

## CHAPTER 13

### FINITE ELEMENTS: STIFFNESS MATRICES

#### 13.1 Introduction

The purpose of this chapter is to use two simple examples to explain the basics of how finite element stiffness matrices are formulated and how static finite element analysis is performed.

Chapter 2 discussed building global stiffness matrices column by column, giving a unit displacement to the dof associated with each column and entering constraint forces for each dof along the column. This chapter will show another method of building global stiffness matrices, based on using **element** stiffness matrices, combining them in an orderly way to generate the global stiffness matrix. The first example uses the lumped parameter 6dof example seen in Section 2.2.4. The second example uses a two-element cantilever. Static condensation is used to prepare for a development of Guyan reduction in the next chapter.

The next chapter will use element mass matrices to assemble global mass matrices and will introduce dynamics using finite elements.

#### 13.2 Six dof Model – Element and Global Stiffness Matrices

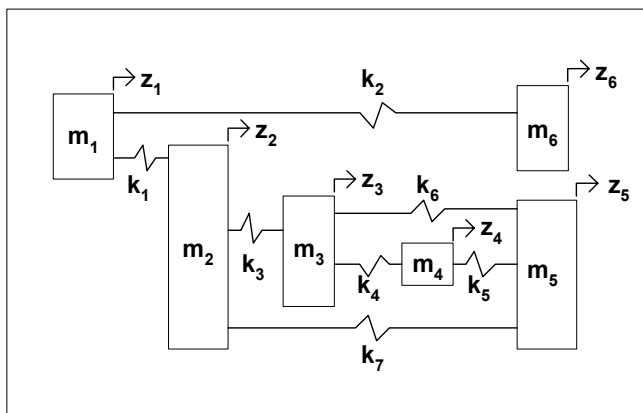


Figure 13.1: Six dof stiffness matrix model.

### 13.2.1 Overview

The global stiffness matrix for the model in Figure 13.1 was defined previously by inspection (Table 2.2). Each column of the matrix was defined by giving a unit displacement to the dof associated with that column and then defining the constraints required to hold the system in that configuration. This method works very well for hand calculations, but creating stiffness and mass matrices with computers requires a different, more systematic approach, where individual element stiffness matrices are developed and combined to give the global stiffness matrix.

We can define an element stiffness matrix for each of the springs in the figure, where the size of the element stiffness matrix is  $(n \times n)$ , and  $n$  is the total number of degrees of freedom associated with the element. For a uni-axial spring, there are two degrees of freedom, the displacements in the “z” direction at both ends, hence a  $2 \times 2$  stiffness matrix.

Each element stiffness matrix can be set up using the “inspection” method, by displacing first the left-hand dof for the first column, and then the right-hand dof for the second column as shown in Figure 13.2.

### 13.2.2 Element Stiffness Matrix

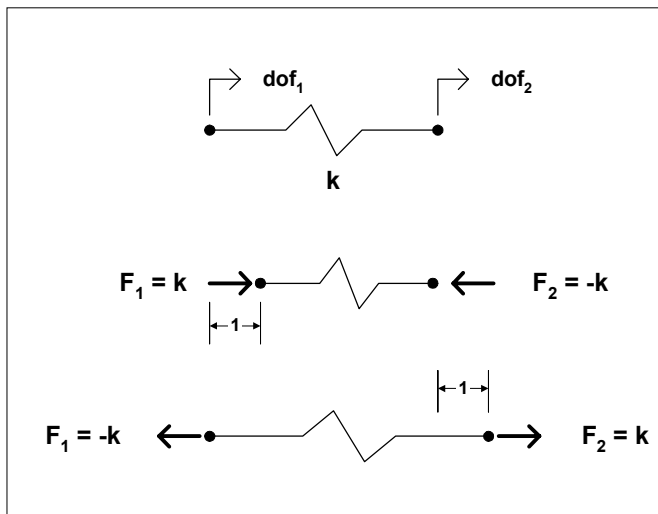


Figure 13.2: Spring element stiffness matrix development.

The resulting element stiffness matrix,  $\mathbf{k}_{el}$ , for a general uni-axial spring element is then:

$$\mathbf{k}_{el,i} = \begin{bmatrix} k_i & -k_i \\ -k_i & k_i \end{bmatrix} \quad (13.1)$$

For spring element 3, for example, the element stiffness matrix would be:

$$\mathbf{k}_{el,3} = \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} \quad (13.2)$$

### 13.2.3 Building Global Stiffness Matrix Using Element Stiffness Matrices

The total number of degrees of freedom for the problem is 6, so the complete system stiffness matrix, the **global** stiffness matrix, is a 6x6 matrix. Each row and column of every element stiffness matrix can be associated with a **global degree of freedom**.

