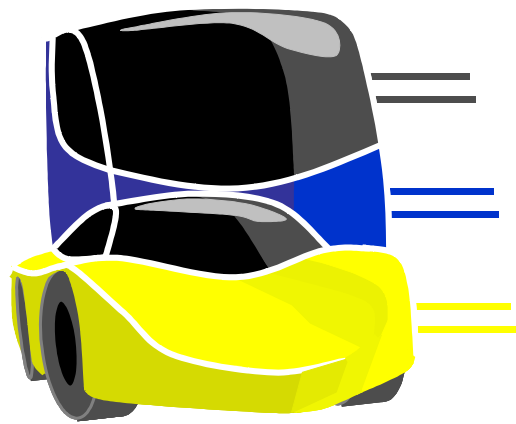




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Solved Problems of Vehicle Dynamics



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Division of Vehicle Dynamics

Preface

This compendium contains solutions to most of the problems discussed during the exercises in the Vehicle Dynamics course (4B1425) at KTH, during the spring 2002.

The problems are taken from the course material, *Exempelsamling i Fordonsdynamik* and *textbook* refers to *Fordonsdynamik*, both compendia are from the Division of Vehicle Dynamics, KTH.

Many people have contributed to the solutions, probably everyone who has been employed at the division. However, any comment or found error is greatly acknowledged by the editor.

KTH, May 2002

Johan Andreasson

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1

Introduction

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Equation (3) in section 1.4.4.

$$\bar{\omega} = \dot{\psi}\hat{Z} + \dot{\theta}\hat{y}^1 + \dot{\varphi}\hat{x}^2 \quad (1.1)$$

Should be used to express the relative angular velocity between the frames A and B with respect to frame A and frame B .

Rewrite the equation

Planar rotation ψ gives

$$\dot{\theta} = \theta = \dot{\varphi} = \varphi = 0 \quad (1.2)$$

and for this case, equation 1.1 can be simplified

$$\bar{\omega} = \dot{\psi}\hat{Z}. \quad (1.3)$$

Transformation into frame A and frame B

Here, frame A equals the inertial frame I and frame B equals frame M_1 . Thus, we can express the angular velocity according to

$$\bar{\omega}^A = \bar{\omega}^I = \dot{\psi}\hat{Z} = \dot{\psi} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } \bar{\omega}^B = \bar{\omega}^{M_1} = {}^{M_1}T_I \bar{\omega}^I = \dot{\psi} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (1.4)$$

Check the result

From equation (4) in section 1.4.4, we have

$$\bar{\omega}^K = \begin{pmatrix} \dot{\varphi} - \dot{\psi}s\theta \\ \dot{\theta}c\varphi + \dot{\psi}s\varphi c\theta \\ -\dot{\theta}s\varphi + \dot{\psi}c\varphi c\theta \end{pmatrix} \quad (1.5)$$

Together with equation 1.2 we can express the angular velocity as

$$\bar{\omega}^B = {}^{M_1}T_K \bar{\omega}^K = \dot{\psi} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } \bar{\omega}^A = {}^IT_B \bar{\omega}^B = {}^IT_{M_1} \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} = \dot{\psi} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (1.6)$$

Difficulties?

Check equation 1.4 by evaluating ${}^{M_1}T_I(0,0,1)^T$ and equation 1.6 by evaluating ${}^KT_{M_1} = {}^KT_{M_2} {}^{M_2}T_{M_1}$. Mind the planar rotation, equation 1.2

SOME DEFINITIONS

${}^A T_B$ is the same as l_{AB} in the textbook.

$c\psi$ is short for $\cos(\psi)$.

$s\psi$ is short for $\sin(\psi)$.

With superscript at a vector, the frame it refers to is indicated, e.g. $\bar{\omega}^K$ means " $\bar{\omega}$ with respect to frame K ".

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From equation (3) in section 1.4.4,

$$\bar{\omega} = \dot{\psi}\hat{Z} + \dot{\theta}\hat{y}^1 + \dot{\varphi}\hat{x}^2, \quad (1.7)$$

equation (4) in section 1.4.4,

$$\bar{\omega}^K = \begin{pmatrix} \dot{\varphi} - \dot{\psi}s\theta \\ \dot{\theta}c\varphi + \dot{\psi}s\varphi c\theta \\ -\dot{\theta}s\varphi + \dot{\psi}c\varphi c\theta \end{pmatrix}. \quad (1.8)$$

should be derived.

Rewrite the equation

Equation 1.7 is expressed in several frames while equation 1.8 is expressed in frame K . Thus, $\hat{Z}$, $\hat{y}^1$ and $\hat{x}^2$ must be transformed into $\hat{z}$, $\hat{y}$ and $\hat{x}$, respectively:

$$\bar{\omega}^K = {}^K T_I \dot{\psi}\hat{Z} + {}^K T_{M_1} \dot{\theta}\hat{y}^1 + {}^K T_{M_2} \dot{\varphi}\hat{x}^2. \quad (1.9)$$

We need to evaluate:

$$\begin{aligned} {}^K T_{M_2} \\ {}^K T_{M_1} &= {}^K T_{M_2} {}^{M_2} T_{M_1} \\ {}^K T_I &= {}^K T_{M_2} {}^{M_1} T_{M_1} {}^{M_1} T_I \end{aligned} \quad (1.10)$$

Transformations

Using the transformations in section 1.4.3, we get

$$\begin{aligned}
 {}^K T_{M_2} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\varphi & -s\varphi \\ 0 & s\varphi & c\varphi \end{pmatrix} \\
 {}^K T_{M_1} &= {}^K T_{M_2} {}^{M_2} T_{M_1} \\
 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\varphi & s\varphi \\ 0 & -s\varphi & c\varphi \end{pmatrix} \begin{pmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{pmatrix} = \begin{pmatrix} c\theta & 0 & -s\theta \\ s\varphi s\theta & c\varphi & s\varphi c\theta \\ c\varphi s\theta & -s\varphi & c\varphi c\theta \end{pmatrix} \quad (1.11)
 \end{aligned}$$

$$\begin{aligned}
 {}^K T_I &= {}^K T_{M_2} {}^{M_2} T_{M_1} {}^{M_1} T_I \\
 &= \begin{pmatrix} c\theta & 0 & -s\theta \\ s\varphi s\theta & c\varphi & s\varphi c\theta \\ c\varphi s\theta & -s\varphi & c\varphi c\theta \end{pmatrix} \begin{pmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} c\theta c\psi & c\theta s\psi & -s\theta \\ s\varphi s\theta c\psi - c\varphi s\psi & c\varphi c\psi + s\varphi s\theta s\psi & s\varphi c\theta \\ s\varphi s\psi + c\varphi s\theta s\psi & c\varphi s\theta s\psi - s\psi & c\varphi c\theta \end{pmatrix}
 \end{aligned}$$

Now, equations 1.11 in equation 1.9 give

$$\bar{\omega}_K = \dot{\psi} \begin{pmatrix} -s\theta \\ s\varphi c\theta \\ c\varphi c\theta \end{pmatrix} + \dot{\theta} \begin{pmatrix} 0 \\ c\varphi \\ -s\varphi \end{pmatrix} + \dot{\varphi} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\dot{\psi}s\theta + \dot{\varphi} \\ \dot{\psi}s\varphi c\theta + \dot{\theta}c\varphi \\ \dot{\psi}c\varphi c\theta - \dot{\theta}s\varphi \end{pmatrix} \quad (1.12)$$

which is the same as equation 1.8.

Comments

Is there a faster way to do the calculations above? Try to evaluate from the right instead, e.g.

$${}^K T_I \dot{\psi} \hat{Z} = ({}^K T_{M_2} ({}^{M_1} T_{M_1} ({}^{M_1} T_I \dot{\psi} \hat{Z}))) \quad (1.13)$$

What is the difference?

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Express the signals $\bar{w}_1$ and $\bar{w}_2$ as well as the velocities $u_x \hat{x}^1 = \bar{u}_x^1$ and $u_z \hat{z}^1 = \bar{u}_z^1$ with respect to the car frame.

Identify the transformation

Since the motion is planar, frame M_2 equals frame K and we can express the transformation matrix as

$${}^K T_{M_1} = {}^{M_2} T_{M_1} = \begin{pmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{pmatrix} \quad (1.14)$$

Transform $\bar{w}_1$ and $\bar{w}_2$

Now, a vector expressed in frame M_2 can be transformed to frame K according to $\bar{v}^K = {}^K T_{M_1} \bar{v}^{M_1}$. For the signals $\bar{w}_1$ and $\bar{w}_2$, this yields

$$\bar{w}_1^K = {}^K T_{M_1} \bar{w}_1^{M_1} = \begin{pmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ w_1 \end{pmatrix} = \begin{pmatrix} -w_1 s\theta \\ 0 \\ w_1 c\theta \end{pmatrix} \quad (1.15)$$

and

$$\bar{w}_2^K = {}^K T_{M_1} \bar{w}_2^{M_1} = \begin{pmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ w_2 \end{pmatrix}^{M_1} = \begin{pmatrix} -w_2 s\theta \\ 0 \\ w_2 c\theta \end{pmatrix} \quad (1.16)$$

Transform $\bar{u}_x$ and $\bar{u}_z$

Like in equations 1.15 and 1.16, we get

$$\bar{u}_x^K = {}^K T_{M_1} \bar{u}_x^1 = \begin{pmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{pmatrix} \begin{pmatrix} u_x \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} u_x c\theta \\ 0 \\ u_x s\theta \end{pmatrix} \quad (1.17)$$

and

$$\bar{u}_z^K = {}^K T_{M_1} \bar{u}_z^1 = \begin{pmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ u_z \end{pmatrix} = \begin{pmatrix} -u_z s\theta \\ 0 \\ u_z c\theta \end{pmatrix} \quad (1.18)$$

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Wheels and tyres

Supplementary problems, Chapter 2

1

Calculate overturning moment for a truck tyre with specifications as below

- with, $b = 0.2\text{m}$
- radius, $R = 0.5\text{m}$
- belt mass, $m = 40\text{kg}$
- lateral force, $F_y = 10\text{kN}$
- normal load, $F_z = 35\text{kN}$

The modes of the tyre are shown in figure 2.

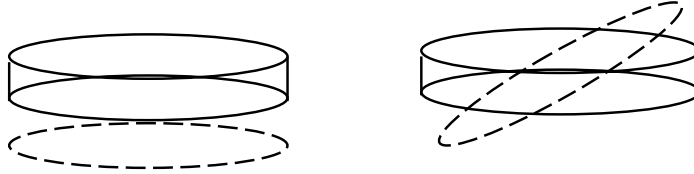


Figure 2.1. The modes: Lateral translation, 30Hz and tilting rotation, 35Hz.

2

Derive an expression for the aligning moment, M_z , for small slip angles. Use the brush model and assume evenly distributed normal pressure, t_z , over the whole contact patch, $2a$.

3

From the brush model, the following expression for the longitudinal force, F_x can be derived:

$$\begin{aligned}
 F_x &= C_s s f(\lambda) \\
 f(\lambda) &= \begin{cases} \lambda(2 - \lambda) & \lambda \leq 1 \\ 1 & \lambda > 1 \end{cases} \\
 \lambda &= \frac{F_z \mu}{2C_s |s|}
 \end{aligned} \tag{2.1}$$

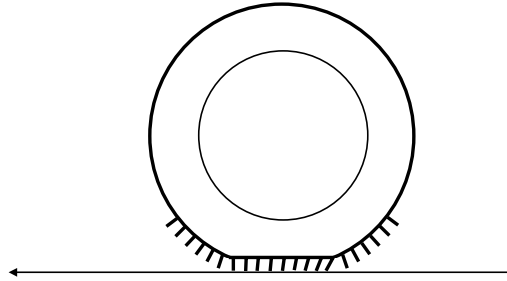


Figure 2.2. The deformation of the brushes over the contact patch.

- a) Derive expression 2.1, assuming that the normal force is evenly distributed over the contact patch.
- b) Sketch the lateral and longitudinal force as a function of longitudinal slip at combined braking and cornering. Use the sketch to describe the principle of the ABS.

Supplementary problem 1

Calculate the overturning moment for a truck tyre.

Side stiffness

The stiffness can be calculated from the frequencies given.

$$f_1 = \frac{\omega_1}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{c_1}{m}} \rightarrow c_1 = m(2\pi f_1)^2 \quad (2.2)$$

and

$$f_2 = \frac{\omega_2}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{c_2}{I}} \rightarrow c_2 = I(2\pi f_2)^2. \quad (2.3)$$

To evaluate this we have to calculate the inertia, I .

Inertia

Assuming that the mass is evenly distributed around the tyre's periphery, the inertia can be calculated according to

$$I = \frac{m}{2}R^2 + \frac{m}{12}b^2 \quad (2.4)$$

Displacement

To find the displacement from x , we need to calculate the total side stiffness of the tyre. The rotational stiffness, c_1 , and translational stiffness, c_2 , are considered to be in series. Transforming, c_2 , to a translational stiffness, c_{2trans} , we can write:

$$c_{tot}^{-1} = c_1^{-1} + c_{2trans}^{-1} = c_1^{-1} + R^2 c_2^{-1} \quad (2.5)$$

The displacement, Δy , is now

$$\Delta y = \frac{F_y}{c_{tot}} \quad (2.6)$$

Overturning moment

We now have the levers needed to calculate the components of the overturning moment:

$$\begin{aligned} \text{Side force induced: } M_{x1} &= F_z \Delta y = 5000 \text{Nm} \\ \text{Normal force induced: } M_{x2} &= F_y R = 595 \text{Nm} \\ \text{Totally: } M_x &= 5.6 \text{KNm.} \end{aligned} \quad (2.7)$$

Supplementary problem 2

Derive an expression for the aligning moment, M_z , for small slip angles, α , assuming evenly distributed normal force.

Aligning moment

This is calculated in section (2.11.1) in the textbook. The answer is

$$M_z = \frac{2}{3}c_p a^3 \tan(\alpha). \quad (2.8)$$

Supplementary problem 3

- Derive an expression for the longitudinal force F_x , with the brush model, assuming evenly distributed normal force.
- Sketch the lateral and longitudinal forces as a function of longitudinal slip during combined turning and braking. Use the sketch to explain the principles of ABS.

a)

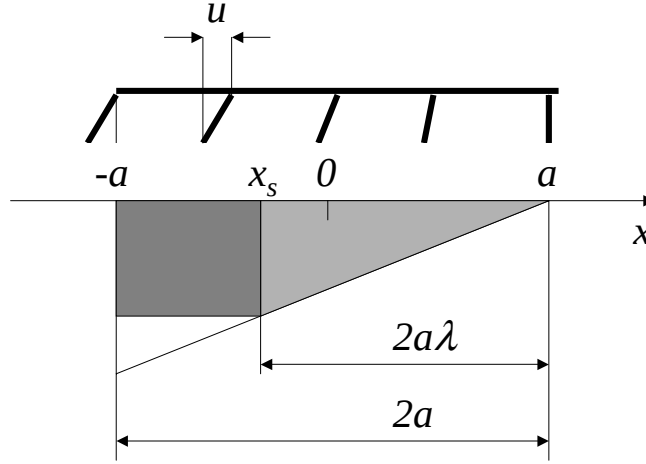


Figure 2.3. Adhesion (light grey) and sliding zone (dark grey) in contact area.

The normal force, F_z is assumed to be evenly distributed over the contact length $2a$, giving a constant vertical pressure.

$$\tau_z = \frac{F_z}{2a} \quad (2.9)$$

Furthermore we know that the lateral and longitudinal force per length unit is limited by the friction:

$$|\tau_{x,y}| \leq \mu \tau_z. \quad (2.10)$$

Thus, sliding occurs when

$$cu_{x,y} \geq \mu \tau_z \quad (2.11)$$

where $u_{x,y}$ is the stretch of the bristle defined according to

$$u_x = x_l - x_u = r\omega\Delta t - v_x\Delta t = \frac{\omega R - v_x}{v_x}(v_x\Delta t) = s(a - x) = -s(x - a) \quad (2.12)$$

$$u_y = \tan(\alpha)(a - x) = -\tan(\alpha)(x - a), \text{ where } \alpha \text{ is the slip angle.}$$

Obviously, the only difference between lateral and longitudinal components is the factors $\tan \alpha$ and s for the lateral and longitudinal parts, respectively. Thus we write

$$u = -\eta(x - a) \begin{cases} \eta = \tan \alpha & \text{lateral} \\ \eta = s & \text{longitudinal} \end{cases} \quad (2.13)$$

We can now rewrite equation 2.14

$$-c\eta(x - a) \geq \mu\tau_z \quad (2.14)$$

giving the point where the sliding starts, x_s , when

$$x \leq x_s = -\frac{\mu\tau_z}{c \tan \alpha} + a. \quad (2.15)$$

Consider the geometry of figure 2.3; the length of the adhesion can be written $2a\lambda = a - x_s$. Thus, we can write

$$\lambda = \frac{a - x_s}{2a} = [\text{use eq. 2.15}] = \frac{\mu\tau_z}{2ac\eta} = [\text{use eq. 2.9}] = \frac{\mu F_z}{4a^2 c\eta} = \frac{\mu F_z}{2C\eta} \quad (2.16)$$

and

$$x_s = a(1 - 2\lambda). \quad (2.17)$$

To get the total force, we integrate the two contributions from sliding and adhesion:

$$\begin{aligned} F &= \int_{-a}^{a(1-2\lambda)} \tau_{x,y} dx + \int_{a(1-2\lambda)}^a \tau_{x,y} dx \\ &= \int_{-a}^{a(1-2\lambda)} \frac{\mu F_z}{2a} dx + \int_{a(1-2\lambda)}^a c\eta(x - a) dx \\ &= \left[-\frac{\mu F_z}{2a} x \right]_{-a}^{a(1-2\lambda)} + \left[c\eta \left(\frac{x^2}{2} - ax \right) \right]_{a(1-2\lambda)}^a \\ &= -2C\eta\lambda(1 - \lambda) - 2\lambda^2 a^2 c\eta \\ &= -2C\eta\lambda(1 - \lambda) - \lambda^2 C\eta \\ &= -C\eta\lambda(2 - \lambda) \end{aligned} \quad (2.18)$$

However, this result itself is not valid. Consider what would happen e.g. if $\lambda \geq 1$. Clearly, λ has to be restricted. We write instead

$$\begin{aligned} F &= -2C\eta f(\lambda) \\ f(\lambda) &= \begin{cases} \lambda(2 - \lambda) & \lambda \leq 1 \\ 1 & \lambda > 1 \end{cases} \end{aligned} \quad (2.19)$$

Now, we complete the expression for the longitudinal force, giving:

$$\begin{aligned} F &= -2C_s s f(\lambda) \\ f(\lambda) &= \begin{cases} \lambda(2 - \lambda) & \lambda \leq 1 \\ 1 & \lambda > 1 \end{cases} \\ \lambda &= \frac{F_z \mu}{2C_s |s|} \end{aligned} \quad (2.20)$$

b)

Figure 2.4 illustrates the lateral and longitudinal forces. There are three objectives aimed at simultaneously:

- Steerability: Maximum F_y .
- Driving stability: Optimal force distribution front/rear.
- Optimal stopping distance: Maximum F_x

Thus the ABS tries to keep the slip within the area indicated in the figure. How this is done is described in section (2.12) in the textbook.

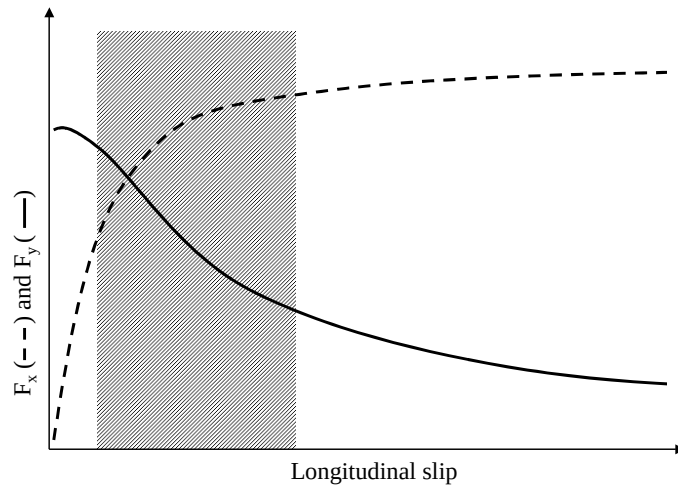


Figure 2.4. The shadowed area shows the preferable range for the ABS system.

