

Chapter 10

Vibration Measurement and Applications

10.1

Voltage sensitivity = $v = 0.098$ volt-meter/Newtonthickness = $t = 2 \text{ mm} = \frac{2}{1000} \text{ m}$ output voltage = 220 volts, pressure applied = $p_x = ?$

$$E = vt p_x \Rightarrow 220 = (0.098) \left(\frac{2}{1000} \right) p_x$$

$$\therefore p_x = 1.1224 \times 10^6 \text{ N/m}^2$$

10.2

 $m = 0.5 \text{ kg}$, $k = 10000 \text{ N/m}$, $c \approx 0$ amplitude = $Y = 4/1000 \text{ m}$ total displacement of mass = $x = 12/1000 \text{ m}$ relative displacement = $z = x - y = 8/1000 \text{ m}$

$$Z = \frac{r^2 Y}{1 - r^2} \quad \text{i.e.,} \quad \frac{8}{1000} = \left\{ \frac{(4/1000) r^2}{1 - r^2} \right\} \Rightarrow r^2 = 2/3$$

$$r = \frac{\omega}{\omega_n} = 0.8165$$

$$\omega = r \omega_n = 0.8165 \sqrt{\frac{10000}{0.5}} = 115.4705 \text{ rad/sec} = 18.3777 \text{ Hz}$$

10.3

$$Z = \frac{r^2 Y}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \quad \text{where} \quad \zeta = \frac{c}{c_{\text{cri}}} = \frac{1}{\sqrt{2}}$$

$$= \frac{r^2 Y}{\sqrt{(1-r^2)^2 + (\sqrt{2} r)^2}} = \frac{r^2 Y}{\sqrt{1+r^4}}$$

10.4

speed range:

$$500 \text{ rpm} = 52.36 \text{ rad/sec} \quad - \quad 1500 \text{ rpm} = 157.08 \text{ rad/sec}$$

$$x = X \cos \omega t ;$$

$$Z = \frac{r^2 Y}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \quad (E_1)$$

CASE (i): Let $\zeta = 0$ Eq. (E₁) gives, for 2% error,

$$\frac{Z}{Y} = \frac{r^2}{|r^2 - 1|} = 1.02 \quad \Rightarrow \quad r = \frac{\omega}{\omega_n} = 7.1414$$

$$\omega_n = \frac{\omega}{7.1414} = \frac{500}{7.1414} \quad \text{or} \quad \frac{1500}{7.1414} = 70.0143 \quad \text{or} \quad 210.0428 \text{ rpm}$$

$$= 1.1669 \quad \text{or} \quad 3.5007 \text{ Hz} = 7.3319 \quad \text{or} \quad 21.9957 \text{ rad/sec}$$

$\therefore \omega_n = 1.1669 \text{ Hz}$

CASE (ii): Let $\zeta = 0.6$

Eg. (E_1) gives $\frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = E = 1.02$

$$\Rightarrow 0.0404 r^4 - 0.5826 r^2 + 1.0404 = 0$$

which gives $r = 2.2562, 3.0546$

Since the quantity E attains maximum at

$$r = \frac{1}{\sqrt{1-2\zeta^2}} = 1.8898 \quad \text{for} \quad \zeta = 0.6$$

we use $r = 2.2562$.

$$\therefore \text{Maximum } \omega_n = \omega/r = 500/(2.2562 \times 60) = 3.6935 \text{ Hz}$$

10.5

Error factor for vibrometer is

$$E = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

Maximum of E occurs at

$$r^* = \frac{1}{\sqrt{1-2\zeta^2}}$$

For $\zeta = 0$, $E = \frac{r^2}{|1-r^2|}$ and $r^* = 1$

Since the range is $4 \leq r < \infty$, we use $r = 4$ for maximum error.

$$E|_{r=4} = \frac{4^2}{|1-4^2|} = 1.0667$$

$$\text{Percent error} = (E-1)100 = 6.67\%$$

10.6

Error factor = $E = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$

E attains maximum at $r^* = 1/\sqrt{1-2\zeta^2}$

For $\zeta = 0.67$, $r^* = 3.1281$ and $E|_{r=r^*, \zeta=0.67} = 1.0053$

$$\text{Percent error} = 0.53\%$$

10.7

Select the vibrometer on the basis of lowest frequency being measured.

(a) $\zeta = 0$:

$$\text{From Eq. (10.19), } \frac{Z}{Y} = 1.03 = \frac{r^2}{|r^2 - 1|}, \quad r^2 = \frac{1.03}{0.03} = 34.3333$$

$$r = \frac{\omega}{\omega_n} = \frac{2\pi(500)}{60\omega_n} = 5.8595$$

$$\omega_n = 8.9359 \text{ rad/sec} = 1.4222 \text{ Hz}$$

(b) $\zeta = 0.6$:

$$\text{From Eq. (10.19), } \frac{Z}{Y} = 1.03 = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = E$$

$$(1.03)^2 \{1 + r^4 - 2r^2 + 4r^2\zeta^2\} = r^4$$

$$\text{i.e., } 0.057404 r^4 - 0.56 r^2 + 1 = 0$$

$$\text{i.e., } r^2 = 2.3535, \quad 7.4018$$

$$\text{i.e., } r = 1.5341^\dagger, \quad 2.7206$$

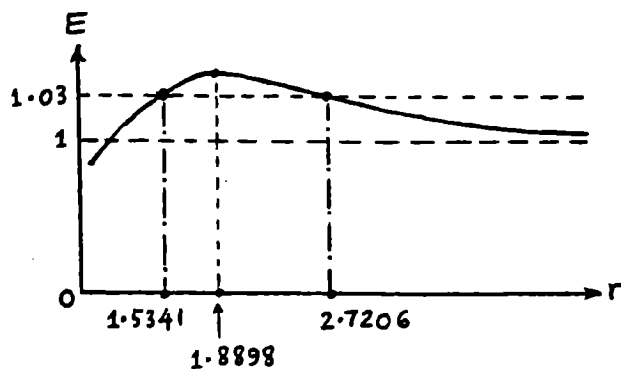
By selecting $r = 2.7206$,

$$\begin{aligned} \omega_n &= \frac{1000\pi}{60(2.7206)} \\ &= 19.2458 \text{ rad/sec} \\ &= 3.0630 \text{ Hz} \end{aligned}$$

[†] The quantity E attains maximum at

$$r = \frac{1}{\sqrt{1 - 2\zeta^2}} = 1.8898$$

for $\zeta = 0.6$. Hence we have to take $r = 2.7206$ to avoid peak of E .



