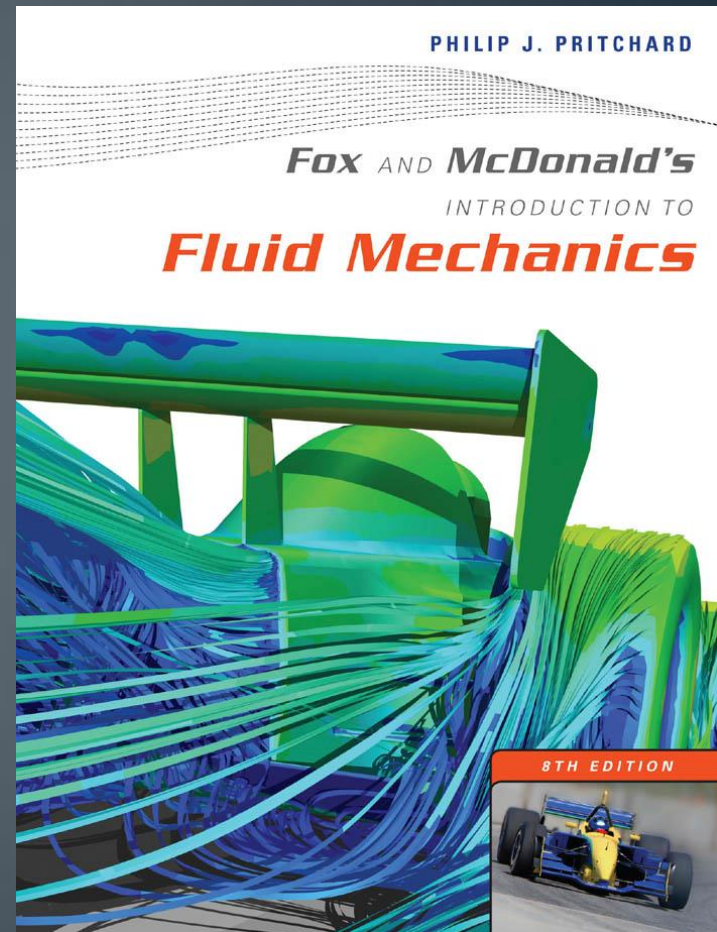
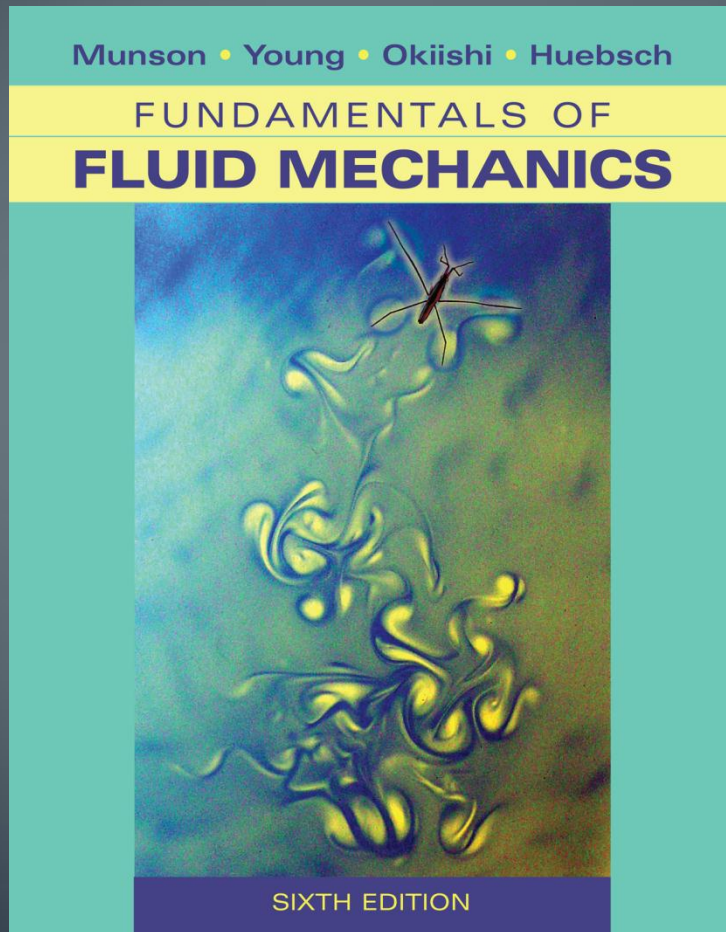


FLUID MECHANICS II

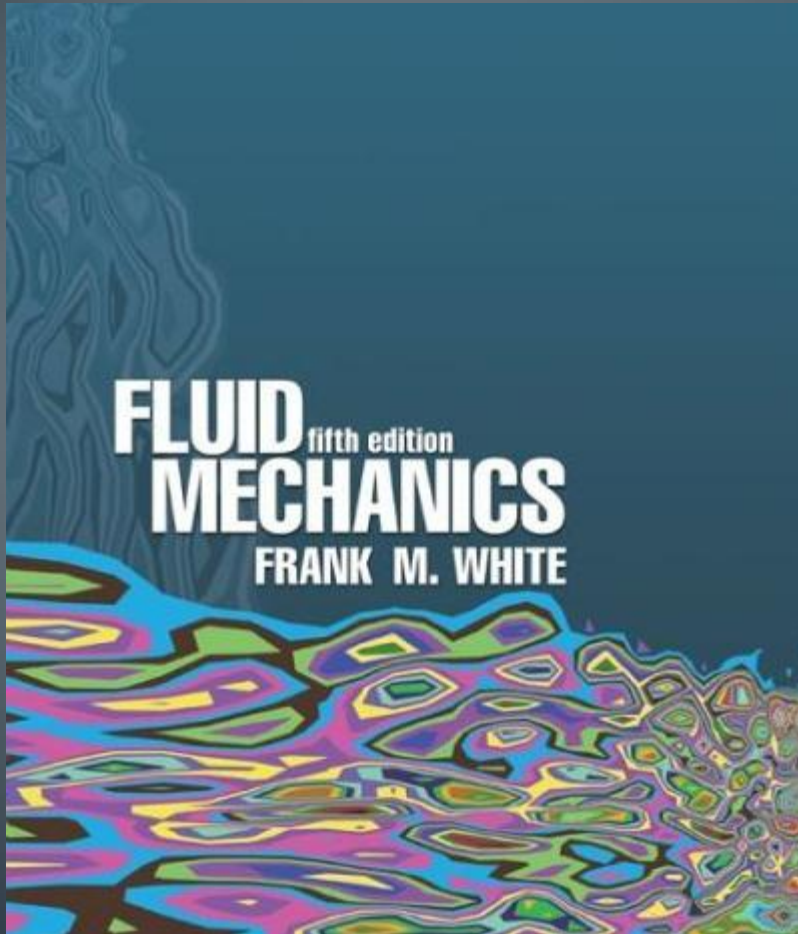
Farbod Fakhrabadi

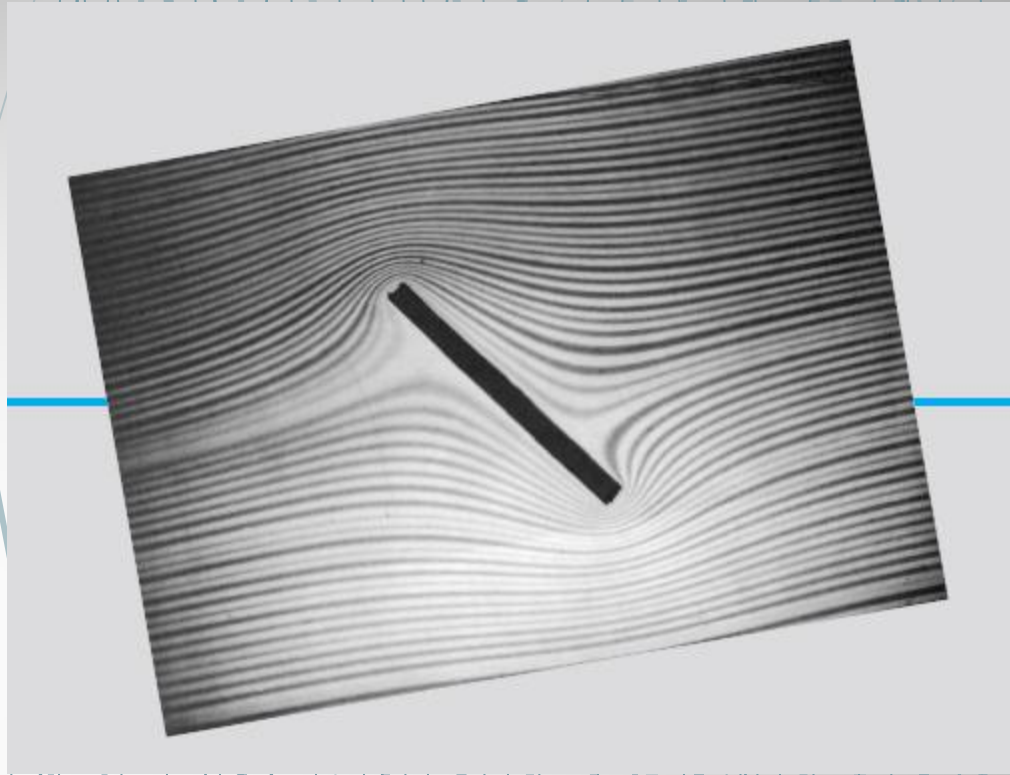
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Textbooks



Textbooks

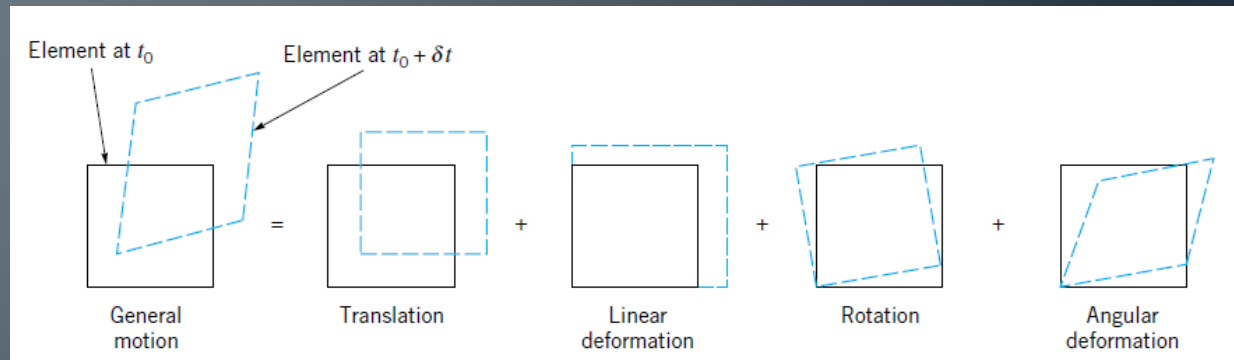




Differential Analysis of Fluid Flow

Fluid Element Kinematics

- Because of the generally complex velocity variation within the field, we expect the element not only to **translate** from one position but also to have its volume changed (**linear deformation**), to **rotate**, and to undergo a change in shape (**angular deformation**).



Fluid Element Kinematics

- Velocity Field

$$\mathbf{V} = u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}}$$

- Acceleration Field

$$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t} + u \frac{\partial \mathbf{V}}{\partial x} + v \frac{\partial \mathbf{V}}{\partial y} + w \frac{\partial \mathbf{V}}{\partial z}$$

$$a_x = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}$$

$$a_y = \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z}$$

$$a_z = \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z}$$

Fluid Element Kinematics

- The acceleration is also concisely expressed as

$$\mathbf{a} = \frac{D\mathbf{V}}{Dt}$$

- where the operator

$$\frac{D()}{Dt} = \frac{\partial()}{\partial t} + u \frac{\partial()}{\partial x} + v \frac{\partial()}{\partial y} + w \frac{\partial()}{\partial z}$$

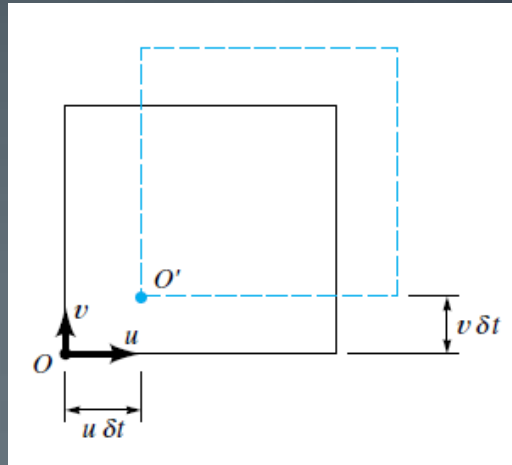
- is termed the **material derivative**, or **substantial derivative**. In vector notation

$$\frac{D()}{Dt} = \frac{\partial()}{\partial t} + (\mathbf{V} \cdot \nabla)()$$

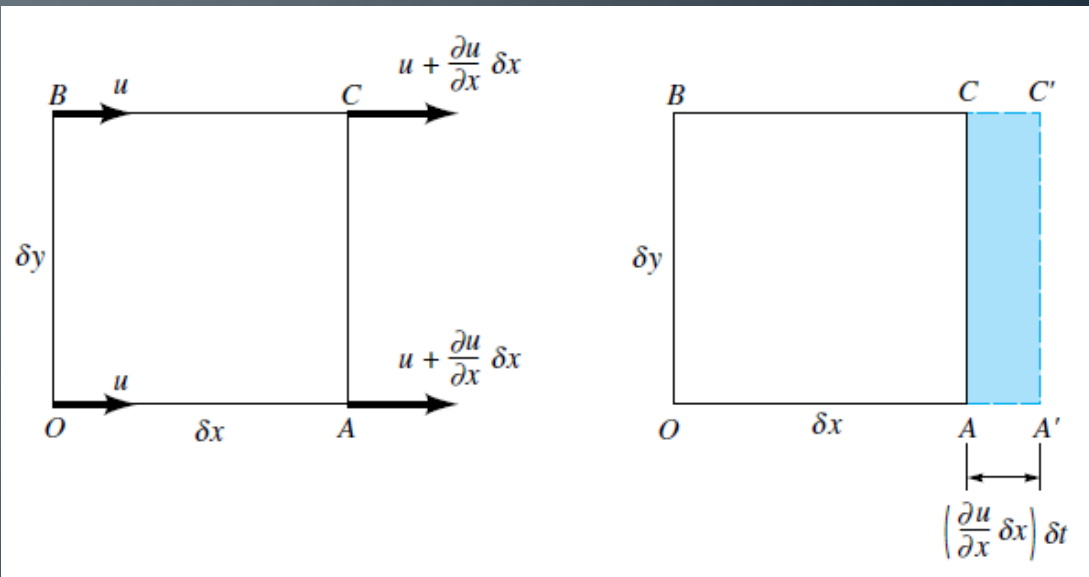
$$\nabla() = \frac{\partial()}{\partial x} \hat{\mathbf{i}} + \frac{\partial()}{\partial y} \hat{\mathbf{j}} + \frac{\partial()}{\partial z} \hat{\mathbf{k}}$$