3

Introduction to dynamics

LAGRANGE'S EQUATION

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} = F_i$$

$L = T - U$ is the Lagrangeian and R is the Rayleigh dissipation.
 q_i is the DOF and F_i is external force acting at that DOF.

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Derive the motion equations for the given system with Lagrange's equation.

Identify the system's DOF

In this case, we only have one Degree Of Freedom, x , indicated in the problem illustration.

Derive expressions for L , R and F

Assume that x refers to the displacement of the mass compared to when the spring is unloaded, we then get:

$$T = \frac{1}{2}m\dot{x}^2, U = \frac{1}{2}kx^2 - mgx \rightarrow$$

$$L = \frac{1}{2}m\dot{x}^2 - \left(\frac{1}{2}kx^2 - mgx \right) \quad (3.1)$$

$$R = \frac{1}{2}c\dot{x}^2$$

$$F = 0.$$

Derivate L and R

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \frac{\partial L}{\partial x} = -kx + mg, \frac{\partial R}{\partial \dot{x}} = c\dot{x} \quad (3.2)$$

Evaluate Lagrange's equation

$$\begin{aligned} \frac{d}{dt}(m\dot{x}) - (-kx + mg) + c\dot{x} &= 0 \\ m\ddot{x} + c\dot{x} + kx &= mg \end{aligned} \quad (3.3)$$

STEINERS LAW (planar motion)

If I^A is the inertia with respect to the centre of gravity, then the inertia with respect to a point B can be written as

$$I^B = I^A + md^2 \quad (3.4)$$

where d is the distance from A to B.

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- a) Calculate the wheel's inertia from the given data.
- b) Evaluate how sensitive the method is to erroneous data.

Calculate the inertia of the pendulum

A momentum relation around the pendulum's centre of rotation gives:

$$-mg\lambda L \sin(\varphi) + J_p \ddot{\varphi} \approx -mg\lambda L \varphi + J_p \ddot{\varphi} = 0 \quad (3.5)$$

Identify the eigenfrequency:

$$\begin{aligned} \ddot{\varphi} = -\omega_0^2 \varphi &= \frac{mg\lambda L}{J_p} \varphi \rightarrow \\ \omega_0^2 &= \frac{mg\lambda L}{J_p} \end{aligned} \quad (3.6)$$

Now, the inertia can be calculated as

$$J_p = \frac{m_p g \lambda L}{\omega_0^2} = \frac{m_p g \lambda L}{\left(\frac{2\pi}{T}\right)^2} \quad (3.7)$$

Calculate the inertia of the wheel

The total inertia of the wheel and pendulum is calculated according to equation 3.7:

$$J_{tot} = \frac{m_p g \lambda L + m_w g L}{\left(\frac{2\pi}{T_{tot}}\right)^2} \text{ kgm}^2 \quad (3.8)$$

With equation 3.4 we can now calculate the wheel's inertia according to

$$J_w = J_{tot} - J_p - m_w d^2 \quad (3.9)$$

Though, the inertia is now around the x-axis while we are looking for the inertia around the y-axis. The wheels symmetry gives $J_{wx} = J_{wz}$ and therefore we can calculate

$$J_{wy} = J_{wx} + J_{wz} = 2J_w = 11.9 \text{ kgm}^2. \quad (3.10)$$

Calculate the wheel's inertia for b)

According to above but with $T_{tot}=2.8$ s and $T_{tot}=2.7$ s we get $J_{wy} = 51.4 \text{ kgm}^2$ and $J_{wy} = 2.2 \text{ kgm}^2$, respectively.

Conclusions?

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Derive the system's motion equations and express them in state space. We use Lagrange's equation for the motion equations.

Identify the system's DOF

There are 2 DOF, the motion of the chariot, x , and the motion of the connection between the spring and the damper, y . We define both positive to the right.

Derive expressions for L , R and F

$$\begin{aligned} T &= \frac{1}{2}m\dot{x}^2, U = \frac{1}{2}kx^2 + \frac{1}{2}k_c y^2 \rightarrow \\ L &= \frac{1}{2}m\dot{x}^2 - \left(\frac{1}{2}kx^2 + \frac{1}{2}k_c y^2\right) \\ R &= \frac{1}{2}c(\dot{x} - \dot{y})^2 \\ F &= F_x. \end{aligned} \tag{3.11}$$

Derivate L and R

Since we have 2 DOF we get four derivatives of L and two of R :

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}} &= m\dot{x}, \frac{\partial L}{\partial x} = -kx, \frac{\partial R}{\partial \dot{x}} = c(\dot{x} - \dot{y}) \\ \frac{\partial L}{\partial \dot{y}} &= 0, \frac{\partial L}{\partial y} = -k_c y, \frac{\partial R}{\partial \dot{y}} = -c(\dot{x} - \dot{y}) \end{aligned} \tag{3.12}$$

Evaluate Lagrange's equation

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \frac{\partial R}{\partial \dot{x}} &= m\ddot{x} + c(\dot{x} - \dot{y}) + kx = F_x \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} + \frac{\partial R}{\partial \dot{y}} &= -c(\dot{x} - \dot{y}) + k_c y = 0 \end{aligned} \tag{3.13}$$

Express in state space

Rewrite equations 3.13:

$$\begin{aligned} \ddot{x} &= \frac{1}{m} (F_x - c(\dot{x} - \dot{y}) - kx) \\ \dot{y} &= \frac{1}{c} (c\dot{x} - k_c y) \end{aligned} \tag{3.14}$$

For this model we get three states, two along x and one along y :

$$\begin{aligned}\dot{u} &= \frac{F_x}{m} - \frac{k}{m}x - \frac{k_c}{m}y \\ \dot{x} &= u \\ \dot{y} &= u - \frac{k}{c}y\end{aligned}\tag{3.15}$$

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The losses in a system due to friction is measured and plotted in the problem. With a viscous damper model, this can be described as

$$W_d = \oint F_d dz = \int_{t_1}^{t_2} (c_e \dot{z}) \dot{z} dt \quad (3.16)$$

The motion can be described as

$$\begin{aligned} z &= z_0 \sin \omega t \\ \dot{z} &= z_0 \omega \cos \omega t \end{aligned} \quad (3.17)$$

Use equation 3.17 in 3.26

$$\begin{aligned} W_d &= \int_{t_1}^{t_2} c_e \dot{z}^2 dt = \int_0^T c_e z_0^2 \omega^2 \cos^2(\omega t) dt \\ &= c_e z_0^2 \omega^2 \frac{1}{2} (1 + \cos 2\omega t) dt = \frac{1}{2} c_e z_0^2 \omega^2 \left[t + \frac{\sin 2\omega t}{2\omega} \right]_0^T \\ &= \frac{1}{2} c_e z_0^2 \omega^2 T \end{aligned} \quad (3.18)$$

Now, since the measured energy loss corresponds to the area of the plotted ellipse

$$W_d \approx 25 \text{ squares} \cdot 1000\text{N} \cdot 0.01\text{m/square} = 250\text{Nm} \quad (3.19)$$

Thus

$$c_e = \frac{2W_d}{z_0^2 \omega^2 T} \approx \frac{2 \cdot 250}{0.0045^2 \cdot (4\pi)^2 \cdot 0.5} = 3131\text{Ns/m} \quad (3.20)$$

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Since the terms referring to the spring are equal and render no energy loss, they are left out. The friction model can be described as

$$F_\mu = pA\mu \text{sign}(\dot{x}) \quad (3.21)$$

Model this with a viscous damper instead of friction

$$F_d = c\dot{x} \quad (3.22)$$

The motion of can be described as

$$\begin{aligned} x &= x_0 \sin \omega t \\ \dot{x} &= x_0 \omega \cos \omega t \end{aligned} \quad (3.23)$$

with $\omega T = 2\pi$. The energy loss in the system per cycle due to friction can be found from

$$\begin{aligned} W_\mu &= \oint F_\mu dx = \int_{t_1}^{t_2} pA\mu \text{sign}(\dot{x}) \dot{x} dt \\ &= \int_0^T pA\mu \text{sign}(\dot{x}) x_0 \omega \cos \omega t dt \\ &= pA\mu \int_0^{t_0} x_0 \omega \cos \omega t dt - pA\mu \int_{t_0}^T x_0 \omega \cos \omega t dt \\ &= pA\mu x_0 \left([\sin \omega t]_0^{\frac{T}{4}} + [\sin \omega t]_{\frac{3T}{4}}^T - [\sin \omega t]_{\frac{3T}{4}}^{\frac{3T}{4}} \right) \\ &= pA\mu x_0 [(\sin 0.5\pi - \sin 0) \\ &\quad + (\sin 2\pi - \sin 1.5\pi) - (\sin 1.5\pi - \sin 0.5\pi)] \\ &= pA\mu x_0 [(1 - 0) + (0 - (-1)) - (-1 - 1)] \\ &= 4pA\mu x_0 \end{aligned} \quad (3.24)$$

The energy loss per cycle in a system with a damper model is

$$\begin{aligned} W_d &= \oint F_d dx = \int_{t_1}^{t_2} c\dot{x} \dot{x} dt \\ &= \int_0^T cx_0^2 \omega^2 \cos \omega t dt \\ &= cx_0^2 \omega^2 \frac{1}{2} (1 + \cos 2\omega t) dt = \frac{1}{2} cx_0^2 \omega^2 \left[t + \frac{\sin 2\omega t}{2\omega} \right]_0^T \\ &= \frac{1}{2} cx_0^2 \omega^2 (T - 0 + 0 - 0) = [T\omega = 2\pi] \\ &= cx_0^2 \omega \pi \end{aligned} \quad (3.25)$$

Now, since the energy loss in both systems should be equal, we write

$$4pA\mu x_0 = cx_0^2 \omega \pi \rightarrow c = \frac{4pA\mu}{\pi x_0 \omega} \quad (3.26)$$

Supplementary problems, Chapter 3

1

Using the nonlinear equation

$$\ddot{x} + x^3 = 0 \quad (3.27)$$

show that if x_1 and x_2 are solutions satisfying the differential equation, their superposition, $x_1 + x_2$, is not a solution.

2

Determine the differential equation of motion for the spring-mass system with the discontinuous stiffness resulting from the free gaps of figure 3.

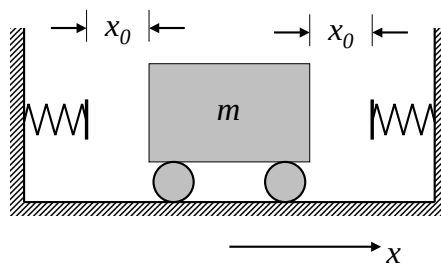


Figure 3.1.

3

Determine the phase plane equation for the damped spring-mass system

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = 0 \quad (3.28)$$

and plot one of the trajectories with $v = y/\omega_n$ and x as coordinates.

Supplementary problem 1

$$\begin{aligned}\ddot{x}_1 + x_1^3 &= 0 \\ \ddot{x}_2 + x_2^3 &= 0\end{aligned}\tag{3.29}$$

However, if $x_3 = x_1 + x_2$, then

$$\begin{aligned}\ddot{x}_3 + x_3^3 &= \ddot{x}_1 + \ddot{x}_2 + x_1^3 + 3x_1^2x_2 + 3x_1x_2^2 + x_2^3 \\ &= \underbrace{\ddot{x}_1 + x_1^3}_{=0} + \underbrace{3x_1^2x_2 + 3x_1x_2^2}_{\neq 0} + \underbrace{\ddot{x}_2 + x_2^3}_{=0} \neq 0\end{aligned}\tag{3.30}$$

Thus, the superposition of x_1 and x_2 is not a solution.

Supplementary problem 2

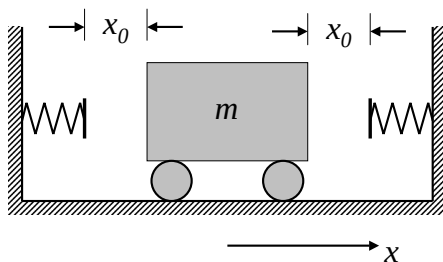


Figure 3.2.

The spring force acting at the mass is

$$f = \begin{cases} -k(x - x_0) & x > x_0 \\ 0 & -x_0 < x < x_0 \\ k(x + x_0) & x < -x_0 \end{cases} \quad (3.31)$$

Thus, Newton's second law gives

$$0 = \begin{cases} m\ddot{x} + k(x - x_0) & x > x_0 \\ m\ddot{x} & -x_0 < x < x_0 \\ m\ddot{x} - k(x + x_0) & x < -x_0 \end{cases} \quad (3.32)$$

Supplementary problem 3

Use state-space to rewrite the equation according to

$$y = \dot{x} \rightarrow \dot{y} + 2\zeta\omega_n y + \omega_n^2 x = 0 \quad (3.33)$$

Substitute

$$\begin{aligned} v = y/\omega_n \rightarrow \\ \dot{v}\omega_n + 2\zeta\omega_n^2 v + \omega_n^2 x = 0 \rightarrow \\ \dot{v} + 2\zeta\omega_n v + \omega_n x = -f(x, y) \end{aligned} \quad (3.34)$$

Find the slope

$$\begin{aligned} \frac{dv}{dx} = \frac{dv/dt}{dx/dt} = \frac{-f(x, y)}{y} = \frac{-2\zeta v\omega_n - \omega_n x}{v\omega_n} = \frac{-2\zeta v - x}{v} = \alpha \rightarrow \\ v = \frac{x}{-\alpha + 2\zeta} \end{aligned} \quad (3.35)$$

Thus, the phase plane equation is

$$v = \frac{x}{-\alpha + 2\zeta} \text{ and } v = \frac{y}{\omega_n} \quad (3.36)$$

In figure 3, a plot for initial values $x_0 = 0, v_0 = 5$ and $\zeta = 0.1, \omega_n = 1$ is shown.

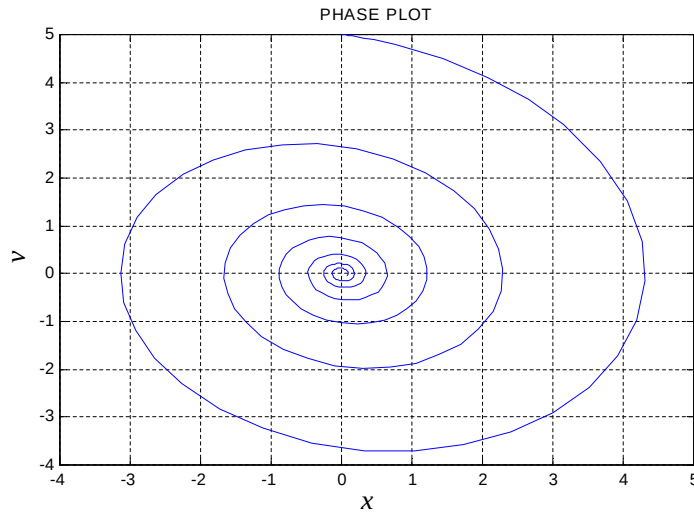


Figure 3.3.

4

Vehicle dynamics in the symmetry plane

401

First the motion equation should be derived. Then the eigen-frequencies and the corresponding dampings for bounce and pitch, should be calculated.

Identify the system's DOF

There are 2 DOF and we choose them as bounce, z , and pitch, θ .

Identify forces

The external forces are induced by w_1 and w_2 where

$$\begin{aligned} F_z &= c\dot{w}_1 + k_1 w_1 + k_2 w_2 \\ M_\theta &= \frac{L}{2}(-c\dot{w}_1 - k_1 w_1 + k_2 w_2) \end{aligned} \quad (4.1)$$

Derive expressions for L , R and F

$$\begin{aligned} T &= \frac{1}{2}m\dot{z}^2 + \frac{1}{2}J\dot{\theta}^2, U = \frac{1}{2}k_1(z - \frac{L}{2}\theta)^2 + \frac{1}{2}k_2(z + \frac{L}{2}\theta)^2 + mgz \rightarrow \\ L &= \frac{1}{2}m\dot{z}^2 + \frac{1}{2}J\dot{\theta}^2 - \left(\frac{1}{2}k_1(z - \frac{L}{2}\theta)^2 + \frac{1}{2}k_2(z + \frac{L}{2}\theta)^2 + mgz\right) \\ R &= \frac{1}{2}c(\dot{z} - \frac{L}{2}\dot{\theta})^2 \\ F &= F_z \text{ and } M_\theta. \end{aligned} \quad (4.2)$$

Derivate L and R

Since we have 2 DOF we get two derivatives of L and two of R .

$$\begin{aligned} \frac{\partial L}{\partial \dot{z}} &= m\dot{z} \\ \frac{\partial L}{\partial z} &= -(k_1(z - \frac{L}{2}\theta) + k_2(z + \frac{L}{2}\theta) + mg) \\ \frac{\partial R}{\partial \dot{z}} &= c(\dot{z} - \frac{L}{2}\dot{\theta}) \\ \frac{\partial L}{\partial \dot{\theta}} &= J\dot{\theta} \\ \frac{\partial L}{\partial \theta} &= -(-\frac{L}{2}k_1(z - \frac{L}{2}\theta) + \frac{L}{2}k_2(z + \frac{L}{2}\theta)) \\ \frac{\partial R}{\partial \theta} &= -\frac{L}{2}c(\dot{z} - \frac{L}{2}\dot{\theta}) \end{aligned} \quad (4.3)$$

Lagrange's equation

We now write Lagrange's equation on matrix form:

$$\begin{pmatrix} m & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} \ddot{z} \\ \ddot{\theta} \end{pmatrix} + \begin{pmatrix} c & -\frac{L}{2}c \\ -\frac{L}{2}c & (\frac{L}{2})^2 \end{pmatrix} \begin{pmatrix} \dot{z} \\ \dot{\theta} \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & \frac{L}{2}(k_2 - k_1) \\ \frac{L}{2}(k_2 - k_1) & (\frac{L}{2})^2(k_1 + k_2) \end{pmatrix} \begin{pmatrix} z \\ \theta \end{pmatrix} =$$

$$\begin{pmatrix} c & 0 \\ -\frac{L}{2}c & 0 \end{pmatrix} \begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix} + \begin{pmatrix} k_1 & k_2 \\ -\frac{L}{2}k_1 & \frac{L}{2}k_2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \quad (4.4)$$

Eigen-frequencies and relative damping

We now only should consider the homogenous version of equation 4.8, this means that there will be no external forces. If we name the mass, damping and stiffness matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$, respectively, we can write:

$$\mathbf{M} \begin{pmatrix} \ddot{z} \\ \ddot{\theta} \end{pmatrix} + \mathbf{C} \begin{pmatrix} \dot{z} \\ \dot{\theta} \end{pmatrix} + \mathbf{K} \begin{pmatrix} z \\ \theta \end{pmatrix} = \mathbf{0} \quad (4.5)$$

Assume $z = z_0 e^{i\omega t}$ and $\theta = \theta_0 e^{i\omega t}$, we can then write:

$$(-\omega^2 \mathbf{M} + i\omega \mathbf{C} + \mathbf{K}) \begin{pmatrix} z_0 \\ \theta_0 \end{pmatrix} = \mathbf{0} \quad (4.6)$$

The determinant equation

$$\| -\omega^2 \mathbf{M} + i\omega \mathbf{C} + \mathbf{K} \| = 0 \quad (4.7)$$

gives, with numerical data

$$\begin{aligned} \omega^4 - 200\omega^2 + 9375 &= 0 \\ -i5\omega^3 + i625\omega &= 0 \end{aligned} \quad (4.8)$$

Solving these equations give

$$\begin{aligned} \omega^2 &= 100 \pm 25 \\ \omega^2 &= 125 \end{aligned} \quad (4.9)$$

Thereby we see that we have a real double root in $\omega^2 = 125$. From polynom division we get that:

$$\frac{\omega^4 - i5\omega^3 - 200\omega^2 + i625\omega + 9375}{\omega^2 - 125} = \omega^2 - i5\omega - 75 \quad (4.10)$$

giving

$$\omega = i\frac{5}{2} \pm \sqrt{\frac{275}{4}} \quad (4.11)$$

Though, we assumed $z = z_0 e^{i\omega t}$ and $\theta = \theta_0 e^{i\omega t}$ but now we have $\omega = a + ib$. This corresponds to $e^{-\zeta\omega_0 t} e^{i\omega_e t}$ where ζ is the relative damping, ω_e is the damped eigen-frequency and ω_0 is the undamped eigen-frequency.

