5) give:

$$C_2 \sin \beta \ell + C_4 \sinh \beta \ell = -\frac{F_0}{\rho A c^2} \left(\frac{\cos \beta \ell + \cosh \beta \ell}{2} - 1 \right) \quad (14)$$

Eqs. (10) and (5) yield:

$$-C_2 \sin \beta \ell + C_4 \sinh \beta \ell = \frac{F_0}{2 \rho A c^2} (\cos \beta \ell - \cosh \beta \ell) \quad (15)$$

Solution of Eqs. (14) and (15) gives:

$$C_4 = \frac{F_0}{2 \rho A c^2 \sinh \beta \ell} (1 - \cosh \beta \ell) = -\frac{F_0}{2 \rho A c^2} \tanh \frac{\beta \ell}{2} \quad (16)$$

$$C_2 = -\frac{F_0}{2 \rho A c^2 \sin \beta \ell} (1 - \cos \beta \ell) = \frac{F_0}{2 \rho A c^2} \tan \frac{\beta \ell}{2} \quad (17)$$

Thus the complete solution becomes

$$w(x,t) = \frac{F_0}{2 \rho A c^2} \left[\left(\cos \beta x + \cosh \beta x \right) + \tan \frac{\beta \ell}{2} \sin \beta x - \tanh \frac{\beta \ell}{2} \sinh \beta x - 2 \right] \sin \omega t$$

8.45

$$\omega = \frac{1000}{60} (2 \pi) = 104.72 \text{ rad/sec.}$$

$$F(t) = m e \omega^2 \sin \omega t = (0.5) (104.72^2) \sin 104.72 t = F_0 \sin \omega t \quad (1)$$

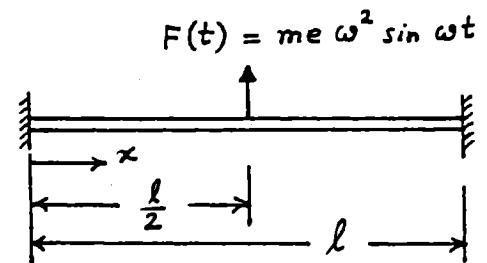
where $F_0 = 5483.1392 \text{ N}$; $\omega = 104.72 \text{ rad/sec}$

Steady state response of the beam can be expressed as:

$$w(x,t) = \sum_{n=1}^{\infty} W_n(x) q_n(t) \quad (2)$$

where the normal modes, $W_n(x)$, are given by (see Fig. 8.15):

$$W_n(x) = \sinh \beta_n x - \sin \beta_n x + \alpha_n (\cosh \beta_n x - \cos \beta_n x) \quad (3)$$



$$\text{and } \alpha_n = \frac{\sinh \beta_n \ell - \sin \beta_n \ell}{\cos \beta_n \ell - \cosh \beta_n \ell} \quad (4)$$

The generalized force, $Q_n(t)$, can be expressed as:

$$Q_n(t) = \int_0^{\ell} f(x,t) W_n(x) dx = F_0 W_n\left(\frac{\ell}{2}\right) \sin \omega t \quad (5)$$

The steady state values of the generalized coordinates, $q_n(t)$, are given by Eq. (8.117):

$$\begin{aligned} q_n(t) &= \frac{1}{\rho A b \omega_n} \int_0^t Q_n(\tau) \sin \omega_n (t - \tau) d\tau \\ &= \frac{F_0 W_n\left(\frac{\ell}{2}\right)}{\rho A \ell \omega_n} \int_0^t \sin \omega \tau \sin \omega_n (t - \tau) d\tau \end{aligned} \quad (6)$$

$$\text{where } b = \int_0^{\ell} W_n^2(x) dx = \ell \quad (7)$$

The integral of Eq. (6) can be evaluated as:

$$\begin{aligned} &\int_0^t \sin \omega \tau \left\{ \sin \omega_n t \cos \omega_n \tau - \cos \omega_n t \sin \omega_n \tau \right\} d\tau \\ &= \sin \omega_n t \int_0^t \frac{1}{2} \sin 2 \omega_n \tau d\tau - \cos \omega_n t \int_0^t \frac{1}{2} (1 - \cos 2 \omega_n \tau) d\tau \\ &= \frac{\sin \omega_n t}{2 \omega_n} - \frac{t \cos \omega_n t}{2} \end{aligned} \quad (8)$$

Thus $q_n(t)$ can be expressed as

$$\begin{aligned} q_n(t) &= \frac{F_0 W_n\left(\frac{\ell}{2}\right)}{\rho A \ell \omega_n} \left\{ \frac{\sin \omega_n t}{2 \omega_n} - \frac{t \cos \omega_n t}{2} \right\} \\ &= \frac{18.2771 W_n\left(\frac{\ell}{2}\right)}{\omega_n} \left\{ \sin \omega_n t - t \cos \omega_n t \right\} \end{aligned} \quad (9)$$

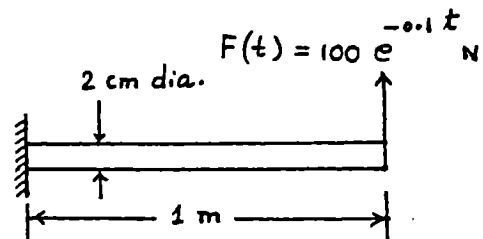
The steady state response of Eq. (2) can be expressed as

$$w(x,t) = 18.2771 \sum_{n=1}^{\infty} \frac{W_n\left(\frac{\ell}{2}\right)}{\omega_n} W_n(x) \left(\sin \omega_n t - t \cos \omega_n t \right) \text{ m} \quad (10)$$

8.46

Steady state displacement of the beam can be expressed as

$$w(x,t) = \sum_{n=1}^{\infty} W_n(x) q_n(t) \quad (1)$$



where the normal modes, $W_n(x)$, are given by (see Fig. 8.15)

$$W_n(x) = C_n \left[\sin \beta_n x - \sinh \beta_n x - \alpha_n (\cos \beta_n x - \cosh \beta_n x) \right] \quad (2)$$

$$\alpha_n = \left(\frac{\sin \beta_n \ell + \sinh \beta_n \ell}{\cos \beta_n \ell + \cosh \beta_n \ell} \right) \quad (3)$$

and the generalized coordinates, $q_n(t)$, are given by

$$\frac{d^2 q_n(t)}{dt^2} + \omega_n^2 q_n(t) = \frac{1}{\rho A b} Q_n(t) \quad (5)$$

$$\text{where } Q_n(t) = \int_0^{\ell} f(x,t) W_n(x) dx = F(t) W_n(\ell) \quad (6)$$

The steady state solution of Eq. (5) is given by

$$q_n(t) = \frac{1}{\rho A b \omega_n} \int_0^t Q_n(\tau) \sin \omega_n (t - \tau) d\tau \quad (7)$$

$$\text{where } b = \int_0^{\ell} W_n^2(x) dx = \ell \quad (8)$$

$$\begin{aligned} \int_0^t Q_n(\tau) \sin \omega_n (t - \tau) d\tau &= 100 W_n(\ell) \int_0^t e^{-0.1 \tau} \sin \omega_n (t - \tau) d\tau \\ &= -100 W_n(\ell) e^{-0.1 t} \int_{t-\tau=t}^{t-\tau=0} e^{0.1(t-\tau)} \sin \omega_n (t - \tau) (-d\tau) \end{aligned} \quad (9)$$

Using the formula

$$\int e^{ax} \sin bx dx = \frac{1}{a^2 + b^2} e^{ax} \left\{ a \sin bx - b \cos bx \right\} \quad (10)$$

Eq. (9) can be evaluated to obtain

$$q_n(t) = \frac{100 W_n(\ell)}{\rho A \ell \omega_n (\omega_n^2 + 0.01)} \left\{ \omega_n e^{-0.1 t} + 0.1 \sin \omega_n t - \omega_n \cos \omega_n t \right\} \quad (11)$$

Using $\rho = 7500 \text{ kg/m}^3$, $\ell = 1 \text{ m}$, $A = \frac{\pi}{4} (0.02^2) = 3.1416 (10^{-4}) \text{ m}^2$, Eq. (11) can be written as

$$q_n(t) = \frac{42.4412 W_n(\ell)}{\omega_n (\omega_n^2 + 0.01)} \left\{ \omega_n e^{-0.1 t} + 0.1 \sin \omega_n t - \omega_n \cos \omega_n t \right\} \quad (12)$$

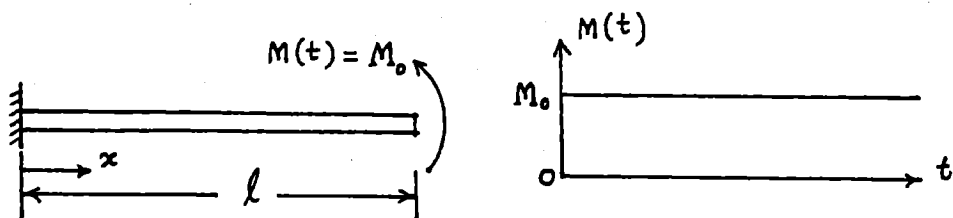
The total steady state solution can be found from Eq. (1).

8.47

The solution is assumed as

$$w(x,t) = \sum_{n=1}^{\infty} W_n(x) q_n(t) \quad (1)$$

where the normal modes of a cantilever beam are given by (Fig. 8.15):



$$W_n(x) = \sin \beta_n x - \sinh \beta_n x - \alpha_n (\cos \beta_n x - \cosh \beta_n x) \quad (2)$$

$$\alpha_n = \left(\frac{\sin \beta_n \ell + \sinh \beta_n \ell}{\cos \beta_n \ell + \cosh \beta_n \ell} \right) \quad (3)$$

where $\beta_n \ell$ are given by the frequency equation:

$$\cos \beta_n \ell \cosh \beta_n \ell = -1 \quad (4)$$

The generalized force $Q_n(t)$ given by Eq. (8.115) becomes

$$Q_n(t) = M_0 \frac{dW_n}{dx} \Big|_{x=\ell} \quad (5)$$

where

$$\frac{dW_n}{dx} \Big|_{x=\ell} = \beta_n (\cos \beta_n \ell - \cosh \beta_n \ell) + \alpha_n \beta_n (\sin \beta_n \ell + \sinh \beta_n \ell) \quad (6)$$

The steady state response of the beam is given by Eq. (1) with

$$\begin{aligned} q_n(t) &= \frac{1}{\rho A b \omega_n} \int_0^t Q_n(\tau) \sin \omega_n (t - \tau) d\tau \\ &= \frac{1}{\rho A b \omega_n} M_0 \frac{dW_n}{dx} \Big|_{x=\ell} \int_0^t \sin \omega_n (t - \tau) d\tau \end{aligned} \quad (7)$$

$$\text{where } b = \int_0^\ell W_n^2(x) dx = \ell \quad (8)$$

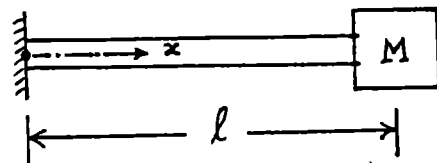
Noting that

$$\int_0^t \sin \omega_n (t - \tau) d\tau = \frac{1}{\omega_n} (1 - \cos \omega_n t) \quad (9)$$

Eq. (7) can be written as

$$q_n(t) = \frac{M_0}{\rho A \ell \omega_n^2} \frac{dW_n}{dx} \Big|_{x=\ell} (1 - \cos \omega_n t) \quad (10)$$

8.48



$$w(x, t) = W(x) \sin \omega t \quad (1)$$

$$\text{where } W(x) = C_1 \cos \beta x + C_2 \sin \beta x + C_3 \cosh \beta x + C_4 \sinh \beta x \quad (2)$$

Boundary conditions:

$$w(0, t) = 0 \quad (3)$$

$$\frac{\partial w}{\partial x}(0, t) = 0 \quad (4)$$

$$\frac{\partial^2 w}{\partial x^2}(x = \ell, t) = 0 \quad (5)$$

$$EI \frac{\partial^3 w}{\partial x^3}(x = \ell, t) = m \frac{\partial^2 w}{\partial t^2}(x = \ell, t) \quad (6)$$

Eq. (2) gives:

$$\frac{dW}{dx} = -\beta C_1 \sin \beta x + \beta C_2 \cos \beta x + \beta C_3 \sinh \beta x + \beta C_4 \cosh \beta x \quad (7)$$

$$\frac{d^2 W}{dx^2} = \beta^2 \left\{ -C_1 \cos \beta x - C_2 \sin \beta x + C_3 \cosh \beta x + C_4 \sinh \beta x \right\} \quad (8)$$

$$\frac{d^3 W}{dx^3} = \beta^3 \left\{ C_1 \sin \beta x - C_2 \cos \beta x + C_3 \sinh \beta x + C_4 \cosh \beta x \right\} \quad (9)$$

Eqs. (2) and (3) give:

$$C_1 + C_3 = 0 \quad (10)$$

Eqs. (2) and (4) lead to:

$$C_2 + C_4 = 0 \quad (11)$$

Eqs. (2) and (5) yield:

$$-C_1 \cos \beta \ell - C_2 \sin \beta \ell + C_3 \cosh \beta \ell + C_4 \sinh \beta \ell = 0 \quad (12)$$