For element 1, which is connected to degrees of freedom 1 and 2:

$$\mathbf{k}_{el,1} = \begin{array}{cc} \begin{matrix} 1^{\text{st}} & \text{and} & 2^{\text{nd}} \\ z_1 & & z_2 \end{matrix} & \text{columns of global stiffness matrix} \\ \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} & \begin{matrix} z_1 & 1^{\text{st}} \text{ row of global stiffness matrix} \\ z_2 & 2^{\text{nd}} \text{ row of global stiffness matrix} \end{matrix} \end{array} \quad (13.3)$$

For element 2, which is connected to degrees of freedom 1 and 6:

$$\mathbf{k}_{el,2} = \begin{array}{cc} \begin{matrix} 1^{\text{st}} & \text{and} & 6^{\text{th}} \\ z_1 & & z_6 \end{matrix} & \text{columns of global stiffness matrix} \\ \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} & \begin{matrix} z_1 & 1^{\text{st}} \text{ row of global stiffness matrix} \\ z_6 & 6^{\text{th}} \text{ row of global stiffness matrix} \end{matrix} \end{array} \quad (13.4)$$

For element 3, which is connected to degrees of freedom 2 and 3:

$$\mathbf{k}_{el,3} = \begin{array}{cc} \begin{matrix} 2^{\text{nd}} & \text{and} & 3^{\text{rd}} \\ z_2 & & z_3 \end{matrix} & \text{columns of global stiffness matrix} \\ \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} & \begin{matrix} z_2 & 2^{\text{nd}} \text{ row of global stiffness matrix} \\ z_3 & 3^{\text{rd}} \text{ row of global stiffness matrix} \end{matrix} \end{array} \quad (13.5)$$

For element 4, which is connected to degrees of freedom 3 and 4:

$$\begin{array}{l}
 \text{3}^{\text{rd}} \text{ and } \text{4}^{\text{th}} \text{ columns of global stiffness matrix} \\
 z_3 \quad z_4 \\
 \mathbf{k}_{\text{el},4} = \begin{bmatrix} k_4 & -k_4 \\ -k_4 & k_4 \end{bmatrix} \quad \begin{array}{l} z_3 \text{ 3}^{\text{rd}} \text{ row of global stiffness matrix} \\ z_4 \text{ 4}^{\text{th}} \text{ row of global stiffness matrix} \end{array}
 \end{array} \quad (13.6)$$

For element 5, which is connected to degrees of freedom 4 and 5:

$$\begin{array}{l}
 \text{4}^{\text{th}} \text{ and } \text{5}^{\text{th}} \text{ columns of global stiffness matrix} \\
 z_4 \quad z_5 \\
 \mathbf{k}_{\text{el},5} = \begin{bmatrix} k_5 & -k_5 \\ -k_5 & k_5 \end{bmatrix} \quad \begin{array}{l} z_4 \text{ 4}^{\text{th}} \text{ row of global stiffness matrix} \\ z_5 \text{ 5}^{\text{th}} \text{ row of global stiffness matrix} \end{array}
 \end{array} \quad (13.7)$$

For element 6, which is connected to degrees of freedom 3 and 5:

$$\begin{array}{l}
 \text{3}^{\text{rd}} \text{ and } \text{5}^{\text{th}} \text{ columns of global stiffness matrix} \\
 z_3 \quad z_5 \\
 \mathbf{k}_{\text{el},6} = \begin{bmatrix} k_6 & -k_6 \\ -k_6 & k_6 \end{bmatrix} \quad \begin{array}{l} z_3 \text{ 3}^{\text{rd}} \text{ row of global stiffness matrix} \\ z_5 \text{ 5}^{\text{th}} \text{ row of global stiffness matrix} \end{array}
 \end{array} \quad (13.8)$$

For element 7, which is connected to degrees of freedom 2 and 5:

$$\begin{array}{l}
 \text{2}^{\text{nd}} \text{ and } \text{5}^{\text{th}} \text{ columns of global stiffness matrix} \\
 z_2 \quad z_5 \\
 \mathbf{k}_{\text{el},7} = \begin{bmatrix} k_7 & -k_7 \\ -k_7 & k_7 \end{bmatrix} \quad \begin{array}{l} z_2 \text{ 2}^{\text{nd}} \text{ row of global stiffness matrix} \\ z_5 \text{ 5}^{\text{th}} \text{ row of global stiffness matrix} \end{array}
 \end{array} \quad (13.9)$$

The global stiffness matrix starts out as a 6x6 null matrix, then each element is cycled through and its elements added to the previous matrix. The initial null matrix is:

$$\mathbf{k}_g = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (13.10)$$

After adding the element stiffness matrix for element 1:

$$\mathbf{k}_g = \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 & 0 \\ -k_1 & k_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (13.11)$$

After adding the element stiffness matrices for elements 1 to 2:

$$\mathbf{k}_g = \begin{bmatrix} k_1 + k_2 & -k_1 & 0 & 0 & 0 & -k_2 \\ -k_1 & k_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -k_2 & 0 & 0 & 0 & 0 & k_2 \end{bmatrix} \quad (13.12)$$

After adding the element stiffness matrices for elements 1 to 3:

$$\mathbf{k}_g = \begin{bmatrix} k_1 + k_2 & -k_1 & 0 & 0 & 0 & -k_2 \\ -k_1 & k_1 + k_3 & -k_3 & 0 & 0 & 0 \\ 0 & -k_3 & k_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -k_2 & 0 & 0 & 0 & 0 & k_2 \end{bmatrix} \quad (13.13)$$

After adding the element stiffness matrices for elements 1 to 4:

$$\mathbf{k}_g = \begin{bmatrix} k_1 + k_2 & -k_1 & 0 & 0 & 0 & -k_2 \\ -k_1 & k_1 + k_3 & -k_3 & 0 & 0 & 0 \\ 0 & -k_3 & k_3 + k_4 & -k_4 & 0 & 0 \\ 0 & 0 & -k_4 & k_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -k_2 & 0 & 0 & 0 & 0 & k_2 \end{bmatrix} \quad (13.14)$$