10.8

$$\delta_{st} = \frac{10}{1000} \text{ m}, \quad \omega = 4000 \text{ rpm} = 418.88 \text{ rad/sec}$$

$$\omega_n = \sqrt{g/\delta_{st}} = \sqrt{9.81 \left(\frac{1000}{10}\right)} = 31.3209 \text{ rad/sec}$$

$$r = \omega/\omega_n = 418.88/31.3209 = 13.3738$$

Let $\zeta = 0$:

$$\begin{aligned} \text{Error factor} &= \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \Big|_{\zeta=0} = \frac{r^2}{|1-r^2|} = \frac{13.3738^2}{|1-13.3738^2|} \\ &= 1.0056 \end{aligned}$$

- (i) Maximum displacement = $Y = Z/1.0056 = 1/1.0056 = 0.9944$ mm
(ii) Maximum velocity = $\omega Y = (418.88)(0.9944) = 416.5473$ mm/sec
(iii) Maximum acceleration = $\omega^2 Y = (418.88)^2(0.9944) = 174483.35$ mm/sec².

10.9 $\frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \quad \text{----- (E}_1\text{)}$

Maximum of $\frac{Z}{Y}$ occurs when $r = r^* = \frac{1}{\sqrt{1-2\zeta^2}}$ (see Eq. (3.84))

For $\zeta = 0.5$, $r^* = \frac{1}{\sqrt{0.5}} = 1.4142$, $(r^*)^2 = 2$

$$\left. \frac{Z}{Y} \right|_{r^*} = \frac{2}{\sqrt{(1-2)^2 + (2 \times 0.5 \times 1.4142)^2}} = \frac{2}{\sqrt{3}} = 1.1547$$

When error is one percent, $\frac{Z}{Y} = 1.01$ or $\frac{Y}{Z} = 0.9901$

Eq. (E₁) can be rewritten as:

$$\left| \frac{Y}{Z} \right|^2 = \frac{(1-r^2)^2 + 4\zeta^2 r^2}{r^4}$$

$$r^4 \left(1 - \left| \frac{Y}{Z} \right|^2 \right) - r^2 (2 - 4\zeta^2) + 1 = 0 \quad \text{----- (E}_2\text{)}$$

For $\frac{Y}{Z} = 0.9901$ and $\zeta = 0.5$, (E₂) becomes

$$0.0197 r^4 - r^2 + 1 = 0$$

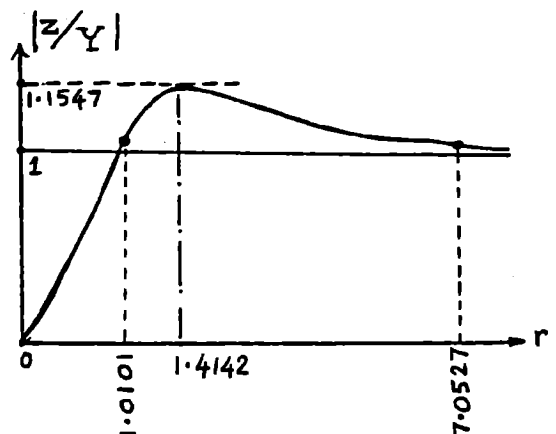
$$r^2 = 1.0203, \quad 49.7411$$

$$r = \frac{\omega}{\omega_n} = 1.0101, \quad 7.0527$$

Lowest frequency for one percent accuracy

$$= 7.0527 (5)$$

$$= 35.2635 \text{ Hz}$$



10.10 Frequency range > 100 Hz, maximum error = 2%

$k = 4000$ N/m, $c = 0 \Rightarrow \zeta = 0$, $m = ?$

For vibrometer with $\zeta = 0$,

$$\frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \Big|_{\zeta=0}$$

$$= \frac{r^2}{|1-r^2|} = 1.02$$

or $r^2/(1-r^2) = -1.02$ since r must be greater than one for higher frequencies

or $r^2 = 51$ i.e., $r = 7.1414$

Minimum impressed frequency = $\omega = 10$ Hz

$$\text{Since } r = \frac{\omega}{\omega_n} = 100/\omega_n = 7.1414, \quad \omega_n = 14.0029 \text{ Hz} = 87.9827 \frac{\text{rad}}{\text{sec}}$$

$$= \sqrt{k/m}$$

$$\therefore m = k/\omega_n^2 = 4000/(87.9827)^2 = 0.5167 \text{ kg}$$

10.11

$$\omega_n = 10 \text{ Hz}, \quad \omega_d = 8 \text{ Hz} = \omega_n \sqrt{1-\zeta^2} = 10 \sqrt{1-\zeta^2} \Rightarrow \zeta = 0.6$$

Let the lowest frequency = $\omega_0 \Rightarrow r_0 = \omega_0/\omega_n$

Error = 2% in the range $r_0 \leq r < \infty$

$$\text{Error} = E = 1.02 = \frac{r^2}{\sqrt{(1-r^2)^2 + (2 \times 0.6 r)^2}}$$

$$\Rightarrow 0.0404 r^4 - 0.5826 r^2 + 1.0404 = 0$$

$$\Rightarrow r = 2.2562, 3.0546$$

Let $r_0 = 2.2562$:

$$\omega_0 = r_0 \omega_n = 2.2562(10) = 22.562 \text{ Hz}$$

Let $r_0 = 3.0546$:

$$\omega_0 = r_0 \omega_n = 3.0546(10) = 30.546 \text{ Hz}$$

\therefore Lowest frequency = 22.562 Hz = 141.7616 rad/sec

10.12

$$\text{Error factor for accelerometer} = E = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