Eqs. (2) and (6) result in:

$$C_1 (\sin \beta \ell + k \cos \beta \ell) + C_2 (-\cos \beta \ell + k \sin \beta \ell) + C_3 (\sinh \beta \ell + k \cosh \beta \ell) + C_4 (\cosh \beta \ell + k \sinh \beta \ell) = 0 \quad (13)$$

where

$$k = \frac{m \omega^2}{EI \beta^3} \quad (14)$$

Eqs. (10) to (13) can be expressed in matrix form as

$$[A] \vec{C} = \vec{0} \quad (15)$$

where

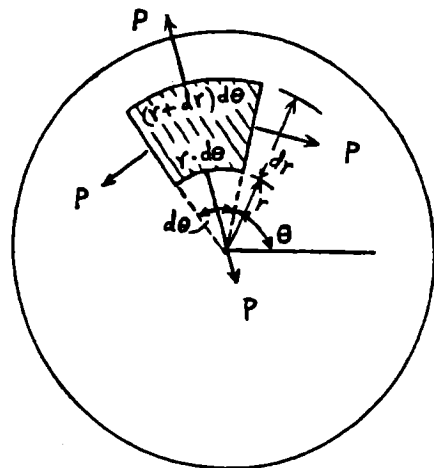
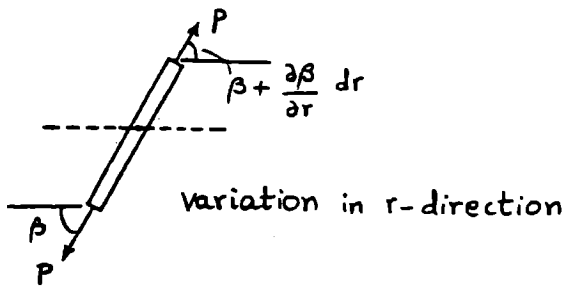
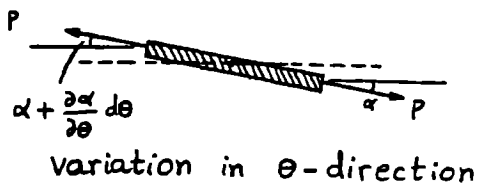
$$[A] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -\cos \beta \ell & -\sin \beta \ell & \cosh \beta \ell & \sinh \beta \ell \\ (\sin \beta \ell + k \cos \beta \ell) & (-\cos \beta \ell + k \sin \beta \ell) & (\sinh \beta \ell + k \cosh \beta \ell) & (\cosh \beta \ell + k \sinh \beta \ell) \end{bmatrix} \quad (16)$$

$$\vec{C} = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} ; \vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

By setting the determinant of the coefficient matrix, $[A]$, to zero, we obtain the frequency equation:

$$|[A]| = 0 \tag{17}$$

8.49



Let $P = \text{tension}$

Forces in radial direction are $P r d\theta$ and $P(r+dr) d\theta$

vertical component of radial forces

$$= P(r+dr) d\theta \left(\beta + \frac{\partial \beta}{\partial r} dr \right) - P r d\theta \beta = P r \left(\frac{\partial \beta}{\partial r} + \frac{1}{r} \beta \right) dr d\theta$$

$$= P r \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) dr d\theta \quad \text{since } \beta = \frac{\partial w}{\partial r}$$

Forces in tangential direction are $P dr$ and $P dr$.

vertical component of tangential forces

$$= P dr \left(\alpha + \frac{\partial \alpha}{\partial \theta} d\theta \right) - P dr \alpha = P \frac{\partial \alpha}{\partial \theta} dr d\theta$$

$$= P \cdot \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} r dr d\theta \quad \text{since } \alpha = \frac{1}{r} \frac{\partial w}{\partial \theta} \text{ and } \frac{\partial \alpha}{\partial \theta} = \frac{1}{r} \frac{\partial^2 w}{\partial \theta^2}$$

Equating the total vertical force to mass times acceleration, we get

$$P r dr d\theta \left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right] = \rho \frac{\partial^2 w}{\partial t^2} r dr d\theta$$

i.e.
$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} = \frac{\rho}{P} \frac{\partial^2 w}{\partial t^2}$$

8.50

For harmonic motion, equation of motion becomes

$$\frac{\partial^2 W}{\partial r^2} + \frac{1}{r} \frac{\partial W}{\partial r} + \frac{1}{r^2} \frac{\partial^2 W}{\partial \theta^2} = -\frac{\rho}{P} \omega^2 W$$

Let $W(r, \theta) = X(r) \cdot Y(\theta)$

$$\frac{r^2}{X} \left(\frac{d^2 X}{dr^2} + \frac{1}{r} \frac{dX}{dr} + \frac{\rho \omega^2}{P} X \right) = -\frac{1}{Y} \frac{d^2 Y}{d\theta^2} = \alpha^2 \text{ (say)}$$

$$\therefore \frac{d^2 Y}{d\theta^2} + \alpha^2 Y = 0 \quad (E_1)$$

$$\frac{d^2 X}{dr^2} + \frac{1}{r} \frac{dX}{dr} + \left(\frac{\rho \omega^2}{P} - \frac{\alpha^2}{r^2} \right) X = 0 \quad (E_2)$$

Solution of (E₂) is $X(r) = C_1 J_m(\gamma \cdot r) + C_2 I_m(\gamma \cdot r)$; $m = 0, 1, 2, \dots$

where J_m and I_m are Bessel functions of the first and second kinds, respectively, and $\gamma = \frac{\rho \omega^2}{P}$

Since $I_m(\gamma \cdot r) \rightarrow \infty$ when $r \rightarrow 0$, $C_2 = 0$ to keep $X(r)$ and hence $w(r, \theta, t)$ finite.

$$\therefore X(r) = C_1 J_m(\gamma \cdot r)$$

$$\text{At } r=R, X(r)=0 \text{ i.e., } J_m(\gamma \cdot R) = 0 \quad (E_3)$$

Eq. (E₃) has several roots: $\gamma_1 R, \gamma_2 R, \dots, \gamma_n R, \dots$

$$\therefore \omega_{mn}^2 = \frac{\gamma_n^2 P}{\rho}$$

8.51

$$(a) \text{ Equation of motion: } \rho \frac{\partial^2 w}{\partial t^2} = P \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + f \quad (E_1)$$

Let the forced response of the rectangular membrane be of the form

$$w(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} T_{mn}(t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (E_2)$$

where $T_{mn}(t)$ is to be determined. Eq. (E₁) and (E₂) give

$$\begin{aligned} \pi^2 P \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) T_{mn}(t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\ + \rho \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{d^2 T_{mn}(t)}{dt^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} - f = 0 \quad (E_3) \end{aligned}$$

Multiply (E₃) by $\sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$ and integrate with respect to x from 0 to a and with respect to y from 0 to b to get

$$\frac{d^2 T_{mn}(t)}{dt^2} + \frac{P \pi^2}{\rho} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) T_{mn}(t) = \frac{4}{ab\rho} F_{mn}(t); \quad (E_4)$$

$m, n = 1, 2, \dots$

where

$$F_{mn}(t) = \int_0^a \int_0^b f(x, y, t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy \quad (E_5)$$

Solution of (E₄) can be expressed as

$$T_{mn}(t) = A_{mn} \cos \pi \sqrt{\frac{P}{\rho} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)} t + B_{mn} \sin \pi \sqrt{\frac{P}{\rho} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)} t \\ + \frac{4}{\pi \rho a b \sqrt{\frac{P}{\rho} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)}} \int_0^t F_{mn}(\tau) \cdot \sin \left\{ \pi \sqrt{\frac{P}{\rho} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)} (t - \tau) \right\} d\tau \quad \text{--- (E}_6\text{)}$$

where A_{mn} and B_{mn} are determined from the known initial conditions of the membrane. Thus the general solution is given by Eq. (E₂) with $T_{mn}(t)$ shown in the last equation, (E₆).

(b) For $f(x, y, t) = f_0$, (E₅) becomes

$$F_{mn}(t) = \int_0^a \int_0^b f_0 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy \\ = \frac{f_0 ab}{\pi^2 mn} (1 - \cos m\pi)(1 - \cos n\pi) \\ = \begin{cases} 0 & \text{for } m \text{ or } n \text{ even} \\ \frac{4 f_0 ab}{\pi^2 mn} & \text{for } m \text{ and } n \text{ odd} \end{cases}$$

Since the membrane is at rest initially, $A_{mn} = B_{mn} = 0$ and Eq. (E₆) gives

$$T_{mn}(t) = \frac{4}{\pi \rho a b \Delta} + \frac{4 f_0 ab}{\pi^2 mn} \int_0^t \sin \pi \Delta (t - \tau) \cdot d\tau \\ = \frac{16 f_0}{\pi^3 mn \rho \Delta} \times \frac{(1 - \cos \pi \Delta t)}{\pi \Delta} \\ = \frac{16 f_0}{\pi^4 mn \rho \Delta^2} (1 - \cos \pi \Delta t) \quad \text{for } m \text{ and } n \text{ odd}$$

where $\Delta = \left\{ \frac{P}{\rho} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \right\}^{1/2}$

Final solution is

$$w(x, y, t) = \frac{16 f_0}{\pi^4 \rho} \sum_{m=1,3,\dots}^{\infty} \sum_{n=1,3,\dots}^{\infty} \frac{1}{mn \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cdot (1 - \cos \pi \Delta t)$$

8.52

From example 8.11,

$$w(x, y, t) = X(x) Y(y) T(t) = (C_1 \cos \alpha x + C_2 \sin \alpha x)(C_3 \cos \beta y + C_4 \sin \beta y) \cdot \\ (A \cos \omega t + B \sin \omega t)$$

Boundary conditions:

$$w(0, y, t) = w(a, y, t) = 0; \quad 0 \leq y \leq b \text{ and } t \geq 0$$

$$w(x, 0, t) = w(x, b, t) = 0; \quad 0 \leq x \leq a \text{ and } t \geq 0$$

i.e. $X(0) = X(a) = 0$, $Y(0) = Y(b) = 0$

i.e. $C_1 = 0, C_3 = 0 \Rightarrow \sin \alpha a = 0, \sin \beta b = 0$

$$\alpha_m = \frac{m\pi}{a} \quad (m = 1, 2, \dots), \quad \beta_n = \frac{n\pi}{b} \quad (n = 1, 2, \dots)$$

$$\therefore \omega_{mn}^2 = c^2 \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right); \quad m, n = 1, 2, \dots$$

and the corresponding displacement solution is

$$w_{mn}(x, y, t) = \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \left(A_{mn} \cos \omega_{mn} t + B_{mn} \sin \omega_{mn} t \right)$$

General solution is

$$w(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn}(x, y, t)$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \left(A_{mn} \cos \omega_{mn} t + B_{mn} \sin \omega_{mn} t \right)$$

8.53

General solution

$$w(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \left(A_{mn} \cos \omega_{mn} t + B_{mn} \sin \omega_{mn} t \right) \quad (E_1)$$

If $w(x, y, 0) = w_0(x, y)$

and $\frac{\partial w}{\partial t}(x, y, 0) = \dot{w}_0(x, y)$ } ; $0 \leq x \leq a,$
 $0 \leq y \leq b$ (E₂)

Eq. (E₁) gives

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = w_0(x, y) \quad (E_3)$$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{mn} B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = \dot{w}_0(x, y) \quad (E_4)$$

These are double Fourier sine series expansions so that

$$A_{mn} = \frac{4}{ab} \int_{x=0}^a \int_{y=0}^b w_0(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy$$

$$= \frac{4}{ab} \int_{x=0}^a w_0 \sin \frac{\pi x}{a} \sin \frac{m\pi x}{a} dx \int_{y=0}^b \sin \frac{\pi y}{b} \sin \frac{n\pi y}{b} dy \quad (E_5)$$

$$B_{mn} = \frac{4}{ab \omega_{mn}} \int_{x=0}^a \int_{y=0}^b \dot{w}_0(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy$$

$$= 0 \text{ for given data} \quad (E_6)$$

Using the relation $\int \sin \alpha t \sin \beta t dt = \frac{1}{2} \left\{ \frac{\sin(\alpha-\beta)t}{\alpha-\beta} - \frac{\sin(\alpha+\beta)t}{\alpha+\beta} \right\}$; $\alpha \neq \beta$
 Eq. (E5) can be simplified as

$$A_{mn} = \frac{4\omega_0}{ab} \left\{ \frac{1}{2} \left[\frac{\sin \frac{(m-1)\pi x}{a}}{\frac{(m-1)\pi}{a}} - \frac{\sin \frac{(m+1)\pi x}{a}}{\frac{(m+1)\pi}{a}} \right]_0^a \right\} \left\{ \frac{1}{2} \left[\frac{\sin \frac{(n-1)\pi y}{b}}{\frac{(n-1)\pi}{b}} - \frac{\sin \frac{(n+1)\pi y}{b}}{\frac{(n+1)\pi}{b}} \right]_0^b \right\}$$

$$= 0 \text{ for } m > 1 \text{ and/or } n > 1 \quad (E7)$$

For $m=1$ and $n=1$, Eq. (E5) can be simplified as

$$A_{11} = \frac{4\omega_0}{ab} \int_0^a \sin^2 \frac{\pi x}{a} dx \int_0^b \sin^2 \frac{\pi y}{b} dy = \omega_0 \quad (E8)$$

$$\therefore w(x, y, t) = \omega_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \cos \omega_{11} t$$

$$\text{with } \omega_{11} = \left\{ c^2 \pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right) \right\}^{1/2}$$

(from the solution of problem 8.52)

8.54

General solution:

$$w(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (A_{mn} \cos \omega_{mn} t + B_{mn} \sin \omega_{mn} t)$$

$$\text{If } w(x, y, 0) =: \omega_0(x, y) \quad \left. \begin{array}{l} \text{and} \\ \frac{\partial w}{\partial t}(x, y, 0) = \dot{\omega}_0(x, y) \end{array} \right\}; \quad 0 \leq x \leq a, \quad 0 \leq y \leq b$$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = \omega_0(x, y)$$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{mn} B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = \dot{\omega}_0(x, y)$$

These are double Fourier sine series expansions of $\omega_0(x, y)$ and $\dot{\omega}_0(x, y)$.