After adding the element stiffness matrices for elements 1 to 5:

$$\mathbf{k}_g = \begin{bmatrix} k_1 + k_2 & -k_1 & 0 & 0 & 0 & -k_2 \\ -k_1 & k_1 + k_3 & -k_3 & 0 & 0 & 0 \\ 0 & -k_3 & k_3 + k_4 & -k_4 & 0 & 0 \\ 0 & 0 & -k_4 & k_4 + k_5 & -k_5 & 0 \\ 0 & 0 & 0 & -k_5 & k_5 & 0 \\ -k_2 & 0 & 0 & 0 & 0 & k_2 \end{bmatrix} \quad (13.15)$$

After adding the element stiffness matrices for elements 1 to 6:

$$\mathbf{k}_g = \begin{bmatrix} k_1 + k_2 & -k_1 & 0 & 0 & 0 & -k_2 \\ -k_1 & k_1 + k_3 & -k_3 & 0 & 0 & 0 \\ 0 & -k_3 & k_3 + k_4 + k_6 & -k_4 & -k_6 & 0 \\ 0 & 0 & -k_4 & k_4 + k_5 & -k_5 & 0 \\ 0 & 0 & -k_6 & -k_5 & k_5 + k_6 & 0 \\ -k_2 & 0 & 0 & 0 & 0 & k_2 \end{bmatrix} \quad (13.16)$$

After adding the element stiffness matrices for elements 1 to 7 we have the final global stiffness matrix.

$$\mathbf{k}_g = \begin{bmatrix} k_1 + k_2 & -k_1 & 0 & 0 & 0 & -k_2 \\ -k_1 & k_1 + k_3 + k_7 & -k_3 & 0 & -k_7 & 0 \\ 0 & -k_3 & k_3 + k_4 + k_6 & -k_4 & -k_6 & 0 \\ 0 & 0 & -k_4 & k_4 + k_5 & -k_5 & 0 \\ 0 & -k_7 & -k_6 & -k_5 & k_5 + k_6 + k_7 & 0 \\ -k_2 & 0 & 0 & 0 & 0 & k_2 \end{bmatrix} \quad (13.17)$$

This checks against the original global stiffness matrix defined by inspection in [Table 2.2](#) and fulfills the symmetry requirement.

$$\begin{array}{cccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
 \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \left[ \begin{array}{cccccc}
 (k_1 + k_2) & -k_1 & 0 & 0 & 0 & -k_2 \\
 -k_1 & (k_1 + k_3 + k_7) & -k_3 & 0 & -k_7 & 0 \\
 0 & -k_3 & (k_3 + k_4 + k_6) & -k_4 & -k_6 & 0 \\
 0 & 0 & -k_4 & (k_4 + k_5) & -k_5 & 0 \\
 0 & -k_7 & -k_6 & -k_5 & (k_5 + k_6 + k_7) & 0 \\
 -k_2 & 0 & 0 & 0 & 0 & k_2 \end{array} \right]
 \end{array}
 \tag{13.18}$$

### 13.3 Two-Element Cantilever Beam

We will now do a static finite element displacement analysis of a two-element cantilever beam. We start by showing the original model and defining the degrees of freedom for the idealized beam, [Figure 13.3](#).

Note that even though the left-hand side node is grounded in the actual beam, there are degrees of freedom associated with the node to allow generating global stiffness and mass matrices for all nodes. The constrained degrees of freedom will be accounted for once the complete global stiffness matrix is available. For this model, each of the three nodes has two degrees of freedom, a translation and a rotation.

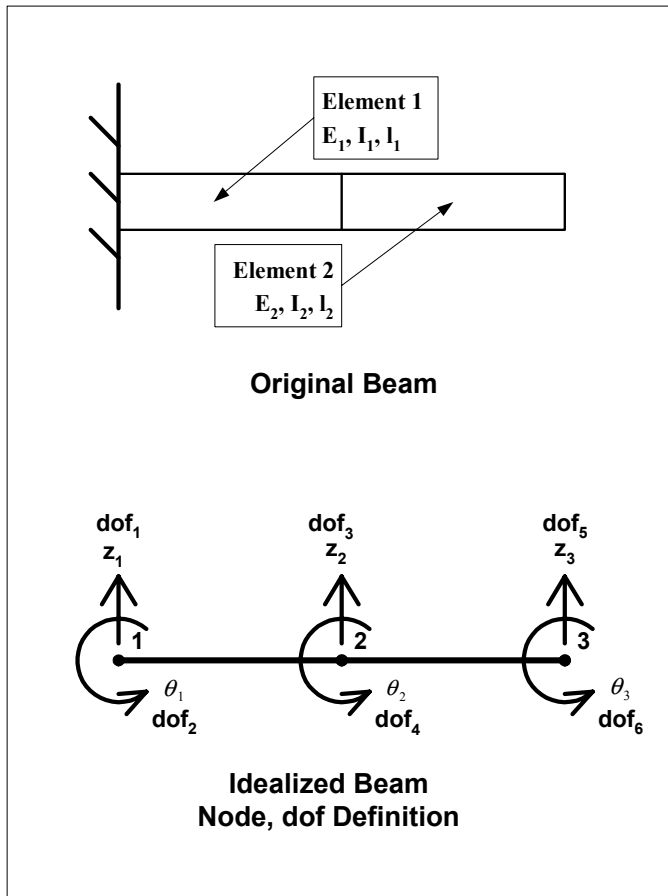


Figure 13.3: Two-element cantilever beam model and node definition.

