8

Structural Dynamic Modification and Sensitivity Analysis

Su Huan Chen
Jilin University

Summary

The matrix perturbation theory for structural dynamic modification and sensitivity analysis is presented in this chapter. The theory covers a broad spectrum of subjects, specifically, matrix perturbation of real modes of complex structures and matrix perturbation of complex modes. The contents include nine sections. Section 8.2 provides the preliminaries to matrix perturbation and sensitivity analysis. Section 8.3 presents the matrix perturbation method including first-order and second-order perturbation. Section 8.4 presents methods for design sensitivity

analysis. In Section 8.5, high-accuracy modal superposition for sensitivity analysis of modes is given. Section 8.6 presents the sensitivity analysis of eigenvectors for free–free structures. In Section 8.7 and Section 8.8, matrix perturbations for repeated modes and closely spaced modes are discussed. In Section 8.9, the matrix perturbation approach for complex modes is presented.

8.1 Introduction

In modern engineering problems, the dynamic design of structures is becoming increasingly important. In order to achieve an optimal design, we repeatedly have to modify the structural parameters and solve the generalized eigenvalue problem. The iterative vibration analysis can be very tedious for large and complex structures. Therefore, it is necessary to seek a fast computation method for sensitivity analysis and reanalysis. The matrix perturbation method is an extremely useful tool for this purpose.

The matrix perturbation method is concerned with how the natural frequencies and modal vectors change if small modifications are imposed on the parameters of structures. Engineering problems often involve many small modifications in the structural parameters, such as material property variations, manufacturing errors, iterative design of structural parameters, design sensitivity analysis, random eigenvalue analysis, robustness analysis of control systems, and so on.

In this chapter, it is assumed that the reader has an undergraduate knowledge in vibration theory and a working knowledge in the finite element method.

The contents of the chapter include the basic preliminaries: vibration equations of the finite element model, eigenvalue problem, modal vectors, orthogonality conditions, modal expansion theorem, and the power series expansion of eigensolutions. The chapter also covers such topics as: the perturbation method for distinct eigenvalues and corresponding eigenvectors; sensitivities of eigenvalues and eigenvectors; the high-accuracy modal superposition method for eigenvector derivatives; eigenvector derivatives for free–free structures; perturbation method for systems with repeated eigenvalues and close eigenvalues; and perturbation method of the complex modes of systems with real unsymmetric matrices.

8.2 Structural Dynamic Modification of Finite Element Model

The finite element method is an important tool to obtain numerical and computational solutions to problems in structural vibration analysis. By applying the finite element method to a structure, a discrete analysis model to idealize the continuum can be obtained. The finite equation of vibrations of a structure in the global coordinate system is

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{Q} \tag{8.1}$$

where $\mathbf{M}$, $\mathbf{K}$, and $\mathbf{C}$ are the mass, stiffness, and damping matrices, respectively, $\ddot{\mathbf{q}}$, $\dot{\mathbf{q}}$, and $\mathbf{q}$ are the acceleration, velocity, and displacement vectors, respectively, and $\mathbf{Q}$ is the external load vector.

Neglecting the damping force and external load vector, Equation 8.1 becomes

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0} \tag{8.2}$$

This is the natural vibration equation for the structure. Its solution (the natural vibration) is harmonic, and is given by

$$\mathbf{q} = \mathbf{u}\cos(\omega t - \varphi) \tag{8.3}$$

where $\mathbf{u}$ is modal vector, and ω the natural frequency of the system. Substituting Equation 8.3 into Equation 8.2, the eigenproblem of structural vibration can be obtained as

$$\mathbf{K}\mathbf{u} = \lambda\mathbf{M}\mathbf{u} \tag{8.4}$$

where λ ($\lambda = \omega^2$) denote the eigenvalues of the system.

In structural vibration analysis, the natural frequencies and the corresponding modal vectors can be obtained by solving the eigenproblem (Equation 8.4). The solutions for n eigenvalues and corresponding eigenvectors satisfy

$$\mathbf{KU} = \mathbf{MU}\boldsymbol{\Lambda} \tag{8.5}$$

where $\mathbf{U}$, which is called the *modal matrix*, is an $(n \times n)$ matrix with its columns equal to the n eigenvectors, and $\boldsymbol{\Lambda}$ is an $(n \times n)$ diagonal matrix consisting of the corresponding eigenvalues as the diagonal elements; specifically

$$\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n] \tag{8.6}$$

$$\boldsymbol{\Lambda} = \text{diag}(\lambda_i), \quad i = 1, 2, \ldots, n \tag{8.7}$$

An important relation for eigenvectors is that of $\mathbf{M}$ orthogonality and $\mathbf{K}$ orthogonality; that is, we have

$$\mathbf{u}_i^{\mathrm{T}}\mathbf{M}\mathbf{u}_j = \delta_{ij} \tag{8.8}$$

$$\mathbf{u}_i^{\mathrm{T}}\mathbf{K}\mathbf{u}_j = \lambda_i\delta_{ij} \tag{8.9}$$

where δ_{ij} is the Kronecker delta. For n eigenpairs, Equation 8.8 and Equation 8.9 can be written as

$$\mathbf{U}^{\mathrm{T}}\mathbf{MU} = \mathbf{I} \tag{8.10}$$

$$\mathbf{U}^{\mathrm{T}}\mathbf{KU} = \boldsymbol{\Lambda} \tag{8.11}$$

Since the modal vectors are independent, an arbitrary displacement vector, $\mathbf{u}$, can be expressed as a linear combination of $\mathbf{u}_i$, $i = 1, 2, \ldots, n$; that is

$$\mathbf{u} = \sum_{r=1}^{n} c_r\mathbf{u}_r = \mathbf{UC} \tag{8.12}$$

where c_r is a constant. Each constant c_r can be determined by

$$c_r = \mathbf{u}_r^{\mathrm{T}}\mathbf{Mu}, \quad r = 1, 2, \ldots, n \tag{8.13}$$

This is known as the *expansion theorem*.

Suppose the physical parameter of a given structure is given a small modification. This will cause a small change in the matrices $\mathbf{K}_0$ and $\mathbf{M}_0$; that is

$$\mathbf{M} = \mathbf{M}_0 + \varepsilon\mathbf{M}_1, \quad \mathbf{K} = \mathbf{K}_0 + \varepsilon\mathbf{K}_1 \tag{8.14}$$

where ε is a small parameter, $\mathbf{K}_0$ and $\mathbf{M}_0$ are the original mass and stiffness matrices, respectively, and $\varepsilon\mathbf{M}_1$ and $\varepsilon\mathbf{K}_1$ are the corresponding modifications. It is obvious that if $\mathbf{M}_0$ and $\mathbf{K}_0$ are symmetric, the matrices $\mathbf{M}_1$ and $\mathbf{K}_1$ are also symmetric.

If $\varepsilon\mathbf{M}_1$ and $\varepsilon\mathbf{K}_1$ are small, the changes of eigenvalues and eigenvectors of the structure are also small. According to the matrix perturbation theory, the eigensolutions of Equation 8.4 can be expressed in the form of a power series in ε; thus

$$\mathbf{u}_i = \mathbf{u}_{0i} + \varepsilon\mathbf{u}_{1i} + \varepsilon^2\mathbf{u}_{2i} + \cdots \tag{8.15}$$

$$\lambda_i = \lambda_{0i} + \varepsilon\lambda_{1i} + \varepsilon^2\lambda_{2i} + \cdots \tag{8.16}$$

where $\mathbf{u}_{0i}$ and λ_{0i} are the eigensolutions of the original structure, λ_{1i} and λ_{2i} are the first- and the second-order perturbations of the eigenvalues, and $\mathbf{u}_{1i}$ and $\mathbf{u}_{2i}$ are the first- and the second-order perturbation of the eigenvectors.

Since the eigensolutions of the original structure, $\mathbf{u}_{0i}$ and λ_{0i}, are known, only the first- and the second-order perturbations of the eigensolutions are required without solving Equation 8.4.

8.3 Perturbation Method of Vibration Modes

The perturbation methods of vibration modes are well developed (Fox and Kapoor, 1968; Rogers, 1977; Chen and Wada, 1979; Hu, 1987; Chen, 1993). In this section, it is assumed that all eigenvalues of the original structure are distinct.

8.3.1 First-Order Perturbation of Distinct Modes

According to the expansion theorem, the first-order perturbation, $\mathbf{u}_{1i}$, can be expanded by the modal vectors, $\mathbf{u}_{0s}$, of the original structure as

$$\mathbf{u}_{1i} = \sum_{s=1}^{n} c_{1s}\mathbf{u}_{0s} \tag{8.17}$$

where

$$c_{1s} = \frac{1}{\lambda_{0i} - \lambda_{0s}}(\mathbf{u}_{0s}^{\mathrm{T}} K_1 \mathbf{u}_{0i} - \lambda_{0i}\mathbf{u}_{0s}^{\mathrm{T}}\mathbf{M}_1\mathbf{u}_{0i}), \quad s \neq i \tag{8.18}$$

$$c_{1i} = -\tfrac{1}{2}\mathbf{u}_{0i}^{\mathrm{T}}\mathbf{M}_1\mathbf{u}_{0i} \tag{8.19}$$

The first-order perturbation of the eigenvalues is

$$\lambda_{1i} = \mathbf{u}_{0i}^{\mathrm{T}}\mathbf{K}_1\mathbf{u}_{0i} - \lambda_{0i}\mathbf{u}_{0i}^{\mathrm{T}}\mathbf{M}_1\mathbf{u}_{0i} \tag{8.20}$$

8.3.2 Second-Order Perturbation of Distinct Modes

If the parameter modification is fairly large, in order to obtain high computing accuracy, the second-order perturbation must be used. According to the expansion theorem, the second-order perturbation, $\mathbf{u}_{2i}$, can be expanded by the modal vectors, $\mathbf{u}_{0s}$, of the original structure as

$$\mathbf{u}_{2i} = \sum_{s=1}^{n} c_{2j}\mathbf{u}_{0s} \tag{8.21}$$

where

$$c_{2s} = \frac{1}{\lambda_{0i} - \lambda_{0s}}(\mathbf{u}_{0s}^{\mathrm{T}}\mathbf{K}_1\mathbf{u}_{1i} - \lambda_{0i}\mathbf{u}_{0s}^{\mathrm{T}}\mathbf{M}_1\mathbf{u}_{1i} - \lambda_{1i}\mathbf{u}_{0s}^{\mathrm{T}}\mathbf{M}_0\mathbf{u}_{1i} - \lambda_{1i}\mathbf{u}_{0s}^{\mathrm{T}}\mathbf{M}_1\mathbf{u}_{0i}), \quad s \neq i \tag{8.22}$$

$$c_{2s} = -\frac{1}{2}(\mathbf{u}_{1i}^{\mathrm{T}}\mathbf{M}_0\mathbf{u}_{1i} + \mathbf{u}_{0i}^{\mathrm{T}}\mathbf{M}_1\mathbf{u}_{1i} + \mathbf{u}_{1i}^{\mathrm{T}}\mathbf{M}_1\mathbf{u}_{0i}) \tag{8.23}$$

The second perturbation of the eigenvalues is

$$\lambda_{2i} = \mathbf{u}_{0i}^{\mathrm{T}}\mathbf{K}_1\mathbf{u}_{1i} - \lambda_{0i}\mathbf{u}_{0i}^{\mathrm{T}}\mathbf{M}_1\mathbf{u}_{1i} - \lambda_{1i}\mathbf{u}_{0i}^{\mathrm{T}}\mathbf{M}_0\mathbf{u}_{1i} - \lambda_{1i}\mathbf{u}_{0i}^{\mathrm{T}}\mathbf{M}_1\mathbf{u}_{0i} \tag{8.24}$$

8.3.3 Numerical Examples

As illustrations of the matrix perturbation method, several numerical examples are given now.

Example 8.1

Consider the five-degree-of-freedom (five-DoF) system shown in Figure 8.1. The physical parameters are given as

$$m_1 = m_2 = m_3 = m_4 = 1.0, \quad m_5 = 0.5, \quad k_1 = k_2 = k_3 = k_4 = k_5 = 1.0$$

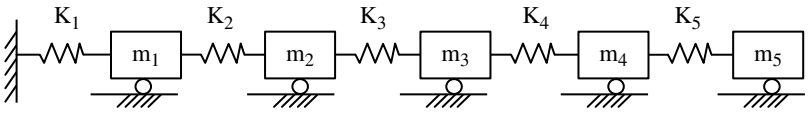

FIGURE 8.1 Mass–spring system for Example 8.1.

In order to study the computing accuracy of first- and second-order perturbations, let us assume that the fifth mass undergoes a decrement of 5 to 30%, and the stiffness of the first spring undergoes a decrement of 5 to 30%.

The computed results for the natural frequencies are presented in Table 8.1, in which the initial solutions mean the eigensolutions of the original structure.