$$A_{mn} = \frac{4}{ab} \int_0^a \int_0^b \omega_0(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy$$

$$= 0 \text{ for given data}$$

$$B_{mn} = \frac{4}{ab \omega_{mn}} \int_0^a \int_0^b \dot{\omega}_0(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy$$

$$= \frac{4 \dot{\omega}_0}{ab \omega_{mn}} \int_0^a \sin \frac{m\pi x}{a} \sin \frac{\pi x}{a} dx \int_0^b \sin \frac{n\pi y}{b} \sin \frac{2\pi y}{b} dy \dots (E1)$$

using the relation

$$\int \sin \alpha t \cdot \sin \beta t \cdot dt = \frac{1}{2} \left\{ \frac{\sin(\alpha-\beta)t}{\alpha-\beta} - \frac{\sin(\alpha+\beta)t}{\alpha+\beta} \right\} \text{ for } \alpha \neq \beta,$$

Eg. (E₁) becomes

$$B_{mn} = \frac{4 \dot{w}_0}{ab \omega_{mn}} \left\{ \frac{1}{2} \left(\frac{\sin \frac{(m-1)\pi x}{a}}{m-1} - \frac{\sin \frac{(m+1)\pi x}{a}}{m+1} \right) \Big|_0^a \right\} x$$

$$\left\{ \frac{1}{2} \left(\frac{\sin \frac{(n-2)\pi y}{b}}{n-2} - \frac{\sin \frac{(n+2)\pi y}{b}}{n+2} \right) \Big|_0^b \right\}$$

= 0 for $m \neq 1$ and $n \neq 2$.

$$B_{12} = \frac{4 \dot{w}_0}{ab \omega_{mn}} \left(\int_0^a \sin^2 \frac{\pi x}{a} dx \right) \left(\int_0^b \sin^2 \frac{2\pi y}{b} dy \right)$$

$$= \frac{4 \dot{w}_0}{ab \omega_{mn}} \left\{ \left(\frac{x}{2} - \frac{\sin \frac{\pi x}{a} \cos \frac{\pi x}{a}}{\left(\frac{2\pi}{a}\right)} \right) \Big|_0^a \right\} \left\{ \left(\frac{y}{2} - \frac{\sin \frac{2\pi y}{b} \cos \frac{2\pi y}{b}}{\left(\frac{4\pi}{b}\right)} \right) \Big|_0^b \right\}$$

$$= \frac{4 \dot{w}_0}{ab \omega_{12}} \left(\frac{a}{2} \right) \left(\frac{b}{2} \right) = \frac{\dot{w}_0}{\omega_{12}}$$

$$\therefore w(x, y, t) = \frac{\dot{w}_0}{\omega_{12}} \sin \frac{\pi x}{a} \sin \frac{2\pi y}{b} \sin \omega_{12} t$$

8.55

Fundamental natural frequency of transverse vibration is given by:

$$\omega_{11}^2 = c^2 \pi^2 \left(\frac{1}{a_1^2} + \frac{1}{b_1^2} \right) \quad \text{for rectangular membrane of sides } \dots (E_1)$$

a_1 and b_1

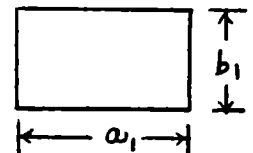
$$\text{with } c^2 = P/\rho \quad \dots (E_2)$$

$$(a) \text{ For } a_1 = b_1 = a, \quad \omega_{11}^2 = c^2 \pi^2 \left(\frac{2}{a^2} \right) \quad \dots (E_3)$$

$$(c) \text{ For } a_1 = 2b_1, \quad \text{area} = a_1 b_1 = a^2 = 2b_1^2$$

$$\therefore b_1 = a/\sqrt{2}, \quad a_1 = \sqrt{2} a$$

$$\omega_{11}^2 = c^2 \pi^2 \left(\frac{1}{a_1^2} + \frac{1}{b_1^2} \right) = c^2 \pi^2 \left(\frac{5}{2a^2} \right) \quad \dots (E_4)$$



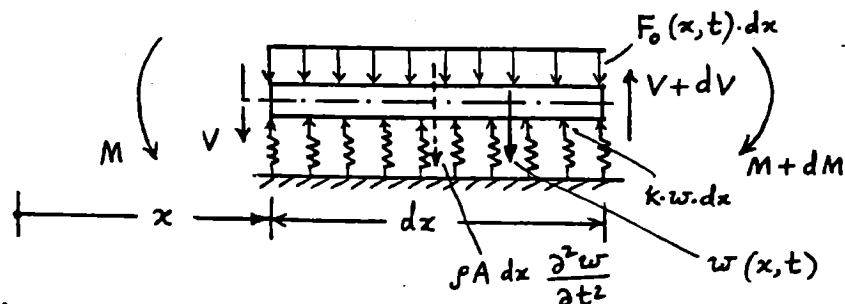
(b) From solution of problem 8.50, the fundamental natural frequency of a circular membrane of radius R is given by

$$\omega_{01}^2 = \frac{\gamma_1^2 P}{\rho} = c^2 \gamma_1^2 \quad \text{where } \gamma_1 R = 2.404 = \text{first zero of Bessel function of the first kind.}$$

$$\text{Since } \pi R^2 = a^2, \quad R = a/\sqrt{\pi} \quad \text{and}$$

$$\omega_{01}^2 = c^2 \left(\frac{2.404}{R} \right)^2 = c^2 \left(\frac{4.2610}{a} \right)^2 \quad \dots (E_5)$$

8.56



(a) If load due to the car moves with a constant velocity v_0 in the x -direction,

$$F(x,t) = F(x - v_0 t) \quad (E_1)$$

Equilibrium equations are

$$(V + dV) - V - F_0(x,t) dx + k w dx = -\rho A dx \frac{\partial^2 w}{\partial t^2}$$

$$\text{or } \frac{\partial V}{\partial x} - F_0(x,t) + k w = -\rho A \frac{\partial^2 w}{\partial t^2} \quad (E_2)$$

$$\text{and } (M + dM) - M - (V + dV) dx + F_0(x,t) dx \cdot \frac{dx}{2} = 0$$

$$\text{or } \frac{\partial M}{\partial x} - V = 0 \quad (E_3)$$

Using (E₃), (E₂) can be rewritten as

$$\frac{\partial^2 M}{\partial x^2} - F_0(x,t) + k w = -\rho A \frac{\partial^2 w}{\partial t^2} \quad (E_4)$$

$$\text{But } M = EI \frac{\partial^2 w}{\partial x^2} \quad (E_5)$$

Eqs. (E₄) and (E₅) lead to

$$\frac{\partial^2}{\partial x^2} (EI \frac{\partial^2 w}{\partial x^2}) - F_0(x,t) + k w = -\rho A \frac{\partial^2 w}{\partial t^2} \quad (E_6)$$

For a uniform beam, (E₆) simplifies to, in view of (E₁),

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} + k w = F_0(x - v_0 t) \quad (E_7)$$

(b) Solution:

$$\text{Defining } y = x - v_0 t \quad (E_8)$$

Eq. (E₇) can be expressed as

$$EI \frac{d^4 w}{dy^4} + \rho A v_0^2 \frac{d^2 w}{dy^2} + k w = F_0(y) \quad (E_9)$$

Let $F_0 =$ concentrated load. Then the governing equation at all points of the beam, except at $y=0$, is

$$EI \frac{d^4 w}{dy^4} + \rho A v_0^2 \frac{d^2 w}{dy^2} + k w = 0 \quad (E_{10})$$

Solution of (E₁₀) can be expressed as

$$w(y) = e^{\alpha y} \quad (E_{11})$$

Substitution of (E₁₁) into (E₁₀) gives

$$EI \alpha^4 + \rho A v_0^2 + k = 0 \quad (E_{12})$$

whose roots can be expressed as

$$\alpha_{1,2} = \pm (a + ib) \quad , \quad \alpha_{3,4} = \pm (a - ib) \quad (E_{13})$$

with

$$\left. \begin{aligned} a &= \sqrt{1-c} d \quad , \quad b = \sqrt{1+c} d \quad , \quad c = \frac{v_0^2}{\sqrt{\frac{4EI k}{\rho^2 A^2}}} \\ \text{and } d &= \left(\frac{k}{4EI}\right)^{1/4} \end{aligned} \right\} \quad (E_{14})$$

Thus the solution of (E₁₀) becomes

$$w(y) = A_1 e^{\alpha_1 y} + A_2 e^{\alpha_2 y} + A_3 e^{\alpha_3 y} + A_4 e^{\alpha_4 y} \quad (E_{15})$$

where A_i ($i=1,2,3,4$) are constants which can be determined from the following conditions:

$$\left. \begin{aligned} w &= 0 \quad \text{at } y = \infty \\ \frac{d^2 w}{dy^2} &= 0 \quad \text{at } y = \infty \end{aligned} \right\} \begin{array}{l} \text{deflection \& bending moment are} \\ \text{zero at } y = \infty \end{array}$$

$$\left. \frac{dw}{dy} = 0 \quad \text{at } y = 0 \right\} \text{slope is zero under the load}$$

$$\left. \begin{aligned} EI \frac{d^3 w}{dy^3} \Big|_{y=0^+} - EI \frac{d^3 w}{dy^3} \Big|_{y=0^-} &= P \\ \text{i.e., } EI \frac{d^2 w}{dy^2} &= \frac{P}{2} \end{aligned} \right\} \begin{array}{l} \text{shear force has} \\ \text{discontinuity} \\ \text{under the load} \end{array}$$

8.57

$$W(x) = \frac{c_0 x^2}{24 EI} (l-x)^2 = \frac{c_0}{24 EI} (x^2 l^2 + x^4 - 2l x^3)$$

$$\frac{dW}{dx} = \frac{c_0}{24 EI} (2x l^2 + 4x^3 - 6l x^2)$$

$$\frac{d^2 W}{dx^2} = \frac{c_0}{24 EI} (2l^2 + 12x^2 - 12lx)$$

$$N = EI \int_0^l \left(\frac{d^2 W}{dx^2}\right)^2 dx = 4EI \left(\frac{c_0}{24 EI}\right)^2 \int_0^l (l^2 + 6x^2 - 6lx)^2 dx$$

$$= \frac{c_0^2 l^5}{720 EI} \quad (E_1)$$

$$D = \rho A \int_0^l (W(x))^2 dx = \rho A \left(\frac{c_0}{24 EI}\right)^2 \int_0^l (x^2 l^2 + x^4 - 2l x^3)^2 dx$$

$$= \frac{\rho A l^9 c_0^2}{362880 E^2 I^2} \quad (E_2)$$

$$\omega^2 = \frac{N}{D} = \left(\frac{c_0^2 l^5}{720 EI} \right) \left(\frac{362880 E^2 I^2}{\rho A l^3 c_0^2} \right) = 504 \frac{EI}{\rho A l^4}$$

$$\therefore \omega = 22.4499 \sqrt{\frac{EI}{\rho A l^4}}$$

This can be compared with the exact solution (Fig. 8.15):

$$\omega_{\text{exact}} = (4.730041)^2 \sqrt{\frac{EI}{\rho A l^4}} = 22.3733 \sqrt{\frac{EI}{\rho A l^4}}$$

8.58

$$W(x) = c_0 \left(1 - \cos \frac{2\pi x}{l} \right)$$

$$\frac{dW}{dx} = c_0 \frac{2\pi}{l} \sin \frac{2\pi x}{l} \quad ; \quad \frac{d^2W}{dx^2} = c_0 \left(\frac{2\pi}{l} \right)^2 \cos \frac{2\pi x}{l}$$

$$N = EI \int_0^l \left(\frac{d^2W}{dx^2} \right)^2 dx = EI c_0^2 \left(\frac{2\pi}{l} \right)^4 \int_0^l \cos^2 \frac{2\pi x}{l} dx$$

$$= 8EI c_0^2 \pi^4 / l^3$$

$$D = \rho A \int_0^l (W(x))^2 dx = \rho A c_0^2 \int_0^l \left(1 + \cos^2 \frac{2\pi x}{l} - 2 \cos \frac{2\pi x}{l} \right) dx$$

$$= 3c_0^2 \rho A l / 2$$

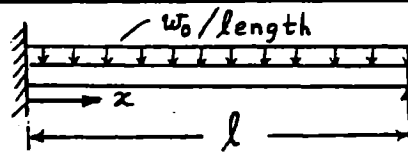
$$\omega^2 = \frac{N}{D} = \frac{16\pi^4}{3} \cdot \frac{EI}{\rho A l^4}$$

$$\therefore \omega = 22.7930 \sqrt{\frac{EI}{\rho A l^4}} \quad \text{Compare this with}$$

$$\omega_{\text{exact}} = (4.730041)^2 \sqrt{\frac{EI}{\rho A l^4}} = 22.3733 \sqrt{\frac{EI}{\rho A l^4}}$$

8.59

From strength of materials, the static deflection curve is given by



$$W(x) = \frac{w_0 x^2}{48EI} (l-x)(2x-3l) = \frac{c_0}{48EI} (5lx^3 - 2x^4 - 3l^2x^2)$$

$$\frac{dW}{dx} = \frac{c_0}{48EI} (15lx^2 - 8x^3 - 6l^2x)$$

$$\frac{d^2W}{dx^2} = \frac{c_0}{8EI} (5lx - 4x^2 - l^2)$$

$$N = EI \int_0^l \left(\frac{d^2W}{dx^2} \right)^2 dx = \frac{c_0^2 l^5}{320 EI}$$

$$D = \rho A \int_0^l (W(x))^2 dx = 1.3090 \times 10^{-5} \frac{c_0^2 \rho A l^9}{E^2 I^2}$$

$$\omega^2 = \frac{N}{D} = 238.7319 \frac{EI}{\rho A l^4}$$

$$\therefore \omega = 15.4510 \sqrt{\frac{EI}{\rho A l^4}}$$

This can be compared with the exact value

$$\omega_{\text{exact}} = (3.926602)^2 \sqrt{\frac{EI}{\rho A l^4}} = 15.4182 \sqrt{\frac{EI}{\rho A l^4}}$$

