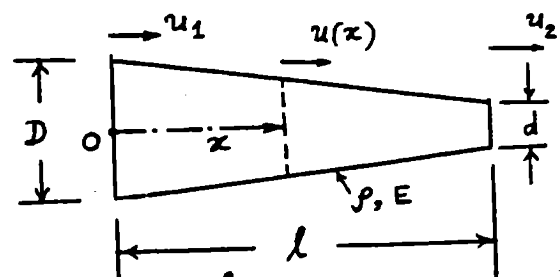
Chapter 12

Finite Element Method

12.1 As diameter $h(x) = D$ at $x=0$ and $= d$ at $x = l$, we have

$$h(x) = D + \left(\frac{d-D}{l}\right)x$$



(i) stiffness matrix:

$V(t)$ = strain energy of element $= \frac{1}{2}\int_0^l EA\left(\frac{\partial u}{\partial x}\right)^2 dx$

with

$$u(x,t) = \left(1-\frac{x}{l}\right)\cdot u_1(t) + \left(\frac{x}{l}\right)\cdot u_2(t)$$

and

$$A(x) = \frac{\pi h^2}{4} = \frac{\pi}{4}\left[D^2 + \left(\frac{d-D}{l}\right)^2 x^2 + 2D\left(\frac{d-D}{l}\right)x\right]$$

Thus strain erergy expression becomes

$$V = \frac{\pi E}{24 l}(D^2 + d^2 + dD)(u_1^2 + u_2^2 - 2u_1u_2)$$

$$= \frac{1}{2}\vec{u}^T[k]\vec{u} \equiv \frac{1}{2}(u_1 \quad u_2)\begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22}\end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

This gives the element stiffness matrix as

$$[k] = \frac{\pi E}{12 l}(D^2 + d^2 + dD)\begin{bmatrix} 1 & -1 \\ -1 & 1\end{bmatrix}$$

(ii) consistent mass matrix:

$T(t)$ = kinetic energy of element $= \frac{1}{2}\int_0^l \rho A\left(\frac{\partial u}{\partial t}\right)^2 dx$

with

$$\dot{u}(x,t) \equiv \frac{\partial u}{\partial t}(x,t) = \left(1-\frac{x}{l}\right)\dot{u}_1(t) + \left(\frac{x}{l}\right)\dot{u}_2(t)$$

Substituting for $A(x)$ and $\dot{u} = \frac{\partial u}{\partial t}$, kinetic energy expression can be derived as

$$T = \frac{1}{2}\int_0^{\ell} \rho\ \frac{\pi}{4}\left\{D^2 + \left(\frac{d-D}{\ell}\right)^2 x^2 + 2D\left(\frac{d-D}{\ell}\right)x\right\}\left\{\dot{u}_1^2 + x\left(-\frac{2}{\ell}\dot{u}_1^2 + \frac{2}{\ell}\dot{u}_1\dot{u}_2\right) + x^2\left(\frac{\dot{u}_1^2}{\ell^2} + \frac{\dot{u}_2^2}{\ell^2} - \frac{2\dot{u}_1\dot{u}_2}{\ell^2}\right)\right\} dx$$

$$= \frac{\pi\rho\ell}{8}\left\{\dot{u}_1^2\left(\frac{D^2}{5} + \frac{d^2}{30} + \frac{Dd}{10}\right) + \dot{u}_2^2\left(\frac{D^2}{30} + \frac{d^2}{5} + \frac{Dd}{10}\right) + 2\dot{u}_1\dot{u}_2\left(\frac{D^2}{20} + \frac{d^2}{20} + \frac{Dd}{15}\right)\right\}$$

$$= \frac{1}{2}\vec{\dot{u}}^T[m_c]\vec{\dot{u}} \equiv \frac{1}{2}(\dot{u}_1\ \ \dot{u}_2)\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22}\end{bmatrix}\begin{Bmatrix}\dot{u}_1 \\ \dot{u}_2\end{Bmatrix}$$

This gives the consistent mass matrix as

$$[m_c] = \frac{\pi\rho\ell}{4}\begin{bmatrix}\left(\frac{D^2}{5} + \frac{d^2}{30} + \frac{Dd}{10}\right) & \left(\frac{D^2}{20} + \frac{d^2}{20} + \frac{Dd}{15}\right) \\ \left(\frac{D^2}{20} + \frac{d^2}{20} + \frac{Dd}{15}\right) & \left(\frac{D^2}{30} + \frac{d^2}{5} + \frac{Dd}{10}\right)\end{bmatrix}$$

(iii) Lumped mass matrix:

$$\text{Total mass of element} = \frac{\pi\rho}{4}\int_0^{\ell} h^2(x)\cdot dx$$

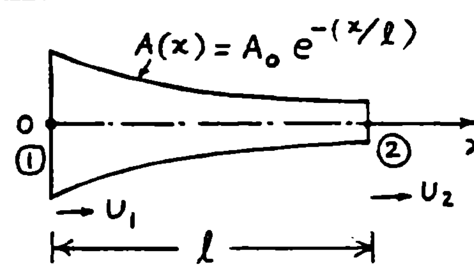
$$= \frac{\pi\rho}{4}\int_0^{\ell}\left\{D^2 + \left(\frac{d-D}{\ell}\right)^2 x^2 + 2D\left(\frac{d-D}{\ell}\right)x\right\}\cdot dx$$

$$= \frac{\pi\rho\ell}{12}\left(D^2 + d^2 + Dd\right)$$

Distributing the mass at the two nodes,

$$[m_\ell] = \frac{\pi\rho\ell}{24}\begin{bmatrix}(D^2 + d^2 + Dd) & 0 \\ 0 & (D^2 + d^2 + Dd)\end{bmatrix}$$

12.2 Assume linear displacement model



$$u(x) = \alpha_1 + \alpha_2 x = U_1 + \left(\frac{U_2 - U_1}{\ell}\right)x$$

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{U_2 - U_1}{\ell}$$

$$\sigma_x = E\,\epsilon_x = E\left(\frac{U_2 - U_1}{\ell}\right)$$

$$\text{strain energy} = \iiint_{V^{(e)}} \frac{1}{2}\sigma_x \epsilon_x\, dV = \frac{1}{2}\int_{x=0}^{\ell} E\left(\frac{U_2 - U_1}{\ell}\right)^2 A(x)\, dx$$

$$= \frac{E}{2\ell^2}(U_2 - U_1)^2 \int_{x=0}^{\ell} A_0 \cdot e^{-\left(\frac{x}{\ell}\right)} . dx = \frac{E}{2\ell^2}(U_2-U_1)^2 A_0 \left[-\ell \cdot e^{-\frac{x}{\ell}}\right]_0^{\ell}$$

$$= \frac{E}{2\ell^2}(U_2 - U_1)^2 \, \ell A_0 \left(1 - \frac{1}{e}\right) = \frac{1}{2}\frac{E A_0}{\ell}(0.6321)(U_2 - U_1)^2$$

$$\equiv \frac{1}{2}\vec{U}^T [k] \vec{U} = \frac{1}{2}(U_1 \quad U_2)[k]\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix}$$

$$\therefore \; [k] = \frac{E A_0}{\ell}(0.6321)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

12.3

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C==========================================================================
C
C      PROBLEM 12.3   PROGRAM FOR STRESS ANALYSIS OF PLANAR TRUSSES
C
C==========================================================================
C      DATA FOR PROBLEM 12.10 (TEST EXAMPLE)
       DIMENSION A(4),EL(4),GS(8,3),PP(8,1),P(8,1),GSS(4,4)
      2 ,STRS(4,1),X(4),Y(4),LOC(4,2),IFIX(4)
       DOUBLE PRECISION DIFF(2)
       DATA LOC/1,3,3,2,3,2,4,4/
       M=1
       DATA NN,NE,ND,NB,NFIX,E/4,4,8,4,4,30.0E+6/
       DATA IFIX/1,2,3,4/
       DATA X/0.,100.,50.,200./
       DATA Y/0.,0.,25.,100./
       DO 10 I=1,8
 10    P(I,1)=0.0
       P(8,1)=-1000.0
       DATA A/2.,2.,1.,1./
C      END OF PROBLEM-DEPENDENT DATA
       CALL TRUSS (NN,NE,ND,NB,M,LOC,X,Y,E,A,EL,NFIX,IFIX,P,GS,DIFF,
      2 GSS,ND2)
       PRINT 21
 21    FORMAT (2X,20H NODAL DISPLACEMENTS,/)
       DO 11 I=1,ND
       PP(I,1)=P(I,1)
 11    PRINT 12, I, P(I,1)
 12    FORMAT (4X,I5,2X,E15.6)
       DO 45 I=1,NFIX
       DO 45 J=1,M
       II=IFIX(I)
       PP(II,J)=0.0
 45    CONTINUE
       PRINT 22
 22    FORMAT (//,2X, 21H STRESSES IN ELEMENTS,/)
       DO 80 K=1,NE
       DO 70 KK=1,M
       I=LOC(K,1)
       J=LOC(K,2)
       I1=2*I-2+1
       I2=I1+1
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```
      I3=2*J-2+1
      I4=I3+1
      XL=(X(J)-X(I))/EL(K)
      XM=(Y(J)-Y(I))/EL(K)
      STRS(K,KK)=(E/EL(K))*(XL*(PP(I3,KK)-PP(I1,KK))+XM*(PP(I4,KK)-
     2 PP(I2,KK)))
 70   CONTINUE
 80   CONTINUE
      DO 120 I=1,NE
 120  PRINT 130, I,LOC(I,1),LOC(I,2),STRS(I,1)
 130   FORMAT (2X,3I4,2X,2E15.8)
      STOP
      END
C ======================================================================
C
C SUBROUTINE TRUSS
C
C ======================================================================
C      NN = NUMBER OF NODES,  NE = NUMBER OF ELEMENTS
C      ND = NUMBER OF DEGREES OF FREEDOM,  NB = SEMI-BANDWIDTH
C      M = NUMBER OF LOAD CONDITIONS, LOC = NODE CONNECTIVITY MATRIX
C      X,Y = X - AND Y- COORDINATES OF NODES
C      E = YOUNGS MODULUS, A = AREAS OF CROSS SECTION OF ELEMENTS
C      EL = LENGTHS OF ELEMENTS, NFIX = NUMBER OF D.O.F. WHICH ARE FIXED
C      IFIX = FIXED DEGREES OF FREEDOM NUMBERS, P = LOAD VECTOR
      SUBROUTINE TRUSS (NN,NE,ND,NB,M,LOC,X,Y,E,A,EL,NFIX,IFIX,P,GS,
     2 DIFF,GSS,ND2)
      DIMENSION LOC(NE,2),X(NN),Y(NN),A(NE),EL(NE),IFIX(NFIX),
     2 P(ND,M),GS(ND,NB),GSS(ND2,ND2)
      DIMENSION B(4,4),N(4)
      DOUBLE PRECISION DIFF(M)
      DO 5 I=1,ND
      DO 5 J=1,NB
 5    GS(I,J)=0.0
      DO 6 I=1,ND2
      DO 6 J=1,ND2
 6    GSS(I,J)=0.0
      DO 200 K=1,NE
      I=LOC(K,1)
      J=LOC(K,2)
      EL(K)=SQRT((X(J)-X(I))**2+(Y(J)-Y(I))**2)
      CON=A(K)*E/EL(K)
      XL=(X(J)-X(I))/EL(K)
      XM=(Y(J)-Y(I))/EL(K)
      B(1,1)=XL**2
      B(1,2)=XL*XM
      B(1,3)=-B(1,1)
      B(1,4)=-B(1,2)
      B(2,2)=XM**2
      B(2,3)=-XL*XM
      B(2,4)=-XM**2
      B(3,3)=XL**2
      B(3,4)=XL*XM
      B(4,4)=XM**2
      DO 10 II=1,4
      DO 10 JJ=1,II
```

```
10    B(II,JJ)=B(JJ,II)
      DO 20 II=1,4
      DO 20 JJ=1,4
20    B(II,JJ)=B(II,JJ)*CON
      DO 90 II=1,2
      N(II)=2*I-2+II
90    N(II+2)=2*J-2+II
      DO 100 II=1,4
      DO 100 JJ=1,4
      IK=N(II)
      JK=N(JJ)
      IF (IK .GT. ND2 .OR. JK .GT. ND2) GO TO 91
      GSS(IK,JK)=GSS(IK,JK)+B(II,JJ)
91    CONTINUE
      IN=JK-IK+1
      IF (IN .LE. 0) GO TO 100
      GS(IK,IN)=GS(IK,IN)+B(II,JJ)
100   CONTINUE
200   CONTINUE
      DO 110 II=1,NFIX
      IX=IFIX(II)
110   GS(IX,1)=GS(IX,1)*1.0E6
      CALL DECOMP (ND,NB,GS)
      CALL SOLVE (ND,NB,M,GS,P,DIFF)
      RETURN
      END
```

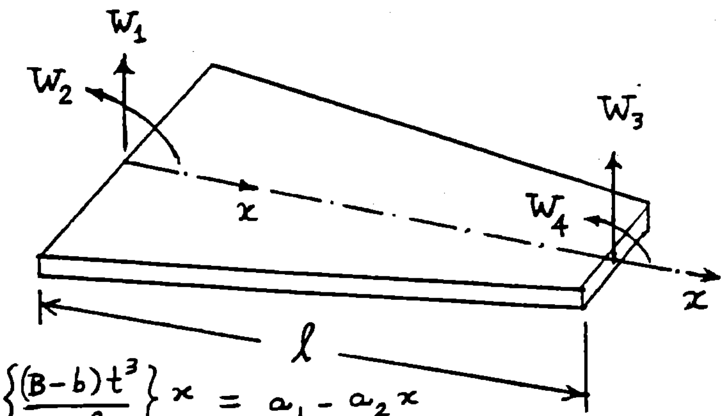
```
NODAL DISPLACEMENTS

     1    -0.116462e-14
     2     0.232924e-09
     3    -0.128993e-14
     4     0.170199e-07
     5     0.116462e-02
     6     0.232925e-02
     7     0.514656e-01
     8    -0.703219e-01
```