We get ω_e from the real part of ω and since $\zeta\omega_0$ is the imaginary part of ω , ζ can be calculated from:

$$\omega_e^2 = \omega_0^2(1 - \zeta^2) \rightarrow \zeta = \frac{Im[\omega]}{\sqrt{\omega_e^2 + Im[\omega]}} = 0.29 \quad (4.12)$$

We now have

$$\omega = \sqrt{125} = 11.18 \text{ rad/s and } \zeta = 0$$

$$\omega = \sqrt{275/4} = 8.29 \text{ rad/s and } \zeta = 0.29$$

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Eigen-frequencies

From equation (1.2) in problem 410 we get the motion matrix equation, but in this case, there is no damping:

$$\begin{pmatrix} m & 0 \\ 0 & \frac{J}{L^2} \end{pmatrix} \begin{pmatrix} \ddot{z} \\ \ddot{\theta}L \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & k_2(1 - \lambda) - k_1\lambda \\ k_2(1 - \lambda) - k_1\lambda & k_1\lambda^2 + k_2(1 - \lambda)^2 \end{pmatrix} \begin{pmatrix} z \\ \theta L \end{pmatrix} = \begin{pmatrix} f_z \\ \frac{M_\theta}{L} \end{pmatrix} \quad (4.13)$$

Next step is to find the eigen frequencies by solving

$$\begin{aligned} |\mathbf{K} - \omega^2 \mathbf{M}| &= |\mathbf{M}^{-1} \mathbf{K} - \omega^2 \mathbf{I}| = 0 \rightarrow \\ (48.6667 - \omega^2)(82.6019 - \omega^2) - 3.3548 \cdot 22.3889 &= 0 \end{aligned} \quad (4.14)$$

giving $\omega_1^2 = 46.5814 \rightarrow f_1 = 1.086$ Hz and $\omega_2^2 = 84.6871 \rightarrow f_1 = 1.4646$ Hz.

Decoupled masses

We search $\lambda_e L_e$ and $(1 - \lambda_e)L_e$. Consider figure 4 – 14 in the text book and the equations on page 4 : 16 and 4 : 17:

$$\begin{aligned} \xi &= \frac{\frac{J}{mL^2} + (1 - \lambda)^2 - 1 + l}{2(l - \lambda)} = -0.2095 \\ L_e &= L \cdot 2 \cdot \sqrt{(1 - l - \xi)^2 + l(1 - l)} = 5.2762 \\ \lambda_e &= \frac{1}{2} - (1 - \lambda - \xi) \frac{L}{L_e} = 0.0547 \end{aligned} \quad (4.15)$$

I.e. $\lambda_e L_e = 0.2887$ m and $(1 - \lambda_e)L_e = 4.9874$ m .

406

Calculate steady-state of a braking car. Before we can derive and solve the motion equation we need to identify the forces that act on the car.

Identify forces

Since the car brakes, there are longitudinal forces acting at the wheels, $F = mr$. The gravitational force mg is not relevant in this case.

Identify the system's DOF

There are 2 DOF: bounce, z , and pitch, θ .

Derive expressions for L , R and F

$$\begin{aligned}
 T &= \frac{1}{2}m\dot{z}^2 + \frac{1}{2}J\dot{\theta}^2, U = \frac{1}{2}k_1(z - \lambda L\theta)^2 + \frac{1}{2}k_2(z + (1 - \lambda)L\theta)^2 \rightarrow \\
 L &= \frac{1}{2}m\dot{z}^2 + \frac{1}{2}J\dot{\theta}^2 - \left(\frac{1}{2}k_1(z - \lambda L\theta)^2 + \frac{1}{2}k_2(z + (1 - \lambda)L\theta)^2\right) \\
 R &= \frac{1}{2}c_1(\dot{z} - \lambda L\dot{\theta})^2 + \frac{1}{2}c_2(\dot{z} + (1 - \lambda)L\dot{\theta})^2 \\
 F_z &= 0, M_\theta = \kappa Lmr.
 \end{aligned} \tag{4.16}$$

where λL is the longitudinal distance from the front wheel to the CoG and κL is the CoG height over ground.

Derivate L and R

Since we have 2 DOF we get two derivatives of L and two of R .

$$\begin{aligned}
 \frac{\partial L}{\partial \dot{z}} &= m\dot{z} \\
 \frac{\partial L}{\partial z} &= -k_1(z - \lambda L\theta) - k_2(z + (1 - \lambda)L\theta) \\
 \frac{\partial R}{\partial \dot{z}} &= c_1(\dot{z} - \lambda L\dot{\theta}) + c_2(\dot{z} + (1 - \lambda)L\dot{\theta}) \\
 \frac{\partial L}{\partial \dot{\theta}} &= J\dot{\theta} \\
 \frac{\partial L}{\partial \theta} &= \lambda Lk_1(z - \lambda L\theta) - (1 - \lambda)Lk_2(z + (1 - \lambda)L\theta) \\
 \frac{\partial R}{\partial \dot{\theta}} &= -\lambda Lc_1(\dot{z} - \lambda L\dot{\theta}) + (1 - \lambda)Lc_2(\dot{z} + (1 - \lambda)L\dot{\theta})
 \end{aligned} \tag{4.17}$$

Solve the Lagrange equation

We are only interested in steady-state, i.e. $\dot{\theta} = \ddot{\theta} = \dot{z} = \ddot{z}$

$$\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} + \frac{\partial R}{\partial \dot{z}} = k_1(z - \lambda L\theta) + k_2(z + (1 - \lambda)L\theta) = 0 \quad (4.18)$$

$$\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} + \frac{\partial R}{\partial \dot{\theta}} = -\lambda L k_1(z - \lambda L\theta) + (1 - \lambda)L k_2(z + (1 - \lambda)L\theta) = \kappa L m r$$

With values we get:

$$1.1 \cdot 10^5 z + 7.8 \cdot 10^4 \theta = 0, 7.8 \cdot 10^4 z + 2.8 \cdot 10^5 \theta = 1260 \quad (4.19)$$

$$\theta = 0.022 \text{ rad} = 1.26^\circ, z = -1.56 \cdot 10^{-2} \text{ m}. \quad (4.20)$$

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Motion matrix equation

Write the motion equations:

$$\begin{aligned} m\ddot{z} + (c_1 + c_2)\dot{z} + (c_2(1 - \lambda)L - c_1\lambda L)\dot{\theta} \\ + (k_1 + k_2)z + (-k_1\lambda + k_2(1 - \lambda))\theta L = f_z \end{aligned} \quad (4.21)$$

$$\begin{aligned} J\ddot{\theta} + (c_2(1 - \lambda)L - c_1\lambda L)\dot{z} + (c_1\lambda^2 L^2 + c_2(1 - \lambda)^2 L^2)\dot{\theta} \\ + (k_2(1 - \lambda)L - k_1\lambda L)z + (k_1\lambda^2 L^2 + k_2(1 - \lambda)^2 L^2)\theta = M_\theta \end{aligned}$$

Set the DOF to z and θL , rewrite equation 7.91 as a matrix equation and divide row two by L :

$$\begin{aligned} \begin{pmatrix} m & 0 \\ 0 & \frac{J}{L^2} \end{pmatrix} \begin{pmatrix} \ddot{z} \\ \ddot{\theta} L \end{pmatrix} + \begin{pmatrix} c_1 + c_2 & c_2(1 - \lambda) - c_1\lambda \\ c_2(1 - \lambda) - c_1\lambda & c_1\lambda^2 + c_2(1 - \lambda)^2 \end{pmatrix} \begin{pmatrix} \dot{z} \\ \dot{\theta} L \end{pmatrix} \\ + \begin{pmatrix} k_1 + k_2 & k_2(1 - \lambda) - k_1\lambda \\ k_2(1 - \lambda) - k_1\lambda & k_1\lambda^2 + k_2(1 - \lambda)^2 \end{pmatrix} \begin{pmatrix} z \\ \theta L \end{pmatrix} = \begin{pmatrix} f_z \\ \frac{M_\theta}{L} \end{pmatrix} \end{aligned} \quad (4.22)$$

Decouple the motion equation

Start by checking that the damping can be written as

$$\mathbf{C} = \alpha \mathbf{M} + \beta \mathbf{K}. \quad (4.23)$$

From positions (1,2) and (1,1), we get

$$\begin{aligned} c_2(1 - \lambda) - c_1\lambda = 0 + (k_2(1 - \lambda) - k_1\lambda)\beta \rightarrow \\ 1200 = 20000\beta \rightarrow \beta = 0.06 \end{aligned} \quad (4.24)$$

$$\begin{aligned} c_1 + c_2 = \alpha m + \beta(k_1 + k_2) \rightarrow \\ 6000 = 1000\alpha + 100000 \cdot 0.06 \rightarrow \alpha = 0 \end{aligned}$$

Checking this against position (2,2)

$$\begin{aligned} c_1\lambda^2 + c_2(1 - \lambda)^2 = \frac{J}{L^2}\alpha + k_1\lambda^2 + k_2(1 - \lambda)^2\beta \rightarrow \\ 1680 = 128 \cdot 0 + 28000 \cdot 0.06 = 1680. \end{aligned} \quad (4.25)$$

Thus, the equations can be decoupled. Next step is to find the eigen-vectors and eigen frequencies by solving

$$|\mathbf{K} - \omega^2 \mathbf{M}| = 0 \quad (4.26)$$

and

$$(\mathbf{K} - \omega^2 \mathbf{M})\bar{\varphi} = \bar{0}. \quad (4.27)$$

From equation 4.45 we get

$$(100000 - 1000\omega^2)(28000 - 128\omega^2) - 20000^2 = 0, \quad (4.28)$$

giving $\omega_1^2 = 77.8251$ and $\omega_2^2 = 240.9249$. Now, we can solve equation 4.46

$$(100000 - 1000\omega^2)\varphi_1 + 20000\varphi_2 = 0 \rightarrow \begin{cases} \varphi_1 = -0.902\varphi_2 & w_1 \\ \varphi_1 = 0.142\varphi_2 & w_2 \end{cases} \quad (4.29)$$

Choose the transformation where $|\bar{\varphi}| = 1$, i.e:

$$\bar{\varphi}_1 = \begin{pmatrix} 0.6698 \\ -0.7426 \end{pmatrix} \text{ and } \bar{\varphi}_2 = \begin{pmatrix} 0.1405 \\ 0.9901 \end{pmatrix} \quad (4.30)$$

The equivalent system is now given by:

$$\begin{aligned} \mathbf{M}_{eq} &= \Phi^T \mathbf{M} \Phi \\ \mathbf{C}_{eq} &= \Phi^T \mathbf{C} \Phi \\ \mathbf{K}_{eq} &= \Phi^T \mathbf{K} \Phi \\ \bar{\mathbf{f}}_{eq} &= \Phi^T \bar{\mathbf{f}} \end{aligned} \quad (4.31)$$

where $\Phi = (\bar{\varphi}_1, \bar{\varphi}_2)$. In this case, we get $m_1 = 520$ kg, $m_2 = 145$ kg, $c_1 = 2.42$ KNs/m, $c_2 = 2.10$ KNs/m, $k_1 = 40.4$ KN/m, $k_2 = 35.0$ KN/m, $f_1 = 186$ N and $f_2 = -248$ N.

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Motion equations

The system is decoupled and thus, each DOF can be analysed individually:

$$\begin{aligned} m_1 \ddot{z}_1 + c_1 \dot{z}_1 + k_1 z_1 &= c_1 \dot{w}_1 + k_1 w_1 \\ m_2 \ddot{z}_2 + c_2 \dot{z}_2 + k_2 z_2 &= c_2 \dot{w}_2 + k_2 w_2 \end{aligned} \quad (4.32)$$

The excitation, w , can be written $w_2 = h \cdot U(t)$ and $w_1 = h \cdot U(t - \frac{L}{v})$. Introduce $2\xi\omega_0 = \frac{c}{m}$ and $\omega_0^2 = \frac{k}{m}$ and rewrite equation 4.32

$$\ddot{z}_1 + 2\xi_1\omega_{01}\dot{z}_1 + \omega_{01}^2 z_1 = 2\xi_1\omega_{01}\dot{w}_1 + \omega_{01}^2 w_1 \quad (4.33)$$

$$\ddot{z}_2 + 2\xi_2\omega_{02}\dot{z}_2 + \omega_{02}^2 z_2 = 2\xi_2\omega_{02}\dot{w}_2 + \omega_{02}^2 w_2 \quad (4.34)$$

Solve with Laplace

Transform the excitation:

$$\begin{aligned} w_1(t) &= h \cdot U(t) \rightarrow W_1(s) = \frac{h}{s} \\ \dot{w}_1(t) &\rightarrow \dot{W}_1(s) = s \frac{h}{s} \\ w_2(t) &= h \cdot U(t - \frac{L}{v}) \rightarrow W_2(s) = \frac{h}{s} e^{-\frac{L}{v}s} \\ \dot{w}_2(t) &\rightarrow \dot{W}_2(s) = s \frac{h}{s} e^{-\frac{L}{v}s} \end{aligned} \quad (4.35)$$

Transform equation 9.1

$$\begin{aligned} s^2 Z_1 + 2\xi_1\omega_{01}s Z_1 + \omega_{01}^2 Z_1 &= 2\xi_1\omega_{01}h + \omega_{01}^2 \frac{h}{s} \rightarrow \\ Z_1(s^2 + 2\xi_1\omega_{01}s + \omega_{01}^2) &= \frac{h}{s}(2\xi_1\omega_{01}s + \omega_{01}^2) \rightarrow \\ Z_1 &= \frac{h}{s} \frac{2\xi_1\omega_{01}s + \omega_{01}^2}{s^2 + 2\xi_1\omega_{01}s + \omega_{01}^2} \end{aligned} \quad (4.36)$$

and equation 9.2

$$\begin{aligned} s^2 Z_2 + 2\xi_2\omega_{02}s Z_2 + \omega_{02}^2 Z_2 &= 2\xi_2\omega_{02}h e^{-\frac{L}{v}s} + \omega_{02}^2 \frac{h}{s} e^{-\frac{L}{v}s} \rightarrow \\ Z_2(s^2 + 2\xi_2\omega_{02}s + \omega_{02}^2) &= \frac{h}{s}(2\xi_2\omega_{02}s + \omega_{02}^2) e^{-\frac{L}{v}s} \rightarrow \\ Z_2(s) &= \frac{h}{s} \frac{2\xi_2\omega_{02}s + \omega_{02}^2}{s^2 + 2\xi_2\omega_{02}s + \omega_{02}^2} e^{-\frac{L}{v}s} \end{aligned} \quad (4.37)$$

Rewrite $Z_1(s)$ and $Z_2(s)$ so that they can be transformed back to the time-domain (Beta handbook).

$$Z_1(s) = h \left[\frac{1}{s} - \frac{s + \xi_1 \omega_{01}}{(s + \xi_1 \omega_{01})^2 + \omega_{01}^2 (1 - \xi_1^2)} + \frac{\xi_1 \omega_{01}}{(s + \xi_1 \omega_{01})^2 + \omega_{01}^2 (1 - \xi_1^2)} \right] \quad (4.38)$$

$$Z_2(s) = e^{-\frac{L}{v}s} h \left[\frac{1}{s} - \frac{s + \xi_2 \omega_{02}}{(s + \xi_2 \omega_{02})^2 + \omega_{02}^2 (1 - \xi_2^2)} + \frac{\xi_2 \omega_{02}}{(s + \xi_2 \omega_{02})^2 + \omega_{02}^2 (1 - \xi_2^2)} \right]$$

Transform $Z_1(s)$ and $Z_2(s)$:

$$z_1(t) = h \cdot U(t) \left[1 - e^{-\xi_1 \omega_{01} t} (\cos(\omega_{01} \sqrt{1 - \xi_1^2} t) - \frac{\xi_1}{(1 - \xi_1^2)} \sin(\omega_{01} \sqrt{1 - \xi_1^2} t)) \right] \quad (4.39)$$

$$z_2(t) = U(t - \frac{L}{v}) \left[1 - e^{-\xi_2 \omega_{02} (t - \frac{L}{v})} (\cos(\omega_{02} \sqrt{1 - \xi_2^2} (t - \frac{L}{v})) - \frac{\xi_2}{(1 - \xi_2^2)} \sin(\omega_{02} \sqrt{1 - \xi_2^2} (t - \frac{L}{v}))) \right]$$

These are the bounce motion of the two masses m_1 and m_2 . Assuming small angles, the pitch motion can be written:

$$\theta(t) = \frac{z_2(t) - z_1(t)}{L} \quad (4.40)$$

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Decouple the motion equation

From equation (1.2) in problem 410 we get the motion matrix equation:

$$\begin{pmatrix} m & 0 \\ 0 & \frac{J}{L^2} \end{pmatrix} \begin{pmatrix} \ddot{z} \\ \ddot{\theta}L \end{pmatrix} + \begin{pmatrix} c_1 + c_2 & c_2(1 - \lambda) - c_1\lambda \\ c_2(1 - \lambda) - c_1\lambda & c_1\lambda^2 + c_2(1 - \lambda)^2 \end{pmatrix} \begin{pmatrix} \dot{z} \\ \dot{\theta}L \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & k_2(1 - \lambda) - k_1\lambda \\ k_2(1 - \lambda) - k_1\lambda & k_1\lambda^2 + k_2(1 - \lambda)^2 \end{pmatrix} \begin{pmatrix} z \\ \theta L \end{pmatrix} = \bar{f}. \quad (4.41)$$

Start by checking that the damping can be written as

$$\mathbf{C} = \alpha \mathbf{M} + \beta \mathbf{K}. \quad (4.42)$$

From positions (1,2) and (1,1), we get

$$\begin{aligned} c_2(1 - \lambda) - c_1\lambda &= 0 + (k_2(1 - \lambda) - k_1\lambda)\beta \rightarrow \\ 1200 &= 25000\beta \rightarrow \beta = 0.08 \end{aligned} \quad (4.43)$$

$$\begin{aligned} c_1 + c_2 &= \alpha m + \beta(k_1 + k_2) \rightarrow \\ 8000 &= 1200\alpha + 100000 \cdot 0.08 \rightarrow \alpha = 0 \end{aligned}$$

Checking this against position (2,2)

$$\begin{aligned} c_1\lambda^2 + c_2(1 - \lambda)^2 &= \frac{J}{L^2}\alpha + k_1\lambda^2 + k_2(1 - \lambda)^2\beta \rightarrow \\ 2320 &= 108 \cdot 0 + 29000 \cdot 0.08 = 2320. \end{aligned} \quad (4.44)$$

Thus, the equations can be decoupled. Next step is to find the eigen-vectors and eigen frequencies by solving

$$|\mathbf{K} - \omega^2 \mathbf{M}| = 0 \quad (4.45)$$

and

$$(\mathbf{K} - \omega^2 \mathbf{M})\bar{\varphi} = 0. \quad (4.46)$$

From equation 4.45 we get

$$(100000 - 1200\omega^2)(29000 - 108\omega^2) - 25000^2 = 0, \quad (4.47)$$

giving $\omega_1^2 = 60.1852$ and $\omega_2^2 = 291.1667$. Now, we can solve equation 4.46

$$(100000 - 1200\omega^2)\varphi_1 + 25000\varphi_2 = 0 \rightarrow \begin{cases} \varphi_1 = -0.9000\varphi_2 & w_1 \\ \varphi_1 = 0.1000\varphi_2 & w_2 \end{cases} \quad (4.48)$$

Choose the transformation:

$$\bar{\varphi}_1 = \begin{pmatrix} 0.9000 \\ -1.0000 \end{pmatrix} \text{ and } \bar{\varphi}_2 = \begin{pmatrix} 0.1000 \\ 1.0000 \end{pmatrix} \quad (4.49)$$

With $\Phi = [\varphi] = (\bar{\varphi}_1, \bar{\varphi}_2)$, the system can be decoupled and this allows studying each DOF separately.

418

Identify the DOF

Since the motion of the road as well as the hook is known, there is only one DOF. Choose the pitch motion, θ .