TABLE 8.1 Comparison of Natural Frequencies

Mode Number		Changes of Structural Parameter (%)					
		5	10	15	20	25	30
1	A	0.3022	0.2922	0.2821	0.2724	0.2629	0.2536
	B	0.3128	0.3128	0.3128	0.3128	0.312	0.3128
	C	3.52	7.14	10.9	14.38	18.97	23.3
	D	0.3017	0.2903	0.2783	0.2658	0.2527	0.2388
	C	0.14	0.58	1.34	2.43	3.9	5.81
	E	0.3022	0.2922	0.2827	0.2740	0.2660	0.2588
	C	0.0033	0.068	0.24	0.58	0.93	2.08
2	A	0.8788	0.8512	0.8249	0.7998	0.7756	0.7523
	B	0.9079	0.98079	0.9079	0.98079	0.9079	0.98079
	C	3.31	6.66	10.1	13.52	17.06	20.68
	D	0.8775	0.8460	0.8133	0.7792	0.7435	0.7060
	C	0.15	0.61	1.41	2.58	4.14	6.15
	E	0.3789	0.8518	0.8267	0.8089	0.7835	0.7659
	C	0.0076	0.062	0.21	0.52	0.91	1.80
3	A	1.3732	1.3348	1.2989	1.2650	1.2332	1.2031
	B	1.1421	1.1421	1.1421	1.1421	1.1421	1.1421
	C	2.99	5.94	8.88	11.79	14.7	17.5
	D	1.371	1.3266	1.2806	1.2328	1.1832	1.1313
	C	0.15	0.62	1.41	2.54	4.05	5.96
	E	1.3733	1.3356	1.3015	1.2712	1.2449	1.2231
	C	0.0074	0.060	0.20	0.49	0.74	1.66
4	A	1.7355	1.6923	1.6520	1.6143	1.5790	1.5457
	B	1.7820	1.7820	1.7820	1.7820	1.7820	
	C	2.68	5.30	7.78	10.38	12.85	15.28
	D	1.7331	1.6828	1.6319	1.5773	1.5209	1.5209
	C	0.14	0.56	1.28	2.29	3.62	5.27
	E	1.7356	1.6933	1.6552	1.6216	1.5830	1.5695
	C	0.0070	0.057	0.19	0.45	0.68	1.53
5	A	1.9273	1.8825	1.8408	1.8017	1.7649	1.7304
	B	1.9753	1.9753	1.9753	1.9753	1.9753	1.9753
	C	2.49	4.93	7.31	9.63	11.92	14.16
	D	1.9248	1.8929	1.896	1.7696	1.7079	1.6492
	C	0.1300	0.51	1.15	2.06	3.23	4.69
	E	1.9274	1.8835	1.8439	1.8090	1.7790	1.7541
	C	0.0064	0.051	0.17	0.41	0.83	1.37

TABLE 8.2 Comparison of Natural Frequencies

Mode No.	A (Hz)	B (Hz)	F (%)	D (Hz)	G (%)	E (Hz)	H (%)
1	27.78	25.45	8.39	29.27	5.36	28.14	1.29
2	109.1	107.7	1.28	110.39	1.18	109.35	0.28
3	157.4	153.2	2.67	159.53	1.35	158.79	0.88
4	230.5	233.2	1.17	231.24	0.32	231.02	0.022
5	320.7	325.6	1.53	320.58	0.04	320.89	0.04
6	391.1	393.7	0.66	392.06	0.24	319.35	0.06

As can be seen from the results, if the change of the structural parameter is 15%, the average change of the natural frequencies is 9.0%. Using the first-order perturbation, the average error of the frequencies is reduced to 1.32%.

If the change of the structural parameter is 30%, the average error of the natural frequencies is 18%. Using the first-order perturbation, the average error of natural frequencies is reduced to 1.6%.

The notation used in Table 8.1 and Table 8.2 is as follows:

- A: the exact solutions of the modified structure
- B: the initial solutions of the original structure
- C: percent error
- D: the first-order perturbation solutions
- E: the second-order perturbation solutions
- F: the percent errors of the initial solutions
- G: the percent errors of the first-order perturbation
- H: the percent errors of the second perturbation

Example 8.2

Consider a truss structure (as shown in Figure 8.2) with 20 rods. The cross section area of the second rod is changed from 1.0 to 2.0 cm^2. The results calculated are listed in Table 8.2.

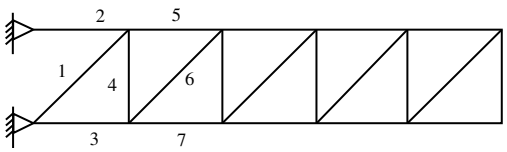

FIGURE 8.2 Truss structure for Example 8.2.

Example 8.3

Consider a torsional vibration system with five disks, as shown in Figure 8.3. The physical parameters of the system are as follows:

- $I_1 = 10.78$ kg cm sec^2
- $I_2 = 82.82$ kg cm sec^2
- $I_3 = 14.27$ kg cm sec^2
- $I_4 = 29.56$ kg cm sec^2
- $I_5 = 21.66$ kg cm sec^2
- $K_1 = 10.48 \times 10^4$ kg cm/rad
- $K_2 = 34.30 \times 10^4$ kg cm/rad
- $K_3 = 24.40 \times 10^4$ kg cm/rad
- $K_4 = 40.60 \times 10^4$ kg cm/rad

The corresponding constrained system is shown in Figure 8.3b, in which the hung stiffness is $K_s = 4060$ kg cm/rad.

The exact eigensolutions of the constrained system are taken as the initial results. Using matrix

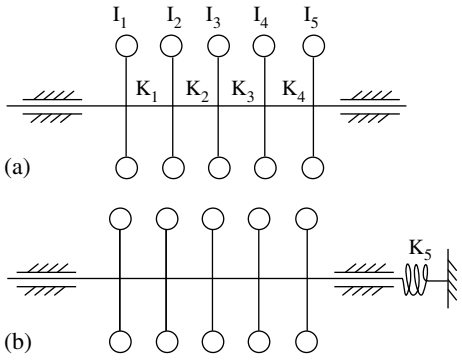

FIGURE 8.3 Torsional vibration system for Example 8.3.

TABLE 8.3 Comparison for Natural Frequencies

	No.				
	1	2	3	4	5
ω_0	0.000000	62.554934	105.668169	177.680405	224.361444
ω_x	1.595950	62.604815	105.669360	177.704224	224.386235
δ		0.079700	0.001070	0.013406	0.011000
$\varepsilon\omega_1$	−1.525707	−0.049895	−0.001132	−0.023834	−0.002508
ω'_0	0.080243	62.554920	105.448168	177.680390	224.383727
δ'		2.24×10^{-5}	9.46×10^{-7}	8.22×10^{-8}	9.93×10^{-5}

TABLE 8.4 Comparison for Eigenvectors

No.	u_0^i	u_x^i	εu_1^i	u'^i_0
1	0.079283	0.097363	−0.000080	0.079823
	0.079283	0.079342	−0.000059	0.079283
	0.079283	0.079287	−0.000004	0.079283
	0.079283	0.079198	0.000085	0.079283
	0.079283	0.079189	0.000153	0.079342
2	−0.095647	−0.095611	0.000036	−0.095647
	−0.057148	−0.057064	−0.000084	−0.057148
	0.008612	0.008719	−0.000107	0.008612
	0.099081	0.099192	−0.000111	0.099081
	0.125223	0.125259	0.000036	0.125223
3	0.277894	0.277884	0.000010	0.277894
	−0.041278	−0.041284	0.000005	−0.41279
	−0.027509	−0.027497	−0.000013	−0.027510
	0.009810	0.009840	−0.000030	0.009810
	0.024263	0.024279	−0.000016	0.024263
4	−0.009038	−0.009033	−0.000005	−0.009038
	0.020312	−0.020309	0.000002	0.020312
	−0.125560	−0.125580	0.000024	−0.125556
	−0.098791	0.098735	−0.000053	−0.098788
	0.144375	0.144413	−0.000038	0.144375
5	0.004823	0.004823	0.000001	0.004824
	−0.020156	−0.020154	−0.0000002	−0.020156
	0.217244	0.217227	0.000018	0.217245
	−0.088717	−0.088725	0.000008	−0.088717
	0.052618	0.052652	−0.000034	0.052618

perturbation, the eigensolutions of the free–free system can be obtained. The results are listed in Table 8.3 and Table 8.4.

The notation used in Table 8.3 is as follows:

- ω_0: exact solution of the natural frequency of the free–free system (l/sec)
- ω'_0: perturbation solutions of the natural frequency of the free–free system (l/sec)
- ω_x: natural frequency of the constrained system (l/sec)

$\varepsilon\omega_1 = \omega'_0 - \omega_x$ the perturbation of the natural frequency

$$\delta = \frac{|\omega_0 - \omega_x|}{\omega_0} \quad (\%)$$

$$\delta' = \frac{|\omega_0 - \omega'_0|}{\omega_0} \quad (\%)$$

The notation used in Table 8.4 is as follows:

- $\mathbf{u}_0^i$: exact solution of eigenvectors of the free–free system
- $\mathbf{u}'^i_0$: perturbation solution of eigenvectors of the free–free system
- $\mathbf{u}_x^i$: eigenvector of the constrained system
- $\varepsilon\mathbf{u}_1^i$: first-order perturbation of eigenvectors

As can be seen from Table 8.3, the natural frequencies of the free–free system are increased by the hung elastic elements. For example, the frequency of the rigid mode is increased to 1.595950 (l/sec), and the frequency of the first elastic mode is increased by 0.8124%. By modifying the eigensolutions with the perturbation method, the frequency of the rigid mode is reduced to 0.079700 (l/sec), which is nearly equal to zero, and all the frequencies of the elastic modes become almost exact solutions. The results in Table 8.4 show that the mode shapes of the free–free system, $\mathbf{u}'^i_0$, are close to the exact solution, u_0^i.

8.4 Design Sensitivity Analysis of Structural Vibration Modes

In the optimization of structural analysis, the design sensitivity analysis of eigenvalues and eigenvectors plays an essential role. The designer can use this information directly in an interactive computer-aided design procedure as a valuable guide. Significant work has been done in this area (Haug et al., 1985; Adelmen and Haftka, 1986; Chen and Pan, 1986; Wang, 1991).

8.4.1 Direct Differential Method for Sensitivity Analysis

Design sensitivity analysis of eigenvalues and eigenvectors will reveal how the changes in some design parameters in the system affect the dynamic characteristics of the structure.

Let $\lambda_{i,j}$ and $\mathbf{u}_{i,j}$ denote the sensitivity of the eigenvalue, λ_i, and the eigenvector, $\mathbf{u}_i$, respectively, with respect to the design variables b_j ($j = 1, 2, \ldots, L$), and let $\mathbf{K}_{,j}$ and $\mathbf{M}_{,j}$ denote the derivative of the stiffness and mass matrices, respectively, with respect to b_j. The design sensitivity of the eigenvalue is

$$\lambda_{i,j} = \mathbf{u}_i^{\mathrm{T}}(\mathbf{K}_{,j} - \lambda_i\mathbf{M}_{,j})\mathbf{u}_i \tag{8.25}$$

The sensitivity of the eigenvector, $\mathbf{u}_{i,j}$, can be expressed as the following series:

$$\mathbf{u}_{i,j} = \sum_{s=1}^{n} c_{ijs}\mathbf{u}_s \tag{8.26}$$

where

$$c_{ijs} = \frac{1}{\lambda_i - \lambda_s}\mathbf{u}_s^{\mathrm{T}}(\mathbf{K}_{,j} - \lambda_i\mathbf{M}_{,j})\mathbf{u}_i, \quad i \neq s, \quad i, s = 1, 2, \ldots, n, \quad i \neq s \tag{8.27}$$

$$c_{iji} = -\frac{1}{2}\mathbf{u}_i^{\mathrm{T}}\mathbf{M}_{,j}\mathbf{u}_i \tag{8.28}$$

8.4.2 Perturbation Sensitivity Analysis

Let $\Delta\mathbf{K}$ and $\Delta\mathbf{M}$ denote the increments of the stiffness and the mass matrices resulting from an incremental change of the design variable, Δb_j, and let $\Delta\lambda_i$ and $\Delta\mathbf{u}_i$ denote the corresponding perturbations of the eigenvalue and eigenvector, respectively. The direct differential method of design sensitivity analysis of vibration modes can now be put into perturbation form, approximately as

$$\lambda_{i,j} = \frac{\Delta\lambda_i}{\Delta b_j} \tag{8.29}$$

$$\mathbf{u}_{i,j} = \frac{\Delta\mathbf{u}_i}{\Delta b_j} \tag{8.30}$$

where $\Delta\lambda_i$ and $\Delta\mathbf{u}_i$ can be evaluated by the perturbation formulas presented in this chapter. In practical analysis, the design variables could be the cross-sectional area of the truss members, bending moment of inertia, equivalent torsional moment of inertia of a beam, the thickness of a plate, or other variable. In some complex structures, a mass, m_r, may be placed at a node point and moving in the direction of the rth DoF, or an elastic support with spring stiffness, K_r, may be placed at a certain node point. It is also possible that an elastic connector of stiffness, K_j, might exist between two components. They can also be considered as design variables. In finite element analysis, $\Delta\mathbf{K}$ and $\Delta\mathbf{M}$ are known to be the sum of the element increments, $\Delta\mathbf{K}^e$ and $\Delta\mathbf{M}^e$; thus

$$\Delta\mathbf{K} = \sum_e \Delta\mathbf{K}^e \tag{8.31}$$

$$\Delta\mathbf{M} = \sum_e \Delta\mathbf{M}^e \tag{8.32}$$

Hence, the sensitivity formulas of vibration modes as given above can be transformed into the finite element perturbation form (Chen and Pan, 1986)

$$\lambda_{i,j} = \frac{1}{\Delta b_j} \sum_e \bar{\mathbf{u}}_i^{\mathrm{T}} (\Delta\mathbf{K}^e - \lambda_i \Delta\mathbf{M}^e) \bar{\mathbf{u}}_i \tag{8.33}$$

and

$$\mathbf{u}_{i,j} = \frac{1}{\Delta b_j} \sum_e \left(\sum_{\substack{s=1 \\ s\neq i}}^{n} \frac{1}{\lambda_i - \lambda_s} \bar{\mathbf{u}}_s^{\mathrm{T}} (\Delta\mathbf{K}^e - \lambda_i \Delta\mathbf{M}^e) \bar{\mathbf{u}}_i \mathbf{u}_s - \frac{1}{2} \bar{\mathbf{u}}_i^{\mathrm{T}} \Delta\mathbf{M}^e \bar{\mathbf{u}}_i \mathbf{u}_i \right) \tag{8.34}$$

In these formulas, the overbar signifies that the eigenvector concerned contains only the components needed for the eth finite element. It is important to observe that, in Equation 8.33 and Equation 8.34, calculations are done on the element basis, and as a result, the calculations are greatly simplified.

Using the shorthand notations

$$\lambda_{i,j}^e = \frac{1}{\Delta b_j} \bar{\mathbf{u}}_i^{\mathrm{T}} \Delta\mathbf{K}^e - \lambda_i \Delta\mathbf{M}^e \bar{\mathbf{u}}_i \tag{8.35}$$

$$\mathbf{u}_{i,j}^e = \frac{1}{\Delta b_j} \left(\sum_{\substack{s=1 \\ s\neq i}}^{n} \frac{1}{\lambda_i - \lambda_s} \bar{\mathbf{u}}_s^{\mathrm{T}} \Delta\mathbf{K}^e - \lambda_i \Delta\mathbf{M}^e \bar{\mathbf{u}}_i \mathbf{u}_s - \frac{1}{2} \bar{\mathbf{u}}_i^{\mathrm{T}} \Delta\mathbf{M}^e \bar{\mathbf{u}}_i \mathbf{u}_i \right) \tag{8.36}$$

Equation 8.33 and Equation 8.34 can be written as

$$\lambda_{i,j} = \sum_e \lambda_{i,j}^e \tag{8.37}$$

and

$$\mathbf{u}_{i,j} = \sum_e \mathbf{u}_{i,j}^e \tag{8.38}$$

where $\lambda_{i,j}^e$ and $\mathbf{u}_{i,j}^e$ are the design sensitivity of the eth element for the eigenvalue λ_i and the eigenvector $\mathbf{u}_i$, respectively. Let us consider the following important cases.