### 13.3.1 Element Stiffness Matrix

The element stiffness matrix can be developed by using basic strength of materials techniques to analyze the forces required to displace each degree of freedom a unit value in the positive direction:

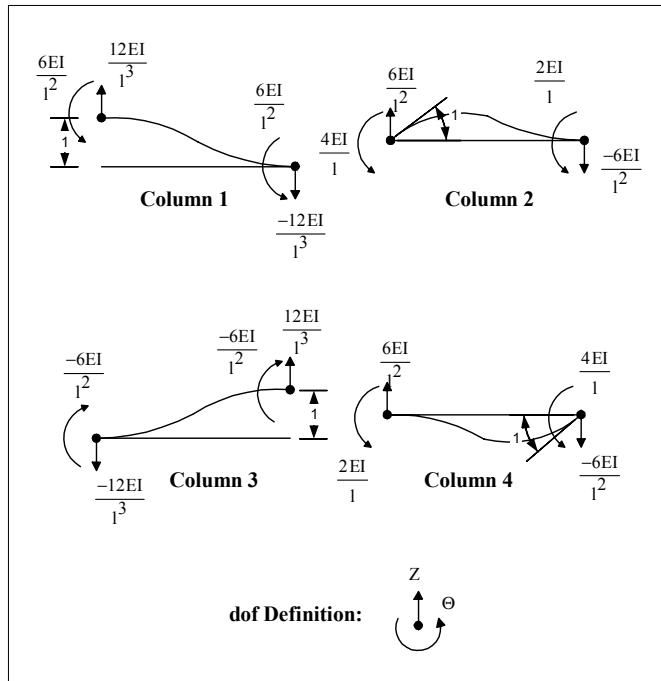


Figure 13.4: Beam element stiffness matrix terms.

### 13.3.2 Degree of Freedom Definition – Beam Stiffness Matrix

Using the degrees of freedom in Figure 13.5 results in the following element stiffness matrix:

$$\mathbf{k}_{el,i} = E_i I_i \begin{bmatrix} \frac{12}{l_i^3} & \frac{6}{l_i^2} & -\frac{12}{l_i^3} & \frac{6}{l_i^2} \\ \frac{6}{l_i^2} & \frac{4}{l_i} & -\frac{6}{l_i^2} & \frac{2}{l_i} \\ -\frac{12}{l_i^3} & -\frac{6}{l_i^2} & \frac{12}{l_i^3} & -\frac{6}{l_i^2} \\ \frac{6}{l_i^2} & \frac{2}{l_i} & -\frac{6}{l_i^2} & \frac{4}{l_i} \end{bmatrix} \quad (13.19)$$

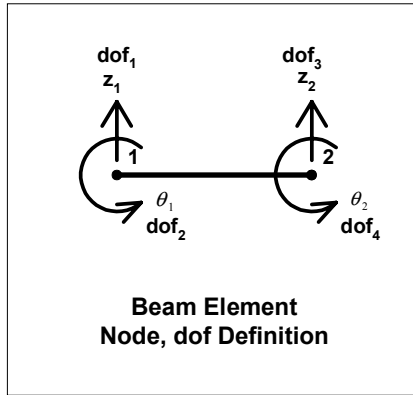


Figure 13.5: Beam element node and degree of freedom definition.

### 13.3.3 Building Global Stiffness Matrix Using Element Stiffness Matrices

To build the global stiffness matrix, we start with a 6x6 null matrix, with the six degrees of freedom being the translation and rotation of each of the three nodes, again including the constrained node 1 degrees of freedom:

$$\mathbf{k}_g = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \text{displacement of node 1} \\ \text{rotation of node 1} \\ \text{displacement of node 2} \\ \text{rotation of node 2} \\ \text{displacement of node 3} \\ \text{rotation of node 3} \end{array} \quad (13.20)$$

The two 4x4 element stiffness matrices are:

$$\mathbf{k}_{el,1} = E_1 I_1 \begin{bmatrix} \frac{12}{l_1^3} & \frac{6}{l_1^2} & \frac{-12}{l_1^3} & \frac{6}{l_1^2} \\ \frac{6}{l_1^2} & \frac{4}{l_1} & \frac{-6}{l_1^2} & \frac{2}{l_1} \\ \frac{-12}{l_1^3} & \frac{-6}{l_1^2} & \frac{12}{l_1^3} & \frac{-6}{l_1^2} \\ \frac{6}{l_1^2} & \frac{2}{l_1} & \frac{-6}{l_1^2} & \frac{4}{l_1} \end{bmatrix} \quad (13.21)$$

$$\mathbf{k}_{el,2} = E_2 I_2 \begin{bmatrix} \frac{12}{l_2^3} & \frac{6}{l_2^2} & \frac{-12}{l_2^3} & \frac{6}{l_2^2} \\ \frac{6}{l_2^2} & \frac{4}{l_2} & \frac{-6}{l_2^2} & \frac{2}{l_2} \\ \frac{-12}{l_2^3} & \frac{-6}{l_2^2} & \frac{12}{l_2^3} & \frac{-6}{l_2^2} \\ \frac{6}{l_2^2} & \frac{2}{l_2} & \frac{-6}{l_2^2} & \frac{4}{l_2} \end{bmatrix} \quad (13.22)$$