Motion equations

Introduce the help variables z_1 and F_1 for the displacement of, and the force acting at the trailer where the spring-damper is attached. F_p is the force at the hook. We can now write:

$$F_1 + c(\dot{z}_1 - \dot{w}) + k(z_1 - w) = 0 \quad (4.50)$$

$$F_1 + F_p = m\ddot{z} \quad (4.51)$$

$$-F_p(1 - \lambda)L + F_1\lambda L = J\ddot{\theta} \quad (4.52)$$

$$z = p + \theta(1 - \lambda)L \quad (4.53)$$

$$z_1 = p + \theta L \quad (4.54)$$

We now have 4 unknowns (F_1 , F_p , z_1 and z) one DOF (θ) and five equations.

Reduce the unknowns

Rewrite eqn 9.3:

$$F_p = \frac{F_1\lambda L - J\ddot{\theta}}{(1 - \lambda)L} = \frac{\lambda}{1 - \lambda}F_1 - \frac{J}{(1 - \lambda)L}\ddot{\theta} \quad (4.55)$$

and insert in eqn 9.2

$$m\ddot{z} = F_1 + \frac{\lambda}{1 - \lambda}F_1 - \frac{J}{(1 - \lambda)L}\ddot{\theta} = \frac{1}{1 - \lambda}F_1 - \frac{J}{(1 - \lambda)L}\ddot{\theta}. \quad (4.56)$$

Use eqn 8.20 in eqn 9.1:

$$(1 - \lambda)(m\ddot{z} + \frac{J}{(1 - \lambda)L}\ddot{\theta}) + c(\dot{z}_1 - \dot{w}) + k(z_1 - w) = 0 \quad (4.57)$$

and insert eqn 9.4 and eqn 8.52:

$$(1 - \lambda)(m(\ddot{p} + \ddot{\theta}(1 - \lambda)L) + \frac{J}{(1 - \lambda)L}\ddot{\theta}) + c((\dot{p} + \dot{\theta}L) - \dot{w}) + k((p + \theta L) - w) = 0 \quad (4.58)$$

$$[(1 - \lambda)^2 m + \frac{J}{L^2}]\ddot{\theta}L = -c\dot{\theta}L - k\theta L - (1 - \lambda)m\ddot{p} - c\dot{p} - kp + c\dot{w} + kw \quad (4.59)$$

State space

Introduce $\theta_p = \dot{\theta}$ and write eqn 4.59 on state space form:

$$\begin{aligned}\dot{\theta}_p &= -\frac{c\theta_p L + k\theta L + (1-\lambda)m\ddot{p} + c\dot{p} + kp - c\dot{w} - kw}{(1-\lambda)^2 mL + \frac{J}{L}} \\ \dot{\theta} &= \theta_p\end{aligned}\tag{4.60}$$

5

Stochastic excitation and comfort

Supplementary problems, Chapter 5

1

The accelerations of the driver seat in a wheel loader have been measured. It turns out that the pitch motion dominates the reduction of comfort. The pitch acceleration can be approximated with

$$\ddot{\theta} = A \sin(2\pi f_1 t), \text{ where } A = 2.0 \text{ rad/s} \text{ and } f_1 = 3.0 \text{ Hz} \quad (5.1)$$

- a) Calculate the spectral density of the pitch acceleration.
- b) Calculate the un-weighted RMS value of the pitch acceleration.
- c) Calculate the comfort-weighted RMS value of the pitch acceleration.
- b) How should the comfort be evaluated? (see table 5.9-3 in the textbook.)

2

The Unevenness of a road can be described with the spectral density:

$$G_d(n) = G_d(n_0) \frac{n^{-u}}{n_0} \text{ where } G_d(n_0) = 0.001 \text{ m}^3, n_0 = 0.1 \text{ m}^{-1} \text{ and } u = 4 \quad (5.2)$$

- a) How does the comfort depend on the vehicle's speed?
- b) Are there differences between the comfort related to bounce and pitch, respectively?
- c) Show these differences/equalities with a simple half-vehicle model.

Supplementary problem 1

$$\begin{aligned}\ddot{\Theta}(t) &= A \sin(2\pi f_1 t) \\ A &= 2.0 \text{ rad/s}^2 \\ f_z &= 3.0 \text{ Hz}\end{aligned}\tag{5.3}$$

Unweighted RMS value

$$\begin{aligned}\ddot{\Theta}_{RMS} &= \sqrt{\frac{1}{T} \int_0^T (\ddot{\Theta}(t))^2 dt} = \sqrt{\frac{1}{T} \int_0^T (A \sin(2\pi f_1 t))^2 dt} \\ &= A \sqrt{\frac{1}{T} \left[\frac{t}{2} - \frac{\sin(4\pi f_1 t)}{4 \cdot 2\pi f_1} \right]_0^T} = A \sqrt{\frac{1}{T} \left[\frac{T}{2} - \frac{\sin(4\pi f_1 T)}{4 \cdot 2\pi f_1} - 0 + \frac{\sin(0)}{4 \cdot 2\pi f_1} \right]_0^T} \\ &= A \sqrt{\frac{1}{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} \text{ rad/s}^2\end{aligned}\tag{5.4}$$

Spectral density

There is only one frequency, f_1 , within the spectrum and thus

$$\begin{aligned}\ddot{\Theta}(f) &\Rightarrow \hat{\ddot{\Theta}} = B\delta(f - f_1) \\ G_{\ddot{\Theta}\ddot{\Theta}} &= C\delta(f - f_1) \text{ when } T \rightarrow \infty\end{aligned}\tag{5.5}$$

From equation (5.9.8) in the textbook, we get

$$\begin{aligned}\ddot{\Theta}_{RMS} &= \sqrt{\int_0^\infty G_{\ddot{\Theta}\ddot{\Theta}}(f) df} \Leftrightarrow \\ \ddot{\Theta}_{RMS}^2 &= \int_0^\infty G_{\ddot{\Theta}\ddot{\Theta}}(f) df = \int_0^\infty C\delta(f - f_1) df = C\end{aligned}\tag{5.6}$$

Equations 5.4 and 5.6 now gives $C = \sqrt{2}^2 = 2$ and thus

$$G_{\ddot{\Theta}\ddot{\Theta}} = 2\delta(f - f_1)\tag{5.7}$$

Comfort-weighted RMS value

From table 5.9-1 in the textbook we see that we should use curve W_e in figure 5.9-3. Since $f_1 = 30 \text{ Hz}$ we get $W_e = 0.35$. Equation 5.9.9 in the textbook gives

$$\ddot{\Theta}_w = \sqrt{\int_0^\infty G_{\ddot{\Theta}\ddot{\Theta}}(f) |w(f)|^2 df} = [\text{one freq.}] = \sqrt{2 \cdot 0.35^2} = 0.495\tag{5.8}$$

Estimate comfort

From equation 5.9.11 in the textbook we get

$$\ddot{\Theta}_{wk} = \sqrt{\sum (k\ddot{\Theta}_w)^2} = k\ddot{\Theta}_w = 0.40 \cdot 0.495 = 0.198 \quad (5.9)$$

Comparing this to table 5.9-3, we see that $\ddot{\Theta}_{wk} = 0.198 < 0.315$ and thus it is not uncomfortable.

Supplementary problem 2

The road's spectral density

$$\begin{aligned}
 G_d(n) &= G_d(n_0) \left(\frac{n}{n_0}\right)^{-n} \\
 G_d(n_0) &= 0.001 \text{m}^3 \\
 n_0 &= 0.11/\text{m} \\
 n &= 4 \\
 G_d(n) &= 0.001 \left(\frac{n}{0.1}\right)^{-4} = \frac{1}{10^7 n^4}
 \end{aligned} \tag{5.10}$$

From table 5.6-2 with $G_d(n_0) = 0.001 \text{m}^3$ we see that it is a class D road, which is "fairly bad".

The spectrum of the road disturbance depends on the road profile as well as the vehicle speed.

$$\begin{aligned}
 G_{ww}(f) &= \frac{1}{V} G_d(n) = [nV = f] = \frac{1}{V} 0.1^{-7} \left(\frac{f}{V}\right)^{-4} \\
 &= 10^{-7} \frac{V^3}{f^4}
 \end{aligned} \tag{5.11}$$

Study the the vertical acceleration

$$G_{\ddot{w}\ddot{w}}(f) = \omega^4 G_{ww}(f) = (2\pi f)^4 \cdot 10^{-7} \frac{V^3}{f^4} = (2\pi)^4 \cdot 10^{-7} V^3 \tag{5.12}$$

Now, study the example in section 5.9 in the textbook, starting at page 5:41. The spectral density for $\ddot{z}$ and $\ddot{\theta}$ are given at page 5:45

$$\begin{aligned}
 G_{\ddot{z}\ddot{z}}(f) &= 8\pi^4 G_d(n_0) n_0^u V^{u-1} f^{4-u} \underbrace{\left(1 + \cos \frac{2\pi f L}{V}\right)}_{\text{max for } \cos \frac{2\pi f L}{V}=1} \frac{f_{01}^4 + 4\zeta_1^2 f_{01}^2 f^2}{f_{01}^4 + (4\zeta_1^2 - 2)f_{01}^2 f^2 + f^4} \\
 G_{\ddot{\theta}\ddot{\theta}}(f) &= \frac{32}{L^2} \pi^4 G_d(n_0) n_0^u V^{u-1} f^{4-u} \underbrace{\left(1 - \cos \frac{2\pi f L}{V}\right)}_{\text{min for } \cos \frac{2\pi f L}{V}=1} \frac{f_{02}^4 + 4\zeta_2^2 f_{02}^2 f^2}{f_{02}^4 + (4\zeta_2^2 - 2)f_{02}^2 f^2 + f^4}
 \end{aligned} \tag{5.13}$$

Thus, the wheelbase, L and the frequency f filter differently for z and θ . This is illustrated as a periodicity in e.g figure 5.9-7 in the textbook. This is often referred to as wheel base filtering.

6

Vehicle dynamics in the asymmetry plane

701

Motion equation

The following refers to the textbook. Consider figure 7.1-1, the $x - y$ and $X - Y$ systems correspond to the M_1 and I systems in section 1.4.2, respectively. We have

$$\begin{aligned}\dot{\vec{r}}_{M_1} &= v_x x^1 + v_y y^1 = (v_x, v_y, 0) \\ \ddot{\vec{r}}_{M_1} &= (\dot{v}_x, \dot{v}_y, 0) + \vec{\omega} \times (v_x, v_y, 0) = (\dot{v}_x, \dot{v}_y, 0) + (0, 0, \dot{\psi}) \times (v_x, v_y, 0) \\ &= (\dot{v}_x - v_y \dot{\psi}, \dot{v}_y + v_x \dot{\psi}, 0)\end{aligned}\quad (7.1)$$

Now we can write the force and torque equilibrium:

$$\begin{aligned}ma_y &= F_{12} \cos \delta + F_{34}, \quad a_y = \dot{v}_y + v_x \dot{\psi} \quad \text{and} \quad F_i = -C_i \alpha_i \rightarrow \\ m(\dot{v}_y + v_x \dot{\psi}) &= -\cos \delta C_{12} \alpha_{12} - C_{34} \alpha_{34}\end{aligned}\quad (7.2)$$

$$J\ddot{\psi} = fF_{12} \cos \delta - bF_{34} = -f \cos \delta C_{12} \alpha_{12} + bC_{34} \alpha_{34}$$

Assume small δ , i.e. $\cos \delta \approx 1$ and rewrite equations 7.91

$$\begin{aligned}m(\dot{v}_y + v_x \dot{\psi}) &= -C_{12} \alpha_{12} + C_{34} \alpha_{34} \\ J\ddot{\psi} &= fF_{12} \cos \delta - bF_{34} = -fC_{12} \alpha_{12} + bC_{34} \alpha_{34}\end{aligned}\quad (7.3)$$

Now, the slip angle, α , must be calculated at the front and rear wheels, respectively. Thus we need to express the velocity at each wheel, v_f and v_r as functions of v_x , v_y and $\dot{\psi}$.

$$\begin{aligned}\vec{r}_f &= \vec{r} + fx^1 \rightarrow \\ \dot{\vec{r}}_f &= \dot{\vec{r}} + \vec{\omega} \times fx^1 = v_x x^1 + v_y y^1 + f\dot{\psi} y^1 = (v_x, v_y + f\dot{\psi}, 0)\end{aligned}\quad (7.4)$$

$$\begin{aligned}\vec{r}_r &= \vec{r} - bx^1 \rightarrow \\ \dot{\vec{r}}_r &= \dot{\vec{r}} - \vec{\omega} \times bx^1 = v_x x^1 + v_y y^1 - b\dot{\psi} y^1 = (v_x, v_y - b\dot{\psi}, 0)\end{aligned}$$

The slip angle can now be calculated as

$$\begin{aligned}\alpha &= \arctan\left(\frac{\dot{\vec{r}}_f y^1}{\dot{\vec{r}}_f x^1}\right) \approx \frac{\dot{\vec{r}}_f y^1}{\dot{\vec{r}}_f x^1} \rightarrow \\ \alpha_{12} &= \frac{v_y + f\dot{\psi}}{v_x} - \delta \\ \alpha_{34} &= \frac{v_y - b\dot{\psi}}{v_x}\end{aligned}\quad (7.5)$$

Use equations 7.5 in equations 7.3:

$$\begin{aligned}m(\dot{v}_y + v_x \dot{\psi}) &= -C_{12} \left(\frac{v_y + f\dot{\psi}}{v_x} - \delta \right) - C_{34} \frac{v_y - b\dot{\psi}}{v_x} \\ J\ddot{\psi} &= C_{34} \frac{v_y - b\dot{\psi}}{v_x} b - C_{12} \left(\frac{v_y + f\dot{\psi}}{v_x} - \delta \right) f\end{aligned}\quad (7.6)$$

Now, we rewrite equation 7.6 on matrix form with δ as input.

$$\begin{pmatrix} m & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} \dot{v}_y \\ \dot{\psi} \end{pmatrix} + \begin{pmatrix} \frac{C_{12}+C_{34}}{v_x} & mv_x + \frac{fC_{12}-bC_{34}}{v_x} \\ \frac{fC_{12}-bC_{34}}{v_x} & \frac{f^2C_{12}+b^2C_{34}}{v_x} \end{pmatrix} \begin{pmatrix} v_y \\ \psi \end{pmatrix} = \begin{pmatrix} C_{12} \\ C_{12}f \end{pmatrix} \delta \quad (7.7)$$

702

Use the result from problem 701, in this case we have constant v_y and $\dot{\psi}$, i.e.

$$\begin{pmatrix} \frac{C_{12}+C_{34}}{v_x} & mv_x + \frac{fC_{12}-bC_{34}}{v_x} \\ \frac{fC_{12}-bC_{34}}{v_x} & \frac{f^2C_{12}+b^2C_{34}}{v_x} \end{pmatrix} \begin{pmatrix} v_y \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} C_{12} \\ C_{12}f \end{pmatrix} \delta \quad (7.8)$$

To avoid messy calculations, we rewrite equation ?? as

$$\mathbf{A} \begin{pmatrix} v_y \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} v_y \\ \dot{\psi} \end{pmatrix} = \mathbf{C} \delta = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \delta \quad (7.9)$$

We want to calculate $\dot{\psi}/\delta$ and v_y/δ . Rewrite equation 7.9

$$\begin{pmatrix} v_y \\ \dot{\psi} \end{pmatrix} \frac{1}{\delta} = \mathbf{A}^{-1} \mathbf{C} \quad (7.10)$$

Thus, we need to evaluate

$$\begin{aligned} \mathbf{A}^{-1} \mathbf{C} &= \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= \frac{\begin{pmatrix} a_{22}c_1 - a_{12}c_2 \\ -a_{21}c_1 + a_{11}c_2 \end{pmatrix}}{a_{11}a_{22} - a_{12}a_{21}} \end{aligned} \quad (7.11)$$

With expressions from equation ??

$$\begin{aligned} \mathbf{A}^{-1} \mathbf{C} &= \frac{v_x \begin{pmatrix} (f^2C_{12} + b^2C_{34})C_{12} - (mv_x^2 + fC_{12} - bC_{34})C_{12}f \\ -(fC_{12} - bC_{34})C_{12} + (C_{12} + C_{34})C_{12}f \end{pmatrix}}{(C_{12} + C_{34})(f^2C_{12} + b^2C_{34}) - (mv_x^2 + fC_{12} - bC_{34})(fC_{12} - bC_{34})} \\ &= \frac{v_x \begin{pmatrix} f^2C_{12}^2 + b^2C_{34}C_{12} - mv_x^2C_{12}f - f^2C_{12}^2 + bC_{34}C_{12}f \\ -fC_{12}^2 + bC_{34}C_{12} + fC_{12}^2 + C_{34}C_{12}f \end{pmatrix}}{b^2C_{12}C_{34} + f^2C_{12}C_{34} - mv_x^2fC_{12} - mv_x^2bC_{34} + 2fbC_{12}C_{34}} \\ &= \frac{v_x \begin{pmatrix} LbC_{12}C_{34} - mv_x^2C_{12}f \\ LC_{12}C_{34} \end{pmatrix}}{L^2C_{12}C_{34} + mv_x^2(bC_{34} - fC_{12})} \end{aligned} \quad (7.12)$$

Evaluation of equation 7.12 with $m = 1100\text{kg}$, $f = 1.3\text{m}$, $b = L - f = 1.8\text{m}$, $C_{12} = 2 \cdot 40\text{kN/rad}$, $C_{34} = 2 \cdot 38.6\text{kN/rad}$ and $v_x = 25\text{m/s}$ gives

$$\begin{pmatrix} v_y \\ \dot{\psi} \end{pmatrix} \frac{1}{\delta} = \begin{pmatrix} -11.1043\text{m/s} \\ 5.74001/\text{s} \end{pmatrix} \quad (7.13)$$

703

Use the result from problem 701, in this case we have constant v_y and $\dot{\psi}$, i.e.

$$\begin{pmatrix} \frac{C_{12}+C_{34}}{v_x} & mv_x + \frac{fC_{12}-bC_{34}}{v_x} \\ \frac{fC_{12}-bC_{34}}{v_x} & \frac{f^2C_{12}+b^2C_{34}}{v_x} \end{pmatrix} \begin{pmatrix} v_y \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} C_{12} \\ C_{12}f \end{pmatrix} \delta \quad (7.14)$$

From problem 702, equation 7.12, we know that

$$\begin{pmatrix} v_y \\ \dot{\psi} \end{pmatrix} \frac{1}{\delta} = \frac{v_x \begin{pmatrix} LbC_{12}C_{34} - mv_x^2C_{12}f \\ LC_{12}C_{34} \end{pmatrix}}{L^2C_{12}C_{34} + mv_x^2(bC_{34} - fC_{12})} \quad (7.15)$$

For the stationary case $a_y = v_x\dot{\psi}$. The steering ratio gives $\delta_R = i_s\delta$. We use row two of equation 7.15:

$$\frac{a_y}{\delta_R} = \frac{\dot{\psi}v_x}{\delta i_s} = \frac{v_x}{i_s} \frac{v_x(LC_{12}C_{34})}{L^2C_{12}C_{34} + mv_x^2(bC_{34} - fC_{12})} = \frac{v_x^2}{i_sL + i_sK_{us}v_x^2} \quad (7.16)$$

With values $m = 900\text{kg}$, $f = 1.3\text{m}$, $b = L - f = 1.1\text{m}$, $C_{12} = 2 \cdot 59\text{kN/rad}$, $C_{34} = 2 \cdot 72\text{kN/rad}$ and $i_s = 20$.

$$\frac{a_y}{\delta_R} = \frac{v_x^2}{48 + 2.21 \cdot 10^{-3}v_x^2} \frac{\text{m/s}^2}{\text{rad}} \quad (7.17)$$

704**Derive K_{us}**

From problem 702, equation 7.12, we know that

$$\left(\frac{v_y}{\dot{\psi}}\right) \frac{1}{\delta} = \frac{v_x \left(\frac{LbC_{12}C_{34} - mv_x^2 C_{12}f}{LC_{12}C_{34}} \right)}{L^2 C_{12}C_{34} + mv_x^2 (bC_{34} - fC_{12})} \quad (7.18)$$

We use row two of equation 7.18:

$$\begin{aligned} \frac{\dot{\psi}}{\delta} &= \frac{v_x(LC_{12}C_{34})}{L^2 C_{12}C_{34} + mv_x^2 (bC_{34} - fC_{12})} \rightarrow \\ \delta &= \frac{\dot{\psi}L^2 C_{12}C_{34} + \dot{\psi}mv_x^2 (bC_{34} - fC_{12})}{v_x(LC_{12}C_{34})} = \frac{\dot{\psi}L}{v_x} + \dot{\psi}v_x m \frac{bC_{34} - fC_{12}}{LC_{12}C_{34}} \end{aligned} \quad (7.19)$$

Assume steady-state, i.e. $a_y = v_x \dot{\psi}$ and $\dot{\psi}R = v_x$, then equation 7.18 can be written

$$\delta = \frac{L}{R} + a_y m \frac{bC_{34} - fC_{12}}{LC_{12}C_{34}} = \delta_A + m \frac{bC_{34} - fC_{12}}{LC_{12}C_{34}} a_y \quad (7.20)$$

Thus, K_{us} can be identified as

$$K_{us} = m \frac{bC_{34} - fC_{12}}{LC_{12}C_{34}} \quad (7.21)$$

 K_{us} for 702 and 703

Use the data from problems 702 and 703:

$$\begin{aligned} 702: K_{us} &= m \frac{bC_{34} - fC_{12}}{LC_{12}C_{34}} = \frac{1100(1.8 \cdot 77200 - 1.3 \cdot 80000)}{3.1 \cdot 80000 \cdot 77000} = 2.0 \cdot 10^{-3} \\ 703: K_{us} &= m \frac{bC_{34} - fC_{12}}{LC_{12}C_{34}} = \frac{900(1.1 \cdot 144000 - 1.3 \cdot 118000)}{2.4 \cdot 118000 \cdot 144000} = 1.1 \cdot 10^{-4} \end{aligned} \quad (7.22)$$

I.e. both cars understeer but the car in 703 understeers less.

