APPENDIX 2

LAPLACE TRANSFORMS

This appendix presents a short introduction to Laplace transforms, the basic tool used in analyzing continuous systems in the frequency domain. The Laplace transform converts linear ordinary differential equations (LODE's) into algebraic equations, making them easy to solve for their frequency and time-domain behavior. There are many excellent presentations of the Laplace transform, as in Oppenheim [1997], for those who would like more information.

A2.1 Definitions

The Laplace transform is a generalized Fourier transform, where given any function f(t), the Fourier transform $F(\omega)$ is defined as:

$$F(\omega) = \mathcal{F}\{f(\cdot)\}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{j\omega t} dt \qquad \text{(A2.1)}$$

where $\omega = 2\pi f$ and f is frequency, in hz.

In the same spirit, we can define the Laplace transform as:

$$F(s) = \mathcal{L}\{f(\cdot)\}(s) = \int_{0^-}^{+\infty} f(t)\, e^{-st} dt \qquad \text{(A2.2)}$$

where s is complex:

$$s = \sigma + j\omega, \qquad \text{(A2.3)}$$

σ and ω are real numbers which define the locations of "s" in the complex plane, see Figure A2.1 below. Also, $\omega = 2\pi f$ as above.

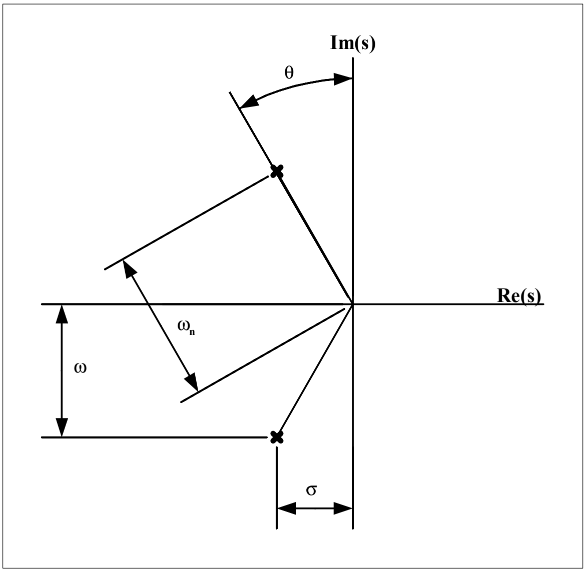

Figure A2.1: σ and ω definitions in complex plane.

Remarks:

1) if $f(t) \equiv 0$ for $t < 0$, then

$$\mathcal{F}\{f(\cdot)\}(\omega) = \mathcal{L}\{f(\cdot)\}(j\omega) \tag{A2.4}$$

2) The "0^-" limit in the Laplace transform definition takes care of $f(t)$'s which contain the δ function.

3) The integral in the definition of the Laplace transform need not be finite, i.e. $\mathcal{L}\{f\}(s)$ may not exist for all $s \in \mathbb{C}$. However, if f(t) is bounded by some exponential:

$$|f(t)| \le M e^{\sigma_0 t} \tag{A2.5}$$

then $\mathcal{L}\{f\}(s)$ will make sense for $s \in \mathbb{C}$ such that $\mathrm{Re}\{s\} > \sigma_0$.

4) The Laplace transform is linear:

$$\mathcal{L}\{a_1 f_1 + a_2 f_2\} = a_1 \mathcal{L}\{f_1\} + a_2 \mathcal{L}\{f_2\} \qquad \text{(A2.6)}$$

A2.2 Examples, Laplace Transform Table

1) Exponential

$$f(t) = e^{-at} 1(t)$$

$$F(s) = \int_{0^-}^{\infty} e^{-at} 1(t) e^{-st} dt = \int_{0^-}^{\infty} e^{-(s+a)t}\, dt = \frac{1}{s+a} \qquad [s > a] \qquad \text{(A2.7a,b)}$$