For a concentrated mass, m_r, placed at a node point and moved in the direction of the rth DoF, Equation 8.33 and Equation 8.34 become

$$\lambda_{i,r} = \frac{\Delta\lambda_i}{\Delta m_r} = -\lambda_i u_{ir}^2 \tag{8.39}$$

and

$$\mathbf{u}_{i,r} = \frac{\Delta \mathbf{u}_i}{\Delta m_r} = \sum_{\substack{s=1 \\ s \neq i}}^{n} \frac{-\lambda_i}{\lambda_i - \kappa_s} u_{sr} u_{ir} \mathbf{u}_s - \frac{1}{2} u_{ir}^2 \mathbf{u}_i \tag{8.40}$$

where u_{ir} is the rth element of the ith eigenvector $\mathbf{u}_i$.

For an elastic connector with stiffness K_j between the rth and the lth DoF of two components, Equation 8.33 and Equation 8.34 become

$$\lambda_{i,j} = \frac{\Delta \lambda_i}{\Delta k_j} = (u_{ir} - u_{il})^2 \tag{8.41}$$

and

$$\mathbf{u}_{i,j} = \frac{\Delta \mathbf{u}_i}{\Delta k_j} = \sum_{\substack{s=1 \\ s \neq i}}^{n} \frac{1}{\lambda_i - \lambda_s} (u_{sr} u_{ir} - u_{sl} u_{ir} - u_{sr} u_{il} + u_{sl} u_{il}) \mathbf{u}_s \tag{8.42}$$

For an elastic support with spring stiffness K_r placed in the direction of the rth DoF, Equation 8.33 and Equation 8.34 become

$$\lambda_{i,r} = \frac{\Delta \lambda_i}{\Delta k_r} = u_{ir}^2 \tag{8.43}$$

and

$$\mathbf{u}_{i,r} = \frac{\Delta \mathbf{u}_i}{\Delta k_r} = \sum_{\substack{s=1 \\ s \neq i}}^{n} \frac{1}{\lambda_i - \lambda_s} u_{sr} u_{ir} \mathbf{u}_s \tag{8.44}$$

8.4.3 Numerical Example

The design sensitivity analysis of an automotive chassis is presented here as an illustration of the method.

Example 8.4

The finite element model of an automobile chassis consists of 39 beam elements involving 30 nodal points and 180 DoF (Figure 8.4).

The design variables for the sensitivity analysis of eigenvalues in this example are the equivalent torsional moment of inertia, J, and the bending moment of inertia, I_y, of the beam element of

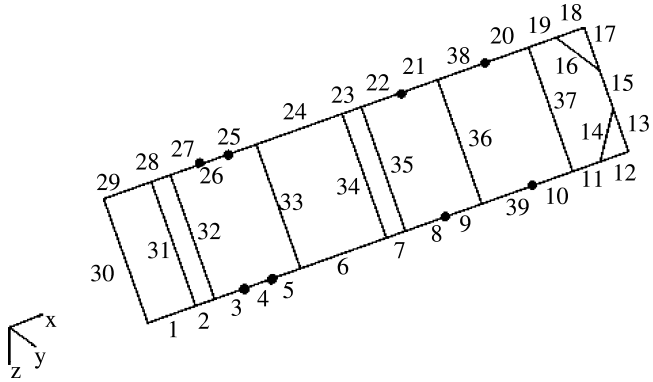

FIGURE 8.4 Finite element model of the automotive chassis for Example 8.4.

TABLE 8.5 Sensitivities of the First Four Chassis Frequencies

1	NE	15	11	19	24	6	37	34	36
	λ^e_{1J}	9.97	5.23	5.23	3.64	3.64	3.34	3.32	3.31
	NE	24	6	23	7	22	8	33	21
	λ^e_{1Iy}	0.015	0.15×10^{-2}	0.33×10^{-3}	0.33×10^{-3}	0.28×10^{-3}	0.19×10^{-3}	0.11×10^{-3}	0.11×10^{-3}
2	NE	16	14	11	19	18	12	13	17
	λ^e_{2J}	6.67×10^{-2}	0.67×10^{-2}	0.45×10^{-2}	0.45×10^{-2}	0.11×10^{-2}	0.11×10^{-2}	0.53×10^{-3}	0.53×10^{-3}
	NE	38	10	21	9	22	8	24	6
	λ^e_{2Iy}	2.92	2.92	1.78	1.78	1.49	1.49	1.44	1.44
3	NE	15	19	11	37	31	30	32	1
	λ^e_{3J}	95.9	47.2	47.2	26.5	25.4	25.2	23.8	12.5
	NE	10	38	9	21	8	22	6	24
	λ^e_{3Iy}	2.96	2.96	1.97	1.97	1.74	1.74	1.70	1.70
4	NE	14	16	19	11	12	18	17	13
	λ^e_{4J}	0.19	0.19	0.12	0.12	0.03	0.03	0.015	0.015
	NE	10	38	39	20	5	25	6	24
	λ^e_{4Iy}	22.4	22.4	15.7	15.7	11.8	11.8	10.9	10.9

the structure. Results for the sensitivity of eigenvalues with respect to J and I_y are given in Table 8.5, in which NE denotes the number of the element. Only the highest eight values are given, and they are listed in descending order.

From Table 8.5, it is seen that for this particular chassis, the sensitivities of the first natural frequency, λ^e_{1Iy}, are much smaller than λ^e_{1J}. This indicates that there is very little effect of the change of bending moment of inertia, I_y, of the beams on the vibration of the chassis at its first natural frequency. Thus, we can conclude that the first mode is a torsional mode. Similarly, the results indicate that the third mode is also a torsional mode. On the other hand, the second and the fourth modes are recognized to be bending modes. This information is very useful to the designer when deciding on a change in the design. For example, if he wants to increase the first torsional frequency, the efficient way is for him to increase the equivalent torsional moments of inertia of beam elements 15, 11, 19, 24, 6, and so on.

It should be noted that only the first low-frequency modes are available and can be used as basis vectors of eigenvector derivatives in Equation 8.26. However, modal truncation induces errors, and the errors become significant if more high-frequency modes are truncated. An improvement to truncated modal superposition representation of eigenvector derivatives is presented in the next section.

8.4.4 Concluding Remarks

As can be seen from the numerical examples given above, the matrix perturbation method is an extremely useful tool for fast reanalysis of a modified structure. It is widely used in a range of structural modifications, such as the modification of various types of elements, local modification of structures, sensitivity analysis of vibration modes, and so on. Therefore, matrix perturbation plays an important role in dynamic analysis and optimization of structures.

8.5 High-Accuracy Modal Superposition for Sensitivity Analysis of Modes

The modal superposition method is often used to compute the derivatives of modal vectors. Because of the cost of generating computer solutions for a dynamic analysis, it is impractical to obtain all modes.

Therefore, only the first L low-frequency modes are computed and are used as basis vectors of eigenvector derivatives. However, as noted above, modal truncation induces errors, which can be significant if more high-frequency modes are truncated. An explicit method to improve the truncated modal superposition representation of eigenvector derivatives is presented (Wang, 1991), in which a residual static mode is used to approximate the contribution due to unavailable high-frequency modes (method one).

In this section a more accurate modal superposition method (method two; Chen, 1993a; Liu and Chen, 1994a) than method one is given. In this method, the contribution of the truncated modes to the eigenvector derivatives is expressed exactly, as a convergent series that can be evaluated by a simple iterative procedure.

8.5.1 Method One

The modal sensitivity can be expressed as

$$u_{i,j} = \sum_{s=1}^{N} c_s u_s = \sum_{j=1}^{L} c_j \mathbf{u}_j + \mathbf{S}_R \tag{8.45}$$

where

$$\mathbf{S}_R = \sum_{j=L+1}^{N} c_j \mathbf{u}_j \tag{8.46}$$

Since $\lambda_i \ll \lambda_{L+1}$, Equation 8.46 can be approximated as

$$\mathbf{S}_R \approx \mathbf{S}_{RA} = \bar{\mathbf{H}}_0 - \bar{\mathbf{W}}_0 \tag{8.47}$$

where

$$\bar{\mathbf{H}}_0 = \mathbf{K}^{-1}(-\mathbf{K}_{,j} + \lambda_{i,j}\mathbf{M} + \lambda_i \mathbf{M}_{,j}) \tag{8.48}$$

$$\bar{\mathbf{W}}_0 = \sum_{j=1}^{L} \frac{1}{\lambda_j} \mathbf{u}_j^{\mathrm{T}}(-\mathbf{K}_{,j} + \lambda_{i,j}\mathbf{M} + \lambda_i \mathbf{M}_{,j})\mathbf{u}_j \tag{8.49}$$

8.5.2 Method Two

The contribution of $u_{i,j}$, S_R due to truncated high-frequencies modes is as follows:

$$\mathbf{S}_R = \sum_{j=0}^{\infty} \lambda_i^j (\mathbf{H}_j - \mathbf{W}_j) \tag{8.50}$$

where

$$\mathbf{W}_j = \mathbf{U}_L \boldsymbol{\Lambda}_L^{-j-1} \mathbf{U}_L^{\mathrm{T}}(-\mathbf{K}_{,j} + \lambda_{i,j}\mathbf{M} + \lambda_i \mathbf{M}_{,j}) \tag{8.51}$$

$$\mathbf{U}_L = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_L] \tag{8.52}$$

$\mathbf{H}_j$ can be obtained with the following iterative procedure:

$$\left.\begin{aligned} \mathbf{H}_0 &= \mathbf{K}^{-1}(-\mathbf{K}_{,j} + \mathbf{K}_{i,j}\mathbf{M} + \lambda_i \mathbf{M}_{,j}) \\ \mathbf{F}'_{j-1} &= \mathbf{M}\mathbf{H}_{j-1}, \quad j \geq 1 \\ \mathbf{H}_j &= \mathbf{K}^{-1}\mathbf{F}'_{j-1} \end{aligned}\right\} \tag{8.53}$$

Define $\mathbf{S}_R(k)$ as

$$\mathbf{S}_R(k) = \sum_{j=0}^{k} \lambda_i^j (\mathbf{H}_j - \mathbf{W}_j) \tag{8.54}$$

Using this definition, the given iterative process can be terminated if the following inequality

$$\|\mathbf{S}_R(k) - \mathbf{S}_R(k-1)\|_2 \leq \varepsilon \tag{8.55}$$

is satisfied, where ε is a specified accuracy requirement.

It should be noted that, if only the first term in the series (Equation 8.50) is retained with all the other terms neglected, then method two is reduced to method one. In addition, the series (Equation 8.50) can be used to estimate the errors induced by the modal truncation.

8.6 Sensitivity of Eigenvectors for Free–Free Structures

As can be seen from Equation 8.48 and Equation 8.53, both method one and method two fail to deal with the free–free structures with rigid-body modes because they involve the inversion of the stiffness matrix. However, we can transform the eigenproblem with a singular stiffness matrix into its equivalent eigenproblem with a nonsingular stiffness matrix, in the sense that these two eigenproblems have the same derivatives of eigenvalues and eigenvectors (Liu and Chen, 1994b).

Consider the eigenvalue problem

$$\bar{\mathbf{K}}\bar{\mathbf{u}}_i = \bar{\lambda}_i \mathbf{M}\bar{\mathbf{u}}_i \tag{8.56}$$

where

$$\bar{\mathbf{K}} \equiv \mathbf{K} - \mu\mathbf{M} \tag{8.57}$$

Here, μ is a nonzero scalar parameter and $\bar{\mathbf{K}}$ is nonsingular if $\mu \neq \lambda_i$ $(i = 1, 2, \ldots, n)$

It can be shown that

$$\bar{\lambda}_i = \lambda_i - \mu \tag{8.58}$$

$$\bar{\mathbf{u}}_i = \mathbf{u}_i, \quad i = 1, 2, \ldots, N \tag{8.59}$$

and

$$\frac{\mathrm{d}\bar{\lambda}_i}{\mathrm{d}b} = \frac{\mathrm{d}\lambda_i}{\mathrm{d}b} \tag{8.60}$$

$$\frac{\mathrm{d}\bar{\mathbf{u}}_i}{\mathrm{d}b} = \frac{\mathrm{d}\mathbf{u}_i}{\mathrm{d}b} \tag{8.61}$$

The derivatives $\mathrm{d}\mathbf{u}_i/\mathrm{d}b$ can be obtained from the derivatives $\mathrm{d}\bar{\mathbf{u}}_i/\mathrm{d}b$ of the eigenproblem of Equation 8.56, in which $\bar{\mathbf{K}}$ is nonsingular. In this context, both method one and method two, discussed in Section 8.5.1 and Section 8.5.2, can be applied to deal with the free–free structures with rigid-body modes.

To achieve a faster average convergent speed for all the first m eigenvector derivatives, μ can be determined as

$$\begin{cases} \mu = \dfrac{\left(\sum_{j=1}^{m} \lambda_j\right)}{m}, & j = 1, 2, \ldots, m \\ \mu \neq \lambda_j, \end{cases} \tag{8.62}$$

8.7 Matrix Perturbation Theory for Repeated Modes

8.7.1 Basic Equations

In this section, let us consider the case of repeated eigenvalues, namely, $\lambda_{0i} = \lambda_{0i+1} = \cdots = \lambda_{0i+m-1}$. The system is known as a *degenerate system*. In engineering, many complex and large structures, such as airplanes, rockets, tall towers, bridges, and ocean platforms, often have multiple or cluster eigenvalues. The matrix perturbation for the repeated modes is presented in Haug, et al. (1980), Chen and Pan (1986), Hu (1987), Mills-Curran (1988), Ojalvo (1988), Dailey (1989), Lim et al. (1989) and Shaw and Jayasuriya (1992).