705

Critical and characteristic speed

Study figure 7.6-1 in the textbook. The critical speed is reached when

$$\frac{\dot{\psi}}{\delta} = \frac{v_x(LC_{12}C_{34})}{L^2C_{12}C_{34} + mv_x^2(bC_{34} - fC_{12})} = \frac{v_x}{L + K_{us}v_x^2} \rightarrow \infty \quad (7.23)$$

i.e when

$$L + K_{us}v_x^2 = 0 \rightarrow v_{crit} = \sqrt{\frac{-L}{K_{us}}} \quad (7.24)$$

The characteristic speed is reached when

$$\frac{\partial \frac{\dot{\psi}}{\delta}}{\partial v_x} = 0 \rightarrow v_{char} = \sqrt{\frac{L}{K_{us}}} \quad (7.25)$$

K_{us} , v_{crit} and v_{char}

Use the data to calculate K_{us} for the original and modified car:

Original:

$$K_{us} = m \frac{bC_{34} - fC_{12}}{LC_{12}C_{34}} = \frac{2000(1.7 \cdot 76000 - 1.5 \cdot 78000)}{3.1 \cdot 78000 \cdot 76000} = 1.29 \cdot 10^{-3} \quad (7.26)$$

Modified:

$$K_{us} = m \frac{bC_{34} - fC_{12}}{LC_{12}C_{34}} = \frac{2000(1.7 \cdot 76000 - 1.5 \cdot 96000)}{3.1 \cdot 96000 \cdot 76000} = -1.27 \cdot 10^{-3}$$

Normally, the car understeers and we get the characteristic speed, $v_{char} = \sqrt{L/K_{us}} = 49.88\text{m/s} = 179.6\text{km/h}$. With modified front tyres, the car oversteers with critical speed $v_{crit} = \sqrt{-L/K_{us}} = 50.24\text{m/s} = 180.9\text{km/h}$.

Plot

We should plot

$$\frac{\dot{\psi}}{\delta}(v_x) = \frac{v_x}{L + K_{us}v_x^2} \quad (7.27)$$

this is illustrated in figure 7.1.

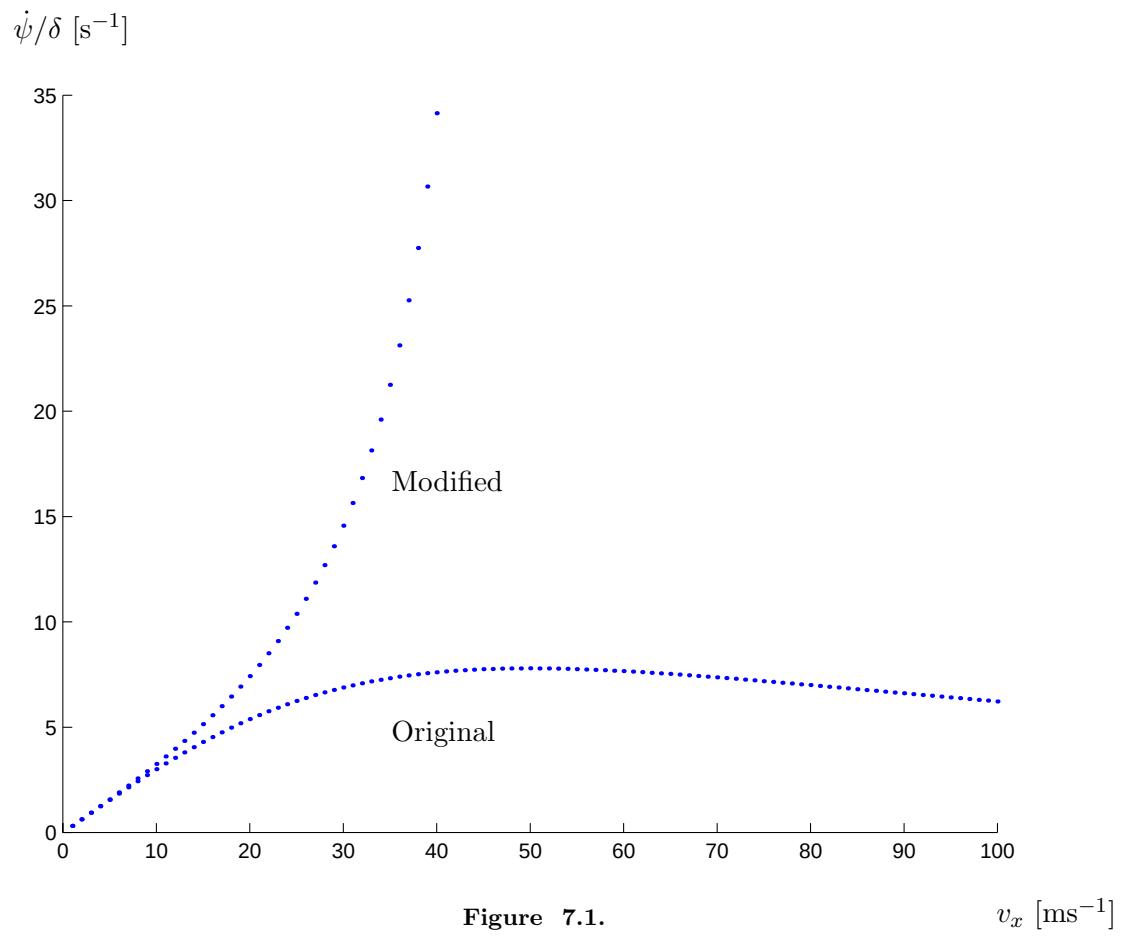


Figure 7.1.

 v_x [ms⁻¹]

706

i_s

Consider equation 7.3.6 in the textbook. Introduce the gear ratio i_s and rewrite the equation:

$$\frac{\delta_H}{i_s} = \delta_A + K_{us}a_y = \quad (7.28)$$

Since we know that $\delta_A = L/R$ we get for $a_y = 0$

$$\begin{aligned} \frac{\delta_{H(a_y=0)}}{i_s} &= \frac{L}{R} \rightarrow \\ i_s &= \frac{\delta_{H(a_y=0)}R}{L} = [\text{from diagram, } a_y = 0] \\ &= \frac{37 \cdot 75}{2.7} = 17.94 \end{aligned} \quad (7.29)$$

K_{us}

From equation 7.28 we get

$$\begin{aligned} \delta_H &= i_s\delta_A + i_sK_{us}a_y \rightarrow \\ K_{us} &= \frac{\delta_H R - i_s L}{i_s a_y R} = [\text{from diagram, } a_y = 6] \\ &= \frac{45 \cdot \pi / 180 - 17.94 \cdot 2.7}{17.94 \cdot 6} \text{rads}^2/\text{m} = 1.297 \cdot 10^{-3} \text{rads}^2/\text{m} \end{aligned} \quad (7.30)$$

v_{char}

Since $K_{us} > 0$ the car understeers and we can calculate the characteristic speed

$$v_{char} = \sqrt{\frac{L}{K_{us}}} = 45.62 \text{m/s} = 164.2 \text{km/h} \quad (7.31)$$

708

Study the relative damping for the acceleration in the y-direction of the car.

a_y

To study the damping, we need the transfer function from δ to a_y . According to the rule at page 7.2 in the textbook, we can write the acceleration of the car relative to the ground frame with respect to the car.

$$\bar{a} = \frac{\partial \bar{v}}{\partial t} + \bar{\omega} \times \bar{v} = \left(\frac{\partial}{\partial t} + \bar{\omega} \times \right) (v_x, v_y, 0) = (\dot{v}_x - v_y \dot{\psi}, \dot{v}_y + v_x \dot{\psi}, 0) \quad (7.32)$$

In the Laplace plane, using the transfer function from Δ to $\dot{\Psi}$ and V_y , we can write

$$A_y = \dot{V}_y + v_x \dot{\Psi} = sG_{\Delta}^{V_y} \Delta + v_x G_{\Delta}^{\dot{\Psi}} \Delta \quad (7.33)$$

Relative damping

To find the relative damping, we only need to study the denominator of equation 7.70, i.e. $|\mathbf{A}|$. Since we have a second order system, this can be expressed as

$$|\mathbf{A}| = s^2 + 2\zeta\omega_0 s + \omega_0^2 \quad (7.34)$$

where ζ is the relative damping. Comparing equation 7.45 to equation 7.8.7 in the textbook gives

$$\sigma = \zeta\omega_0 \rightarrow \zeta = \frac{\sigma}{\omega_0} \quad (7.35)$$

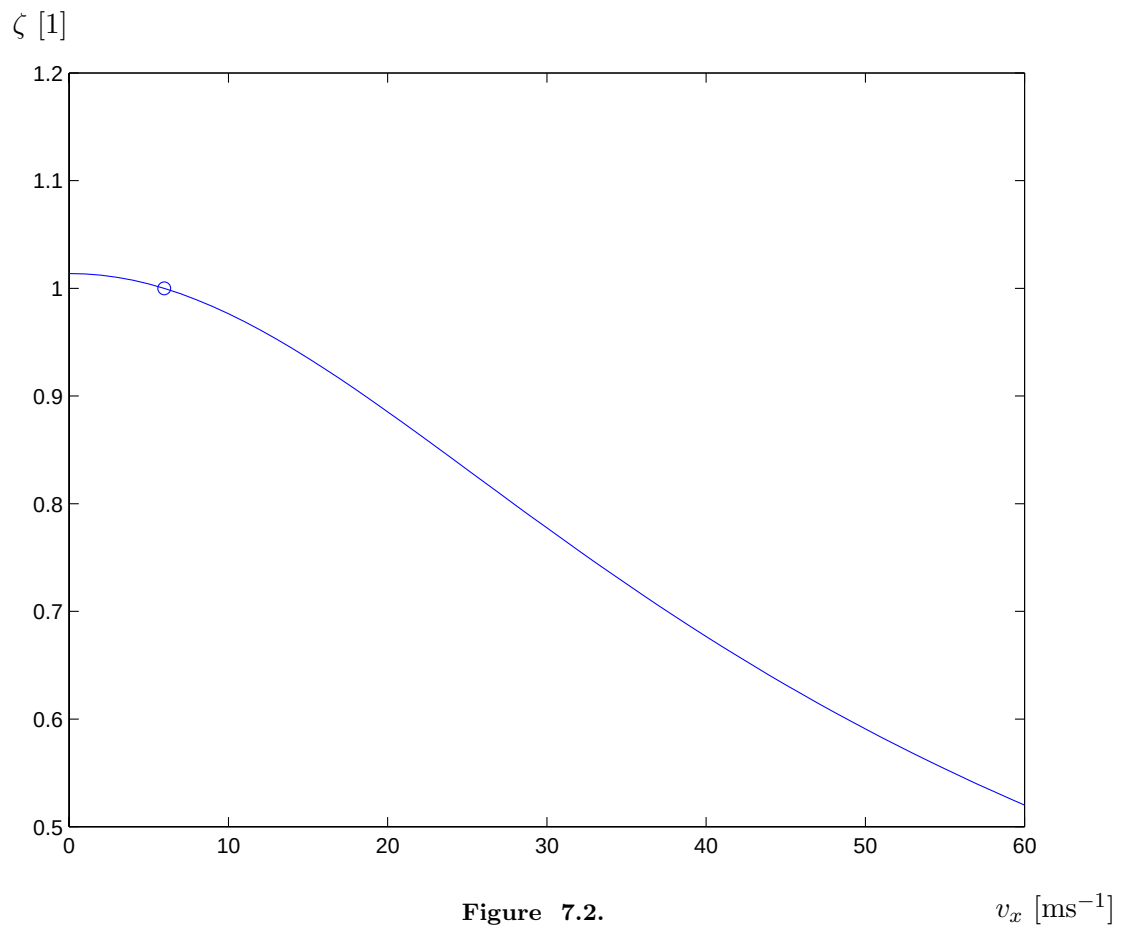
where

$$\begin{aligned} \sigma &= \frac{C_{12}f^2 + C_{34}b^2 + J(C_{12} + C_{34})}{2Jmv_x} \\ \omega_0 &= \sqrt{\frac{C_{12}C_{34}L(K_{us}v_x^2 + L)}{Jmv_x^2}} \end{aligned} \quad (7.36)$$

With data, $m = 1500\text{kg}$, $j = 2800\text{kgm}^2$, $f = 1.2\text{m}$, $b = 1.4\text{m}$, $C_{12} = 2 \cdot 55\text{kN/rad}$ and $C_{34} = 2 \cdot 65\text{kN/rad}$ we get

$$\zeta = \frac{153.8}{\sqrt{17.9v_x^2 + 23016}} \quad (7.37)$$

The plot is shown in figure 7.2.

**Figure 7.2.**

709

Critical speed

We search the speed when the car gets instable, i.e. when the transfer functions get poles in the right half-plane. The poles are given by

$$\begin{aligned} |A| = s^2 + 2\sigma\omega_0s + \omega_0^2 = 0 \rightarrow \\ s = -\sigma \pm \sqrt{\sigma^2 - \omega_0^2} \end{aligned} \quad (7.38)$$

We look for v_x so that $\Re(s) > 0$.

$$\begin{aligned} -\sigma + \sqrt{\sigma^2 - \omega_0^2} > 0 \rightarrow \sigma^2 < \omega_0^2 \rightarrow \\ \omega_0^2 = \frac{C_{12}C_{34}L(K_{us}v_x^2 + L)}{Jmv_x^2} < 0 \end{aligned} \quad (7.39)$$

With data from problem 708, we get

$$\begin{aligned} \omega_0^2 = \frac{-61.5v_x^2 + 48334}{4.2v_x^2} < 0 \rightarrow \\ v_x > 28\text{m/s} \end{aligned} \quad (7.40)$$

710

Acceleration

If we can express the lateral acceleration at the hook, a_{yh} as a function of v_x and $\dot{\psi}$, we can reuse the transfer functions derived earlier. According to the rule at page 7:2 in the textbook, we write

$$\begin{aligned}
 \bar{v}_d &= \bar{v}_{COG} + \bar{\omega} \times (-d)x^1 = (v_x, v_y - d\dot{\psi}, 0) \\
 \bar{a}_d &= \frac{d\bar{v}_d}{dt} = \frac{\partial \bar{v}_d}{\partial t} + \bar{\omega} \times \bar{v}_d \\
 &= (\dot{v}_x, \dot{v}_y - d\ddot{\psi}, 0) + (-v_y\dot{\psi} + d\dot{\psi}^2, v_x\dot{\psi}, 0) \\
 &= (\dot{v}_x - v_y\dot{\psi} + d\dot{\psi}^2, \dot{v}_y - d\ddot{\psi} + v_x\dot{\psi}, 0)
 \end{aligned} \tag{7.41}$$

Transfer function

Here, we are only interested in the y-component of $\bar{a}_d$, a_{yd} . In the Laplace plane, we write

$$\begin{aligned}
 A_{yd} &= sV_y - ds\dot{\Psi} + v_x\dot{\Psi} \rightarrow \\
 G_{\delta}^{a_{yd}} &= sG_{\delta}^{V_y} - dsG_{\delta}^{\dot{\Psi}} + v_xG_{\delta}^{\dot{\Psi}} \\
 &= sG_{\delta}^{V_y} + (v_x - ds)G_{\delta}^{\dot{\Psi}} \\
 &= \frac{s(JC_{12}s - fC_{12}mv_x + \frac{LbC_{12}C_{34}}{v_x}) + (v_x - ds)(mfC_{12} + \frac{LC_{12}C_{34}}{v_x})}{(ms + \frac{C_{12}+C_{34}}{v_x})(Js + \frac{f^2C_{12}+b^2C_{12}}{v_x}) - (mv_x + \frac{fC_{12}-bC_{34}}{v_x})(\frac{fC_{12}-bC_{34}}{v_x})} \tag{7.42} \\
 &= \frac{(JC_{12} - mfd)s^2 + \frac{LC_{12}C_{34}}{v_x}(b-d)s + LC_{12}C_{34}}{mJ(s^2 + 2\sigma s + \omega_0^2)}
 \end{aligned}$$

711

We want to calculate $|G_{\delta_R}^{a_y}|$ and $\arg G_{\delta_R}^{a_y}$

Transfer function

Use the result from problem 710 and insert $d=0$, since $\delta_R = i_s \delta$ we can write.

$$G_{\delta_R}^{a_{yd}} = \frac{1}{i_s} \frac{JC_{12}s^2 + \frac{LbC_{12}C_{34}}{v_x}s + LC_{12}C_{34}}{mJ(s^2 + 2\sigma s + \omega_0^2)} \quad (7.43)$$

Since δ_R is sinusoidal we can write

$$G_{\delta_R}^{a_{yd}}(j\omega) = \frac{1}{i_s} \frac{-JC_{12}\omega^2 + \frac{jLbC_{12}C_{34}}{v_x}\omega + LC_{12}C_{34}}{mJ(-\omega^2 + j2\sigma\omega + \omega_0^2)} \quad (7.44)$$

Now we write

$$|G_{\delta_R}^{a_y}| = \frac{\sqrt{(LC_{12}C_{34} - JC_{12}\omega^2)^2 + (\frac{LbC_{12}C_{34}}{v_x})^2}}{i_s m J \sqrt{(\omega_0^2 - \omega^2)^2 + (2\sigma\omega)^2}} \quad (7.45)$$

$$\arg G_{\delta_R}^{a_y} = \arctan \frac{LbC_{12}C_{34}}{v_x(LC_{12}C_{34} - JC_{12}\omega^2)} - \arctan \frac{2\sigma\omega}{\omega_0^2 - \omega^2}$$

With data we get $\sigma = 11.34$ and $\omega_0^2 = 159.8$. Furthermore, ω for $f = 0.1, 1, 10$ Hz are $0.2\pi, 2\pi$ and 20π , respectively. For the sketch, we also calculate a_y and t .

$$\begin{aligned} \omega = 0.2\pi: |G_{\delta_R}^{a_y}| &= 8.268 \text{m/s}^2 \text{rad} \\ \arg G_{\delta_R}^{a_y} &= 0.0409 - 0.0892 = -0.0483 \text{rad} \\ a_y &= \frac{30\pi}{180} 8.268 = 4.33 \text{m/s}^2 \\ t &= \frac{-0.0483}{2\pi} = -0.0769 \text{s} \end{aligned} \quad (7.46)$$

$$\begin{aligned} \omega = 2\pi: |G_{\delta_R}^{a_y}| &= 6.251 \text{m/s}^2 \text{rad} \\ \arg G_{\delta_R}^{a_y} &= 0.4827 - 0.8696 = -0.3869 \text{rad} \\ a_y &= \frac{30\pi}{180} 6.25 = 3.27 \text{m/s}^2 \\ t &= \frac{-0.3869}{2\pi} = -0.0616 \text{s} \end{aligned} \quad (7.47)$$

$$\begin{aligned} \omega = 20\pi: |G_{\delta_R}^{a_y}| &= 7.027 \text{m/s}^2 \text{rad} \\ \arg G_{\delta_R}^{a_y} &= -0.1915 + 0.3598 = 0.1683 \text{rad} \\ a_y &= \frac{30\pi}{180} 7.027 = 3.68 \text{m/s}^2 \\ t &= \frac{0.1683}{2\pi} = 0.00268 \text{s} \end{aligned} \quad (7.48)$$