Maximum of E occurs at $r^* = \sqrt{1-2\zeta^2}$

$$\text{When } \zeta = 0, \quad r^* = 1 \text{ and } E = \frac{1}{|1-r^2|}$$

Since the range is $0 \leq r \leq 0.65$, we use $r = 0.65$:

$$E|_{r=0.65} = \frac{1}{|1-0.65^2|} = 1.7316$$

$$\text{Percent error} = (E-1)100 = 73.16\%$$

10.13

$$\text{Error factor} = E = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

Value of r at which E attains maximum is $r^* = \sqrt{1-2\zeta^2}$

$$\text{When } \zeta = 0.75, \quad E = \frac{1}{\sqrt{(1-r^2)^2 + 2.25 r^2}} \quad \text{and } r^* = \sqrt{1-1.125} = \text{imaginary}$$

Since the range is $0 \leq r \leq 0.6$, we use $r = 0.6$

$$E|_{r=0.6} = \frac{1}{\sqrt{(1-0.36)^2 + 2.25(0.6)^2}} = 0.9055$$

$$\text{Percent error} = (E-1)100 = -9.45\%$$

10.14

$m = 0.05 \text{ kg}$, max error = 3% over frequency range of 0 to 100 Hz
Find k and c .

For accelerometer, error factor = $E = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$

E attains maximum at $r = r^* = \sqrt{1-2\zeta^2}$

and $E_{\max} = E|_{r^*} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$

(i) Consideration of maximum error:

let error = $e = E - 1 = 0.03 = \frac{1}{2\zeta\sqrt{1-\zeta^2}} - 1$

upon rearrangement, this leads to $\zeta^4 - \zeta^2 + 0.23565 = 0$

or $\zeta = 0.6164, 0.7874$

r^* at $\zeta = 0.6164 = \sqrt{1 - 2(0.6164)^2} = 0.49$

r^* at $\zeta = 0.7874 = \sqrt{1 - 2(0.7874)^2} = \text{imaginary}$

(ii) Consideration of minimum error:

let error = $e = E - 1 = -0.03 = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} - 1$ (E₁)

with $\zeta = 0.6164$, Eq. (E₁) can be simplified as

$r^4 - 0.4802 r^2 - 0.0628 = 0$

$\Rightarrow r^2 = 0.5871, -0.1069$

or $r = 0.7662$

At the maximum frequency,

$\omega = 2\pi(100) = 628.32 \text{ rad/sec}$

$\omega_n = \omega/r = 628.32/0.7662$

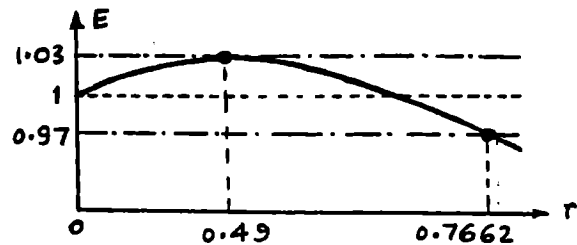
$= 820.0470 \text{ rad/sec}$

$k = m\omega_n^2 = 0.05(820.047)^2$

$= 33623.8541 \text{ N/m}$

$c_c = 2m\omega_n = \frac{c}{\zeta} \Rightarrow c = 2m\omega_n\zeta = 2(0.05)(820.047)(0.6164)$

$= 50.5477 \text{ N-s/m}$



10.15

$m = 0.1 \text{ kg}$, $k = 10000 \text{ N/m}$, $c = 0 \Rightarrow \zeta = 0$

$\omega_n = \sqrt{k/m} = \sqrt{10000/0.1} = 316.2278 \text{ rad/sec}$

Engine speed = $\omega = 1000 \text{ rpm} = 104.72 \text{ rad/sec}$

$r = \omega/\omega_n = 104.72/316.2278 = 0.3312$

peak-to-peak travel of mass = 10 mm

Find: Y , ωY , $\omega^2 Y$.

We have, from Eq. (10.19),

$$\frac{Z}{Y} \Big|_{\zeta=0} = \frac{r^2}{|1-r^2|} = \frac{0.3312^2}{|1-0.3312^2|} = 0.1232$$

Since peak-to-peak travel of mass = 10 mm, $Z = 5$ mm

$$Y = Z/0.1232 = 5/0.1232 = 40.5844 \text{ mm}$$

Max displacement of foundation = $Y = 40.5844$ mm

Max velocity of foundation = $\omega Y = 4249.9984$ mm/sec

Max acceleration of foundation = $\omega^2 Y = 445059.8291$ mm/sec²

10.16

Maximum speed = 3000 rpm = 50 Hz ; $r = \frac{\omega}{\omega_n} = \frac{50}{100} = 0.5$

For accelerometer, $\frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = 1 + \text{error} = 0.9$ (E₁)

$$\text{Here } c = 20 \text{ N-s/m ; } \zeta = \frac{c}{2m\omega_n} = \frac{20}{2m(100 \times 2\pi)} = \frac{0.015915}{m}$$