Linear Motion and Deformation

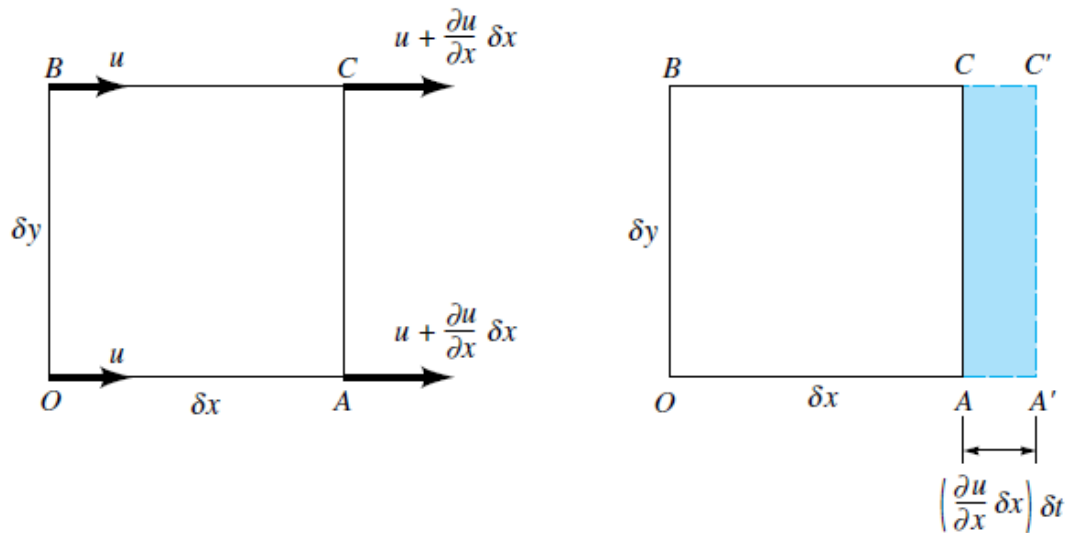
- Linear Motion



- Linear Deformation



Linear Motion and Deformation

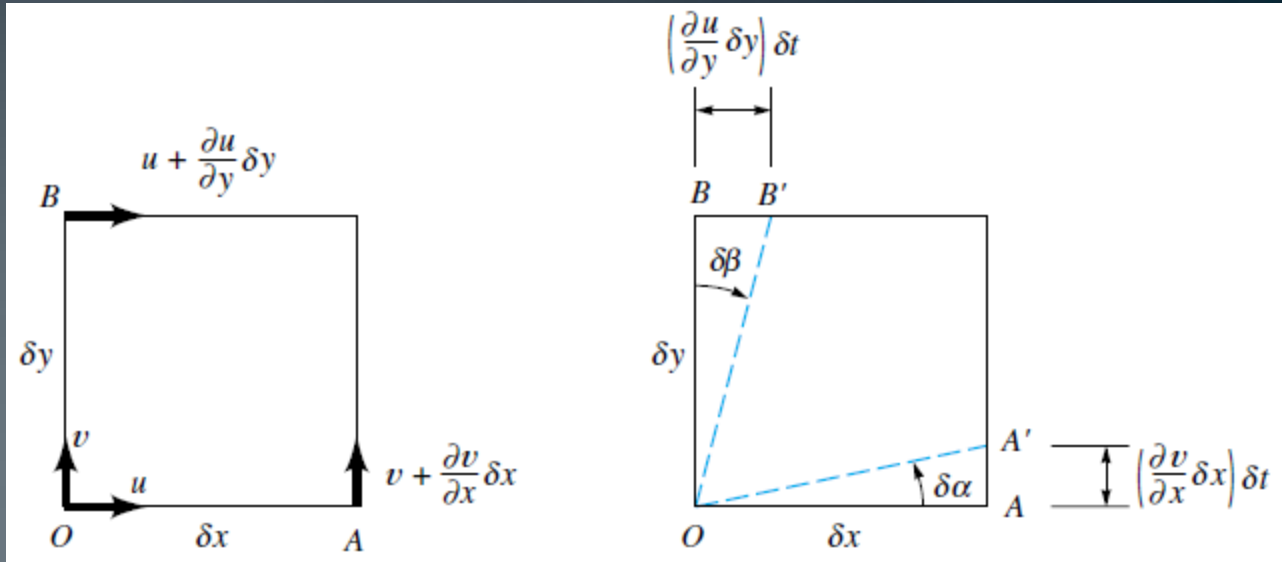


$$\text{Change in } \delta \Psi = \left(\frac{\partial u}{\partial x} \delta x \right) (\delta y \delta z) (\delta t)$$

$$\frac{1}{\delta \Psi} \frac{d(\delta \Psi)}{dt} = \lim_{\delta t \rightarrow 0} \left[\frac{(\partial u / \partial x) \delta t}{\delta t} \right] = \frac{\partial u}{\partial x}$$

$$\frac{1}{\delta \Psi} \frac{d(\delta \Psi)}{dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \mathbf{V}$$

Angular Motion and Deformation



$$\omega_{OA} = \lim_{\delta t \rightarrow 0} \frac{\delta\alpha}{\delta t}$$

$$\tan \delta\alpha \approx \delta\alpha = \frac{(\partial v / \partial x) \delta x \delta t}{\delta x} = \frac{\partial v}{\partial x} \delta t$$

$$\omega_{OA} = \lim_{\delta t \rightarrow 0} \left[\frac{(\partial v / \partial x) \delta t}{\delta t} \right] = \frac{\partial v}{\partial x}$$

Angular Motion and Deformation

- Rotation

$$\omega_z = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

$$\omega_x = \frac{1}{2} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right)$$

$$\omega_y = \frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right)$$

- Vorticity

$$\boldsymbol{\omega} = \omega_x \hat{\mathbf{i}} + \omega_y \hat{\mathbf{j}} + \omega_z \hat{\mathbf{k}}$$

$$\boldsymbol{\omega} = \frac{1}{2} \text{curl } \mathbf{V} = \frac{1}{2} \nabla \times \mathbf{V}$$

$$\boldsymbol{\zeta} = 2 \boldsymbol{\omega} = \nabla \times \mathbf{V}$$

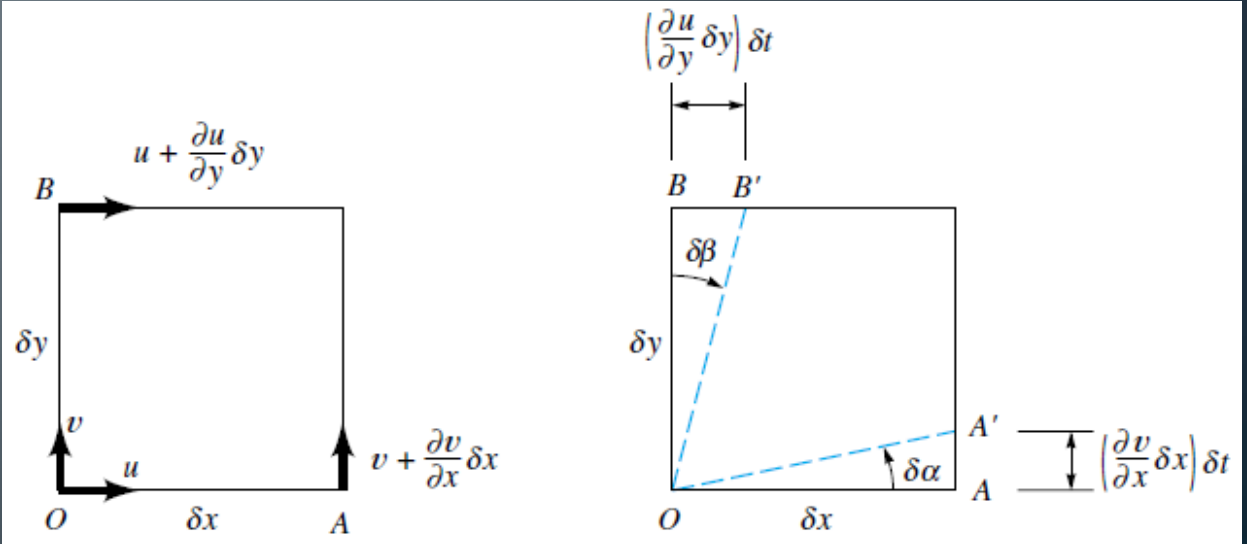
- Angular Deformation

Angular Motion and Deformation

$$\delta\gamma = \delta\alpha + \delta\beta$$

$$\dot{\gamma} = \lim_{\delta t \rightarrow 0} \frac{\delta\gamma}{\delta t} = \lim_{\delta t \rightarrow 0} \left[\frac{(\partial v / \partial x) \delta t + (\partial u / \partial y) \delta t}{\delta t} \right]$$

$$\dot{\gamma} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

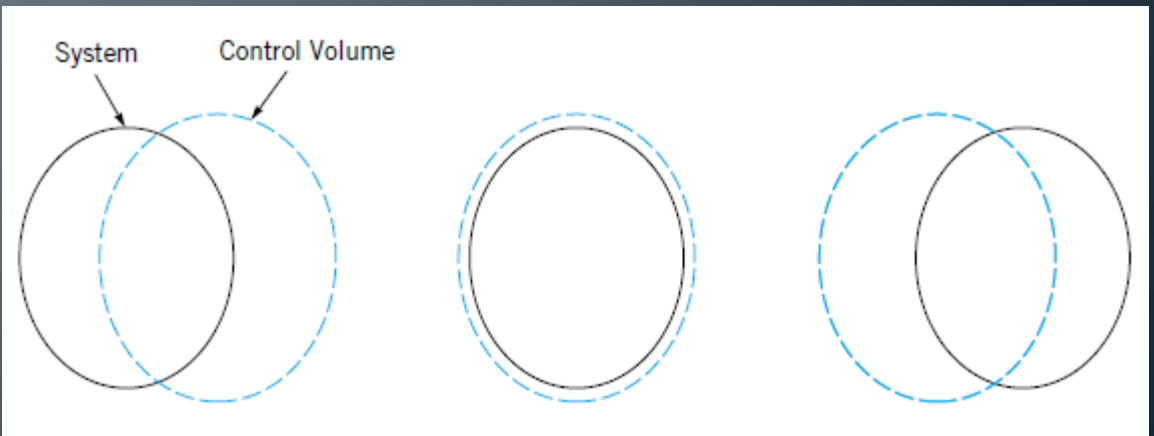


Conservation of Mass

- Conservation of mass requires that the mass, M , of a system remain constant as the system moves through the flow field. In equation form this principle is expressed as

$$\frac{DM_{\text{sys}}}{Dt} = 0$$

$$\frac{\partial}{\partial t} \int_{\text{cv}} \rho dV + \int_{\text{cs}} \rho \mathbf{V} \cdot \hat{\mathbf{n}} dA = 0$$

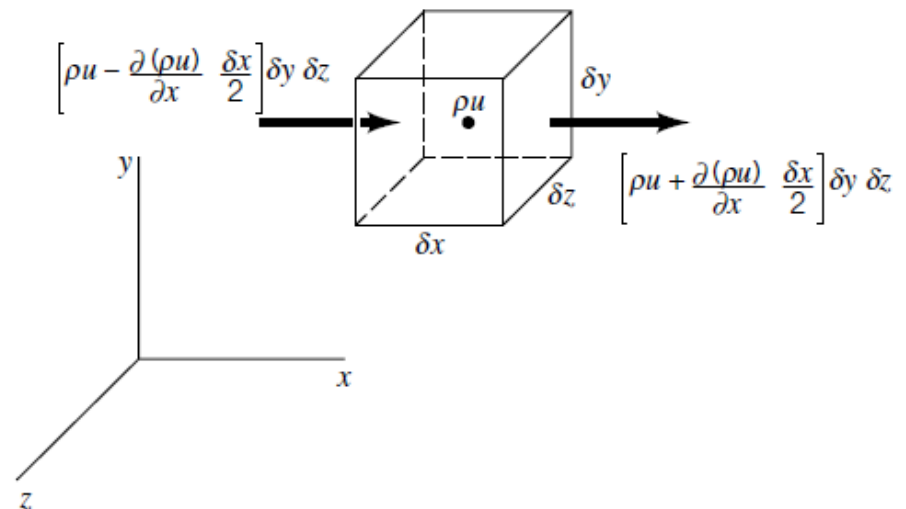
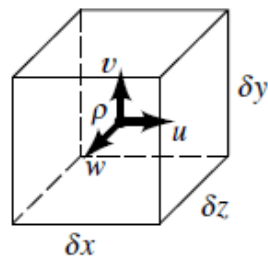


Differential Form of Continuity Equation

$$\frac{DM_{\text{sys}}}{Dt} = 0$$

$$\frac{\partial}{\partial t} \int_{\text{cv}} \rho dV + \int_{\text{cs}} \rho \mathbf{V} \cdot \hat{\mathbf{n}} dA = 0$$

$$\frac{\partial}{\partial t} \int_{\text{cv}} \rho dV \approx \frac{\partial \rho}{\partial t} \delta x \delta y \delta z$$



Differential Form of Continuity Equation

Net rate of mass
outflow in x direction

$$= \left[\rho u + \frac{\partial(\rho u)}{\partial x} \frac{\delta x}{2} \right] \delta y \delta z - \left[\rho u - \frac{\partial(\rho u)}{\partial x} \frac{\delta x}{2} \right] \delta y \delta z = \frac{\partial(\rho u)}{\partial x} \delta x \delta y \delta z$$

$$\text{Net rate of mass outflow in } y \text{ direction} = \frac{\partial(\rho v)}{\partial y} \delta x \delta y \delta z$$

$$\text{Net rate of mass outflow in } z \text{ direction} = \frac{\partial(\rho w)}{\partial z} \delta x \delta y \delta z$$

$$\text{Net rate of mass outflow} = \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right] \delta x \delta y \delta z$$

Differential Form of Continuity Equation

- The differential equation for conservation of mass is

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0$$

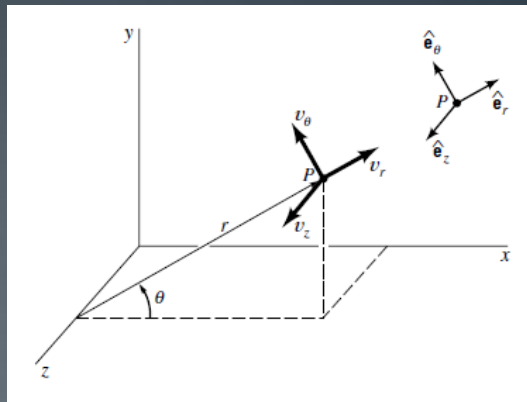
- For **steady** flow of **compressible** fluids

$$\nabla \cdot \rho \mathbf{V} = 0$$

- For **incompressible** fluids

$$\nabla \cdot \mathbf{V} = 0$$

Cylindrical Polar Coordinates



$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial(r\rho v_r)}{\partial r} + \frac{1}{r} \frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

- For **steady** flow of **compressible** fluids

$$\frac{1}{r} \frac{\partial(r\rho v_r)}{\partial r} + \frac{1}{r} \frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

- For **incompressible** fluids

$$\frac{1}{r} \frac{\partial(r v_r)}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

The Stream Function

- Steady, incompressible, plane, two-dimensional flow represents one of the simplest types of flow of practical importance. For this flow the continuity equation reduces to

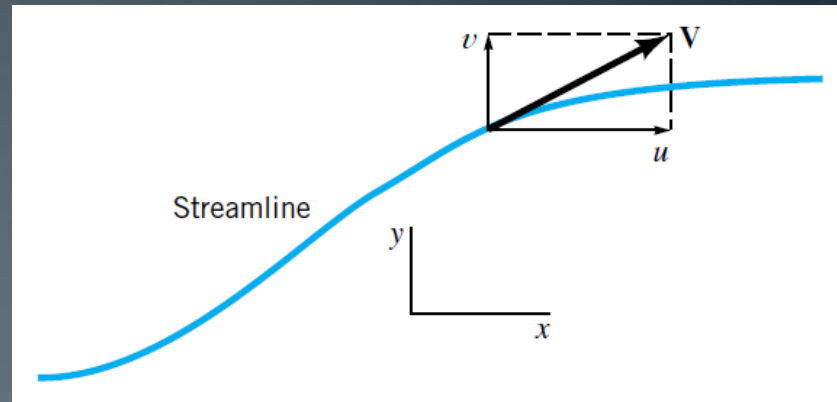
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

- The **stream function**, which relates the velocities as

$$u = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial x}$$

$$\frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial y} \left(-\frac{\partial \psi}{\partial x} \right) = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

- Lines along which ψ is constant are streamlines.