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Acceleration in frequency domain

We want an expression for the lateral acceleration of the COG relative to ground, expressed in the car's coordinate system (M_1). According to e.g. problem 710 with $d = 0$, we know that

$$G_\delta^{a_y} = sG_\delta^{V_y} + v_x G_\delta^{\dot{\Psi}} \quad (7.49)$$

In this case, we have a step input at the steering wheel with an amplitude of 40° , i.e.

$$\begin{aligned} \delta(t) &= \frac{40\pi}{180i_s} H(t) \rightarrow \\ \Delta(s) &= \frac{40\pi}{180i_s} \frac{1}{s} \end{aligned} \quad (7.50)$$

Now, we use the result from problem 710 and write the acceleration in the frequency domain

$$\begin{aligned} A_y(s) &= G_\Delta^{A_y} \Delta(s) \\ &= \frac{(JC_{12} - mfd)s^2 + \frac{LC_{12}C_{34}}{v_x}(b-d)s + LC_{12}C_{34}}{mJ(s^2 + 2\sigma s + \omega_0^2)} \frac{40\pi}{180i_s} \frac{1}{s} \end{aligned} \quad (7.51)$$

Transformation to time domain

To get back to the time domain, equation 7.51 is rewritten to fit the inverse transforms in Beta. The right hand side of equation 7.51 is on the form

$$\frac{a_1 s^2 + b_1 s + c_1}{s(s^2 + 2\sigma s + \omega_0^2)} = \frac{a_1 s + b_1}{s^2 + 2\sigma s + \omega_0^2} + \frac{c_1}{s(s^2 + 2\sigma s + \omega_0^2)} \quad (7.52)$$

where

$$\begin{aligned} a_1 &= \frac{JC_{12} - mfd}{mJ} \frac{40\pi}{180i_s} \\ b_1 &= \frac{LC_{12}C_{34}(b-d)}{v_x mJ} \frac{40\pi}{180i_s} \\ c_1 &= \frac{LC_{12}C_{34}}{mJ} \frac{40\pi}{180i_s} \end{aligned} \quad (7.53)$$

The first term of equation 7.53 can be transformed directly with

$$\begin{aligned} L\left(\frac{1}{b} e^{at} \cos bt\right) &= \frac{s-a}{(s-a)^2 + b^2} \\ L\left(\frac{1}{b} e^{at} \sin bt\right) &= \frac{1}{(s-a)^2 + b^2} \end{aligned} \quad (7.54)$$

where

$$s^2 + 2\sigma s + \omega_0^2 = (s + \sigma)^2 + \omega_0^2 - \sigma^2 \rightarrow a = -\sigma \text{ and } b^2 = \omega_0^2 - \sigma^2 \quad (7.55)$$

We now write

$$\begin{aligned}\frac{a_1 s + b_1}{s^2 + 2\sigma s + \omega_0^2} &= A_1 \frac{s - (-\sigma)}{(s - (-\sigma))^2 + (\omega_0^2 - \sigma^2)} + B_1 \frac{1}{(s - (-\sigma))^2 + (\omega_0^2 - \sigma^2)} \rightarrow \\ A_1 &= a_1 \\ B_1 &= b_1 - A_1 a = b_1 - a_1 \sigma\end{aligned}\tag{7.56}$$

The second term of equation 7.53 has to be rewritten according to

$$\begin{aligned}\frac{c_1}{s(s^2 + 2\sigma s + \omega_0^2)} &= \frac{A}{s} + \frac{Bs + C}{s^2 + 2\sigma s + \omega_0^2} \rightarrow \\ A\omega_0^2 &= c \rightarrow A = \frac{c_1}{\omega_0^2} \\ A + B &= 0 \rightarrow B = -\frac{c_1}{\omega_0^2} \\ 2\sigma A + C &= 0 \rightarrow C = -\frac{2\sigma c_1}{\omega_0^2}\end{aligned}\tag{7.57}$$

Now we can write

$$\frac{A}{s} + \frac{Bs + C}{s^2 + 2\sigma s + \omega_0^2} = \frac{c_1}{\omega_0^2} \frac{1}{s} - \frac{c_1}{\omega_0^2} \frac{s + \sigma}{s^2 + 2\sigma s + \omega_0^2} + \sigma \frac{1}{s^2 + 2\sigma s + \omega_0^2} + \frac{2\sigma c_1}{\omega_0^2} \frac{1}{(s + \sigma)^2 + \omega_0^2 - \sigma^2}\tag{7.58}$$

use the results from equations 7.56 and 7.58 in equation 7.51

$$\begin{aligned}A_y(s) &= \frac{(JC_{12} - mfd)s^2 + \frac{LC_{12}C_{34}}{v_x}(b - d)s + LC_{12}C_{34}}{mJ(s^2 + 2\sigma s + \omega_0^2)} \frac{40\pi}{180i_s} \frac{1}{s} \\ &= \frac{c_1}{\omega_0^2} \frac{1}{s} + (a_1 - \frac{c_1}{\omega_0^2}) \frac{s + \sigma}{(s + \sigma)^2 + (\omega_0^2 - \sigma^2)} \\ &\quad + (b_1 + (1 - a_1)\sigma) \frac{1}{(s + \sigma)^2 + (\omega_0^2 - \sigma^2)}\end{aligned}\tag{7.59}$$

Equation 7.59 can now easily be transformed to the time domain, giving

$$a_y(t) = \frac{c_1}{\omega_0^2} + (a_1 - \frac{c_1}{\omega_0^2}) \frac{1}{b} e^{at} \cos bt + (b_1 + (1 - a_1)\sigma) \frac{1}{b} e^{at} \sin bt\tag{7.60}$$

Calculate $\sigma = 9.049$ and $\omega_0^2 = 119.8 \text{ rad}^2/\text{s}^2$ according to equation 7.8.7 in the textbook. Together with the given data, $L = 2.6 \text{ m}$, $b = 1.5 \text{ m}$, $m = 1000 \text{ kg}$, $i_s = 20$, $v_x = 25 \text{ m/s}$, $C_{12} = 104 \text{ kN/rad}$, $C_{34} = 144 \text{ kN/rad}$ and $J = 2200 \text{ kgm}^2$, we get

$$a_y(t) = 5.16 + 1.52e^{-9.05t} \cos 6.15t + 6.90e^{-9.05t} \sin 6.15t\tag{7.61}$$

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Yaw in frequency domain

We want to calculate

$$\dot{\psi} = v_x G_\delta^\psi \delta(t), \delta(t) = H(t) \frac{30\pi}{180} \sin(0.5 \cdot 2\pi t) \quad (7.62)$$

Use L4 and L24 in Beta to transform $\delta(t)$ to $\Delta(s)$

$$\begin{aligned} \delta(t) &= H(t) \frac{30\pi}{180} \sin(\pi t) \rightarrow \\ \Delta(s) &= \frac{\pi}{6i_s} \frac{\pi}{s^2 + \pi^2} \end{aligned} \quad (7.63)$$

To find G_Δ^Ψ we use the motion equation in the laplace plane

$$\begin{pmatrix} ms + \frac{C_{12}+C_{34}}{v_x} & mv_x + \frac{fC_{12}-bC_{34}}{v_x} \\ \frac{fC_{12}-bC_{34}}{v_x} & Js + \frac{f^2C_{12}+b^2C_{12}}{v_x} \end{pmatrix} \begin{pmatrix} V_y \\ \Psi \end{pmatrix} = \begin{pmatrix} C_{12} \\ C_{12}f \end{pmatrix} \Delta \quad (7.64)$$

We want row two of $\mathbf{A}^{-1}\mathbf{C}$, i.e.

$$\begin{aligned} G_\Delta^\Psi(s) &= \mathbf{A}^{-1}\mathbf{C} = \frac{-A_{21}C_1 + A_{11}C_2}{A_{22}A_{11} - A_{21}A_{12}} \\ &= \frac{mv_x f C_{12} s + C_{12} C_{34} L}{mJv_x(s^2 + 2\sigma s + \omega_0^2)} \end{aligned} \quad (7.65)$$

Express the yaw velocity in the Laplace plane as

$$\dot{\Psi} = \frac{\pi^2 f C_{12}}{6i_s J_z} \frac{s + \frac{C_{34}L}{v_x m f}}{s^2 + 2\sigma s + \omega_0^2} \frac{1}{s^2 + \pi^2} \quad (7.66)$$

This needs to be modified before we can transform it back to the time plane, assume

$$\begin{aligned} \frac{As + B}{s^2 + 2\sigma s + \omega_0^2} + \frac{Cs + D}{s^2 + \pi^2} &= \frac{s + \frac{C_{12}C_{34}L}{v_x m f}}{s^2 + 2\sigma s + \omega_0^2} \frac{\pi}{s^2 + \pi^2} \rightarrow \\ s^3 : A + C &= 0 \\ s^2 : B + 2\sigma C + D &= 0 \\ s : A\pi^2 + C\omega_0^2 + 2\sigma D &= 1 \\ 1 : B\pi^2 + D\omega_0^2 &= \frac{LC_{34}}{v_x m f} \end{aligned} \quad (7.67)$$

Transform equation 7.74 back to the time plane, introduce $\omega_e = \sqrt{\omega_0^2 - \sigma^2}$

$$\begin{aligned}\dot{\Psi}(s) &= \frac{\pi^2 f C_{12}}{6i_s J_z} \left[\frac{As + B}{s^2 + 2\sigma s + \omega_0^2} + \frac{Cs + D}{s^2 + \pi^2} \right] \\ \dot{\psi}(t) &= \frac{\pi^2 f C_{12}}{6i_s J_z} \left[A e^{-\sigma t} \cos(t\omega_e) \right. \\ &\quad \left. + (B - A\sigma) \frac{1}{\omega_e} e^{-\sigma t} \sin(t\omega_e) + C \cos(\pi t) + \frac{D}{\pi} \sin(\pi t) \right]\end{aligned}\tag{7.68}$$

Use the data from problem 715, giving $\omega_0^2 = 74.9425$, $\sigma = 5.6591$, $\omega_2 = 6.5539$, $A = 5.672 \cdot 10^{-3}$, $B = 5.687 \cdot 10^{-2}$, $C = -5.672 \cdot 10^{-3}$ and $D = 1.210 \cdot 10^{-1}$. Thus

$$\begin{aligned}\dot{\psi}(t) &= 4.2768 [0.005672 e^{-5.6591t} \cos(6.5539t) \\ &\quad - 0.01357 e^{-5.6591t} \sin(6.5539t) - 0.005672 \cos(\pi t) + 0.0385 \sin(\pi t)]\end{aligned}\tag{7.69}$$

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Yaw in frequency domain

We want to calculate

$$\psi = v_x G_\delta^\psi \delta(t), \quad \delta(t) = \frac{60\pi}{180i_s} \sin(\pi t) [H(t) - H(t-2)] \quad (7.70)$$

Use L4 and L24 in Beta to transform $\delta(t)$ to $\Delta(s)$

$$\begin{aligned} \delta(t) &= \frac{60\pi}{180i_s} \sin(\pi t) [H(t) - H(t-2)] \rightarrow \\ \Delta(s) &= \frac{\pi}{3i_s} \frac{\pi}{s^2 + \pi^2} (1 - e^{-2s}) \end{aligned} \quad (7.71)$$

To find G_Δ^Ψ we use the motion equation in the laplace plane

$$\begin{pmatrix} ms + \frac{C_{12}+C_{34}}{v_x} & mv_x + \frac{fC_{12}-bC_{34}}{v_x} \\ \frac{fC_{12}-bC_{34}}{v_x} & Js + \frac{f^2C_{12}+b^2C_{12}}{v_x} \end{pmatrix} \begin{pmatrix} V_y \\ \dot{\Psi} \end{pmatrix} = \begin{pmatrix} C_{12} \\ C_{12}f \end{pmatrix} \Delta \quad (7.72)$$

We want row two of $\mathbf{A}^{-1}\mathbf{C}$, i.e.

$$G_\Delta^\Psi(s) = \mathbf{A}^{-1}\mathbf{C} = \frac{-A_{21}C_1 + A_{11}C_2}{A_{22}A_{11} - A_{21}A_{12}} = \frac{mv_x f C_{12}s + C_{12}C_{34}L}{mJv_x(s^2 + 2\sigma s + \omega_0^2)} \quad (7.73)$$

Now we can express the yaw in the Laplace plane as

$$s\Psi(s) = \dot{\Psi}(s) = \frac{\pi^2}{3i_s J m} \frac{mfC_{12}s + \frac{C_{12}C_{34}L}{v_x}}{s^2 + 2\sigma s + \omega_0^2} \frac{1 - e^{-2s}}{s^2 + \pi^2} \quad (7.74)$$

The change in direction is given by $\psi(t \rightarrow \infty) - \psi(t \rightarrow 0)$.

$$\begin{aligned} \lim_{t \rightarrow 0} \psi(t) &= \lim_{s \rightarrow \infty} s\Psi(s) = \lim_{s \rightarrow \infty} \frac{\pi^2}{3i_s J m} \frac{mfC_{12}s + \frac{C_{12}C_{34}L}{v_x}}{s^2 + 2\sigma s + \omega_0^2} \frac{1 - e^{-2s}}{s^2 + \pi^2} = 0 \\ \lim_{t \rightarrow \infty} \psi(t) &= \lim_{s \rightarrow 0} s\Psi(s) = \lim_{s \rightarrow 0} \frac{\pi^2}{3i_s J m} \frac{mfC_{12}s + \frac{C_{12}C_{34}L}{v_x}}{s^2 + 2\sigma s + \omega_0^2} \frac{1 - e^{-2s}}{s^2 + \pi^2} = 0 \end{aligned} \quad (7.75)$$

Thus $\psi(t \rightarrow \infty) - \psi(t \rightarrow 0) = 0 - 0 = 0$ and the car returns to the initial course after the manoeuvre.

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We want to calculate

$$\lim_{t \rightarrow \infty} \psi(t) \quad (7.76)$$

Yaw in frequency domain

Use the motion equations in the laplace plane

$$\begin{pmatrix} ms + \frac{C_{12}+C_{34}}{v_x} & mv_x + \frac{fC_{12}-bC_{34}}{v_x} \\ \frac{fC_{12}-bC_{34}}{v_x} & Js + \frac{f^2C_{12}+b^2C_{34}}{v_x} \end{pmatrix} \begin{pmatrix} V_y \\ \dot{\Psi} \end{pmatrix} = \begin{pmatrix} C_{12} \\ C_{12}f \end{pmatrix} \Delta \quad (7.77)$$

In this case, $\Delta = 0$ but an external force is applied at a distance d behind the front axis giving

$$\mathbf{A} \begin{pmatrix} v_y \\ \psi \end{pmatrix} = \mathbf{B} F_{wind}, \quad \mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -1 \\ d-f \end{pmatrix} \quad (7.78)$$

From $\mathbf{A}^{-1}\mathbf{B}$ we get the transfer function from F_{wind} to ψ

$$G_{F_{wind}}^{\psi} = \frac{v_x m(d-f)s + C_{12}d + C_{34}(d-L)}{v_x mJ(s^2 + 2\sigma s + \omega_0^2)} \quad (7.79)$$

In the time domain, $F_w = F_w(t) = F_m(H(t) - H(t-t_0))$, giving

$$F_{wind}(s) = \frac{F_m}{s}(1 - e^{-t_0 s}) \quad (7.80)$$

We can now write

$$\begin{aligned} \Psi &= \frac{\dot{\Psi}}{s} = \frac{1}{s} G_{F_{wind}}^{\psi} F_{wind} \\ &= \frac{v_x m(d-f)s + C_{12}d + C_{34}(d-L)}{v_x mJ(s^2 + 2\sigma s + \omega_0^2)} \frac{F_m}{s^2} (1 - e^{-t_0 s}) \end{aligned} \quad (7.81)$$

$t \rightarrow \infty$

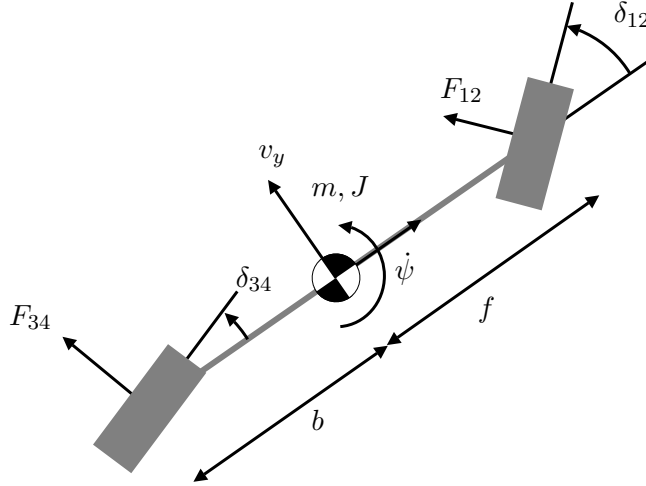
We use that

$$\begin{aligned} \lim_{t \rightarrow \infty} \psi(t) &= \lim_{s \rightarrow 0} s\Psi(s) \\ &= s \frac{v_x m(d-f)s + C_{12}d + C_{34}(d-L)}{v_x mJ(s^2 + 2\sigma s + \omega_0^2)} \frac{F_m}{s^2} (1 - e^{-t_0 s}) = \frac{1-1}{0} \end{aligned} \quad (7.82)$$

Thus, we can use l'Hospitals rule and write

$$\begin{aligned} \lim_{s \rightarrow 0} s\Psi(s) &= \lim_{s \rightarrow 0} \frac{\frac{\partial}{\partial s}[(v_x m(d-f)s + C_{12}d + C_{34}(d-L))(F_m(1 - e^{-t_0 s}))]}{\frac{\partial}{\partial s}[v_x mJs(s^2 + 2\sigma s + \omega_0^2)]} \\ &= \frac{C_{12}d + C_{34}(d-L)F_m t_0}{v_x mJ\omega_0^2} = -0.05881 \text{rad} \end{aligned} \quad (7.83)$$

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We want to find

$$\mathbf{A} \begin{pmatrix} v_y \\ \dot{\psi} \end{pmatrix} = \mathbf{C} \begin{pmatrix} \delta_{12} \\ \delta_{34} \end{pmatrix} \quad (7.84)$$

Tyre forces

The slip angle is given by

$$\alpha_{12} = \arctan \frac{v_y + f\dot{\psi}}{v_x} - \delta_{12} \alpha_{34} = \arctan \frac{v_y - b\dot{\psi}}{v_x} - \delta_{34} \quad (7.85)$$

Assuming linear tyres and small angles, the tyre forces are

$$F_{12} = -C_{12} \left(\frac{v_y + f\dot{\psi}}{v_x} - \delta_{12} \right) F_{34} = -C_{34} \left(\frac{v_y - b\dot{\psi}}{v_x} - \delta_{34} \right) \quad (7.86)$$

Accelerations

To set up the motion equation, we need the accelerations of the COG relative to the ground, with respect to the car.

$$\left(\frac{\partial}{\partial t} + \bar{\omega} \times \right) (v_x, v_y, 0) = (v_x - v_y \dot{\psi}, v_y + v_x \dot{\psi}, 0) \rightarrow a_y = \dot{v}_y + v_x \dot{\psi} \quad (7.87)$$

Motion equations

Assume small δ , in y -direction we get

$$\begin{aligned} ma_y &= m(\dot{v}_y + v_x \dot{\psi}) = F_{12} \cos \delta_{12} + F_{34} \cos \delta_{34} \rightarrow \\ mv_y + \frac{C_{12} + C_{34}}{v_x} v_y + (mv_x + \frac{fC_{12} - bC_{34}}{v_x}) \dot{\psi} &= (C_{12} + kC_{34}) \delta_{12} \end{aligned} \quad (7.88)$$

and around ψ

$$\begin{aligned} J\ddot{\psi} &= -C_{12}f\left(\frac{v_y + f\ddot{\psi}}{v_x} - \delta_{12}\right) + C_{34}b\left(\frac{v_y + f\ddot{\psi}}{v_x} - k\delta_{12}\right) \rightarrow \\ J\ddot{\psi} + \frac{f^2C_{12} + b^2C_{34}}{v_x} \dot{\psi} + \frac{fC_{12} - bC_{34}}{v_x} v_y &= (fC_{12} - bkC_{34}) \delta_{12} \end{aligned} \quad (7.89)$$