```
STRESSES IN ELEMENTS

  1   1   3    0.11180378e+04
  2   3   2    0.38109762e-02
  3   3   4    0.22360752e+04
  4   2   4   -0.28284324e+04
```

12.4 (a) Derivation of element stiffness matrix:



At x,

thickness $= t$

width $= w(x) = B - \left(\frac{B-b}{\ell}\right) x$

$$I(x) = \frac{1}{12}\, w(x)\, t^3 = \frac{B t^3}{12} - \left\{\frac{(B-b)t^3}{12\ell}\right\} x = a_1 - a_2 x$$

with $a_1 = \frac{B t^3}{12}$ and $a_2 = \frac{(B-b)t^3}{12\ell}$

Deflection of beam:
$$w(x) = \sum_{i=1}^{4} N_i(x) \cdot W_i \qquad (E_1)$$

where $N_i(x)$ are defined by Eqs. (12.33) – (12.36).

Strain energy of element is given by

$$V = \frac{1}{2} \int_0^\ell E\, I(x) \left\{\frac{d^2 w(x)}{dx^2}\right\}^2 dx \qquad (E_2)$$

with $\frac{d^2 N_1}{dx^2} = -\frac{6}{\ell^2} + \frac{12}{\ell^3} x$, $\frac{d^2 N_2}{dx^2} = -\frac{4}{\ell} + \frac{6}{\ell^2} x$,

$\frac{d^2 N_3}{dx^2} = \frac{6}{\ell^2} - \frac{12}{\ell^3} x$, $\frac{d^2 N_4}{dx^2} = -\frac{2}{\ell} + \frac{6}{\ell^2} x$

$$\left\{\frac{d^2 w(x)}{dx^2}\right\}^2 = c_1^2 + c_2^2 x^2 + 2 c_1 c_2 x = (c_1 + c_2 x)^2 \qquad (E_3)$$

where $c_1 = -\frac{6}{\ell^2} W_1 - \frac{4}{\ell} W_2 + \frac{6}{\ell^2} W_3 - \frac{2}{\ell} W_4$

and $c_2 = \frac{12}{\ell^3} W_1 + \frac{6}{\ell^2} W_2 - \frac{12}{\ell^3} W_3 + \frac{6}{\ell^2} W_4$

Integration in Eq. (E_2) gives

$$V = \frac{1}{2} E \Big\{ W_1^2 \Big[\frac{36}{\ell^4}(a_1 \ell - a_2 \ell^2/2) - \frac{72}{\ell^5}(a_1 \ell^2 - 2 a_2 \ell^3/3)$$
$$+ \frac{144}{\ell^6}(a_1 \ell^3/3 - a_2 \ell^4/4)\Big] + W_2^2 \Big[\frac{16}{\ell^2}(a_1 \ell - a_2 \ell^2/2)$$
$$- \frac{24}{\ell^3}(a_1 \ell^2 - 2 a_2 \ell^3/3) + \frac{36}{\ell^4}(a_1 \ell^3/3 - a_2 \ell^4/4)\Big]$$
$$+ W_3^2 \Big[\frac{36}{\ell^4}(a_1 \ell - a_2 \ell^2/2) - \frac{72}{\ell^5}(a_1 \ell^2 - 2 a_2 \ell^3/3)$$
$$+ \frac{144}{\ell^6}(a_1 \ell^3/3 - a_2 \ell^4/4)\Big] + W_4^2 \Big[\frac{4}{\ell^2}(a_1 \ell - a_2 \ell^2/2)$$
$$- \frac{12}{\ell^3}(a_1 \ell^2 - 2 a_2 \ell^3/3) + \frac{36}{\ell^4}(a_1 \ell^3/3 - a_2 \ell^4/4)\Big]$$

$$+ 2W_1W_2\Big[\frac{24}{\ell^3}(a_1\ell - a_2\ell^2/2) - \frac{42}{\ell^4}(a_1\ell^2 - 2a_2\ell^3/3)$$
$$+ \frac{72}{\ell^5}(a_1\ell^3/3 - a_2\ell^4/4)\Big] + 2W_1W_3\Big[-\frac{36}{\ell^4}(a_1\ell - a_2\ell^2/2)$$
$$+ \frac{72}{\ell^5}(a_1\ell^2 - 2a_2\ell^3/3) - \frac{144}{\ell^6}(a_1\ell^3/3 - a_2\ell^4/4)\Big]$$
$$+ 2W_1W_4\Big[\frac{12}{\ell^3}(a_1\ell - a_2\ell^2/2) - \frac{30}{\ell^4}(a_1\ell^2 - 2a_2\ell^3/3)$$
$$+ \frac{72}{\ell^5}(a_1\ell^3/3 - a_2\ell^4/4)\Big] + 2W_2W_3\Big[-\frac{24}{\ell^3}(a_1\ell - a_2\ell^2/2)$$
$$+ \frac{42}{\ell^4}(a_1\ell^2 - 2a_2\ell^3/3) - \frac{72}{\ell^5}(a_1\ell^3/3 - a_2\ell^4/4)\Big]$$
$$+ 2W_2W_4\Big[\frac{8}{\ell^2}(a_1\ell - a_2\ell^2/2) - \frac{18}{\ell^3}(a_1\ell^2 - 2a_2\ell^3/3)$$
$$+ \frac{36}{\ell^4}(a_1\ell^3/3 - a_2\ell^4/4)\Big] + 2W_3W_4\Big[-\frac{12}{\ell^3}(a_1\ell - a_2\ell^2/2)$$
$$+ \frac{30}{\ell^4}(a_1\ell^2 - 2a_2\ell^3/3) - \frac{72}{\ell^5}(a_1\ell^3/3 - a_2\ell^4/4)\Big]\Big\} \qquad (E_4)$$

By writing $V = \frac{1}{2}\vec{W}^T[k]\vec{W}$

with $\vec{W} = \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{Bmatrix}$, the stiffness matrix can be identified.

Defining $a_1 = \frac{Bt^3}{12}$, $d_1 = \frac{(B-b)t^3}{12}$, $a_2 = \frac{d_1}{\ell}$, $a_2\ell = d_1$,

the elements of $[k]$ can be expressed as:

$k_{11} = E\{a_1(\frac{12}{\ell^3}) - d_1(\frac{6}{\ell^3})\}$, $\quad k_{22} = E\{a_1(\frac{4}{\ell}) - d_1(\frac{1}{\ell})\}$,

$k_{33} = E\{a_1(\frac{12}{\ell^3}) - d_1(\frac{6}{\ell^3})\}$, $\quad k_{44} = E\{a_1(\frac{4}{\ell}) - d_1(\frac{3}{\ell})\}$,

$k_{12} = E\{a_1(\frac{6}{\ell^2}) - d_1(\frac{2}{\ell^2})\}$, $\quad k_{13} = E\{a_1(-\frac{12}{\ell^3}) + d_1(\frac{6}{\ell^3})\}$,

$k_{14} = E\{a_1(\frac{6}{\ell^2}) - d_1(\frac{4}{\ell^2})\}$, $\quad k_{23} = E\{a_1(-\frac{6}{\ell^2}) + d_1(\frac{2}{\ell^2})\}$,

$k_{24} = E\{a_1(\frac{2}{\ell}) - d_1(\frac{1}{\ell})\}$, $\quad k_{34} = E\{a_1(-\frac{6}{\ell^2}) + d_1(\frac{4}{\ell^2})\}$

(b) Stresses induced in the beam:

$B = 0.25$ m, $b = 0.10$ m, $t = 0.025$ m, $\ell = 2$ m, $E = 2.07\times10^{11}$ N/m^2,
$P = 1000$ N, $a_1 = 32552.0833\times10^{-11}$, $d_1 = 19531.25\times10^{-11}$.

stiffness matrix can be computed as:

$$[k] = \begin{bmatrix} 70750 & 80860 & -70750 & 60640 \\ 80860 & 114600 & -80860 & 47170 \\ -70750 & -80860 & 70750 & -60640 \\ 60640 & 47170 & -60640 & 74120 \end{bmatrix} \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{matrix}$$

(columns: $w_1 \quad w_2 \quad w_3 \quad w_4$)

Equilibrium equations:

$$[K] \vec{W} = \vec{F}$$

i.e.

$$\begin{bmatrix} 70750 & -60640 \\ -60640 & 74120 \end{bmatrix} \begin{Bmatrix} W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1000 \end{Bmatrix}$$

The solution of these equations is

$$W_3 = 0.03871 \text{ m}, \quad W_4 = 0.04516 \text{ rad}$$

stress at root:

$$\sigma_{max}\Big|_{x=0} = \frac{M\, y_{max}}{I}\Big|_{x=0} = \frac{E I(x) \dfrac{d^2 w(x)}{dx^2}\, y_{max}}{I(x)}\Bigg|_{x=0}$$

$$= E \frac{d^2 w(x)}{dx^2} \frac{t}{2}\Big|_{x=0}$$

$$= \frac{Et}{2}\left[-\frac{6}{\ell^2} W_1^{\nearrow 0} - \frac{4}{\ell} W_2^{\nearrow 0} + \frac{6}{\ell^2} W_3 - \frac{2}{\ell} W_4\right]$$

$$= \frac{Et}{2}\left[\frac{6}{\ell^2} W_3 - \frac{2}{\ell} W_4\right]$$

$$= \frac{(2.07 \times 10^{11})(0.025)}{2}\left\{\frac{6}{4} \times 0.03871 - \frac{2}{2} \times 0.04516\right\}$$

$$= 3.3392 \times 10^7 \text{ N/m}^2$$

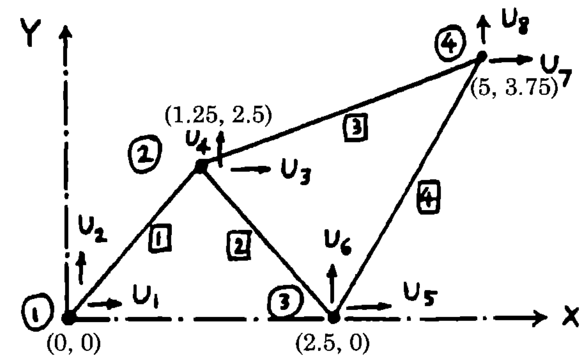
12.5 $A^{(i)} = 13 \times 10^{-4}$ m^2, i = 1, 2, 3, 4, $E = 200 \times 10^9$ N/m^2

$$\ell^{(1)} = \sqrt{\{1.25^2 + 2.50^2\}} = 2.795085 \text{ m}$$

$$\ell^{(2)} = \sqrt{\{1.25^2 + 2.50^2\}} = 2.795085 \text{ m}$$

$$\ell^{(3)} = \sqrt{\{3.75^2 + 1.25^2\}} = 3.952847 \text{ m}$$

$$\ell^{(4)} = \sqrt{\{2.5^2 + 3.75^2\}} = 4.506939 \text{ m}$$



$$[k^i] = \frac{E^{(i)} A^{(i)}}{\ell^{(i)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

From Eq. (12.52), the global stiffness matrix of element i is

$$\left[\bar{k}^{(i)}\right] = [\lambda^{(i)}]^T [k^{(i)}] [\lambda^{(i)}] \qquad (E_1)$$

where $[\lambda^{(i)}] = \begin{bmatrix} \cos\theta_i & \sin\theta_i & 0 & 0 \\ 0 & 0 & \cos\theta_i & \sin\theta_i \end{bmatrix}$

and θ_i is the angle made by the element with respect to x axis. Thus (E_1) gives

$$\left[\bar{k}^{(i)}\right] = \frac{E^{(i)} A^{(i)}}{\ell^{(i)}} \begin{bmatrix} \cos^2\theta_i & \cos\theta_i \sin\theta_i & -\cos^2\theta_i & -\cos\theta_i \sin\theta_i \\ \cos\theta_i \sin\theta_i & \sin^2\theta_i & -\cos\theta_i \sin\theta_i & -\sin^2\theta_i \\ -\cos^2\theta_i & -\cos\theta_i \sin\theta_i & \cos^2\theta_i & \cos\theta_i \sin\theta_i \\ -\cos\theta_i \sin\theta_i & -\sin^2\theta_i & \cos\theta_i \sin\theta_i & \sin^2\theta_i \end{bmatrix}$$

Here $\theta_1 = 63.4349°$ (line ① ②) $\theta_2 = 116.5651°$ (line ③ ②)
$\theta_3 = 18.4350°$ (line ② ④) $\theta_4 = 56.3099°$ (line ③ ④)

$$\frac{E^{(1)} A^{(1)}}{\ell^{(1)}} = \frac{(200 \times 10^9)(13 \times 10^{-4})}{2.795085} = 93.0204 \times 10^6 \text{ N/m},$$

$\cos\theta_1 = 0.4472$, $\sin\theta_1 = 0.8944$

$$\left[\bar{k}^{(1)}\right] = 10^6 \begin{bmatrix} 18.6041 & 37.2082 & -18.6041 & -37.2082 \\ 37.2082 & 74.4162 & -37.2082 & -74.4162 \\ -18.6041 & -37.2082 & 18.6041 & 37.2082 \\ -37.2082 & -74.4162 & 37.2082 & 74.4162 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{matrix}$$

(columns correspond to U_1, U_2, U_3, U_4)

$$\frac{E^{(2)} A^{(2)}}{\ell^{(2)}} = 93.0204 \times 10^6 \text{ N/m}, \ \cos\theta_2 = -0.4472, \ \sin\theta_2 = 0.8944$$

$$\left[\overline{k}^{(2)}\right] = 10^6 \begin{array}{c} \\ \end{array}\begin{matrix} U_5 & U_6 & U_3 & U_4 \end{matrix}$$

$$\left[\overline{k}^{(2)}\right] = 10^6 \begin{bmatrix} 18.6041 & -37.2082 & -18.6041 & 37.2082 \\ -37.2082 & 74.4162 & 37.2082 & -74.4162 \\ -18.6041 & 37.2082 & 18.6041 & -37.2082 \\ 37.2082 & -74.4162 & -37.2082 & 74.4162 \end{bmatrix} \begin{matrix} U_5 \\ U_6 \\ U_3 \\ U_4 \end{matrix}$$

$$\frac{E^{(3)} A^{(3)}}{\ell^{(3)}} = \frac{(200 \times 10^9)(13 \times 10^{-4})}{3.952847} = 65.7754 \times 10^6 \text{ N/m}, \cos\theta_3 = 0.9487,$$

$\sin\theta_3 = 0.3162$

$$\begin{matrix} U_3 & U_4 & U_7 & U_8 \end{matrix}$$

$$\left[\overline{k}^{(3)}\right] = 10^6 \begin{bmatrix} 59.1978 & 19.7326 & -59.1978 & -19.7326 \\ 19.7326 & 6.57757 & -19.7326 & -6.57757 \\ -59.1978 & -19.7326 & 59.1978 & 19.7326 \\ -19.7326 & -6.57757 & 19.7326 & 6.57757 \end{bmatrix} \begin{matrix} U_3 \\ U_4 \\ U_7 \\ U_8 \end{matrix}$$

$$\frac{E^{(4)} A^{(4)}}{\ell^{(4)}} = \frac{(200 \times 10^9)(13 \times 10^{-4})}{4.506939} = 57.6888 \times 10^6 \text{ N/m}, \cos\theta_4 = 0.5547,$$

$\sin\theta_4 = 0.8321$

$$\begin{matrix} U_5 & U_6 & U_7 & U_8 \end{matrix}$$

$$\left[\overline{k}^{(4)}\right] = 10^6 \begin{bmatrix} 17.7504 & 26.6256 & -17.7504 & -26.6256 \\ 26.6256 & 39.9384 & -26.6256 & -39.9384 \\ -17.7504 & -26.6256 & 17.7504 & 26.6256 \\ -26.6256 & -39.9384 & 26.6256 & 39.9384 \end{bmatrix} \begin{matrix} U_5 \\ U_6 \\ U_7 \\ U_8 \end{matrix}$$

12.6

$$\vec{U}^{(1)} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ \vdots \\ U_7 \\ U_8 \end{Bmatrix} \equiv [A^{(1)}]\, \vec{\underset{\sim}{U}}$$

$$\vec{U}^{(2)} = \begin{Bmatrix} U_5 \\ U_6 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ \vdots \\ U_7 \\ U_8 \end{Bmatrix} \equiv [A^{(2)}]\, \vec{\underset{\sim}{U}}$$

$$\vec{U}^{(3)} = \begin{Bmatrix} U_3 \\ U_4 \\ U_7 \\ U_8 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ \vdots \\ U_7 \\ U_8 \end{Bmatrix} \equiv [A^{(3)}] \vec{\underset{\sim}{U}}$$

$$\vec{U}^{(4)} = \begin{Bmatrix} U_5 \\ U_6 \\ U_7 \\ U_8 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ \vdots \\ U_7 \\ U_8 \end{Bmatrix} \equiv [A^{(4)}] \vec{\underset{\sim}{U}}$$

Assembled stiffness matrix of the truss is given by Eq. (12.63):

$$[\underset{\sim}{K}] = \sum_{e=1}^{4} [A^{(e)}]^T \left[\overline{k}^{(e)}\right] [A^{(e)}]$$

$10^6 \times$

	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8
U_1	18.6041	37.2082	−18.6041	−37.2082	0	0	0	0
U_2	37.2082	74.4162	−37.2082	−74.4162	0	0	0	0
U_3	−18.6041	−37.2082	18.6041 +18.6041 +59.1978	37.2082 −37.2082 +19.7326	−18.6041	37.2082	−59.1978	−19.7326
U_4	−37.2082	−74.4162	37.2082 −37.2082 +19.7326	74.4162 +74.4162 +6.57757	37.2082	−74.4162	−19.7326	−6.57757
U_5	0	0	−18.6041	37.2082	18.6041 +17.7504	−37.2082 +26.6256	−17.7504	−26.6256
U_6	0	0	37.2082	−74.4162	−37.2082 26.6256	74.4162 39.9384	−26.6256	−39.9384
U_7	0	0	−59.1978	−19.7326	−17.7504	−26.6256	59.1978 +17.7504	19.7326 +26.6256
U_8	0	0	−19.7326	−6.57757	−26.6256	−39.9384	19.7326 +26.6256	6.57757 +39.9384

Since nodes ① & ③ are fixed, $U_1 = U_2 = U_5 = U_6 = 0$ and the final equilibrium equations can be expressed as

$[K]\vec{U} = \vec{F}$ where

$$[K] = \begin{bmatrix} 96.406 & 19.7326 & -59.1978 & -19.7326 \\ 19.7326 & 155.4100 & -19.7326 & -6.57757 \\ -59.1978 & -19.7326 & 76.9482 & 46.3582 \\ -19.7326 & -6.57757 & 42.3582 & 46.5160 \end{bmatrix} 10^6 \text{ N/m}, \qquad \vec{U} = \begin{Bmatrix} U_3 \\ U_4 \\ U_7 \\ U_8 \end{Bmatrix} \text{ m}$$

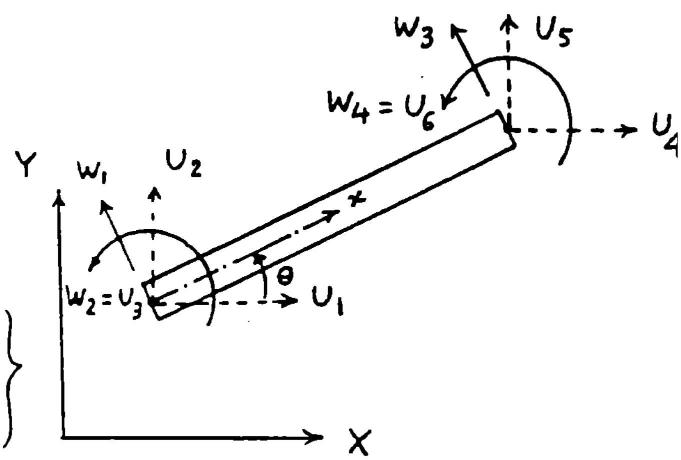
$$\text{and } \vec{F} = \begin{Bmatrix} F_3 \\ F_4 \\ F_7 \\ F_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -5000 \end{Bmatrix} \text{ N}$$

12.7 $W_1 = U_1 \cos(90+\theta) + U_2 \cos\theta + U_3(0)$

$W_2 = U_1(0) + U_2(0) + U_3(1)$

i.e.,

$$\begin{Bmatrix} W_1 \\ W_2 \end{Bmatrix} = \begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix}$$



$$\vec{W} = [\lambda]\vec{U}$$

where $\vec{W} = \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{Bmatrix}$, $\vec{U} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix}$, and

$$\underset{4\times 6}{[\lambda]} = \begin{bmatrix} -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For a beam element with degrees of freedom $\vec{W}$,

$$[k^{(e)}] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

and

$$[m^{(e)}] = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$$

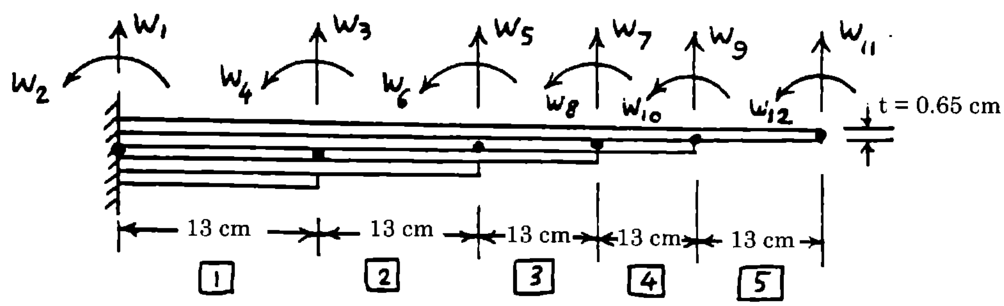
Matrices in global system (X, Y system) are given by

$$[\bar{k}^{(e)}] = [\lambda]^T [k^{(e)}] [\lambda]$$

and

$$[\bar{m}^{(e)}] = [\lambda]^T [m^{(e)}] [\lambda]$$

(12.8)



Width = 3.8 cm

$$[k^{(e)}] = \frac{E^{(e)} I^{(e)}}{\ell^{(e)^3}}$$

$$= \begin{bmatrix} 12 & 6\,\ell^{(e)} & -12 & 6\,\ell^{(e)} \\ 6\,\ell^{(e)} & 4\,\ell^{(e)^2} & -6\,\ell^{(e)} & 2\,\ell^{(e)^2} \\ -12 & -6\,\ell^{(e)} & 12 & -6\,\ell^{(e)} \\ 6\,\ell^{(e)} & 2\,\ell^{(e)^2} & -6\,\ell^{(e)} & 4\,\ell^{(e)^2} \end{bmatrix}$$

$E^{(e)} = E = 200 \times 10^9$ N/m^2; e = 1 to 5
$\ell^{(e)} = \ell = 0.13$ m; e = 1 to 5

$$I^{(1)} = \frac{1}{12}\,(0.038)\,(0.0325)^3 = 108.706 \times 10^{-9}\ \text{m}^{-4} = 125\ I_5$$

$$I^{(2)} = \frac{1}{12}\,(0.038)\,(0.026)^3 = 64\ I_5$$

$I^{(3)} = 27\ I_5$
$I^{(4)} = 8\ I_5$

$$I^{(5)} = \frac{1}{12}\,(0.038)\,(0.0065)^3 = 8.69646 \times 10^{-10}\ \text{m}^{-4}$$

Assembled stiffness matrix, after applying the boundary conditions, is given by Eq. (E_2).

Load vector:

$$\vec{F} = \begin{Bmatrix} F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \\ F_{10} \\ F_{11} \\ F_{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -8900 \\ 0 \end{Bmatrix} \text{N} \qquad (E_1)$$

$$[K] = \frac{E\,I_5}{\ell^3} \begin{array}{c} \begin{array}{cccccccccc} W_3 & W_4 & W_5 & W_6 & W_7 & W_8 & W_9 & W_{10} & W_{11} & W_{12} \end{array} \\ \begin{bmatrix} 2268 & -366\,\ell & -768 & 384\,\ell & 0 & 0 & 0 & 0 & 0 & 0 \\ -366\,\ell & 756\,\ell^2 & -384\,\ell & 128\,\ell^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -768\,\ell & -384\,\ell & 1092 & -222\,\ell & -324 & 162\,\ell & 0 & 0 & 0 & 0 \\ 384\,\ell & 128\,\ell^2 & -222\,\ell & 364\,\ell^2 & -162\,\ell & 54\,\ell^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -324 & -162\,\ell & 420 & -114\,\ell & -96 & 48\,\ell & 0 & 0 \\ 0 & 0 & 162\,\ell & 54\,\ell^2 & -114\,\ell & 140\,\ell^2 & -48\,\ell & 16\,\ell^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -96 & -48\,\ell & 108 & -42\,\ell & -12 & 6\,\ell \\ 0 & 0 & 0 & 0 & 48\,\ell & 16\,\ell^2 & -42\,\ell & 36\,\ell^2 & -6\,\ell & 2\,\ell^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -12 & -6\,\ell & 12 & -6\,\ell \\ 0 & 0 & 0 & 0 & 0 & 0 & 6\,\ell & 2\,\ell^2 & -6\,\ell & 4\,\ell^2 \end{bmatrix} \end{array} \begin{array}{c} W_3 \\ W_4 \\ W_5 \\ W_6 \\ W_7 \\ W_8 \\ W_9 \\ W_{10} \\ W_{11} \\ W_{12} \end{array}$$

with $E = 200 \times 10^9$ Pa, $I_5 = 8.69646 \times 10^{-10}$ m^{-4} and $\ell = 0.13$ m

Solution of equilibrium equations, $[K]\ \vec{W} = \vec{F}$, gives

$$\vec{W}^T = \{W_3, W_4, W_5, W_6, W_7, W_8, W_9, W_{10}, W_{11}, W_{12}\}^T$$

$$= \{-0.002099 \text{ m}, -0.031132, -0.009366 \text{ m}, -0.078424, -0.025113 \text{ m}, -0.158496, -0.057428 \text{ m}, -0.320642, -0.136585 \text{ m}, -0.753031\}$$

12.9 Assembled stiffness matrix is given in the solution of Problem 12.8. For the assembled mass matrix, we note

$$[m^{(e)}] = \frac{\rho^{(e)} A^{(e)} \ell^{(e)}}{420} \begin{bmatrix} 156 & 22\,\ell^{(e)} & 54 & -13\,\ell^{(e)} \\ 22\,\ell^{(e)} & 4\,\ell^{(e)^2} & 13\,\ell^{(e)} & -3\,\ell^{(e)^2} \\ 54 & 13\ell^{(e)} & 156 & -22\,\ell^{(e)} \\ -13\,\ell^{(e)} & -3\,\ell^{(e)^2} & -22\,\ell^{(e)} & 4\,\ell^{(e)^2} \end{bmatrix}$$

with

$\rho^{(e)} = 7800$ kg/m^3, $\ell = 1$ to 5

$\ell^{(e)} = \ell = 0.13$ m, $\ell = 1$ to 5

$A^{(1)} = 5\,A_5$; $A^{(2)} = 4\,A_5$; $A^{(3)} = 3\,A_5$; $A^{(4)} = 2\,A_5$; $A^{(5)} = A_5 = 2.47 \times 10^{-5}$ m^2

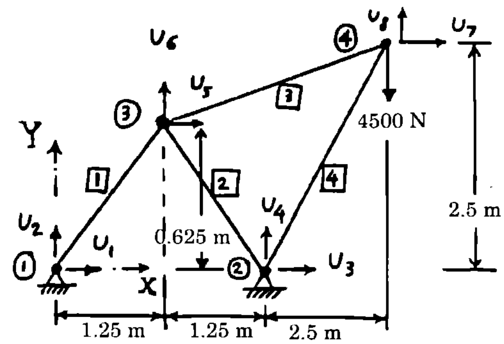
Assembled mass matrix, after applying boundary conditions, is:

$$[M] = \frac{\rho\,\ell\,A_5}{420} \begin{array}{c} \begin{matrix} W_3 & W_4 & W_5 & W_6 & W_7 & W_8 & W_9 & W_{10} & W_{11} & W_{12} \end{matrix} \\ \begin{bmatrix} 1404 & -22\,\ell & 216 & -52\,\ell & 0 & 0 & 0 & 0 & 0 & 0 \\ -22\,\ell & 36\,\ell^2 & 52\,\ell & -12\,\ell^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 216 & 52\,\ell & 1092 & -22\,\ell & 162 & -39\,\ell & 0 & 0 & 0 & 0 \\ -52\,\ell & -12\,\ell^2 & -22\,\ell & 28\,\ell^2 & 39\,\ell & -9\,\ell^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 162 & 39\,\ell & 780 & -22\,\ell & 108 & -26\,\ell & 0 & 0 \\ 0 & 0 & -39\,\ell & -9\,\ell^2 & -22\,\ell & 20\,\ell^2 & 26\,\ell & -6\,\ell^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 108 & 26\,\ell & 468 & -22\,\ell & 54 & -13\,\ell \\ 0 & 0 & 0 & 0 & -22\,\ell & -6\,\ell^2 & -22\,\ell & 12\,\ell^2 & 13\,\ell & -3\,\ell^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 54 & 13\,\ell & 156 & -22\,\ell \\ 0 & 0 & 0 & 0 & 0 & 0 & -13\,\ell & -3\,\ell^2 & -22\,\ell & 4\,\ell^2 \end{bmatrix} \end{array}$$

with $\rho = 7800$ kg/m^3, $\ell = 0.13$ m and $A_5 = 2.47 \times 10^{-5}$ m^2

12.10

element e	area of c/s, $A^{(e)}$	length $\ell^{(e)}$	Young's Modulus $E^{(e)}$
1	13×10^{-4} m^2	1.39754 m	200×10^9 Pa
2	13×10^{-4} m^2	1.39754 m	200×10^9 Pa
3	6.5×10^{-4} m^2	4.19263 m	200×10^9 Pa
4	6.5×10^{-4} m^2	3.53553 m	200×10^9 Pa



element e	global node corresponding to local node 1	2	coordinates of local nodes (m) X_i	Y_i	X_j	Y_j	direction cosines ℓ_{ij}	m_{ij}
1	1	3	0	0	1.25	0.625	0.8944	0.4472
2	3	2	1.25	0.625	2.5	0	0.8944	– 0.4472
3	3	4	1.25	0.625	5	2.5	0.8944	0.4472
4	2	4	2.5	0	5	2.5	0.7071	0.7071

$$[k^{(e)}] = \frac{A^{(e)}\,E^{(e)}}{\ell^{(e)}} \begin{bmatrix} \ell_{ij}^2 & \ell_{ij}\,m_{ij} & -\ell_{ij}^2 & -\ell_{ij}\,m_{ij} \\ \ell_{ij}\,m_{ij} & m_{ij}^2 & -\ell_{ij}\,m_{ij} & -m_{ij}^2 \\ -\ell_{ij}^2 & -\ell_{ij}\,m_{ij} & \ell_{ij}^2 & \ell_{ij}\,m_{ij} \\ -\ell_{ij}\,m_{ij} & -m_{ij}^2 & \ell_{ij}\,m_{ij} & m_{ij}^2 \end{bmatrix}$$

$$[k^{(1)}] = \frac{(13 \times 10^{-4})\,(200 \times 10^9)}{1.39754} \begin{matrix} & U_1 & U_2 & U_5 & U_6 & \\ & \left[\begin{matrix} 0.8 \\ 0.4 \\ -0.8 \\ -0.4 \end{matrix}\right. & \begin{matrix} 0.4 \\ 0.2 \\ -0.4 \\ -0.2 \end{matrix} & \begin{matrix} -0.8 \\ -0.4 \\ 0.8 \\ 0.4 \end{matrix} & \left.\begin{matrix} -0.4 \\ -0.2 \\ 0.4 \\ 0.2 \end{matrix}\right] & \begin{matrix} U_1 \\ U_2 \\ U_5 \\ U_6 \end{matrix} \end{matrix}$$