For $r=0.5$, Eq. (E₁) gives $\zeta = 0.8198$

$$c_c = \frac{c}{\zeta} = 20/0.8198 = 24.3962 \text{ N-s/m}$$

$$m = \frac{c_c}{2\omega_n} = \frac{24.3962}{2(100 \times 2\pi)} = 0.01941 \text{ kg} = 19.41 \text{ grams}$$

$$k = m\omega_n^2 = 0.01941(100 \times 2\pi)^2 = 7622.7967 \text{ N/m}$$

10.17

$$\omega_n = 2\pi(0.5) = \pi \text{ rad/sec ; } \omega_d = \omega_n \sqrt{1-\zeta^2}$$

$$\sqrt{1-\zeta^2} = \omega_d/\omega_n = 0.48/0.5 \Rightarrow \zeta = 0.28$$

$$r_1 = \omega_1/\omega_n = 4\pi/\pi = 4 ; \quad r_2 = \omega_2/\omega_n = 8 ; \quad r_3 = \omega_3/\omega_n = 12$$

$$\phi_i = \tan^{-1} \left(\frac{2\zeta r_i}{1-r_i^2} \right)$$

$$\phi_1 = \tan^{-1} \left(\frac{2(0.28)4}{1-16} \right) = -8.4934^\circ$$

$$\phi_2 = \tan^{-1} \left(\frac{2 \times 0.28 \times 8}{1-64} \right) = -4.0675^\circ$$

$$\phi_3 = \tan^{-1} \left(\frac{2 \times 0.28 \times 12}{1-144} \right) = -2.6905^\circ$$

$$\frac{r_1^2}{\sqrt{(1-r_1^2)^2 + (2\zeta r_1)^2}} \times 20 = 21.0994$$

$$\frac{r_2^2}{\sqrt{(1-r_2^2)^2 + (2\zeta r_2)^2}} \times 10 = 10.1331$$

$$\frac{r_3^2}{\sqrt{(1-r_3^2)^2 + (2\zeta r_3)^2}} \times 5 = 5.0294$$

Record indicated by vibrometer is given by

$$z(t) = 21.0994 \sin(4\pi t + 8.4934^\circ) + 10.1331 \sin(8\pi t + 4.0675^\circ) + 5.0294 \sin(12\pi t + 2.6905^\circ) \text{ mm}$$

$$\textcircled{10.18} \quad x(t) = 20 \sin 50t + 5 \sin 150t \quad \text{mm} \quad (E_1)$$

$$\ddot{x}(t) = -20(50)^2 \sin 50t - 5(150)^2 \sin 150t \quad \text{mm/sec}^2$$

$$= -50000 \sin 50t - 112500 \sin 150t \quad \text{mm/sec}^2 \quad (E_2)$$

$$\omega_n = 100 \text{ rad/sec}, \quad \omega_d = \omega_n \sqrt{1-\zeta^2} = 80 \Rightarrow \zeta = 0.6$$

$$r_1 = \frac{\omega_1}{\omega_n} = \frac{50}{100} = 0.5; \quad r_2 = \frac{\omega_2}{\omega_n} = \frac{150}{100} = 1.5$$

$$\phi_1 = \tan^{-1} \left(\frac{2\zeta r_1}{1-r_1^2} \right) = \tan^{-1} \left(\frac{2 \times 0.6 \times 0.5}{1-0.25} \right) = 38.6598^\circ$$

$$\phi_2 = \tan^{-1} \left(\frac{2\zeta r_2}{1-r_2^2} \right) = \tan^{-1} \left(\frac{2 \times 0.6 \times 1.5}{1-2.25} \right) = -55.2222^\circ$$

$$\frac{50000}{\sqrt{(1-r_1^2)^2 + (2\zeta r_1)^2}} = 52057.9206$$

$$\frac{112500}{\sqrt{(1-r_2^2)^2 + (2\zeta r_2)^2}} = 51335.6229$$

output of the accelerometer is given by

$$\ddot{z}(t) = -52057.9206 \sin(50t - 38.6598^\circ) - 51335.6229 \sin(150t + 55.2222^\circ) \quad \text{mm/sec}^2 \quad (E_3)$$

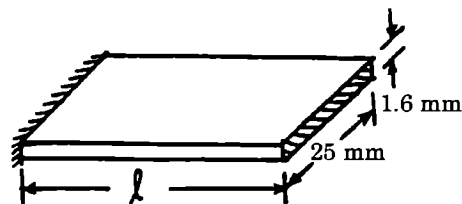
It can be seen that E_3 is substantially different from E_2 .

10.19 For given beam,

$$A = (1.6 \times 25) \text{ mm}^2 = 40 \times 10^{-6} \text{ m}^2,$$

$$I = \frac{1}{12} (0.025) (0.0016)^3 = 8.53333 \times 10^{-12} \text{ m}^4,$$

$$\ell = 0.05 \text{ m to } 0.25 \text{ m.}$$



$$\text{For a cantilever beam, Fig. 8.15 gives } \omega_n = (\beta_n \ell)^2 \left(\frac{EI}{\rho A \ell^4} \right)^{\frac{1}{2}}$$

$$\text{where } (\beta_1 \ell)^2 = (1.875104)^2 = 3.516015$$

$$(\beta_2 \ell)^2 = (4.694091)^2 = 22.03449$$

$$(\beta_3 \ell)^2 = (7.854757)^2 = 61.69721$$

$$(\beta_4 \ell)^2 = (10.995541)^2 = 120.90192$$

For spring steel, $E = 200 \times 10^9 \text{ Pa}$, $\rho = 7800 \text{ kg/m}^3$

$$\left(\frac{EI}{\rho A \ell^4} \right)^{\frac{1}{2}} = \left\{ \frac{(200 \times 10^9) (8.53333 \times 10^{-12})}{(7800) (40 \times 10^{-6}) \ell^4} \right\}^{\frac{1}{2}} = \frac{2.33882}{\ell^2}$$

$$\omega_n = (\beta_n \ell)^2 \left(\frac{2.33882}{\ell^2} \right)$$

The first four frequencies are given below:

	ℓ^2	ω_1	ω_2	ω_3	ω_4
$\ell = 0.05 \text{ m}$	0.0025	3289.33	20613.88	57719.47	113107.13
$\ell = 0.25 \text{ m}$	0.0625	131.573	824.6996	2308.78	4524.29

Hence, the range of frequencies that can be measured is given by $\omega > 131.573 \text{ rad/sec}$. However, for first mode only (which is easiest to excite), the range of frequencies is $131.573 \text{ rad/sec} \leq \omega \leq 3289.33 \text{ rad/sec}$.