Assume that $\lambda_0 = \lambda_{01} = \lambda_{02} = \cdots = \lambda_{0m}$; that is, λ_0 is a repeated eigenvalue with multiplicity equal to m, and $\mathbf{u}_{01}, \mathbf{u}_{02}, \cdots, \mathbf{u}_{0m}$ are the eigenvectors associated with λ_0. Then, a linear combination of $\mathbf{u}_{0j}$ $(j = 1, 2, \ldots, m)$, denoted as $\mathbf{U}_0$, will also be the eigenvector associated with λ_0:

$$\mathbf{U}_0 = \mathbf{U}_{0m}\alpha \tag{8.63}$$

where

$$\mathbf{U}_{0m} = [\mathbf{u}_{01}, \mathbf{u}_{02}, \ldots, \mathbf{u}_{0m}] \tag{8.64}$$

$$\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha} = \mathbf{I} \tag{8.65}$$

and

$$\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_m]^{\mathrm{T}} \tag{8.66}$$

Note that $\boldsymbol{\alpha}$ is a constant matrix to be determined.

According to the matrix perturbation method, the eigenvalues and eigenvectors of the structure with repeated eigenvalues for the perturbed structure can be expressed as

$$\boldsymbol{\Lambda}_m = \boldsymbol{\Lambda}_0 + \varepsilon\boldsymbol{\Lambda}_1 \tag{8.67}$$

$$\mathbf{U}_m = \mathbf{U}_{0m}\alpha + \varepsilon(\mathbf{U}_0\mathbf{C}_m + \mathbf{U}_A\mathbf{C}_A) = \mathbf{U}_{0m}\alpha + \varepsilon(\mathbf{U}_{0m}\alpha\mathbf{C}_m + \mathbf{U}_A\mathbf{C}_A) \tag{8.68}$$

where $\mathbf{U}_A$ is the $n \times (n - m)$ modal matrix containing all the eigenvectors except $\mathbf{U}_{0m}$, $\boldsymbol{\Lambda}_m$ is the $m \times m$ eigenvalue diagonal matrix of the perturbed structure, $\boldsymbol{\Lambda}_1$ is the $m \times m$ diagonal matrix with its diagonal elements equal to the first-order perturbations of eigenvalues, $\mathbf{C}_m$ is an $m \times m$ matrix to be determined, and $\mathbf{C}_A$ is an $(n - m) \times (n - m)$ matrix to be determined.

8.7.2 The First-Order Perturbation of Eigensolutions

$\boldsymbol{\Lambda}_1$ and α can be computed from the following $(m \times m)$ eigenproblem:

$$\mathbf{W}\boldsymbol{\alpha} = \boldsymbol{\alpha}\boldsymbol{\Lambda}_1, \quad \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha} = \mathbf{I} \tag{8.69}$$

where

$$\mathbf{W} = \mathbf{U}_{0m}^{\mathrm{T}}(\mathbf{K}_1 - \lambda_0\mathbf{M}_1)\mathbf{U}_{0m} \tag{8.70}$$

Solving the $m \times m$ eigenproblem of Equation 8.69 can produce $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\alpha}$.

If matrix $\mathbf{W}$ has no repeated eigenvalues, $\boldsymbol{\alpha}$ can be uniquely determined; if matrix $\mathbf{W}$ has repeated eigenvalues, $\boldsymbol{\alpha}$ can be determined using the higher order perturbation equations. Here, we assume that matrix $\mathbf{W}$ has no repeated eigenvalues; that is, $\lambda_{1i} \neq \lambda_{1j}$, $(i \neq j)$, where λ_{1k} $(0 < k \leq m)$ are the elements of the diagonal matrix $\boldsymbol{\Lambda}_1$.

The matrix $\mathbf{C}_A$ is

$$\mathbf{C}_A = (\boldsymbol{\Lambda}_A - \lambda_0\mathbf{I})^{-1}\mathbf{U}_A^{\mathrm{T}}(\lambda_0\mathbf{M}_1 - \mathbf{K}_1)\mathbf{U}_{0m}\boldsymbol{\alpha} \tag{8.71}$$

The elements of C_m are

$$C_{ij}^{m} = \frac{R_{ij}}{\lambda_{jm}^{(1)} - \lambda_{im}^{(1)}}, \quad i \neq j \quad i,j = 1,2,\ldots,m \tag{8.72}$$

where R_{ij} are the elements of $\mathbf{R}$ given by

$$\mathbf{R} = -\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{U}_{0m}^{\mathrm{T}}\mathbf{M}_1\mathbf{U}_{0m}\boldsymbol{\alpha}\boldsymbol{\Lambda}_1 + \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{U}_{0m}\mathbf{K}_1\mathbf{U}_A\mathbf{C}_A - \lambda_0\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{U}_{0m}\mathbf{M}_1\mathbf{U}_A\mathbf{C}_A - \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{U}_{0m}^{\mathrm{T}}\mathbf{M}_0\mathbf{U}_A\mathbf{C}_A\boldsymbol{\Lambda}_1 \tag{8.73}$$

and

$$C_{ii}^{m} = \frac{1}{2}Q_{ii} \tag{8.74}$$

where Q_{ii} is the diagonal elements of $\mathbf{Q}$, given by

$$\mathbf{Q} = -\alpha^{\mathrm{T}}\mathbf{U}_{0m}^{\mathrm{T}}\mathbf{M}_1\mathbf{U}_{0m}\alpha \tag{8.75}$$

8.7.3 High-Accuracy Modal Superposition for the First-Order Perturbation of Repeated Modes

In Section 8.5, the high-accuracy modal superposition for the first-order perturbation of eigenvectors of distinct eigenvalues is given. In this section, we extend these methods to the situation with repeated modes.

8.7.3.1 Method One for Computing $\mathbf{U}_1$

Assuming $\mathbf{U}_{AL}$ and $\boldsymbol{\Lambda}_{AL}$ are the first L modes and eigenvalues excluding the repeated modes, the first-order perturbation of eigenvectors is

$$\mathbf{U}_1 = \mathbf{U}_{0m}\alpha\mathbf{C}_m + \mathbf{U}_{AL}\mathbf{C}_{AL} + \mathbf{S}_R \tag{8.76}$$

$$\mathbf{S}_R = \mathbf{U}_{\mathrm{S}} - [\mathbf{U}_{0m}\vdots\mathbf{U}_{AL}]\mathrm{diag}(\lambda_0^{-1}, \boldsymbol{\Lambda}_{AL}^{-1})[\mathbf{U}_{0m}\vdots\mathbf{U}_{AL}]^{\mathrm{T}}\mathbf{T} \tag{8.77}$$

where $\mathbf{U}_{\mathrm{S}}$ is the static displacement obtained by

$$\mathbf{K}\mathbf{U}_{\mathrm{S}} = \mathbf{T} \tag{8.78}$$

and

$$\mathbf{T} = \mathbf{M}_0\mathbf{U}_{0m}\boldsymbol{\alpha}\boldsymbol{\Lambda}_1 + \lambda_0\mathbf{M}_1\mathbf{U}_{0m}\boldsymbol{\alpha} - \mathbf{K}_1\mathbf{U}_{0m}\boldsymbol{\alpha} \tag{8.79}$$

In Equation 8.79, $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\alpha}$ can be obtained from Equation 8.69.

The matrix $\mathbf{C}_{AL}$ is given by

$$\mathbf{C}_{AL} = (\boldsymbol{\Lambda}_{AL} - \lambda_0\mathbf{I})^{-1}\mathbf{U}_{AL}^{\mathrm{T}}(\lambda_0\mathbf{M}_1 - \mathbf{K}_1)\mathbf{U}_{0m}\boldsymbol{\alpha} \tag{8.80}$$

and the elements of matrix $\mathbf{C}_m$ are

$$C_{ij}^{m} = \frac{R_{ij}}{\lambda_{jm}^{(1)} - \lambda_{im}^{(1)}}, \quad i \neq j, \quad i,j = 1,2,\ldots,m \tag{8.81}$$

where $\mathbf{R}$ is given by

$$\mathbf{R} = \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{U}_{0m}^{\mathrm{T}}\mathbf{M}_1\mathbf{U}_{0m}\boldsymbol{\Lambda}_1 - \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{U}_{0m}^{\mathrm{T}}(\lambda_0\mathbf{M}_1 - \mathbf{K}_1)(\mathbf{U}_{AL}\mathbf{C}_{AL} + \mathbf{S}_R) - \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{U}_{0m}^{\mathrm{T}}\mathbf{M}_0\mathbf{S}_R\boldsymbol{\Lambda}_1 \tag{8.82}$$

The diagonal elements of $\mathbf{C}_m$ are

$$C_{ii}^{m} = \frac{1}{2}Q_{ii} \tag{8.83}$$

where

$$\mathbf{Q} = -\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{U}_{0m}^{\mathrm{T}}\mathbf{M}_1\mathbf{U}_{0m}\boldsymbol{\alpha} - \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{U}_{0m}^{\mathrm{T}}\mathbf{M}_0\mathbf{S}_R - \mathbf{S}_R^{\mathrm{T}}\mathbf{M}_0\mathbf{U}_{0m}\boldsymbol{\alpha} \tag{8.84}$$

8.7.3.2 Method Two for Computing $\mathbf{U}_1$

The first-order perturbation of eigenvectors can be expressed as

$$\mathbf{U}_1 = \mathbf{U}_{0m}\boldsymbol{\alpha}\mathbf{C}_m + \mathbf{U}_{AL}\mathbf{C}_{AL} + \mathbf{S}_R \tag{8.85}$$

where $\mathbf{C}_{AL}$ can also be calculated using Equation 8.80; that is

$$\mathbf{C}_{AL} = (\boldsymbol{\Lambda}_{AL} - \lambda_0\mathbf{I})^{-1}\mathbf{U}_{AL}^{\mathrm{T}}(\lambda_0\mathbf{M}_1 - \mathbf{K}_1)\mathbf{U}_{0m}\boldsymbol{\alpha}$$

and $\mathbf{S}_R$ is given by

$$\mathbf{S}_R = \sum_{j=0}^{\infty}\lambda_0^j(\mathbf{H}_j - \mathbf{W}_j) \tag{8.86}$$

where

$$\mathbf{W}_j = [\mathbf{U}_{0m}\vdots\mathbf{U}_{AL}]\boldsymbol{\Lambda}_0^{-j-1}[\mathbf{U}_{0m}\vdots\mathbf{U}_{AL}]^{\mathrm{T}}\mathbf{T}, \quad j \geq 0 \tag{8.87}$$

$$\mathbf{T} = \mathbf{M}_0\mathbf{U}_{0m}\boldsymbol{\alpha}\boldsymbol{\Lambda}_1 + \lambda_0\mathbf{M}_1\mathbf{U}_{0m}\boldsymbol{\alpha} - \mathbf{K}_1\mathbf{U}_{0m}\boldsymbol{\alpha} \tag{8.88}$$

The iterative method for computing $\mathbf{H}_j$ is as follows:

$$\begin{aligned}
&\mathbf{H}_0 = \mathbf{K}^{-1}\mathbf{T},\\
&\mathbf{F}'_{j-1} = \mathbf{M}\mathbf{H}_{j-1}, \quad j \geq 1\\
&\mathbf{H}_j = \mathbf{K}^{-1}\mathbf{F}'_{j-1},
\end{aligned} \tag{8.89}$$

This iterative process can be terminated according to the accuracy requirement. If we define $\mathbf{S}_R(k)$ as

$$\mathbf{S}_R(k) = \sum_{j=0}^{k}\lambda_0^j(\mathbf{H}_j - \mathbf{W}_j) \tag{8.90}$$

the termination condition can be stated as

$$\|S_R(k) - S_R(k-1)\|_2 \leq \varepsilon, \quad j = 1, 2, \ldots, m \tag{8.91}$$

where ε is a specified accuracy requirement.

The computation method for $\mathbf{C}_m$ in Equation 8.85 is similar to that of Equation 8.81 to Equation 8.84. The only difference is that $\mathbf{S}_R$ in Equation 8.82 and Equation 8.84 can be replaced with $\mathbf{S}_R(k)$ in Equation 8.90.

8.8 Matrix Perturbation Method for Closely Spaced Eigenvalues

The vibration modes with close frequencies, that is, with clusters of frequencies, often occur in certain structural systems including large space structures, multispan beams, and in some nearly periodic structures and symmetric structures. Therefore, it is important here to present the perturbation method for vibration modes with close eigenvalues (Liu, 2000).

The perturbation analysis of close eigenvalues can be transformed into a problem with a repeated eigenvalue, which is equal to the average value of the close eigenvalues (Chen, 1993).

8.8.1 Method One of Perturbation Analysis for Close Eigenvalues

Consider vibration eigenproblem

$$\mathbf{K}_0[\mathbf{U}_0 \vdots \mathbf{U}_A] = \mathbf{M}_0[\mathbf{U}_0 \vdots \mathbf{U}_A]\text{diag}(\mathbf{\Lambda}_0, \mathbf{\Lambda}_A) \tag{8.92}$$

$$[\mathbf{U}_0 \vdots \mathbf{U}_A]^{\mathrm{T}}\mathbf{M}_0[\mathbf{U}_0 \vdots \mathbf{U}_A] = \mathbf{I} \tag{8.93}$$

where $\mathbf{K}_0$ and $\mathbf{M}_0$ are $n \times n$ real symmetric matrices, and $\mathbf{\Lambda}_0$ and $\mathbf{U}_0$ are the $m \times m$ diagonal matrix of close eigenvalues and the corresponding $n \times m$ modal matrix.

Using the spectral decomposition of $\mathbf{K}_0$, the problem can be expressed as

$$\mathbf{K}_0 = \bar{\mathbf{K}}_0 + \varepsilon\,\delta\mathbf{K}_0 \tag{8.94}$$

where

$$\bar{\mathbf{K}}_0 = \mathbf{M}_0(\lambda_0\mathbf{U}_0\mathbf{U}_0^{\mathrm{T}})\mathbf{M}_0 + \mathbf{M}_0(\mathbf{U}_A\mathbf{\Lambda}_A\mathbf{U}_A^{\mathrm{T}})\mathbf{M}_0 \tag{8.95}$$

$$\varepsilon\,\delta\mathbf{K}_0 = \mathbf{M}_0(\mathbf{U}_0(\varepsilon\,\delta\mathbf{\Lambda}_0)\mathbf{U}_0^{\mathrm{T}})\mathbf{M}_0 \tag{8.96}$$

$$\varepsilon\,\delta\mathbf{\Lambda}_0 = \mathbf{\Lambda}_0 - \lambda_0\mathbf{I} = \mathbf{\Lambda}_0 - \left(\frac{\sum_{i=1}^{m}\lambda_{0i}}{m}\right)\mathbf{I} \tag{8.97}$$

It can be seen that $\bar{\mathbf{K}}_0$ given by Equation 8.95 satisfies

$$\bar{\mathbf{K}}_0[\mathbf{U}_0 \vdots \mathbf{U}_A] = \mathbf{M}_0[\mathbf{U}_0 \vdots \mathbf{U}_A]\text{diag}(\lambda_0\mathbf{I}, \mathbf{\Lambda}_A) \tag{8.98}$$

$$[\mathbf{U}_0 \vdots \mathbf{U}_A]^{\mathrm{T}}\mathbf{M}_0[\mathbf{U}_0 \vdots \mathbf{U}_A] = \mathbf{I} \tag{8.99}$$

This indicates that λ_0 and $\mathbf{U}_0$ are the repeated eigenvalues and the corresponding eigenvector subspace with multiplicity m of the eigenproblem (Equation 8.92), and $\mathbf{\Lambda}_A$ and $\mathbf{U}_A$ are also the eigensolution of eigenproblem (Equation 8.92).