$$[k^{(2)}] = \frac{(13 \times 10^{-4})\,(200 \times 10^9)}{1.39754} \begin{matrix} & U_5 & U_6 & U_3 & U_4 & \\ & \left[\begin{matrix} 0.8 \\ -0.4 \\ -0.8 \\ 0.4 \end{matrix}\right. & \begin{matrix} -0.4 \\ 0.2 \\ 0.4 \\ -0.2 \end{matrix} & \begin{matrix} -0.8 \\ 0.4 \\ 0.8 \\ -0.4 \end{matrix} & \left.\begin{matrix} 0.4 \\ -0.2 \\ -0.4 \\ 0.2 \end{matrix}\right] & \begin{matrix} U_5 \\ U_6 \\ U_3 \\ U_4 \end{matrix} \end{matrix}$$

$$[k^{(3)}] = \frac{(6.5 \times 10^{-4})\,(200 \times 10^9)}{4.19263} \begin{matrix} & U_5 & U_6 & U_7 & U_8 & \\ & \left[\begin{matrix} 0.8 \\ 0.4 \\ -0.8 \\ -0.4 \end{matrix}\right. & \begin{matrix} 0.4 \\ 0.2 \\ -0.4 \\ -0.2 \end{matrix} & \begin{matrix} -0.8 \\ -0.4 \\ 0.8 \\ 0.4 \end{matrix} & \left.\begin{matrix} -0.4 \\ -0.2 \\ 0.4 \\ 0.2 \end{matrix}\right] & \begin{matrix} U_5 \\ U_6 \\ U_7 \\ U_8 \end{matrix} \end{matrix}$$

$$[k^{(4)}] = \frac{(6.5 \times 10^{-4})\,(200 \times 10^9)}{3.53553} \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \begin{matrix} U_3 \\ U_4 \\ U_7 \\ U_8 \end{matrix}$$

(columns: U_3, U_4, U_7, U_8)

Assembled stiffness matrix, after deleting the rows and columns corresponding to degrees of freedom U_1, U_2, U_3 and U_4, is

$$[K] = 10^6 \begin{bmatrix} 322.471 & 12.4027 & -24.8054 & -12.4027 \\ 12.4027 & 80.6178 & -12.4027 & -6.20136 \\ -24.8054 & -12.4027 & 43.1902 & 30.7875 \\ -12.4027 & -6.20136 & 30.7875 & 24.5861 \end{bmatrix} \begin{matrix} U_5 \\ U_6 \\ U_7 \\ U_8 \end{matrix}$$

(columns: U_5, U_6, U_7, U_8)

Load vector: $(\vec{F})^T = \{0, 0, 0, -4500\text{ N}\}$, $(\vec{U})^T = \{U_5, U_6, U_7, U_8\}$

Equilibrium eq. : $[K]\,\vec{U} = \vec{F}$

Solution of these equations is

$(\vec{U})^T = \{3.02426 \times 10^{-5},\ 6.04849 \times 10^{-5},\ 0.00133645,\ -0.0018261\}$ m

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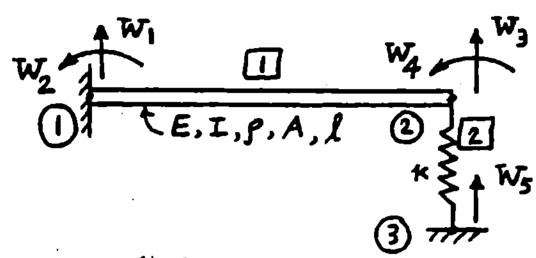
12.11 (a) ONE ELEMENT IDEALIZATION



$$[k^{(1)}] = \frac{EI}{l^3} \left[\begin{array}{cc|cc} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ \hline -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{array}\right] \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{matrix} \qquad (E_1)$$

(columns: w_1, w_2, w_3, w_4)

where $I = \frac{1}{12} b h^3 = \frac{1}{12}\left(\frac{50}{100}\right)\left(\frac{25}{1000}\right)^3 = 6.5104 \times 10^{-8}\ m^4$.

$$[K^{(2)}] = k \left[\begin{array}{c|c} 1 & -1 \\ \hline -1 & 1 \end{array}\right] \begin{matrix} w_5 \\ w_3 \end{matrix} \qquad (E_2)$$

(columns: w_5, w_3)

To incorporate the boundary conditions $w_1 = w_2 = w_5 = 0$, we delete the first two rows and columns in (E_1) and first row and column in (E_2). The assembled matrix becomes

$$[K] = \frac{EI}{l^3}\begin{bmatrix} \left(12 + k\frac{l^3}{EI}\right) & -6l \\ -6l & 4l^2 \end{bmatrix} \begin{matrix} w_3 \\ w_4 \end{matrix}$$

(columns: w_3, w_4)

$$= \frac{(2.07\times10^{11})(6.5104\times10^{-8})}{(0.25)^3}\begin{bmatrix} 12+\frac{(10^5)(0.25)^3}{(2.07\times10^{11})(6.5104\times10^{-8})} & -6(0.25) \\ -6(0.25) & 4(0.25)^2 \end{bmatrix}$$

$$= 8.6250\times10^5 \begin{bmatrix} 12.1159 & -1.5 \\ -1.5 & 0.25 \end{bmatrix}$$

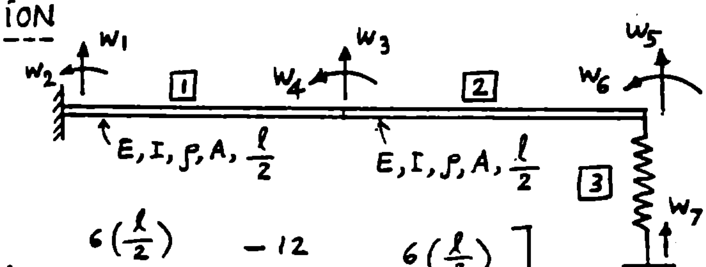
Load vector is $\vec{P} = \begin{Bmatrix} P_3 \\ P_4 \end{Bmatrix} = \begin{Bmatrix} 500 \\ 0 \end{Bmatrix}$

Equilibrium equations become $[K]\vec{W} = \vec{P}$

i.e., $\begin{bmatrix} 12.1159 & -1.5 \\ -1.5 & 0.25 \end{bmatrix}\begin{Bmatrix} W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} 57.971\times10^{-5} \\ 0 \end{Bmatrix}$

The solution is given by $W_3 = 1.8605\times10^{-4}$ m; $W_4 = 11.1629\times10^{-4}$ rad.

(b) TWO ELEMENT IDEALIZATION



$$[k^{(1)}] = [k^{(2)}] = \frac{EI}{\left(\frac{l}{2}\right)^3}\begin{bmatrix} 12 & 6\left(\frac{l}{2}\right) & -12 & 6\left(\frac{l}{2}\right) \\ 6\left(\frac{l}{2}\right) & 4\left(\frac{l}{2}\right)^2 & -6\left(\frac{l}{2}\right) & 2\left(\frac{l}{2}\right)^2 \\ -12 & -6\left(\frac{l}{2}\right) & 12 & -6\left(\frac{l}{2}\right) \\ 6\left(\frac{l}{2}\right) & 2\left(\frac{l}{2}\right)^2 & -6\left(\frac{l}{2}\right) & 4\left(\frac{l}{2}\right)^2 \end{bmatrix} \quad (E_1)$$

Substituting $E = 2.07\times10^{11}$, $I = 6.5104\times10^{-8}$ and $l = 0.25$, Eq. (E_1) becomes

(columns: W_3, W_4, W_5, W_6 ---- for $i = 2$; W_1, W_2, W_3, W_4 --- for $i = 1$)

$$[K^{(i)}] = 69\times10^5 \begin{bmatrix} 12 & 0.75 & -12 & 0.75 \\ 0.75 & 0.0625 & -0.75 & 0.03125 \\ -12 & -0.75 & 12 & -0.75 \\ 0.75 & 0.03125 & -0.75 & 0.0625 \end{bmatrix} \begin{matrix} W_1 & W_3 \\ W_2 & W_4 \\ W_3 & W_5 \\ W_4 & W_6 \end{matrix}$$

$$[K^{(3)}] = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} W_7 \\ W_5 \end{matrix} \quad (\text{columns } W_7, W_5)$$

Assembled stiffness matrix is

$$[K] = 69 \times 10^5 \begin{bmatrix} (12+12) & (-0.75+0.75) & -12 & 0.75 \\ (-0.75+0.75) & (0.0625+0.0625) & -0.75 & 0.03125 \\ -12 & -0.75 & \left(12+\frac{10^5}{69\times 10^5}\right) & -0.75 \\ 0.75 & 0.03125 & -0.75 & 0.0625 \end{bmatrix} \begin{matrix} W_3 \\ W_4 \\ W_5 \\ W_6 \end{matrix} \quad (\text{columns } W_3, W_4, W_5, W_6)$$

Load vector is

$$\vec{P} = \begin{Bmatrix} P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 500 \\ 0 \end{Bmatrix}$$

Equilibrium equations are $[K]\vec{W} = \vec{P}$

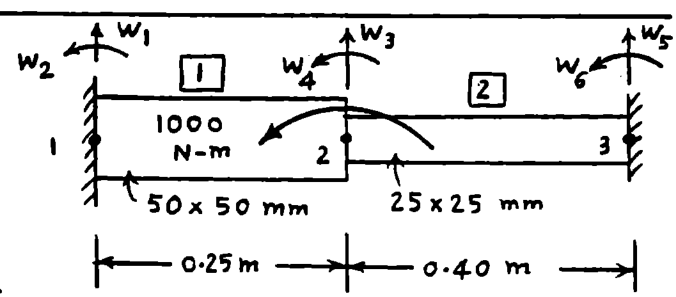
i.e.,

$$69 \times 10^5 \begin{bmatrix} 24 & 0 & -12 & 0.75 \\ 0 & 0.125 & -0.75 & 0.03125 \\ -12 & -0.75 & 12.0145 & -0.75 \\ 0.75 & 0.03125 & -0.75 & 0.0625 \end{bmatrix} \begin{Bmatrix} W_3 \\ W_4 \\ W_5 \\ W_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 500 \\ 0 \end{Bmatrix}$$

The solution is given by :

$W_3 = 0.5814 \times 10^{-4}$ m, $W_4 = 8.372 \times 10^{-4}$ rad, $W_5 = 1.86 \times 10^{-4}$ m, $W_6 = 11.16 \times 10^{-4}$ rad.

(12.12) Element 1 :



$$I = \frac{1}{12}\left(\frac{50}{1000}\right)\left(\frac{50}{1000}\right)^3$$

$$= 0.5208 \times 10^{-6} \text{ m}^4$$

$$\frac{EI}{l^3} = \frac{(2.1\times 10^{11})(0.5208 \times 10^{-6})}{(0.25)^3}$$

$$= 0.7 \times 10^7$$

$$[K^{(1)}] = 0.7\times10^{7} \begin{bmatrix} 12 & 1.5 & -12 & 1.5 \\ 1.5 & 0.25 & -1.5 & 0.125 \\ -12 & -1.5 & 12 & -1.5 \\ 1.5 & 0.125 & -1.5 & 0.25 \end{bmatrix} \begin{matrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{matrix}$$

(columns W_1, W_2, W_3, W_4)

$$= 0.7\times10^{7} \begin{bmatrix} 12 & -1.5 \\ -1.5 & 0.25 \end{bmatrix} \begin{matrix} W_3 \\ W_4 \end{matrix}$$

(columns W_3, W_4) after applying boundary conditions

Element 2:

$$I = \frac{1}{12}\left(\frac{25}{1000}\right)\left(\frac{25}{1000}\right)^3 = 0.3255\times10^{-7}\ m^4$$

$$\frac{EI}{l^3} = \frac{(2.1\times10^{11})(0.3255\times10^{-7})}{(0.4)^3} = 10.6805\times10^{4}$$

$$[K^{(2)}] = 10.6805\times10^{4} \begin{bmatrix} 12 & 2.4 & -12 & 2.4 \\ 2.4 & 0.64 & -2.4 & 0.32 \\ -12 & -2.4 & 12 & -2.4 \\ 2.4 & 0.32 & -2.4 & 0.64 \end{bmatrix} \begin{matrix} W_3 \\ W_4 \\ W_5 \\ W_6 \end{matrix}$$

(columns W_3, W_4, W_5, W_6)

$$= 10.6805\times10^{4} \begin{bmatrix} 12 & 2.4 \\ 2.4 & 0.64 \end{bmatrix} \begin{matrix} W_3 \\ W_4 \end{matrix}$$

(columns W_3, W_4) after applying boundary conditions

Assembled stiffness matrix:

$$[K] = \begin{bmatrix} (8.4\times10^{7} + 0.1282\times10^{7}) & (-1.05\times10^{7} + 0.0256\times10^{7}) \\ (-1.05\times10^{7} + 0.0256\times10^{7}) & (0.175\times10^{7} + 0.0068\times10^{7}) \end{bmatrix}$$

Equilibrium equations:

$$10^{7} \begin{bmatrix} 8.5282 & -1.0244 \\ -1.0244 & 0.1818 \end{bmatrix} \begin{Bmatrix} W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10^{3} \end{Bmatrix}$$

Solution is given by

$$W_3 = 2.0446\times10^{-4}\ m, \quad W_4 = 1.7021\times10^{-3}\ rad$$

Stresses in elements:

Bending moment $= M = EI \dfrac{d^2 w(x)}{dx^2}$

with $\quad w(x) = \sum_{i=1}^{4} W_i \; N_i(x)$

$$\sigma_{max} = \frac{M \cdot c}{I} = \frac{M h}{2 I} = \frac{E h}{2} \frac{d^2 w}{dx^2}$$

$$= \frac{E h}{2}\left[\frac{W_1}{l^3}(12x - 6l) + \frac{W_2}{l^2}(6x - 4l) + \frac{W_3}{l^3}(6l - 12x) + \frac{W_4}{l^2}(6x - 2l)\right] \quad \text{---}(E_1)$$

For element 1,

$W_1 = 0$, $W_2 = 0$, $W_3 = 2.0446 \times 10^{-4}$, $W_4 = 1.7021 \times 10^{-3}$,

$l = 0.25$, $h = 0.05$, $E = 2.1 \times 10^{11}$. Hence Eq. (E_1) gives

$$\sigma_{max}\Big|_{fixed\ end} = \sigma_{max}(x=0) = 3.1560 \times 10^{7} \ N/m^2$$

$$\sigma_{max}\Big|_{loaded\ end} = \sigma_{max}(x = 0.25) = 3.9929 \times 10^{7} \ N/m^2$$

For element 2,

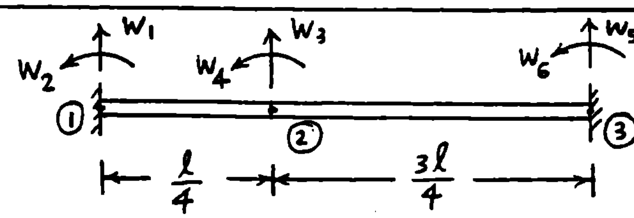
$W_1 = 2.0446 \times 10^{-4}$, $W_2 = 1.7021 \times 10^{-3}$, $W_3 = 0$, $W_4 = 0$,

$l = 0.4$, $h = 0.025$, $E = 2.1 \times 10^{11}$. Hence Eq. (E_1) gives

$$\sigma_{max}\Big|_{loaded\ end} = \sigma_{max}(x=0) = -6.4807 \times 10^{7} \ N/m^2$$

$$\sigma_{max}\Big|_{fixed\ end} = \sigma_{max}(x=0.4) = 4.2467 \times 10^{7} \ N/m^2$$

12.13



For element 1:

$$[K^{(1)}] = \frac{EI}{(l/4)^3}\begin{bmatrix} 12 & \frac{3l}{2} & -12 & \frac{3l}{2} \\ \frac{3l}{2} & \frac{l^2}{4} & -\frac{3l}{2} & \frac{l^2}{8} \\ -12 & -\frac{3l}{2} & 12 & -\frac{3l}{2} \\ \frac{3l}{2} & \frac{l^2}{8} & -\frac{3l}{2} & \frac{l^2}{4} \end{bmatrix} \begin{matrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{matrix}$$

(columns: W_1, W_2, W_3, W_4)

For element 2:

$$[K^{(2)}] = \frac{EI}{(3\ell/4)^3} \begin{bmatrix} 12 & \frac{9\ell}{2} & -12 & \frac{9\ell}{2} \\ \frac{9\ell}{2} & \frac{9\ell^2}{4} & -\frac{9\ell}{2} & \frac{9\ell^2}{8} \\ -12 & -\frac{9\ell}{2} & 12 & -\frac{9\ell}{2} \\ \frac{9\ell}{2} & \frac{9\ell^2}{8} & -\frac{9\ell}{2} & \frac{9\ell^2}{4} \end{bmatrix} \begin{matrix} W_3 \\ W_4 \\ W_5 \\ W_6 \end{matrix}$$

(columns: W_3, W_4, W_5, W_6)

Assembled stiffness matrix:

$$[K] = \frac{64\,EI}{\ell^3} \begin{bmatrix} \left(12+\frac{12}{27}\right) & \left(-\frac{3\ell}{2}+\frac{9\ell}{54}\right) \\ \left(-\frac{3\ell}{2}+\frac{9\ell}{54}\right) & \left(\frac{\ell^2}{4}+\frac{9\ell^2}{108}\right) \end{bmatrix} = \frac{64EI}{3\ell^3} \begin{bmatrix} \frac{112}{3} & -4\ell \\ -4\ell & \ell^2 \end{bmatrix} \begin{matrix} W_3 \\ W_4 \end{matrix}$$

(columns: W_3, W_4)

Equilibrium equations: $[K]\vec{W} = \vec{P}$

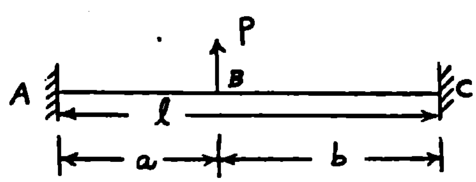
i.e.,

$$\frac{64\,EI}{3\,\ell^3} \begin{bmatrix} 112/3 & -4\ell \\ -4\ell & \ell^2 \end{bmatrix} \begin{Bmatrix} W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$

Solution is given by

$$W_3 = \frac{9\;P\ell^3}{4096\;EI} \quad \text{and} \quad W_4 = \frac{9\;P\ell^2}{1024\;EI}.$$

<u>Simple beam deflection formula:</u>



$$y_{AB} = -\frac{P b^2 x^2}{6EI\ell^3}\left[x(3a+b) - 3a\ell\right]$$

at $x = a$,

$$y_B = -\frac{P b^2 a^2}{6EI\ell^3}\left[3a^2 + ab - 3a\ell\right]$$

when $a = \ell/4$ and $b = 3\ell/4$,

$$y_B = \frac{9}{4096}\,\frac{P\ell^3}{EI}$$

$$\text{slope} = \frac{dy_{AB}}{dx} = -\frac{Pb^2}{6EI\ell^2}\left[3x^2(3a+b) - 6xa\ell\right]$$

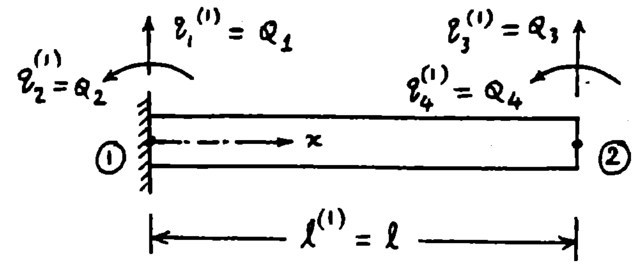
at $x = a$, $$\left.\frac{dy}{dx}\right|_B = -\frac{Pb^2}{6EI\ell^3}\left[9a^3 + 3a^2 b - 6a^2\ell\right]$$

When $a = \ell/4$ and $b = 3\ell/4$, $\left.\frac{dy}{dx}\right|_B = \frac{9P\ell^2}{1024\;EI}$

∴ Both results are same. Reason: shape functions = static deflection relations.

12.14

$$[\underset{\sim}{K}] = [K^{(1)}] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$



$$[\underset{\sim}{M}] = [M^{(1)}] = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$$

Since node ① is fixed, $Q_1 = Q_2 = 0$. By deleting the first two rows and columns in $[\underset{\sim}{K}]$ and $[\underset{\sim}{M}]$, the eigenvalue problem becomes

$$\left[[K] - \omega^2 [M] \right] \begin{Bmatrix} Q_3 \\ Q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Frequency equation is

$$\left| \frac{EI}{l^3}\begin{bmatrix} 12 & -6l \\ -6l & 4l^2 \end{bmatrix} - \frac{\rho A l \omega^2}{420}\begin{bmatrix} 156 & -22l \\ -22l & 4l^2 \end{bmatrix} \right| = 0$$

Let $\lambda = \dfrac{\rho A l^4 \omega^2}{420\, EI}$. Then $\begin{vmatrix} 12 - 156\lambda & -6l + 22l\lambda \\ -6l + 22l\lambda & 4l^2 - 4l^2\lambda \end{vmatrix} = 0$

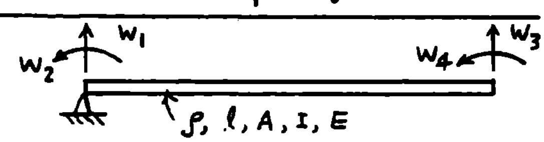
i.e. $140\, l^2 \lambda^2 - 408\, l^2 \lambda + 12\, l^2 = 0$

i.e. $\lambda = 0.029715,\ 2.884571$

i.e. $\omega_1^2 = 12.4803 \dfrac{EI}{\rho A l^4}$, $\quad \omega_2^2 = 1211.5198 \dfrac{EI}{\rho A l^4}$

$\therefore \quad \omega_1 = 3.5327 \sqrt{EI/\rho A l^4}\,, \quad \omega_2 = 34.8069 \sqrt{EI/\rho A l^4}$

12.15



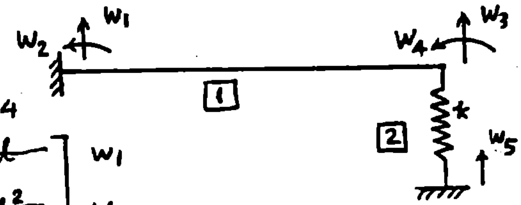
$$[\underset{\sim}{K}] = [K^{(1)}] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \qquad (E_1)$$

$$[\underset{\sim}{M}] = [M^{(1)}] = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \qquad (E_2)$$

Applying the boundary condition $W_1 = 0$, the eigenvalue problem can be expressed as
$$-\omega^2 [M] \vec{W} + [K] \vec{W} = \vec{0}$$
i.e.,
$$\left[-\omega^2 \frac{\rho A l}{420} \begin{bmatrix} 4l^2 & 13l & -3l^2 \\ 13l & 156 & -22l \\ -3l^2 & -22l & 4l^2 \end{bmatrix} + \frac{EI}{l^3} \begin{bmatrix} 4l^2 & -6l & 2l^2 \\ -6l & 12 & -6l \\ 2l^2 & -6l & 4l^2 \end{bmatrix} \right] \begin{Bmatrix} W_2 \\ W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$
i.e.,
$$\begin{bmatrix} (4l^2 - \lambda l^2) & (-6l - 13l\lambda) & (2l^2 + 3l^2\lambda) \\ (-6l - 13l\lambda) & (12 - 156\lambda) & (-6l + 22l\lambda) \\ (2l^2 + 3l^2\lambda) & (-6l + 22l\lambda) & (4l^2 - 4l^2\lambda) \end{bmatrix} \begin{Bmatrix} W_2 \\ W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (E_3)$$
where $\lambda = \dfrac{\omega^2 \rho A l^4}{420\, EI}$.

Eq. (E_3) gives the frequency equation
$$\begin{vmatrix} 4l^2(1-\lambda) & -l(6+13\lambda) & l^2(2+3\lambda) \\ -l(6+13\lambda) & 12(1-13\lambda) & l(-6+22\lambda) \\ l^2(2+3\lambda) & l(-6+22\lambda) & 4l^2(1-\lambda) \end{vmatrix} = 0$$
which reduces to
$$-\lambda (196\lambda^2 - 2436\lambda + 1680) = 0$$
Roots are:
$$\lambda_1 = 0; \quad \lambda_{2,3} = \frac{2436 \pm \sqrt{(2436)^2 - 4(196)(1680)}}{2(196)}$$
$$= 0.7329; \quad 11.6957$$
Thus
$$\omega_1 = 0$$
$$\omega_2 = 17.5447 \sqrt{\frac{EI}{\rho A l^4}}$$
$$\omega_3 = 70.0870 \sqrt{\frac{EI}{\rho A l^4}}$$
These values can be compared with the exact values (see Fig. 8.15): $\omega_1 = 0$, $\omega_2 = 15.4182 \sqrt{\dfrac{EI}{\rho A l^4}}$, $\omega_3 = 49.9649 \sqrt{\dfrac{EI}{\rho A l^4}}$.

12.16 Element matrices:



$$[K^{(1)}] = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \begin{matrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{matrix}$$

(columns W_1, W_2, W_3, W_4; rows and columns for W_1, W_2 struck out)

$$[K^{(2)}] = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} W_3 \\ W_5 \end{matrix}$$

(columns W_3, W_5; row and column for W_5 struck out)

$$[M^{(1)}] = \frac{\rho A l}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \begin{matrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{matrix}$$

(columns W_1, W_2, W_3, W_4; rows and columns for W_1, W_2 struck out)

$$[M^{(2)}] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} W_3 \\ W_5 \end{matrix}$$

(columns W_3, W_5) (mass of spring is assumed to be zero)

Assembled matrices:

$$[K] = \frac{EI}{l^3} \begin{bmatrix} \left(12 + \frac{k l^3}{EI}\right) & -6l \\ -6l & 4l^2 \end{bmatrix} \begin{matrix} W_3 \\ W_4 \end{matrix}$$

$$[M] = \frac{\rho A l}{420} \begin{bmatrix} 156 & -22l \\ -22l & 4l^2 \end{bmatrix} \begin{matrix} W_3 \\ W_4 \end{matrix}$$

(columns W_3, W_4)

Eigenvalue equation:

$$\left| -\frac{\rho A l \omega^2}{420} \begin{bmatrix} 156 & -22l \\ -22l & 4l^2 \end{bmatrix} + \frac{EI}{l^3} \begin{bmatrix} \left(12 + \frac{k l^3}{EI}\right) & -6l \\ -6l & 4l^2 \end{bmatrix} \right| = 0$$

i.e.,

$$\begin{vmatrix} 12 + \frac{k l^3}{EI} - 156\lambda & -6l + 22l\lambda \\ -6l + 22l\lambda & 4l^2 - 4l^2\lambda \end{vmatrix} = 0$$

where $\lambda = \left(\frac{\rho A l^4 \omega^2}{420 E I} \right)$

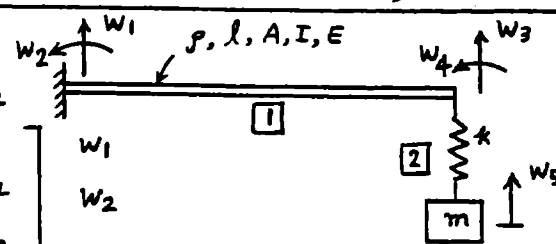
Upon expansion, the frequency equation reduces to

$$35 \lambda^2 - \left(102 + \frac{k l^3}{E I}\right) \lambda + \left(3 + \frac{k l^3}{E I}\right) = 0$$

which implies that

$$\lambda_{1,2} = \left\{ \frac{\left(102 + \frac{k l^3}{EI}\right) \pm \left(9984 + 64 \frac{k l^3}{EI} + \frac{k^2 l^6}{E^2 I^2}\right)^{\frac{1}{2}}}{70} \right\}$$

12.17 Element matrices:



$$[K^{(1)}] = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{matrix}$$

(columns: w_1, w_2, w_3, w_4)

$$[M^{(1)}] = \frac{\rho A l}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{matrix}$$

(columns: w_1, w_2, w_3, w_4)

$$[K^{(2)}] = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} w_3 \\ w_5 \end{matrix}, \quad [M^{(2)}] = \begin{bmatrix} 0 & 0 \\ 0 & m \end{bmatrix} \begin{matrix} w_3 \\ w_5 \end{matrix}$$

(columns: w_3, w_5)

Assembled matrices (after applying boundary conditions):

$$[K] = \frac{EI}{l^3} \begin{bmatrix} 12 + \frac{k l^3}{EI} & -6l & -\frac{k l^3}{EI} \\ -6l & 4l^2 & 0 \\ -\frac{k l^3}{EI} & 0 & \frac{k l^3}{EI} \end{bmatrix} \begin{matrix} w_3 \\ w_4 \\ w_5 \end{matrix}$$

(columns: w_3, w_4, w_5)

$$= \frac{EI}{l^3} \begin{bmatrix} 13 & -6l & -1 \\ -6l & 4l^2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$[M] = \frac{\rho A l}{420} \begin{bmatrix} 156+0 & -22l & 0 \\ -22l & 4l^2 & 0 \\ 0 & 0 & m\left(\frac{420}{\rho A l}\right) \end{bmatrix} \begin{matrix} w_3 \\ w_4 \\ w_5 \end{matrix}$$

(columns: w_3, w_4, w_5)

$$= \frac{\rho A l}{420} \begin{bmatrix} 156 & -22l & 0 \\ -22l & 4l^2 & 0 \\ 0 & 0 & 420 \end{bmatrix}$$

Frequency equation:

$$\left| -\omega^2 [M] + [K] \right| = 0$$

i.e.,

$$\left| -\frac{\rho A l \omega^2}{420} \begin{bmatrix} 156 & -22l & 0 \\ -22l & 4l^2 & 0 \\ 0 & 0 & 420 \end{bmatrix} + \frac{EI}{l^3} \begin{bmatrix} 13 & -6l & -1 \\ -6l & 4l^2 & 0 \\ -1 & 0 & 1 \end{bmatrix} \right| = 0$$

i.e.,

$$\begin{vmatrix} 13-156\lambda & -6l+22l\lambda & -1 \\ -6l+22l\lambda & 4l^2-4l^2\lambda & 0 \\ -1 & 0 & 1-420\lambda \end{vmatrix} = 0 \qquad (E_1)$$

where $\lambda = \left(\frac{\rho A l^4 \omega^2}{420 EI} \right)$

Eq. (E_1) can be simplified to obtain

$$-58\,800\,\lambda^3 + 173\,180\,\lambda^2 - 7128\,\lambda + 12 = 0 \qquad (E_2)$$

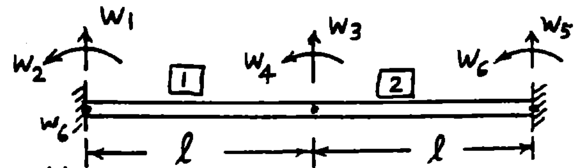
Roots of Eq. (E_2) given by:

$\lambda_1 = 0.00175555 \Rightarrow \omega_1 = 0.8587 \sqrt{EI/(\rho A l^4)}$

$\lambda_2 = 0.039951 \Rightarrow \omega_2 = 4.0965 \sqrt{EI/(\rho A l^4)}$

$\lambda_3 = 2.90351 \Rightarrow \omega_3 = 34.9210 \sqrt{EI/(\rho A l^4)}$

12.18 Element matrices:



for e = 2 --- W_3 W_4 W_5 W_6

for e = 1 --- W_1 W_2 W_3 W_4