Building up the global stiffness matrix, element by element, inserting element 1 first:

$$\mathbf{k}_g = \begin{bmatrix} \frac{12E_1 I_1}{l_1^3} & \frac{6E_1 I_1}{l_1^2} & \frac{-12E_1 I_1}{l_1^3} & \frac{6E_1 I_1}{l_1^2} & 0 & 0 \\ \frac{6E_1 I_1}{l_1^2} & \frac{4E_1 I_1}{l_1} & \frac{-6E_1 I_1}{l_1^2} & \frac{2E_1 I_1}{l_1} & 0 & 0 \\ \frac{-12E_1 I_1}{l_1^3} & \frac{-6E_1 I_1}{l_1^2} & \frac{12E_1 I_1}{l_1^3} & \frac{-6E_1 I_1}{l_1^2} & 0 & 0 \\ \frac{6E_1 I_1}{l_1^2} & \frac{2E_1 I_1}{l_1} & \frac{-6E_1 I_1}{l_1^2} & \frac{4E_1 I_1}{l_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (13.23)$$

Inserting the element 2 terms leaves  $\mathbf{k}_g$  :

$$\begin{bmatrix}
\frac{12E_1I_1}{I_1^3} & \frac{6E_1I_1}{I_1^2} & \frac{-12E_1I_1}{I_1^3} & \frac{6E_1I_1}{I_1^2} & 0 & 0 \\
\frac{6E_1I_1}{I_1^2} & \frac{4E_1I_1}{I_1} & \frac{-6E_1I_1}{I_1^2} & \frac{2E_1I_1}{I_1} & 0 & 0 \\
\frac{-12E_1I_1}{I_1^3} & \frac{-6E_1I_1}{I_1^2} & \left( \frac{12E_1I_1}{I_1^3} + \frac{12E_2I_2}{I_2^3} \right) & \left( \frac{-6E_1I_1}{I_1^2} + \frac{6E_2I_2}{I_2^2} \right) & \frac{-12E_2I_2}{I_2^3} & \frac{6E_2I_2}{I_2^2} \\
\frac{6E_1I_1}{I_1^2} & \frac{2E_1I_1}{I_1} & \left( \frac{-6E_1I_1}{I_1^2} + \frac{6E_2I_2}{I_2^2} \right) & \left( \frac{4E_1I_1}{I_1} + \frac{4E_2I_2}{I_2} \right) & \frac{-6E_2I_2}{I_2^2} & \frac{2E_2I_2}{I_2} \\
0 & 0 & \frac{-12E_2I_2}{I_2^3} & \frac{-6E_2I_2}{I_2^2} & \frac{12E_2I_2}{I_2^3} & \frac{-6E_2I_2}{I_2^2} \\
0 & 0 & \frac{6E_2I_2}{I_2^2} & \frac{2E_2I_2}{I_2} & \frac{-6E_2I_2}{I_2^2} & \frac{4E_2I_2}{I_2}
\end{bmatrix}$$

(13.24)

Note how the contributions for the stiffness elements for node 2 from the left-hand and right-hand beams add together.

### 13.3.4 Eliminating Constraint Degrees of Freedom from Stiffness Matrix

We now have the entire global stiffness matrix, including the degrees of freedom which are constrained, the translation and rotation of node 1 (the first two rows and columns of  $\mathbf{k}_g$ ). To eliminate the constrained degrees of freedom, we eliminate the rows and columns which correspond to the constrained global degrees of freedom, reducing the global stiffness matrix to a 4x4 matrix:

$$\mathbf{k}_g = \begin{bmatrix}
\left( \frac{12E_1I_1}{I_1^3} + \frac{12E_2I_2}{I_2^3} \right) & \left( \frac{-6E_1I_1}{I_1^2} + \frac{6E_2I_2}{I_2^2} \right) & \frac{-12E_2I_2}{I_2^3} & \frac{6E_2I_2}{I_2^2} \\
\left( \frac{-6E_1I_1}{I_1^2} + \frac{6E_2I_2}{I_2^2} \right) & \left( \frac{4E_1I_1}{I_1} + \frac{4E_2I_2}{I_2} \right) & \frac{-6E_2I_2}{I_2^2} & \frac{2E_2I_2}{I_2} \\
\frac{-12E_2I_2}{I_2^3} & \frac{-6E_2I_2}{I_2^2} & \frac{12E_2I_2}{I_2^3} & \frac{-6E_2I_2}{I_2^2} \\
\frac{6E_2I_2}{I_2^2} & \frac{2E_2I_2}{I_2} & \frac{-6E_2I_2}{I_2^2} & \frac{4E_2I_2}{I_2}
\end{bmatrix} \quad (13.25)$$

To facilitate hand calculations, we will make the two-beam elements identical, with the same E, I and lengths, l. The global stiffness matrix can then be rewritten as:

$$\mathbf{k}_g = EI \begin{bmatrix} \frac{24}{l^3} & 0 & \frac{-12}{l^3} & \frac{6}{l^2} \\ 0 & \frac{8}{l} & \frac{-6}{l^2} & \frac{2}{l} \\ \frac{-12}{l^3} & \frac{-6}{l^2} & \frac{12}{l^3} & \frac{-6}{l^2} \\ \frac{6}{l^2} & \frac{2}{l} & \frac{-6}{l^2} & \frac{4}{l} \end{bmatrix} \quad (13.26)$$

### 13.3.5 Static Solution: Force Applied at Tip

We have all the information required to solve a static problem. For example, we could solve for the displacements of the system for a z direction force applied at the tip of the beam. The equation for static equilibrium of the system is:

$$\mathbf{k}_g \mathbf{z} = \mathbf{F} \quad (13.27)$$

Expanding:

$$\begin{bmatrix} k_{g11} & k_{g12} & k_{g13} & k_{g14} \\ k_{g21} & k_{g22} & k_{g23} & k_{g24} \\ k_{g31} & k_{g32} & k_{g33} & k_{g34} \\ k_{g41} & k_{g42} & k_{g43} & k_{g44} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \quad (13.28)$$