10.20

$$\frac{k X_R}{F_0} = \frac{1 - r^2}{(1 - r^2)^2 + (2 \zeta r)^2} = \frac{N}{D} \text{ (assume)}$$

$$\frac{d}{dr} \left(\frac{k X_R}{F_0} \right) = \frac{D \frac{dN}{dr} - N \frac{dD}{dr}}{D^2} = 0 \text{ i.e., } D \frac{dN}{dr} - N \frac{dD}{dr} = 0$$

$$\text{or } \left\{ (1 - r^2)^2 + (2 \zeta r)^2 \right\} (-2r) - (1 - r^2) \left\{ 2(1 - r^2)(-2r) + 2(2 \zeta r)(2 \zeta) \right\} = 0$$

This equation can be simplified as

$$r^4 - 2r^2 + (1 - 4\zeta^2) = 0$$

and its solution is given by

$$r^2 = 1 \pm 2\zeta \text{ or } r = \sqrt{1 + 2\zeta}; \sqrt{1 - 2\zeta}$$

Since

$$\frac{k X_R}{F_0} \Big|_{r = \sqrt{1 + 2\zeta}} = -\frac{1}{4\zeta(1 + \zeta)} \quad (1)$$

and

$$\frac{k X_R}{F_0} \Big|_{r = \sqrt{1 - 2\zeta}} = \frac{1}{4\zeta(1 - \zeta)} \quad (2)$$

we note that $r = R_1 = \sqrt{1 - 2\zeta}$ corresponds to a maximum and $r = R_2 = \sqrt{1 + 2\zeta}$ corresponds to a minimum of X_R .

10.21

$$\frac{k X_I}{F_0} = \frac{-2 \zeta r}{(1-r^2)^2 + 4 \zeta^2 r^2}$$

$$\frac{d}{dr} \left(\frac{k X_I}{F_0} \right) = \frac{D \frac{dN}{dr} - N \frac{dD}{dr}}{D^2} = 0 \quad (1)$$

$$\text{where } \frac{dN}{dr} = -2 \zeta \quad \text{and} \quad \frac{dD}{dr} = -2(1-r^2)(2r) + 8 \zeta^2 r$$

By setting the numerator of Eq. (1) equal to zero, we obtain

$$\left\{ 1 + r^4 - 2r^2 + 4 \zeta^2 r^2 \right\} (-2 \zeta) - (-2 \zeta r) \left\{ -4r + 4r^3 + 8 \zeta^2 r \right\} = 0$$

which can be simplified to obtain

$$3r^4 + (4\zeta^2 - 2)r^2 - 1 = 0 \quad (2)$$

The roots of Eq. (2) are given by

$$r^2 = \frac{1 - 2\zeta^2 - 2\sqrt{\zeta^4 - \zeta^2 + 1}}{3} ; \frac{1 - 2\zeta^2 + 2\sqrt{\zeta^4 - \zeta^2 + 1}}{3} \quad (3)$$

Since it is difficult to determine, from Eq. (3), the correct value of r that corresponds to the minimum of X_I , we use a numerical computation. For $\zeta = 0.1$, for example, Eq. (3) gives

$$r^2 = -0.0030 ; 0.6583$$

This shows that

$$r = \left\{ \frac{1 - 2\zeta^2 + 2\sqrt{\zeta^4 - \zeta^2 + 1}}{3} \right\}^{\frac{1}{2}} \quad (4)$$

corresponds to the minimum of X_I . For small values of ζ , $\zeta^2 \ll 1$ and Eq. (4) gives

$$r \approx 1 \quad (5)$$

Thus X_I attains its minimum value close to $r = 1$.

10.22

Response of a single d.o.f. system with hysteretic damping is given by Eq. (3.106):

$$\frac{X}{F_0} = \frac{1}{k - m \omega^2 + i k \beta}$$

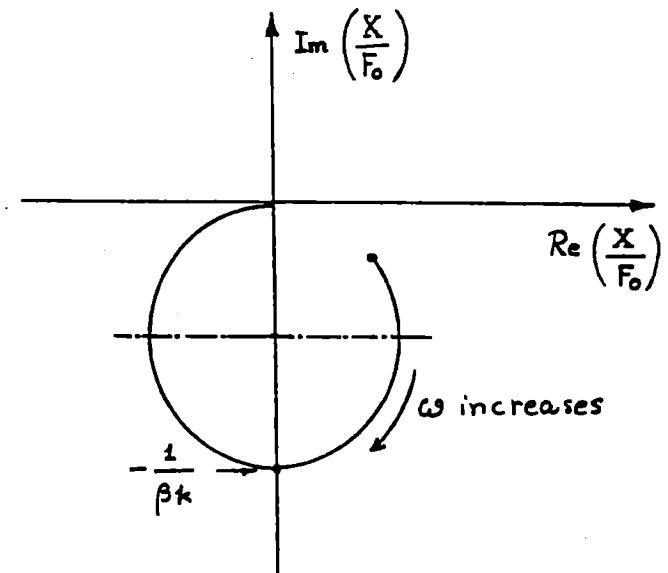
$$\operatorname{Re} \left(\frac{X}{F_0} \right) = \frac{k - m \omega^2}{(k - m \omega^2)^2 + (k \beta)^2}, \quad \operatorname{Im} \left(\frac{X}{F_0} \right) = \frac{-k \beta}{(k - m \omega^2)^2 + (k \beta)^2}$$

$$\left[\operatorname{Re} \left(\frac{X}{F_0} \right) \right]^2 + \left[\operatorname{Im} \left(\frac{X}{F_0} \right) \right]^2 = \frac{1}{(k - m \omega^2)^2 + (k \beta)^2} \quad (1)$$

It can be verified that Eq. (1) can be rewritten as

$$\left[\operatorname{Re} \left(\frac{X}{F_0} \right) \right]^2 + \left[\operatorname{Im} \left(\frac{X}{F_0} \right) + \frac{1}{2 \beta k} \right]^2 = \left(\frac{1}{2 \beta k} \right)^2 \quad (2)$$

Eq. (2) shows that the locus of $\left(\frac{X}{F_0} \right)$ as ω increases from zero is part of a circle, with center $\left(0, -\frac{1}{2 k \beta} \right)$ and radius $\frac{1}{2 k \beta}$, as shown in the following figure.