8.60 $W(x) = \begin{cases} \frac{x^2}{48EI} (-4x+3l) ; & 0 \leq x \leq \frac{l}{2} \\ \frac{(l-x)^2}{48EI} (4x-l) ; & \frac{l}{2} \leq x \leq l \end{cases}$



$$\frac{d^2 W}{dx^2} = \begin{cases} \frac{1}{8EI} (-4x+l) ; & 0 \leq x \leq \frac{l}{2} \\ \frac{1}{8EI} (4x-3l) ; & \frac{l}{2} \leq x \leq l \end{cases}$$

$$\begin{aligned} T_{\text{max of beam}} &= \frac{\omega^2}{2} \int_0^l \rho A [W(x)]^2 dx \\ &= \frac{\omega^2 \rho A}{2} \left(\frac{1}{48EI} \right)^2 \left[\int_0^{l/2} x^4 (16x^2 + 9l^2 - 24xl) dx + \int_{l/2}^l (l-x)^4 (4x-l)^2 dx \right] \\ &= \frac{\rho A \omega^2}{4608 E^2 I^2} \left\{ \left[\frac{16x^7}{7} + \frac{9l^2 x^5}{5} - \frac{24lx^6}{6} \right]_0^{l/2} \right. \\ &\quad \left. + \left[\frac{16x^7}{7} - \frac{72lx^6}{6} + \frac{129l^2 x^5}{5} - \frac{116l^3 x^4}{4} + \frac{54l^4 x^3}{3} - \frac{12l^5 x^2}{2} + l^6 x \right]_{l/2}^l \right\} \\ &= 5 \times 10^{-6} \left(\frac{\rho A \omega^2 l^7}{E^2 I^2} \right) \end{aligned}$$

$$T_{\text{max of mass}} = \frac{1}{2} m \dot{w}_{\text{max}} \Big|_{x=l/2} = \frac{1}{2} m \omega^2 W^2(x=l/2)$$

$$= \frac{1}{2} m \omega^2 \left(\frac{l^3}{192EI} \right)^2 = 13.6 \times 10^{-6} \left(\frac{m \omega^2 l^6}{E^2 I^2} \right)$$

$$T_{\text{max total}} = 5 \times 10^{-6} \frac{\rho A \omega^2 l^7}{E^2 I^2} + 13.6 \times 10^{-6} \frac{m \omega^2 l^6}{E^2 I^2}$$

$$\begin{aligned}
 V_{\max} &= \frac{1}{2} \int_0^l EI \left(\frac{d^2W}{dx^2} \right)^2 dx \\
 &= \frac{1}{128 EI} \left[\int_0^{\frac{l}{2}} (16x^2 + l^2 - 8lx) dx + \int_{\frac{l}{2}}^l (16x^2 + 9l^2 - 24lx) dx \right] \\
 &= \frac{1}{128 EI} \left[\left(\frac{16}{3} x^3 + l^2 x - \frac{8lx^2}{2} \right) \Big|_0^{\frac{l}{2}} + \left(\frac{16}{3} x^3 + 9l^2 x - \frac{24lx^2}{2} \right) \Big|_{\frac{l}{2}}^l \right] \\
 &= \frac{l^3}{384 EI}
 \end{aligned}$$

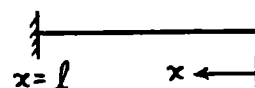
$T_{\max} \text{ total} = V_{\max}$ gives

$$\omega^2 = \frac{l^3}{384 EI} \cdot \frac{E^2 I^2}{(5 \times 10^{-6} \rho A l^7 + 13.6 \times 10^{-6} m l^6)} = \frac{2590.7 EI}{l^3 (5 \rho A l + 13.6 m)}$$

$$\therefore \omega = 50.8987 \sqrt{\frac{EI}{l^3 (5 M_b + 13.6 m)}} \quad \text{where } M_b = \rho A l = \text{mass of beam}$$

8.61

Let $W(x) = c_0 \left(1 - \frac{x}{l}\right)^2 = c_0 \left(1 + \frac{x^2}{l^2} - \frac{2x}{l}\right)$



$$\frac{d^2W}{dx^2} = 2c_0 \left(\frac{1}{l^2}\right)$$

$$\begin{aligned}
 T_{\max} &= \frac{\omega^2}{2} \int_0^l \rho A(x) [W(x)]^2 dx = \frac{\rho \omega^2 A_0 c_0^2}{2l} \int_0^l x \left(1 + \frac{x^2}{l^2} - \frac{2x}{l}\right)^2 dx \\
 &= \frac{\rho \omega^2 A_0 c_0^2}{2l} \left[\frac{x^2}{2} + \frac{x^6}{6l^4} + \frac{4x^4}{4l^2} + \frac{2x^4}{4l^2} - \frac{4x^5}{5l^3} - \frac{4x^3}{3l} \right]_0^l
 \end{aligned}$$

$$= \frac{\rho \omega^2 A_0 c_0^2 l}{60}$$

$$V_{\max} = \frac{1}{2} \int_0^l EI(x) \left(\frac{d^2W}{dx^2} \right)^2 dx = \frac{E}{2} \int_0^l \left(\frac{I_0 x}{l} \right) \left(\frac{4c_0^2}{l^4} \right) dx = \frac{EI_0 c_0^2}{l^3}$$

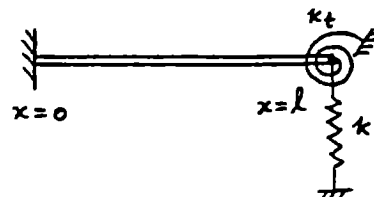
$T_{\max} = V_{\max}$ gives

$$\omega^2 = \frac{EI_0 c_0^2}{l^3} \cdot \frac{60}{\rho A_0 c_0^2 l} = \frac{60 EI_0}{\rho A_0 l^4}$$

$$\omega = 7.7460 \sqrt{\frac{EI_0}{\rho A_0 l^4}}$$

8.62

Let $W(x) = \frac{c_0}{24EI} (x^4 - 4lx^3 + 6l^2x^2)$



$$\frac{d^2W}{dx^2} = \frac{c_0}{2EI} (x^2 - 2lx + l^2)$$

$$V_{\max} = \frac{EI}{2} \int_0^l \left(\frac{d^2W}{dx^2} \right)^2 dx + \frac{1}{2} k_t [W(x=l)]^2 + \frac{1}{2} k_t \left[\frac{dW}{dx}(x=l) \right]^2$$

$$\text{Since } \frac{\kappa}{2} [W(x=l)]^2 = \frac{\kappa}{2} C_0^2 \frac{l^8}{64 E^2 I^2},$$

$$\frac{\kappa_t}{2} \left[\frac{dW}{dx}(x=l) \right]^2 = \frac{\kappa_t}{2} C_0^2 \frac{l^6}{36 E^2 I^2},$$

$$\text{and } \frac{1}{2} EI \int_0^l \left(\frac{d^2 W}{dx^2} \right)^2 dx = \frac{C_0^2 l^5}{40 EI} \quad (\text{from solution of problem 8.57})$$

$$V_{\max} = \frac{C_0^2 l^5}{40 EI} + \frac{\kappa C_0^2 l^8}{128 E^2 I^2} + \frac{\kappa_t C_0^2 l^6}{72 E^2 I^2}$$

$$T_{\max} = \frac{\omega^2 \rho A}{2} \int_0^l [W(x)]^2 dx = \omega^2 \rho A \left(\frac{13 C_0^2 l^9}{6480 E^2 I^2} \right) \quad \text{from problem 8.57}$$

$$V_{\max} = T_{\max} \text{ gives}$$

$$\omega^2 = \left(12.4615 \frac{EI}{\rho A l^4} + 3.8942 \frac{\kappa}{\rho A l} + 6.9231 \frac{\kappa_t}{\rho A l^4} \right)^{1/2}$$

8.63

$$W(x) = C_1 \left(1 - \cos \frac{2\pi x}{l} \right), \quad \frac{d^2 W}{dx^2} = C_1 \left(\frac{2\pi}{l} \right)^2 \cos \frac{2\pi x}{l}$$

$$T_{\max} = \frac{\omega^2}{2} \int_0^l \rho A [W(x)]^2 dx = \frac{\omega^2 \rho A C_1^2}{2} \int_0^l \left(1 + \cos^2 \frac{2\pi x}{l} - 2 \cos \frac{2\pi x}{l} \right) dx$$

$$= \frac{\rho A \omega^2 C_1^2}{2} \left[x + \left(\frac{x}{2} + \frac{l}{4\pi} \sin \frac{4\pi x}{l} \right) + \left(\frac{l}{\pi} \sin \frac{2\pi x}{l} \right) \right]_0^l$$

$$= \frac{3}{4} \rho A \omega^2 C_1^2 l$$

$$V_{\max} = \frac{EI}{2} \int_0^l \left(\frac{d^2 W}{dx^2} \right)^2 dx = \frac{EI}{2} \int_0^l C_1^2 \left(\frac{2\pi}{l} \right)^4 \cos^2 \frac{2\pi x}{l} dx$$

$$= \frac{8EI C_1^2 \pi^4}{l^4} \left(\frac{x}{2} + \frac{l}{8\pi} \sin \frac{4\pi x}{l} \right)_0^l = \frac{4EI C_1^2 \pi^4}{l^3}$$

$$T_{\max} = V_{\max} \text{ gives}$$

$$\omega^2 = \frac{4EI C_1^2 \pi^4}{l^3} \cdot \frac{4}{3 \rho A C_1^2 l} = \frac{519.52 EI}{\rho A l^4}$$

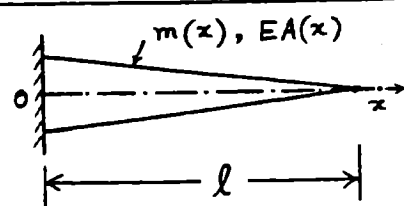
$$\omega = 22.7930 \sqrt{\frac{EI}{\rho A l^4}}$$

8.64

$$U(x) = C_1 \sin \frac{\pi x}{2l}$$

$$\frac{dU}{dx} = C_1 \frac{\pi}{2l} \cos \frac{\pi x}{2l}$$

$$V_{\max} = \frac{1}{2} \int_0^l EA(x) \left(\frac{dU}{dx} \right)^2 dx$$



$$\begin{aligned}
&= \frac{1}{2} \int_0^l 2EA_0 \left(1 - \frac{x}{l}\right) c_1^2 \left(\frac{\pi}{2l}\right)^2 \cos^2 \frac{\pi x}{2l} \cdot dx \\
&= EA_0 c_1^2 \left(\frac{\pi}{2l}\right)^2 \left[\int_0^l \cos^2 \frac{\pi x}{2l} dx - \int_0^l \frac{x}{l} \cos^2 \frac{\pi x}{2l} dx \right] \\
&= EA_0 c_1^2 \left(\frac{\pi}{2l}\right)^2 \left[\left(\frac{x}{2} + \frac{l}{2\pi} \sin \frac{\pi x}{l}\right)_0^l - \frac{1}{l} \left(\frac{x^2}{4} + \frac{x \sin \frac{\pi x}{l}}{\left(\frac{2\pi}{l}\right)} + \frac{\cos \frac{\pi x}{l}}{\left(\frac{8\pi^2}{4l^2}\right)}\right)_0^l \right] \\
&= \frac{EA_0 c_1^2 \pi^2}{16l} \left(1 + \frac{4}{\pi^2}\right) \\
T_{\max} &= \frac{\omega^2}{2} \int_0^l m(x) U^2 dx = \frac{\omega^2}{2} \int_0^l 2m_0 \left(1 - \frac{x}{l}\right) c_1^2 \sin^2 \frac{\pi x}{2l} \cdot dx \\
&= \omega^2 m_0 c_1^2 \left[\int_0^l \sin^2 \frac{\pi x}{2l} dx - \int_0^l \frac{x}{l} \sin^2 \frac{\pi x}{2l} dx \right] \\
&= \omega^2 m_0 c_1^2 \left[\left(\frac{x}{2} - \frac{l}{2\pi} \sin \frac{\pi x}{l}\right)_0^l - \frac{1}{l} \left(\frac{x^2}{4} - \frac{x \sin \frac{\pi x}{l}}{\left(\frac{2\pi}{l}\right)} - \frac{\cos \frac{\pi x}{l}}{\left(\frac{8\pi^2}{4l^2}\right)}\right)_0^l \right] \\
&= \omega^2 m_0 c_1^2 \frac{l}{4} \left(1 - \frac{4}{\pi^2}\right) \\
V_{\max} &= T_{\max} \text{ gives} \\
\omega^2 &= \frac{EA_0 \pi^2}{16l} \frac{\left(1 + \frac{4}{\pi^2}\right) 4}{m_0 l \left(1 - \frac{4}{\pi^2}\right)} = 5.8303 \frac{EA_0}{m_0 l^2} \\
\therefore \omega_1 &= 2.4146 \sqrt{\frac{EA_0}{m_0 l^2}}
\end{aligned}$$