Where:

$z_1$  is translation of node 2

$z_2$  is rotation of node 2

$z_3$  is translation of node 3

$z_4$  is rotation of node 3

$F_1$  is z force applied to node 2

$F_2$  is y moment applied to node 2

$F_3$  is z force applied to node 3

$F_4$  is y moment applied to node 3

## 13.4 Static Condensation

### 13.4.1 Derivation

Solving the static equation is trivial using a computer, but doing a 4x4 inverse by hand is difficult, so we will reduce the problem to a 2x2 problem using static condensation. Static condensation is not typically used for static problems, but is the precursor for Guyan reduction (dynamic condensation), which will be introduced in the eigenvalue analysis in the next chapter.

Static condensation involves separating the degrees of freedom into “master” and “slave” degrees of freedom. If master dof’s are chosen such that they include all degrees of freedom where forces/moments are applied and also degrees of freedom where displacements are desired, the resulting solution is exact. If the slave dof set includes dof’s where forces/moments are applied and/or where displacements are desired, the technique will create errors.

For an **exact static solution**, master dof’s are chosen as dof’s where forces/moments are applied and where displacements/rotations are desired.

For dynamic problems master degrees of freedom are typically chosen as displacements of the higher mass nodes and rotations of the higher mass moment of inertia nodes, with slave degrees of freedom being the displacements and rotations of the relatively lower inertia nodes.

For the two-element cantilever, we will solve for the two translations of node 2 and node 3 as master degrees of freedom, and will condense (reduce out) the two rotations. We will develop the theory first, then will substitute our cantilever example.

The first step is to rearrange the degrees of freedom, rows and columns of the stiffness matrix, into dependent (slave) displacements to be reduced,  $\mathbf{z}_a$ , and independent (master) displacements,  $\mathbf{z}_b$ . This involves moving the second and fourth rows and columns of the cantilever stiffness matrix up to become the first and second rows and columns, which moves the first and third rows and columns down to the second and fourth positions.

$$\mathbf{kz} = \mathbf{F} \quad (13.29)$$

$$\begin{bmatrix} \mathbf{k}_{aa} & \mathbf{k}_{ab} \\ \mathbf{k}_{ba} & \mathbf{k}_{bb} \end{bmatrix} \begin{bmatrix} \mathbf{z}_a \\ \mathbf{z}_b \end{bmatrix} = \begin{bmatrix} \mathbf{F}_a \\ \mathbf{F}_b \end{bmatrix} \quad (13.30)$$

Multiplying out the first matrix equation:

$$\mathbf{k}_{aa}\mathbf{z}_a + \mathbf{k}_{ab}\mathbf{z}_b = \mathbf{F}_a \quad (13.31)$$

Solving for  $\mathbf{z}_a$  :

$$\mathbf{z}_a = \mathbf{k}_{aa}^{-1}(\mathbf{F}_a - \mathbf{k}_{ab}\mathbf{z}_b) \quad (13.32)$$

If no forces (moments) are applied at the dependent (slave) degrees of freedom,  $\mathbf{F}_a = [0]$ , and the equation above becomes:

$$\mathbf{z}_a = \mathbf{k}_{aa}^{-1}(-\mathbf{k}_{ab}\mathbf{z}_b) = -\mathbf{k}_{aa}^{-1}\mathbf{k}_{ab}\mathbf{z}_b \quad (13.33)$$

We can now rewrite the displacement vector in terms of  $\mathbf{z}_b$  only:

$$\mathbf{z} = \begin{bmatrix} \mathbf{z}_a \\ \mathbf{z}_b \end{bmatrix} = \begin{bmatrix} -\mathbf{k}_{aa}^{-1}\mathbf{k}_{ab} \\ \mathbf{I} \end{bmatrix} \mathbf{z}_b = \begin{bmatrix} -\mathbf{k}_{aa}^{-1}\mathbf{k}_{ab}\mathbf{z}_b \\ \mathbf{z}_b \end{bmatrix} \quad (13.34)$$

Defining a transformation matrix for brevity:

$$\mathbf{z} = \begin{bmatrix} \mathbf{z}_a \\ \mathbf{z}_b \end{bmatrix} = \begin{bmatrix} -\mathbf{k}_{aa}^{-1}\mathbf{k}_{ab} \\ \mathbf{I} \end{bmatrix} \mathbf{z}_b = \begin{bmatrix} \mathbf{T}_{ab} \\ \mathbf{I} \end{bmatrix} \mathbf{z}_b = \mathbf{Tz}_b \quad (13.35)$$

Where:

$$\mathbf{T}_{ab} = -\mathbf{k}_{aa}^{-1}\mathbf{k}_{ab} \quad (13.36)$$

Substituting back into the original static equilibrium equation:

$$\mathbf{kz} = \mathbf{k}(\mathbf{Tz}_b) = \mathbf{F} \quad (13.37)$$

Multiplying both sides by  $\mathbf{T}^T$  to reduce the number of degrees of freedom from (a + b) to b:

$$(\mathbf{T}^T \mathbf{k} \mathbf{T}) \mathbf{z}_b = \mathbf{T}^T \mathbf{F} \quad (13.38)$$