On matrix form

$$\begin{pmatrix} mD + \frac{C_{12} + C_{34}}{v_x} & mv_x + \frac{fC_{12} - bC_{34}}{v_x} \\ \frac{fC_{12} - bC_{34}}{v_x} & JD + \frac{f^2C_{12} + b^2C_{34}}{v_x} \end{pmatrix} \begin{pmatrix} v_y \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} C_{12} + kC_{34} \\ fC_{12} - bkC_{34} \end{pmatrix} \delta_{12} \quad (7.90)$$

Laplace plane

We now are to study the transfer function from δ_{12} to $\dot{\psi}$ to find the time delay. Transform equation 7.91

$$\begin{pmatrix} ms + \frac{C_{12} + C_{34}}{v_x} & mv_x + \frac{fC_{12} - bC_{34}}{v_x} \\ \frac{fC_{12} - bC_{34}}{v_x} & Js + \frac{f^2C_{12} + b^2C_{34}}{v_x} \end{pmatrix} \begin{pmatrix} v_y \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} C_{12} + kC_{34} \\ fC_{12} - bkC_{34} \end{pmatrix} \delta_{12} \quad (7.91)$$

We have

$$G_{\delta_{12}}^{\dot{\psi}}(s) = \mathbf{A}^{-1}(1, :) \mathbf{C} = \frac{-A_{21}C_1 + A_{11}C_2}{A_{22}A_{11} - A_{21}A_{12}} \quad (7.92)$$

Evaluating equation 7.92 gives

$$G_{\delta_{12}}^{\dot{\psi}}(s) = \mathbf{A}^{-1}(1, :) \mathbf{C} = \frac{mv_x(fC_{12} - bkC_{34})s + C_{12}C_{34}L(1 - k)}{mJv_x(s^2 + 2\sigma s + \omega_0^2)} \quad (7.93)$$

Since we have sinusoidal motion, we can write $s = j\omega$ and we get

$$G_{\delta_{12}}^{\dot{\psi}}(j\omega) = \frac{mv_x(fC_{12} - bkC_{34})j\omega + C_{12}C_{34}L(1 - k)}{mJv_x(-\omega^2 + 2\sigma j\omega + \omega_0^2)} \rightarrow \quad (7.94)$$

$$\angle G_{\delta_{12}}^{\dot{\psi}}(j\omega) = \arctan \frac{mv_x(fC_{12} - bkC_{34})\omega}{C_{12}C_{34}L(1 - k)} - \arctan \frac{2\sigma\omega}{\omega^2 + \omega_0^2}$$

With data we get

$$\begin{aligned} k = 0 : \angle G_{\delta_{12}}^{\dot{\psi}}(j\omega) &= -0.2360 \text{ rad} \rightarrow \\ \Delta t &= \frac{\angle G_{\delta_{12}}^{\dot{\psi}}(j\omega)}{\omega} = -75.1 \text{ ms} \end{aligned} \quad (7.95)$$

$$\begin{aligned}
 k = 0.1 : \angle G_{\delta_{12}}^{(v_y, \dot{\psi})}(j\omega) &= -0.2539\text{rad} \rightarrow \\
 \Delta t &= \frac{\angle G_{\delta_{12}}^{(v_y, \dot{\psi})}(j\omega)}{\omega} = -80.8\text{ms}
 \end{aligned} \tag{7.96}$$

$$\begin{aligned}
 k = -0.1 : \angle G_{\delta_{12}}^{(v_y, \dot{\psi})}(j\omega) &= -0.2215\text{rad} \rightarrow \\
 \Delta t &= \frac{\angle G_{\delta_{12}}^{(v_y, \dot{\psi})}(j\omega)}{\omega} = -70.1\text{ms}
 \end{aligned} \tag{7.97}$$

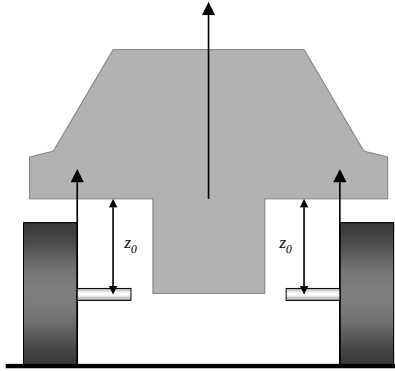
8

Vehicle stability

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Camber angles

Standing still



Driving through a curve

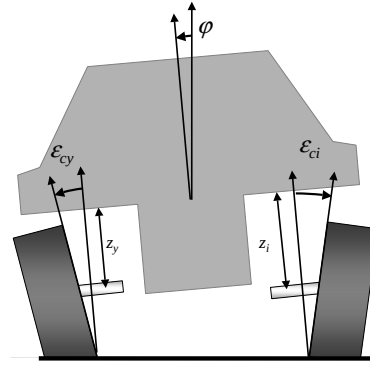


Figure 8.1.

Consider figure 8.1. The roll of the var $\varphi = (2^\circ, 4^\circ, 6^\circ)$ is known. ε_{cy} and ε_{ci} can be calculated from $z_y - z_0$ and $z_i - z_0$, respectively. From the track width, w_t , we get

$$\begin{aligned} z_i - z_0 &= \Delta z = \frac{w_t}{2} \tan \varphi \approx \frac{w_t}{2} \varphi \\ z_y - z_0 &= -\Delta z = -\frac{w_t}{2} \tan \varphi \approx -\frac{w_t}{2} \varphi \end{aligned} \quad (8.1)$$

With φ and ε , we can now calculate the camber angle as $\gamma_i = \varphi - \varepsilon_{ci}$ and $\gamma_y = \varphi + \varepsilon_{cy}$.

Settings 1

φ [°]	Δz [mm]	ε_{ci} [°]	ε_{cy} [°]	γ_i	γ_y
2	26	-0.5	-0.9	2.5	1.1
4	52	-0.1	-0.8	4.1	3.2
6	78	0.5	-0.5	5.5	5.5

Settings 2

φ [°]	Δz [mm]	ε_{ci} [°]	ε_{cy} [°]	γ_i	γ_y
2	26	0.5	-0.3	1.5	1.7
4	52	1.1	-0.4	2.9	3.6
6	78	1.9	-0.3	4.1	5.7

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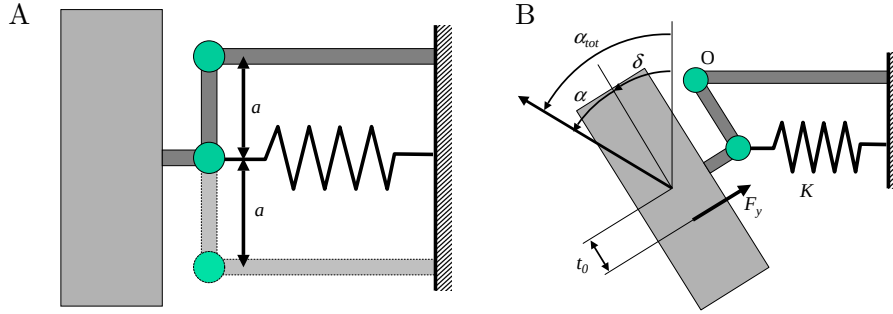


Figure 8.2.

The steering rod is either mounted in front of or behind the McPherson suspension as shown in figure 8.2, A. The steering lever and pushrod are considered to be rigid and all angle are small.

Study figure 8.2, B. The lateral force generated by the tyre, F_y can be written

$$F_y = -C_\alpha \alpha = -C_{\alpha_{tot}} \alpha_{tot}, \text{ where } \alpha_{tot} = \alpha + \delta \quad (8.2)$$

A momentum equilibrium around O gives

$$-F_y(a - t_0) - K \cdot a^2 \delta \quad (8.3)$$

Rewrite eqn 8.2

$$\alpha = \frac{-F_y}{C_\alpha} \text{ and } \alpha_{tot} = \frac{-F_y}{C_{\alpha_{tot}}} \quad (8.4)$$

and eqn 8.3

$$\delta = \frac{-F_y(a - t_0)}{K \cdot a^2} \quad (8.5)$$

giving

$$\begin{aligned} \frac{-F_y}{C_{\alpha_{tot}}} &= \frac{-F_y}{C_\alpha} - \frac{F_y(a - t_0)}{K \cdot a^2} \rightarrow \\ C_{\alpha_{tot}} &= \frac{1}{\frac{1}{C_\alpha} + \frac{a-t_0}{K \cdot a^2}} \end{aligned} \quad (8.6)$$

With values we get

$$\begin{aligned} \text{Case A: } C_{\alpha_{tot}} &= 40.1 \text{ kN/rad} \\ \text{Case B: } C_{\alpha_{tot}} &= 57.1 \text{ kN/rad} \end{aligned} \quad (8.7)$$

Since the tyres themselves has $C_\alpha = 50 \text{ kN/rad}$, case A will increase the understeer-
ing of the car and therefore this case should be chosen.

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Side stiffness

Write the stiffness of the steering axis with respect to the wheel rotation, C_w . Then the momentum equation around the steering pin can be written

$$-F_y(n_c - t_0) - \frac{1}{2}C_w\delta = 0 \quad (8.8)$$

Furthermore

$$F_y = -C_\alpha\alpha = -C_{\alpha_{tot}}\alpha_{tot}, \text{ where } \alpha_{tot} = \alpha + \delta \quad (8.9)$$

Rewrite equations 8.8 and 8.9 according to problem 803, giving

$$\begin{aligned} \frac{-F_y}{C_{\alpha_{tot}}} &= \frac{F_y}{C_\alpha} - \frac{2F_y(a - t_0)}{K \cdot a^2} \rightarrow \\ C_{\alpha_{tot}} &= \frac{2}{\frac{F_y}{C_\alpha} - \frac{F_y(n_c - t_0)}{C_w}} \end{aligned} \quad (8.10)$$

Steering axis stiffness

Use solid mechanics to find the the stiffness of the axis

$$C = \frac{GK}{L}, \text{ where } K = \frac{\pi(D^4 - d^4)}{32} \quad (8.11)$$

Transform C to C_w through the gear ratio i_s :

$$C_w = i_s^2 C = i_s^2 \frac{G\pi(D^4 - d^4)}{32L} \quad (8.12)$$

Total side stiffness

The total side stiffness for the whole front suspension is now given by equation 8.10 and 8.12

$$C_{\alpha_{tot}} = \frac{2}{\frac{F_y}{C_\alpha} - \frac{F_y(n_c - t_0)32L}{i_s^2 G\pi(D^4 - d^4)}} \quad (8.13)$$

With data, this yields 89.6kN/rad.

Assume a displacement of the spring, δ and the corresponding displacement and stiffness at the lower spindle joint, δ_1 and k_1 , respectively. A momentum equilibrium yields

$$k\delta f_F = k_1\delta_1 f_L \quad (8.17)$$

Geometry gives

$$\frac{f_F}{\delta} = \frac{f_L}{\delta_1} \quad (8.18)$$

Equations 8.17 and 8.18 now gives

$$k_1 = k\left(\frac{f_F}{f_L}\right)^2 \quad (8.19)$$

Step two is to go from the lower spindle joint to the wheel's contact point. According to above, with a momentum equilibrium around MP , we get

$$k_2 = k_1\left(\frac{b}{a}\right)^2 = k\left(\frac{f_F}{f_L}\right)^2\left(\frac{b}{a}\right)^2 \quad (8.20)$$

where a is the distance from MP to the wheel's contact point.

The total roll stiffness, C_φ , can now be calculated according to

$$C_\varphi = 2\frac{M}{\varphi} = 2\frac{k_2\delta_2 d}{\varphi} = [\delta_2 = d\varphi] = 2k\left(\frac{f_F b d}{f_L a}\right)^2 \quad (8.21)$$

where d is the distance from the wheel's contact point to the roll centre.

Now we need expressions for a , b and c .

$$\begin{aligned} a &= \sqrt{(R-r)^2 + (F+x-S/2)^2} = 0.982\text{m} \\ b &= x - f_L = 1.111\text{m} \\ d &= \sqrt{(t_{RC})^2 + (S/2)^2} = 0.766\text{m} \end{aligned} \quad (8.22)$$

Thus, the total roll stiffness is

$$C_\varphi = 2 \cdot 20000 \left(\frac{0.15 \cdot 1.111 \cdot 0.766}{0.3 \cdot 0.982} \right)^2 = 7510\text{Nm/rad} \quad (8.23)$$

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Roll angle

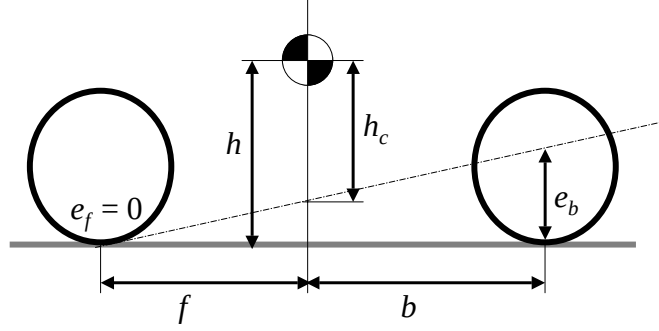


Figure 8.4.

Consider figure 8.4. Geometry gives

$$h_c = h - e_f - \frac{e_b - e_f}{f + b} f = h - \frac{e_b}{f + b} f \quad (8.24)$$

Momentum equilibrium around the roll axis gives

$$(C_f + C_b)\varphi = mgh_c \sin \varphi + ma_y h_c \cos \varphi \quad (8.25)$$

where C_f and C_b are roll the front and rear roll stiffness, respectively. φ is the roll angle, m is the mass of the car, g is the gravitational acceleration and a_y is the lateral acceleration. Assume that φ is small and rewrite eqn 8.25

$$(C_f + C_b)\varphi = mgh_c \varphi + ma_y h_c \quad (8.26)$$

In steady-state, $a_y = \frac{v_x^2}{R}$ and thus, we can rewrite eqn 9.3

$$\varphi = \frac{mh_c}{C_f + C_b - mgh_c} \frac{v_x^2}{R} \quad (8.27)$$

Use the given data to calculate h_c and φ

$$\begin{aligned} h_c &= h - \frac{e_b}{f + b} f = 0.6 - \frac{0.325}{1 + 1.6} 1000 = 0.475 \text{m} \\ \varphi &= \frac{mh_c}{C_f + C_b - mgh_c} \frac{v_x^2}{R} = \frac{1200 \cdot 0.475}{22000 + 18000 - 1200 \cdot 9.82 \cdot 0.475} \frac{30^2}{400} \\ &= 0.03728 \text{rad} = 2.136^\circ \end{aligned} \quad (8.28)$$

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 K_{us}

We should calculate

$$K_{us} = m \frac{bC_{34} - fC_{12}}{LC_{12}C_{34}} \quad (8.29)$$

In this case, we have to calculate C_{12} and C_{34} , the rest is known.

 C_{34}

The rear suspension is a rigid axis and thus only the tyre stiffness contributes

$$C_{34} = 2 \cdot C_{\alpha} = 2 \cdot 40000 = 80\text{kN/rad} \quad (8.30)$$

 C_{12}

The front suspension contains several stiffnesses in series

$$C_{12} = \left(\sum_i C_i \right)^{-1} = \left[\frac{1}{C_{tyre}} + \frac{1}{C_{steering}} + \frac{1}{C_{susp}} + \frac{1}{C_{roll}} + \frac{1}{C_{\gamma 1}} + \frac{1}{C_{\gamma 2}} \right]^{-1} \quad (8.31)$$

The contributions from tyres, steering and suspension are:

$$\text{Tyres: } C_{tyre} = 2C_{\alpha} = 80\text{kN/rad}$$

$$\text{Steering: } C_{steering} = \frac{i_s^2 C_s}{n_c - t_0} = \frac{20^2 \cdot 65}{0.025 - (-0.04)} = 400\text{kN/rad} \quad (8.32)$$

$$\text{Suspension: } C_{susp} = \frac{2C_{su}}{n_{su} - t_0} = \frac{20 \cdot 38000}{0.08 - (-0.04)} = 633\text{kN/rad}$$

Left to calculate are the contributions from the roll steering, camber due to suspension and camber due to lateral forces.

Roll steering The roll steering is the influence on the steering angle due to roll, i.e. $\frac{\partial \delta}{\partial \varphi}$

$$\frac{\partial \delta}{\partial \varphi} = \frac{\partial \delta}{\partial z} \frac{\partial z}{\partial \varphi} \quad (8.33)$$

Assume symmetric roll, i.e. the deflection at each suspension is Δz in opposite directions. Then

$$\frac{\partial z}{\partial \varphi} = \frac{\Delta z}{\Delta \varphi} = \left[\frac{2\Delta z}{t_{wf}} = \Delta \varphi \right] = \frac{t_{wf}}{2} = 0.75\text{m} \quad (8.34)$$

With the given data we can write

$$\frac{\partial \delta}{\partial \varphi} = \frac{\partial \delta}{\partial z} \frac{\partial z}{\partial \varphi} = -4 \frac{\pi}{180} 0.75 = -0.05236 \quad (8.35)$$

To calculate the roll angle we need to find $\frac{\partial \varphi}{\partial a_y}$. According to problem 807 we can write

$$\begin{aligned}\varphi &= \frac{mh_c}{C_f + C_b - mgh_c} a_y \rightarrow \\ \frac{\partial \varphi}{\partial a_y} &= \frac{mh_c}{(C_f + C_b) - mgh_c}\end{aligned}\quad (8.36)$$

with

$$h_c = h - \frac{fe_b + bef}{L} = 0.75 - \frac{1.2 \cdot 0.31 + 1.4 \cdot 0.05}{2.6} = 0.58\text{m} \quad (8.37)$$

To get the stiffness, we also must know the relation to the side force, F_{12} . A momentum equilibrium around the rear wheel's centre gives

$$\begin{aligned}\frac{\partial a_y}{\partial F_{12}} &= \frac{L}{bm} \rightarrow \\ \frac{\partial \varphi}{\partial F_{12}} &= \frac{\partial \varphi}{\partial a_y} \frac{\partial a_y}{\partial F_{12}} = \frac{h_c L}{(C_f + C_b - mgh_c)b} \\ &= \frac{0.58 \cdot 2.6}{(28000 + 20000 - 1250 \cdot 9.82 \cdot 9.58)1.4} = 2.635 \cdot 10^{-5} \text{rad/N}\end{aligned}\quad (8.38)$$

Now, we can write

$$\begin{aligned}C_{roll} &= \frac{-\partial F_{12}}{\partial \delta} = \frac{-1}{\frac{\partial \varphi}{\partial F_{12}} \frac{\partial \delta}{\partial \varphi}} \\ &= \frac{-1}{\frac{\partial \varphi}{\partial F_{12}} \frac{\partial \delta}{\partial \varphi}} = \frac{-1}{-0.05236 \cdot 2.635} 10^5 = 725 \text{kN/rad}\end{aligned}\quad (8.39)$$

Camber from suspension We want to find the effect in the side stiffness from the camber change due to suspension i.e.

$$C_{\gamma 1} = -\frac{\partial F_{12}}{\partial \delta_\epsilon} = -\frac{\partial F_{12}}{\partial \varphi} \frac{\partial \varphi}{\partial \epsilon} \frac{\partial \epsilon}{\partial \delta_\epsilon} \quad (8.40)$$

The textbook, page 8:5 gives

$$\frac{\partial \epsilon}{\partial \varphi} = 1 + \frac{t_{wf}}{2} \frac{\partial \epsilon}{\partial z} \quad (8.41)$$

We also need to find $\frac{\partial \epsilon}{\partial \delta_\epsilon}$ which is given from the ratio between the tyre and camber stiffnesses, i.e. $\frac{-C_\alpha}{C_\epsilon}$

$$C_{\gamma 1} = -\frac{\partial F_{12}}{\partial \varphi} \frac{\partial \varphi}{\partial \epsilon} \frac{\partial \epsilon}{\partial \delta_\epsilon} = -\frac{-1 \cdot 10^5}{2.635} \frac{1}{1 + \frac{1.5}{2}(-12 \frac{\pi}{180})} \frac{-40000}{2000} = 656 \text{kN/rad} \quad (8.42)$$

Camber from steering The effect in the side stiffness from the camber change due to steering is searched i.e.

$$C_{\gamma 2} = -\frac{\partial F_{12}}{\partial \delta_\epsilon} = -\frac{\partial F_{12}}{\partial \epsilon} \frac{\partial \epsilon}{\partial \delta_\epsilon} = \frac{-180}{260\pi} \frac{-40000}{2000} = 4407 \text{kN/rad} \quad (8.43)$$

Gathering all The total stiffness of the front suspension, C_{12} is now given according to eqn. 8.31

$$\begin{aligned}
 C_{12} &= \left(\sum_i \frac{1}{C_i} \right)^{-1} = \left[\frac{1}{C_{tyre}} + \frac{1}{C_{steering}} + \frac{1}{C_{susp}} + \frac{1}{C_{roll}} + \frac{1}{C_{\gamma 1}} + \frac{1}{C_{\gamma 2}} \right]^{-1} \\
 &= 10^3 \left[\underbrace{\frac{1}{80}}_{63.4\%} + \underbrace{\frac{1}{400}}_{12.7\%} + \underbrace{\frac{1}{633}}_{8.0\%} + \underbrace{\frac{1}{725}}_{7.0\%} + \underbrace{\frac{1}{656}}_{7.7\%} + \underbrace{\frac{1}{4407}}_{1.2\%} \right]^{-1} = 50.7 \text{Kn/rad}
 \end{aligned} \tag{8.44}$$

Values in K_{us}

Now we have both C_{12} and C_{34} and can calculate K_{us} according to eqn. 8.29

$$\begin{aligned}
 K_{us} &= m \frac{bC_{34} - fC_{12}}{LC_{12}C_{34}} \\
 &= 1250 \frac{1.4 \cdot 80000 - 1.2 \cdot 50700}{2.6 \cdot 80000 \cdot 50700} = 6.06 \cdot 10^{-3} \text{rads}^2/\text{m}
 \end{aligned} \tag{8.45}$$

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Handling diagram

The plots mentioned below are found in figure 8.5. Since the rear axis is rigid, we know that the stiffness of the rear suspension equals the stiffness of the tyres. The front suspension is flexible up to $\delta_{max}=0.04$ rad. To plot the stiffness, we need to find the μ at this point.

$$\mu_{max} = \frac{2C_{su}\delta_{max}}{F_{z12}} = \frac{2 \cdot 6000 \cdot 0.04}{9600} = 0.5 \quad (8.46)$$

This is added to the plot as a red dash-dotted line. The over-all stiffness of the front suspension is calculated by adding the plots as $\alpha = f(F_y/F_z)$. The result is shown as a black, dashed line.