If $\mathbf{\Lambda}_0 \rightarrow \lambda_0\mathbf{I}$, $\varepsilon\,\delta\mathbf{\Lambda}_0 \rightarrow \mathbf{0}$, and $\bar{\mathbf{K}}_0 \rightarrow \mathbf{K}_0$, and if the small parameter modifications $\varepsilon\mathbf{K}_1$ and $\varepsilon\mathbf{M}_1$ are introduced to the matrices $\mathbf{K}_0$ and $\mathbf{M}_0$, the eigenproblem with close eigenvalues becomes

$$(\mathbf{K}_0 + \varepsilon\mathbf{K}_1)\mathbf{U} = (\mathbf{M}_0 + \varepsilon\mathbf{M}_1)\mathbf{U}\mathbf{\Lambda} \tag{8.100}$$

$$\mathbf{U}(\mathbf{M}_0 + \varepsilon\mathbf{M}_1)\mathbf{U}^{\mathrm{T}} = \mathbf{I} \tag{8.101}$$

Substituting Equation 8.94 into Equation 8.100, we obtain

$$(\bar{\mathbf{K}}_0 + \varepsilon\bar{\mathbf{K}}_1)\mathbf{U} = (\mathbf{M}_0 + \varepsilon\mathbf{M}_1)\mathbf{U}\mathbf{\Lambda} \tag{8.102}$$

$$\mathbf{U}(\mathbf{M}_0 + \varepsilon\mathbf{M}_1)\mathbf{U}^{\mathrm{T}} = \mathbf{I} \tag{8.103}$$

where

$$\varepsilon\bar{\mathbf{K}}_1 = \varepsilon\,\delta\mathbf{K}_0 + \varepsilon\mathbf{K}_1 \tag{8.104}$$

$$\mathbf{\Lambda} = \lambda_0\mathbf{I} + \varepsilon\mathbf{\Lambda}_1 + \varepsilon^2\mathbf{\Lambda}_2 + \cdots \tag{8.105}$$

$$\mathbf{U} = \mathbf{U}_0\alpha + \varepsilon\mathbf{U}_1 + \varepsilon^2\mathbf{U}_2 + \cdots \tag{8.106}$$

Therefore, the eigenproblem of Equation 8.102 and Equation 8.103 can be considered to be a perturbed eigenproblem with the perturbation matrices equal to $(\delta\mathbf{K}_0 + \mathbf{K}_1)$ and $\mathbf{M}_1$, respectively. The eigensolutions, $\mathbf{\Lambda}$ and $\mathbf{U}$, can be obtained from Equation 8.102 and Equation 8.103 by using the perturbation method for repeated eigenvalues as discussed in Section 8.7. Accordingly, the perturbation problem of modes with close eigenvalues is transformed into one of the repeated eigenvalues.

The complete algorithm for $\mathbf{\Lambda}$ and $\mathbf{U}$ is given below.

(1) Compute

$$\lambda_0 = \frac{\sum_{i=1}^{m} \lambda_{0i}}{m}$$

(2) Compute

$$\mathbf{W} = \mathbf{U}_0^{\mathrm{T}}(\delta\mathbf{K}_0 + \mathbf{K}_1 - \lambda_0\mathbf{M}_1)\mathbf{U}_0$$

(3) Solve the eigenvalue problem

$$\mathbf{W}\boldsymbol{\alpha} = \boldsymbol{\alpha}\mathbf{\Lambda}_1$$

$$\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha} = \mathbf{I}$$

(4) Compute the perturbed eigenvalues of the close eigenvalues

$$\mathbf{\Lambda} = \lambda_0\mathbf{I} + \mathbf{\Lambda}_1$$

(5) Compute the new eigenvectors $\mathbf{U}_0\boldsymbol{\alpha}$ corresponding to λ_0.

(6) Compute the matrix $\mathbf{C}_A$

$$\mathbf{C}_A = (\mathbf{\Lambda}_A - \lambda_0\mathbf{I})^{-1}\mathbf{U}_A^T(\lambda_0\mathbf{M}_1 - \mathbf{K}_1 - \delta\mathbf{K}_0)\mathbf{U}_0\boldsymbol{\alpha}$$

(7) Compute

$$\mathbf{R} = [R_{ij}]$$

$$\mathbf{R} = -\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{U}_0^{\mathrm{T}}\mathbf{M}_1\mathbf{U}_0\boldsymbol{\alpha}\mathbf{\Lambda}_1 - \lambda_0\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{U}_0^{\mathrm{T}}\mathbf{M}_1\mathbf{U}_A\mathbf{C}_A - \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{U}_0^{\mathrm{T}}(\delta\mathbf{K}_0 + \mathbf{K}_1)\mathbf{U}_A\mathbf{C}_A - \boldsymbol{\alpha}\mathbf{U}_0^{\mathrm{T}}\mathbf{M}_0\mathbf{U}_A\mathbf{C}_A\mathbf{\Lambda}_1$$

(8) Compute

$$\mathbf{C}_m = [C_{ij}^m]$$

$$C_{ij}^m = \frac{R_{ij}}{\lambda_{1j} - \lambda_{1i}}, \quad i \neq j, \quad i, j = 1, 2, \ldots, m$$

$$C_{ii}^m = \frac{1}{2}Q_{ii}$$

$$\mathbf{Q} = -\alpha^{\mathrm{T}}\mathbf{U}_0^{\mathrm{T}}\mathbf{M}_1\mathbf{U}_0\alpha$$

(9) Compute the perturbed eigenvectors $\mathbf{U}$

$$\mathbf{U} = \mathbf{U}_0\boldsymbol{\alpha} + \mathbf{U}_0\boldsymbol{\alpha}\mathbf{C}_m + \mathbf{U}_A\mathbf{C}_A$$

8.8.2 Method Two of Perturbation Analysis for Close Eigenvalues

Because of the importance of the problem in both theory and practice, we now present method two of perturbation analysis for close eigenvalues, which is equivalent to method one given above.

Using the spectral decomposition of $\mathbf{M}_0$, the problem can be expressed as

$$\mathbf{M}_0 = \bar{\mathbf{M}}_0 + \varepsilon\,\delta\mathbf{M}_0 \tag{8.107}$$

Then, the following equations hold:

$$\mathbf{K}_0[\mathbf{U}_0 \dot{:} \mathbf{U}_A] = \bar{\mathbf{M}}_0[\mathbf{U}_0 \dot{:} \mathbf{U}_A]\mathrm{diag}(\lambda_0\mathbf{I}, \boldsymbol{\Lambda}_A) \tag{8.108}$$

$$[\mathbf{U}_0 \dot{:} \mathbf{U}_A]^{\mathrm{T}}\mathbf{M}_0[\mathbf{U}_0 \dot{:} \mathbf{U}_A] = \mathbf{I} \tag{8.109}$$

where

$$\bar{\mathbf{M}} = \lambda_0^{-2}\mathbf{K}_0\mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\mathbf{K}_0 + \mathbf{K}_0[\mathbf{U}_A(\boldsymbol{\Lambda}_A^{-1})^2\mathbf{U}_A^{\mathrm{T}}]\mathbf{K}_0 \tag{8.110}$$

$$\varepsilon\,\delta\mathbf{M}_0 = \mathbf{K}_0[\mathbf{U}_0\varepsilon\,\delta(\boldsymbol{\Lambda}_0^{-1})^2\mathbf{U}_0^{\mathrm{T}}]\mathbf{K}_0 \tag{8.111}$$

$$\varepsilon\,\delta\boldsymbol{\Lambda}_0^{-2} = \boldsymbol{\Lambda}_0^{-2} - \lambda_0^{-2}\mathbf{I} \tag{8.112}$$

and

$$\lambda_0 = \frac{\sum_{i=1}^{m}\lambda_{0i}}{m} \tag{8.113}$$

It can be seen that $\bar{\mathbf{M}}_0$ and $\varepsilon\,\delta\mathbf{M}_0$ given by Equation 8.110 and Equation 8.111 satisfy Equation 8.108 and Equation 8.109; that is, λ_0 and $\mathbf{U}_0$ are the repeated eigenvalues and corresponding modal matrix of Equation 8.108 and Equation 8.109. $\boldsymbol{\Lambda}_A$ and $\mathbf{U}_A$ are the eigenvalue diagonal matrix and the corresponding modal matrix excluding $\boldsymbol{\Lambda}_0$ and $\mathbf{U}_0$, respectively.

If $\mathbf{K}_0$ and $\mathbf{M}_0$ are modified to $\mathbf{K}_0 + \varepsilon\mathbf{K}_1$ and $\mathbf{M}_0 + \varepsilon\mathbf{M}_1$, the eigenvalue problem becomes

$$(\mathbf{K}_0 + \varepsilon\mathbf{K}_1)\mathbf{U} = (\mathbf{M}_0 + \varepsilon\mathbf{M}_1)\mathbf{U}\boldsymbol{\Lambda} \tag{8.114}$$

$$\mathbf{U}(\mathbf{M}_0 + \varepsilon\mathbf{M}_1)\mathbf{U}^{\mathrm{T}} = \mathbf{I} \tag{8.115}$$

Substituting Equation 8.107 into Equation 8.114 yields

$$(\mathbf{K}_0 + \varepsilon\mathbf{K}_1)\mathbf{U} = (\bar{\mathbf{M}}_0 + \varepsilon\bar{\mathbf{M}}_1)\mathbf{U}\boldsymbol{\Lambda} \tag{8.116}$$

$$\mathbf{U}(\bar{\mathbf{M}}_0 + \varepsilon\bar{\mathbf{M}}_1)\mathbf{U}^{\mathrm{T}} = \mathbf{I} \tag{8.117}$$

where

$$\varepsilon\bar{\mathbf{M}}_1 = \varepsilon\,\delta\mathbf{M}_0 + \varepsilon\mathbf{M}_1 \tag{8.118}$$

Thus, Equation 8.116 and Equation 8.117 can be considered to be a perturbed eigenproblem with repeated eigenvalues, and the perturbation method for repeated eigenvalues can be used to obtain the perturbed eigensolutions of Equation 8.116 and Equation 8.117:

$$\boldsymbol{\Lambda} = \lambda_0\mathbf{I} + \varepsilon\boldsymbol{\Lambda}_1 + \cdots \tag{8.119}$$

$$\mathbf{U} = \mathbf{U}_0\alpha + \varepsilon\mathbf{U}_1 + \cdots \tag{8.120}$$

Example 8.5

For the six-DoF mass–spring system shown in Figure 8.5, the stiffness and mass matrices $\mathbf{K}_0$ and $\mathbf{M}_0$ are given by

$$\mathbf{K}_0 = \begin{bmatrix} 1500 & -1000 & & & & \\ -1000 & 1200 & -200 & & & \\ & -200 & 15,200 & -5000 & -5000 & -5000 \\ & & -5000 & 5000 & & \\ & & -5000 & & 5000 & \\ & & -5000 & & & 5000 \end{bmatrix} \text{ (N/m)}$$

$$\mathbf{M}_0 = \text{diag}(200, 300, 50, 20, 20, 20.004) \text{ (kg)}$$

The perturbation eigensolutions are computed for the following three cases:

Case 1

$$\varepsilon\mathbf{K}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ (N/m)}$$

$$\varepsilon\mathbf{M}_1 = \text{diag}(0, \ldots, 0) \text{ (kg)}$$

Case 2

$$\varepsilon\mathbf{K}_1 = \begin{bmatrix} 5.00 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ (N/m)}$$

$$\varepsilon\mathbf{M}_1 = \text{diag}(0, 0, 0, 0, 0, 0.5) \text{ (kg)}$$

Case 3

$$\varepsilon\mathbf{K}_1 = \mathbf{0},\ \varepsilon\boldsymbol{M}_1 = \text{diag}(0, 0, 0, 0, 2.0, 0) \text{ (kg)}$$

The unperturbed eigensolutions have a single pair of close eigenvalues given by

$$\boldsymbol{\Lambda}_0 = \text{diag}(249.966642, 250.000000)$$

$$\mathbf{U}_0^{\mathrm{T}} = \begin{bmatrix} 0.00000 & 0.00000 & -0.00012 & -0.091283 & -0.091283 & 0.182560 \\ 0.00000 & 0.00000 & 0.00000 & -0.158114 & 0.158114 & 0.00000 \end{bmatrix}$$

The other unperturbed eigensolutions are as follows:

$$\boldsymbol{\Lambda}_A = \text{diag}(0.594885, 2.478725, 10.234656, 552.175102)$$

$$\mathbf{U}_A = \begin{bmatrix} -0.058595 & -0.027542 & 0.028427 & -0.000001 \\ 0.032047 & -0.027659 & 0.039259 & 0.000127 \\ -0.006732 & 0.074593 & 0.058387 & -0.104793 \\ -0.007020 & 0.075340 & 0.058527 & 0.086699 \\ -0.007020 & 0.075340 & 0.058527 & 0.086699 \\ -0.007020 & 0.075340 & 0.058527 & 0.086667 \end{bmatrix}$$

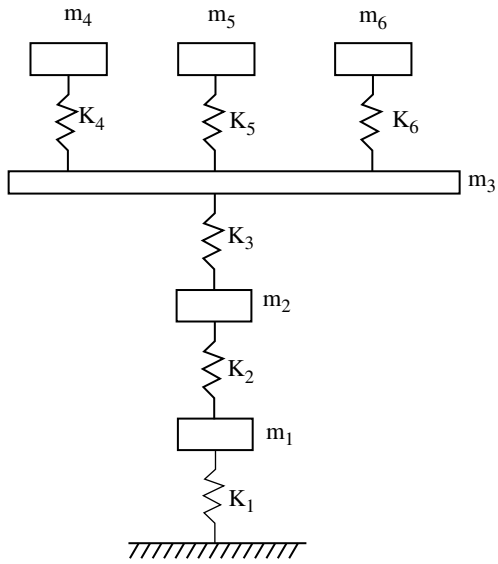

FIGURE 8.5 Six-DoF mass–spring system for Example 8.5.