The Stream Function

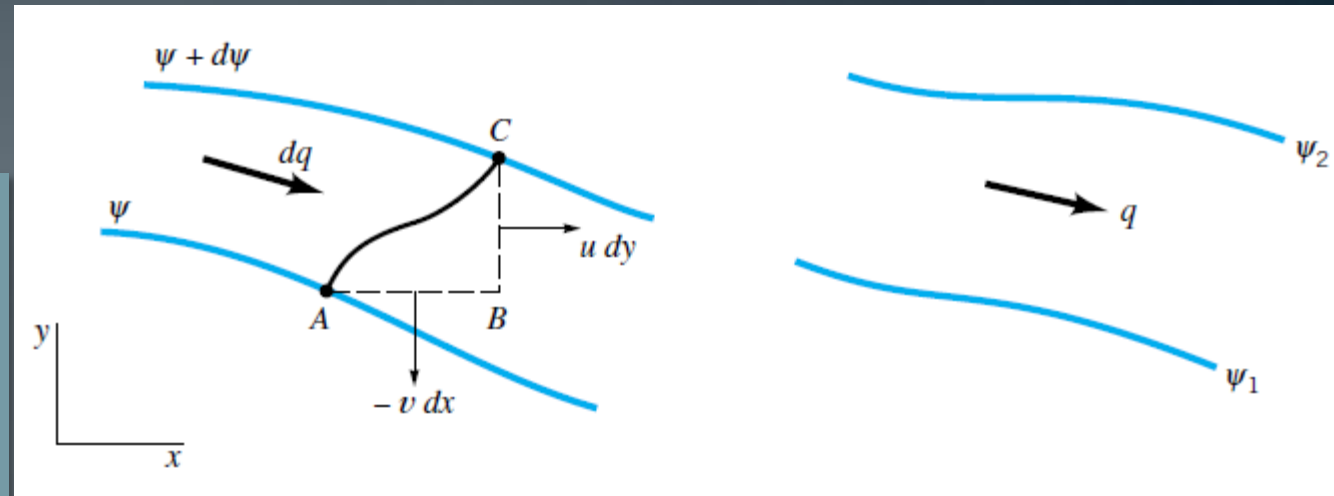
$$\frac{dy}{dx} = \frac{v}{u}$$

$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = -v dx + u dy$$

$$-v dx + u dy = 0$$

$$\frac{dy}{dx} = \frac{v}{u}$$

- Volume rate of flow



The Stream Function

$$dq = u dy - v dx$$

$$dq = \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx$$

$$dq = d\psi$$

$$q = \int_{\psi_1}^{\psi_2} d\psi = \psi_2 - \psi_1$$

The Stream Function

- In cylindrical coordinates the continuity equation for incompressible, plane, two-dimensional flow reduces to

$$\frac{1}{r} \frac{\partial(rv_r)}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} = 0$$

$$v_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \quad v_\theta = -\frac{\partial \psi}{\partial r}$$

Problems

6.4 The three components of velocity in a flow field are given by

$$u = x^2 + y^2 + z^2$$

$$v = xy + yz + z^2$$

$$w = -3xz - z^2/2 + 4$$

(a) Determine the volumetric dilatation rate and interpret the results. **(b)** Determine an expression for the rotation vector. Is this an irrotational flow field?

Problems

(a) Volumetric dilatation rate = $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ (Eq. 6.9)

Thus, for velocity components given

$$\text{volumetric dilatation rate} = 2x + (x+z) + (-3x-z) = \underline{\underline{0}}$$

This result indicates that there is no change in the volume of a fluid element as it moves from one location to another.

(b) From Eqs. 6.12, 6.13, and 6.14 with the velocity components given:

$$\omega_z = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = \frac{1}{2} (y - 2y) = -\frac{y}{2}$$

$$\omega_x = \frac{1}{2} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) = \frac{1}{2} [0 - (y + 2z)] = -\left(\frac{y}{2} + z\right)$$

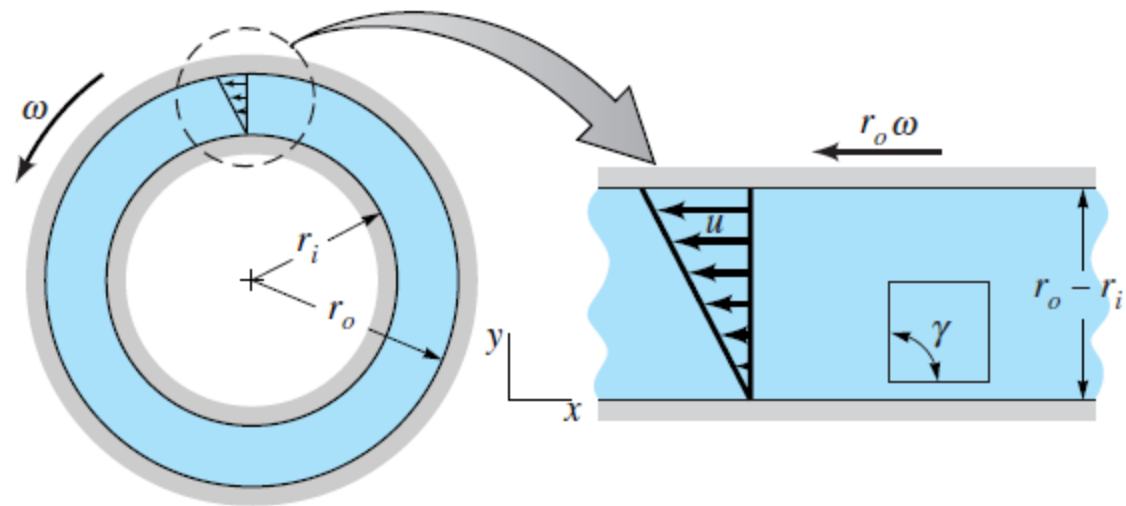
$$\omega_y = \frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) = \frac{1}{2} [2z - (-3z)] = \frac{5z}{2}$$

Thus,

$$\underline{\underline{\vec{\omega} = -\left(\frac{y}{2} + z\right) \hat{i} + \frac{5z}{2} \hat{j} - \frac{y}{2} \hat{k}}}$$

Problems

6.9 A viscous fluid is contained in the space between concentric cylinders. The inner wall is fixed, and the outer wall rotates with an angular velocity ω . (See Fig. P6.9a and [Video V6.1](#).) Assume that the velocity distribution in the gap is linear as illustrated in Fig. P6.9b. For the small rectangular element shown in Fig. P6.9b, determine the rate of change of the right angle γ due to the fluid motion. Express your answer in terms of r_0 , r_i , and ω .



Problems

6.12 For each of the following stream functions, with units of m^2/s , determine the magnitude and the angle the velocity vector makes with the x -axis at $x = 1 \text{ m}$, $y = 2 \text{ m}$. Locate any stagnation points in the flow field.

(a) $\psi = xy$

(b) $\psi = -2x^2 + y$

Problems

From the definition of the stream function,

$$u = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial x} \quad (\text{Eqs. 6.37})$$

so that for the velocity distribution given,

$$\frac{\partial \psi}{\partial y} = x^2 \quad (1)$$

$$\frac{\partial \psi}{\partial x} = 2xy - x \quad (2)$$

Integrate Eq.(1) with respect to y to obtain

$$\int d\psi = \int x^2 dy + f_1(x)$$

or

$$\psi = x^2 y + f_1(x) \quad (3)$$

Similarly, integrate Eq.(2) with respect to x to obtain

$$\int d\psi = \int (2xy - x) dx + f_2(y)$$

or

$$\psi = x^2 y - \frac{x^2}{2} + f_2(y) \quad (4)$$

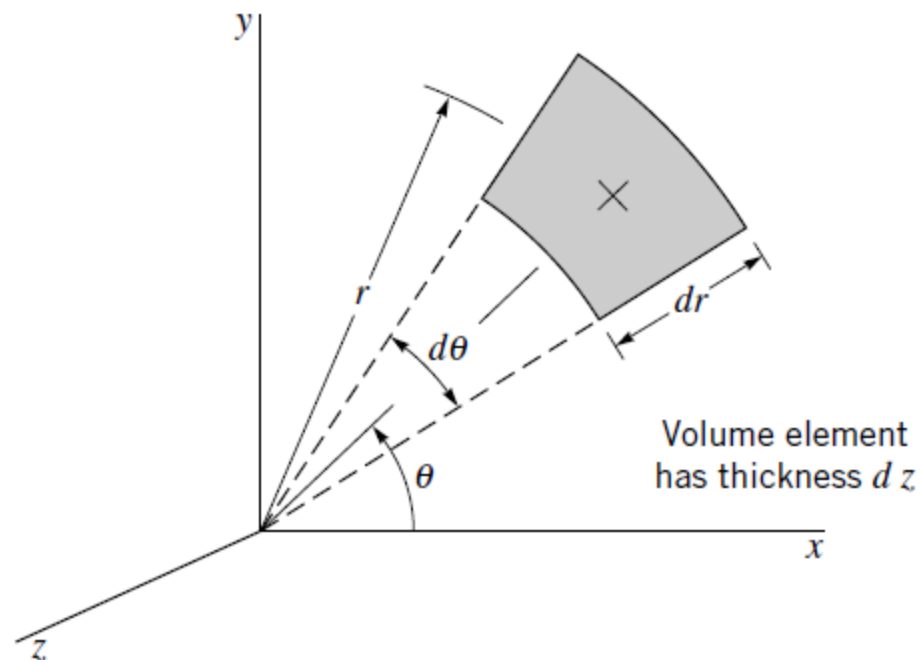
Thus, to satisfy both Eqs. (3) and (4)

$$\underline{\underline{\psi = x^2 y - \frac{x^2}{2} + C}}$$

where C is an arbitrary constant.

Problems

6.17 Make use of the control volume shown in Fig. P6.17 to derive the continuity equation in cylindrical coordinates (Eq. 6.33 in text).



Problems

6.19 In a certain steady, two-dimensional flow field the fluid density varies linearly with respect to the coordinate x ; that is, $\rho = Ax$ where A is a constant. If the x component of velocity u is given by the equation $u = y$, determine an expression for v .

Problems

For a variable density flow,

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} \quad (\text{Eq. 6.29})$$

With

$$\rho u = (Ax)(y) = Axy$$

it follows that

$$\frac{\partial(\rho u)}{\partial x} = Ay$$

Thus,

$$\frac{\partial(\rho v)}{\partial y} = -Ay \quad (1)$$

Integrate Eq.(1) with respect to y to obtain

$$\int d(\rho v) = - \int Ay dy + f_1(x)$$

or

$$\rho v = - \frac{Ay^2}{2} + f_1(x)$$

With $\rho = Ax$

$$v = - \left(\frac{1}{Ax} \right) \left(\frac{Ay^2}{2} \right) + \frac{f_1(x)}{Ax}$$

or

$$v = \underline{\underline{- \frac{y^2}{2x} + f(x)}}$$

where $f(x)$ is an arbitrary function of x .

Conservation of Linear Momentum

- To develop the differential, linear momentum equations we can start with the linear momentum equation

$$\mathbf{F} = \left. \frac{D\mathbf{P}}{Dt} \right|_{\text{sys}}$$

$$\mathbf{P} = \int_{\text{sys}} \mathbf{V} \, dm$$

$$\sum \mathbf{F}_{\text{contents of the control volume}} = \frac{\partial}{\partial t} \int_{\text{cv}} \mathbf{V} \rho \, dV + \int_{\text{cs}} \mathbf{V} \rho \mathbf{V} \cdot \hat{\mathbf{n}} \, dA$$

$$\delta \mathbf{F} = \frac{D(\mathbf{V} \, \delta m)}{Dt}$$

$$\delta \mathbf{F} = \delta m \frac{D\mathbf{V}}{Dt}$$

$$\delta \mathbf{F} = \delta m \, \mathbf{a}$$

Description of Forces Acting on the Differential Element

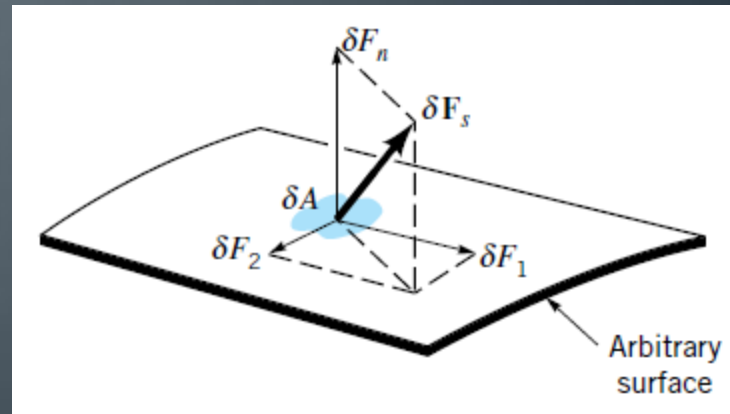
- In general, two types of forces need to be considered: **surface forces**, which act on the surface of the differential element, and **body forces**, which are distributed throughout the element.