$$[K^{(e)}] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \begin{matrix} W_1 & W_3 \\ W_2 & W_4 \\ W_3 & W_5 \\ W_4 & W_6 \end{matrix}$$

(rows: for e = 1 → W_1, W_2, W_3, W_4; for e = 2 → W_3, W_4, W_5, W_6)

Columns: W_3, W_4, W_5, W_6 ----- for e = 2; W_1, W_2, W_3, W_4 -- for e = 1

$$[M^{(e)}] = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \begin{matrix} W_1 & W_3 \\ W_2 & W_4 \\ W_3 & W_5 \\ W_4 & W_6 \end{matrix}$$

Assembled matrices, after incorporating boundary conditions $W_1 = W_2 = W_5 = W_6 = 0$:

$$[K] = \frac{EI}{l^3}\begin{bmatrix} 12+12 & -6l+6l \\ -6l+6l & 4l^2+4l^2 \end{bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 24 & 0 \\ 0 & 8l^2 \end{bmatrix} \begin{matrix} W_3 \\ W_4 \end{matrix}$$

(columns W_3, W_4)

$$[M] = \frac{\rho A l}{420}\begin{bmatrix} 156+156 & -22l+22l \\ -22l+22l & 4l^2+4l^2 \end{bmatrix} = \frac{\rho A l}{420}\begin{bmatrix} 312 & 0 \\ 0 & 8l^2 \end{bmatrix} \begin{matrix} W_3 \\ W_4 \end{matrix}$$

(columns W_3, W_4)

Eigenvalue problem:

$$\left[-\omega^2 [M] + [K]\right] \vec{W} = \vec{0}$$

For natural frequencies,

$$\left| -\omega^2 [M] + [K] \right| = 0$$

i.e.,

$$\left| -\frac{\omega^2 \rho A l}{420}\begin{bmatrix} 312 & 0 \\ 0 & 8l^2 \end{bmatrix} + \frac{EI}{l^3}\begin{bmatrix} 24 & 0 \\ 0 & 8l^2 \end{bmatrix} \right| = 0$$

i.e.,

$$\begin{vmatrix} (24 - 312\lambda) & 0 \\ 0 & 8l^2(1-\lambda) \end{vmatrix} = 0 \quad \text{where } \lambda = \left(\frac{\rho A l^4 \omega^2}{420\, EI}\right).$$

i.e., $192\,\ell^2(1-\lambda)(1-13\lambda)=0$

$\therefore \quad \lambda_1 = \frac{1}{13}, \quad \omega_1 = 22.736\sqrt{\frac{EI}{\rho A L^4}}$ with $L = 2\ell$

$\lambda_2 = 1, \quad \omega_2 = 81.9756\sqrt{\frac{EI}{\rho A L^4}}$

For mode shapes,

$$\left[-\omega_i^2[M] + [K]\right]\vec{W}^{(i)} = \vec{0} \; ; \quad i = 1, 2$$

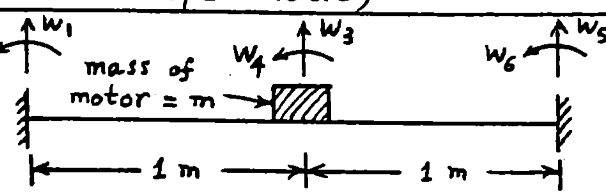
i.e.,

$$\begin{bmatrix} (24-312\,\lambda_i) & 0 \\ 0 & 8\ell^2(1-\lambda_i) \end{bmatrix} \begin{Bmatrix} W_3^{(i)} \\ W_4^{(i)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

For $\lambda_1 = \frac{1}{13}$, $W_3^{(1)}$ can have any value (transverse displacement mode)

For $\lambda_2 = 1$, $W_4^{(2)}$ can have any value (rotation or slope mode)

12.19 $m = 100$ kg, $\ell_1 = \ell_2 = 1$ m



From solution of problem 12.18, we have

$$[K] = \frac{EI}{\ell^3}\begin{bmatrix} 24 & 0 \\ 0 & 8\ell^2 \end{bmatrix} \begin{matrix} W_3 \\ W_4 \end{matrix} \qquad (E_1)$$

(columns W_3, W_4)

$$[M] = \frac{\rho A \ell}{420}\begin{bmatrix} 312 & 0 \\ 0 & 8\ell^2 \end{bmatrix} \begin{matrix} W_3 \\ W_4 \end{matrix} \qquad (E_2)$$

(columns W_3, W_4)

When the mass of motor is added to d.o.f. W_3, the mass matrix becomes

$$[M'] = \frac{\rho A \ell}{420}\begin{bmatrix} 312 + m\left(\frac{420}{\rho A \ell}\right) & 0 \\ 0 & 8\,\ell^2 \end{bmatrix} \qquad (E_3)$$

Eqs. (E_1) and (E_3) yield the frequency equation

$$\left|-\omega^2[M'] + [K]\right| = 0$$

i.e.,

$$\left| -\frac{\rho A l \omega^2}{420}\begin{bmatrix}\left(312+\frac{42000}{\rho A l}\right) & 0 \\ 0 & 8l^2\end{bmatrix} + \frac{EI}{l^3}\begin{bmatrix}24 & 0 \\ 0 & 8l^2\end{bmatrix}\right| = 0$$

i.e.,

$$\begin{vmatrix} 24-\left(312+\frac{42000}{\rho A l}\right)\lambda & 0 \\ 0 & 8l^2(1-\lambda)\end{vmatrix} = 0$$

i.e.,

$$192\, l^2 (1-\lambda)\left\{1-\left(13+\frac{1750}{\rho A l}\right)\lambda\right\} = 0$$

where $\lambda = \left(\frac{\rho A l^4 \omega^2}{420\, E I}\right)$.

$$\therefore\ \lambda_1 = 1 \ \Rightarrow\ \omega_1^2 = \frac{420\, E I\, \lambda_1}{\rho A l^4} = \frac{420\, E I}{\rho A l^4}$$

$$\lambda_2 = \frac{1}{\left(13+\frac{1750}{\rho A l}\right)} \ \Rightarrow\ \omega_2^2 = \frac{420\, E I}{\rho A l^4\left(13+\frac{1750}{\rho A l}\right)}$$

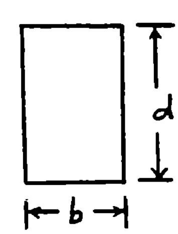
For the steel beam,

$E = 2.1 \times 10^{11}$ Pa, $l = 1$ m, $\rho = 7.8 \times 10^3$ kg/m^3

$$\omega_1^2 = \frac{420(2.1\times10^{11})\, I}{(7.8\times10^3)\, A\, (1)^4} \ \Rightarrow\ \omega_1 = 10.6338 \times 10^4 \sqrt{\frac{I}{A}} \ \text{rad/sec} \quad (E_4)$$

$$\omega_2^2 = \frac{420\,(2.1\times10^{11})\, I}{13(7.8\times10^3)\, A\, (1)^4 + 1750\,(1)^3} = \frac{88.2\times10^{12}\, I}{101.4\times10^3\, A + 1750}$$

$$\Rightarrow\ \omega_2 = \frac{9.3915\times10^6\, \sqrt{I}}{(101400\, A + 1750)^{1/2}} \ \text{rad/sec} \quad (E_5)$$



Let depth = twice width for cross-section.

Then $A = 2b^2$ and $I = \frac{1}{12}(b)(2b)^3 = 0.6667\, b^4$

Operating speed of motor $= \Omega = \frac{1800\,(2\pi)}{60}$

$= 188.496$ rad/sec

Assume ω_1 = smaller than ω_2.

For $\omega_1 > 188.496$, we need to have

$$10^4 (10.6338) \sqrt{\frac{0.6667\ b^4}{2 b^2}} > 188.496$$

i.e., $b > 3.07 \times 10^{-3}$ m

Let $b = 5$ mm and $d = 10$ mm so that

$A = 50 \times 10^{-6}\ m^2$ and $I = 416.6667 \times 10^{-12}\ m^4$

This gives

$$\omega_2 = \frac{9.3915 \times 10^6 \left(416.6667 \times 10^{-12}\right)^{1/2}}{\left(101400 \times 50 \times 10^{-6} + 1750\right)^{1/2}} = 4.5760 \ \frac{rad}{sec}$$

This violates the original assumption of $\omega_1 > \omega_2$.

Assume ω_2 = smaller than ω_1.

For $\omega_2 > 188.496$, we need to have

$$\frac{9.3915 \times 10^6 \left(0.6667\ b^4\right)^{1/2}}{\left\{101400 \left(2 b^2\right) + 1750\right\}^{1/2}} > 188.496$$

i.e.,

$$1654.8624\ b^4 - 0.2028\ b^2 - 0.00175 > 0 \qquad (E_6)$$

By setting the inequality in (E_6) to equality and solving for b^2, we find

$b^2 = 0.00109145$ or $b = 0.03304$ m

Let $b = 35$ mm and $d = 70$ mm, so that the left side of inequality (E_6) becomes

$(0.002483 - 0.000248 - 0.001750)$ which is positive.

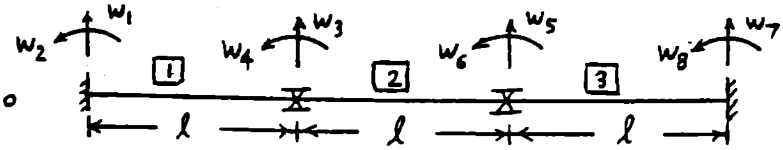
Hence the final design is given by

$b = 0.035$ m

$d = 0.070$ m

12.20 Boundary conditions:

$W_1 = W_2 = W_3 = W_5 = W_7 = W_8 = 0$



$W_5 \quad W_6 \quad W_7 \quad W_8$ ------ for $i = 3$

$W_3 \quad W_4 \quad W_5 \quad W_6$ ---- for $i = 2$

$W_1 \quad W_2 \quad W_3 \quad W_4$ -- for $i = 1$

$$[K^{(i)}] = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \begin{matrix} W_1 & W_3 & W_5 \\ W_2 & W_4 & W_6 \\ W_3 & W_5 & W_7 \\ W_4 & W_6 & W_8 \end{matrix}$$

$W_5 \quad W_6 \quad W_7 \quad W_8$ --------- for $i = 3$

$W_3 \quad W_4 \quad W_5 \quad W_6$ ---- for $i = 2$

$W_1 \quad W_2 \quad W_3 \quad W_4$ -- for $i = 1$

$$[M^{(i)}] = \frac{\rho A l}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \begin{matrix} W_1 & W_3 & W_5 \\ W_2 & W_4 & W_6 \\ W_3 & W_5 & W_7 \\ W_4 & W_6 & W_8 \end{matrix}$$

Assembled stiffness matrix, after applying boundary conditions:

$$[K] = \frac{EI}{l^3} \begin{bmatrix} (4l^2 + 4l^2) & 2l^2 \\ 2l^2 & (4l^2 + 4l^2) \end{bmatrix} \begin{matrix} W_4 \\ W_6 \end{matrix}$$

(columns: W_4, W_6)

$$[M] = \frac{\rho A l}{420} \begin{bmatrix} (4l^2 + 4l^2) & -3l^2 \\ -3l^2 & (4l^2 + 4l^2) \end{bmatrix} \begin{matrix} W_4 \\ W_6 \end{matrix}$$

(columns: W_4, W_6)

Frequency equation:

$$\left| -\frac{\rho A l \omega^2}{420} \begin{bmatrix} 8l^2 & -3l^2 \\ -3l^2 & 8l^2 \end{bmatrix} + \frac{EI}{l^3} \begin{bmatrix} 8l^2 & 2l^2 \\ 2l^2 & 8l^2 \end{bmatrix} \right| = 0 \qquad (E_1)$$

Defining $\quad \lambda = \left(\frac{\rho A l^4 \omega^2}{420 EI} \right)$, Eq. (E1) can be expressed as

$$\begin{vmatrix} 8(1-\lambda) & (2+3\lambda) \\ (2+3\lambda) & 8(1-\lambda) \end{vmatrix} = 0$$

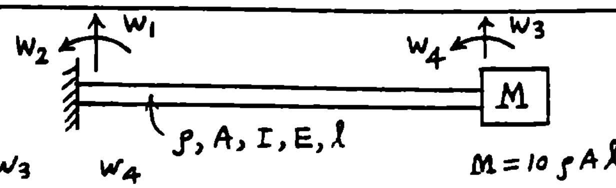
or

$$11\lambda^2 - 28\lambda + 12 = 0$$

$$\therefore \quad \lambda_{1,2} = \frac{28 \pm \sqrt{784 - 4(11)(12)}}{2(11)} = \frac{6}{11}, \ 2$$

i.e., $\omega_1 = 15.1357 \sqrt{\frac{EI}{\rho A l^4}}$ and $\omega_2 = 28.9828 \sqrt{\frac{EI}{\rho A l^4}}$

(12.21) Element matrices:



$$[K^{(1)}] = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{matrix}$$

(columns: w_1, w_2, w_3, w_4)

$$[M^{(1)}] = \frac{\rho A l}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{matrix}$$

(columns: w_1, w_2, w_3, w_4)

Assembled matrices (after applying boundary conditions $w_1 = w_2 = 0$ and adding mass M at d.o.f. w_3):

$$[K] = \frac{EI}{l^3} \begin{bmatrix} 12 & -6l \\ -6l & 4l^2 \end{bmatrix} \begin{matrix} w_3 \\ w_4 \end{matrix}$$

(columns: w_3, w_4)

$$[M] = \frac{\rho A l}{420} \begin{bmatrix} 156 + 4200 & -22l \\ -22l & 4l^2 \end{bmatrix} \begin{matrix} w_3 \\ w_4 \end{matrix}$$

(columns: w_3, w_4)

Defining $\lambda = \left(\frac{\rho A l^4 \omega^2}{420 EI} \right)$, the frequency equation can be written as

$$\begin{vmatrix} (12 - 4356\,\lambda) & (-6\ell + 22\,\ell\,\lambda) \\ (-6\ell + 22\,\ell\,\lambda) & (4\ell^2 - 4\ell^2\,\lambda) \end{vmatrix} = 0$$

or

$$4235\,\lambda^2 - 4302\,\lambda + 3 = 0$$

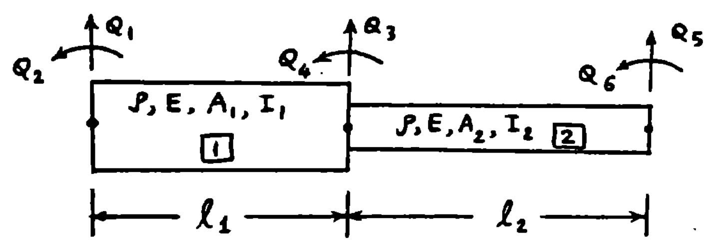
This gives

$$\lambda_{1,2} = \frac{4302 \pm (4302^2 - 4 \times 4235 \times 3)^{\frac{1}{2}}}{2(4235)}$$

$$= 0.0006978\,,\quad 1.01512272$$

$$\therefore\ \omega_1 = 0.5414\sqrt{\frac{EI}{\rho A \ell^4}}\,,\qquad \omega_2 = 20.6483\sqrt{\frac{EI}{\rho A \ell^4}}$$

12.22



$$[k^{(e)}] = [\bar{k}^{(e)}] = \frac{E^{(e)} I^{(e)}}{\ell^{(e)^3}} \begin{bmatrix} 12 & 6\ell^{(e)} & -12 & 6\ell^{(e)} \\ 6\ell^{(e)} & 4\ell^{(e)^2} & -6\ell^{(e)} & 2\ell^{(e)^2} \\ -12 & -6\ell^{(e)} & 12 & -6\ell^{(e)} \\ 6\ell^{(e)} & 2\ell^{(e)^2} & -6\ell^{(e)} & 4\ell^{(e)^2} \end{bmatrix};\ e = 1,2$$

$$[m^{(e)}] = [\bar{m}^{(e)}] = \frac{\rho^{(e)} A^{(e)} \ell^{(e)}}{420} \begin{bmatrix} 156 & 22\ell^{(e)} & 54 & -13\ell^{(e)} \\ 22\ell^{(e)} & 4\ell^{(e)^2} & 13\ell^{(e)} & -3\ell^{(e)^2} \\ 54 & 13\ell^{(e)} & 156 & -22\ell^{(e)} \\ -13\ell^{(e)} & -3\ell^{(e)^2} & -22\ell^{(e)} & 4\ell^{(e)^2} \end{bmatrix};\ e = 1,2$$

$$[A^{(1)}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},\qquad [A^{(2)}] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[\underset{\sim}{K}] = \sum_{e=1}^{2} [A^{(e)}]^T [\bar{k}^{(e)}] [A^{(e)}]$$

$$= E \begin{bmatrix} 12 I_1/\ell_1^3 & 6I_1/\ell_1^2 & -12 I_1/\ell_1^3 & 6I_1/\ell_1^2 & 0 & 0 \\ 6I_1/\ell_1^2 & 4I_1/\ell_1 & -6I_1/\ell_1^2 & 2I_1/\ell_1 & 0 & 0 \\ -12I_1/\ell_1^3 & -6I_1/\ell_1^2 & \left(\frac{12 I_1}{\ell_1^3}+\frac{12 I_2}{\ell_2^3}\right) & \left(-\frac{6I_1}{\ell_1^2}+\frac{6I_2}{\ell_2^2}\right) & -12 I_2/\ell_2^3 & 6I_2/\ell_2^2 \\ 6I_1/\ell_1^2 & 2I_1/\ell_1 & \left(-\frac{6I_1}{\ell_1^2}+\frac{6I_2}{\ell_2^2}\right) & \left(\frac{4I_1}{\ell_1}+\frac{4I_2}{\ell_2}\right) & -6I_2/\ell_2^2 & 2I_2/\ell_2 \\ 0 & 0 & -12I_2/\ell_2^3 & -6I_2/\ell_2^2 & 12I_2/\ell_2^3 & -6I_2/\ell_2^2 \\ 0 & 0 & 6I_2/\ell_2^2 & 2I_2/\ell_2 & -6I_2/\ell_2^2 & 4I_2/\ell_2 \end{bmatrix}$$

$$[\underset{\sim}{M}] = \sum_{e=1}^{2} [A^{(e)}]^T [\bar{m}^{(e)}] [A^{(e)}]$$

$$= \frac{\rho}{420} \begin{bmatrix} 156 A_1\ell_1 & 22 A_1\ell_1^2 & 54 A_1\ell_1 & -13 A_1\ell_1^2 & 0 & 0 \\ 22 A_1\ell_1^2 & 4A_1\ell_1^3 & 13 A_1\ell_1^2 & -3A_1\ell_1^3 & 0 & 0 \\ 54A_1\ell_1 & 13 A_1\ell_1^2 & 156(A_1\ell_1+A_2\ell_2) & 22(-A_1\ell_1^2+A_2\ell_2^2) & 54A_2\ell_2 & -13A_2\ell_2^2 \\ -13A_1\ell_1^2 & -3A_1\ell_1^3 & 22(-A_1\ell_1^2+A_2\ell_2^2) & 4(A_1\ell_1^3+A_2\ell_2^3) & 13A_2\ell_2^2 & -3A_2\ell_2^3 \\ 0 & 0 & 54A_2\ell_2 & 13A_2\ell_2^2 & 156A_2\ell_2 & -22A_2\ell_2^2 \\ 0 & 0 & -13A_2\ell_2^2 & -3A_2\ell_2^3 & -22A_2\ell_2^2 & 4A_2\ell_2^3 \end{bmatrix}$$

where $I^{(e)} = I_e$ and $\ell^{(e)} = \ell_e$; $e = 1,2$

Since $Q_1 = Q_2 = Q_5 = Q_6 = 0$, rows and columns 1,2,5 and 6 in $[\underset{\sim}{K}]$ and $[\underset{\sim}{M}]$ are deleted to obtain the frequency equation as

$$\left| [K] - \omega^2 [M] \right| = 0$$

i.e.

$$\begin{vmatrix} \left\{12E\left(\frac{I_1}{\ell_1^3}+\frac{I_2}{\ell_2^3}\right)-\frac{156\rho\omega^2}{420}(A_1\ell_1+A_2\ell_2)\right\} & \left\{6E\left(-\frac{I_1}{\ell_1^2}+\frac{I_2}{\ell_2^2}\right)-\frac{22\rho\omega^2}{420}\left(-A_1\ell_1^2+A_2\ell_2^2\right)\right\} \\ \left\{6E\left(-\frac{I_1}{\ell_1^2}+\frac{I_2}{\ell_2^2}\right)-\frac{22\rho\omega^2}{420}\left(-A_1\ell_1^2+A_2\ell_2^2\right)\right\} & \left\{4E\left(\frac{I_1}{\ell_1}+\frac{I_2}{\ell_2}\right)-\frac{4\rho\omega^2}{420}\left(A_1\ell_1^3+A_2\ell_2^3\right)\right\} \end{vmatrix} = 0$$

Roots of this equation give ω_1 and ω_2.

<u>Load vector:</u>

$$\vec{\bar{f}}^{(1)} = \vec{f}^{(1)} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} = \begin{Bmatrix} \int_0^{\ell_1} f_1(x,t)\, N_1(x)\, dx \\ \int_0^{\ell_1} f_2(x,t)\, N_2(x)\, dx \\ \int_0^{\ell_1} f_3(x,t)\, N_3(x)\, dx \\ \int_0^{\ell_1} f_4(x,t)\, N_4(x)\, dx \end{Bmatrix} = p \begin{Bmatrix} \int_0^{\ell_1} N_1 \cdot dx \\ \int_0^{\ell_1} N_2 \cdot dx \\ \int_0^{\ell_1} N_3 \cdot dx \\ \int_0^{\ell_1} N_4 \cdot dx \end{Bmatrix}$$

$$= p \begin{Bmatrix} l_1/2 \\ l_1^2/12 \\ l_1/2 \\ l_1^2/12 \end{Bmatrix}$$

$$\vec{\bar{f}}^{(2)} = \vec{f}^{(2)} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

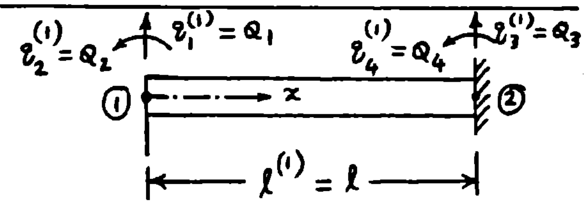
$$\vec{\underset{\sim}{F}} = \sum_{e=1}^{2} [A^{(e)}]^T \, \vec{\bar{f}}^{(e)} = p \begin{Bmatrix} l_1/2 \\ l_1^2/12 \\ l_1/2 \\ l_1^2/12 \\ 0 \\ 0 \end{Bmatrix}$$

After applying the boundary conditions, load vector becomes

$$\vec{F} = \begin{Bmatrix} p\, l_1/2 \\ p\, l_1^2/12 \end{Bmatrix}.$$

12.23

$$[\underset{\sim}{K}] = [K^{(1)}] = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$



$$[\underset{\sim}{M}] = [M^{(1)}] = \frac{\rho A l}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$$

Since node ① is pin connected and node ② is fixed, $Q_1 = Q_3 = Q_4 = 0$. By deleting the corresponding rows and columns in $[\underset{\sim}{K}]$ and $[\underset{\sim}{M}]$, the eigenvalue problem becomes

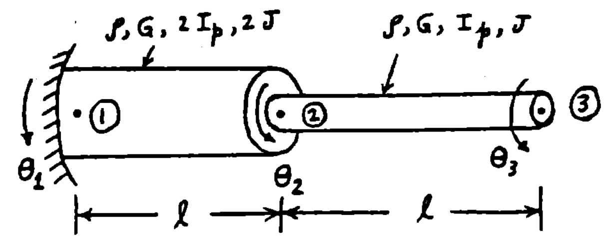
$$\left[[K] - \omega^2 [M] \right] \vec{Q} = \vec{0}$$

Frequency equation is

$$\left| \frac{EI}{l^3}(4l^2) - \omega^2 \frac{\rho A l}{420}(4l^2) \right| = 0$$

which gives $\omega_1^2 = \dfrac{420\, EI}{\rho A l^4}$ or $\omega_1 = 20.4939 \sqrt{\dfrac{EI}{\rho A l^4}}$.

12.24



$$[\overline{m}^{(1)}] = [m^{(1)}] = \frac{\rho_1 I_{p_1} l_1}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{\rho I_p l}{12}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

$$[\overline{m}^{(2)}] = [m^{(2)}] = \frac{\rho_2 I_{p_2} l_2}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{\rho I_p l}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[\overline{k}^{(1)}] = [k^{(1)}] = \frac{2 G_1 J_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{GJ}{l}\begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}$$

$$[\overline{k}^{(2)}] = [k^{(2)}] = \frac{2 G_2 J_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{GJ}{l}\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$[A^{(1)}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad [A^{(2)}] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\underset{\sim}{M}] = \sum_{e=1}^{2} [A^{(e)}]^T [\overline{m}^{(e)}] [A^{(e)}] = \frac{\rho I_p l}{12}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$[\underset{\sim}{K}] = \sum_{e=1}^{2} [A^{(e)}]^T [\overline{k}^{(e)}] [A^{(e)}] = \frac{2GJ}{l}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Since $\theta_1 = 0$, $[M] = \frac{\rho I_p l}{12}\begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix}$ and $[K] = \frac{2GJ}{l}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$

Frequency equation is

$$\left| \frac{2GJ}{l}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} - \frac{\rho I_p l \omega^2}{12}\begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix} \right| = 0$$

i.e. $\left| \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix} \right| = 0$ where $\lambda = \frac{\rho I_p l^2 \omega^2}{24\, GJ}$

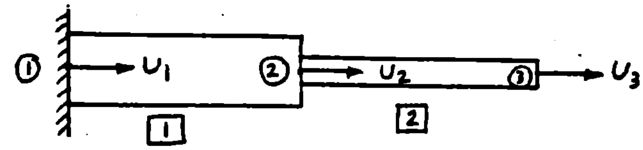
i.e. $11\lambda^2 - 14\lambda + 2 = 0$

i.e. $\lambda_1 = 0.1640$, $\lambda_2 = 1.1087$

$\omega_1^2 = 3.9360\, GJ/(\rho I_p l^2)$, $\omega_2^2 = 26.6088\, GJ/(\rho I_p l^2)$

$\therefore\ \omega_1 = 1.983935 \sqrt{\frac{GJ}{\rho I_p l^2}}$ and $\omega_2 = 5.158372 \sqrt{\frac{GJ}{\rho I_p l^2}}$

12.25 Element matrices:



$$[K^{(1)}] = \frac{A_1 E_1}{l_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{4AE}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{matrix}U_1\\ U_2\end{matrix}$$

$$[K^{(2)}] = \frac{A_2 E_2}{l_2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{AE}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{matrix}U_2\\ U_3\end{matrix}$$

$$[M^{(1)}] = \frac{\rho_1 A_1 l_1}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} = \frac{2\rho A l}{3}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{matrix}U_1\\ U_2\end{matrix}$$

$$[M^{(2)}] = \frac{\rho_2 A_2 l_2}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} = \frac{\rho A l}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{matrix}U_2\\ U_3\end{matrix}$$

Assembled matrices (with $U_1 = 0$):

$$[K] = \frac{AE}{l}\begin{bmatrix}5 & -1\\ -1 & 1\end{bmatrix}\begin{matrix}U_2\\ U_3\end{matrix}, \quad [M] = \frac{\rho A l}{6}\begin{bmatrix}10 & 1\\ 1 & 2\end{bmatrix}\begin{matrix}U_2\\ U_3\end{matrix}$$

Eigenvalue problem:

$$\left[-\omega^2 [M] + [K]\right]\vec{U} = \vec{0} \qquad (E_1)$$

For natural frequencies,

$$\left| -\frac{\rho A l \omega^2}{6}\begin{bmatrix}10 & 1\\ 1 & 2\end{bmatrix} + \frac{AE}{l}\begin{bmatrix}5 & -1\\ -1 & 1\end{bmatrix}\right| = 0$$

i.e.,

$$\begin{vmatrix}5 - 10\lambda & -1-\lambda\\ -1-\lambda & 1-2\lambda\end{vmatrix} = 0 \qquad (E_2)$$

where $\lambda = \left(\frac{\rho l^2 \omega^2}{6E}\right)$ $\qquad (E_3)$

i.e., $19\lambda^2 - 22\lambda + 4 = 0$

This gives

$$\lambda_1 = 0.2259, \quad \omega_1 = 1.1642\,\{E/(\rho l^2)\} \qquad (E_4)$$

$$\lambda_2 = 0.9320, \quad \omega_2 = 2.3648\,\{E/(\rho l^2)\} \qquad (E_5)$$

For eigenvectors,

Use of (E_2) leads to

$$(5 - 10\lambda_1)\,U_2 - (1 + \lambda_1)\,U_3 = 0$$

or $U_3 = \left(\frac{5 - 10\lambda_1}{1 + \lambda_1} \right) U_2 = 2.2359\ U_2$ with $\lambda_1 = 0.2259$

$$\therefore \quad \underset{\sim}{\vec{U}}^{(1)} = \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 1.0 \\ 2.2359 \end{Bmatrix} \qquad (E_6)$$

Similarly,

$U_3 = \left(\frac{5 - 10\lambda_2}{1 + \lambda_2} \right) U_2 = -2.2360\ U_2$ with $\lambda_2 = 0.9320$

$$\therefore \quad \underset{\sim}{\vec{U}}^{(2)} = \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 1.0 \\ -2.2360 \end{Bmatrix} \qquad (E_7)$$

Orthonormalization of normal modes with [M]-matrix:

Let $\vec{U}^{(1)} = a_1\, \underset{\sim}{\vec{U}}^{(1)}$ and $\vec{U}^{(2)} = a_2\, \underset{\sim}{\vec{U}}^{(2)}$ $\qquad (E_8)$