Expanding the term in parentheses above, and redefining it to be  $\mathbf{k}_{bb}^*$  :

$$\begin{aligned} \mathbf{k}_{bb}^* &= \mathbf{T}^T \mathbf{k} \mathbf{T} = \begin{bmatrix} \mathbf{T}_{ab}^T & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{k}_{aa} & \mathbf{k}_{ab} \\ \mathbf{k}_{ba} & \mathbf{k}_{bb} \end{bmatrix} \begin{bmatrix} \mathbf{T}_{ab} \\ \mathbf{I} \end{bmatrix} \\ &= \begin{bmatrix} (\mathbf{T}_{ab}^T \mathbf{k}_{aa} + \mathbf{k}_{ba}) & (\mathbf{T}_{ab}^T \mathbf{k}_{ab} + \mathbf{k}_{bb}) \end{bmatrix} \begin{bmatrix} \mathbf{T}_{ab} \\ \mathbf{I} \end{bmatrix} \\ &= (\mathbf{T}_{ab}^T \mathbf{k}_{aa} + \mathbf{k}_{ba}) \mathbf{T}_{ab} + (\mathbf{T}_{ab}^T \mathbf{k}_{ab} + \mathbf{k}_{bb}) \mathbf{I} \\ &= \mathbf{T}_{ab}^T \mathbf{k}_{aa} \mathbf{T}_{ab} + \mathbf{k}_{ba} \mathbf{T}_{ab} + \mathbf{T}_{ab}^T \mathbf{k}_{ab} + \mathbf{k}_{bb} \\ &= (-\mathbf{k}_{ba} \mathbf{k}_{aa}^{-1}) \mathbf{k}_{aa} (-\mathbf{k}_{aa}^{-1} \mathbf{k}_{ab}) + \mathbf{k}_{ba} (-\mathbf{k}_{aa}^{-1} \mathbf{k}_{ab}) + (-\mathbf{k}_{ba} \mathbf{k}_{aa}^{-1}) \mathbf{k}_{ab} + \mathbf{k}_{bb} \\ &= \mathbf{k}_{ba} \mathbf{k}_{aa}^{-1} \mathbf{k}_{ab} - \mathbf{k}_{ba} \mathbf{k}_{aa}^{-1} \mathbf{k}_{ab} - \mathbf{k}_{ba} \mathbf{k}_{aa}^{-1} \mathbf{k}_{ab} + \mathbf{k}_{bb} \\ &= \mathbf{k}_{bb} - \mathbf{k}_{ba} \mathbf{k}_{aa}^{-1} \mathbf{k}_{ab} \end{aligned} \quad (13.39)$$

where:  $\mathbf{T}_{ab} = -\mathbf{k}_{aa}^{-1} \mathbf{k}_{ab}$  and  $\mathbf{T}_{ab}^T = -\mathbf{k}_{ba} \mathbf{k}_{aa}^{-1}$ .

So, the original (a + b) degree of freedom problem now can be transformed to a “b” degree of freedom problem by partitioning into dependent and independent degrees of freedom, and solving for the reduced stiffness matrix  $\mathbf{k}_{bb}^*$  and reduced force vector  $\mathbf{F}_b^*$  :

$$\begin{aligned} \mathbf{F}_b^* &= \mathbf{T}^T \mathbf{F} \\ &= \begin{bmatrix} \mathbf{T}_{ab}^T & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{F}_a \\ \mathbf{F}_b \end{bmatrix} = \mathbf{T}_{ba} \mathbf{F}_a + \mathbf{F}_b \\ &= \mathbf{F}_b - \mathbf{k}_{ba} \mathbf{k}_{aa}^{-1} \mathbf{F}_a \end{aligned} \quad (13.40)$$

Then the reduced problem becomes:

$$\mathbf{k}_{bb}^* \mathbf{z}_b = \mathbf{F}_b^* \quad (13.41)$$

After the  $\mathbf{z}_b$  degrees of freedom are known, the  $\mathbf{z}_a$  degrees of freedom can be expanded from the  $\mathbf{z}_b$  masters using, if  $\mathbf{F}_a = [0]$ :

$$\mathbf{z}_a = -\mathbf{k}_{aa}^{-1} \mathbf{k}_{ab} \mathbf{z}_b \quad (13.42)$$

### 13.4.2 Solving Two-Element Cantilever Beam Static Problem

We will now solve the example cantilever for a force applied at the tip. Earlier we showed that the stiffness matrix is:

$$\mathbf{k}_g = EI \begin{bmatrix} \frac{24}{l^3} & 0 & \frac{-12}{l^3} & \frac{6}{l^2} \\ 0 & \frac{8}{l} & \frac{-6}{l^2} & \frac{2}{l} \\ \frac{-12}{l^3} & \frac{-6}{l^2} & \frac{12}{l^3} & \frac{-6}{l^2} \\ \frac{6}{l^2} & \frac{2}{l} & \frac{-6}{l^2} & \frac{4}{l} \end{bmatrix} \quad (13.43)$$

Rearranging rows, 1 to 3, 2 to 1, 3 to 4 and 4 to 2:

$$\mathbf{k}_g = EI \begin{bmatrix} 0 & \frac{8}{l} & \frac{-6}{l^2} & \frac{2}{l} \\ \frac{6}{l^2} & \frac{2}{l} & \frac{-6}{l^2} & \frac{4}{l} \\ \frac{24}{l^3} & 0 & \frac{-12}{l^3} & \frac{6}{l^2} \\ \frac{-12}{l^3} & \frac{-6}{l^2} & \frac{12}{l^3} & \frac{-6}{l^2} \end{bmatrix} \quad (13.44)$$