$$\delta \mathbf{F}_b = \delta m \mathbf{g}$$

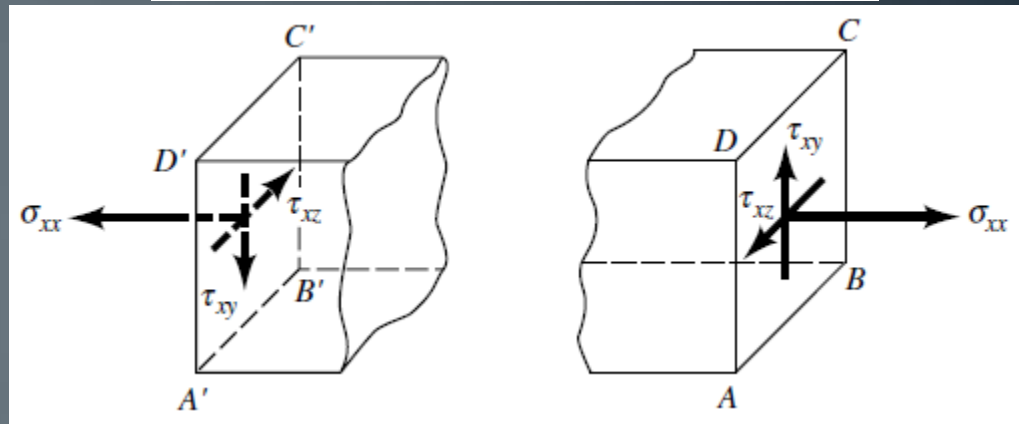
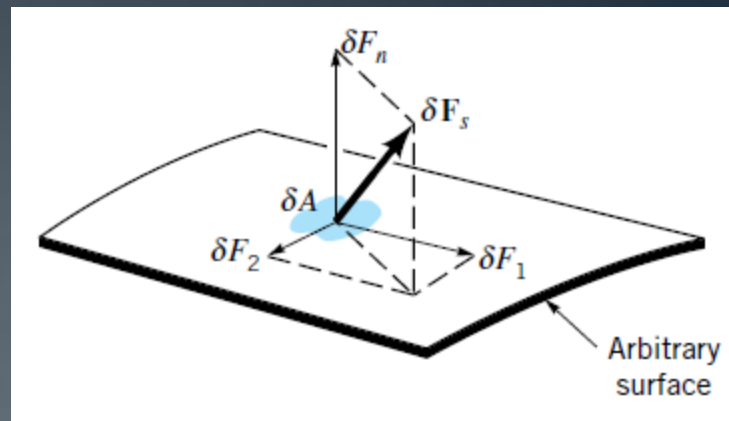
$$\delta F_{bx} = \delta m g_x$$

$$\delta F_{by} = \delta m g_y$$

$$\delta F_{bz} = \delta m g_z$$



Description of Forces Acting on the Differential Element

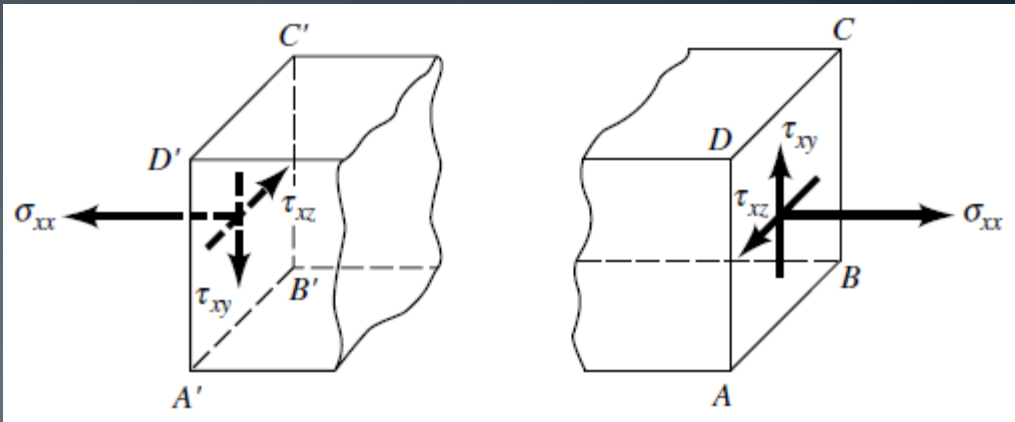


$$\sigma_n = \lim_{\delta A \rightarrow 0} \frac{\delta F_n}{\delta A}$$

$$\tau_1 = \lim_{\delta A \rightarrow 0} \frac{\delta F_1}{\delta A}$$

$$\tau_2 = \lim_{\delta A \rightarrow 0} \frac{\delta F_2}{\delta A}$$

Description of Forces Acting on the Differential Element



$$\delta F_{sx} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) \delta x \delta y \delta z$$

$$\delta F_{sy} = \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) \delta x \delta y \delta z$$

$$\delta F_{sz} = \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \right) \delta x \delta y \delta z$$

$$\delta \mathbf{F}_s = \delta F_{sx} \hat{\mathbf{i}} + \delta F_{sy} \hat{\mathbf{j}} + \delta F_{sz} \hat{\mathbf{k}}$$

$$\delta \mathbf{F} = \delta \mathbf{F}_s + \delta \mathbf{F}_b$$

Equations of Motion

- The expressions for the body and surface forces can now be used to develop the equations of motion.

$$\delta F_x = \delta m a_x$$

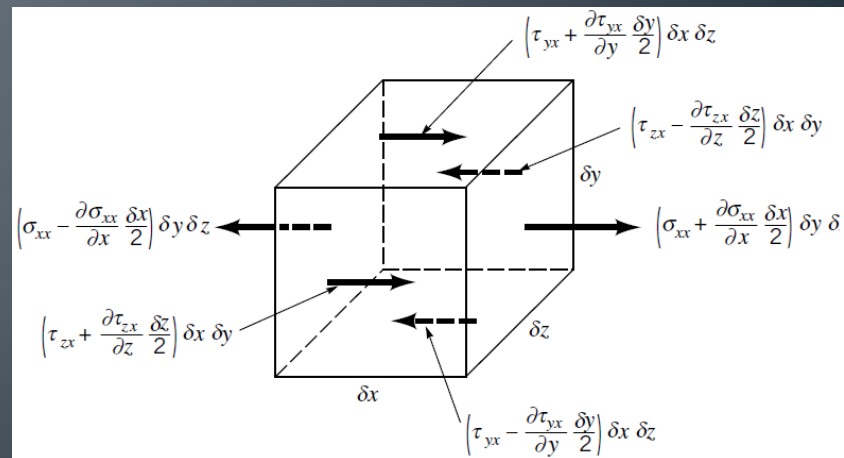
$$\delta F_y = \delta m a_y$$

$$\delta F_z = \delta m a_z$$

$$\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right)$$

$$\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right)$$

$$\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right)$$



Inviscid Flow

- Flow fields in which the shearing stresses are assumed to be negligible are said to be **inviscid**, **nonviscous**, or **frictionless**.

$$-p = \sigma_{xx} = \sigma_{yy} = \sigma_{zz}$$

- Euler's Equations of Motion**

$$\begin{aligned}\rho g_x - \frac{\partial p}{\partial x} &= \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) \\ \rho g_y - \frac{\partial p}{\partial y} &= \rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) \\ \rho g_z - \frac{\partial p}{\partial z} &= \rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right)\end{aligned}$$

Inviscid Flow

- **The Bernoulli Equation**

$$\rho \mathbf{g} - \nabla p = \rho(\mathbf{V} \cdot \nabla)\mathbf{V}$$

$$\mathbf{g} = -g\nabla z$$

$$(\mathbf{V} \cdot \nabla)\mathbf{V} = \frac{1}{2}\nabla(\mathbf{V} \cdot \mathbf{V}) - \mathbf{V} \times (\nabla \times \mathbf{V})$$

$$-\rho g \nabla z - \nabla p = \frac{\rho}{2} \nabla(\mathbf{V} \cdot \mathbf{V}) - \rho \mathbf{V} \times (\nabla \times \mathbf{V})$$

$$\frac{\nabla p}{\rho} + \frac{1}{2} \nabla(V^2) + g \nabla z = \mathbf{V} \times (\nabla \times \mathbf{V})$$

- **The Bernoulli Equation**

$$\frac{\nabla p}{\rho} \cdot d\mathbf{s} + \frac{1}{2} \nabla(V^2) \cdot d\mathbf{s} + g \nabla z \cdot d\mathbf{s} = [\mathbf{V} \times (\nabla \times \mathbf{V})] \cdot d\mathbf{s}$$

$$[\mathbf{V} \times (\nabla \times \mathbf{V})] \cdot d\mathbf{s} = 0$$

$$\frac{dp}{\rho} + \frac{1}{2} d(V^2) + g dz = 0$$

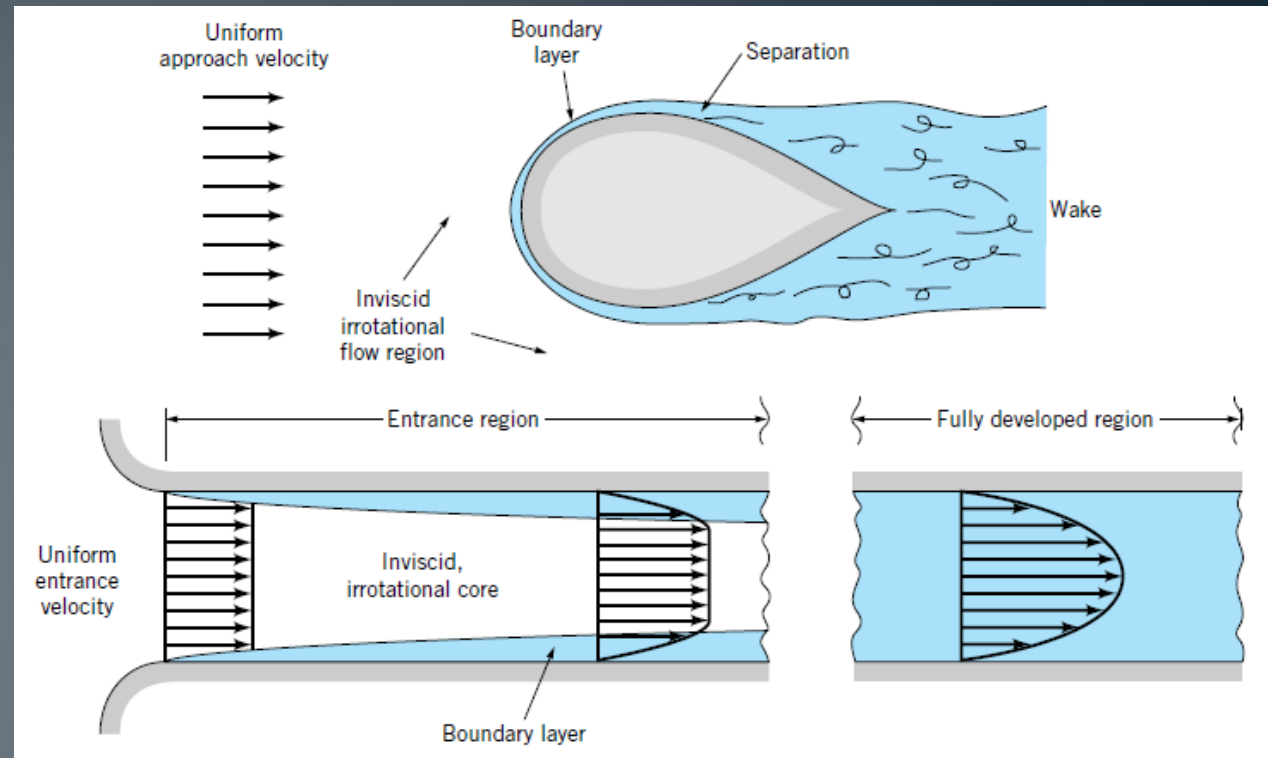
Inviscid Flow

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

- inviscid flow
- steady flow
- incompressible flow
- flow along a streamline

Inviscid Flow

- Irrotational Flow**



$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$$

Inviscid Flow

- The Velocity Potential**

$$u = \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y} \quad w = \frac{\partial \phi}{\partial z}$$

$$\mathbf{V} = \nabla \phi$$

- For an incompressible fluid we know from conservation of mass that

$$\nabla \cdot \mathbf{V} = 0$$

$$\nabla^2 \phi = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

Inviscid Flow

- **The Velocity Potential**
- For some problems it will be convenient to use cylindrical coordinates,

$$\nabla() = \frac{\partial()}{\partial r} \hat{\mathbf{e}}_r + \frac{1}{r} \frac{\partial()}{\partial \theta} \hat{\mathbf{e}}_\theta + \frac{\partial()}{\partial z} \hat{\mathbf{e}}_z$$

$$\nabla\phi = \frac{\partial\phi}{\partial r} \hat{\mathbf{e}}_r + \frac{1}{r} \frac{\partial\phi}{\partial \theta} \hat{\mathbf{e}}_\theta + \frac{\partial\phi}{\partial z} \hat{\mathbf{e}}_z$$

$$\mathbf{V} = v_r \hat{\mathbf{e}}_r + v_\theta \hat{\mathbf{e}}_\theta + v_z \hat{\mathbf{e}}_z$$

$$v_r = \frac{\partial\phi}{\partial r} \quad v_\theta = \frac{1}{r} \frac{\partial\phi}{\partial \theta} \quad v_z = \frac{\partial\phi}{\partial z}$$

- Also, Laplace's equation in cylindrical coordinates is

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial\phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2\phi}{\partial \theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

Some Basic, Plane Potential Flows

- A major advantage of Laplace's equation is that it is a **linear partial differential equation**.
- The practical implication of this result is that if we have certain basic solutions we can combine them to obtain more complicated and interesting solutions.

$$u = \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y}$$

$$v_r = \frac{\partial \phi}{\partial r} \quad v_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta}$$

$$u = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial x}$$

$$v_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \quad v_\theta = -\frac{\partial \psi}{\partial r}$$

Some Basic, Plane Potential Flows

- We know that by defining the velocities in terms of the stream function, conservation of mass is identically satisfied. If we now impose the condition of irrotationality, it follows that

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$
$$\frac{\partial}{\partial y} \left(\frac{\partial \psi}{\partial y} \right) = \frac{\partial}{\partial x} \left(-\frac{\partial \psi}{\partial x} \right)$$
$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$\left. \frac{dy}{dx} \right|_{\text{along } \psi = \text{constant}} = \frac{v}{u}$$
$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = u dx + v dy$$
$$\left. \frac{dy}{dx} \right|_{\text{along } \phi = \text{constant}} = -\frac{u}{v}$$