2) Impulse

$$f(t) = \delta(t)$$

$$F(s) = \int_{0^-}^{\infty} \delta(t) e^{-st} dt = e^{-0} = 1 \qquad [\text{for any } s] \qquad \text{(A2.8a,b)}$$

3) Step

$$f(t) = 1(t)$$

$$F(s) = \int_{0^-}^{\infty} e^{-st} dt = \frac{-\left[e^{-s(\infty)} - e^{-s(0)}\right]}{s} = \frac{1}{s} \qquad [s > 0] \qquad \text{(A2.9a,b)}$$

Table A2.1 below contains Laplace transforms for a few selected functions in the time domain. The “Region of Convergence” or “ROC” is defined as the range of values of “s” for which the integral in the definition of the Laplace transform (A2.2) converges (Oppenheim 1997).

	f(t)	Laplace Transform	Region of Convergence
1)	$\delta(t)$	1	all s
2)	$\delta(t-T)$	e^{-sT}	all s
3)	$1(t)$	$\frac{1}{s}$	$\text{Re}\{s\} > 0$
4)	$\frac{1}{m!}t^m 1(t)$	$\frac{1}{s^{m+1}}$	$\text{Re}\{s\} > 0$
5)	$e^{-at}1(t)$	$\frac{1}{s+a}$	$\text{Re}\{s\} > \text{Re}\{a\}$
6)	$\frac{1}{(m-1)!}t^{m-1}e^{-at}1(t)$	$\frac{1}{(s+a)^m}$	$\text{Re}\{s\} > \text{Re}\{a\}$
7)	$(1-e^{-at})1(t)$	$\frac{a}{s(s+a)}$	$\text{Re}\{s\} > \max\{0, \text{Re}\{a\}\}$
8)	$(e^{-at}-a^{-bt})1(t)$	$\frac{b-a}{(s+a)(s+b)}$	$\text{Re}\{s\} > \max\{\text{Re}\{a\}, \text{Re}\{b\}\}$
9)	$\sin(at)\ 1(t)$	$\frac{a}{s^2+a^2}$	$\text{Re}\{s\} > 0$
10)	$\cos(at)1(t)$	$\frac{s}{s^2+a^2}$	$\text{Re}\{s\} > 0$
11)	$e^{-at}\sin(bt)1(t)$	$\frac{b}{(s+a)^2+b^2}$	$\text{Re}\{s\} > a$
12)	$e^{-at}\cos(bt)1(t)$	$\frac{s+a}{(s+a)^2+b^2}$	$\text{Re}\{s\} > a$

Table A2.1: Laplace transform table.

A2.3 Duality

The following duality conditions exist:

$$t\, f(t) \;\Leftrightarrow\; -\frac{d}{ds}F(s)$$

$$\frac{d}{dt}f(t) \;\Leftrightarrow\; s\,F(s) \qquad \text{(A2.10a,b)}$$

A2.4 Differentiation and Integration

Differentiation and the Laplace transform: Suppose

$$\mathcal{L}\{x\}(s) = X(s) \qquad \text{(A2.11)}$$

then

$$\mathcal{L}\{\dot{x}\}(s) = sX(s) - x(0^-)\,, \qquad \text{(A2.12)}$$

so we can interpret "s" as a differentiation operator:

$$\frac{d}{dt} \leftrightarrow s \qquad \text{(A2.13)}$$

Integration and the Laplace transform: Suppose

$$\mathcal{L}\{x\}(s) = X(s)\,, \qquad \text{(A2.14)}$$

then

$$L\left\{\int_0^t x(\tau)d\tau\right\}(s) = \frac{1}{s}X(s)\,, \qquad \text{(A2.15)}$$

and we can interpret "1/s" as an integration operator:

$$\frac{1}{s} \leftrightarrow \int_0^t dt \qquad \text{(A2.16)}$$

A2.5 Applying Laplace Transforms to LODE's with Zero Initial Conditions

Assume we have a linear ordinary differential equation as shown in (A2.17):

$$\dddot{y}(t)+a_1\ddot{y}(t)+a_2\dot{y}(t)+a_3y(t) = b_1\ddot{u}(t)+b_2\dot{u}(t)+b_3u(t) \qquad \text{(A2.17)}$$