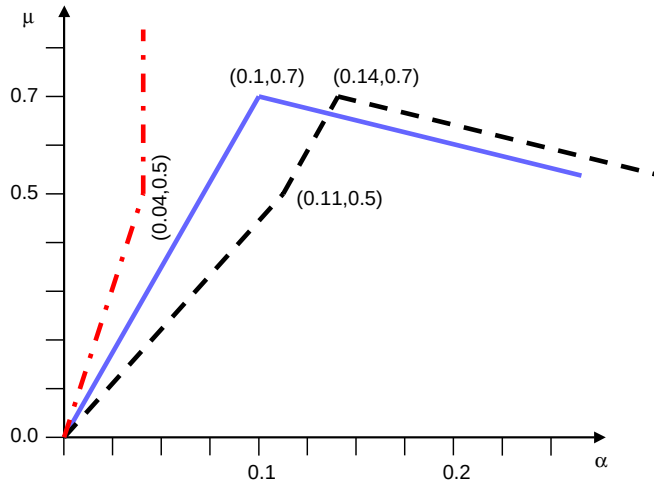


Figure 8.5. Characteristics of the tyre pair and the rear suspension (blue, filled), the front linkage (red, dash-dotted) and the front suspension (black, dashed).

The handling diagram is drawn as $\alpha_{34} - \alpha_{12}$ on the horizontal axis and F_y/F_z on the vertical axis. Thus, the stable plot in the handling diagram of the entire car equals the plot of the front linkage, dash-dotted red in the plot above.

Steer input

The required steer input at a certain speed and radius is found from

$$\begin{aligned} \frac{F_y}{F_z} &= \frac{a_y}{g} = \frac{v_x^2}{Rg} = \frac{v_x^2}{Lg} \frac{L}{R} = 30.58 \frac{L}{R} \\ \alpha_{34} - \alpha_{12} &= \delta - \frac{L}{R} \end{aligned} \quad (8.47)$$

R=150m

$L/R = 3/150 = 0.02 \rightarrow a_y/g = 0.6116$ and from the handling diagram we find that this corresponds to 0.04 rad and thus $\delta_{R150} = 0.02 + 0.04 = 0.06\text{rad}$

R=300m

$L/R = 3/300 = 0.01 \rightarrow a_y/g = 0.3058$ and from the handling diagram we find that this corresponds to 0.024 rad and thus $\delta_{R150} = 0.01 + 0.024 = 0.034\text{rad}$

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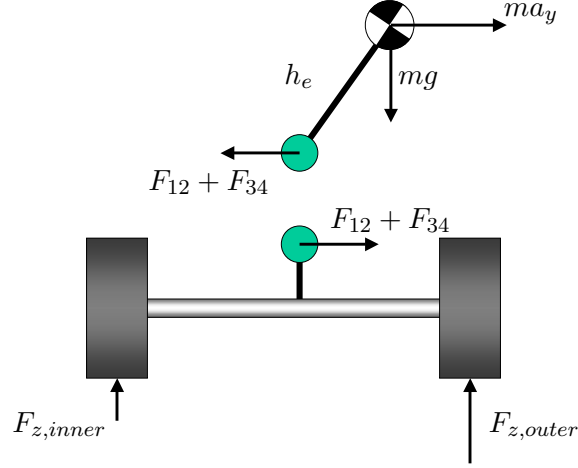


Figure 8.6.

From figure 8 we get

$$\begin{aligned}
 F_{z,outer} &= \frac{F_{z12}}{2} + \Delta F_z \\
 \Delta F_z &= \frac{C_f \psi + F_{12} e_f}{s_f} \\
 C_{tot} \psi &= mgh_e \underbrace{\sin \psi}_{\approx \psi} + ma_y h_e \underbrace{\cos \psi}_{\approx 1}
 \end{aligned} \tag{8.48}$$

Thus, we need to calculate h_e , C_{tot} , F_{12} , F_{z12} and a_y .

C_{tot}

Study figure 8. The front and rear roll stiffnesses are calculated according to

$$\begin{aligned}
 C_{tot} &= C_a + C_b \\
 C_f &= 2k_1 \left(\frac{b_f d_f}{a_f} \right)^2 + C_{\psi f} \\
 C_b &= 2k_3 \left(\frac{F_{Fr} d_r}{a_r} \right)^2
 \end{aligned} \tag{8.49}$$

Where following values are measurements from the suspension drawings

$$\begin{aligned}
 a_f &= 1.3\text{m} & b_f &= 0.67\text{m} & d_f &= 1.1\text{m} \\
 a_r &= 0.7\text{m} & F_{Fr} &= 0.35\text{m} & d_r &= 0.54\text{m}
 \end{aligned}$$

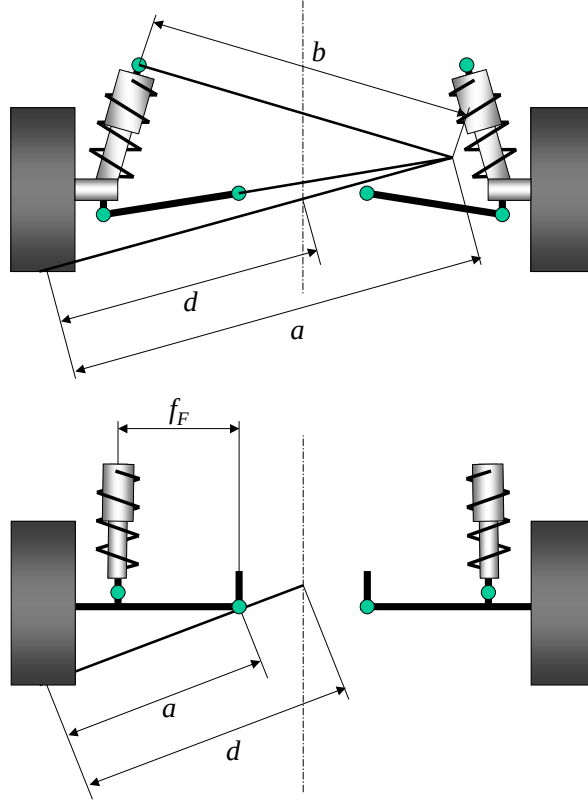


Figure 8.7.

h_e

From the suspension drawings we also get $e_f = 0.12\text{m}$, $e_r = 0.24\text{m}$ and now we can calculate

$$h_e = h - (e_f + \frac{e_b - e_f}{L} \lambda L) = h - e_f - \lambda(e_b - e_f) \quad (8.50)$$

ψ

From equation 8.48

$$\begin{aligned} C_{tot}\psi &= mgh_e \underbrace{\sin \psi}_{\approx \psi} + ma_y h_e \underbrace{\cos \psi}_{\approx 1} \rightarrow \\ \psi(C_{tot} - mgh_e) &= ma_y h_e \rightarrow \\ \psi &= \frac{ma_y h_e}{C_{tot} - mgh_e} \end{aligned} \quad (8.51)$$

F_{12} , F_{z12} and a_y

Since we are driving in a curve in steady-state, F_{12} , F_{z12} and a_y are all constant and can be calculated according to

$$\begin{aligned} F_{12} &= (1 - \lambda)ma_y \\ F_{z12} &= (1 - \lambda)mg \\ a_y &= \frac{v_x^2}{R} \end{aligned} \tag{8.52}$$

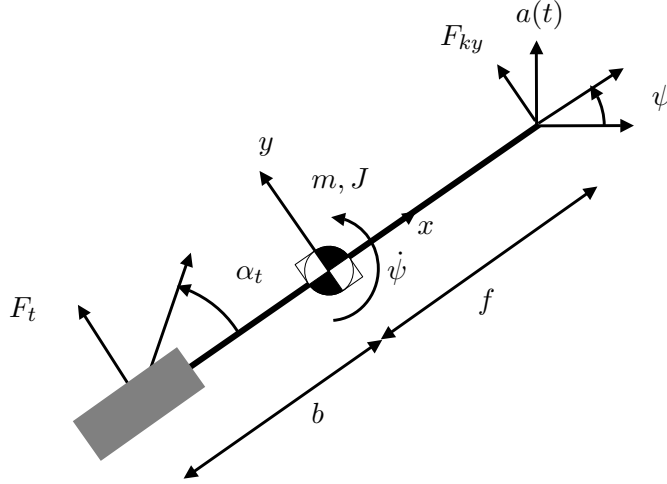
Now

$$\begin{aligned} F_{z,outer} &= \frac{F_{z12}}{2} + \Delta \\ &= \frac{(1 - \lambda)mg}{2} + \frac{C_f\psi + F_{12}e_f}{s_f} = 5416\text{N} \end{aligned} \tag{8.53}$$

9

Trailers

901



Equations of motion

$$\begin{aligned}
 J\ddot{\psi} &= F_{ky}f - F_tb \\
 v_x &= u \cos \psi + \dot{a}(t) \sin \psi \\
 v_y &= -u \sin \psi + \dot{a}(t) \cos \psi
 \end{aligned} \tag{9.1}$$

and

$$\begin{aligned}
 \bar{v} &= (v_x, v_y, 0) = (u, v_y, 0) \\
 \bar{\omega} &= (0, 0, \omega) \\
 \dot{\bar{v}} &= \left(\frac{\partial}{\partial t} + \bar{\omega} \times \right) \bar{v} = (\dot{v}_x - v_y \dot{\psi}, \dot{v}_y + v_x \dot{\psi}, 0)
 \end{aligned} \tag{9.2}$$

Force equilibrium

$$\begin{aligned}
 ma_y &= m(\dot{v}_y + v_x \dot{\psi}) = F_t + F_{ky} \rightarrow \\
 F_{ky} &= -F_t + m(\dot{v}_y + v_x \dot{\psi})
 \end{aligned} \tag{9.3}$$

Equations 9.1 and 9.3 give

$$\begin{aligned}
 J\ddot{\psi} &= (-F_t + m(\dot{v}_y + v_x \dot{\psi}))f - F_tb \\
 &= mf(\dot{v}_y + v_x \dot{\psi}) - Ftb
 \end{aligned} \tag{9.4}$$

The lateral acceleration, a_y

$$\begin{aligned}
 \dot{v}_y &= -u\dot{\psi} \cos \psi + \ddot{a}(t) \cos \psi - \dot{a}(t)\dot{\psi} \sin \psi - f\ddot{\psi} \\
 a_y &= \dot{v}_y + v_x \dot{\psi} \\
 &= (\ddot{a}(t) - u\dot{\psi}) \cos \psi - \dot{a}(t)\dot{\psi} \sin \psi - f\ddot{\psi} + u\dot{\psi}) \cos \psi + \dot{a}(t)\dot{\psi} \sin \psi \\
 &= \ddot{a}(t) \cos \psi - f\ddot{\psi}
 \end{aligned} \tag{9.5}$$

Tyre force, F_t

$$\begin{aligned} F_t &= -C_t \alpha_t = -C_t \frac{v_{yt}}{v_{xt}}, \quad v_{xt} = v_x \text{ and } v_{yt} = v_y - b\dot{\psi} \rightarrow \\ F_t &= -C_t \frac{v_y - b\dot{\psi}}{v_x} = -C_t \frac{-u \sin \psi + \dot{a}(t) \cos \psi - L\dot{\psi}}{u \cos \psi + \dot{a}(t) \sin \psi} \end{aligned} \quad (9.6)$$

Use equations 9.4, 9.5 and 9.6 to write

$$J\ddot{\psi} = mf(\ddot{a}(t) \cos \psi - f\ddot{\psi} + LC_t \frac{-u \sin \psi + \dot{a}(t) \cos \psi - L\dot{\psi}}{u \cos \psi + \dot{a}(t) \sin \psi}) \quad (9.7)$$

Now assume small ψ so that $\cos \psi \approx 1$, $\sin \psi \approx \psi$ and $|u \cos \psi| \gg |\dot{a}(t) \sin \psi|$. Then,

$$\begin{aligned} J\ddot{\psi} &= mf(\ddot{a}(t) - f\ddot{\psi} + LC_t \frac{-u\psi + \dot{a}(t) - L\dot{\psi}}{u}) \rightarrow \\ (J + mf^2)\ddot{\psi} + \frac{C_t L^2}{u}\dot{\psi} + C_t L\psi &= mf\ddot{a}(t) + \frac{C_t L}{u}\dot{a}(t) \rightarrow \\ \ddot{\psi} + \underbrace{\frac{C_t L^2}{u(J + mf^2)}}_{2\zeta\omega_0} \dot{\psi} + \underbrace{\frac{C_t L}{J + mf^2}}_{\omega_0^2} \psi &= \underbrace{\frac{mf}{J + mf^2}}_{K_1} \ddot{a}(t) + \underbrace{\frac{C_t L}{u(J + mf^2)}}_{K_2} \dot{a}(t) \rightarrow \\ \ddot{\psi} + 2\zeta\omega_0\dot{\psi} + \omega_0^2\psi &= K_1\ddot{a}(t) + K_2\dot{a}(t) \end{aligned} \quad (9.8)$$

902

Use the result from problem 901.

$$\ddot{\psi} + 2\zeta\omega_0\dot{\psi} + \omega_0^2\psi = K_1\ddot{a}(t) + K_2\dot{a}(t) \quad (9.9)$$

Transform into Laplace plane

$$s^2\Psi + 2\zeta\omega_0s\Psi + \omega_0^2\Psi = K_1s^2A + K_2sA \quad (9.10)$$

Form the transfer function from the lateral displacement of the coupler, a , to the yaw angle of the trailer, ψ :

$$G_A^\Psi(s) = \frac{K_1s^2 + K_2s}{s^2 + 2\zeta\omega_0s + \omega_0^2} \quad (9.11)$$

Consider the case when $a(t)$ is a sinusoidal function, i.e. when $s = j\omega$:

$$G_A^\Psi(j\omega) = \frac{-K_1\omega^2 + jK_2\omega}{-\omega^2 + j2\zeta\omega_0\omega + \omega_0^2} \quad (9.12)$$

Insert the numerical values given

$$\begin{aligned} 2\zeta\omega_0 &= \frac{C_t L^2}{u(J + mf^2)} = \frac{30000 \cdot (3 + 2)^2}{25(1800 + 1000 \cdot 3^2)} = 2.78 \\ \omega_0^2 &= \frac{C_t L}{J + mf^2} = \frac{30000(2 + 3)}{1800 + 1000 \cdot 3^2} = 13.9 \\ K_1 &= \frac{mf}{J + mf^2} = \frac{1000 \cdot 3}{1800 + 1000 \cdot 3^2} = 0.278 \\ K_2 &= \frac{C_t L}{u(J + mf^2)} = \frac{30000(2 + 3)}{1800 + 1000 \cdot 3^2} = 0.556 \end{aligned} \quad (9.13)$$

The transfer function is then

$$G_A^\Psi(j\omega) = \frac{-0.278\omega^2 + j0.556\omega}{-\omega^2 + j2.78\omega + 13.9} \quad (9.14)$$

Use equation 9.14 to create bode plots, see figure 9.1.

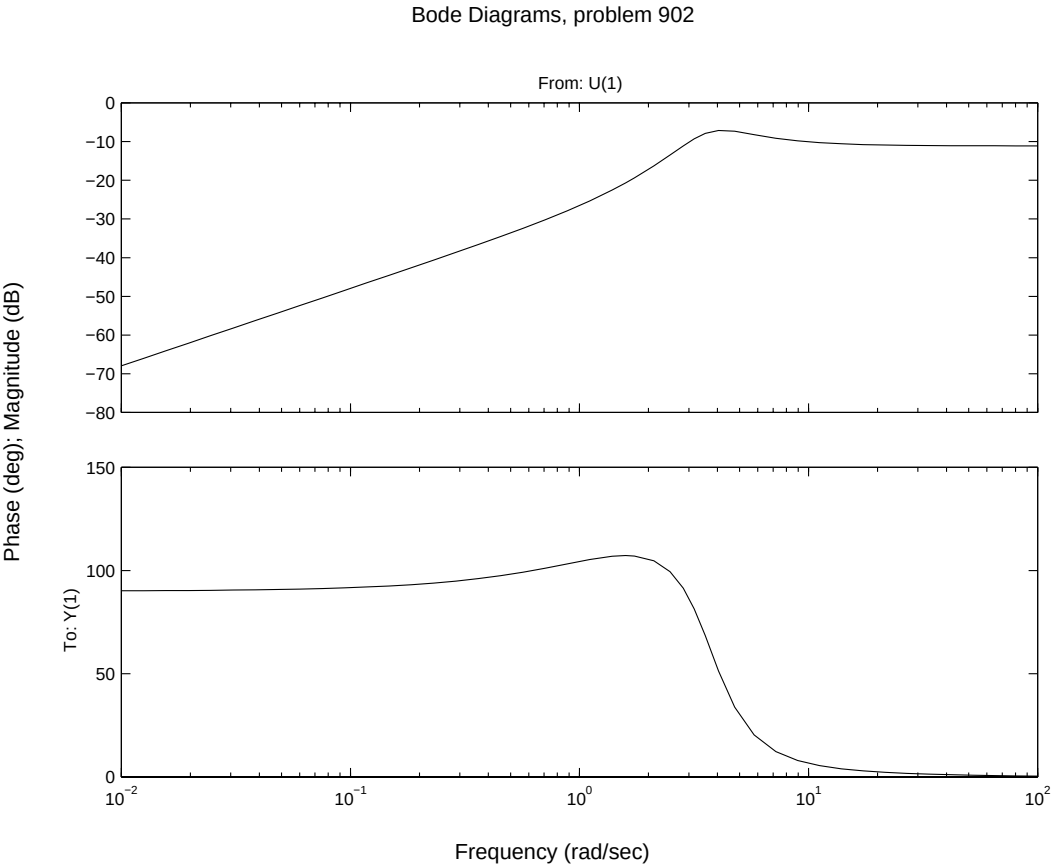


Figure 9.1.