- Uniform Flow**

Some Basic, Plane Potential Flows

$$\frac{\partial \phi}{\partial x} = U \quad \frac{\partial \phi}{\partial y} = 0$$

$$\phi = Ux + C$$

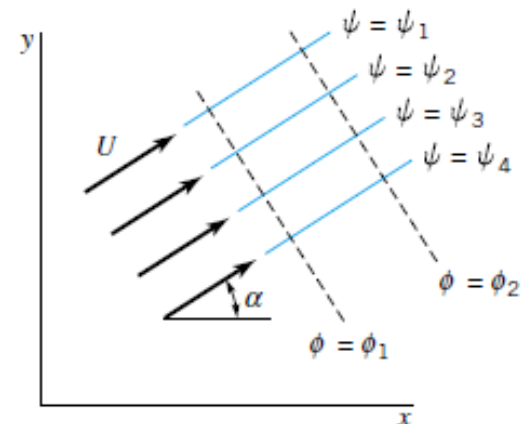
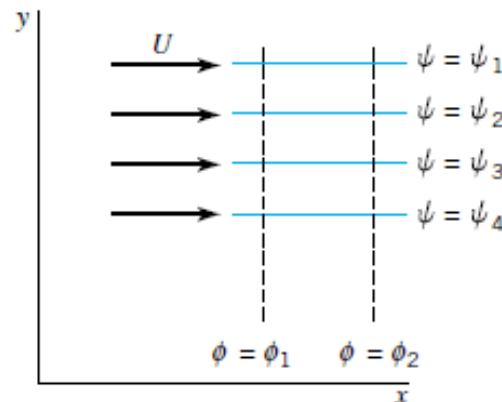
$$\phi = Ux$$

$$\frac{\partial \psi}{\partial y} = U \quad \frac{\partial \psi}{\partial x} = 0$$

$$\psi = Uy$$

$$\phi = U(x \cos \alpha + y \sin \alpha)$$

$$\psi = U(y \cos \alpha - x \sin \alpha)$$



- Source and Sink

Some Basic, Plane Potential Flows

$$(2\pi r)v_r = m$$

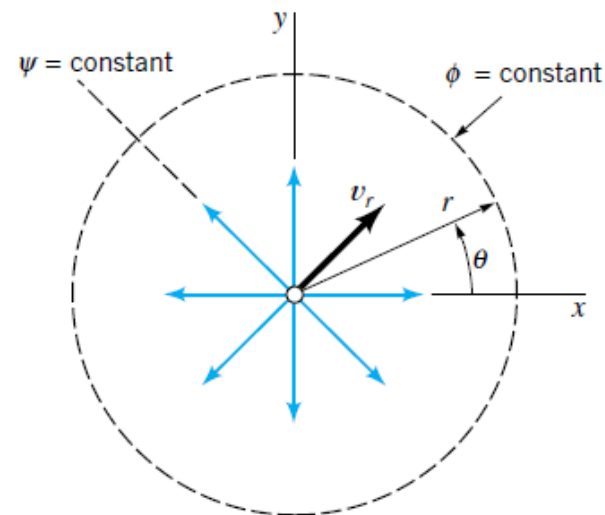
$$v_r = \frac{m}{2\pi r}$$

$$\frac{\partial \phi}{\partial r} = \frac{m}{2\pi r} \quad \frac{1}{r} \frac{\partial \phi}{\partial \theta} = 0$$

$$\phi = \frac{m}{2\pi} \ln r$$

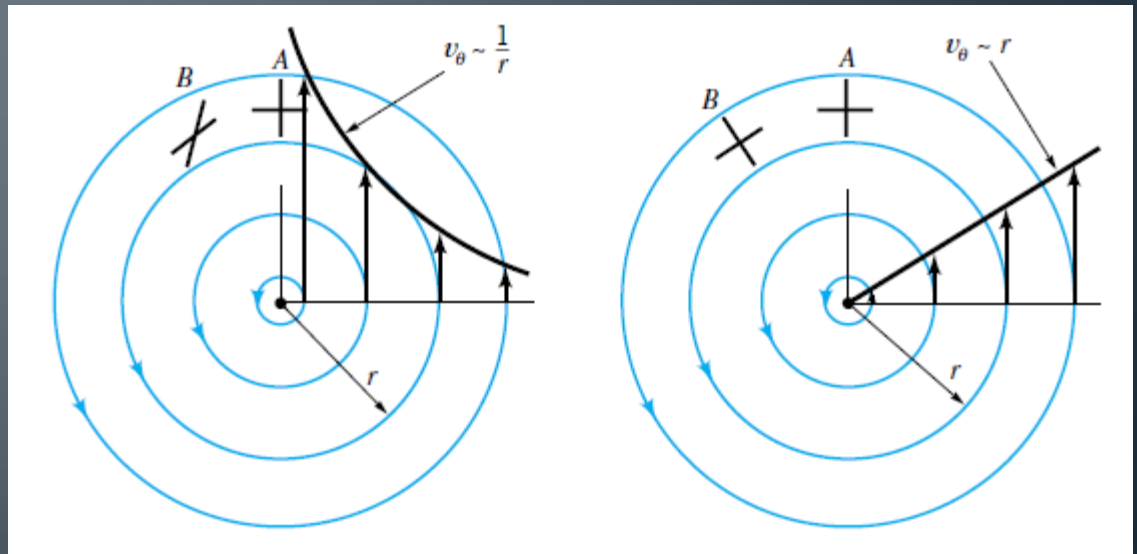
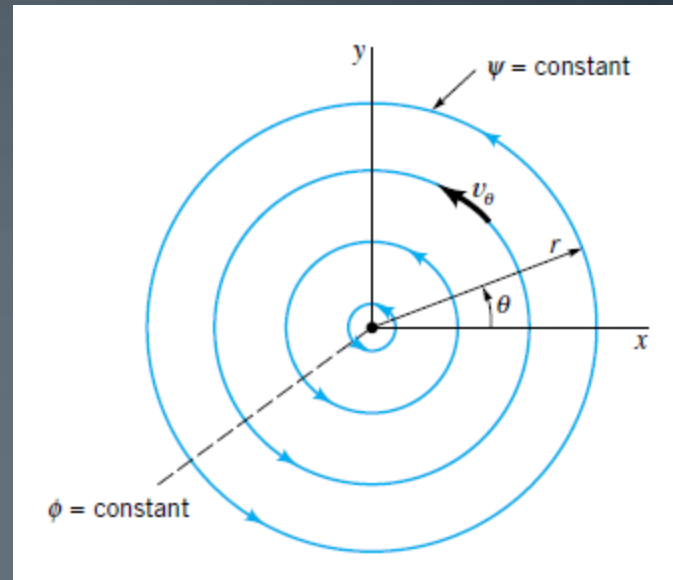
$$\frac{1}{r} \frac{\partial \psi}{\partial \theta} = \frac{m}{2\pi r} \quad \frac{\partial \psi}{\partial r} = 0$$

$$\psi = \frac{m}{2\pi} \theta$$



- **Vortex**

Some Basic, Plane Potential Flows



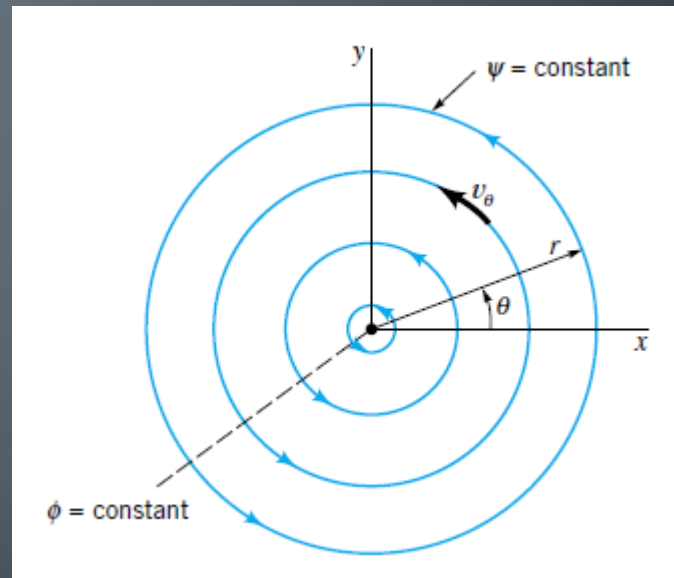
- **Vortex**

Some Basic,
Plane Potential
Flows

$$\phi = K\theta$$

$$\psi = -K \ln r$$

$$v_{\theta} = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = \frac{K}{r}$$



Some Basic, Plane Potential Flows

- **Vortex**
- A mathematical concept commonly associated with vortex motion is that of **circulation**.

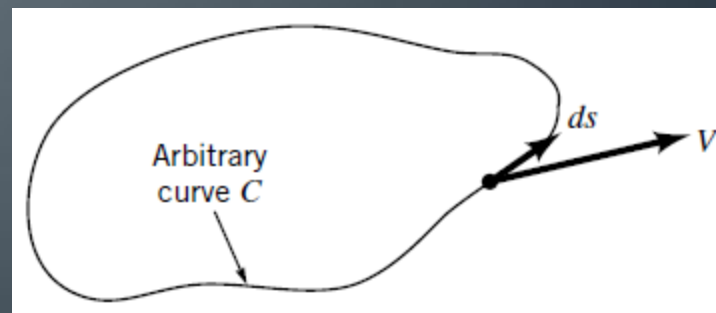
$$\Gamma = \oint_C \mathbf{V} \cdot d\mathbf{s}$$

$$\Gamma = \oint_C d\phi = 0$$

$$\Gamma = \int_0^{2\pi} \frac{K}{r} (r d\theta) = 2\pi K$$

$$\phi = \frac{\Gamma}{2\pi} \theta$$

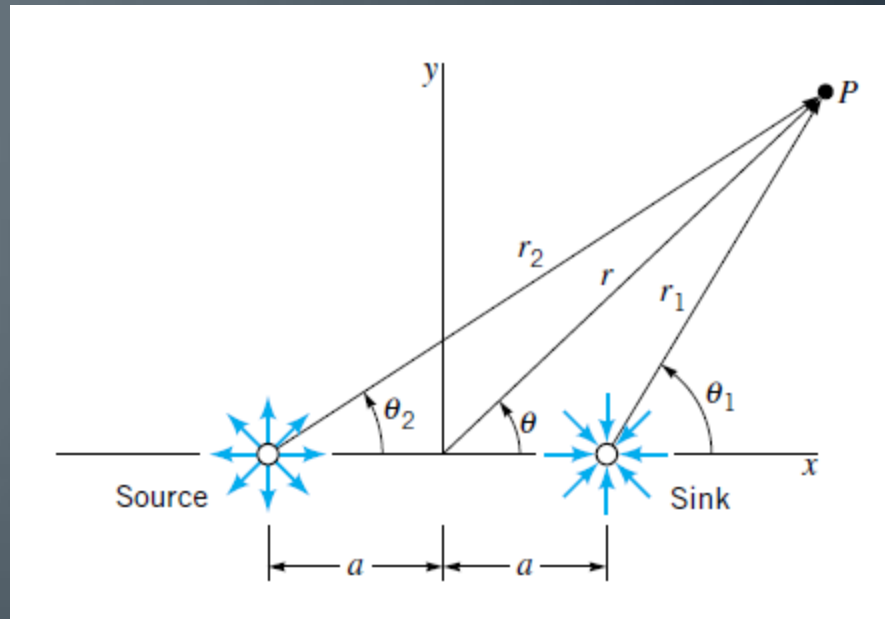
$$\psi = -\frac{\Gamma}{2\pi} \ln r$$



- **Doublet**

Some Basic, Plane Potential Flows

$$\psi = -\frac{m}{2\pi} (\theta_1 - \theta_2)$$



- **Doublet**

$$\psi = -\frac{m}{2\pi} (\theta_1 - \theta_2)$$

$$\tan\left(-\frac{2\pi\psi}{m}\right) = \tan(\theta_1 - \theta_2) = \frac{\tan \theta_1 - \tan \theta_2}{1 + \tan \theta_1 \tan \theta_2}$$

$$\tan \theta_1 = \frac{r \sin \theta}{r \cos \theta - a}$$

$$\tan \theta_2 = \frac{r \sin \theta}{r \cos \theta + a}$$

Some Basic,
Plane Potential
Flows

$$\tan\left(-\frac{2\pi\psi}{m}\right) = \frac{2ar \sin \theta}{r^2 - a^2}$$

$$\psi = -\frac{m}{2\pi} \tan^{-1}\left(\frac{2ar \sin \theta}{r^2 - a^2}\right)$$

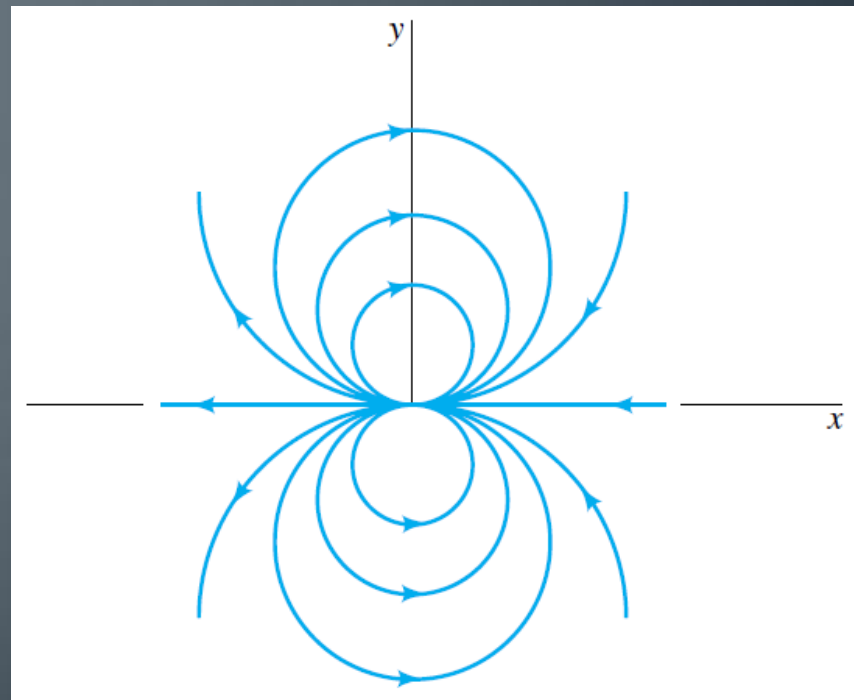
$$\psi = -\frac{m}{2\pi} \frac{2ar \sin \theta}{r^2 - a^2} = -\frac{mar \sin \theta}{\pi(r^2 - a^2)}$$