Assume $\ddot{y}(t)=0,\ \dot{y}(t)=0,\ y(t)=0$ and take the Laplace transform of both sides, using the linearity property (A2.6):

$$\mathcal{L}\{\dddot{y}\}(s)+a_1\mathcal{L}\{\ddot{y}\}(s)+a_2\mathcal{L}\{\dot{y}\}(s)+a_3\mathcal{L}\{y\}(s) = b_1\mathcal{L}\{\ddot{u}\}(s)+b_2\mathcal{L}\{\dot{u}\}(s)+b_3\mathcal{L}\{u\}(s) \qquad \text{(A2.18)}$$

Recalling that "s" is the differentiation operator, replace "dots" with "s":

$$s^3Y(s)+a_1s^2Y(s)+a_2sY(s)+a_3Y(s) = b_1s^2U(s)+b_2sU(s)+b_3U(s) \qquad \text{(A2.19)}$$

We are now left with a polynomial equation in "s" that can be factored into terms multiplying Y(s) and U(s):

$$\left[s^3+a_1s^2+a_2s+a_3\right]Y(s) = \left[b_1s^2+b_2s+b_3\right]U(s) \qquad \text{(A2.20)}$$

Solving for Y(s):

$$Y(s) = \frac{\left[b_1s^2+b_2s+b_3\right]}{\left[s^3+a_1s^2+a_2s+a_3\right]}U(s) \qquad \text{(A2.21)}$$

It can be shown that the terms in the numerator and denominator above are the Laplace transform of the impulse response, H(s):

$$Y(s) = H(s)U(s)\,, \qquad \text{(A2.22)}$$

$$H(s) = \mathcal{L}\left[h(\cdot)\right](s)\,, \qquad \text{(A2.23)}$$

and $h(\cdot)$ is the impulse response. For the example LODE (A2.17) the Laplace transform of the impulse response is:

$$H(s) = \frac{\left[b_1s^2+b_2s+b_3\right]}{\left[s^3+a_1s^2+a_2s+a_3\right]} \qquad \text{(A2.24)}$$

A2.6 Transfer Function Definition

It can be shown that the transfer function of a system described by a LODE is the Laplace transform of its impulse response, H(s), (A2.23).

Taking the Laplace transform of the LODE has provided the Laplace transform of the impulse response. If we could inverse-transform H(s) we could get the impulse response h(t) without having to integrate the differential equation. Typically the inverse transform is found by simplifying/expanding H(s) into terms which can be found in tables, such as Table A2.1, and than inverting "by inspection."

A2.7 Frequency Response Definition

Having obtained H(s) directly from the LODE by replacing "dots" by "s," we can obtain the frequency response of the system (the Fourier transform of the impulse response) by substituting " $j\omega$" for "s" in H(s).

$$H(j\omega) = H(s)\big|_{s=j\omega} \tag{A2.25}$$

A2.8 Applying Laplace Transforms to LODE's with Initial Conditions

In A2.5 we looked at applying Laplace transforms to LODE's with zero initial conditions, which led to transfer function and frequency response definitions. Since transfer functions and frequency responses deal with steady state sinusoidal excitation response of the system, initial conditions are of no significance, as it is assumed that all measurements of the system undergoing sinusoidal excitation are taken over a long enough period of time that transients have died out.

On the other hand, if we are solving for the transient response of a system defined by a LODE that has initial conditions, obviously the initial conditions will not be zero. We will use the basic definition of the differentiation operation from (A2.12) to define the Laplace transform of 1st and 2nd order differential equations with initial conditions $x(0)$ and $\dot{x}(0)$:

1st Order: $$\mathcal{L}\{\dot{x}(t)\} = sX(s) - x(0) \tag{A2.26}$$

2nd Order: $$\mathcal{L}\{\ddot{x}(t)\} = s^2X(s) - sx(0) - \dot{x}(0) \tag{A2.27}$$

A2.9 Applying Laplace Transform to State Space

We defined the form of state space equations in Chapter 5 as below:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \qquad \text{(A2.28)}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t) \qquad \text{(A2.29)}$$

where the initial conditions are set by $\mathbf{x}(0) = \mathbf{x}_o$. The general block diagram for a SISO state space system is shown in Figure A2.1.