where a_1 and a_2 are constants to be determined.

$\vec{U}^{(1)T} [M]\, \vec{U}^{(1)} = 1$ gives $\quad a_1^2 = \dfrac{1}{\underset{\sim}{\vec{U}}^{(1)T} [M]\, \underset{\sim}{\vec{U}}^{(1)}}$

Since $\underset{\sim}{\vec{U}}^{(1)T} [M]\, \underset{\sim}{\vec{U}}^{(1)} = (1.0 \quad 2.2359)\, \dfrac{\rho A l}{6} \begin{bmatrix} 10 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} 1.0 \\ 2.2359 \end{Bmatrix}$

$= 4.0784\ \rho A l,$

$a_1 = 0.4952 / \sqrt{\rho A l}$, and

$$\vec{U}^{(1)} = \frac{1}{\sqrt{\rho A l}} \begin{Bmatrix} 0.4952 \\ 1.1072 \end{Bmatrix} \qquad (E_9)$$

Similarly, $\quad a_2^2 = \dfrac{1}{\underset{\sim}{\vec{U}}^{(2)T} [M]\, \underset{\sim}{\vec{U}}^{(2)}}$

where $\underset{\sim}{\vec{U}}^{(2)T} [M]\, \underset{\sim}{\vec{U}}^{(2)} = (1.0 \quad -2.236)\, \dfrac{\rho A l}{6} \begin{bmatrix} 10 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} 1.0 \\ -2.236 \end{Bmatrix}$

$= 2.5879\ \rho A l$

$a_2 = 0.6216 / \sqrt{\rho A l}$, and

$$\vec{U}^{(2)} = \frac{1}{\sqrt{\rho A l}} \begin{Bmatrix} 0.6216 \\ -1.3900 \end{Bmatrix} \qquad (E_{10})$$

Modal matrix:

$$[U] = \frac{1}{\sqrt{\rho A l}} \begin{bmatrix} 0.4952 & 0.6216 \\ 1.1072 & -1.3900 \end{bmatrix} \qquad (E_{11})$$

with

$$[U]^T[M][U] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } [U]^T[K][U] = \frac{E}{\rho l^2}\begin{bmatrix} 1.3554 & 0 \\ 0 & 5.5921 \end{bmatrix}$$

Forced vibration equations:

Equations of motion are $[M]\ddot{\vec{U}} + [K]\vec{U} = \vec{P}$ (E_{12})

Let $\vec{U}(t) = [U]\,\vec{\eta}(t)$ (E_{13})

where $[U]$ = modal matrix and

$\vec{\eta}(t) = \begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \end{Bmatrix}$ = vector of generalized coordinates

Substituting (E_{13}) into (E_{12}) and premultiplying by $[U]^T$ gives the uncoupled equations of motion:

$$\ddot{\eta}_i(t) + \omega_i^2\,\eta_i(t) = Q_i(t) \quad ; \quad i = 1,2 \qquad (E_{14})$$

Where the generalized loads $Q_i(t)$ are given by

$$\vec{Q} = [U]^T\,\vec{P} = \frac{1}{\sqrt{\rho A l}}\begin{bmatrix} 0.4952 & 0.6216 \\ 1.1072 & -1.3900 \end{bmatrix}\begin{Bmatrix} 0 \\ P(t) \end{Bmatrix}$$

$$= \frac{1}{\sqrt{\rho A l}}\begin{Bmatrix} 0.6216 \\ -1.3900 \end{Bmatrix} P(t) \qquad (E_{15})$$

Hence equations of motion become

$$\ddot{\eta}_1 + 1.3554\left(\frac{E}{\rho l^2}\right)\eta_1 = \frac{0.6216}{\sqrt{\rho A l}}\,P(t) \qquad (E_{16})$$

$$\ddot{\eta}_2 + 5.5921\left(\frac{E}{\rho l^2}\right)\eta_2 = -\frac{1.3900}{\sqrt{\rho A l}}\,P(t) \qquad (E_{17})$$

Assume all initial conditions to be zero:

$$\left.\begin{aligned} \vec{U}(t=0) &= [U]\,\vec{\eta}(0) = 0 \\ \dot{\vec{U}}(t=0) &= [U]\,\dot{\vec{\eta}}(0) = 0 \end{aligned}\right\} \Rightarrow \vec{\eta}(0) = \dot{\vec{\eta}}(0) = \vec{0} \qquad (E_{18})$$

Thus the solution of Eqs. (E_{16}) and (E_{17}) can be expressed as

$$\eta_1(t) = \frac{1}{\omega_1}\int_0^t Q_1(\tau)\,\sin\omega_1(t-\tau)\,d\tau$$

$$= \sqrt{\frac{l}{AE}}\,(0.5339)\int_0^t P(\tau)\,\sin\left\{1.1642\sqrt{\frac{E}{\rho l^2}}\,(t-\tau)\right\}.d\tau \qquad (E_{19})$$

$$\eta_2(t) = \frac{1}{\omega_2} \int_0^t Q_2(\tau) \sin \omega_2 (t-\tau) \, d\tau$$

$$= -\sqrt{\frac{l}{AE}} (0.5878) \int_0^t P(\tau) \cdot \sin\left\{\sqrt{\frac{E}{\rho l^2}} \, 2.3648 \, (t-\tau)\right\} d\tau \qquad (E_{20})$$

since

$$P(t) = \begin{cases} P_0 & \text{for } 0 \le t \le t_0 \\ 0 & \text{for } t > t_0 \end{cases} \qquad (E_{21})$$

we can express the solution as

$$\vec{U}(t) = \begin{Bmatrix} U_2(t) \\ U_3(t) \end{Bmatrix} = [U] \vec{\eta}(t) = \frac{1}{\sqrt{\rho A l}} \begin{Bmatrix} 0.4952 \, \eta_1(t) + 0.6216 \, \eta_2(t) \\ 1.1072 \, \eta_1(t) - 1.3900 \, \eta_2(t) \end{Bmatrix}$$

which becomes, in view of Eqs. $(E_{19}) - (E_{21})$:

For $t \le t_0$:

$$U_2(t) = \frac{P_0 l}{AE} \left\{ 0.3816 - 0.2271 \cos 1.1642 \sqrt{\frac{E}{\rho l^2}} \, t - 0.1545 \cos 2.3648 \sqrt{\frac{E}{\rho l^2}} \, t \right\}$$

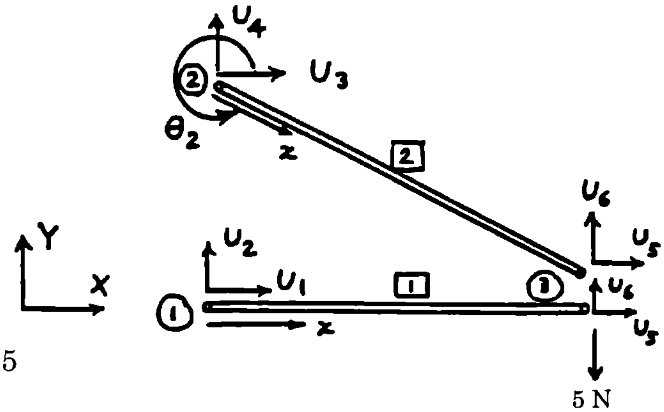
$$U_3(t) = \frac{P_0 l}{AE} \left\{ 0.1622 - 0.5078 \cos 1.1642 \sqrt{\frac{E}{\rho l^2}} \, t + 0.3456 \cos 2.3648 \sqrt{\frac{E}{\rho l^2}} \, t \right\} \qquad (E_{22})$$

For $t > t_0$:

$$U_2(t) = \frac{P_0 l}{AE} \left\{ 0.2271 \cos 1.1642 \sqrt{\frac{E}{\rho l^2}} \, (t-t_0) - 0.2271 \cos 1.1642 \sqrt{\frac{E}{\rho l^2}} \, t + 0.1547 \cos 2.3648 \sqrt{\frac{E}{\rho l^2}} \, (t-t_0) - 0.1547 \cos 2.3648 \sqrt{\frac{E}{\rho l^2}} \, t \right\}$$

$$U_3(t) = \frac{P_0 l}{AE} \left\{ 0.5078 \cos 1.1642 \sqrt{\frac{E}{\rho l^2}} \, (t-t_0) - 0.5078 \cos 1.1642 \sqrt{\frac{E}{\rho l^2}} \, t - 0.3456 \cos 2.3648 \sqrt{\frac{E}{\rho l^2}} \, (t-t_0) + 0.3456 \cos 2.3648 \sqrt{\frac{E}{\rho l^2}} \, t \right\} \qquad (E_{23})$$

12.26 $E^{(1)} = E^{(2)} = 200$ GPa,
$A^{(1)} = A^{(2)} = 0.5 \times 10^{-3}$ m^2,
$\ell^{(1)} = 0.5$ m,



$$\ell^{(2)} = \sqrt{0.2^2 + 0.5^2} = 0.5385 \text{ (m)}$$

$\cos\theta_1 = \cos 0 = 1$,
$\sin\theta_1 = 0$,

$$\cos\theta_2 = (X_3 - X_2)/\ell^{(2)} = \frac{0.5}{0.5385} = 0.9285$$

$$\sin\theta_2 = (Y_3 - Y_2)/\ell^{(2)} = \frac{0 - 10}{0.5385} = -0.3714$$

Element stiffness matrices:

$$[k^{(1)}] = \frac{A^{(1)} E^{(1)}}{\ell^{(1)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \ [\lambda^{(1)}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$[k^{(1)}] = [\lambda^{(1)}]^T [k^{(1)}] [\lambda^{(1)}] = 200 \times 10^6 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \\ U_5 \\ U_6 \end{matrix}$$

(columns: $U_1 \ U_2 \ U_5 \ U_6$)

$$[k^{(2)}] = \frac{A^{(2)} E^{(2)}}{\ell^{(2)}} \begin{bmatrix} +1 & -1 \\ -1 & 1 \end{bmatrix} = 185.7010 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[\lambda^{(2)}] = \begin{bmatrix} 0.9285 & -0.3714 & 0 & 0 \\ 0 & 0 & 0.9285 & -0.3714 \end{bmatrix}$$

$$[k^{(2)}] = [\lambda^{(2)}]^T [k^{(2)}] [\lambda^{(2)}]$$

(columns: $U_3 \ U_4 \ U_5 \ U_6$)

$$= 185.7010 \times 10^6 \begin{bmatrix} 0.8621 & -0.3448 & -0.8621 & 0.3448 \\ -0.3448 & 0.1379 & 0.3448 & -0.1379 \\ -0.8621 & 0.3448 & 0.8621 & -0.3448 \\ 0.3448 & -0.1379 & -0.3448 & 0.1379 \end{bmatrix} \begin{matrix} U_3 \\ U_4 \\ U_5 \\ U_6 \end{matrix}$$

$$[K] = \begin{bmatrix} (200 \times 10^6 + 185.7010 \times 10^6 \times 0.8621) & (0 - 0.3448 \times 185.7010 \times 10^6) \\ (0 - 0.3448 \times 185.7010 \times 10^6) & (0 + 0.1379 \times 185.7010 \times 10^6) \end{bmatrix}$$

$$= \begin{bmatrix} 359.8886 & -64.0297 \\ -64.0297 & 25.6082 \end{bmatrix} \times 10^6$$

Load vector: $\vec{F} = \begin{Bmatrix} 0 \\ -5 \end{Bmatrix}$

Equilibrium equations:

$[K]\ \vec{U} = \vec{F}$

i.e.,

$$\left.\begin{aligned} 359.8886\ U_5 - 64.0297\ U_6 &= 0 \\ -64.0297\ U_5 + 25.6082\ U_6 &= -5 \times 10^{-6} \end{aligned}\right\} \qquad (E_1)$$

Solution is:

$U_5 = -0.0626 \times 10^{-6}$ m

$U_6 = -0.3517 \times 10^{-6}$ m

Axial displacements of elements:

$$\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix}^{(1)} = [\lambda^{(1)}]\ \vec{U}^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} U_1 = 0 \\ U_2 = 0 \\ U_5 \\ U_6 \end{bmatrix} = \begin{Bmatrix} 0 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -0.0626 \times 10^{-6} \end{Bmatrix} \text{ m}$$

$$\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix}^{(2)} = [\lambda^{(2)}]\ \vec{U}^{(2)} = \begin{bmatrix} -0.9285 & -0.3714 & 0 & 0 \\ 0 & 0 & 0.9285 & -0.3714 \end{bmatrix} \begin{bmatrix} U_3 = 0 \\ U_4 = 0 \\ U_5 \\ U_6 \end{bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0.07250 \times 10^{-6} \end{Bmatrix} \text{ m}$$

Stresses in elements:

$$\sigma^{(1)} = E^{(1)}\ \varepsilon^{(1)} = E^{(1)}\ (U_2^{(1)} - U_1^{(1)})/\ell^{(1)} = \frac{200 \times 10^9\ (-0.0626 \times 10^{-6})}{0.5} = -25.04 \text{ KPa}$$

$$\sigma^{(2)} = E^{(2)}\ \varepsilon^{(2)} = E^{(2)}(U_2^{(2)} - U_1^{(2)})/\ell^{(2)} = \frac{(200 \times 10^9)\ (0.07250 \times 10^{-6})}{0.5385} = 26.9266 \text{ KPa}$$

12.27 $E_i = 2.1 \times 10^{11}$ Pa ; $i = 1,2,3$

$l_1 = 2.0$ m, $l_2 = 0.4$ m, $l_3 = 2.4$ m

$$I_1 = I_2 = \frac{\pi}{64}\left[\left(\frac{830}{1000}\right)^4 - \left(\frac{800}{1000}\right)^4\right]$$

$$= 3.189853 \times 10^{-3} \ m^4$$

$$I_3 = \frac{1}{12}\left[350(550)^3 - 320(520)^3\right] \times 10^{-12}$$

$$= 1.103058 \times 10^{-3} \ m^4$$

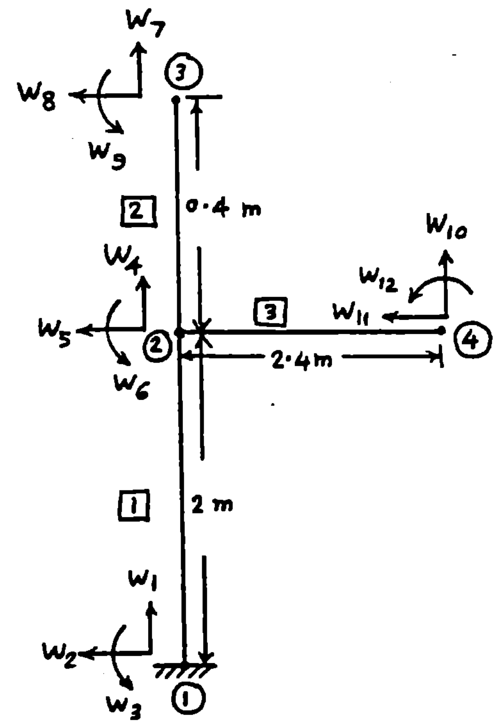
$\rho_i = 7.8 \times 10^3$ kg/m^3 ; $i = 1,2,3$

$$A_1 = A_2 = \frac{\pi}{4}\left[\left(\frac{830}{1000}\right)^2 - \left(\frac{800}{1000}\right)^2\right]$$

$$= 0.038406 \ m^2$$

$$A_3 = (350 \times 550 - 320 \times 520) \times 10^{-6}$$

$$= 0.0261 \ m^2$$



$$\frac{E_1 I_1}{l_1^3} = 8373.3641 \times 10^4 \ ; \quad \frac{E_2 I_2}{l_2^3} = 1046670.516 \times 10^4 \ ;$$

$$\frac{E_3 I_3}{l_3^3} = 1675.6523 \times 10^4$$

Element stiffness matrices:

$$[K^{(e)}] = \frac{E_e I_e}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}$$

Columns: W_2, W_3, W_5, W_6; rows: W_2, W_3, W_5, W_6

$$[K^{(1)}] = \begin{bmatrix} 10.0480 & 10.0480 & -10.0480 & 10.0480 \\ 10.0480 & 13.3974 & -10.0480 & 6.6987 \\ -10.0480 & -10.0480 & 10.0480 & -10.0480 \\ 10.0480 & 6.6987 & -10.0480 & 13.3974 \end{bmatrix} \times 10^8$$

$$
\begin{array}{c} \\ \left[K^{(2)}\right] = \end{array}
\begin{array}{cccc} w_5 & w_6 & w_8 & w_9 \\ \end{array}
$$

$$
\left[K^{(2)}\right] = \begin{bmatrix} 1256.0052 & 251.2010 & -1256.0052 & 251.2010 \\ 251.2010 & 66.9869 & -251.2010 & 33.4935 \\ -1256.0052 & -251.2010 & 1256.0052 & -251.2010 \\ 251.2010 & 33.4935 & -251.2010 & 66.9869 \end{bmatrix} \times 10^8 \quad \begin{matrix} w_5 \\ w_6 \\ w_8 \\ w_9 \end{matrix}
$$

$$
\begin{array}{cccc} w_4 & w_6 & w_{10} & w_{12} \end{array}
$$

$$
\left[K^{(3)}\right] = \begin{bmatrix} 2.0108 & 2.4129 & -2.0108 & 2.4129 \\ 2.4129 & 3.8607 & -2.4129 & 1.9303 \\ -2.0108 & -2.4129 & 2.0108 & -2.4129 \\ 2.4129 & 1.9303 & -2.4129 & 3.8607 \end{bmatrix} \times 10^8 \quad \begin{matrix} w_4 \\ w_6 \\ w_{10} \\ w_{12} \end{matrix}
$$

Axial stiffnesses are given by

$$
\frac{A_1 E_1}{l_1} = 40.3263 \times 10^8, \quad \frac{A_2 E_2}{l_2} = 201.6315 \times 10^8, \quad \frac{A_3 E_3}{l_3} = 22.8375 \times 10^8
$$

Considering the axial stiffnesses of elements 1 and 2 at degree of freedom w_4, the assembled stiffness matrix can be derived as

$$
\begin{array}{ccccccc} w_4 & w_5 & w_6 & w_8 & w_9 & w_{10} & w_{12} \end{array}
$$

$$
[K] = 10^8 \begin{bmatrix} 243.9686 & 0 & 2.4129 & 0 & 0 & -2.0108 & 2.4129 \\ 0 & 1266.0532 & 241.1530 & -1256.0052 & 251.2010 & 0 & 0 \\ 2.4129 & 241.1530 & 84.245 & -251.2010 & 33.4935 & -2.4129 & 1.9303 \\ 0 & -1256.0052 & -251.2010 & 1256.0052 & -251.2010 & 0 & 0 \\ 0 & 251.2010 & 33.4935 & -251.2010 & 66.9869 & 0 & 0 \\ -2.0108 & 0 & -2.4129 & 0 & 0 & 2.0108 & -2.4129 \\ 2.4129 & 0 & 1.9303 & 0 & 0 & -2.4129 & 3.8607 \end{bmatrix} \begin{matrix} w_4 \\ w_5 \\ w_6 \\ w_8 \\ w_9 \\ w_{10} \\ w_{12} \end{matrix}
$$

$$
\frac{\rho_1 A_1 l_1}{420} = 1.4265, \quad \frac{\rho_2 A_2 l_2}{420} = 0.2853, \quad \frac{\rho_3 A_3 l_3}{420} = 1.1633
$$

Element mass matrices:

$$
\left[M^{(e)}\right] = \frac{\rho_e A_e l_e}{420} \begin{bmatrix} 156 & 22 l_e & 54 & -13 l_e \\ 22 l_e & 4 l_e^2 & 13 l_e & -3 l_e^2 \\ 54 & 13 l_e & 156 & -22 l_e \\ -13 l_e & -3 l_e^2 & -22 l_e & 4 l_e^2 \end{bmatrix}
$$

$$[M^{(1)}] = \begin{bmatrix} 222.534 & 62.766 & 77.031 & -37.089 \\ 62.766 & 22.824 & 37.089 & -17.118 \\ 77.031 & 37.089 & 222.534 & -62.766 \\ -37.089 & -17.118 & -62.766 & 22.824 \end{bmatrix} \begin{matrix} W_2 \\ W_3 \\ W_5 \\ W_6 \end{matrix}$$

(columns: W_2, W_3, W_5, W_6)

$$[M^{(2)}] = \begin{bmatrix} 44.5068 & 2.5106 & 15.4062 & -1.4836 \\ 2.5106 & 0.1826 & 1.4836 & -0.1369 \\ 15.4062 & 1.4836 & 44.5068 & -2.5106 \\ -1.4836 & -0.1369 & -2.5106 & 0.1826 \end{bmatrix} \begin{matrix} W_5 \\ W_6 \\ W_8 \\ W_9 \end{matrix}$$

(columns: W_5, W_6, W_8, W_9)

$$[M^{(3)}] = \begin{bmatrix} 181.4748 & 61.4222 & 62.8182 & -36.2950 \\ 61.4222 & 26.8024 & 36.2950 & -20.1018 \\ 62.8182 & 36.2950 & 181.4748 & -61.4222 \\ -36.2950 & -20.1018 & -61.4222 & 26.8024 \end{bmatrix} \begin{matrix} W_4 \\ W_6 \\ W_{10} \\ W_{12} \end{matrix}$$

(columns: W_4, W_6, W_{10}, W_{12})

For axial motion,

$$\frac{\rho_1 A_1 l_1}{3} = 199.71, \qquad \frac{\rho_2 A_2 l_2}{3} = 39.942, \qquad \frac{\rho_3 A_3 l_3}{3} = 162.864.$$

Considering mass matrix terms (corresponding to axial motion of elements 1 and 2) at degree of freedom W_4, the assembled mass matrix can be obtained as:

$$[M] = \begin{bmatrix} 421.1268 & 0 & 61.4222 & 0 & 0 & 62.8182 & -36.2950 \\ 0 & 267.0408 & -60.2554 & 15.4062 & -1.4836 & 0 & 0 \\ 61.4222 & -60.2554 & 49.8090 & 1.4836 & -0.1369 & 36.2950 & -20.1018 \\ 0 & 15.4062 & 1.4836 & 44.5068 & -2.5106 & 0 & 0 \\ 0 & -1.4836 & -0.1369 & -2.5106 & 0.1826 & 0 & 0 \\ 62.8182 & 0 & 36.2950 & 0 & 0 & 181.4748 & -61.4222 \\ -36.2950 & 0 & -20.1018 & 0 & 0 & -61.4222 & 26.8024 \end{bmatrix} \begin{matrix} W_4 \\ W_5 \\ W_6 \\ W_8 \\ W_9 \\ W_{10} \\ W_{12} \end{matrix}$$

(columns: W_4, W_5, W_6, W_8, W_9, W_{10}, W_{12})

Once [K] and [M] are known, the natural frequencies can be found by solving the eigenvalue problem.

(12.28) Force vector:

$\vec{F}^T = \{F_4, F_5, F_6, F_8, F_9, F_{10}, F_{10}\} = \{0, 0, 0, 0, 0, 5000, 500\}$

Equilibrium equations: $[K]\vec{W} = \vec{F}$ ---(E_1)

where $[K]$ is given in the solution of Problem 12.27. The solution of Eqs. (E_1) is given by

$W_4 = 0.2066 \times 10^{-6}$ m, $W_5 = 0.3732 \times 10^{-4}$ m, $W_6 = 0.3731 \times 10^{-4}$ rad,
$W_8 = 0.5224 \times 10^{-4}$ m, $W_9 = 0.3731 \times 10^{-4}$ rad, $W_{10} = 0.1954 \times 10^{-3}$ m,
$W_{12} = 0.1046 \times 10^{-3}$ rad.

Bending stress in element "e":

$$\sigma_{max} = \left(E_e I_e \left.\frac{d^2 w}{dx^2}\right|_e\right) \cdot y_{max,e} \Big/ I_e = E_e \, y_{max,e} \left.\frac{d^2 w}{dx^2}\right|_e$$

$$= E_e \, y_{max,e} \sum_{i=1}^{4} W_i^{(e)} \frac{d^2 N_i^{(e)}}{dx^2}$$

$$= E_e \, y_{max,e} \Big[W_1^{(e)} \left(-\frac{6}{\ell^2} + \frac{12}{\ell^3} x\right) + W_2^{(e)} \left(-\frac{4}{\ell} + \frac{6}{\ell^2} x\right)$$
$$+ W_3^{(e)} \left(\frac{6}{\ell^2} - \frac{12}{\ell^3} x\right) + W_4^{(e)} \left(-\frac{2}{\ell} + \frac{6}{\ell^2} x\right) \Big] \quad ---(E_2)$$

Bending stress in element 1 (e = 1):

$W_1^{(1)} = W_2 = 0$, $W_2^{(1)} = W_3 = 0$, $W_3^{(1)} = W_5 = 0.3732 \times 10^{-4}$ m,
$W_4^{(1)} = W_6 = 0.3731 \times 10^{-4}$ rad

$\ell_1 = 2$ m, $y_{max,1} = 0.415$ m, $E_1 = 2.1 \times 10^{11}$ Pa

At $x = 0$ (at node ①):

$$\sigma_{max} = (2.1 \times 10^{11})(0.415)\left[0.3732 \times 10^{-4} \left(\frac{6}{4}\right) + 0.3731 \times 10^{-4} \left(-\frac{2}{2}\right)\right]$$
$$= 1.6262 \times 10^{6} \text{ Pa}$$

At $x = \ell_1 = 2$ m (at node ②):

$$\sigma_{max} = (2.1 \times 10^{11})(0.415)\left[0.3732 \times 10^{-4} \left(\frac{6}{4} - \frac{12}{4}\right) + 0.3731 \times 10^{-4} \left(-\frac{2}{2} + \frac{6}{2}\right)\right]$$
$$= 1.6245 \times 10^{6} \text{ Pa}$$

Axial stress in element 1 (at node ①):

$$\sigma = \left(\frac{W_4 - W_1}{\ell_1}\right) E_1 = \epsilon_1 E_1 = \left[(0.2066 \times 10^{-6} - 0)/2\right](2.1 \times 10^{11})$$
$$= 0.0217 \times 10^{6} \text{ Pa}$$

Total stress in element 1 $= (1.6262 + 0.0217)10^6 = 1.6479 \times 10^6$ Pa

Bending stress in element 2 ($e = 2$):

$W_1^{(2)} = W_5 = 0.3732 \times 10^{-4}$ m, $\quad W_2^{(2)} = W_6 = 0.3731 \times 10^{-4}$ rad,

$W_3^{(2)} = W_8 = 0.5224 \times 10^{-4}$ m, $\quad W_4^{(2)} = W_9 = 0.3731 \times 10^{-4}$ rad,

$l_2 = 0.4$ m, $\quad y_{max,2} = 0.415$ m, $\quad E_2 = 2.1 \times 10^{11}$ Pa

At $x = 0$:

$$\sigma_{max} = (2.1 \times 10^{11})(0.415)\left[0.3732\left(-\frac{6}{0.16}\right) + 0.3731\left(-\frac{4}{0.4}\right) + 0.5224\left(\frac{6}{0.16}\right) - 0.3731\left(-\frac{2}{0.4}\right)\right]10^{-4} = 32.503 \times 10^6 \text{ Pa}$$

Bending stress in element 3 ($e = 3$):

$W_1^{(3)} = W_4 = 0.2066 \times 10^{-6}$ m, $\quad W_2^{(3)} = W_6 = 0.3731 \times 10^{-4}$ rad,

$W_3^{(3)} = W_{10} = 0.1954 \times 10^{-3}$ m, $\quad W_4^{(3)} = W_{12} = 0.1046 \times 10^{-3}$ rad

$l_3 = 2.4$ m, $\quad y_{max,3} = 0.275$ m, $\quad E_3 = 2.1 \times 10^{11}$ Pa

At $x = 0$ (node ②):

$$\sigma_{max} = (2.1 \times 10^{11})(0.275)\left[0.2066 \times 10^{-6}\left(-\frac{6}{5.76}\right) + 0.3731 \times 10^{-4}\left(-\frac{4}{2.4}\right) + 0.1954 \times 10^{-3}\left(\frac{6}{5.76}\right) + 0.1046 \times 10^{-3}\left(-\frac{2}{2.4}\right)\right]$$

$$= 3.1172 \times 10^6 \text{ Pa}$$

12.29

Fig. 1

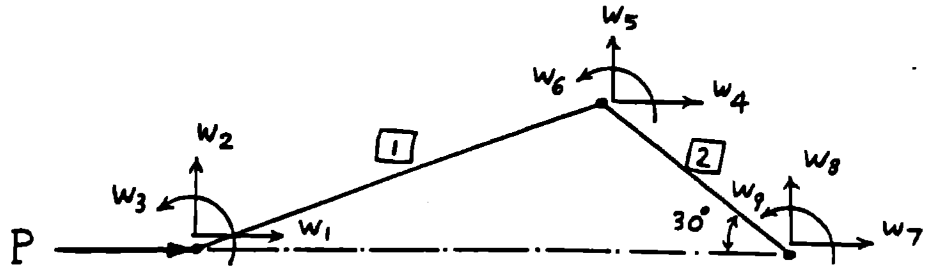
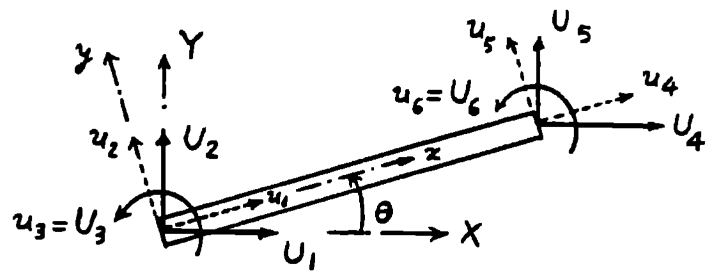


Fig. 2
General beam element in a plane



Pressure on the piston 1 MPa, diameter of the piston 0.3 m, the length of the crank 0.3 m, the length of the connecting rod 1.2 m.

Solution:
For a general beam element in XY plane, we consider four axial nodal displacements U_1, U_2, U_4 and U_5, and two bending nodal displacements U_3 and U_6. By superimposing the stiffness matrices of a bar element and a beam element, the stiffness matrix of the element shown in Fig. 2 can be found as

$$[K] = \frac{E\,I}{\ell^3}\begin{bmatrix} \frac{A\,\ell^2}{I} & 0 & 0 & -\frac{A\,\ell^2}{I} & 0 & 0 \\ 0 & 12 & 6\,\ell & 0 & -12 & 6\,\ell \\ 0 & 6\,\ell & 4\,\ell^{©} & \mathcal{W} & -6\,\ell & 2\,\ell^{©} \\ -\frac{A\,\ell^2}{I} & 0 & 0 & \frac{A\,\ell^2}{I} & 0 & 0 \\ 0 & -12 & -6\,\ell & \mathcal{W} & \mathcal{N}© & -6\,\ell \\ 0 & 6\,\ell & 2\,\ell^{©} & 0 & -6\,\ell & 4\,\ell^{©} \end{bmatrix}\begin{matrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{matrix} \qquad (E_1)$$

$$\begin{matrix} U_1 & U_2 & U_3 & U_4 & U_5 & U_6 \end{matrix}$$

If U_i, i = 1, … , 6 denote the global nodal displacements, we find from Fig. 2,

$$\begin{aligned} u_1 &= U_1 \cos\theta + U_2 \sin\theta \\ u_2 &= -U_1 \sin\theta + U_2 \cos\theta \\ u_3 &= U_3 \\ u_4 &= U_4 \cos\theta + U_5 \sin\theta \\ u_5 &= -U_4 \sin\theta + U_5 \cos\theta \\ u_6 &= U_6 \end{aligned} \qquad (E_2)$$

Defining $\lambda = \cos\theta$ and $\mu = \sin\theta$, Eq. (E_2) can be expressed as

$$\vec{u} = [\underset{\sim}{\lambda}]\ \vec{U}$$

where