- **Doublet**

Some Basic,
Plane Potential
Flows

$$\psi = -\frac{K \sin \theta}{r}$$

$$\phi = \frac{K \cos \theta}{r}$$



- Source in a Uniform Stream

Superposition of Basic, Plane Potential Flows

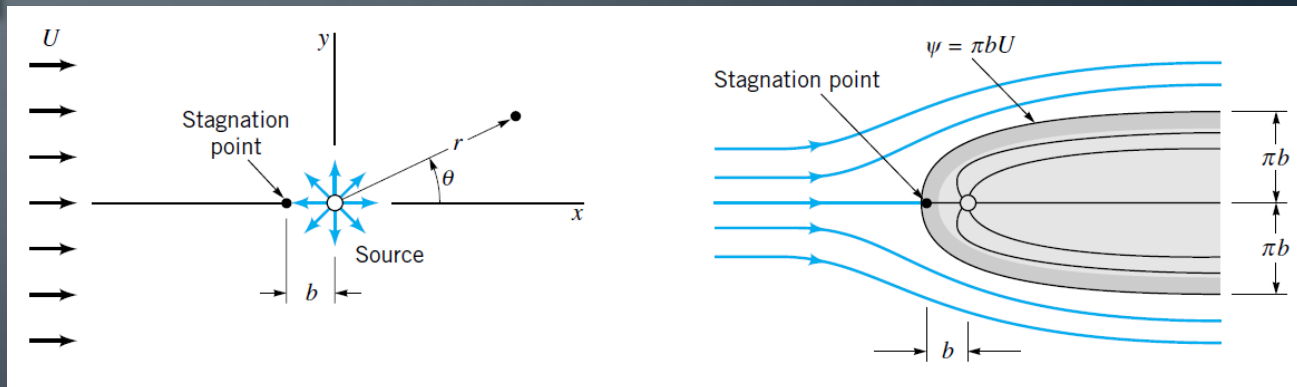
$$\begin{aligned}\psi &= \psi_{\text{uniform flow}} + \psi_{\text{source}} \\ &= Ur \sin \theta + \frac{m}{2\pi} \theta\end{aligned}$$

$$\phi = Ur \cos \theta + \frac{m}{2\pi} \ln r$$

$$v_r = \frac{m}{2\pi r}$$

$$U = \frac{m}{2\pi b}$$

$$b = \frac{m}{2\pi U}$$



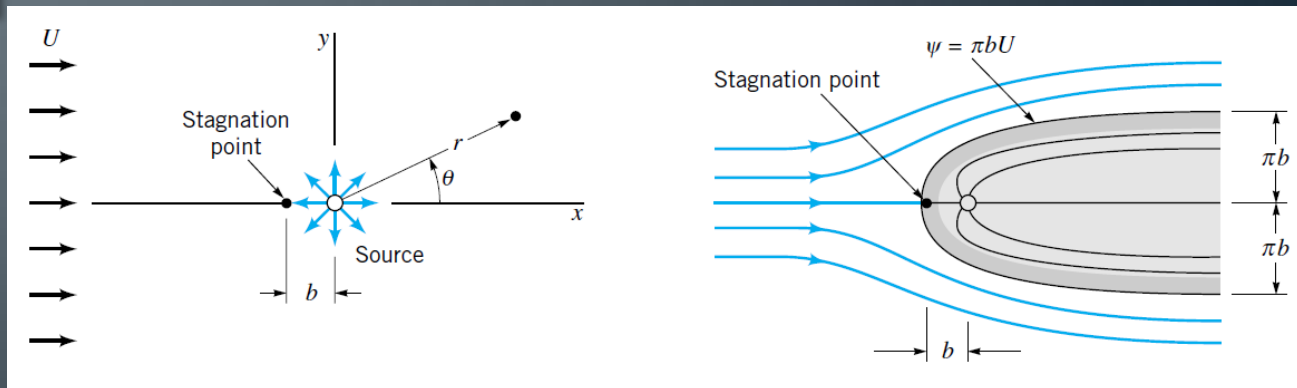
- Source in a Uniform Stream

Superposition of Basic, Plane Potential Flows

$$\psi_{\text{stagnation}} = \frac{m}{2}$$

$$\pi b U = U r \sin \theta + b U \theta$$

$$r = \frac{b(\pi - \theta)}{\sin \theta}$$



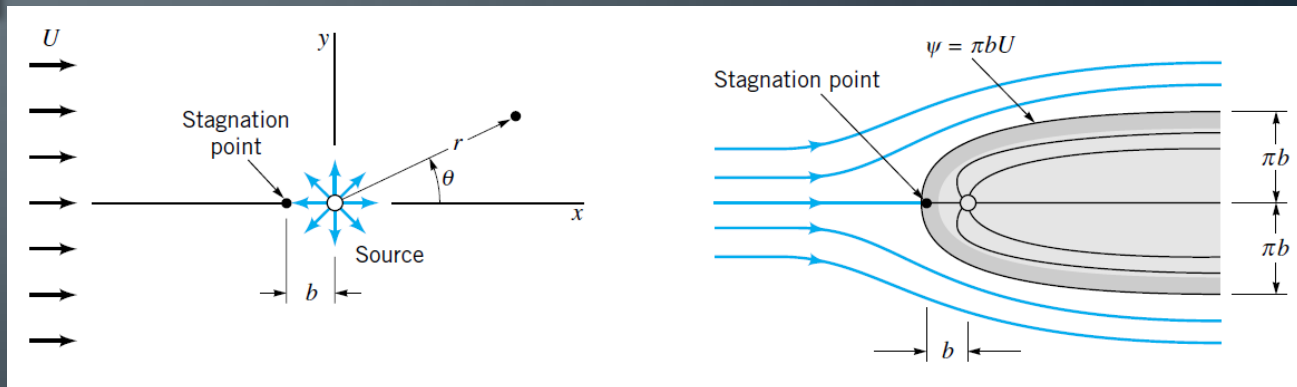
- Source in a Uniform Stream

Superposition of Basic, Plane Potential Flows

$$y = b(\pi - \theta)$$

$$v_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = U \cos \theta + \frac{m}{2\pi r}$$

$$v_\theta = -\frac{\partial \psi}{\partial r} = -U \sin \theta$$



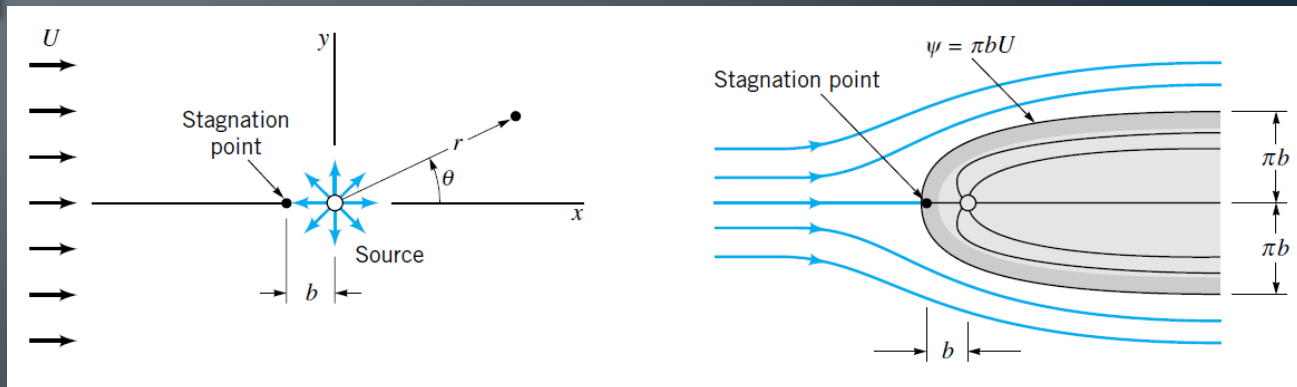
- **Source in a Uniform Stream**

Superposition of Basic, Plane Potential Flows

$$V^2 = v_r^2 + v_\theta^2 = U^2 + \frac{Um \cos \theta}{\pi r} + \left(\frac{m}{2\pi r} \right)^2$$

$$V^2 = U^2 \left(1 + 2 \frac{b}{r} \cos \theta + \frac{b^2}{r^2} \right)$$

$$p_0 + \frac{1}{2}\rho U^2 = p + \frac{1}{2}\rho V^2$$



- Rankine Ovals

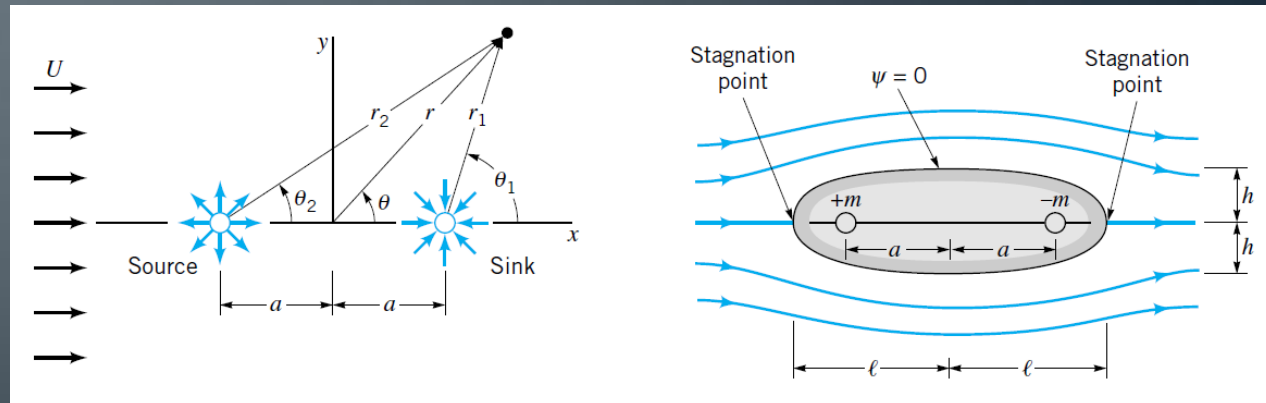
$$\psi = Ur \sin \theta - \frac{m}{2\pi} (\theta_1 - \theta_2)$$

$$\phi = Ur \cos \theta - \frac{m}{2\pi} (\ln r_1 - \ln r_2)$$

Superposition of
Basic, Plane
Potential Flows

$$\psi = Ur \sin \theta - \frac{m}{2\pi} \tan^{-1} \left(\frac{2ar \sin \theta}{r^2 - a^2} \right)$$

$$\psi = Uy - \frac{m}{2\pi} \tan^{-1} \left(\frac{2ay}{x^2 + y^2 - a^2} \right)$$



- Rankine Ovals

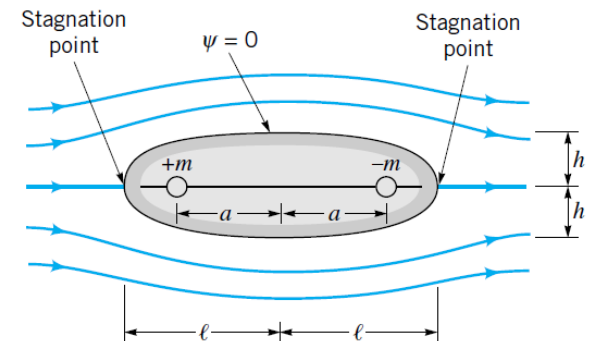
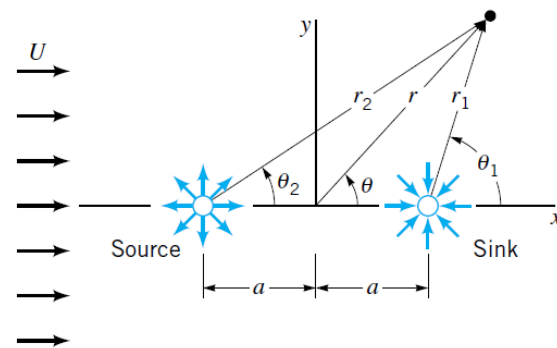
Superposition of Basic, Plane Potential Flows

$$\ell = \left(\frac{ma}{\pi U} + a^2 \right)^{1/2}$$

$$\frac{\ell}{a} = \left(\frac{m}{\pi Ua} + 1 \right)^{1/2}$$

$$h = \frac{h^2 - a^2}{2a} \tan \frac{2\pi U h}{m}$$

$$\frac{h}{a} = \frac{1}{2} \left[\left(\frac{h}{a} \right)^2 - 1 \right] \tan \left[2 \left(\frac{\pi Ua}{m} \right) \frac{h}{a} \right]$$

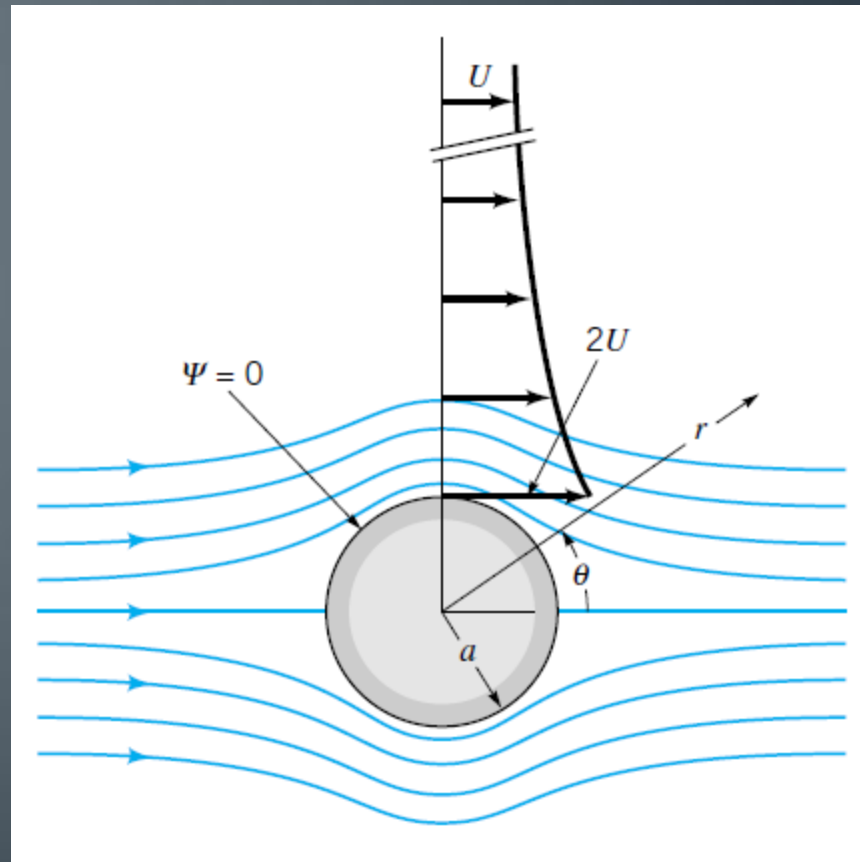


- **Flow Around a Circular Cylinder**

$$\psi = Ur \sin \theta - \frac{K \sin \theta}{r}$$

$$\phi = Ur \cos \theta + \frac{K \cos \theta}{r}$$

Superposition of
Basic, Plane
Potential Flows



- **Flow Around a Circular Cylinder**

Superposition of
Basic, Plane
Potential Flows

$$\psi = \left(U - \frac{K}{r^2} \right) r \sin \theta$$

$$U - \frac{K}{a^2} = 0$$

$$\psi = Ur \left(1 - \frac{a^2}{r^2} \right) \sin \theta$$

$$\phi = Ur \left(1 + \frac{a^2}{r^2} \right) \cos \theta$$

$$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = U \left(1 - \frac{a^2}{r^2} \right) \cos \theta$$

$$v_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = -U \left(1 + \frac{a^2}{r^2} \right) \sin \theta$$