8.65

We take the deflection curve satisfying the boundary conditions

$$w(0, y) = w(a, y) = w(x, 0) = w(x, b) = 0 \text{ as}$$

$$w(x, y) = c_1 x y (x-a)(y-b)$$

$$\frac{\partial w}{\partial x} = c_1 y (x-a)(y-b) + c_1 x y (y-b)$$

$$\frac{\partial w}{\partial y} = c_1 x (x-a)(y-b) + c_1 x y (x-a)$$

$$\begin{aligned}
\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 &= c_1^2 y^2 (y-b)^2 (x-a)^2 + c_1^2 x^2 y^2 (y-b)^2 \\
&\quad + c_1^2 x^2 (x-a)^2 (y-b)^2 + c_1^2 x^2 y^2 (x-a)^2
\end{aligned}$$

$$\begin{aligned}
&= c_1^2 \left[2x^2 y^4 + x^2 y^2 (2a^2 + 2b^2) + x^2 y^3 (-4b) + y^4 (a^2) + y^2 (a^2 b^2) \right. \\
&\quad + y^3 (-2ba^2) + xy^4 (-2a) + xy^2 (-2ab^2) + xy^3 (4ab) \\
&\quad + 2x^4 y^2 + x^4 (b^2) + x^4 y (-2b) + x^2 (a^2 b^2) + x^2 y (-2ba^2) \\
&\quad \left. + x^3 y^2 (-2a) + x^3 (-2ab^2) + x^3 y (4ab) + x^3 y^2 (-2a) \right]
\end{aligned}$$

$$\int_0^a \int_0^b \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] dx dy = c_1^2 \left(\frac{a^3 b^5}{45} + \frac{a^5 b^3}{45} \right)$$

$$w^2(x, y) = c_1^2 \left[x^4 y^4 + a^2 x^2 y^4 + b^2 x^4 y^2 + a^2 b^2 x^2 y^2 - 2a x^3 y^4 - 2b x^4 y^3 + 2ab x^3 y^3 + 2ab x^3 y^3 - 2a^2 b x^2 y^3 - 2ab^2 x^3 y^2 \right]$$

$$\int_0^a \int_0^b w^2 dx dy = c_1^2 a^5 b^5 / 900$$

$$V_{\max} = \frac{P}{2} \frac{c_1^2}{45} (a^3 b^3) (a^2 + b^2)$$

$$T_{\max} = \frac{\omega^2 \rho}{2} \cdot \frac{c_1^2 a^5 b^5}{900}$$

$$V_{\max} = T_{\max} \text{ gives } \frac{\omega^2 \rho}{P} = 20 \left(\frac{1}{a^2} + \frac{1}{b^2} \right)$$

$$\therefore \omega_1 = 4.4721 \sqrt{\frac{P}{\rho} \left(\frac{1}{a^2} + \frac{1}{b^2} \right)}$$

$$\text{Exact value of } \omega_1 = \pi \sqrt{\frac{P}{\rho} \left(\frac{1}{a^2} + \frac{1}{b^2} \right)}$$

8.66

For a shaft under torsion, the shear stress (τ) induced at a radius r from the center of the cross section is given by

$$\tau = \frac{M_t(x) r}{J} \quad (1)$$

where $M_t(x)$ = torque at section x . The potential energy of the shaft is given by the strain energy:

$$V = \frac{1}{2} \int \frac{1}{G} \tau^2 dA dx \quad (2)$$

Using Eq. (8.61):

$$M_t(x) = GJ \frac{\partial \theta}{\partial x} \quad (3)$$

Eq. (2) can be expressed as

$$V = \frac{1}{2} \int_0^L GJ \left(\frac{\partial \theta}{\partial x} \right)^2 dx \quad (4)$$

The kinetic energy of the shaft can be written as

$$T = \frac{1}{2} \int_0^L \rho J \left(\frac{\partial \theta}{\partial t} \right)^2 dx \quad (5)$$

By assuming a harmonic variation of $\theta(x, t)$ as

$$\theta(x, t) = \Theta(x) \cos \omega t \quad (6)$$

and equating V_{\max} to T_{\max} , we obtain

$$\omega^2 = \frac{\int_0^{\ell} GJ \left(\frac{d\Theta}{dx} \right)^2 dx}{\int_0^{\ell} \rho J \Theta^2 dx} \quad (7)$$

For the shaft shown in Fig. 8.36,

$$G = 80 (10^9) \text{ N/m}^2 \quad ; \quad \rho g = 76.5 \text{ kN/m}^3$$

$$J = \frac{\pi}{32} d^4 = \frac{\pi}{32} (0.05^4) = 61.3594 (10^{-8}) \text{ m}^4$$

Let $\Theta(x)$ be assumed to vary linearly on either side of the disc for simplicity:

$$\begin{aligned} \Theta(x) &= \frac{\theta_0 x}{0.8} \quad ; \quad 0 \leq x \leq 0.8 \\ &= \frac{\theta_0 (1-x)}{0.2} \quad ; \quad 0.8 \leq x \leq 1 \end{aligned} \quad (8)$$

where θ_0 is the angular displacement of the disc. Equation (8) satisfies the boundary conditions and gives:

$$\begin{aligned} \frac{d\Theta}{dx} &= \frac{\theta_0}{0.8} \quad ; \quad 0 \leq x \leq 0.8 \\ &= -\frac{\theta_0}{0.2} \quad ; \quad 0.8 \leq x \leq 1.0 \end{aligned} \quad (9)$$

$$\int_0^{\ell} GJ \left(\frac{d\Theta}{dx} \right)^2 dx = 306\,797 \theta_0^2 \quad ; \quad \int_0^{\ell} \rho J \Theta^2 dx = 159.5384 (10^{-5}) \theta_0^2$$

Thus Eq. (7) gives the approximate natural frequency as:

$$\omega^2 \approx \frac{306797 \theta_0^2}{159.5384 (10^{-5}) \theta_0^2} = 1923.0292 (10^5)$$

$$\omega \approx 13,867.3328 \text{ rad/sec}$$

For comparison, the torsional natural frequency of a fixed-fixed shaft with no intermediate disc, is given by (see Fig. 8.12):

$$\omega_1 = \frac{\pi c}{\ell} = \frac{\pi}{\ell} \sqrt{\frac{G}{\rho}} = \frac{\pi}{1.0} \sqrt{\frac{80 (10^9) (9.81)}{76500}} = 10,062.3557 \text{ rad/sec}$$

8.67 (a) $W(x) = c_1 x(l-x)$, Let $P = \text{tension}$

$$V_{\max} = \frac{1}{2} \int_0^l P \left(\frac{dW}{dx} \right)^2 dx = \frac{1}{2} c_1^2 P \int_0^l (l-2x)^2 dx$$

$$= \frac{1}{2} P c_1^2 \left[l^2 x + \frac{4}{3} x^3 - 2lx^2 \right]_0^l = \frac{1}{6} P c_1^2 l^3$$

$$T_{\max} = \frac{\omega^2}{2} \int_0^l \rho [W]^2 dx = \frac{1}{2} \rho c_1^2 \omega^2 \int_0^l (x^2 l^2 + x^4 - 2lx^3) dx$$

$$= \frac{1}{2} \rho c_1^2 \omega^2 \left[\frac{1}{3} x^3 l^2 + \frac{1}{5} x^5 - \frac{1}{2} l x^4 \right]_0^l = \frac{1}{60} \omega^2 \rho c_1^2 l^5$$

$V_{\max} = T_{\max}$ gives

$$\omega = \left(\frac{10 P}{\rho l^2} \right)^{1/2} = 3.1623 \sqrt{\frac{P}{\rho l^2}}$$

Exact value is $\omega_1 = \pi \sqrt{\frac{P}{\rho l^2}}$.

(b) $W(x) = c_1 x(l-x) + c_2 x^2(l-x)^2$

$$\frac{dW}{dx} = c_1(l-2x) + c_2(2l^2x + 4x^3 - 6lx^2)$$

$$V_{\max} = \frac{1}{2} \int_0^l P \left(\frac{dW}{dx} \right)^2 dx = \frac{P}{2} \int_0^l [c_1^2(l^2 + 4x^2 - 4lx) + c_2^2(4l^4x^2 + 16x^6 + 36l^2x^4 + 16l^2x^4 - 48lx^5 - 24l^3x^3) + 2c_1c_2(2l^3x + 4lx^3 - 6l^2x^2 - 4l^2x^2 - 8x^4 + 12lx^3)] dx$$

$$= \frac{P}{2} \left(\frac{1}{3} c_1^2 l^3 + \frac{2}{105} c_2^2 l^7 + \frac{2}{15} c_1 c_2 l^5 \right)$$

$$T_{\max} = \frac{\omega^2}{2} \int_0^l \rho [W(x)]^2 dx = \frac{\rho \omega^2}{2} \int_0^l [c_1^2(l^2x^2 + x^4 - 2lx^3) + c_2^2(l^4x^4 + x^8 + 6l^2x^6 - 4lx^7 - 4l^3x^5) + c_1c_2(2l^3x^3 + 6lx^5 - 6l^2x^4 - 2x^6)] dx$$

$$= \frac{\rho \omega^2}{2} \left[\frac{1}{30} c_1^2 l^5 + \frac{1}{630} c_2^2 l^9 + \frac{1}{70} c_1 c_2 l^7 \right]$$

$$X = P \left(\frac{1}{3} c_1^2 l^3 + \frac{2}{105} c_2^2 l^7 + \frac{2}{15} c_1 c_2 l^5 \right)$$

$$Y = \rho \left(\frac{1}{30} c_1^2 l^5 + \frac{1}{630} c_2^2 l^9 + \frac{1}{70} c_1 c_2 l^7 \right)$$

$$\frac{\partial X}{\partial c_1} = P \left(\frac{2}{3} c_1 l^3 + \frac{2}{15} c_2 l^5 \right) , \quad \frac{\partial X}{\partial c_2} = P \left(\frac{4}{105} c_2 l^7 + \frac{2}{15} c_1 l^5 \right)$$

$$\frac{\partial Y}{\partial c_1} = \rho \left(\frac{1}{15} c_1 l^5 + \frac{1}{70} c_2 l^7 \right) , \quad \frac{\partial Y}{\partial c_2} = \rho \left(\frac{1}{315} c_2 l^9 + \frac{1}{70} c_1 l^7 \right)$$

$$\frac{\partial X}{\partial c_1} - \omega^2 \frac{\partial Y}{\partial c_1} = 0 , \quad \frac{\partial X}{\partial c_2} - \omega^2 \frac{\partial Y}{\partial c_2} = 0 \quad \text{give}$$

$$P \left(c_1 \frac{2}{3} l^3 + c_2 \frac{2}{15} l^5 \right) - \omega^2 \rho \left(c_1 \frac{l^5}{15} + c_2 \frac{l^7}{70} \right) = 0$$

$$P \left(c_2 \frac{4l^7}{105} + c_1 \frac{2l^5}{15} \right) - \omega^2 \rho \left(c_2 \frac{l^9}{315} + c_1 \frac{l^7}{70} \right) = 0$$

$$\text{i.e., } c_1 \left(\frac{2}{3} P l^3 - \frac{\rho \omega^2 l^5}{15} \right) + c_2 \left(\frac{2}{15} P l^5 - \frac{\rho \omega^2 l^7}{70} \right) = 0$$

$$c_1 \left(\frac{2}{15} P l^5 - \frac{\rho \omega^2 l^7}{70} \right) + c_2 \left(\frac{4}{105} P l^7 - \frac{\rho \omega^2 l^9}{315} \right) = 0$$

Setting the determinant of the coefficients of c_1 and c_2 to zero gives

$$\left(\frac{2}{3} P l^3 - \frac{1}{15} \rho \omega^2 l^5 \right) \left(\frac{4}{105} P l^7 - \frac{1}{315} \rho \omega^2 l^9 \right) - \left(\frac{2}{15} P l^5 - \frac{1}{70} \rho \omega^2 l^7 \right) \left(\frac{2}{15} P l^5 - \frac{1}{70} \rho \omega^2 l^7 \right) = 0$$

$$\text{i.e., } \omega^4 - 112 \left(\frac{P}{\rho l^2} \right) \omega^2 + 1008 \left(\frac{P}{\rho l^2} \right)^2 = 0$$

$$\omega^2 = 9.8697 \frac{P}{\rho l^2}, \quad 102.1302 \frac{P}{\rho l^2}$$

$$\therefore \omega_1 = 3.1416 \sqrt{\frac{P}{\rho l^2}}, \quad \omega_2 = 10.1059 \sqrt{\frac{P}{\rho l^2}}$$

First mode Third mode

Exact solution is :

$$\omega_1 = 3.1416 \sqrt{\frac{P}{\rho l^2}}, \quad \omega_2 = 6.2832 \sqrt{\frac{P}{\rho l^2}}, \quad \omega_3 = 9.4248 \sqrt{\frac{P}{\rho l^2}}$$

$$\text{8.68 (a) } V_{\max} = \frac{1}{2} \int_0^l EA \left(\frac{dU}{dx} \right)^2 dx \quad \left| \begin{array}{l} U(x) = \frac{c_1}{l} x \\ \frac{dU}{dx} = \frac{c_1}{l} \end{array} \right.$$

$$= \frac{1}{2} \int_0^l EA \left(\frac{c_1}{l} \right)^2 dx$$

$$= \frac{EA c_1^2}{2l} \text{ for uniform beam}$$

$$T_{\max} = \frac{\rho \omega^2}{2} \int_0^l A U^2 dx = \frac{\rho \omega^2}{2} \int_0^l A \frac{c_1^2}{l^2} x^2 dx$$

$$= \frac{\omega^2 \rho A c_1^2 l}{6} \text{ for uniform beam}$$

$$T_{\max} = V_{\max} \text{ gives } \omega^2 = \frac{3E}{\rho l^2}$$

$$\therefore \omega_1 = 1.73205 \sqrt{\frac{E}{\rho l^2}}$$

$$(b) V_{\max} = \frac{EA}{2} \int_0^l \left(\frac{dU}{dx} \right)^2 dx$$

$$= \frac{EA}{2} \left[\frac{c_1^2}{l} + \frac{4}{3} \frac{c_2^2}{l} + \frac{2c_1 c_2}{l} \right]$$

$$T_{\max} = \frac{\rho \omega^2}{2} \int_0^l A \cdot U^2 dx$$

$$= \frac{\rho A \omega^2}{2} \left[\frac{1}{3} c_1^2 l + \frac{1}{5} c_2^2 l + \frac{1}{2} c_1 c_2 l \right]$$

$$U(x) = \frac{c_1 x}{l} + \frac{c_2 x^2}{l^2}$$

$$\frac{dU}{dx} = \frac{c_1}{l} + \frac{2c_2 x}{l^2}$$

Eqs. (E.1) and (E.2) of Example 8.12 give

$$\frac{\partial X}{\partial c_1} - \omega^2 \frac{\partial Y}{\partial c_1} = 0 \quad \text{and} \quad \frac{\partial X}{\partial c_2} - \omega^2 \frac{\partial Y}{\partial c_2} = 0 \quad (E.1)$$

$$\text{Where } X = \frac{EA}{2l} \left[c_1^2 + \frac{4}{3} c_2^2 + 2c_1 c_2 \right] \quad (E.2)$$

$$\text{and } Y = \frac{\rho A l}{2} \left[\frac{1}{3} c_1^2 + \frac{1}{5} c_2^2 + \frac{1}{2} c_1 c_2 \right] \quad (E.3)$$