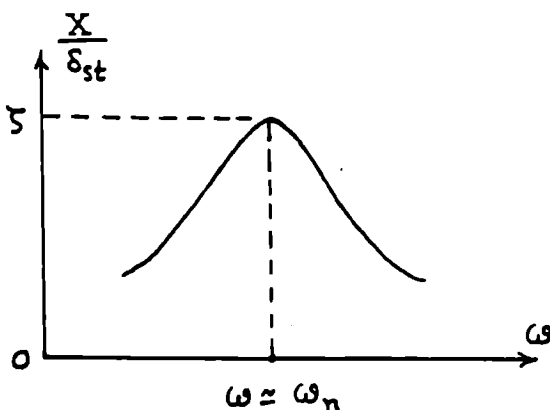


10.23 The peak of Bode diagram is equal to $\approx \frac{1}{2\zeta}$.

In the present case, peak-to-peak value is plotted; hence $X \approx \frac{11.43}{2} = 5.715 \mu\text{ m}$.

$$\frac{X}{\delta_{st}} = \frac{5.715}{1.27} = 4.5 = \frac{1}{2\zeta}$$

or $\zeta = 0.1111$.



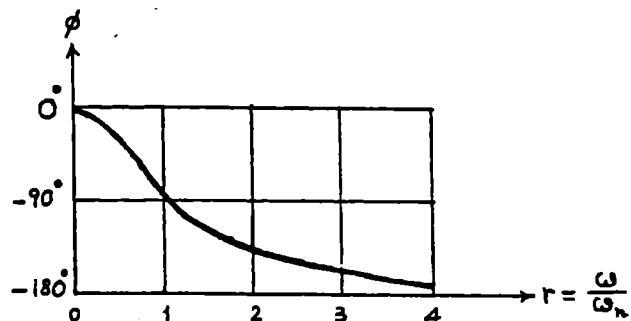
10.24 Reduction in amplitude from 173 mm/s to 20 mm/s in 7 cycles or 22 ms.

Eq. 2.92 gives:

$$\frac{1}{7} \ln \left(\frac{x_1}{x_8} \right) = \delta \quad \text{or} \quad \delta = \frac{1}{7} \ln \left(\frac{173}{20} \right) = 0.308223$$

$$\text{Hence } \zeta = \frac{\delta}{2\pi} = 0.049055$$

10.25 Typical Bode plot of phase angle is shown in the figure.



(a) $\phi = 90^\circ$ at $r = \frac{\omega}{\omega_n} \approx 1$. Hence the value of ω_n can be determined from the value of r corresponding to $\phi = 90^\circ$.

(b) Since

$$\phi = \tan^{-1} \left(-\frac{2\zeta r}{1-r^2} \right) = \tan^{-1} \left(-\frac{c\omega}{k-m\omega^2} \right)$$

we find

$$-\frac{c\omega_1}{k-m\omega_1^2} = -1 \quad \text{and} \quad -\frac{c\omega_2}{k-m\omega_2^2} = 1$$

where ω_1 and ω_2 correspond to the half power points. Hence, by finding the values of ω corresponding to $\phi = -45^\circ$ and $\phi = -135^\circ$, we obtain ω_1 and ω_2 . From these values, the damping ratio can be found using the relation:

$$\zeta = \frac{\omega_2 - \omega_1}{2\omega_n}$$

10.26

10.27

10.28

Characteristic	Problem 10.26	Problem 10.27	Problem 10.28
n	16	18	18
N	750 rpm	1000 rpm	1500 rpm
d	15 mm	2 cm	10 mm
D	100 mm	15 cm	80 mm
α	30°	20°	40°
$\frac{d}{D} \cos \alpha$	0.1299	0.1253	0.09576
Dominant frequency of vibration			
Inner race defect	6779.4 cycles/min (1078.97 Hz)	10127.7 cycles/min (1611.87 Hz)	14792.8 cycles/min (2354.33 Hz)
Outer race defect	5220.6 cycles/min (830.88 Hz)	7872.3 cycles/min (1252.91 Hz)	12207.2 cycles/min (1942.84 Hz)
Ball or roller defect	1214.6 cycles/min (193.31 Hz)	1761.7 cycles/min (280.39 Hz)	2188.1 cycles/min (348.25 Hz)
Cage defect	326.3 cycles/min (51.93 Hz)	437.3 cycles/min (69.61 Hz)	678.2 cycles/min (107.93 Hz)

10.29

$$f(x) = \frac{1}{4} ; 1 \leq x \leq 5$$

$$\bar{x} = \text{mean value of } x = \int_1^5 f(x) x \, dx = \frac{1}{4} \left(\frac{x^2}{2} \right)_1^5 = 3$$

$$\sigma^2 = (\text{standard deviation})^2 = \int_1^5 (x - \bar{x})^2 f(x) \, dx$$

$$= \frac{1}{4} \int_1^5 (x - 3)^2 \, dx = \frac{1}{4} \left(\frac{x^3}{3} - 3x^2 + 9x \right)_1^5 = \frac{4}{3}$$

$$k = \text{kurtosis} = \frac{1}{\sigma^4} \int_{-\infty}^{\infty} (x - \bar{x})^4 f(x) \, dx = \frac{9}{16} \int_1^5 (x - 3)^4 \left(\frac{1}{4} \right) \, dx$$