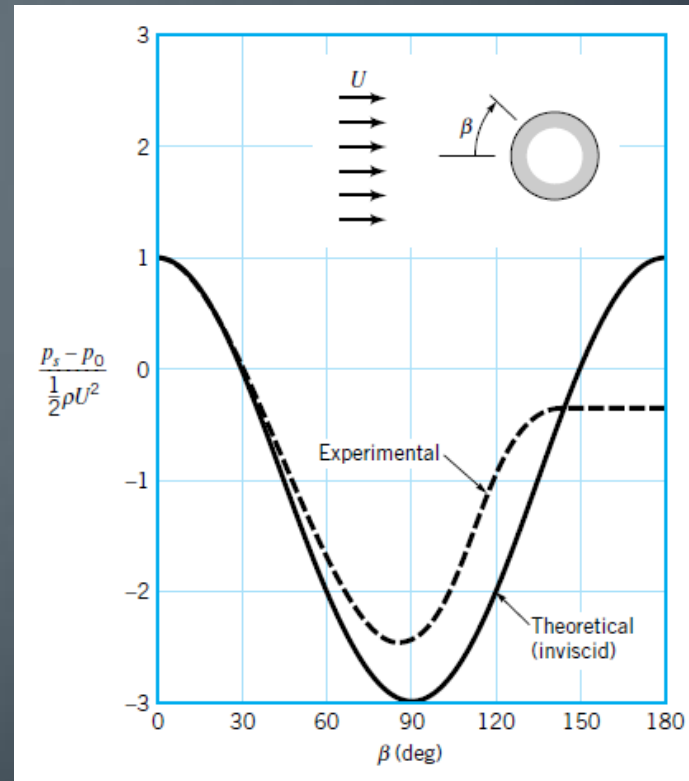
- Flow Around a Circular Cylinder

$$v_{\theta s} = -2U \sin \theta$$

$$p_0 + \frac{1}{2}\rho U^2 = p_s + \frac{1}{2}\rho v_{\theta s}^2$$

$$p_s = p_0 + \frac{1}{2}\rho U^2(1 - 4 \sin^2 \theta)$$

Superposition of
Basic, Plane
Potential Flows

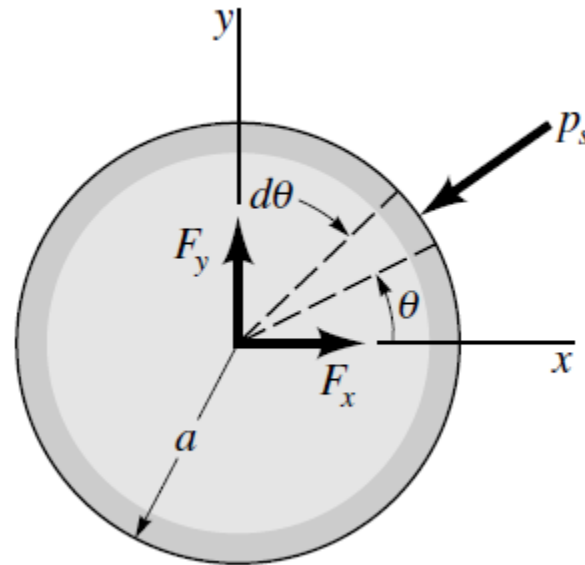


- **D'Alembert's paradox**

Superposition of
Basic, Plane
Potential Flows

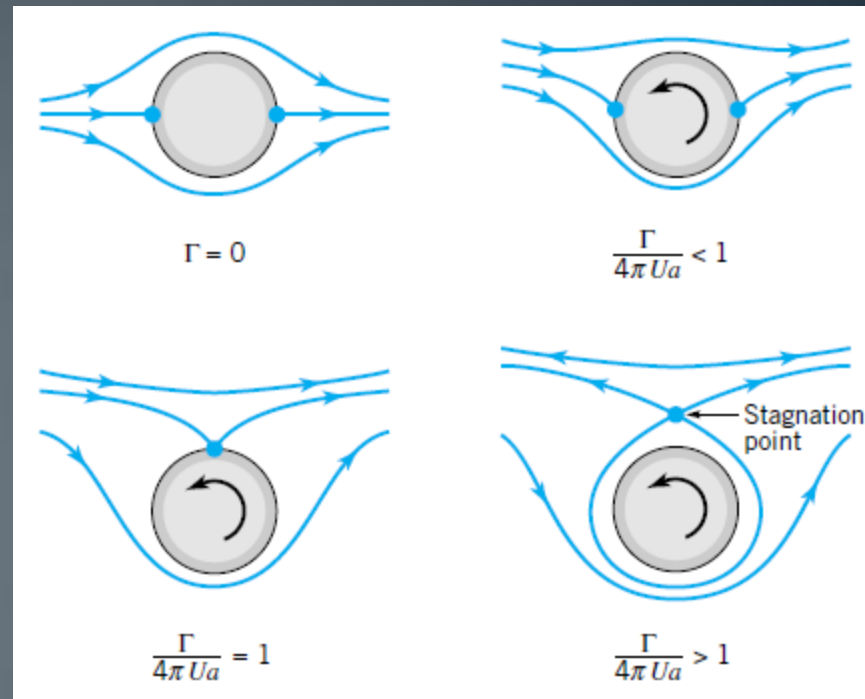
$$F_x = - \int_0^{2\pi} p_s \cos \theta a d\theta$$

$$F_y = - \int_0^{2\pi} p_s \sin \theta a d\theta$$



- Flow Around a Circular Cylinder

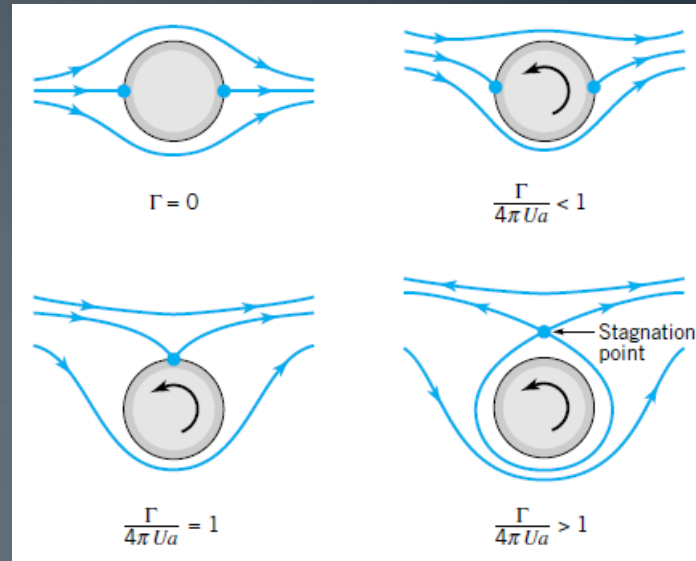
Superposition of Basic, Plane Potential Flows



$$\psi = Ur \left(1 - \frac{a^2}{r^2} \right) \sin \theta - \frac{\Gamma}{2\pi} \ln r$$

$$\phi = Ur \left(1 + \frac{a^2}{r^2} \right) \cos \theta + \frac{\Gamma}{2\pi} \theta$$

- Flow Around a Circular Cylinder



Superposition of
Basic, Plane
Potential Flows

$$p_0 + \frac{1}{2} \rho U^2 = p_s + \frac{1}{2} \rho \left(-2U \sin \theta + \frac{\Gamma}{2\pi a} \right)^2$$

$$p_s = p_0 + \frac{1}{2} \rho U^2 \left(1 - 4 \sin^2 \theta + \frac{2\Gamma \sin \theta}{\pi a U} - \frac{\Gamma^2}{4\pi^2 a^2 U^2} \right)$$

$$F_x = 0$$

$$F_y = -\rho U \Gamma$$

EXAMPLE 6.6



V6.2 Vortex in a beaker

A liquid drains from a large tank through a small opening as illustrated in Fig. E6.6. A vortex forms whose velocity distribution away from the tank opening can be approximated as that of a free vortex having a velocity potential

$$\phi = \frac{\Gamma}{2\pi} \theta$$

Determine an expression relating the surface shape to the strength of the vortex as specified by the circulation Γ .

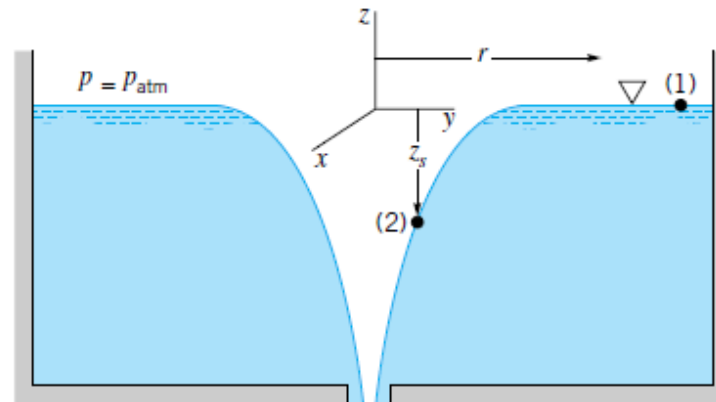


FIGURE E6.6

SOLUTION

Since the free vortex represents an irrotational flow field, the Bernoulli equation

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2$$

can be written between any two points. If the points are selected at the free surface, $p_1 = p_2 = 0$, so that

$$\frac{V_1^2}{2g} = z_s + \frac{V_2^2}{2g} \quad (1)$$

where the free surface elevation, z_s , is measured relative to a datum passing through point (1).

The velocity is given by the equation

$$v_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = \frac{\Gamma}{2\pi r}$$

We note that far from the origin at point (1), $V_1 = v_\theta \approx 0$ so that Eq. 1 becomes

$$z_s = -\frac{\Gamma^2}{8\pi^2 r^2 g} \quad (\text{Ans})$$

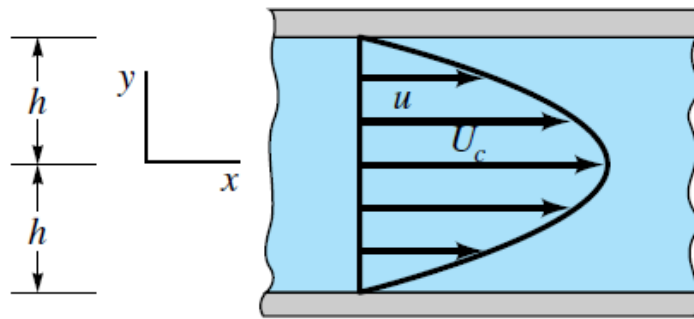
which is the desired equation for the surface profile. The negative sign indicates that the surface falls as the origin is approached as shown in Fig. E6.6. This solution is not valid very near the origin since the predicted velocity becomes excessively large as the origin is approached.

Problems

6.31 It is known that the velocity distribution for two-dimensional flow of a viscous fluid between wide parallel plates (Fig. P6.31) is parabolic; that is,

$$u = U_c \left[1 - \left(\frac{y}{h} \right)^2 \right]$$

with $v = 0$. Determine, if possible, the corresponding stream function and velocity potential.



Problems

To determine the stream function let

$$u = \frac{\partial \psi}{\partial y} = U_c \left[1 - \left(\frac{y}{h} \right)^2 \right]$$

and integrate with respect to y to obtain

$$\int d\psi = \int U_c \left[1 - \left(\frac{y}{h} \right)^2 \right] dy$$

or

$$\psi = U_c \left[y - \frac{y^3}{3h^2} \right] + f_1(x)$$

Since $v = -\frac{\partial \psi}{\partial x} = 0$, ψ is not a function of x so that

$$\psi = U_c y \left[1 - \frac{1}{3} \left(\frac{y}{h} \right)^2 \right] + C$$

where C is an arbitrary constant.

To determine the velocity potential let

$$u = \frac{\partial \phi}{\partial x} = U_c \left[1 - \left(\frac{y}{h} \right)^2 \right]$$

and integrate with respect to x to obtain

$$\int d\phi = \int U_c \left[1 - \left(\frac{y}{h} \right)^2 \right] dx$$

or

$$\phi = U_c \left[x - \left(\frac{y}{h} \right)^2 x \right] + f_2(y)$$

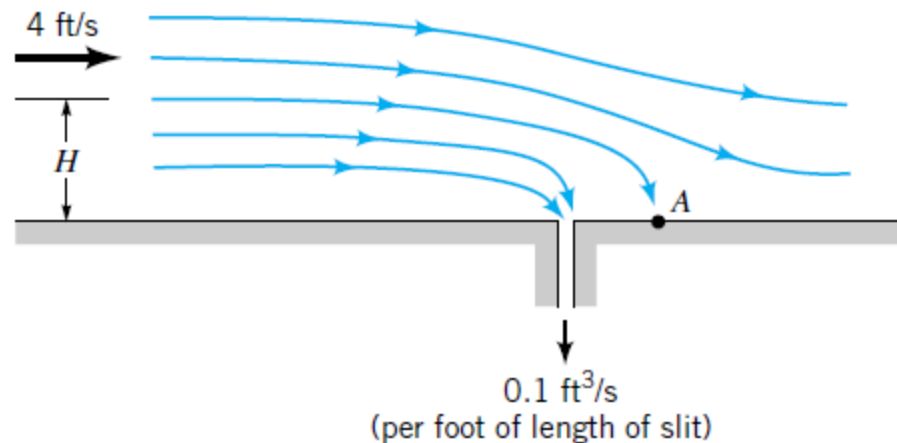
However,

$$v = \frac{\partial \phi}{\partial y} = 0 = -\frac{2 U_c x y}{h^2} + \frac{\partial f_2(y)}{\partial y}$$

and this relationship cannot be satisfied for all values of x and y . Thus, there is not a velocity potential that describes this flow (the flow is not irrotational).

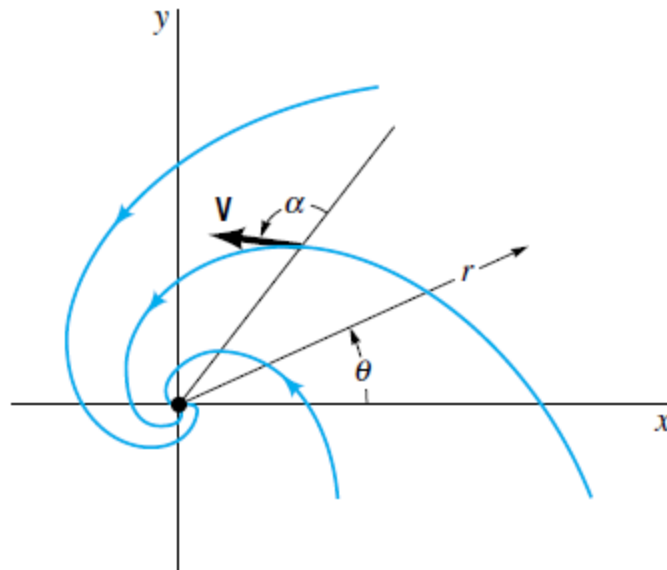
Problems

6.47 Water flows over a flat surface at 4 ft/s, as shown in Fig. P6.47. A pump draws off water through a narrow slit at a volume rate of $0.1 \text{ ft}^3/\text{s}$ per foot length of the slit. Assume that the fluid is incompressible and inviscid and can be represented by the combination of a uniform flow and a sink. Locate the stagnation point on the wall (point A) and determine the equation for the stagnation streamline. How far above the surface, H , must the fluid be so that it does not get sucked into the slit?



Problems

6.49 The velocity potential for a spiral vortex flow is given by $\phi = (\Gamma/2\pi) \theta - (m/2\pi) \ln r$, where Γ and m are constants. Show that the angle, α , between the velocity vector and the radial direction is constant throughout the flow field (see Fig. P6.49).



Problems

For the velocity potential given,

$$v_r = \frac{\partial \phi}{\partial r} = -\frac{m}{2\pi r} \quad v_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = \frac{\Gamma}{2\pi r}$$

Since $\vec{V} \cdot \hat{e}_r = |\vec{V}| \cos \alpha$

and $\vec{V} = v_r \hat{e}_r + v_\theta \hat{e}_\theta$

then
$$\cos \alpha = \frac{\vec{V} \cdot \hat{e}_r}{|\vec{V}|} = \frac{v_r}{\sqrt{v_r^2 + v_\theta^2}}$$

$$= \frac{1}{\sqrt{1 + \left(\frac{v_\theta}{v_r}\right)^2}} = \frac{1}{\sqrt{1 + \frac{\left(\frac{\Gamma}{2\pi r}\right)^2}{\left(-\frac{m}{2\pi r}\right)^2}}}$$

$$= \frac{1}{\sqrt{1 + \left(\frac{\Gamma}{m}\right)^2}}$$

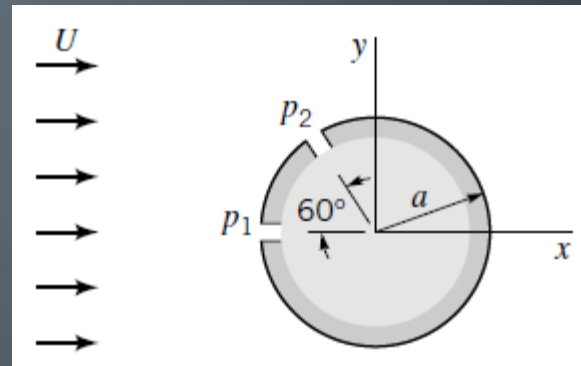
Thus, for a given Γ and m the angle α is a constant.