The perturbed eigensolutions associated with the single pair of close eigenvalues for the three cases are summarized in Table 8.6. These results show that the perturbation analysis of distinct eigenvalues is not only inaccurate but also misleading when applied to close eigenvalues, and that the perturbed eigensolutions given by the present method are in good agreement with the exact solutions.

For example, in Case 3, the eigenvalue errors induced by the present method are reduced to 1.047950 and 0.000000, while the errors induced by the perturbation of distinct eigenvalues are 4.174100 and 3.025595. The eigenvectors obtained by the perturbation method of distinct eigenvalues are not only

TABLE 8.6 Comparison of Eigensolutions with Close Eigenvalues

	Exact		Perturbation Method of Distinct Eigenvalues		Perturbation Method of Close Eigenvalues	
			Case 1			
Eigenvalues	249.973872	250.159351	250.008324	250.125000	249.973872	250.159451
Eigenvectors	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
	0.000018	−0.000066	−0.000043	0.000079	0.000011	0.000005
	0.150492	0.103411	−0.433574	0.039471	0.150501	−0.103354
	0.014254	−0.181965	0.251058	0.355699	0.014251	0.182015
	−0.164753	0.078701	0.182585	−0.395285	−0.164745	−0.078658
			Case 2			
Eigenvalues	243.878599	247.932109	244.759777	246.875000	243.725997	247.908461
Eigenvectors	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
	0.000000	−0.000004	0.000002	0.000005	0.000000	0.000000
	−0.000015	0.001506	−0.000772	−0.001976	0.000011	0.000006
	−0.000622	0.182090	8.467433	−5.098718	0.001103	0.183060
	−0.155859	−0.091163	−8.648272	−4.784467	−0.157555	−0.092069
	0.156472	−0.090082	0.180521	9.882118	0.158052	−0.090988
			Case 3			
Eigenvalues	234.474405	249.975015	245.800915	237.500000	233.326455	249.975015
Eigenvectors	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
	0.000016	0.000000	−0.000008	0.000021	0.000000	−0.000000
	−0.005558	−0.000015	0.003030	−0.007905	0.000006	0.000010
	−0.089507	−0.158189	34.141638	−19.920531	−0.091153	0.158189
	0.175420	0.000158	−34.321586	−19.612210	0.182572	−0.000158
	−0.089778	0.158022	0.181586	39.528471	−0.091415	−0.158025

inaccurate but also misleading, while good agreement with the exact eigenvectors has been obtained by the present method.

8.8.3 Concluding Remarks

Perturbation analysis of vibration modes with close frequencies is presented in this section. It can be regarded as a general treatment of perturbation analysis, because the perturbation analysis of both distinct eigenvalues and repeated eigenvalues is contained in the present method. The results obtained by this method allow one to analyze the influence of parameter changes in a system on the dynamic characteristics of the system, which is very important for effective structural design.

8.9 Matrix Perturbation Theory for Complex Modes

In Section 8.2 to Section 8.8, the matrix perturbation for real modes of systems with real symmetric mass and stiffness matrices, **M** and **K**, was given. However, in many engineering problems such as systems with nonproportional damping (see Chapter 1), dynamic systems under nonconservative forces, analysis of aero-elastic flutter, and structural vibration control systems, the system matrices are not symmetric and may not be diagonalizable. In this case, the matrix perturbation for real modes cannot be used, and we must use the matrix perturbation for complex modes (Murthy and Haftka, 1988; Chen, 1993; Liu, 1999; Adhikari and Friswell, 2001). In the following, we assume that the system is not defective; that is, the system has a complete eigenvector set to span the eigenspace. The discussion in this chapter is limited to the nondefective systems.

8.9.1 Basic Equations

The vibration equation of a linear system with n-DoFs is given by

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{Q}(t) \tag{8.121}$$

where the matrices **M**, **C**, and **K**, are assumed to be real and unsymmetric. The free vibration equation of the system is

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0} \tag{8.122}$$

The corresponding right eigenvalues problem is

$$(\mathbf{M}s^2 + \mathbf{C}s + \mathbf{K})\mathbf{x} = \mathbf{0} \tag{8.123}$$

and its adjoint eigenvalue problem is

$$(\mathbf{M}s^2 + \mathbf{C}s + \mathbf{K})^{\mathrm{T}}\mathbf{y} = 0$$

$$\mathbf{y}^{\mathrm{T}}(\mathbf{M}s^2 + \mathbf{C}s + \mathbf{K}) = 0 \tag{8.124}$$

It is common in literature to call **y** the *left eigenvector*, while **x** in the original system, a column vector, is called the *right eigenvector*.

Let us introduce a state vector

$$\mathbf{u} = \begin{Bmatrix} s\mathbf{x} \\ \mathbf{x} \end{Bmatrix} = \mathbf{T}\mathbf{x} \tag{8.125}$$

where **T** is the state transformation matrix

$$\mathbf{T} = \begin{Bmatrix} s\mathbf{I} \\ \mathbf{I} \end{Bmatrix} \tag{8.126}$$

Similarly, we introduce the state vector

$$\mathbf{v} = \begin{Bmatrix} s\mathbf{y} \\ \mathbf{y} \end{Bmatrix} = \mathbf{T}\mathbf{y} \tag{8.127}$$

Hence, Equation 8.123 and Equation 8.124 become

$$(\mathbf{A}s + \mathbf{B})\mathbf{u} = \mathbf{0} \tag{8.128}$$

$$(\mathbf{A}s + \mathbf{B})^{\mathrm{T}}\mathbf{v} = \mathbf{0} \tag{8.129}$$

or

$$\mathbf{v}^{\mathrm{T}}(\mathbf{A}s + \mathbf{B}) = \mathbf{0}$$

where

$$\mathbf{A} = \begin{bmatrix} -\mathbf{C} & -\mathbf{K} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

It is well known that the eigenvalues of the adjoint eigenproblem (Equation 8.129) are identical to that of the original eigenproblem (Equation 8.128). The *characteristic equation* is

$$\det(\mathbf{A} + s\mathbf{B}) = 0$$

This characteristic determinant is a polynomial of $2n$ order in s, and $2n$ eigenvalues s_i ($i = 1, 2, \ldots, 2n$) can be found in the complex domain. The left and right modal vectors, $\mathbf{v}_i$ and $\mathbf{u}_i$, corresponding to s_i satisfy

$$\mathbf{A}\mathbf{u}_i = s_i\mathbf{B}\mathbf{u}_i \tag{8.130}$$

and

$$\mathbf{A}^{\mathrm{T}}\mathbf{v}_i = s_i\mathbf{B}^{\mathrm{T}}\mathbf{v}_i \tag{8.131}$$

The *orthogonality conditions* are

$$\mathbf{v}_j^{\mathrm{T}}\mathbf{B}\mathbf{u}_i = \mathbf{0} \tag{8.132}$$

$$\mathbf{v}_j^{\mathrm{T}}\mathbf{A}\mathbf{u}_i = \mathbf{0} \tag{8.133}$$

The *normalization conditions* are

$$\begin{aligned} \mathbf{v}_i^{\mathrm{T}}\mathbf{B}\mathbf{u}_i &= 1 \\ \mathbf{u}_i^{\mathrm{T}}\mathbf{B}\mathbf{u}_i &= 1 \end{aligned} \tag{8.134}$$

Therefore, the orthogonality conditions can be written as

$$\begin{aligned} \mathbf{v}_j^{\mathrm{T}}\mathbf{B}\mathbf{u}_i &= \delta_{ij} \\ \mathbf{v}_j^{\mathrm{T}}\mathbf{A}\mathbf{u}_i &= s_i\delta_{ij} \end{aligned} \tag{8.135}$$

8.9.2 Matrix Perturbation Method for Distinct Modes

If small changes are made on the structural parameters, the mass, damping, and stiffness matrices of the system also have small changes given by

$$\begin{aligned} \mathbf{M} &= \mathbf{M}_0 + \varepsilon\mathbf{M}_1 \\ \mathbf{C} &= \mathbf{C}_0 + \varepsilon\mathbf{C}_1 \\ \mathbf{K} &= \mathbf{K}_0 + \varepsilon\mathbf{K}_1 \end{aligned} \tag{8.136}$$

and we have

$$\mathbf{A} = \mathbf{A}_0 + \varepsilon \mathbf{A}_1 \tag{8.137}$$

$$\mathbf{B} = \mathbf{B}_0 + \varepsilon \mathbf{B}_1 \tag{8.138}$$

where ε is a small parameter.

In the following, we first consider the case of distinct eigenvalues, s_{0i}, of the original system. According to the matrix perturbation theory, the eigenvalues and eigenvectors can be expressed as a power series in ε, that is

$$\mathbf{S} = \mathbf{S}_0 + \varepsilon \mathbf{S}_1 + \varepsilon^2 \mathbf{S}_2 + \cdots \tag{8.139}$$

$$\mathbf{U} = \mathbf{U}_0 + \varepsilon \mathbf{U}_1 + \varepsilon^2 \mathbf{U}_2 + \cdots \tag{8.140}$$

$$\mathbf{V} = \mathbf{V}_0 + \varepsilon \mathbf{V}_1 + \varepsilon^2 \mathbf{V}_2 + \cdots \tag{8.141}$$

where $\mathbf{S}_0$, $\mathbf{U}_0$, and $\mathbf{V}_0$ are the eigensolutions of the original system; $\mathbf{S}_1$, $\mathbf{U}_1$, and $\mathbf{V}_1$ are the first-order perturbations of eigensolutions; and $\mathbf{S}_2$, $\mathbf{U}_2$, and $\mathbf{V}_2$ the second-order perturbations.

$\mathbf{U}_1$ can be expressed as a linear combination of the right eigenvectors of the original system as

$$\mathbf{U}_1 = \mathbf{U}_0 \mathbf{C}^1 \tag{8.142}$$

where $\mathbf{C}^1$ is to be the determined matrix given by

$$C_{ij}^1 = \frac{1}{S_{0j} - S_{0i}} P_{ij}^1, \quad j \neq i, \quad i, j = 1, 2, \ldots \tag{8.143}$$

Also

$$\mathbf{S}_1 = \text{diag}(P_{11}^1, P_{22}^1, \ldots) \tag{8.144}$$

where P_{ij}^1 are the elements of $\mathbf{P}^1$ given by

$$\mathbf{P}^1 = \mathbf{V}_0^{\mathrm{T}}(-\mathbf{A}_1 \mathbf{U}_0 + \mathbf{B}_1 \mathbf{U}_0 \mathbf{S}_0) \tag{8.145}$$

The $\mathbf{V}_1$ can be expressed as the expansion of $\mathbf{V}_0$

$$\mathbf{V}_1 = \mathbf{V}_0 \mathbf{D}^1 \tag{8.146}$$

where $\mathbf{D}^1$ is to be the determined coefficient matrix given by

$$D_{ij}^1 = \frac{1}{S_{0j} - S_{0i}} R_{ij}^1, \quad j \neq i, \quad i, j = 1, 2, \ldots \tag{8.147}$$

and R_{ij}^1 are the nondiagonal elements of $\mathbf{R}^1$

$$\mathbf{R}^1 = \mathbf{U}_0^{\mathrm{T}}(\mathbf{B}_1^{\mathrm{T}} \mathbf{V}_0 \mathbf{S}_0 - \mathbf{A}_1^{\mathrm{T}} \mathbf{V}_0) \tag{8.148}$$

If the modification of the parameter is fairly large, the second-order perturbation must be used to obtain high accuracy. According to the expansion theorem, the second-order perturbation of eigenvectors, $\mathbf{U}_2$, can be expressed as

$$\mathbf{U}_2 = \mathbf{U}_0 \mathbf{C}^2 \tag{8.149}$$

In a similar manner, $\mathbf{S}_2$ and the elements C_{ij}^2 can be obtained as

$$\mathbf{S}_2 = \text{diag}(P_{11}^2, P_{22}^2, \ldots) \tag{8.150}$$

$$C_{ij}^2 = \frac{1}{S_{0j} - S_{0i}} P_{ij}^2, \quad j \neq i, i, \quad j = 1, 2, \ldots \tag{8.151}$$

where P_{ij}^2 are the nondiagonal elements of $\mathbf{P}^2$

$$\mathbf{P}^2 = \mathbf{V}_0^{\mathrm{T}} \mathbf{B}_0 \mathbf{U}_1 \mathbf{S}_1 + \mathbf{V}_0^{\mathrm{T}} \mathbf{B}_1 \mathbf{U}_0 \mathbf{S}_1 + \mathbf{V}_0^{\mathrm{T}} \mathbf{B}_1 \mathbf{U}_1 \mathbf{S}_0 - \mathbf{V}_0^{\mathrm{T}} \mathbf{A}_1 \mathbf{U}_1 \tag{8.152}$$

$\mathbf{V}_2$ can be expressed as

$$\mathbf{V}_2 = \mathbf{V}_0\mathbf{D}^2 \tag{8.153}$$

where $\mathbf{D}^2$ is to be the determined coefficient matrix given by

$$D_{ij}^2 = \frac{1}{S_{0j} - S_{0i}} R_{ij}^2, \quad j \neq i, \quad i,j = 1,2,\ldots \tag{8.154}$$

and R_{ij}^2 are the nondiagonal elements of $\mathbf{R}^2$

$$\mathbf{R}^2 = \mathbf{U}_0^{\mathrm{T}}(\mathbf{B}_0^{\mathrm{T}}\mathbf{V}_1\mathbf{S}_1 + \mathbf{B}_1^{\mathrm{T}}\mathbf{V}_0\mathbf{S}_1 + \mathbf{B}_1^{\mathrm{T}}\mathbf{V}_1\mathbf{S}_0 - \mathbf{A}_1^{\mathrm{T}}\mathbf{V}_1) \tag{8.155}$$