Rearranging columns, 1 to 3, 2 to 1, 3 to 4 and 4 to 2:

$$\mathbf{k}_g = EI \begin{bmatrix} \frac{8}{l} & \frac{2}{l} & 0 & \frac{-6}{l^2} \\ \frac{2}{l} & \frac{4}{l} & \frac{6}{l^2} & \frac{-6}{l^2} \\ 0 & \frac{6}{l^2} & \frac{24}{l^3} & \frac{-12}{l^3} \\ \frac{-6}{l^2} & \frac{-6}{l^2} & \frac{-12}{l^3} & \frac{12}{l^3} \end{bmatrix} \quad (13.45)$$

Breaking out and identifying the four submatrices of dependent (a) and independent (b) degrees of freedom:

$$\begin{aligned} \mathbf{k}_{aa} &= \frac{EI}{l} \begin{bmatrix} 8 & 2 \\ 2 & 4 \end{bmatrix} & \mathbf{k}_{ab} &= \frac{EI}{l^2} \begin{bmatrix} 0 & -6 \\ 6 & -6 \end{bmatrix} \\ \mathbf{k}_{ba} &= \frac{EI}{l^2} \begin{bmatrix} 0 & 6 \\ -6 & -6 \end{bmatrix} & \mathbf{k}_{bb} &= \frac{EI}{l^3} \begin{bmatrix} 24 & -12 \\ -12 & 12 \end{bmatrix} \end{aligned} \quad (13.46a-d)$$

Finding the inverse of  $\mathbf{k}_{aa}$  :

$$\mathbf{k}_{aa}^{-1} = \frac{1}{14EI} \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} \quad (13.47)$$

$$-\mathbf{k}_{aa}^{-1} \mathbf{k}_{ab} = \frac{-1}{14l} \begin{bmatrix} -6 & -6 \\ 24 & -18 \end{bmatrix} \quad (13.48)$$

$$\mathbf{k}_{ba} \mathbf{k}_{aa}^{-1} \mathbf{k}_{ab} = \frac{EI}{14l^3} \begin{bmatrix} 144 & -108 \\ -108 & 144 \end{bmatrix} \quad (13.49)$$

$$\begin{aligned}
\mathbf{k}_{bb}^* &= \mathbf{k}_{bb} - \mathbf{k}_{ba} \mathbf{k}_{aa}^{-1} \mathbf{k}_{ab} \\
&= \frac{EI}{14l^3} \left\{ \begin{bmatrix} 336 & -168 \\ -168 & 168 \end{bmatrix} - \begin{bmatrix} 144 & -108 \\ -108 & 144 \end{bmatrix} \right\} \\
&= \frac{EI}{14l^3} \begin{bmatrix} 192 & -60 \\ -60 & 24 \end{bmatrix}
\end{aligned} \tag{13.50}$$

$$\mathbf{k}_{bb}^{*-1} = \frac{14l^3}{1008EI} \begin{bmatrix} 24 & 60 \\ 60 & 192 \end{bmatrix} = \frac{l^3}{72EI} \begin{bmatrix} 24 & 60 \\ 60 & 192 \end{bmatrix} \tag{13.51}$$

Solving for the two displacements,  $\mathbf{z}_b$  for a tip force of magnitude P:

$$\begin{aligned}
\mathbf{z}_b &= \begin{bmatrix} z_2 \\ z_3 \end{bmatrix} = \mathbf{k}_{bb}^{*-1} \mathbf{F}_{bb}^* \\
&= \frac{l^3}{72EI} \begin{bmatrix} 24 & 60 \\ 60 & 192 \end{bmatrix} \begin{bmatrix} 0 \\ P \end{bmatrix} \\
&= \frac{Pl^3}{72EI} \begin{bmatrix} 60 \\ 192 \end{bmatrix} = \frac{Pl^3}{EI} \begin{bmatrix} \frac{60}{72} \\ \frac{192}{72} \end{bmatrix} = \frac{Pl^3}{EI} \begin{bmatrix} \frac{5}{6} \\ \frac{8}{3} \end{bmatrix}
\end{aligned} \tag{13.52}$$

The tip displacement is:

$$z_3 = \frac{8Pl^3}{3EI} \tag{13.53}$$

The well-known solution for the displacement of the tip of a cantilever is:

$$z_{\text{tip}} = \frac{PL^3}{3EI} \tag{13.54}$$

Knowing that the total length of the cantilever, L, is 2l:

$$z_{\text{tip}} = \frac{PL^3}{3EI} = \frac{P(2l)^3}{3EI} = \frac{8Pl^3}{3EI} \quad (13.55)$$

The reduced problem has provided the correct solution. Once again, normally we would not solve a reduced static problem except during a hand calculation, but the derivation of static condensation will be useful in the next chapter when dynamic condensation, Guyan reduction, is introduced.

### Problems

P13.1 Assemble the global mass and stiffness matrices for [Figure P2.1](#) element by element. Compare results with P2.1 results.

P13.2 In Section 13.4.2 we solved for the displacements of a two-element cantilever beam with a tip load by reducing out the rotations of the beam. Solve the problem by reducing out the rotations of the middle and tip nodes and the displacement of the middle node. Use a symbolic algebra program to invert the  $3 \times 3$   $\mathbf{k}_{aa}$  matrix.