$$\vec{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}, \quad \vec{U} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} \quad \text{and} \quad [\underset{\sim}{\lambda}] = \begin{bmatrix} \lambda & \mu & 0 & 0 & 0 & 0 \\ -\mu & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & \mu & 0 \\ 0 & 0 & 0 & -\mu & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (E_3)$$

Globl stiffness matrix of the general beam element can be found as

$$[K^{(e)}] = [\underset{\sim}{\lambda}]^T\ [k]\ [\underset{\sim}{\lambda}] \qquad (E_4)$$

For element 1 :

$E = 200 \times 10^9$ Pa, $\ell = 1.2$ m

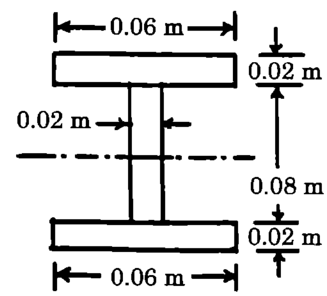
$$I = 2\left[\frac{1}{12}(0.06)(0.02)^3 + (0.06 \times 0.02)(0.04 + 0.01)^2\right] + \frac{1}{12}(0.02)(0.08)^3$$

$$= 6.9333 \times 10^{-6}\ m^4$$

$A = 2\,(0.02 \times 0.06) + 0.02 \times 0.08 = 0.004\ m^2$

$\lambda = \cos 7.1808° = 0.9922$

$\mu = \sin 7.1808° = 0.1250$



For element 2 :

$E = 200 \times 10^9$ Pa, $I = 6.9333 \times 10^{-6}\ m^4$

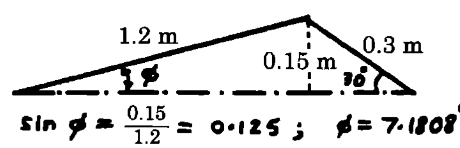
$A = 0.004\ m^2$, $\ell = 0.3$ m,

$\lambda = \cos 330° = 0.866$

$\mu = \sin 330° = -0.5$

Boundary conditions:

$W_2 = W_7 = W_8 = 0$



Load on piston $= \dfrac{\pi}{4}(0.3)^2(10^6) = 70.6858 \times 10^3$ (N)

Load vector:

$$\vec{F} = \begin{Bmatrix} F_1 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_9 \end{Bmatrix} = \begin{Bmatrix} 70685.8 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Element matrices in global system (given by Eq. (E_4)):

$$[k^{(1)}] = 10^8 \times \begin{bmatrix} 6.5640 & 0.8149 & -0.0072 & -6.5640 & -0.8149 & -0.0072 \\ & 0.1990 & 0.0573 & -0.8149 & -0.1990 & 0.0573 \\ & & 0.0462 & 0.0072 & -0.0573 & 0.0231 \\ & \text{symmetric} & & 6.5640 & 0.8149 & 0.0072 \\ & & & & 0.1990 & -0.0573 \\ & & & & & 0.0462 \end{bmatrix} \begin{matrix} W_1 \\ W_2 \\ W_3 \\ W_4 \\ W_5 \\ W_6 \end{matrix}$$

(columns: W_1, W_2, W_3, W_4, W_5, W_6)

$$[k^{(2)}] = 10^9 \times \begin{bmatrix} 2.1541 & -0.8878 & 0.0462 & -2.1541 & 0.8878 & 0.0462 \\ & 1.1289 & 0.0801 & 0.8878 & -1.1289 & 0.0801 \\ & & 0.0185 & -0.0462 & -0.0801 & 0.0092 \\ & & & 2.1541 & -0.8878 & -0.0462 \\ & & & & 1.1281 & -0.0801 \\ & & & & & 0.0185 \end{bmatrix} \begin{matrix} W_4 \\ W_5 \\ W_6 \\ W_7 \\ W_8 \\ W_9 \end{matrix}$$

(columns: W_4, W_5, W_6, W_7, W_8, W_9)

Assembled stiffness matrix (after incorporating boundary conditions):

$$[K] = 10^9 \times \begin{bmatrix} 0.6564 & -0.0007 & -0.6564 & -0.0815 & -0.0007 & 0 \\ & 0.0046 & 0.0007 & -0.0057 & 0.0023 & 0 \\ & & 2.8105 & -0.8064 & 0.0469 & 0.0462 \\ & & & 1.1488 & 0.0743 & 0.0801 \\ & & & & 0.0231 & 0.0092 \\ & & & & & 0.0185 \end{bmatrix} \begin{matrix} W_1 \\ W_3 \\ W_4 \\ W_5 \\ W_6 \\ W_9 \end{matrix}$$

(columns: W_1, W_3, W_4, W_5, W_6, W_9)

Solution of the equilibrium equations $[k]\ \vec{W} = \vec{F}$ gives the following:

$W_1 = 0.0007$ m

$W_3 = 0.0022$ rad

$W_4 = 0.0005$ m

$W_5 = 0.0008$ m

$W_6 = -0.0024$ rad

$W_9 = -0.0035$ rad

Element displacements in local coordinate system:

Element 1: $\lambda = 0.9922$, $\mu = 0.1250$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}^{(1)} = \begin{bmatrix} \lambda & \mu & 0 & 0 & 0 & 0 \\ -\mu & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & \mu & 0 \\ 0 & 0 & 0 & -\mu & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{(1)} \begin{Bmatrix} W_1 = 0.0007 \text{ m} \\ W_2 = 0 \\ W_3 = 0.0022 \text{ rad} \\ W_4 = 0.0005 \text{ m} \\ W_5 = 0.0008 \text{ m} \\ W_6 = 0.0024 \text{ rad} \end{Bmatrix} = \begin{Bmatrix} 0.0007 \text{ m} \\ -0.0001 \text{ m} \\ 0.0022 \text{ rad} \\ 0.0006 \text{ m} \\ 0.0007 \text{ m} \\ -0.0024 \text{ rad} \end{Bmatrix}$$

Element 2: $\lambda = 0.866$, $\mu = -0.5$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}^{(2)} = \begin{bmatrix} \lambda & \mu & 0 & 0 & 0 & 0 \\ -\mu & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & \mu & 0 \\ 0 & 0 & 0 & -\mu & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{(2)} \begin{Bmatrix} W_4 = 0.0005 \text{ m} \\ W_5 = 0.0008 \text{ m} \\ W_6 = 0.0024 \text{ rad} \\ W_7 = 0 \\ W_8 = 0 \\ W_9 = -0.0035 \text{ rad} \end{Bmatrix} = \begin{Bmatrix} 0.00003 \\ 0.0009 \\ -0.0024 \\ 0 \\ 0 \\ -0.0035 \end{Bmatrix}$$

Axial stresses:

Element 1:

$$\sigma(x) = E\left(-\frac{1}{\ell} \quad \frac{1}{\ell}\right)\begin{Bmatrix} u_1 \\ u_4 \end{Bmatrix}^{(1)} = \frac{E}{\ell}(-u_1 + u_4)^{(1)} = \frac{200 \times 10^9}{1.2}(-0.0007 + 0.0006)$$

$$= -16.67 \text{ MPa}$$

Element 2:

$$\sigma(x) = \frac{E}{\ell}(-u_1 + u_4)^{(2)} = \frac{200 \times 10^9}{0.3}(-0.00003 + 0) = -20 \text{ MPa}$$

Bending stresses:

Element 1: $Y_{max} = 0.05$ m, $\ell = 1.2$ m.

$$\sigma(x) = E\,\frac{d^2w(x)}{dx^2}\,Y_{max} = E\,Y_{max}\left\{\left(-\frac{6}{\ell^2} + \frac{12x}{\ell^3}\right)u_2 + \left(-\frac{4}{\ell^2} + \frac{6x}{\ell^3}\right)\ell\,u_3\right.$$

$$\left. + \left(\frac{6}{\ell^2} + \frac{12x}{\ell^3}\right)u_5 + \left(-\frac{2}{\ell^2} + \frac{6x}{\ell^3}\right)\ell\,u_6\right\}$$

at $x = 0$, $\sigma(0) = -7.7984 \times 10^{-7}$ Pa

at $x = 1.2$ m, $\sigma(1.2) = -7.6463 \times 10^{7}$ Pa

Element 2: $Y_{max} = 0.05$ m, $\ell = 0.3$ m.

$$\sigma(x) = E \quad Y_{max} \left\{ \left(-\frac{6}{\ell^2} + \frac{12\,x}{\ell^3} \right) u_2 + \left(-\frac{4}{\ell^2} + \frac{6\,x}{\ell^3} \right) \ell\, u_3 + \left(\frac{6}{\ell^2} - \frac{12\,x}{\ell^3} \right) u_5 + \left(-\frac{2}{\ell^2} + \frac{6\,x}{\ell^3} \right) \ell\, u_6 \right\}$$

at $x = 0$, $\sigma(0) = -7.6463 \times 10^7$ Pa

at $x = 0$, $\sigma(0.3) = -1.4901 \times 10^{-8}$ Pa

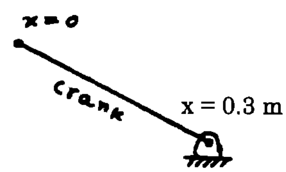
Maximum stresses:

In element 1:

σ_{max} = max (axial ± bending)

$= -1.667 \times 10^7 \pm 7.6463 \times 10^7$

$= -9.3133 \times 10^7$ Pa



In element 2:

σ_{max} = max (axial ± bending)

$= -2 \times 10^7 \pm 7.6463 \times 10^7$

$= -9.6463 \times 10^7$ Pa

(12.30) $m = 5000$ kg, $\ell = 12$ m, $d = 0.6$ m, $t = 0.02$ m, $P_{max} = 700$ kPa

Solution:

$\ell = 12$ m, $E = 200 \times 10^9$ Pa, $P = 7750$ kg/m^3

$$I = \frac{\pi}{64} \{(d + 2\,t)^4 - d^4\} = \frac{\pi}{64} \{(0.6 + 0.04)^4 - 0.6^4\} = 0.0018738 \text{ m}^4$$

$$A = \frac{\pi}{4} \{(d + 2\,t)^2 - d^2\} = \frac{\pi}{4} (0.64^2 - 0.6^2) = 0.038956 \text{ m}^2$$

$M = 5000$ kg, $P_{max} = 700$ kPa

$$\text{Pressure at } x = \left(\frac{x}{\ell} P_{max} \right) \text{ Pa}$$

$$\text{load at } x = \frac{x}{\ell} P_{max} (d + 2\,t) \text{ N/m} = 37333\,x \text{ N/m}$$

Stiffness matrix:

$$[\underset{\sim}{k}] = [k^{(1)}] = \frac{E\,I}{\ell^3} \begin{bmatrix} 12 & 6\,\ell & -12 & 6\,\ell \\ 6\,\ell & 4\,\ell^2 & -6\,\ell & 2\,\ell^2 \\ -12 & -6 & 12 & -6\,\ell \\ 6\,\ell & 2\,\ell^2 & -6\,\ell & 4\,\ell^2 \end{bmatrix} \begin{matrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{matrix}$$

(columns: W_1, W_2, W_3, W_4)

$$\text{where} \quad \frac{E\,I}{\ell^3} = \frac{200 \times 10^9 \times 0.0018738}{12^3} = 216875 \text{ N/m}$$

$$[k] = \frac{EI}{\ell^3}\begin{bmatrix} 12 & -6\ell \\ -6\ell & 4\ell^2 \end{bmatrix} = 216875 \begin{array}{c} \begin{matrix} W_3 & W_4 \end{matrix} \\ \begin{bmatrix} 12 & -72 \\ -72 & 576 \end{bmatrix} \end{array} \begin{matrix} W_3 \\ W_4 \end{matrix}$$

Mass matrix:

$$[\underset{\sim}{M}] = [m^{(1)}] = \frac{\rho A \ell}{420} \begin{array}{c} \begin{matrix} W_1 & W_2 & W_3 & W_4 \end{matrix} \\ \begin{bmatrix} 156 & 22\ell^2 & 54 & -13\ell \\ 22\ell & 4\ell^2 & 13\ell & -3\ell^2 \\ 54 & 13\ell & 156 & -22\ell \\ -13\ell & -3\ell^2 & -22\ell & 4\ell^2 \end{bmatrix} \end{array} \begin{matrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{matrix}$$

where $\dfrac{\rho A \ell}{420} = \dfrac{7.75 \times 10^3 \times 0.038956 \times 12}{420} = 8.6260$ kg

Mass matrix after incorporating $W_1 = W_2 = 0$,

$$[M] = \frac{\rho A \ell}{420} = \begin{bmatrix} 156 & -22\ell \\ -22\ell & 4\ell^2 \end{bmatrix} = 8.6260 \begin{array}{c} \begin{matrix} W_3 & W_4 \end{matrix} \\ \begin{bmatrix} 156 & -264 \\ -264 & 576 \end{bmatrix} \end{array} \begin{matrix} W_3 \\ W_4 \end{matrix}$$

Consistent load vector:

$f_1 = \int_0^{\ell} f(x)\, N_1(x)\, dx$ where $f(x)$ = distributed transverse load

$$= \int_0^{\ell} (37333\, x)\left(1 - 3\frac{x^2}{\ell^2} + 2\frac{x^3}{\ell^3}\right) dx = 37333\left(\frac{3}{20}\ell^2\right)$$

$= 806\ 392.8$ N

$$f_3 = \int_0^{\ell} f(x)\, N_3(x)\, dx = \int_0^{\ell} (37333\, x)\left[\left(3\frac{x}{\ell}\right)^2 - 2\left(\frac{x}{\ell}\right)^3\right] dx$$

$$= 37333\left(\frac{7\ell^2}{20}\right) = 1881583.2 \text{ N}$$

As there is no distributed bending moment, $f_2 = f_4 = 0$.

(a) STATIC ANALYSIS

Equilibrium equations are $[k]\vec{w} = \vec{f}$

i.e.,

$$216875\begin{bmatrix} 12 & -72 \\ -72 & 576 \end{bmatrix}\begin{bmatrix} W_3 \\ W_4 \end{bmatrix} = \begin{bmatrix} f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} 1881583.2 \\ 0 \end{bmatrix}$$

Solution is

$$w_3 = 2.8920 \text{ (m)}$$
$$w_4 = 0.3615 \text{ (rad)}$$

Since $w(x) = N_1(x)\,W_1 + N_2(x)\,W_2 + N_3(x)\,W_3 + N_4(x)\,W_4$

$$\text{Stress} = \sigma = \frac{M\,C}{I} = E\,I\left(\frac{d^2 W}{d\,x^2}\right)\frac{C}{I}$$

with $\dfrac{d^2w}{dx^2} = \sum_{i=1}^{4} \dfrac{d^2 N_i(x)}{d\,x^2}\,W_i$

Since $W_1 = W_2 = 0$, $N_3 = 3\left(\dfrac{x}{\ell}\right)^2 - 2\left(\dfrac{x}{\ell}\right)^3$ and $N_4 = -\dfrac{x^2}{\ell} + \dfrac{x^3}{\ell^2}$ and stress will be maximum at $x = 0$,

$$\sigma_{max}\Big|_{x=0} = E\,c\,\frac{d^2w}{dx^2}\Big|_{x=0} = (200 \times 10^9)\left(\frac{d + 2\,t}{2}\right)\frac{d^2w}{dx^2}\Big|_{x=0}$$
$$= (200 \times 10^9)\,(0.32)\,(0.06025)$$
$$= 3.856 \times 10^9 \text{ (Pa)}$$

Direct compressive stress due to load W:

$$\sigma_c = \frac{W}{A} = \frac{5000 \times 9.8}{0.038956} = 1.257829 \times 10^6 \text{ Pa}$$

∴ Max. compressive stress $= \sigma_c + \sigma_{max} = 3.85726 \times 10^9$ Pa

Max. tensile stress $= \sigma_c + \sigma_{max} = 3.85474 \times 10^9$ Pa

(b) NATURAL FREQUENCY ANALYSIS

By adding the mass of tank, the mass matrix becomes

$$[M] = 8.6260\begin{bmatrix} 156 & -264 \\ -264 & 576 \end{bmatrix} + \begin{bmatrix} 5000 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6345.656 & -2277.264 \\ -2277.264 & 4968.576 \end{bmatrix}$$

Natural frequencies are given by:

$$\left|-\omega^2\,[M] + [K]\right| = 0$$

i.e.,

$$\begin{vmatrix} 2.6025 \times 10^6 - 6345.656\,\omega^2 & -15.615 \times 10^6 + 2277.264\,\omega^2 \\ -15.615 \times 10^6 + 2277.264\,\omega^2 & 124.92 \times 10^6 - 4968.576\,\omega^2 \end{vmatrix} = 0$$

∴ $\omega_1 = 166.6493$ rad/sec

$\omega_2 = 10.5356$ rad/sec

12.31 $A_1 = 0.001\ m^2$, $A_2 = 0.0006\ m^2$,
$E = 200 \times 10^9$ Pa,
$\rho = 7.75 \times 10^3\ kg/m^3$, $\ell_1 = \ell_2 = 1$ m.

Solution:
Consistent mass matrices

$$[m^{(1)}]_c = \frac{\rho_1 A_1 \ell_1}{6} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{7.75 \times 10^3 \times 0.001 \times 1}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= 1.2917 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \end{matrix} \quad (\text{columns } U_1\ U_2)$$

$$[m^{(2)}]_c = \frac{\rho_2 A_2 \ell_2}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{7.75 \times 10^3 \times 0.0006 \times 1}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= 0.775 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{matrix} U_2 \\ U_3 \end{matrix} \quad (\text{columns } U_2\ U_3)$$

Lamped mass matrices

$$[m^{(1)}]_\ell = \frac{\rho_1 A_1 \ell_1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 3.8751 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \end{matrix} \quad (\text{columns } U_1\ U_2)$$

$$[m^{(2)}]_\ell = \frac{\rho_2 A_2 \ell_2}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2.325 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{matrix} U_2 \\ U_3 \end{matrix} \quad (\text{columns } U_2\ U_3)$$

Stiffness matrices

$$[k^{(1)}] = \frac{A_1 E_1}{\ell_1} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{0.001 \times 200 \times 10^9}{1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 0.2 \times 10^9 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \end{matrix} \quad (\text{columns } U_1\ U_2)$$

$$[k^{(2)}] = \frac{A_2 E_2}{\ell_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{0.0006 \times 200 \times 10^9}{1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 0.12 \times 10^9 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} U_2 \\ U_3 \end{matrix} \quad (\text{columns } U_2\ U_3)$$

Assembled matrices (before applying boundary conditions)

$$[\underset{\sim}{K}] = 10^9 \begin{array}{c} \begin{matrix} U_1 & U_2 & U_3 \end{matrix} \\ \begin{bmatrix} 0.2 & -0.2 & 0 \\ -0.2 & 0.32 & -0.12 \\ 0 & -0.12 & 0.12 \end{bmatrix} \end{array} \begin{matrix} \\ U_1 \\ U_2 \\ U_3 \end{matrix}$$

$$[\underset{\sim}{M}]_c = \begin{array}{c} \begin{matrix} U_1 & U_2 & U_3 \end{matrix} \\ \begin{bmatrix} 2.5834 & 1.2917 & 0 \\ 1.2917 & 4.1334 & 0.775 \\ & 0.775 & 1.55 \end{bmatrix} \end{array} \begin{matrix} \\ U_1 \\ U_2 \\ U_3 \end{matrix}$$

$$[\underset{\sim}{M}]_\ell = \begin{array}{c} \begin{matrix} U_1 & U_2 & U_3 \end{matrix} \\ \begin{bmatrix} 3.8751 & 0 & 0 \\ 0 & 6.2001 & 0 \\ 0 & 0 & 2.325 \end{bmatrix} \end{array} \begin{matrix} \\ U_1 \\ U_2 \\ U_3 \end{matrix}$$

Assembled matrices after applying boundary conditions

$$[K] = 10^9 \begin{bmatrix} 0.32 & -0.12 \\ -0.12 & 0.12 \end{bmatrix}$$

$$[M]_c = \begin{bmatrix} 4.1334 & 0.775 \\ 0.775 & 1.55 \end{bmatrix}$$

$$[\underset{\sim}{M}]_\ell = \begin{bmatrix} 6.2001 & 0 \\ 0 & 2.325 \end{bmatrix}$$

Natural frequency with consistent mass matrices:

$$\left| -\omega^2 \begin{bmatrix} 4.1334 & 0.775 \\ 0.775 & 1.55 \end{bmatrix} + 10^9 \begin{bmatrix} 0.32 & -0.12 \\ -0.12 & 0.12 \end{bmatrix} \right| = 0$$

$\therefore \quad \omega_2 = 13.4131 \times 10^3$ rad/sec

$\omega_1 = 4.7937 \times 10^3$ rad/sec

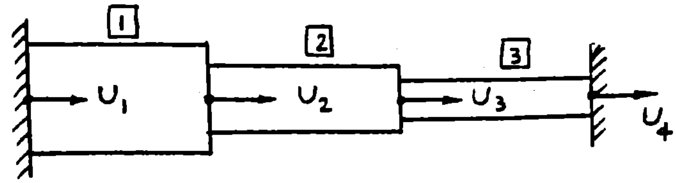
Natural frequencies with lumped mass matrices:

$$\left| -\omega^2 \begin{bmatrix} 6.2001 & 0 \\ 0 & 2.325 \end{bmatrix} + 10^9 \begin{bmatrix} 0.32 & -0.12 \\ -0.12 & 0.12 \end{bmatrix} \right| = 0$$

$\therefore \quad \omega_2 = 9.1224 \times 10^3$ rad/sec

$\omega_1 = 4.4728 \times 10^3$ rad/sec

12.32 With consistent mass matrices:



$$[M^{(e)}]_c = \frac{\rho^{(e)} A^{(e)} \ell^{(e)}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[M^{(1)}]_c = \frac{(7.8 \times 10^3)(0.4 \times 10^{-3})(0.2)}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 0.104 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[M^{(2)}]_c = \frac{(7.8 \times 10^3)(0.2 \times 10^{-3})(0.2)}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 0.052 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[M^{(3)}]_c = \frac{(7.8 \times 10^3)(0.1 \times 10^{-3})(0.2)}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 0.026 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[\underset{\sim}{M}]_c = 0.026 \begin{bmatrix} 8 & 4 & 0 & 0 \\ 4 & 8+4 & 2 & 0 \\ 0 & 2 & 4+2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{matrix}$$

(columns: U_1, U_2, U_3, U_4)

$$[M]_c = 0.026 \begin{bmatrix} 12 & 2 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 0.312 & 0.052 \\ 0.052 & 0.156 \end{bmatrix} \begin{matrix} U_2 \\ U_3 \end{matrix}$$

(columns: U_2, U_3)

$$[k^{(e)}] = \frac{A^{(e)} E^{(e)}}{\ell^{(e)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k^{(1)}] = \frac{(0.4 \times 10^{-3})(2.1 \times 10^{11})}{0.2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 4.2 \times 10^8 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k^{(2)}] = \frac{(0.2 \times 10^{-3})(2.1 \times 10^{11})}{0.2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 2.1 \times 10^8 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k^{(3)}] = \frac{(0.1 \times 10^{-3})(2.1 \times 10^{11})}{0.2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 1.05 \times 10^8 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[\underset{\sim}{K}] = 1.05 \times 10^8 \begin{bmatrix} 4 & -4 & 0 & 0 \\ -4 & 4+2 & -2 & 0 \\ 0 & -2 & 2+1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{matrix}$$

(columns: U_1, U_2, U_3, U_4)

$$[K] = 1.05 \times 10^8 \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix} = 10^8 \begin{bmatrix} 6.3 & -2.1 \\ -2.1 & 3.15 \end{bmatrix} \begin{matrix} U_2 \\ U_3 \end{matrix}$$

(columns U_2, U_3)

Natural frequencies are given by

$$\left| -\omega^2 [M]_c + [K] \right| = 0$$

i.e.

$$\begin{vmatrix} 6.3 - 0.312\lambda & -2.1 - 0.052\lambda \\ -2.1 - 0.052\lambda & 3.15 - 0.156\lambda \end{vmatrix} = 0$$

where $\lambda = 10^{-8}\,\omega^2$

i.e.,

$$0.045968\,\lambda^2 - 2.184\,\lambda + 15.435 = 0$$

This gives $\lambda_1 = 8.6372$, $\lambda_2 = 38.8721$

or $\omega_1 = 2.9389 \times 10^4$ rad/sec , $\omega_2 = 6.2347 \times 10^4$ rad/sec.

With lumped mass matrices:

$$[M^{(e)}]_\ell = \frac{\rho A \ell}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[M^{(1)}]_\ell = \frac{(7.8 \times 10^3)(0.4 \times 10^{-3})(0.2)}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0.312 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[M^{(2)}]_\ell = \frac{(7.8 \times 10^3)(0.2 \times 10^{-3})(0.2)}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0.156 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[M^{(3)}]_\ell = \frac{(7.8 \times 10^3)(0.1 \times 10^{-3})(0.2)}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0.078 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(columns U_1, U_2, U_3, U_4)

$$[\underset{\sim}{M}]_\ell = 0.078 \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4+2 & 0 & 0 \\ 0 & 0 & 2+1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{matrix}$$

(columns U_2, U_3)

$$[M]_\ell = 0.078 \begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0.468 & 0 \\ 0 & 0.234 \end{bmatrix} \begin{matrix} U_2 \\ U_3 \end{matrix}$$

Natural frequencies are given by

$$\left| -\omega^2 [M]_\ell + [K] \right| = 0$$

i.e.,

$$\left| -\omega^2 \begin{bmatrix} 0.468 & 0 \\ 0 & 0.234 \end{bmatrix} + 10^8 \begin{bmatrix} 6.3 & -2.1 \\ -2.1 & 3.15 \end{bmatrix} \right| = 0$$

i.e.,

$$\begin{vmatrix} 6.3 - 0.468\,\lambda & -2.1 \\ -2.1 & 3.15 - 0.234\,\lambda \end{vmatrix} = 0$$

where $\lambda = 10^{-8}\,\omega^2$

i.e.,

$$0.109512\,\lambda^2 - 2.9484\,\lambda + 15.435 = 0$$

This gives $\lambda_1 = 7.1157$, $\lambda_2 = 19.8074$

or $\omega_1 = 2.6675 \times 10^4$ rad/sec , $\omega_2 = 4.4506 \times 10^4$ rad/sec.

12.33

12.34