If $j = i$, the coefficients C_{ii}^1, D_{ii}^1, C_{ii}^2, and D_{ii}^2 can be computed as

$$C_{ii}^1 = \frac{-1}{\mathbf{u}_{0i}^{\mathrm{T}}(\mathbf{B}_0 + \mathbf{B}_0^{\mathrm{T}})\mathbf{u}_{0i}} \left(\mathbf{u}_{0i}^{\mathrm{T}}\mathbf{B}_1\mathbf{u}_{0i} + \sum_{\substack{j=1 \\ j\neq i}}^{n} C_{ij}\mathbf{u}_{0j}^{\mathrm{T}}(\mathbf{B}_0 + \mathbf{B}_0^{\mathrm{T}})\mathbf{u}_{0i} \right) \tag{8.156}$$

$$D_{ii}^1 = Q_{ii}^1 - C_{ii}^1 \tag{8.157}$$

where Q_{ii}^1 is the diagonal element of $\mathbf{Q}_1$

$$\mathbf{Q}_1 = -\mathbf{V}_0^{\mathrm{T}}\mathbf{B}_1\mathbf{U}_0 \tag{8.158}$$

$$C_{ii}^2 = \frac{-\mathbf{u}_{1i}^{\mathrm{T}}\mathbf{B}_0\mathbf{u}_{1i} + \mathbf{u}_{0i}^{\mathrm{T}}\mathbf{B}_1\mathbf{u}_{1i} + \mathbf{u}_{1i}^{\mathrm{T}}\mathbf{B}_1\mathbf{u}_{0i} - \sum_{\substack{j=1 \\ j\neq i}}^{n} C_{ij}\mathbf{u}_{0j}^{\mathrm{T}}(\mathbf{B}_0 + \mathbf{B}_0^{\mathrm{T}})\mathbf{u}_{0i}}{\mathbf{u}_{0i}^{\mathrm{T}}(\mathbf{B}_0 + \mathbf{B}_0^{\mathrm{T}})\mathbf{u}_{0i}} \tag{8.159}$$

$$D_{ii}^2 = Q_{ii}^2 - C_{ii}^2 \tag{8.160}$$

and Q_{ii}^2 is the diagonal element of $\mathbf{Q}^2$

$$\mathbf{Q}^2 = -\mathbf{V}_0^{\mathrm{T}}\mathbf{B}_0\mathbf{U}_1 - \mathbf{V}_1^{\mathrm{T}}\mathbf{B}_0\mathbf{U}_1 - \mathbf{V}_1^{\mathrm{T}}\mathbf{B}_1\mathbf{U}_0 \tag{8.161}$$

8.9.3 High-Accuracy Modal Superposition for Eigenvector Derivatives

For a large-scale structure, only a small number of the first lower L modes are extracted, and the higher modes are truncated in order to reduce the computational cost. The modal superposition method may not only give inaccurate result, but also may be misleading if the truncation is considerable. In this section, we give a high-accuracy modal superposition method for derivatives of the complex mode of nonsymmetric matrices.

8.9.3.1 Improved Modal Superposition

An improved modal superposition (IMS) to reduce the computation errors by modal truncation was proposed (Lim et al., 1989). The derivatives of modes can be expressed as

$$\frac{\partial \mathbf{u}_i}{\partial \mathbf{b}} = \bar{\alpha}_{ii}\mathbf{u}_i + \bar{z}_i \tag{8.162}$$

$$\bar{z}_i = \sum_{\substack{j=1 \\ j\neq i}}^{L} \frac{\mathbf{v}_j^{\mathrm{T}}\mathbf{F}_i}{S_i - S_j}\mathbf{u}_j + \mathbf{A}^{-1}\mathbf{F}_i + \sum_{j=1}^{L} \frac{\mathbf{v}_j^{\mathrm{T}}\mathbf{F}_i}{S_j}\mathbf{u}_j \tag{8.163}$$

$$\mathbf{F}_i = \left(\frac{\partial \mathbf{A}}{\partial \mathbf{b}} - \frac{\partial S_i}{\partial \mathbf{b}}\mathbf{B} - S_i\frac{\partial \mathbf{B}}{\partial \mathbf{b}} \right)\mathbf{u}_i \tag{8.164}$$

where $\mathbf{A}^{-1}\mathbf{F}_i$ is the contribution of the truncated higher modes to the derivatives of modes, as given by

$$\bar{\alpha}_{ii} = -\frac{1}{2}\left(\mathbf{u}_i^{\mathrm{T}}\frac{\partial \mathbf{B}}{\partial \mathbf{b}}\mathbf{u}_i + \mathbf{u}_i^{\mathrm{T}}\mathbf{B}\bar{z}_i + \bar{z}_i^{\mathrm{T}}\mathbf{B}\mathbf{u}_i\right) \tag{8.165}$$

8.9.3.2 High-Accuracy Modal Superposition

Assume that the eigenvalues are ordered according to their modular magnitude, and satisfy the following condition:

$$|S_i| < |S_j|, \quad j > L \tag{8.166}$$

The derivatives of modes can be expressed as

$$\frac{\partial \mathbf{u}_i}{\partial \mathbf{b}} = \bar{\alpha}_{ii}\mathbf{u}_i + \bar{z}_i = \bar{\alpha}_{ii}\mathbf{u}_i + z_{iL} + z_{iH} \tag{8.167}$$

where

$$\bar{z}_i = z_{iL} + z_{iH} \tag{8.168}$$

$$z_{iL} = \sum_{\substack{j=1 \\ j\neq i}}^{L} \frac{\mathbf{v}_j^{\mathrm{T}}\mathbf{F}_i}{S_i - S_j}\mathbf{u}_j \tag{8.169}$$

$$z_{iH} = -\sum_{j=1}^{K}\left(\mathbf{A}^{-1}(\mathbf{B}\mathbf{A}^{-1})^{j-1} - \mathbf{U}_L((\mathbf{S}_L)^{-1})^j\mathbf{V}_L^{\mathrm{T}}\right)\mathbf{F}_i \tag{8.170}$$

Also, $\bar{\alpha}_{ii}$ can be obtained from Equation 8.165, and K denotes the number of terms used in series (Equation 8.170).

It can be shown that for $K = 1$, Equation 8.167 is equivalent to Equation 8.162.

8.9.3.3 Numerical Example

Example 8.6

Consider a 20-DoF system, as shown in Figure 8.6, with the parameters given by

$$m_1 = m_2 = \cdots = m_{19} = 2m, \quad m_{20} = m = 1.0 \text{ kg}$$

$$k_1 = k_2 = \cdots = k_{21} = 1.0 \times 10^3 \text{ N/m}$$

$$c_1 = c_2 = \cdots = c_7 = 3c, \quad c_8 = c_9 = \cdots = c_{14} = 2c$$

$$c_{15} = c_{16} = \cdots = c_{21} = c = 0.1 \text{ N sec/m}$$

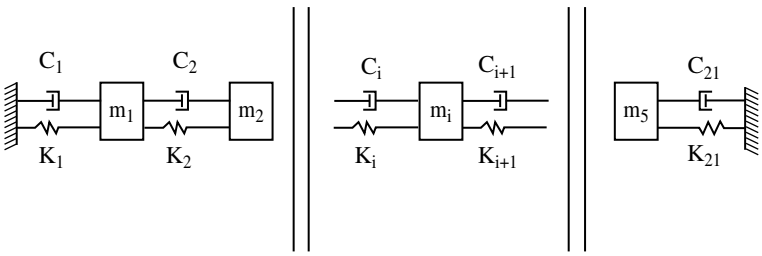

FIGURE 8.6 The 20-DoF system for Example 8.6.

TABLE 8.7 Errors of Eigenvector Derivatives (%)

	Modes used	TMS	IMS	HAMS		
				2	3	4
$\partial u_1/\partial b_1$	4	60.18	13.14	1.41	0.01	0.00
	8	26.66	1.01	0.03	0.00	0.00
	12	16.76	0.66	0.01	0.00	0.00
$\partial u_1/\partial b_1$	4	101.00	50.94	27.72	7.64	1.51
	8	48.65	10.32	1.29	0.38	0.03
	12	23.98	3.37	0.35	0.06	0.01
$\partial u_1/\partial b_1$	4	69.74	14.82	3.07	0.02	0.00
	8	26.60	1.39	0.23	0.00	0.00
	12	25.92	0.85	0.12	0.00	0.00
$\partial u_1/\partial b_1$	4	90.63	43.79	21.33	9.01	2.00
	8	44.86	6.65	1.56	0.35	0.04
	12	23.76	3.23	0.37	0.11	0.02
$\partial u_1/\partial b_1$	4	72.52	15.07	8.46	1.11	0.03
	8	37.95	1.22	0.57	0.03	0.00
	12	28.54	0.52	0.03	0.00	0.00
$\partial u_1/\partial b_1$	4	63.33	9.19	1.92	0.42	0.00
	8	27.56	0.75	0.49	0.21	0.00
	12	25.00	0.47	0.15	0.01	0.00

For the purpose of comparison, the errors of the truncation modal superposition (TMS), the IMS, and the high-accuracy superposition (HAMS) in computing the derivatives of eigenvectors are listed in Table 8.7.

For the sake of simplicity, the errors of eigenvector derivatives are represented by

$$\left|\left(\frac{\partial u_i}{\partial b_j}\right)_{\varepsilon} - \left(\frac{\partial u_i}{\partial b_j}\right)_{\alpha}\right|$$

where $(\partial u_i/\partial b_j)_{\varepsilon}$ denotes the exact solution and $(\partial u_i/\partial b_j)_{\alpha}$ denotes those obtained by the three methods presented above. In the computation of the eigenvector derivatives, the parameters m_1 and m_{10} are functions of the design variable b_1, the parameters c_8 and c_{15} are functions of design variable b_2, and K denotes the number of terms used in series (Equation 8.170).

The results in Table 8.7 confirm that the solution accuracy of the high-accuracy modal superposition is much higher than that of the TMS and the IMS. For example, if only the first four modes are used, the errors of $\partial u_2/\partial b_1$ are 101.00 and 50.94% for the truncated modal superposition and the IMS, and the errors are reduced to about 27.27, 7.64, and 1.51% for the high-accuracy modal superposition. If the first 12 modes are used, the error of $\partial u_1/\partial b_1$ is 16.76% for the truncated modal superposition, and the errors are reduced to 0.01, 0.00 and 0.00% for the case of $K = 2$, 3, 4 in the series (Equation 8.170), where the first four modes are used, respectively.

8.9.4 Matrix Perturbation for Repeated Eigenvalues of Nondefective Systems

8.9.4.1 Basic Equation

Consider a system having repeated eigenvalues, $S_0 = S_1 = \cdots = S_{0m}$, with multiplicity m, and the corresponding right and left modal matrices

$$\mathbf{U}_{0m} = [\,\mathbf{u}_{01} \quad \mathbf{u}_{02} \quad \cdots \quad \mathbf{u}_{0m}\,] \tag{8.171}$$

$$\mathbf{V}_{0m} = [\,\mathbf{v}_{01} \quad \mathbf{v}_{02} \quad \cdots \quad \mathbf{v}_{0m}\,] \tag{8.172}$$

the remaining eigenvalues being distinct.

The repeated eigenvalues satisfy the following equations:

$$\mathbf{A}_0\mathbf{U}_{0m} = \mathbf{B}_0\mathbf{U}_{0m}\mathbf{S}_0 \tag{8.173}$$

$$\mathbf{A}_0^{\mathrm{T}}\mathbf{V}_{0m} = \mathbf{B}_0^{\mathrm{T}}\mathbf{V}_{0m}\mathbf{S}_0 \tag{8.174}$$

$$\mathbf{V}_{0m}^{\mathrm{T}}\mathbf{B}_0\mathbf{U}_{0m} = \mathbf{I} \tag{8.175}$$

$$\mathbf{u}_{0i}^{\mathrm{T}}\mathbf{B}_0\mathbf{u}_{0i} = 1 \tag{8.176}$$

If small changes are made to the parameters, we have

$$\mathbf{A} = \mathbf{A}_0 + \varepsilon\mathbf{A}_1 \tag{8.177}$$

$$\mathbf{B} = \mathbf{B}_0 + \varepsilon\mathbf{B}_1 \tag{8.178}$$

The eigenvalues and eigenvectors of the perturbed system can be expressed as power series expansions in ε:

$$\mathbf{S}_m = \mathbf{S}_0 + \varepsilon\mathbf{S}_1 + \varepsilon^2\mathbf{S}_2 + \cdots \tag{8.179}$$

$$\mathbf{U}_m = \mathbf{U}_0 + \varepsilon\mathbf{U}_1 + \varepsilon^2\mathbf{U}_2 + \cdots \tag{8.180}$$

$$\mathbf{V}_m = \mathbf{V}_0 + \varepsilon\mathbf{V}_1 + \varepsilon^2\mathbf{V}_2 + \cdots \tag{8.181}$$

where

$$\mathbf{U}_0 = \mathbf{U}_{0m}\alpha \tag{8.182}$$

$$\mathbf{V}_0 = \mathbf{V}_{0m}\beta \tag{8.183}$$

and $\alpha_{m\times m}$ and $\beta_{m\times m}$ are to be determined coefficient matrices.

8.9.4.2 The First-Order Perturbation of Eigenvalues

The first-order perturbation diagonal matrix, $\mathbf{S}_1$, of the repeated eigenvalues and the coefficient matrix, $\boldsymbol{\alpha}$, can be obtained from the equations:

$$\mathbf{W}\boldsymbol{\alpha} = \boldsymbol{\alpha}\mathbf{S}_1 \tag{8.184}$$

$$\mathbf{W}^{\mathrm{T}}\beta = \beta\mathbf{S}_1 \tag{8.185}$$

$$\mathbf{W} = \mathbf{V}_{0m}^{\mathrm{T}}(\mathbf{A}_1 - \mathbf{S}_0\mathbf{B}_1)\mathbf{U}_{0m} \tag{8.186}$$

and the normalization conditions

$$\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha} = \mathbf{I} \tag{8.187}$$

$$\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\alpha} = \mathbf{I} \tag{8.188}$$

If matrix $\mathbf{W}$ has no repeated eigenvalues, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ can be uniquely determined. If $\mathbf{W}$ has repeated eigenvalues, we must consider the higher order perturbation equations for determining $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Here, we assume that S_{1i} are distinct eigenvalues, that is, $S_{1i} \neq S_{1j}$ $(i \neq j)$, where S_{1i} are the diagonal elements of $\mathbf{S}_1$.