Let $y = x - 3$ so that $dy = dx$. This gives

$$k = \frac{9}{64} \int_{y=-2}^2 y^4 \, dy = \frac{9}{5}$$

10.30

$$\bar{x} = \text{mean value of } x = \sum_i f(x_i) x_i$$

$$= \frac{1}{32} (1) + \frac{3}{32} (2) + \frac{3}{16} (3) + \frac{6}{16} (4) + \frac{3}{16} (5) + \frac{3}{32} (6) + \frac{1}{32} (7) = 4$$

$$\sigma^2 = (\text{standard deviation})^2 = \sum_i (x_i - \bar{x})^2 f(x_i)$$

$$= (1 - 4)^2 \frac{1}{32} + (2 - 4)^2 \frac{3}{32} + (3 - 4)^2 \frac{3}{16} + (4 - 4)^2 \frac{6}{16}$$

$$+ (5 - 4)^2 \frac{3}{16} + (6 - 4)^2 \frac{3}{32} + (7 - 4)^2 \frac{1}{32} = \frac{27}{16} = 1.6875$$

$$\sum_i (x_i - \bar{x})^4 f(x_i) = (1 - 4)^4 \frac{1}{32} + (2 - 4)^4 \frac{3}{32} + (3 - 4)^4 \frac{3}{16}$$

$$+ (4 - 4)^4 \frac{6}{16} + (5 - 4)^4 \frac{3}{16} + (6 - 4)^4 \frac{3}{32} + (7 - 4)^4 \frac{1}{32} = \frac{135}{16} = 8.4375$$

$$k = \text{kurtosis} = \frac{1}{\sigma^4} \sum_i (x_i - \bar{x})^4 f(x_i) = \frac{8.4375}{1.6875^2} = 2.9630$$

10.31

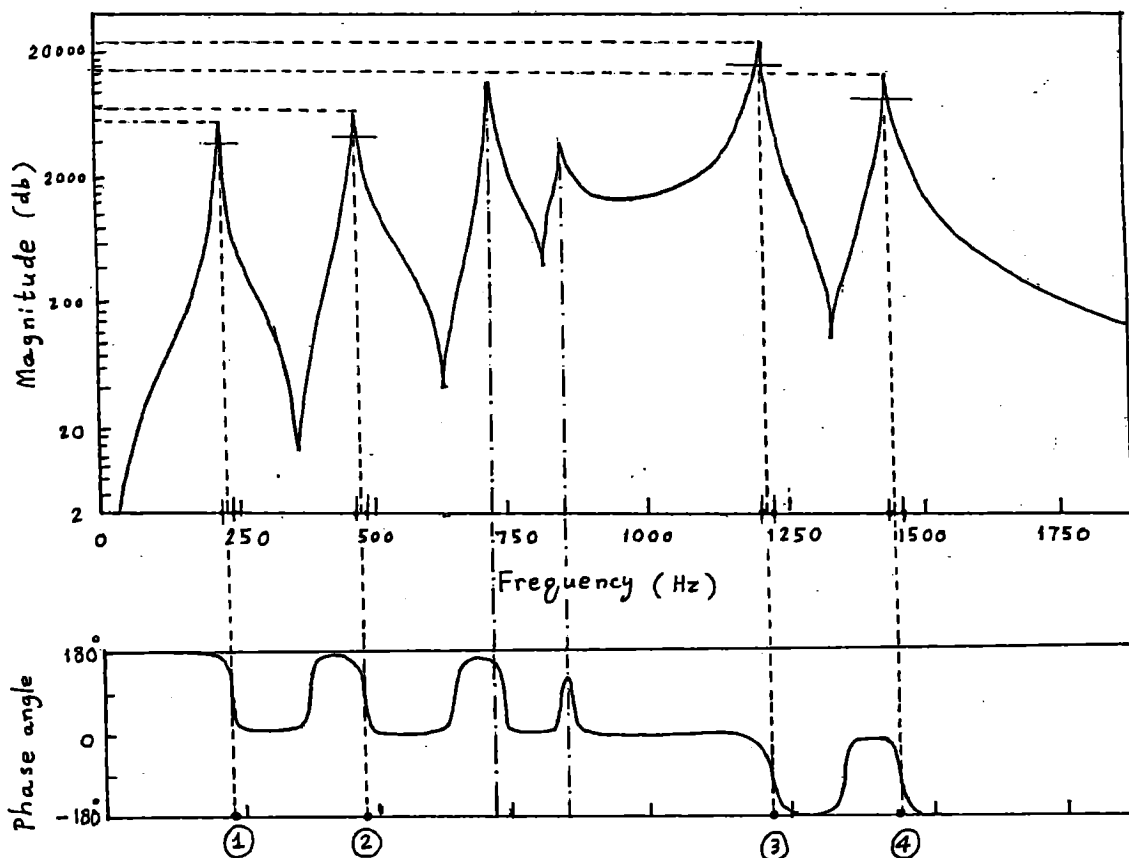


Fig. 10.46

Basic rules:

- At resonance, the magnitude will have a sharp peak.
- At resonance, phase will be 90° and the phase changes by 180° as frequency crosses the natural frequency.

Using these rules, we identify four resonant frequencies in Fig. 10.46.