Eqs. (E.1) to (E.3) give

$$c_1 \left(\frac{EA}{l} - \frac{1}{3} \omega^2 \rho A l \right) + c_2 \left(\frac{EA}{l} - \frac{1}{4} \omega^2 \rho A l \right) = 0$$

$$c_2 \left(\frac{4}{3} \frac{EA}{l} - \frac{1}{5} \omega^2 \rho A l \right) + c_1 \left(\frac{EA}{l} - \frac{1}{4} \omega^2 \rho A l \right) = 0$$

Frequency equation is

$$\begin{vmatrix} \left(1 - \frac{1}{3} \lambda\right) & \left(1 - \frac{1}{4} \lambda\right) \\ \left(1 - \frac{1}{4} \lambda\right) & \left(\frac{4}{3} - \frac{1}{5} \lambda\right) \end{vmatrix} = \frac{\lambda^2}{240} - \frac{13\lambda}{90} + \frac{1}{3} = 0$$

$$\text{where } \lambda = \frac{\omega^2 \rho l^2}{E}$$

$$\lambda = 2.48596, 32.1807$$

$$\therefore \omega_1 = 1.57669 \sqrt{\frac{E}{\rho l^2}}, \quad \omega_2 = 5.67280 \sqrt{\frac{E}{\rho l^2}}$$

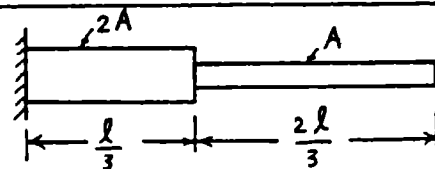
8.69

$$U(x) = c_1 \sin \frac{\pi x}{2l} + c_2 \sin \frac{3\pi x}{2l}$$

$$\frac{dU}{dx} = \frac{\pi}{2l} \left(c_1 \cos \frac{\pi x}{2l} + 3c_2 \cos \frac{3\pi x}{2l} \right)$$

$$V_{\max} = \frac{EA}{2} \left(\frac{\pi}{2l} \right)^2 \left[\int_0^{l/3} 2 \left(c_1 \cos \frac{\pi x}{2l} + 3c_2 \cos \frac{3\pi x}{2l} \right)^2 dx \right. \\ \left. + \int_{l/3}^l \left(c_1 \cos \frac{\pi x}{2l} + 3c_2 \cos \frac{3\pi x}{2l} \right)^2 dx \right]$$

$$\text{Using the relations } \int \cos^2 \frac{\pi x}{2l} dx = \frac{x}{2} + \frac{l}{2\pi} \sin \frac{\pi x}{l},$$



$$\int \cos^2 \frac{3\pi x}{2l} dx = \frac{x}{2} + \frac{l}{6\pi} \sin \frac{3\pi x}{l},$$

$$\int \cos \frac{\pi x}{2l} \cdot \cos \frac{3\pi x}{2l} \cdot dx = \frac{\sin \frac{\pi x}{l}}{(2\pi/l)} + \frac{\sin \frac{2\pi x}{l}}{(4\pi/l)},$$

$$V_{\max} = \frac{EA\pi^2}{8l^2} \left[c_1^2 \left(\frac{2}{3}l + \frac{l}{2\pi} \sin \frac{\pi}{3} \right) + c_2^2 (6l) \right. \\ \left. + c_1 c_2 \left(\frac{3l}{\pi} \sin \frac{\pi}{3} + \frac{3l}{2\pi} \sin \frac{2\pi}{3} \right) \right]$$

$$T_{\max} = \frac{\rho\omega^2}{2} \left[\int_0^{\frac{l}{3}} 2A \left\{ c_1^2 \sin^2 \frac{\pi x}{2l} + c_2^2 \sin^2 \frac{3\pi x}{2l} \right. \right. \\ \left. \left. + 2c_1 c_2 \sin \frac{\pi x}{2l} \cdot \sin \frac{3\pi x}{2l} \right\} dx + \int_{\frac{l}{3}}^l A \left\{ c_1^2 \sin^2 \frac{\pi x}{2l} \right. \right. \\ \left. \left. + c_2^2 \sin^2 \frac{3\pi x}{2l} + 2c_1 c_2 \sin \frac{\pi x}{2l} \sin \frac{3\pi x}{2l} \right\} dx \right]$$

$$\text{But } \int \sin^2 \frac{\pi x}{2l} dx = \frac{x}{2} - \frac{l}{2\pi} \sin \frac{\pi x}{l},$$

$$\int \sin^2 \frac{3\pi x}{2l} dx = \frac{x}{2} - \frac{l}{6\pi} \sin \frac{3\pi x}{l},$$

$$\int \sin \frac{\pi x}{2l} \cdot \sin \frac{3\pi x}{2l} dx = \frac{\sin \frac{\pi x}{l}}{(2\pi/l)} - \frac{\sin \frac{2\pi x}{l}}{(4\pi/l)}$$

$$T_{\max} = \frac{\rho A \omega^2}{2} \left[c_1^2 \left(\frac{2l}{3} - \frac{l}{2\pi} \sin \frac{\pi}{3} \right) + c_2^2 \left(\frac{2l}{3} \right) + c_1 c_2 \left(\frac{l}{\pi} \sin \frac{\pi}{3} - \frac{l}{2\pi} \sin \frac{2\pi}{3} \right) \right]$$

$$X = \frac{EA\pi^2}{4l^2} \left[c_1^2 \left(\frac{2l}{3} + \frac{l}{2\pi} \sin \frac{\pi}{3} \right) + c_2^2 (6l) + c_1 c_2 \left(\frac{3l}{\pi} \sin \frac{\pi}{3} + \frac{3l}{2\pi} \sin \frac{2\pi}{3} \right) \right]$$

$$Y = \rho A \left[c_1^2 \left(\frac{2l}{3} - \frac{l}{2\pi} \sin \frac{\pi}{3} \right) + c_2^2 \left(\frac{2l}{3} \right) + c_1 c_2 \left(\frac{l}{\pi} \sin \frac{\pi}{3} - \frac{l}{2\pi} \sin \frac{2\pi}{3} \right) \right]$$

$$\frac{\partial X}{\partial c_1} = \frac{EA\pi^2}{4l^2} \left[2c_1 \left(\frac{2l}{3} + \frac{l}{2\pi} \sin \frac{\pi}{3} \right) + c_2 \left(\frac{3l}{\pi} \sin \frac{\pi}{3} + \frac{3l}{2\pi} \sin \frac{2\pi}{3} \right) \right]$$

$$= \frac{EA\pi^2}{4l^2} [2c_1(0.8045l) + c_2(1.2405l)]$$

$$\frac{\partial X}{\partial c_2} = \frac{EA\pi^2}{4l^2} [c_2(12l) + c_1 \left(\frac{3l}{\pi} \sin \frac{\pi}{3} + \frac{3l}{2\pi} \sin \frac{2\pi}{3} \right)]$$

$$= \frac{EA\pi^2}{4l^2} [c_2(12l) + c_1(1.2405l)]$$

$$\frac{\partial Y}{\partial c_1} = \rho A \left[2c_1 \left(\frac{2l}{3} - \frac{l}{2\pi} \sin \frac{\pi}{3} \right) + c_2 \left(\frac{l}{\pi} \sin \frac{\pi}{3} - \frac{l}{2\pi} \sin \frac{2\pi}{3} \right) \right]$$

$$= \rho A [2c_1(0.5288l) + c_2(0.1378l)]$$

$$\frac{\partial Y}{\partial c_2} = \rho A \left[c_2 \left(\frac{4l}{3} \right) + c_1 \left(\frac{l}{\pi} \sin \frac{\pi}{3} - \frac{l}{2\pi} \sin \frac{2\pi}{3} \right) \right]$$

$$= \rho A [c_2(1.3333l) + c_1(0.1378l)]$$

$$\frac{\partial X}{\partial c_1} - \omega^2 \frac{\partial Y}{\partial c_1} = 0 \quad \text{and} \quad \frac{\partial X}{\partial c_2} - \omega^2 \frac{\partial Y}{\partial c_2} = 0 \quad \text{give}$$

$$c_1 \left(\frac{EA\pi^2}{l} 0.40225 - \rho A \omega^2 l 1.0576 \right) + c_2 \left(\frac{EA\pi^2}{l} 0.31012 - \rho A \omega^2 l 0.1378 \right) = 0$$

$$c_1 \left(\frac{EA\pi^2}{l} 0.31012 - \rho A \omega^2 l 0.1380 \right) + c_2 \left(\frac{EA\pi^2}{l} 3.0 - \rho A \omega^2 l 1.3333 \right) = 0$$

Frequency equation is

$$\left(0.40225 \frac{EA\pi^2}{l} - \rho A \omega^2 l 1.0576 \right) \left(\frac{EA\pi^2}{l} 3.0 - \rho A \omega^2 l 1.3333 \right) - \left(\frac{EA\pi^2}{l} 0.31012 - \rho A \omega^2 l 0.1380 \right)^2 = 0$$

$$\text{i.e.,} \quad \omega^4 - 25.7090 \omega^2 \left(\frac{E}{\rho l^2} \right) + 77.7692 \left(\frac{E}{\rho l^2} \right)^2 = 0$$

$$\omega^2 = \left(\frac{25.7090 \pm 18.7010}{2} \right) \frac{E}{\rho l^2}$$

$$\therefore \omega_1 = 1.8719 \sqrt{\frac{E}{\rho l^2}}, \quad \omega_2 = 4.7022 \sqrt{\frac{E}{\rho l^2}}$$

8.70

$$U(x) = c_1 \sin \frac{\pi x}{2l} + c_2 \sin \frac{3\pi x}{2l}, \quad \frac{dU}{dx} = \frac{\pi}{2l} \left(c_1 \cos \frac{\pi x}{2l} + 3c_2 \cos \frac{3\pi x}{2l} \right)$$

$$V_{\max} = \frac{1}{2} \int_0^l EA(x) \left(\frac{dU}{dx} \right)^2 dx = \frac{1}{2} \int_0^l 2EA_0 \left(1 - \frac{x}{l} \right) \left(\frac{\pi}{2l} \right)^2 \left(c_1^2 \cos^2 \frac{\pi x}{2l} + 9c_2^2 \cos^2 \frac{3\pi x}{2l} + 6c_1 c_2 \cos \frac{\pi x}{2l} \cos \frac{3\pi x}{2l} \right) dx$$

$$= c_1^2 \frac{EA_0 \pi^2}{16l} \left(1 + \frac{4}{\pi^2} \right) + c_2^2 \frac{9EA_0 \pi^2}{16l} \left(1 + \frac{4}{9\pi^2} \right) + c_1 c_2 \frac{3EA_0 \pi^2}{2l}$$

$$T_{\max} = \frac{\omega^2}{2} \int_0^l m(x) \cdot U^2 \cdot dx = m_0 \omega^2 \int_0^l \left(1 - \frac{x}{l} \right) \left(c_1^2 \sin^2 \frac{\pi x}{2l} + c_2^2 \sin^2 \frac{3\pi x}{2l} + 2c_1 c_2 \sin \frac{\pi x}{2l} \sin \frac{3\pi x}{2l} \right) dx$$

$$= c_1^2 \frac{\omega^2 m_0 l}{4} \left(1 - \frac{4}{\pi^2} \right) + c_2^2 \frac{\omega^2 m_0 l}{4} \left(1 - \frac{4}{9\pi^2} \right)$$

$$+ c_1 c_2 \frac{2\omega^2 m_0 l}{\pi^2}$$

$$X = c_1^2 \frac{EA_0}{l} (0.86685) + c_2^2 \frac{EA_0}{l} (5.80165) + c_1 c_2 \frac{EA_0}{l} (1.5)$$

$$Y = c_1^2 m_0 l (0.14868) + c_2^2 m_0 l (0.23874) + c_1 c_2 m_0 l (0.20264)$$

$$\frac{\partial X}{\partial c_1} = c_1 \frac{EA_0}{l} (1.73370) + c_2 \frac{EA_0}{l} (1.5)$$

$$\frac{\partial X}{\partial c_2} = c_1 \frac{EA_0}{l} (1.5) + c_2 \frac{EA_0}{l} (11.60330)$$

$$\frac{\partial Y}{\partial c_1} = c_1 m_0 l (0.29736) + c_2 m_0 l (0.20264)$$

$$\frac{\partial Y}{\partial c_2} = c_1 m_0 l (0.20246) + c_2 m_0 l (0.47748)$$

$$\frac{\partial X}{\partial c_1} - \omega^2 \frac{\partial Y}{\partial c_1} = 0 \quad \text{and} \quad \frac{\partial X}{\partial c_2} - \omega^2 \frac{\partial Y}{\partial c_2} = 0 \quad \text{give}$$

$$c_1 \left(\frac{EA_0}{l} 1.73370 - \omega^2 m_0 l 0.29736 \right) + c_2 \left(\frac{EA_0}{l} 1.5 - \omega^2 m_0 l 0.20264 \right) = 0$$

$$c_1 \left(\frac{EA_0}{l} 1.5 - \omega^2 m_0 l 0.20246 \right) + c_2 \left(\frac{EA_0}{l} 11.60330 - \omega^2 m_0 l 0.47748 \right) = 0$$