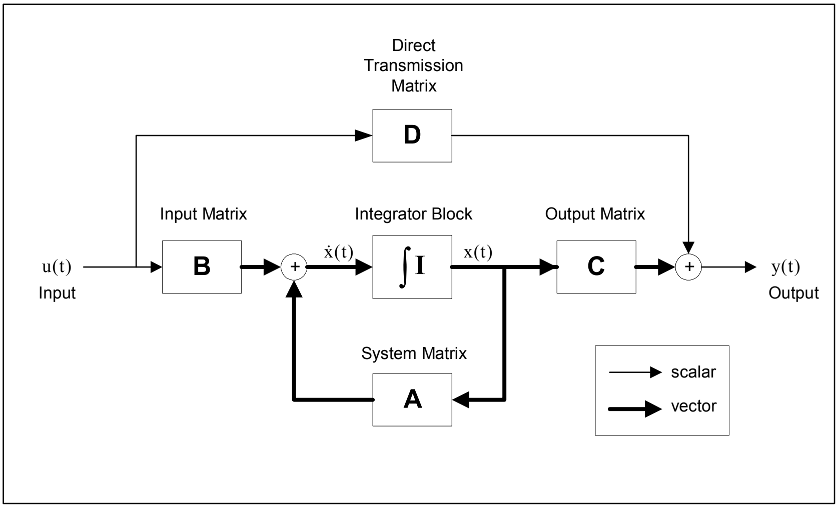

Figure A2.1: State space block diagram.

Taking Laplace transform of (A2.28):

$$\begin{aligned}\mathcal{L}\{\dot{\mathbf{x}}\}(s) &= \mathcal{L}\{\mathbf{A}\mathbf{x}\}(s) + \mathcal{L}\{\mathbf{B}u\}(s) \\ s\mathbf{X}(s) - \mathbf{x}(0^-) &= \mathbf{A}\mathcal{L}\{x\}(s) + \mathbf{B}\mathcal{L}\{u\}(s) \\ &= \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s)\end{aligned} \qquad \text{(A2.30a,b)}$$

Solving for $\mathbf{X}(s)$:

$$\begin{aligned}s\mathbf{X}(s) - \mathbf{A}\mathbf{X}(s) &= \mathbf{x}(0^-) + \mathbf{B}U(s) \\ (s\mathbf{I} - \mathbf{A})\mathbf{X}(s) &= \mathbf{x}(0^-) + \mathbf{B}U(s) \\ \mathbf{X}(s) &= (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0^-) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s)\end{aligned} \qquad \text{(A2.31a,b,c)}$$

The two terms on the right-hand side of (A2.31c) have special significance:

1) $(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}(0^-)$ is the Laplace transform of the homogeneous solution, the initial condition response.

2) $(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}U(s)$ is the Laplace transform of the particular solution, the forced response.

Taking the Laplace transform of (A2.29), the output equation:

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}U(s) \tag{A2.32}$$

Knowing X(s) from (A2.31c) and substituting in (A2.32):

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}(0^-) + \left[\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}\right]U(s) \tag{A2.33}$$

If the initial conditions are zero, $x(0^-) = 0$, then

$$\mathbf{Y}(s) = \left[\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}\right]U(s), \tag{A2.34}$$

with the transfer function for the system being defined by H(s):

$$\mathbf{H}(s) = \left[\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}\right] \tag{A2.35}$$

When the terms in H(s) above are multiplied out, they will result in the following polynomial form:

$$\mathbf{H}(s) = \frac{b(s)}{a(s)} + \mathbf{D} \tag{A2.36}$$