Resonant frequency

$$\omega_1 = 228.4483 \text{ Hz}$$

$$\omega_2 = 474.1379 \text{ Hz}$$

$$\omega_3 = 1215.5172 \text{ Hz}$$

$$\omega_4 = 1448.27579 \text{ Hz}$$

Damping ratio

$$\zeta_1 = \frac{235.3448 - 219.8276}{2(228.4483)}$$

$$= 0.033962$$

$$\zeta_2 = \frac{482.7586 - 458.6207}{2(474.1379)}$$

$$= 0.025454$$

$$\zeta_3 = \frac{1228.4482 - 1198.2758}{2(1215.5172)}$$

$$= 0.012411$$

$$\zeta_4 = \frac{1453.4483 - 1431.0345}{2(1448.2759)}$$

$$= 0.007738$$

10.32

Radius of circle

$$= 1.25 = \frac{1}{4\zeta}$$

$$\therefore \zeta = \frac{1}{4(1.25)} = 0.2$$

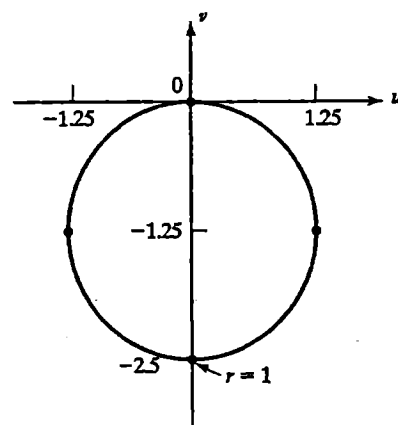


FIGURE 10.47

10.33 Range of $\omega = 62.832$ to 314.16 rad/sec = 600 to 3000 rpm
 Max acceleration level = $10g = 98.1$ m/sec²
 Max weight of specimen = 10 N
 Max vibration amplitude = 0.0025 m

Frequency range:

variable speed electric motor can be used to obtain the frequency range (for a mechanical shaker).

Vibration amplitude:

$$\text{If } y(t) = A \sin \omega t, \quad (E_1)$$

$$\text{acceleration} = A\omega^2.$$

At $\omega = 314.16$ rad/sec, amplitude needed to achieve the maximum acceleration is:

$$\begin{aligned} \text{amplitude (A)} &= \text{acceleration} / \omega^2 \\ &= 98.1 / (314.16)^2 = 0.9939 \times 10^{-3} \text{ m} \end{aligned}$$

At $\omega = 62.832$ rad/sec, amplitude needed to achieve the maximum acceleration is:

$$\begin{aligned} \text{amplitude (A)} &= \text{acceleration} / \omega^2 \\ &= 98.1 / (62.832)^2 = 0.02485 \text{ m} \end{aligned}$$

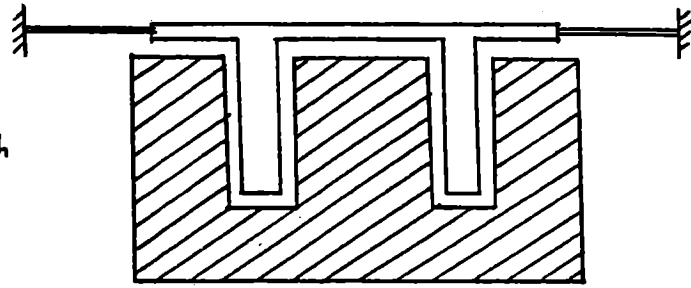
(amplitude is too high; hence direct application of $y(t)$ is not permitted).

Mechanical shaker of the type shown in Fig. 10.18 can be used. Electrodynamic shaker of the type shown in Fig. 10.19 (a) can also be used.

Electromagnetic shaker:

Max force (F_{max})
available depends on:

- (a) magnetic field strength
- (b) number of turns
- (c) coil diameter
- (d) current flowing



Limitations are: (a) material strength, (b) cooling provided.

$$\text{Max acceleration} = \left(\frac{F_{max}}{m_s + m_t} \right)$$

Where m_s = mass of specimen and m_t = mass of shaker table.

10.34

Speed range: 300 - 600 rpm

Frequency range: 31.416 - 62.832 rad/sec

Number of reeds = 12

Uniform spacing of frequencies give the reed frequencies as:

$$\Omega_1, \dots, \Omega_{12} = 31.416, 34.272, 37.128, 39.984, 42.840,$$

$$45.696, 48.552, 51.408, 54.264, 57.120, 59.976, 62.832 \text{ rad/sec}$$

Let each reed be considered as a cantilever beam of cross section $a \times b$ inches. Let lengths of all reeds be same and the material be aluminum for light weight. The fundamental natural frequency of a reed is given by (Fig. 8.15):

$$\begin{aligned} \omega_1 &= (\beta_1 \ell)^2 \left\{ \frac{EI}{\rho A \ell^4} \right\}^{\frac{1}{2}} = (1.8751^2) \left\{ \frac{EI}{\rho A \ell^4} \right\}^{\frac{1}{2}} \\ &= 3.516 \left\{ \frac{(10^7) I}{(0.1/386.4) A \ell^4} \right\}^{\frac{1}{2}} = 69.1142 (10^4) \left\{ \frac{I}{A \ell^4} \right\}^{\frac{1}{2}} \end{aligned} \quad (1)$$

By equating ω_1 given by Eq. (1) to $\Omega_1, \dots, \Omega_{12}$, in turn, the proper value of $\left\{ \frac{I}{A \ell^4} \right\}^{\frac{1}{2}}$

needed for different reeds can be computed. By selecting a common value of ℓ for all reeds, the cross section of any reed can then be found to achieve the required value of

$$\left\{ \frac{I}{A \ell^4} \right\}^{\frac{1}{2}}.$$

10.35

Iterative process is to be used.

1. Select trial values of the design parameters (material of the beam and its dimensions).
2. Model the beam as a spring-mass system with:
 $m = \text{end mass} = 50\%$ of mass of beam:

$$m = \frac{1}{2} \rho A \ell \quad (1)$$

$k = \text{stiffness of a cantilever beam:}$

$$k = \frac{3 E I}{\ell^3} \quad (2)$$

3. Equations of motion:

$$m \ddot{x} + k(x - y) = 0$$

$$\text{or } m \ddot{z} + k z = -m \ddot{y} \quad (3)$$

where $z = \text{relative displacement of end mass.}$

4. Since $\ddot{y}_{\max} = 0.2 \text{ g}$, assume a constant force of $-m(0.2 \text{ g})$ on the right hand side of Eq. (3) and solve the equation to find $z(t)$.
5. From the known z_{\max} value, compute the maximum stress (σ_{\max}) induced in the beam. If σ_{\max} is less than the yield stress of the material, the design is complete. Otherwise, go to step 1 and change one or more design parameters and repeat the procedure until a satisfactory design is found.