Frequency equation is

$$\left(\frac{EA_0}{l} 1.7337 - \omega^2 m_0 l 0.29736 \right) \left(\frac{EA_0}{l} 11.6033 - \omega^2 m_0 l 0.47748 \right) - \left(\frac{EA_0}{l} 1.5 - \omega^2 m_0 l 0.20264 \right)^2 = 0$$

$$\text{or} \quad \omega^4 - 36.369 \omega^2 \left(\frac{EA_0}{m_0 l^2} \right) + 177.04292 \left(\frac{EA_0}{m_0 l^2} \right)^2 = 0$$

$$\omega^2 = \left(\frac{36.369 \pm 24.7898}{2} \right) \frac{EA_0}{m_0 l^2}$$

$$\therefore \omega_1 = 2.4062 \sqrt{\frac{EA_0}{m_0 l^2}}, \quad \omega_2 = 5.5299 \sqrt{\frac{EA_0}{m_0 l^2}}$$

8.71

For a string, $V_{\max} = \frac{1}{2} \int_0^l \rho \left(\frac{dW}{dx} \right)^2 dx$, $T_{\max} = \frac{\rho \omega^2}{2} \int_0^l W^2 dx$

$$\text{Here } W(x) = c_1 x(l-x) + c_2 x^2(l-x)^2$$

$$\frac{dW}{dx} = c_1 l - 2c_1 x + 2c_2 l^2 x - 6c_2 l x^2 + 4c_2 x^3$$

$$\int_0^l \left(\frac{dW}{dx} \right)^2 dx = c_1^2 \frac{l^3}{3} + c_2^2 \frac{2}{105} l^7 + c_1 c_2 \frac{2}{15} l^5$$

$$\int_0^l W^2 dx = c_1^2 \frac{l^5}{30} + c_2^2 \frac{l^9}{630} + c_1 c_2 \frac{l^7}{70}$$

Equating T_{\max} and V_{\max} , we get $\omega^2 = \frac{x}{y}$

$$\text{where } x = \frac{\rho}{2} \left[c_1^2 \frac{l^3}{3} + c_2^2 \frac{2}{105} l^7 + c_1 c_2 \frac{2}{15} l^5 \right]$$

$$\text{and } y = \frac{\rho}{2} \left[c_1^2 \frac{l^5}{30} + c_2^2 \frac{l^9}{630} + c_1 c_2 \frac{l^7}{70} \right]$$

$$\frac{\partial(\omega^2)}{\partial c_1} = \frac{\partial(\omega^2)}{\partial c_2} = 0 \quad \text{yield Eqs. } (E_{11}) \text{ and } (E_{12}) \text{ of Example 8.13.}$$

$$\frac{\partial X}{\partial c_1} = \frac{P}{2} \left[c_1 \frac{2}{3} l^3 + c_2 \frac{2}{15} l^5 \right], \quad \frac{\partial Y}{\partial c_1} = \frac{P}{2} \left[c_1 \frac{l^5}{15} + c_2 \frac{l^7}{70} \right]$$

$$\frac{\partial X}{\partial c_2} = \frac{P}{2} \left[c_1 \frac{2}{15} l^5 + c_2 \frac{4}{105} l^7 \right], \quad \frac{\partial Y}{\partial c_2} = \frac{P}{2} \left[c_1 \frac{l^7}{70} + c_2 \frac{l^9}{315} \right]$$

$$\frac{\partial X}{\partial c_1} - \omega^2 \frac{\partial Y}{\partial c_1} = 0 \Rightarrow c_1 \left(\frac{Pl^3}{3} - \frac{P\omega^2 l^5}{30} \right) + c_2 \left(\frac{Pl^5}{15} - \frac{P\omega^2 l^7}{140} \right) = 0$$

$$\frac{\partial X}{\partial c_2} - \omega^2 \frac{\partial Y}{\partial c_2} = 0 \Rightarrow c_1 \left(\frac{Pl^5}{15} - \frac{P\omega^2 l^7}{140} \right) + c_2 \left(\frac{2Pl^7}{105} - \frac{P\omega^2 l^9}{630} \right) = 0$$

By setting the determinant of the coefficient matrix of c_1 and c_2 to zero, we get the frequency equation as

$$\begin{vmatrix} \left(\frac{1}{3} - \frac{\lambda}{30} \right) & \left(\frac{1}{15} - \frac{\lambda}{140} \right) \\ \left(\frac{1}{15} - \frac{\lambda}{140} \right) & \left(\frac{2}{105} - \frac{\lambda}{630} \right) \end{vmatrix} = \lambda^2 - 112\lambda + 1008 = 0$$

where $\lambda = \frac{\rho l^2 \omega^2}{P}$.

$$\lambda_1 = 9.8722, \quad \lambda_2 = 102.4144$$

$$\therefore \omega_1 = 3.142 \sqrt{\frac{P}{\rho l^2}}, \quad \omega_2 = 10.12 \sqrt{\frac{P}{\rho l^2}}$$

8.72

Results of Ex8_72

1. beta = 0.01

>>program12

Roots of nonlinear equation

Data:

n = 2

xs = 5.000000e-001

xinc = 1.000000e-003

nint = 5000

iter = 10000

eps = 1.000000e-006

Roots

3.144773e+000

6.284776e+000

2. beta = 0.1

>>program12

Roots of nonlinear equation

Data:

n = 2

xs = 5.000000e-001

xinc = 1.000000e-003

nint = 5000

iter = 10000

eps = 1.000000e-006

Roots

3.173097e+000

6.299059e+000

3. beta = 1.0

>>program12

Roots of nonlinear equation

Data:

n = 2

xs = 5.000000e-001

xinc = 1.000000e-003

nint = 5000

iter = 10000

eps = 1.000000e-006

Roots

8.603336e-001

3.425618e+000

4. beta = 10.0
 >>program12
 Roots of nonlinear equation

Data:
 n = 2
 xs = 5.000000e-001
 xinc = 1.000000e-003
 nint = 5000
 iter = 10000
 eps = 1.000000e-006

Roots
 1.428870e+000
 7.228110e+000

5. beta = 100.0
 >>program12
 Roots of nonlinear equation

Data:
 n = 2
 xs = 5.000000e-001
 xinc = 1.000000e-003
 nint = 5000
 iter = 10000
 eps = 1.000000e-006

Roots
 1.555245e+000
 2.644501e+001

8.73

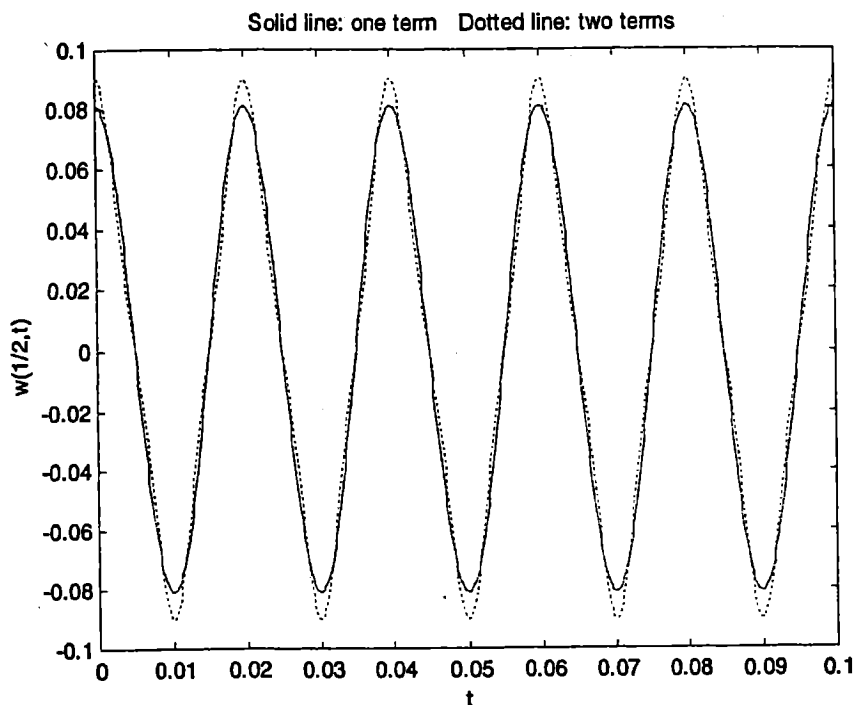
Results of Ex8_73

 >>program12
 Roots of nonlinear equation

Data:
 n = 5
 xs = 2.000000e+000
 xinc = 1.000000e-003
 nint = 50000
 iter = 100000
 eps = 1.000000e-006

Roots
 4.730041e+000
 7.853205e+000
 1.099561e+001
 1.413717e+001
 1.727876e+001

8.74



```

% Ex8_74.m
h = 0.1;
l = 1.0;
c = 100.0;
x = 1.0/2.0;
for i = 1: 501
    t(i) = (i-1)*0.1/500;
    w1(i) = ( 8*h/(pi^2) ) * ( sin(pi*x/l)*cos(pi*c*t(i)/l) );
    w2(i) = ( 8*h/(pi^2) ) * ( sin(pi*x/l)*cos(pi*c*t(i)/l)...
        - sin(3*pi*x/l)*cos(3*pi*c*t(i)/l)/9 );
end
plot(t,w1);
hold on;
plot(t, w2, ':');
xlabel('t');
ylabel('w(1/2,t)')
title('Solid line: one term   Dotted line: two terms');

```

8.75

Results of Ex8_75

1. beta = 0.01

Please input N:

5

Please input XS and XINC:

1.0 0.01

ROOTS OF NONLINEAR EQUATION

DATA:

N = 5

XS = 1

XINC = 0.01

NINT = 5000

ITER = 10000

EPS = 1e-006

ROOTS:

3.14477252

6.28477645

9.42583887

12.56716634

15.70859986

2. beta = 0.1

Please input N:

5

Please input XS and XINC:

1.0 0.01

ROOTS OF NONLINEAR EQUATION

DATA:

N = 5

XS = 1

XINC = 0.01

NINT = 5000

ITER = 10000

EPS = 1e-006

ROOTS:

3.17309718

6.29905936

9.43537598

12.57432316

15.71432680

3. beta = 1.0

Please input N:

5

Please input XS and XINC:

1.0 0.01

ROOTS OF NONLINEAR EQUATION

DATA:

N = 5

XS = 1

XINC = 0.01

NINT = 5000

ITER = 10000

EPS = 1e-006

ROOTS:

3.42561846

6.43729818

9.52933441

12.64528722

15.77128487

4. beta = 10.0

Please input N:

5

Please input XS and XINC:

1.0 0.01

ROOTS OF NONLINEAR EQUATION

DATA:

N = 5

XS = 1

XINC = 0.01

NINT = 5000

ITER = 10000

EPS = 1e-006

ROOTS:

1.42887002

7.22810977

10.20026259

13.21418568

16.25936123

```

5. beta = 100.0
Please input N:
5
Please input XS and XINC:
1.0 0.01

```

ROOTS OF NONLINEAR EQUATION

```

DATA:
N = 5
XS = 1
XINC = 0.01
NINT = 5000
ITER = 10000
EPS = 1e-006

```

```

ROOTS:
1.55524515
35.78466751
45.12916593
51.36179625
54.47903674

```

8.76

```

Results of Ex8_76
*****
Please input N:
5
Please input XS and XINC:
1.0 0.01

```

ROOTS OF NONLINEAR EQUATION

```

DATA:
N = 5
XS = 1
XINC = 0.01
NINT = 5000
ITER = 10000
EPS = 1e-006

```

```

ROOTS:
4.73004075
7.85320462
10.99560784
14.13716549
17.27875966

```

8.77

```

C=====
C
C PROGRAM 12.F
C MAIN PROGRAM FOR CALLING THE SUBROUTINE NONEQN
C
C=====
C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
  DIMENSION X(5)
  DATA N,XS,XINC,NINT,ITER,EPS
  2 /5,1.0,0.01,5000,1000,0.000001/
C END OF PROBLEM-DEPENDENT DATA
  PRINT 10,N,XS,XINC,NINT,ITER,EPS
  10  FORMAT (//,28H ROOTS OF NONLINEAR EQUATION,//,6H DATA: ,/,
  2  2H N,4X,2H =,I4,/,3H XS,3X,2H =,E15.6,/,8H XINC =,E15.6,/,
  3  8H NINT =,I4,/,8H ITER =,I4,/,8H EPS =,E15.6)

```

```

      CALL NONEQN (N,X,XS,XINC,NINT,ITER,EPS)
      PRINT 20,(X(I),I=1,N)
20    FORMAT (//,7H ROOTS:,,/(E15.6))
      STOP
      END

```

```

C=====
C
C SUBROUTINE NONEQN
C
C=====

```

```

      SUBROUTINE NONEQN (N,X,XS,XINC,NINT,ITER,EPS)
      DATA NP,II,CC,DELX /1,0,0.0,0.0/
      DIMENSION X(N)
      A=F(XS)
      IF (A) 13,11,13
11    IF (XS) 13,12,13
12    II=II+1
      X(II)=0.0
      XS=XS+XINC
      A=F(XS)
13    AA=A
      A=F(XS)
      IF (CC .LT. XS) CC=XS
      IF (NP .GT. NINT) GO TO 18
      XS=XS+XINC
      NP=NP+1
      IF (A/AA) 14,16,13
14    XS=XS-XINC
      XINC=XINC/5.0
      I=1
15    DELX=F(XS)*XINC/(F(XS+XINC)-F(XS))
      XS=XS-DELX
      ITER1=I
      IF (ABS(DELX) .LE. EPS) GO TO 16
      IF (CC .LT. XS) CC=XS
      I=I+1
      IF (I-ITER) 15,15,16
16    II=II+1
      X(II)=XS
      XS=CC+XINC
      IF (II-N) 17,18,18
17    A=F(XS)
      XS=XS+XINC
      IF (A .EQ. 0.0) A=F(XS)
      GO TO 13
18    CONTINUE
      RETURN
      END

```

```

C=====
C
C FUNCTION F(Y)
C THIS FUNCTION IS PROBLEM-DEPENDENT
C
C=====
      FUNCTION F(Y)
      F = Y * TAN(Y) - 0.01
      RETURN
      END