```
%------ Program Ex12_33_34.m
%------Initialization of values------------------------
A1 = 256e-4 ;
A2 = 16e-4 ;
A3 = 9e-4 ;

E1 = 20e10 ;
E2 = E1 ;
E3 = E1 ;

R1 = 7.8e3 ;
R2 = R1 ;
R3 = R1 ;

L1 = 3 ;
L2 = 2 ;
L3 = 1 ;

%------Definition of [K]------------------------------

K11 = A1*E1/L1+A2*E2/L2 ;
K12 = -A2*E2/L2 ;
K13 = 0 ;

K21 = K12 ;
K22 = A2*E2/L2+A3*E3/L3 ;
K23 = -A3*E3/L3 ;

K31 = K13 ;
K32 = K23 ;
K33 = A3*E3/L3 ;

K = [ K11 K12 K13; K21 K22 K23; K31 K32 K33 ]
```

```
%-------- Calculation of matrix

P = [ 0 0 500]'

U = inv(K)*P

%------- Definition of [M]-------------------

M11 = (2*R1*A1*L1+2*R2*A2*L2)/6;
M12 = (R2*A2*L2)/6;
M13 = 0;
M21 = M12;
M22 = (2*R2*A2*L2+2*R3*A3*L3)/6;
M23 = R3*A3*L3;

M31 = M13;
M32 = M23;
M33 = 2*M23;

M= [M11 M12 M13; M21 M22 M23; M31 M32 M33 ]

MI = inv(M);

KM = MI*K;

%-------------Calculation of eigen vector and eigenvalue-------------------

[L,V] = eig(KM)
```

```
Results of Ex12_33 and Ex12_34
**********************
>> Ex12_33

K =

  1.0e+009 *

    1.8667   -0.1600         0
   -0.1600    0.3400   -0.1800
         0   -0.1800    0.1800


P =

     0
     0
   500


U =

  1.0e-005 *

    0.0293
    0.3418
    0.6196


M =

  208.0000    4.1600         0
    4.1600   10.6600    7.0200
         0    7.0200   14.0400
```

```
L =

   -0.0253    0.6914    0.0772
    0.8009   -0.1479    0.5712
   -0.5982   -0.7072    0.8172


V =

  1.0e+007 *

    9.0719         0         0
         0    0.9178         0
         0         0    0.2860
```

12.35

```
>> PROGRAM17
NATIONAL FREQUENCIES OF THE STEPPED BEAMS
1.0892e+003 3.5774e+003 7.8776e+003 1.9602e+004
MODE SHAPES
1    2.3905e-004    5.5663e-005   8.4828e-005   -5.4254e-004
2   -1.1061e-005    3.3146e-004   9.7314e-005   -1.1524e-004
3   -2.6780e-005   -3.3777e-004   4.8527e-005    64908e-005
4    9.6166e-006    8.0727e-005   2.8116e-005    5.7820e-004
```

12.36

```
NATURAL FREQUENCIES OF THE STEPPED BEAM
1089.203865 3577.355406 7877.612285 19602.488436
MODE SHAPES
1    0.0002390496    0.0000556632   0.0000848280   -0.0005425426
2   -0.0000110609    0.0003314590   0.0000973142   -0.0001152391
3   -0.0000267802   -0.0003377731   0.0000485271    0.0000649080
4    0.0000096166    0.0000807270   0.0000281164    0.0005781958
```

12.37

```
C
========================================================================
C
C PROGRAM 17.F
C PROGRAM FOR FINITE ELEMENT VIBRATION ANALYSIS OF STEPPED BEAM
C
C
========================================================================
C DIMENSIONS: XL(NE),XI(NE),A(NE),XMAS(NE),BJ(NE,4),XM(4,4),XK(4,4)
C             AI(4,8),AIT(8,4),XKA(4,8),XMA(4,8),AKA(8,8),AMA(8,8),
C             BIGM(N,N),BIGK(N,N),BM(ND,ND),BK(ND,ND)
C             U(ND,ND),UI(ND,ND),UTI(ND,ND),BMU(ND,ND),UMU(ND,ND),
C             XF(ND,ND),EV(ND,ND)
C NE = NUMBER OF ELEMENTS
C N  = TOTAL NUMBER OF DEGREES OF FREEDOM
C ND = TOTAL NUMBER OF DEGREES OF FREEDOM AFTER DELETING ZERO D.O.F.
        DIMENSION XL(3,XI(3),A(3),XMAS(3),BJ(3,4),XM(4,4),XK(4,4),
  2 AI(4,8),AIT(8,4),XKA(4,8),XMA(4,8),AKA(8,8),AMA(8,8),BIGM(8,8,)
  3 BIGK(8,8),BM(4,4), BK(4,4)
    DIMENSION U(4,4), UI(4,4),UTI(4,4),BMU(4,4),UMU(4,4),XF(4,4),
  2 EV(4,4)
    INTEGER BJ
    DATA XL/0.8,0.5,0.2/
    DATA XI/0.0000083333,0.0000034133,0.00000052083/
    DATA A/16.0,9.0,4.0/
    DATA BJ/1,3,5,2,4,6,3,5,7,4,6,8/
    E=70E+09
    RHO=3000
    DO 10 I=1,3
 10    XMAS(I)=A(I)*RHO
       DO 20 I=1,8
       DO 20 J=1,8
       BIGM(I,J)=0.0
 20    BIGK(I,J)=0.0
       DO 100 II=1,3
       DO 30 I=1,4
       DO 30 J=1,8
 30    AI(I,J)=0.0
       I1=BJ(II,1)
       I2=BJ(II,2)
       I3=BJ(II,3)
       I4=BJ(II,4)
       AI(1,I1)=1.0
       AI(2,I2)=1.0
       AI(3,I3)=1.0
       AI(4,I4)=1.0
       XM(1,1)=156.0
       XM(1,2)=22.0*XL(II)
```

```
      XM(1,3)=54.0
      XM(1,4)=-13.0*XL(II)
      XM(2,2)=4.0*(XL(II)**2)
      XM(2,3)=13.0*XL(II)
      XM(2,4)=-3.0*(XL(II)**2)
      XM(3,3)=156.0
      XM(3,4)=-22.0*XL(II)
      XM(4,4)=4.0*(XL(II)**2)
      XK(1,1)=12.0
      XK(1,2)=6.0*XL(II)
      XK(1,3)=-12.0
      XK(1,4)=6.0*XL(II)
      XK(2,2)=4.0*(XL(II)**2)
      XK(2,3)=-6.0*XL(II)
      XK(2,4)=2.0*(XK(II)**2)
      XK(3,3)=12.0
      XK(3,4)=-6.0*XL(II)
      XK(4,4)=4.0*(XL(II)**2)
      DO 40 I=1,4
      DO 40 J=1,4
      XM(J,I)=XM(I,J)
 40   XK(J,I)=XK(I,J)
      DO 50 I=1,4
      DO 50 J=1,4
      XM(I,J)=(XMAS(II)*XL(II)/420.0)*XM(I,J)
 50   XK(I,J)=(E*XI(II)/(XL(II)**3))*XK(I,J)
      DO 60 I=1,8
      DO 60 J=1,4
 60   AIT(I,J)=AI(J,I)
      CALL MATMUL (XKA,XK,AI,4,4,8)
      CALL MATMUL (XMA,XM,AI,4,4,8)
      CALL MATMUL (AKA,AIT,XKA,8,4,8)
      CALL MATMUL (AMA,AIT,XMA,8,4,8)
      DO 70 I=1,8
      DO 70 J=1,4
      BIGM(I,J)=BIGM(I,J)+AMA(I,J)
 70   BIGK(I,J)=BIGK(I,J)+AKA(I,J)
 100  CONTINUE
C APPLICATION OF BOUNDARY CONDITIONS
C ROWS AND COLUMNS CORRESPONDING TO ZERO DISPLACEMENTS ARE DELETED
      DO 110 I=1,4
      DO 110 J=1,4
      BM(I,J)=BIGM(I+2,J+2)
 110  BK(I,J)=BIGK(I+2,J+2)
C DECOMPOSING THE MATRIX [BK] INTO TRIANGULAR MATRICES
      ND=4
      CALL DECOMP (BK,U,ND)
```

```
C FINDING THE INVERSE OF THE UPPER TRIANGULAR MATRIX [U]
      DO 120 I=1,ND
      DO 120 J=1,ND
  120 UI(I,J)=0.0
      DO 130 I=1,ND
  130 UI(I,I)=1.0/U(I,I)
      DO 150 J=1,ND
      DO 150 II=1,ND
      I=ND-II+1
      IF(I .GE. J) GO TO 150
      IP=I+1
      SUM=0.0
      DO 140 K=IP,J
  140 SUM=SUM+U(I,K)*UI(K,J)
      UI(I,J)=-SUM/U(I,I)
  150 CONTINUE
      DO 160 I=1,ND
      DO 160 J=1,ND
  160 UTI(I,J)=UI(J,I)
      CALL MATMUL (BMU,BM,UI,ND,ND,ND)
      CALL MATMUL (UMU,UTI,BMU,ND,ND,ND)
      CALL JACOBI(UMU,ND,EV,1.0E-05,200)
      CALL MATMUL (XF,UI,EV,ND,ND,ND)
      DO 170 I=1,ND
  170 UMU(I,I)=SQRT(1.0/UMU(I,I))
      PRINT 180,(UMU(I,I),I=1,ND)
  180 FORMAT (//,40H NATURAL FREQUENCIES OF THE STEPPED BEAM,//,
     2 4(1X,E15.6))
      PRINT 190
  190 FORMAT(//,12H MODE SHAPES)
      DO 200 J=1,ND
  200 PRINT 210, J, (XF(I,J),I=1,ND)
  210 FORMAT (/.I4,5X,4(1X,E15.6))
      STOP
      END
C ==============================================================
C
C SUBROUTINE MATMUL
C
C ==============================================================
C MATRIX MULTIPLICATION SUBROUTINE: A = B * C
C B(L,M) AND C(M,N) ARE INPUT MATRICES, A(L,N) IS OUTPUT MATRIX
      SUBROUTINE MATMUL (A,B,C,L,M,N)
      DIMENSION A(L,N),B(L,M),C(M,N)
      DO 10 I=1,L
      DO 10 J=1,N
      A(I,J)=0.0
      DO 10 K=1,M
```

```
  10      A(I,J)=A(I,J)+B(I,K)*C(K,J)
          RETURN
          END
C =================================================================
C
C SYBROUTINE JACOBI
C
C =================================================================
          SUBROUTINE JACOBI (D,N,E,EPS,ITMAX)
          DIMENSION D(N,N),E(N,N)
          ITER=0
          DO 110 I=1,N
          DO 110 J=1,N
          E(I,I)=0.0
  110     E(I,J)=1.0
  120     ZZ=0.0
          NM1=N-1
          DO 130 I=1,NM1
          IP1=I+1
          DO 130 J=IP1,N
          IF (ABS(D(I,J)) .LE. ZZ) GO TO 130
          ZZ=ABS(D(I,J))
          IR=I
          IC=J
  130     CONTINUE
          IF (ITER .EQ. 0) YY=ZZ*EPS
          IF (ZZ .LE. YY) GO TO 210
          DIF=D(IR,IR) - D(IC,IC)
          TANZ=(-DIF+SQRT(DIF**2+4.0*ZZ**2))/(2.0*D(IR,IC))
          COSZ=1.0/SQRT(1.0+TANZ**2)
          SONZ=COSZ*TANZ
          DO 140 I=1,N
          ZZZ=E(I,IR)
          E(I,IR)=COSZ*ZZZ+SINZ*E(I,IC)
  140     E(I,IC)=COSZ*E(I,IC)-SINZ*ZZZ
          I=1
  150     IF (I .EQ. IR) GO TO 160
          YYY=D(I,IR)
          D(I,IR)=COSZ*YYY+SINZ*D(I,IC)
          D(I,IC)=COSZ*D(I,IC)-SINZ*YYY
          I=I+1
          GO TO 150
  160     I=IR+1
  170     IF (I .EQ .IC)GO TO 180
          YYY=D(IR,I)
          D(IR,I)=COSZ*YYY+SINZ*D(I,IC)
          D(I,IC)=COSZ*D(I,IC)-SINZ*YYY
```

```
      I=I+1
      GO TO 170
180   I=IC+1
190   IF (I .GT. N) GO TO 200
      ZZZ=D(IR,I)
      D(IR,I)=COSZ*ZZZ+SINZ*D(IC,I)
      D(IC,I)=COSZ*D(IC,I)-SINZ*ZZZ
      I=I+1
      GO TO 190
200   YYY=D(IR,IR)
      D(IR,IR)=YYY*COSZ**2+D(IR,IC)*2.0*COSZ*SINZ+D(IC,IC)
    2 *SINZ**2
      D(IC,IC)=D(IC,IC)*COSZ**2+YYY*SINZ**2-D(IR,IC)*2.0*COSZ*SINZ
      D(IR,IC)=0.0
      ITER=ITER+1
      IF (ITER .LT. ITMAX) GO TO 120
210   RETURN
      END
C ================================================================
C
C SUBROUTINE DECOMP
C
C ================================================================
      SUBROUTINE DECOMP (A,U,N)
      DIMENSION A(N,N),U(N,N)
      DO 10 I=1,N
      DO 10 J=1,N
10    U(I,J)=0.0
      U(1,1)=SQRT(A(1,1))
      DO 90 J=2,N
90    U(1,J)=A(1,J)/U(1,1)
      DO 40 I=2,N
      IM=I-1
      SUM=0.0
      DO 30 K=1,IM
30    SUM=SUM+U(K,I)**2
      U(I,I)=SQRT(A(I,I)-SUM)
      J=I+1
      SUM=0.0
      DO 50 K=1,IM
50    SUM=SUM+U(K,I)*U(K,J)
      U(I,J)=(A(I,J)-SUM)/U(I,I)
40    CONTINUE
      RETURN
      END
```

NATURAL FREQUENCIES OF THE STEPPED BEAM

0.108920E+04 0.357736E+04 0.787761E+04 0.196024E+05

MODE SHAPES

1	0.239050E-03	0.556632E-04	0.848280E-04	-0.542542E-03
2	-0.110609E-04	0.331459E-03	0.973142E-04	-0.115239E-03
3	-0.267802E-04	-0.337773E-03	0.485271E-04	0.649080E-04
4	0.961657E-05	0.807270E-04	0.281164E-04	0.5781196E-03

12.38