8.9.4.3 The First-Order Perturbation of Eigenvectors

According to the modal expansion theorem, the first-order perturbation of the right and left eigenvectors, $\mathbf{U}_1$ and $\mathbf{V}_1$, can be expressed as

$$\mathbf{U}_1 = \mathbf{U}_{0m}\boldsymbol{\alpha}\mathbf{C}_m^1 + \mathbf{U}_A\mathbf{C}_A^1 \tag{8.189}$$

$$\mathbf{V}_1 = \mathbf{V}_{0m}\boldsymbol{\beta}\mathbf{D}_m^1 + \mathbf{V}_A\mathbf{D}_A^1 \tag{8.190}$$

where $\mathbf{C}_m^1$ and $\mathbf{C}_A^1$ are coefficient matrices which are to be determined, and $\mathbf{U}_A$ and $\mathbf{V}_A$ are the right and left modal matrices corresponding to the distinct eigenvalues:

$$\mathbf{C}_A^1 = (\mathbf{S}_A - \mathbf{S}_0)^{-1}\mathbf{V}_A^{\mathrm{T}}(\mathbf{S}_0\mathbf{B}_1 - \mathbf{A}_1)\mathbf{U}_{0m}\boldsymbol{\alpha} \tag{8.191}$$

$$\mathbf{D}_A^1 = (\mathbf{S}_A - \mathbf{S}_0)^{-1}\mathbf{U}_A^{\mathrm{T}}(\mathbf{S}_0\mathbf{B}_1^{\mathrm{T}} - \mathbf{A}_1^{\mathrm{T}})\mathbf{V}_{0m}\boldsymbol{\beta} \tag{8.192}$$

The elements of matrix $\mathbf{C}_m^1$ can be computed by

$$C_{mij}^1 = \frac{R_{ij}^1}{\lambda_{1j} - \lambda_{1i}}, \quad i \neq j, \quad i,j = 1,2,\ldots,m \tag{8.193}$$

where R_{ij}^1 are the nondiagonal elements of $\mathbf{R}^1$:

$$\mathbf{R}^1 = \boldsymbol{\beta}^{\mathrm{T}}\mathbf{V}_{0m}^{\mathrm{T}}\mathbf{B}_0\mathbf{U}_A\mathbf{C}_A^1\mathbf{S}_1 + \boldsymbol{\beta}^{\mathrm{T}}\mathbf{V}_{0m}^{\mathrm{T}}\mathbf{B}_1\mathbf{U}_{0m}\boldsymbol{\alpha}\mathbf{S}_1 + \boldsymbol{\beta}^{\mathrm{T}}\mathbf{V}_{0m}^{\mathrm{T}}\mathbf{B}_1\mathbf{U}_A\mathbf{C}_A^1\mathbf{S}_0 - \boldsymbol{\beta}^{\mathrm{T}}\mathbf{V}_{0m}^{\mathrm{T}}\mathbf{A}_1\mathbf{U}_A\mathbf{C}_A^1 \tag{8.194}$$

$$C_{mii}^1 = \frac{-1}{\mathbf{u}_{0i}^{\mathrm{T}}(\mathbf{B}_0 + \mathbf{B}_0^{\mathrm{T}})\mathbf{u}_{0i}}\left(\mathbf{u}_{0i}^{\mathrm{T}}\mathbf{B}_1\mathbf{u}_{0i} + \sum_{\substack{j=1\\ j\neq i}}^{m} c_{mij}^1\mathbf{u}_{0j}^{\mathrm{T}}(\mathbf{B}_0 + \mathbf{B}_0^{\mathrm{T}})\mathbf{u}_{0i} + \sum_{j=m+1}^{n} c_{Aij}^1\mathbf{u}_{0j}^{\mathrm{T}}(\mathbf{B}_0 + \mathbf{B}_0^{\mathrm{T}})\mathbf{u}_{0i}\right) \tag{8.195}$$

$$D_{mij}^1 = \frac{R_{ij}^2}{S_{1j} - S_{1i}}, \quad i \neq j, \quad i,j = 1,2,\ldots,m \tag{8.196}$$

where R_{ij}^2 are the nondiagonal elements of $\mathbf{R}^2$:

$$\mathbf{R}^2 = \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{U}_{0m}^{\mathrm{T}}\mathbf{B}_0\mathbf{V}_A\mathbf{D}_A^1\mathbf{S}_1 + \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{U}_{0m}^{\mathrm{T}}\mathbf{B}_1^{\mathrm{T}}\mathbf{V}_{0m}\boldsymbol{\beta}\mathbf{S}_1 + \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{U}_{0m}^{\mathrm{T}}\mathbf{B}_1^{\mathrm{T}}\mathbf{V}_A\mathbf{D}_A^1\mathbf{S}_0 - \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{U}_{0m}^{\mathrm{T}}\mathbf{A}_1^{\mathrm{T}}\mathbf{V}_A\mathbf{D}_A^1 \tag{8.197}$$

$$D_{mii}^1 = Q_{ii}^2 - C_{mii}^1 \tag{8.198}$$

where Q_{ii}^2 are the diagonal elements of $\mathbf{Q}^2$:

$$\mathbf{Q}^2 = -\boldsymbol{\beta}^{\mathrm{T}}\mathbf{V}_{0m}^{\mathrm{T}}\mathbf{B}_1\mathbf{U}_{0m}\boldsymbol{\alpha} \tag{8.199}$$

8.9.5 Matrix Perturbation for Close Eigenvalues of Unsymmetric Matrices

Assume that $\mathbf{S}_0$ is a diagonal matrix with m close eigenvalues; $\mathbf{U}_{0n\times m}$ and $\mathbf{V}_{0n\times m}$ are the corresponding right and left eigenvectors matrices; $\mathbf{S}_A$ is the remaining distinct eigenvalue diagonal matrix; $\mathbf{U}_{An\times(n-m)}$ and $\mathbf{V}_{An\times(n-m)}$ are the corresponding right and left eigenvector matrices. They satisfy the following equations:

$$\mathbf{A}_0[\mathbf{U}_0 \vdots \mathbf{U}_A] = \mathbf{B}_0[\mathbf{U}_0 \vdots \mathbf{U}_A]\mathrm{diag}(\mathbf{S}_0, \mathbf{S}_A) \tag{8.200}$$

$$\mathbf{A}_0^{\mathrm{T}}[\mathbf{V}_0 \vdots \mathbf{V}_A] = \mathbf{B}_0^{\mathrm{T}}[\mathbf{V}_0 \vdots \mathbf{V}_A]\mathrm{diag}(\mathbf{S}_0, \mathbf{S}_A) \tag{8.201}$$

$$[\mathbf{V}_0 \vdots \mathbf{V}_A]^{\mathrm{T}}\mathbf{B}_0[\mathbf{U}_0 \vdots \mathbf{U}_A] = \mathbf{I} \tag{8.202}$$

$$\mathbf{u}_{0i}^{\mathrm{T}}\mathbf{B}_0\mathbf{u}_{0i} = 1 \tag{8.203}$$

Construct the matrix $\bar{\mathbf{A}}_0$ as

$$\mathbf{A}_0 = \bar{\mathbf{A}}_0 + \varepsilon\bar{\mathbf{A}}_0 \tag{8.204}$$

where

$$\bar{\mathbf{A}}_0 = \mathbf{B}_0[\mathbf{U}_0 \vdots \mathbf{U}_A]\mathrm{diag}(S_0\mathbf{I}, \mathbf{S}_A)[\mathbf{V}_0 \vdots \mathbf{V}_A]^{\mathrm{T}}\mathbf{B}_0 \tag{8.205}$$

$$\varepsilon\bar{\mathbf{A}}_0 = \mathbf{B}_0\mathbf{U}_0(\varepsilon[\delta\mathbf{S}_0])\mathbf{V}_0^{\mathrm{T}}\mathbf{B}_0 \tag{8.206}$$

$$\varepsilon[\delta\mathbf{S}_0] = \mathbf{S}_0 - S_0\mathbf{I} \tag{8.207}$$

$$S_0 = \frac{1}{m}\left(\sum_{k=1}^{n} S_{0i}\right) \tag{8.208}$$

Here, S_{0i} are the close eigenvalues, and S_0 is the average of S_{0i} $(i = 1, 2, \ldots, m)$.

It can be shown that the following equations hold:

$$\bar{\mathbf{A}}_0\mathbf{U}_0 = \mathbf{B}_0\mathbf{U}_0S_0\mathbf{I} \tag{8.209}$$

$$\bar{\mathbf{A}}_0^{\mathrm{T}}\mathbf{V}_0 = \mathbf{B}_0^{\mathrm{T}}\mathbf{V}_0S_0\mathbf{I} \tag{8.210}$$

These equations indicate that S_0 is the repeated eigenvalue with multiplicity, m, for the eigenproblem defined by Equation 8.209 and Equation 8.210, and $\mathbf{U}_0$ and $\mathbf{V}_0$ are the corresponding right and left modal matrices, respectively.

If small modifications $\varepsilon\mathbf{A}_1$ and $\varepsilon\mathbf{B}_1$ are imposed on the matrices $\mathbf{A}_0$ and $\mathbf{B}_0$, then the eigenproblems of the perturbed system become

$$(\bar{\mathbf{A}}_0 + \varepsilon\bar{\mathbf{A}}_1)\mathbf{U} = (\mathbf{B}_0 + \varepsilon\mathbf{B}_1)\mathbf{U}\mathbf{S} \tag{8.211}$$

$$(\bar{\mathbf{A}}_0 + \varepsilon\bar{\mathbf{A}}_1)^{\mathrm{T}}\mathbf{V} = (\mathbf{B}_0 + \varepsilon\mathbf{B}_1)^{\mathrm{T}}\mathbf{V}\mathbf{S} \tag{8.212}$$

where

$$\varepsilon\bar{\mathbf{A}}_1 = \varepsilon\bar{\mathbf{A}}_0 + \varepsilon\mathbf{A}_1 \tag{8.213}$$

The eigensolutions of Equation 8.211 and Equation 8.212 are given by

$$\mathbf{U} = \mathbf{U}_0\boldsymbol{\alpha} + \varepsilon\mathbf{U}_1 \tag{8.214}$$

$$\mathbf{V} = \mathbf{V}_0\boldsymbol{\alpha} + \varepsilon\mathbf{V}_1 \tag{8.215}$$

$$\mathbf{S} = \mathbf{S}_0 + \varepsilon\mathbf{S}_1 \tag{8.216}$$

It should be noted that Equation 8.211 and Equation 8.212 are the eigenproblem for repeated eigenvalues. That is, the perturbation analysis for close eigenvalues has been transferred into that of repeated eigenvalues. Hence, the methods given by Section 8.9.4 can be used to compute $\mathbf{S}_1$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\mathbf{U}_1$, and $\mathbf{V}_1$ in Equation 8.214 to Equation 8.216.

References

Adelmen, H.M. and Haftka, R.T., Sensitivity analysis of discrete structural systems, *AIAA J.*, 24, 823, 1986.

Adhikari, S. and Friswell, M.I., Eigenderivative analysis of asymmetric non-conservative systems, *Int. J. Numer. Methods Eng.*, 39, 1813, 2001.

Chen, S.H. 1993. *Matrix Perturbation Theory in Structural Dynamics*, International Academic Publishers, Beijing.

Chen, S.H. and Liu, Z.S., High accuracy modal superposition for eigenvector derivatives, *Chin. J. Mech.*, 25, 432, 1993a.

Chen, S.H. and Liu, Z.S., Perturbation analysis of vibration modes with close frequencies, *Commun. Numer. Methods Eng.*, 9, 427, 1993b.

Chen, S.H. and Pan, H.H., Design sensitivity analysis of vibration modes by finite element perturbation, *Proc. of the Fourth IMAC*, 38, 1986.

Chen, J.C. and Wada, B.K., Matrix perturbation for structural dynamics, *AIAA J.*, 15, 1095, 1979.

Dailey, R.L., Eigenvector derivatives with repeated eigenvalues, *AIAA J.*, 27, 486, 1989.

Fox, R.L. and Kapoor, M.P., Rates of change of eigenvalues and eigenvectors, *AIAA J.*, 12, 2426, 1968.

Haug, E.J., Komkov, V., and Choi, K.K. 1985. *Design Sensitivity Analysis of Structural Systems*, Academic Press, Orlando, FL.

Haug, E.J. and Rousselet, B., Design sensitivity analysis in structural dynamics, eigenvalue variations, *J. Struct. Mech.*, 8, 161, 1980.

Hu, H. 1987. *Natural Vibration Theory for Multi-DoF Structures*, Science Press, Beijing (in Chinese).

Lim, K.B., Juang, J.N., and Ghaemmaghani, P., Eigenvector derivatives of repeated eigenvalues using singular value decomposition, *J. Guidance Control Dyn.*, 12, 282, 1989.

Liu, J.K., Perturbation technique for non-self-adjoint systems with repeated eigenvalues, *AIAA J.*, 37, 222, 1999.

Liu, X.L., Derivation of formulas for perturbation analysis with modes of close eigenvalues, *Struct. Eng. Mech.*, 10, 427, 2000.

Liu, Z.S. and Chen, S.H., Contribution of the truncated modes to eigenvector derivatives, *AIAA J.*, 32, 1551, 1994a.

Liu, Z.S. and Chen, S.H., An accurate method for computing eigenvector derivatives for free–free structures, *Int. J. Comput. Struct.*, 52, 1135, 1994b.

Mills-Curran, W.C., Calculation of eigenvector derivatives for structures with repeated eigenvalues, *AIAA J.*, 26, 867, 1988.

Murthy, D.V. and Haftka, R.T., Derivatives of eigenvalues and eigenvectors of a general complex matrix, *Int. J. Numer. Methods Eng.*, 26, 293, 1988.

Ojalvo, E.U., Efficient computation of modal sensitivities for systems with repeated frequencies, *AIAA J.*, 26, 361, 1988.

Rogers, L.C., Derivatives of eigenvalues and eigenvectors, *AIAA J.*, 8, 943, 1977.

Shaw, J. and Jayasuriya, S., Modal sensitivities for repeated eigenvalues and eigenvalues derivatives, *AIAA J.*, 30, 850, 1992.

Wang, B.P., Improved approximate methods for computing eigenvector derivatives in structural dynamics, *AIAA J.*, 29, 1018, 1991.