```

ROOTS OF NONLINEAR EQUATION

```

DATA:
N      =      5
XS     =      0.100000E+01
XINC   =      0.100000E-01
NINT   =5000
ITER   =1000
EPS    =      0.100000E-05

```

```

ROOTS:
  0.314477E+01
  0.628478E+01
  0.942584E+01
  0.125672E+02
  0.157086E+02

```

```

C=====
C
C FUNCTION F(Y)
C THIS FUNCTION IS PROBLEM-DEPENDENT
C
C=====
      FUNCTION F(Y)
      F = Y * TAN(Y) - 0.1
      RETURN
      END

```

ROOTS OF NONLINEAR EQUATION

```

DATA:
N    =    5
XS   =    0.100000E+01
XINC =    0.100000E-01
NINT =5000
ITER =1000
EPS  =    0.100000E-05

```

```

ROOTS:
  0.317310E+01
  0.629906E+01
  0.943538E+01
  0.125743E+02
  0.157143E+02

```

```

C=====
C
C FUNCTION F(Y)
C THIS FUNCTION IS PROBLEM-DEPENDENT
C
C=====
      FUNCTION F(Y)
      F = Y * TAN(Y) - 1.0
      RETURN
      END

```

ROOTS OF NONLINEAR EQUATION

```

DATA:
N    =    5
XS   =    0.100000E+01
XINC =    0.100000E-01
NINT =5000
ITER =1000
EPS  =    0.100000E-05

```

```

ROOTS:
  0.342562E+01
  0.643730E+01
  0.952933E+01
  0.126453E+02
  0.157713E+02

```

```

C=====
C
C FUNCTION F(Y)
C THIS FUNCTION IS PROBLEM-DEPENDENT
C
C=====

```

```

FUNCTION F(Y)
F = Y * TAN(Y) - 10.0
RETURN
END

```

ROOTS OF NONLINEAR EQUATION

DATA:

```

N      = 5
XS     = 0.100000E+01
XINC   = 0.100000E-01
NINT   =5000
ITER   =1000
EPS    = 0.100000E-05

```

ROOTS:

```

0.142887E+01
0.722811E+01
0.102003E+02
0.132142E+02
0.162594E+02

```

```

C=====
C
C FUNCTION F(Y)
C THIS FUNCTION IS PROBLEM-DEPENDENT
C
C=====
      FUNCTION F(Y)
      F = Y * TAN(Y) - 100.0
      RETURN
      END

```

ROOTS OF NONLINEAR EQUATION

DATA:

```

N      = 5
XS     = 0.100000E+01
XINC   = 0.100000E-01
NINT   =5000
ITER   =1000
EPS    = 0.100000E-05

```

ROOTS:

```

0.155525E+01
0.357847E+02
0.607154E+02
0.638345E+02
0.669543E+02

```

8.78

Frequency equation for a fixed-fixed beam is (from Fig. 8.15):

$$\cos \beta l \cosh \beta l - 1 = 0$$

The main program, the function F(Y) and the output are given below.

```

C=====
C
C PROGRAM
C MAIN PROGRAM FOR CALLING THE SUBROUTINE NONEQN
C
C=====
C FOLLOWING LINES CONTAIN PROBLEM-DEPENDENT DATA
  DIMENSION X(5)
  DATA N,NINT,ITER,EPS/5,50,100,0.000001/
  XS=2.0
  XINC=0.5
C END OF PROBLEM-DEPENDENT DATA
  WRITE (12,10) N,XS,XINC,NINT,ITER,EPS
10  FORMAT (//,28H ROOTS OF NONLINEAR EQUATION,//,6H DATA://,
2   2H N,4X,2H =,14,/,3H XS,3X,2H =,E15.6,/,8H XINC =,E15.6,/,
3   8H NINT =,14,/,8H ITER =,14,/,8H EPS =,E15.6)
  CALL NONEQN (N,X,XS,XINC,NINT,ITER,EPS)
  WRITE (12,20) (X(I),I=1,N)
20  FORMAT (//,7H ROOTS://,(E15.6))
  STOP
  END
C=====
C
C FUNCTION F(Y)
C THIS FUNCTION IS PROBLEM-DEPENDENT
C
C=====
  FUNCTION F(Y)
  F=COS(Y)*COSH(Y)-1.0
  RETURN
  END

```

ROOTS OF NONLINEAR EQUATION

DATA:

```

N      =    5
XS     =    0.200000E+01
XINC   =    0.500000E+00
NINT   =    50
ITER   =    100
EPS    =    0.100000E-05

```

ROOTS:

```

0.473004E+01
0.785320E+01
0.109956E+02
0.141372E+02
0.172788E+02

```

8.79 From Example 8.7, the n^{th} mode shape is given by

$$W_n(x) = C_{1n} \left\{ (\cos \beta_n x - \cosh \beta_n x) - \frac{(\cos \beta_n l - \cosh \beta_n l)}{(\sin \beta_n l - \sinh \beta_n l)} (\sin \beta_n x - \sinh \beta_n x) \right\}$$

where $\beta_n l = 3.926602, 7.068583, 10.210176, 13.351768$ for $n = 1, 2, 3, 4$.
The program and results are given below.

```

C =====
C
C PROBLEM 8.79
C
C =====
      DIMENSION X(4),W(4,9)
      N=4
      M=9
      MD=10
C N      = NUMBER OF MODE SHAPES TO BE COMPUTED
C M      = NUMBER OF STATIONS IN THE BEAM AT WHICH VALUE OF DEFLECTION
C        IS REQUIRED
C MD     = NUMHGR OF SEGMENTS IN THE BEAM
C W(N,M) = MATRIX CONTAINING THE MODE SHAPES IN ROWS
      DATA X/3.926602,7.068583,10.210176,13.351768/
      DO 10 I=1,N
      DO 20 J=1,M
      Y=X(I)*REAL(J)/REAL(MD)
      YY=X(I)
20      W(1,J)=(COS(Y)-COSH(Y))-((COS(YY)-COSH(YY))/(SIN(YY)-SINH(YY)))*
      2 (SIN(Y)-SINH(Y))
10      CONTINUE
      WRITE (86,30)
30      FORMAT (//,2X,43H MODE SHAPES OF FIXED-SIMPLY SUPPORTED BEAM,/)
      DO 40 I=1,N
40      WRITE (86,50) I,(W(1,J),J=1,M)
50      FORMAT (/,2X,6H MODE:,,14,4E15.6,/, (12X,4E15.6))
      STOP
      END

```

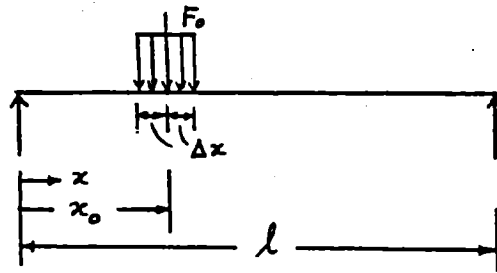
MODE SHAPES OF FIXED-SIMPLY SUPPORTED BEAM

MODE:	1	-0.133996E+00	-0.455738E+00	-0.848519E+00	-0.120675E+01
		-0.144486E+01	-0.150550E+01	-0.136498E+01	-0.103456E+01
		-0.557241E+00			
MODE:	2	-0.382233E+00	-0.107449E+01	-0.149510E+01	-0.131923E+01
		-0.570351E+00	0.422676E+00	0.119881E+01	0.139349E+01
		0.917175E+00			
MODE:	3	-0.690370E+00	-0.147476E+01	-0.112212E+01	0.204393E+00
		0.130049E+01	0.114192E+01	-0.111633E+00	-0.126025E+01
		-0.120557E+01			
MODE:	4	-0.100204E+01	-0.141423E+01	0.927429E-01	0.139201E+01
		0.539917E+00	-0.114441E+01	-0.107568E+01	0.642578E+00
		0.137500E+01			

8.80

Equation of motion is given by Eq. (8.77):

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = f(x, t) \quad (E_1)$$



Representation of load:

Load is $f(x)$ which is zero everywhere except over a length $2\Delta x$ at $x = x_0$, where it is equal to $(F_0/2\Delta x)$. The load $f(x)$ can be expanded into Fourier sine series as

$$f(x) = \sum_{n=1}^{\infty} f_n \sin \frac{n\pi x}{l} \quad (E_2)$$

where

$$\begin{aligned} f_n &= \frac{2}{l} \int_0^{x_0 - \Delta x} (0) \sin \frac{n\pi x}{l} dx + \frac{2}{l} \int_{x_0 - \Delta x}^{x_0 + \Delta x} \frac{F_0}{2\Delta x} \sin \frac{n\pi x}{l} dx \\ &\quad + \frac{2}{l} \int_{x_0 + \Delta x}^l (0) \sin \frac{n\pi x}{l} dx = \frac{F_0}{l \Delta x} \int_{x_0 - \Delta x}^{x_0 + \Delta x} \sin \frac{n\pi x}{l} dx \\ &= \frac{2 F_0}{l} \sin \frac{n\pi x_0}{l} \frac{\sin \left(\frac{n\pi \Delta x}{l} \right)}{\left(\frac{n\pi \Delta x}{l} \right)} \quad (E_3) \end{aligned}$$

As $\Delta x \rightarrow 0$, (E_3) becomes $f_n = \frac{2 F_0}{l} \sin \frac{n\pi x_0}{l} \quad (E_4)$

$$\therefore f(x) = \frac{2 F_0}{l} \sum_{n=1}^{\infty} \sin \frac{n\pi x_0}{l} \sin \frac{n\pi x}{l} \quad (E_5)$$

Since F_0 is moving along the bridge with constant velocity v , at time t , F_0 will be at a distance $x_0 = vt$ from left support. Hence the load distribution of Eq. (E5) can be rewritten as

$$f(x, t) = \frac{2 F_0}{l} \left\{ \sin \frac{\pi vt}{l} \sin \frac{\pi x}{l} + \sin \frac{2\pi vt}{l} \sin \frac{2\pi x}{l} + \dots \right\} \quad (E_6)$$

Setting $\omega = \frac{\pi v}{l}$, (E_6) can be expressed as

$$f(x, t) = \frac{2 F_0}{l} \left\{ \sin \frac{\pi x}{l} \sin \omega t + \sin \frac{2\pi x}{l} \sin 2\omega t + \dots \right\} \quad (E_7)$$

Equation of motion, (E₁), becomes

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = f(x, t) = \sum_{n=1}^{\infty} F_n \sin \frac{n\pi x}{l} \sin n\omega t \quad (E_8)$$

where $F_n = \frac{2 F_0}{l}$; $n = 1, 2, \dots$

We can find the solution of Eq. (E₈) by superposing the solutions of the individual harmonic components. Thus, we need to find, for the n^{th} harmonic, the solution of the following equation:

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = F_n \sin \frac{n\pi x}{l} \sin n\omega t \quad (E_9)$$

The solution (particular integral) of Eq. (E₉) can be taken as

$$w(x, t) = w_n \sin \frac{n\pi x}{l} \sin n\omega t \quad (E_{10})$$

Substitution of (E₁₀) into (E₉) gives

$$w_n = \frac{F_n l^4}{EI (n\pi)^4 \left\{ 1 - \left(\frac{n\omega}{\omega_n} \right)^2 \right\}} \quad (E_{11})$$

where $\omega_n = n^2 \pi^2 \sqrt{\frac{EI}{\rho A l^4}}$ (E₁₂)

Adding the homogeneous solution to the particular integral of Eq. (E₁₀), the general solution can be expressed as

$$w(x, t) = \sum_{i=1}^{\infty} \sin \frac{i\pi x}{l} \left\{ A_i \cos \omega_i t + B_i \sin \omega_i t \right\} + w_n \sin \frac{n\pi x}{l} \sin n\omega t \quad (E_{13})$$

The initial conditions are given by

$$w(x, 0) = 0 \quad , \quad \frac{\partial w}{\partial t}(x, 0) = 0 \quad (E_{14})$$

These initial conditions are satisfied if

$$A_i = 0 \text{ for all } i, \quad B_i = 0 \text{ for all } i \neq n$$

$$B_n = -w_n \left(\frac{n\omega}{\omega_n} \right)$$

$$w(x, t) = \frac{F_n l^4}{EI (n\pi)^4 \left\{ 1 - \left(\frac{n\omega}{\omega_n} \right)^2 \right\}} \sin \frac{n\pi x}{l} \left\{ \sin n\omega t - \frac{n\omega}{\omega_n} \sin \omega_n t \right\} \quad (E_{15})$$

Since $\omega_n = n^2 \omega_1$, where ω_1 is the fundamental frequency of the bridge, Eq. (E15) becomes

$$w(x,t) = \frac{F_n l^4}{EI \pi^4 \left\{ n^4 - \left(\frac{n\omega}{\omega_1} \right)^2 \right\}} \sin \frac{n\pi x}{l} \left\{ \sin n\omega t - \frac{n\omega}{n^2 \omega_1} \sin n^2 \omega_1 t \right\} \quad (E16)$$

Thus the total response of the bridge can be expressed as

$$w(x,t) = \frac{2 F_0 l^3}{EI \pi^4} \left[\frac{\sin \frac{\pi x}{l}}{1 - \left(\frac{\omega}{\omega_1} \right)^2} \left\{ \sin \omega t - \left(\frac{\omega}{\omega_1} \right) \sin \omega_1 t \right\} \right. \\ + \frac{\sin \frac{2\pi x}{l}}{2^4 - \left(\frac{2\omega}{\omega_1} \right)^2} \left\{ \sin 2\omega t - \left(\frac{\omega}{2\omega_1} \right) \sin 4\omega_1 t \right\} \\ \left. + \frac{\sin \frac{3\pi x}{l}}{3^4 - \left(\frac{3\omega}{\omega_1} \right)^2} \left\{ \sin 3\omega t - \left(\frac{\omega}{3\omega_1} \right) \sin 9\omega_1 t \right\} + \dots \right] \quad (E17)$$