```
C ====================================================================
C
C PROBLEM 12.38
C GENERATION OF ASSEMBLED STIFFNESS MATRIX OF A PLANAR TRUSS
C
C ====================================================================
C NE = NUMBER OF ELEMENTS
C N = TOTAL NUMBER OF DEGREES OF FREEDOM
C NN = TOTAL NUMBER OF NODES
C A(I) = AREA OF I TH ELEMENT
C EL(I) = LENGTH OF I TH ELEMENT
C DIMENSIONS:
C      BIGK(N,N),XK(4,4),A(NE),EL(NE),CX(NN),CY(NN),LOC(NE,2),AI(4,N),
C      AIT(N,4),XKA(4,N),AKA(N,N)
C
C ***** INPUT DATA *****
      DIMENSION BIGK(8,8),XK(4,4),A(4),EL(4),CX(4),CY(4),LOC(4,2),
     2 AI(4,8),AIT(8,4),XKA(4,8),AKA(8,8)
      DATA NE,N,NN/4,8,4/
      E=2.0E+6
      DATA A/2.0,2.0,1.0,1.0/
      DATA CX/0.0,50.0,100.0,200.0/
      DATA CY/0.0,50.0,0.0,100.0/
      DATA LOC/1,2,2,3,2,3,4,4/
C ***** END OF INPUT DATA *****
      DO 10 I=1,N
      DO 10 J=1,N
 10   BIGK(I,J)=0.0
      DO 100 I=1,NE
      I1=LOC(I,1)
      I2=LOC(I,2)
      EL(I)=SQRT((CX(I2)-CX(I1))**2+(CY(I2)-CY(I1))**2)
```

```
      XLIJ=(CX(I2)-CX(I1))/EL(I)
      XMIJ=(CY(I2)-CY(I1))/EL(I)
      XK(1,1)=XLIJ**2
      XK(1,2)=XLIJ*XMIJ
      XK(1,3)=-XK(1,1)
      XK(1,4)=-XK(1,2)
      XK(2,1)=XK(1,2)
      XK(2,2)=XMIJ**2
      XK(2,3)=-XK(2,1)
      XK(2,4)=-XK(2,2)
      XK(3,1)=-XK(1,1)
      XK(3,2)=-XK(1,2)
      XK(3,3)=XK(1,1)
      XK(3,4)=XK(1,2)
      XK(4,1)=-XK(2,1)
      XK(4,2)=-XK(2,2)
      XK(4,3)=XK(2,1)
      XK(4,4)=XK(2,2)
      DO 20 II=1,4
      DO 20 JJ=1,4
 20   XK(II,JJ)=XK(II,JJ)*A(I)*E/EL(I)
      DO 30 II=1,4
      DO 30 JJ=1,N
 30   AI(II,JJ)=0.0
      I1=LOC(I,1)*2-1
      I2=LOC(I,1)*2
      I3=LOC(I,2)*2-1
      I4=LOC(I,2)*2
      AI(1,I1)=1.0
      AI(2,I2)=1.0
      AI(3,I3)=1.0
      AI(4,I4)=1.0
      DO 40 II=1,N
      DO 40 JJ=1,4
 40   AIT(II,JJ)=AI(JJ,II)
      CALL MATMUL (XKA,XK,AI,4,4,N)
      CALL MATMUL (AKA,AIT,XKA,N,4,N)
      DO 50 II=1,N
      DO 50 JJ=1,N
 50   BIGK(II,JJ)=BIGK(II,JJ)+AKA(II,JJ)
 100  CONTINUE
      DO 120 II=1,N
 120  PRINT 130, (BIGK(II,JJ),JJ=1,N)
 130  FORMAT (1X,10(1X,E8.2))
      STOP
      END
```

```
0.28E+05 0.28E+05 -.28E+05 -.28E+05 0.00E+00 0.00E+00 0.00E+00 0.00E+00
0.28E+05 0.28E+05 -.28E+05 -.28E+05 0.00E+00 0.00E+00 0.00E+00 0.00E+00
-.28E+05 -.28E+05 0.68E+05 0.38E+04 -.28E+05 0.28E+05 -.11E+05 -.38E+04
-.28E+05 -.28E+05 0.38E+04 0.58E+05 0.28E+05 -.28E+05 -.38E+04 -.13E+04
0.00E+00 0.00E+00 -.28E+05 0.28E+05 0.35E+05 -.21E+05 -.71E+04 -.71E+04
0.00E+00 0.00E+00 0.28E+05 -.28E+05 -.21E+05 0.35E+05 -.71E+04 -.71E+04
0.00E+00 0.00E+00 -.11E+05 -.38E+04 -.71E+04 -.71E+04 0.18E+05 0.11E+05
0.00E+00 0.00E+00 -.38E+04 -.13E+04 -.71E+04 -.71E+04 0.11E+05 0.83E+04
```

12.39 The general program is used to solve the example of section 12.10. The program and output are given.

```
C =====================================================================
C
C PROBLEM 12.39
C PROGRAM FOR FINITE ELEMENT VIBRATION ANALYSIS OF STEPPED BEAM
C
C =====================================================================
C DIMENSIONS: XL(NE),XI(NE),A(NE),XMAS(NE),BJ(NE,4),XM(4,4),XK(4,4),
C             AI(4,N),AIT(N,4),XKA(4,N),XMA(4,N),AKA(N,N),AMA(N,N),
C             BIGM(N,N),BIGK(N,N),BM(ND,ND),BK(ND,ND)
C             U(ND,ND),UI(ND,ND),UTI(ND,ND),BMU(ND,ND),UMU(ND,ND),
C             XF(ND,ND),EV(ND,ND)
C NE = NUMBER OF STEPS = NUMBER OF ELEMENTS
C ND = NUMBER OF DEGREES OF FREEDOM (EXCLUDING THE FIXED D.O.F.)
C N  = TOTAL NUMBER OF DEGREES OF FREEDOM
C XL(I) = LENGTH OF I TH STEP OR ELEMENT
C XI(I) = AREA MOMENT OF INERTIA OF I TH STEP OR ELEMENT
C A(I) = CROSS-SECTIONAL AREA OF I TH STEP OR ELEMENT
C RHO = MASS DENSITY OF THE MATERIAL
C E = YOUNG'S MODULUS
C BJ(I,J) = GLOBAL D.O.F. NUMBER CORRESPONDING TO THE LOCAL J TH
C           D.O.F. OF ELEMENT I
C ***** INPUT DATA *****
      DIMENSION XL(3),XI(3),A(3),XMAS(3),BJ(3,4),XM(4,4),XK(4,4),
     2 AI(4,8),AIT(8,4),XKA(4,8),XMA(4,8),AKA(8,8),AMA(8,8),
     3 BIGM(8,8),BIGK(8,8),BM(4,4),BK(4,4)
      DIMENSION U(4,4),UI(4,4),UTI(4,4),BMU(4,4),UMU(4,4),XF(4,4),
     2 EV(4,4)
      INTEGER BJ
      NE=3
      DATA XL/40.0,32.0,24.0/
      DATA XI/1.333333,6.75,0.083333/
      DATA A/4.0,9.0,1.0/
      DO 2 I=1,NE
      BJ(I,1)=I*2-1
      BJ(I,2)=I*2
      BJ(I,3)=I*2+1
      BJ(I,4)=I*2+2
 2    CONTINUE
      E=30.0E+06
      RHO=0.283/386.4
C ***** END OF INPUT DATA *****
      N=NE*2+2
      ND=NE*2-2
      DO 10 I=1,NE
 10   XMAS(I)=A(I)*RHO
      DO 20 I=1,N
      DO 20 J=1,N
      BIGM(I,J)=0.0
 20   BIGK(I,J)=0.0
      DO 100 II=1,NE
      DO 30 I=1,4
      DO 30 J=1,N
 30   AI(I,J)=0.0
      I1=BJ(II,1)
      I2=BJ(II,2)
      I3=BJ(II,3)
```

```
      I4=BJ(II,4)
      AI(1,I1)=1.0
      AI(2,I2)=1.0
      AI(3,I3)=1.0
      AI(4,I4)=1.0
      XM(1,1)=156.0
      XM(1,2)=22.0*XL(II)
      XM(1,3)=54.0
      XM(1,4)=-13.0*XL(II)
      XM(2,2)=4.0*(XL(II)**2)
      XM(2,3)=13.0*XL(II)
      XM(2,4)=-3.0*(XL(II)**2)
      XM(3,3)=156.0
      XM(3,4)=-22.0*XL(II)
      XM(4,4)=4.0*(XL(II)**2)
      XK(1,1)=12.0
      XK(1,2)=6.0*XL(II)
      XK(1,3)=-12.0
      XK(1,4)=6.0*XL(II)
      XK(2,2)=4.0*(XL(II)**2)
      XK(2,3)=-6.0*XL(II)
      XK(2,4)=2.0*(XL(II)**2)
      XK(3,3)=12.0
      XK(3,4)=-6.0*XL(II)
      XK(4,4)=4.0*(XL(II)**2)
      DO 40 I=1,4
      DO 40 J=1,4
      XM(J,I)=XM(I,J)
 40   XK(J,I)=XK(I,J)
      DO 50 I=1,4
      DO 50 J=1,4
      XM(I,J)=(XMAS(II)*XL(II)/420.0)*XM(I,J)
 50   XK(I,J)=(E*XI(II)/(XL(II)**3))*XK(I,J)
      DO 60 I=1,N
      DO 60 J=1,4
 60   AIT(I,J)=AI(J,I)
      CALL MATMUL (XKA,XK,AI,4,4,N)
      CALL MATMUL (XMA,XM,AI,4,4,N)
      CALL MATMUL (AKA,AIT,XKA,N,4,N)
      CALL MATMUL (AMA,AIT,XMA,N,4,N)
      DO 70 I=1,N
      DO 70 J=1,N
      BIGM(I,J)=BIGM(I,J)+AMA(I,J)
 70   BIGK(I,J)=BIGK(I,J)+AKA(I,J)
 100  CONTINUE
C APPLICATION OF BOUNDARY CONDITIONS
C ROWS AND COLUMNS CORRESPONDING TO ZERO DISPLACEMENTS ARE DELETED
      DO 110 I=1,ND
      DO 110 J=1,ND
      BM(I,J)=BIGM(I+2,J+2)
 110  BK(I,J)=BIGK(I+2,J+2)
C DECOMPOSING THE MATRIX [BK] INTO TRIANGULAR MATRICES
      CALL DECOMP (BK,U,ND)
C FINDING THE INVERSE OF THE UPPER TRIANGULAR MATRIX [U]
      DO 120 I=1,ND
```

```
      DO 120 J=1,ND
120   UI(I,J)=0.0
      DO 130 I=1,ND
130   UI(I,I)=1.0/U(I,I)
      DO 150 J=1,ND
      DO 150 II=1,ND
      I=ND-II+1
      IF (I .GE. J) GO TO 150
      IP=I+1
      SUM=0.0
      DO 140 K=IP,J
140   SUM=SUM+U(I,K)*UI(K,J)
      UI(I,J)=-SUM/U(I,I)
150   CONTINUE
      DO 160 I=1,ND
      DO 160 J=1,ND
160   UTI(I,J)=UI(J,I)
      CALL MATMUL (BMU,BM,UI,ND,ND,ND)
      CALL MATMUL (UMU,UTI,BMU,ND,ND,ND)
      CALL JACOBI (UMU,ND,EV,1.0E-05,200)
      CALL MATMUL (XF,UI,EV,ND,ND,ND)
      DO 170 I=1,ND
170   UMU(I,I)=SQRT(1.0/UMU(I,I))
      PRINT 180, (UMU(I,I),I=1,ND)
180   FORMAT (//,40H NATURAL FREQUENCIES OF THE STEPPED BEAM,//,
     2 4(1X,E15.6))
      PRINT 190
190   FORMAT (//,12H MODE SHAPES)
      DO 200 J=1,ND
200   PRINT 210, J,(XF(I,J),I=1,ND)
210   FORMAT (/,I4,5X,4(1X,E15.6))
      STOP
      END
```

```
NATURAL FREQUENCIES OF THE STEPPED BEAM

   0.160083E+03    0.617460E+03    0.225198E+04    0.712653E+04


MODE SHAPES

   1        0.103333E-01    0.189147E-03    0.141626E-01    0.445258E-04

   2       -0.376593E-02    0.202976E-03    0.471097E-02    0.259495E-03

   3        0.167902E-03   -0.181687E-03    0.135672E-02    0.207580E-03

   4        0.182648E-03    0.607247E-04    0.373775E-03    0.163869E-03
```

12.40 Program 17.F is used by replacing three lines (data XL, XI and A). The new lines are

```
DATA XL/0.2,0.2,0.2/
DATA XI/0.133333E-07,0.133333E-07,0.133333E-07/
DATA A/0.0004,0.0004,0.0004/
```

The density and Young's Modulus, from Section 12.10, are 7880 kg/m³ and 200 GPa respectively.

The results are given below.

NATURAL FREQUENCIES OF THE STEPPED BEAM

0.205922E+04 0.495886E+04 0.134511E+05 0.212338E+05

MODE SHAPES

1	0.446568E-05	0.179093E-04	0.446568E-05	-0.179093E-04
2	-0.216519E-05	0.108159E-04	0.216519E-05	0.108159E-04
3	0.240286E-06	0.207140E-04	0.240167E-06	-0.271826E-04
4	-0.179093E-06	0.108159E-04	0.207230E-06	0.204584E-03

Exact natural frequencies (from chapter 8):

$W_1 = 1807.6675$, $W_2 = 4982.9053$, $W_3 = 8883.9889$, $W_4 = 16147.8019$

12.41 Helicopter cross section 0.02 m × 0.3 m, length 1.2 m.

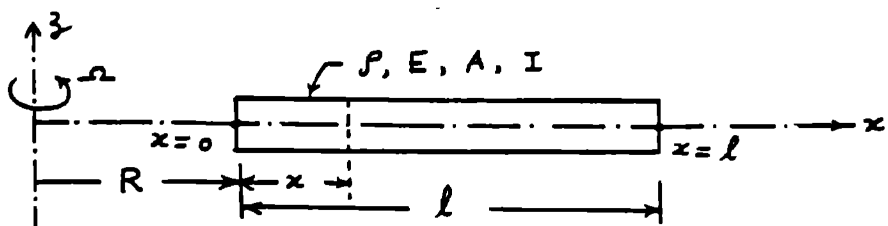


Fig. 1

(i) Rotating beam element strain energy due to rotation:

The rotation of the beam induces an axial force P in the beam due to centrifugal action. If the beam bends in xz-plane as shown in Fig. 2, the change in the horizontal projection of an element of length "dz" is given by

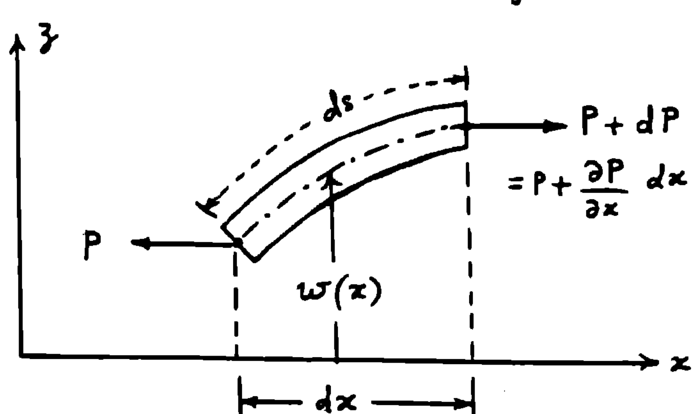


Fig. 2

$$ds - dx = \left\{(d\,x)^2 + \frac{\delta\,w}{\delta\,x}\,dx\right\}^{\frac{1}{2}} - dx \approx \frac{1}{2}\left(\frac{\delta\,w}{\delta\,x}\right)^2 dx... \qquad (E_1)$$

Since the axial force P acts against the change in the horizontal projection, the work done by P is

$$\frac{1}{2}\int_0^{\ell} P\,(x)\left(\frac{\delta\, w}{\delta\, x}\right)^2 dx$$

where

$$P(x) = \int_{x_e + x}^{\ell + x_e} \frac{\rho\, A}{g}\, \Omega^2\, \zeta\;\; d\zeta \quad \text{where } \rho = \text{weight density}$$

$$= \frac{\rho\, A\, \Omega^2}{2\, g}\; [(\ell + R)^2 - (R + x)^2] \tag{E_2}$$

Work done by the transverse distributed force $P_w(x)$ can be expressed as

$$\int_0^{\ell} P_w(x)\; w(x)\; dx \quad \text{where } P_w(x) = \frac{\rho\, A\, \Omega^2}{g}\; w(x) \tag{E_3}$$

If w_i, i = 1, 2, 3, 4 denote the nodal displacements of the beam element, the transverse displacement w(x) can be expressed as

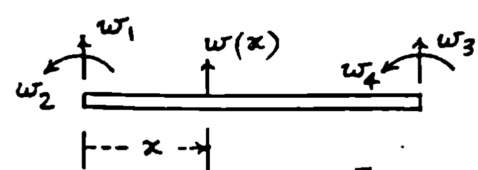


Fig. 3

$$w(x,\, t) = \sum_{i=1}^{4} N_i(x)\; w_i(t) \tag{E_4}$$

where $N_i(x)$ are the shape functions given by Eqs. (12.33) – (12.36). Introducing time variation of displacements, we get
kinetic energy of element is:

$$T(t) = \frac{1}{2}\int_0^{\ell} \rho\, A \left\{\frac{\delta\, w\,(x,\, t)}{\delta\, t}\right\}^2 dx \equiv \frac{1}{2}\; \dot{\vec{w}}^T\, [m]\; \dot{\vec{w}} \tag{E_5}$$

Total strain energy of the element is

$$v(t) = \frac{1}{2}\int_0^{\ell} E\, I \left\{\frac{\delta^2\, w\,(x,\, t)}{\delta\, x^2}\right\}^2 dx + \frac{1}{2}\int_0^{\ell} P(x) \left\{\frac{\delta\, w\,(x,\, t)}{\delta\, x}\right\}^2 dx$$

$$- \int_0^{\ell} P_w(x,\, t)\cdot w(x,\, t)\; dx \equiv \frac{1}{2}\; \vec{w}^T\, [k]\; \vec{w} \tag{E_6}$$

Total virtual work of element is

$$\delta\, W(t) = \int_0^{\ell} f(x,\, t)\; \delta\, w(x,\, t) \equiv \vec{f}\,(t)\; d\vec{w}\,(t) \tag{E_7}$$

where f(x, t) denotes the distributed force (which is zero in the present case). Evaluation of integrals in Eqs. (E_5) – (E_7) enables us to find the mass matrix, stiffness matrix and load vector.

(ii) For the helicopter blade, we can model it as one beam element for simplicity.

$\Omega = 300\ (2\pi)/60 = 31.416$ rad/sec.

$A = 0.006\ m^2,\ I = \frac{1}{12}\ (0.3)\ (0.02)^3 = 0.2 \times 10^{-6}\ m^4,\ \ell = 1.2$ m,

$E = 71$ GPa, $\rho = 2660\ kg/m^3$ (for aluminium)

Boundary conditions:

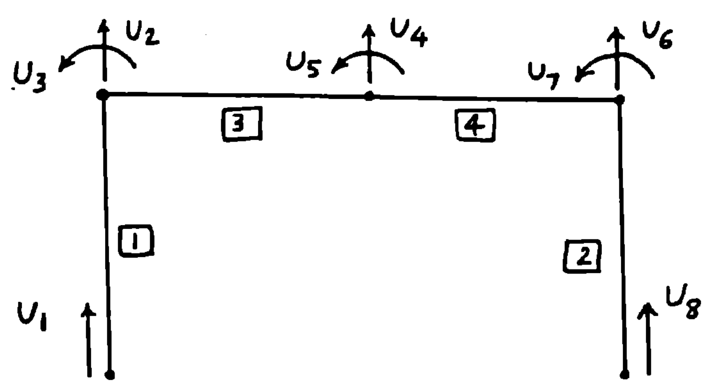
$$W_1 = W_2 = 0$$

Solve the eigenvalue problem

$$[-\omega^2\ [M] + [k]]\ \vec{\omega} = \vec{o} \text{ where } \vec{w} = \begin{Bmatrix} W_3 \\ W_4 \end{Bmatrix}$$

for the natural frequencies and mode shapes.

(12.42) Electric motor mass 500 kg,
girder: E = 200 GPa,
$\rho = 250\ kg/m^3$,
columns: E = 30 GPa,
$\rho = 75\ kg/m^3$
h/b = 2



Solution:

(a) Generate element matrices

$$[k^{(1)}] = \frac{A_1 E_1}{\ell_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \end{matrix},\quad [K^{(2)}] = \frac{A_2 E_2}{\ell_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} U_8 \\ U_6 \end{matrix}$$

(columns of $[k^{(1)}]$: U_1, U_2; columns of $[K^{(2)}]$: U_8, U_6)

$$[k^{(3)}] = \frac{E_3 I_3}{\ell_3^3} \begin{bmatrix} 12 & 6\,\ell_3 & -12 & 6\,\ell_3 \\ 6\,\ell_3 & 4\,\ell_3^2 & -6\,\ell_3 & 2\,\ell_3^2 \\ -12 & -6\,\ell_3 & 12 & -6\,\ell_3 \\ 6\,\ell_3 & 2\,\ell_3^2 & -6\,\ell_3 & 4\,\ell_3^2 \end{bmatrix} \begin{matrix} U_2 \\ U_3 \\ U_4 \\ U_5 \end{matrix}$$

(columns: U_2, U_3, U_4, U_5)

$$[k^{(4)}] = \frac{E_4 I_4}{\ell_4^3} \begin{bmatrix} 12 & 6\,\ell_4 & -12 & 6\,\ell_4 \\ 6\,\ell_4 & 4\,\ell_4^2 & -6\,\ell_4 & 2\,\ell_4^2 \\ -12 & -6\,\ell_4 & 12 & -6\,\ell_4 \\ 6\,\ell_4 & 2\,\ell_4^2 & -6\,\ell_4 & 4\,\ell_4^2 \end{bmatrix} \begin{matrix} U_4 \\ U_5 \\ U_6 \\ U_7 \end{matrix} \quad (\text{columns } U_4\; U_5\; U_6\; U_7)$$

$$[M^{(1)}] = \frac{\rho_1 A_1 \ell_1}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \end{matrix}, \quad [M^{(2)}] = \frac{\rho_2 A_2 \ell_2}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{matrix} U_8 \\ U_6 \end{matrix}$$

(columns $U_1\; U_2$ and $U_8\; U_6$ respectively)

$$[M^{(3)}] = \frac{\rho_3 A_3 \ell_3}{420} \begin{bmatrix} 156 & 22\,\ell_3 & 54 & -13\,\ell_3 \\ 22\,\ell_3 & 4\,\ell_3^2 & 13\,\ell_3 & -3\,\ell_3^2 \\ 54 & 13\,\ell_3 & 156 & -22\,\ell_3 \\ -13\,\ell_3 & -3\,\ell_3^2 & -22\,\ell_3 & 4\,\ell_3^2 \end{bmatrix} \begin{matrix} U_2 \\ U_3 \\ U_4 \\ U_5 \end{matrix} \quad (\text{columns } U_2\; U_3\; U_4\; U_5)$$

$$[M^{(4)}] = \frac{\rho_4 A_4 \ell_4}{420} \begin{bmatrix} 156 & 22\,\ell_4 & 54 & -13\,\ell_4 \\ 22\,\ell_4 & 4\,\ell_4^2 & 13\,\ell_4 & -3\,\ell_4^2 \\ 54 & 13\,\ell_4 & 156 & -22\,\ell_4 \\ -13\,\ell_4 & -3\,\ell_4^2 & -22\,\ell_4 & 4\,\ell_4^2 \end{bmatrix} \begin{matrix} U_4 \\ U_5 \\ U_6 \\ U_7 \end{matrix} \quad (\text{columns } U_4\; U_5\; U_6\; U_7)$$

where

$\rho_1 = \rho_2 = 75$ kg/m^3, $\ell_1 = \ell_2 = 3$ m,

$E_1 = E_2 = 30$ GPa, $A_1 = A_2 = \dfrac{\pi d^2}{4}$, $\rho_3 = \rho_4 = 250$ kg/m^3,

$\ell_3 = \ell_4 = 3$ m, $E_3 = E_4 = 200$ GPa, $A_3 = A_4 = bh = 2\,b^2$

$I_3 = I_4 = \dfrac{1}{12}\, bh^3 = \dfrac{2}{3}\, b^4$.

(b) Find the assembled stiffness and mass matrices after applying the boundary conditions $U_1 = U_8 = 0$.

(c) Select trial values of d and b and find the fundamental natural frequency (ω_1) by solving the eigenvalue problem

$$\left| -\omega^2 [M] + [K] \right| = 0$$

(d) Change d and b until $\omega_1 > \dfrac{1500\,(2\pi)}{60} = 157.08$ rad/sec.