Problems

6.59 Assume that the flow around the long, circular cylinder of Fig. P6.59 is nonviscous and incompressible. Two pressures, p_1 and p_2 , are measured on the surface of the cylinder, as illustrated. It is proposed that the free-stream velocity, U , can be related to the pressure difference $\Delta p = p_1 - p_2$ by the equation

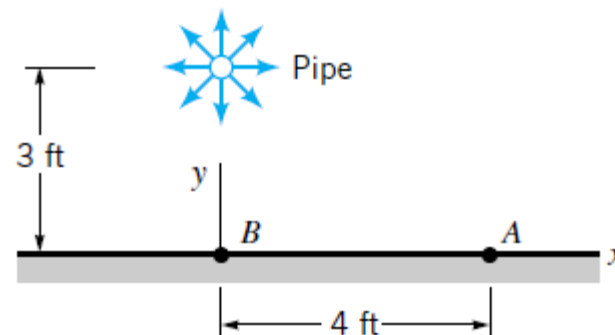
$$U = C \sqrt{\frac{\Delta p}{\rho}}$$

where ρ is the fluid density. Determine the value of the constant C . Neglect body forces.



Problems

6.67 A long, porous pipe runs parallel to a horizontal plane surface, as shown in Fig. P6.67. The longitudinal axis of the pipe is perpendicular to the plane of the paper. Water flows radially from the pipe at a rate of $0.5 \pi \text{ ft}^3/\text{s}$ per foot of pipe. Determine the difference in pressure (in lb/ft^2) between point B and point A . The flow from the pipe may be approximated by a two-dimensional source. *Hint:* To develop the stream function or velocity potential for this type of flow, place (symmetrically) another equal source on the other side of the wall. With this combination there is no flow across the x axis, and this axis can be replaced with a solid boundary. This technique is called the *method of images*.



■ FIGURE P6.67

Problems

For a source,

$$\phi = \frac{m}{2\pi} \ln r = \frac{m}{4\pi} \ln r^2$$

where r is measured from the source. With the coordinate system shown in figure

$$r^2 = x^2 + (y-3)^2 \quad (\text{for upper source})$$

and

$$r^2 = x^2 + (y+3)^2 \quad (\text{for lower source})$$

so that for the combined sources

$$\phi = \frac{m}{4\pi} \left\{ \ln [x^2 + (y-3)^2] + \ln [x^2 + (y+3)^2] \right\}$$

Since,

$$u = \frac{\partial \phi}{\partial x}$$

and

$$\frac{\partial}{\partial x} \ln [x^2 + (y-3)^2] = \frac{2x}{x^2 + (y-3)^2}$$

$$\frac{\partial}{\partial x} \ln [x^2 + (y+3)^2] = \frac{2x}{x^2 + (y+3)^2}$$

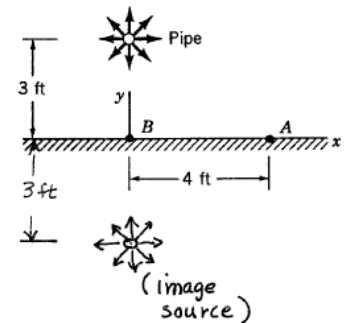
it follows that

$$u = \frac{m}{4\pi} \left[\frac{2x}{x^2 + (y-3)^2} + \frac{2x}{x^2 + (y+3)^2} \right]$$

Along the wall, $y=0$, $v=0$ and therefore

$$V_w = u = \frac{m}{4\pi} \left(\frac{4x}{x^2 + 9} \right)$$

(cont.)



Problems

(cont.)

At point A, $x = 4 \text{ ft}$, and with $m = 0.5\pi \frac{\text{ft}^2}{\text{s}}$,

$$V_{wA} = \frac{0.5\pi \frac{\text{ft}^2}{\text{s}}}{4\pi} \left[\frac{4(4 \text{ ft})}{(4 \text{ ft})^2 + 9 \text{ ft}^2} \right] = \frac{2}{25} \frac{\text{ft}}{\text{s}}$$

At point B, $x = 0$, and

$$V_{wB} = 0$$

Thus, from the Bernoulli equation

$$\frac{p_B}{\gamma} + \frac{V_{wB}^2}{2g} = \frac{p_A}{\gamma} + \frac{V_{wA}^2}{2g}$$

or

$$\begin{aligned} p_B - p_A &= \frac{1}{2} \frac{\gamma}{g} V_{wA}^2 \\ &= \frac{(62.4 \frac{\text{lb}}{\text{ft}^3})}{2 (32.2 \frac{\text{ft}}{\text{s}^2})} \left(\frac{2}{25} \frac{\text{ft}}{\text{s}} \right)^2 = \underline{\underline{0.00620 \text{ psf}}} \end{aligned}$$

Viscous Flow

- **Stress-Deformation Relationships**
- For incompressible Newtonian fluids it is known that the stresses are linearly related to the rates of deformation and can be expressed in Cartesian coordinates as

$$\sigma_{xx} = -p + 2\mu \frac{\partial u}{\partial x}$$

$$\sigma_{yy} = -p + 2\mu \frac{\partial v}{\partial y}$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial w}{\partial z}$$

$$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

$$\tau_{yz} = \tau_{zy} = \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)$$

$$\tau_{zx} = \tau_{xz} = \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)$$

- where p is the pressure, the negative of the average of the three normal stresses

Viscous Flow

- **Stress-Deformation Relationships**
- In cylindrical polar coordinates the stresses for incompressible Newtonian fluids are expressed as

$$\sigma_{rr} = -p + 2\mu \frac{\partial v_r}{\partial r}$$

$$\sigma_{\theta\theta} = -p + 2\mu \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right)$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial v_z}{\partial z}$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu \left[r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right]$$

$$\tau_{\theta z} = \tau_{z\theta} = \mu \left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r} \frac{\partial v_z}{\partial \theta} \right)$$

$$\tau_{zr} = \tau_{rz} = \mu \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right)$$

Viscous Flow

- **The Navier–Stokes Equations**
- The stresses as defined in the preceding section can be substituted into the differential equations of motion and simplified by using the continuity equation to obtain

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

$$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \rho g_y + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)$$

$$\rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \rho g_z + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)$$

Viscous Flow

- **The Navier–Stokes Equations**
- In terms of cylindrical polar coordinates, the Navier–Stokes equation can be written as

$$\begin{aligned} \rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) \\ = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \rho g_\theta + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_\theta}{\partial r} \right) - \frac{v_\theta}{r^2} + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2} \right] \end{aligned}$$

$$\begin{aligned} \rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) \\ = -\frac{\partial p}{\partial r} + \rho g_r + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_r}{\partial r} \right) - \frac{v_r}{r^2} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right] \end{aligned}$$

$$\begin{aligned} \rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) \\ = -\frac{\partial p}{\partial z} + \rho g_z + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right] \end{aligned}$$

Viscous Flow

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$$\begin{aligned} \rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) \\ = -\frac{\partial p}{\partial r} + \rho g_r + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_r}{\partial r} \right) - \frac{v_r}{r^2} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right] \end{aligned}$$

$$\begin{aligned} \rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) \\ = -\frac{\partial p}{\partial z} + \rho g_z + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right] \end{aligned}$$

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$$\tau_{yz} = \tau_{zy} = \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)$$

$$\tau_{zx} = \tau_{xz} = \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)$$

- where p is the pressure, the negative of the average of the three normal stresses

Viscous Flow

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- In cylindrical polar coordinates the stresses for incompressible Newtonian fluids are expressed as

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$$\sigma_{zz} = -p + 2\mu \frac{\partial v_z}{\partial z}$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu \left[r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right]$$

$$\tau_{\theta z} = \tau_{z\theta} = \mu \left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r} \frac{\partial v_z}{\partial \theta} \right)$$

$$\tau_{zr} = \tau_{rz} = \mu \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right)$$

Viscous Flow

- **The Navier–Stokes Equations**
- The stresses as defined in the preceding section can be substituted into the differential equations of motion and simplified by using the continuity equation to obtain

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

$$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \rho g_y + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)$$

$$\rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \rho g_z + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)$$

$$\rho \left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{V}$$

Viscous Flow

- **The Navier–Stokes Equations**
- In terms of cylindrical polar coordinates, the Navier–Stokes equation can be written as

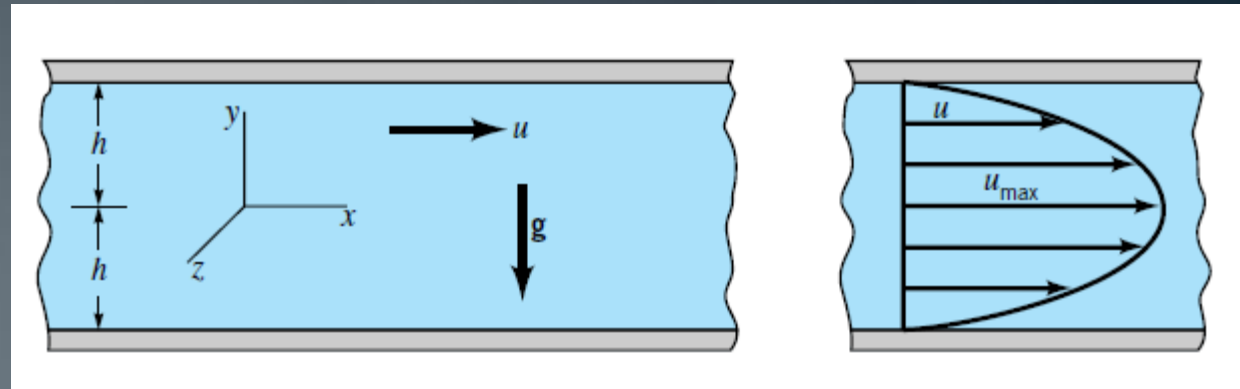
$$\begin{aligned} \rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) \\ = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \rho g_\theta + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_\theta}{\partial r} \right) - \frac{v_\theta}{r^2} + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2} \right] \end{aligned}$$

$$\begin{aligned} \rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) \\ = -\frac{\partial p}{\partial r} + \rho g_r + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_r}{\partial r} \right) - \frac{v_r}{r^2} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right] \end{aligned}$$

$$\begin{aligned} \rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) \\ = -\frac{\partial p}{\partial z} + \rho g_z + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right] \end{aligned}$$

- Steady, Laminar Flow Between Fixed Parallel Plates

Some Simple
Solutions for
Viscous,
Incompressible
Fluids



$$0 = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial y^2} \right)$$

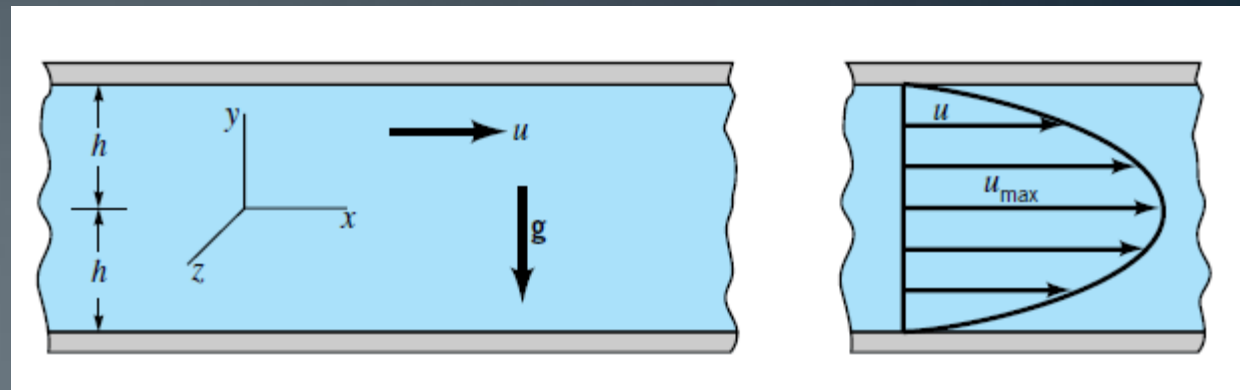
$$0 = -\frac{\partial p}{\partial y} - \rho g$$

$$0 = -\frac{\partial p}{\partial z}$$

$$p = -\rho g y + f_1(x)$$

- Steady, Laminar Flow Between Fixed Parallel Plates

Some Simple
Solutions for
Viscous,
Incompressible
Fluids



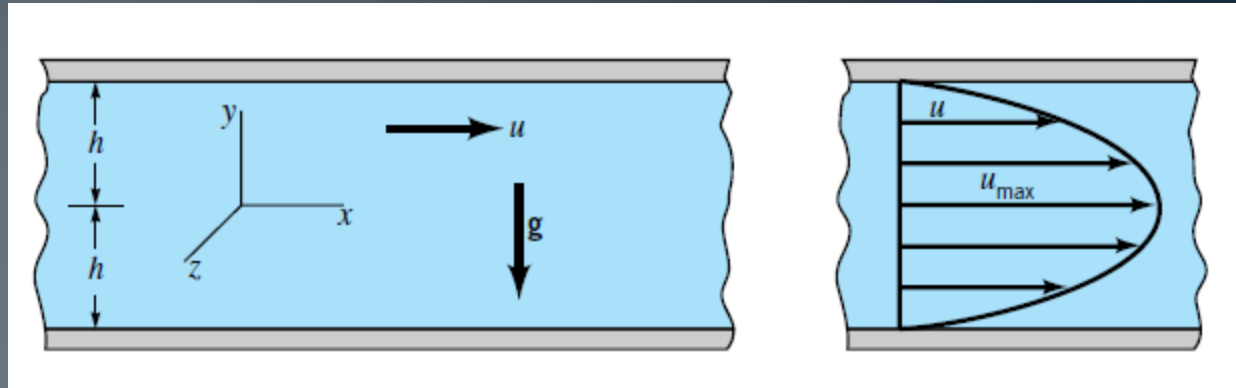
$$\frac{d^2u}{dy^2} = \frac{1}{\mu} \frac{\partial p}{\partial x}$$

$$\frac{du}{dy} = \frac{1}{\mu} \left(\frac{\partial p}{\partial x} \right) y + c_1$$

$$u = \frac{1}{2\mu} \left(\frac{\partial p}{\partial x} \right) y^2 + c_1 y + c_2$$

- Steady, Laminar Flow Between Fixed Parallel Plates

Some Simple
Solutions for
Viscous,
Incompressible
Fluids



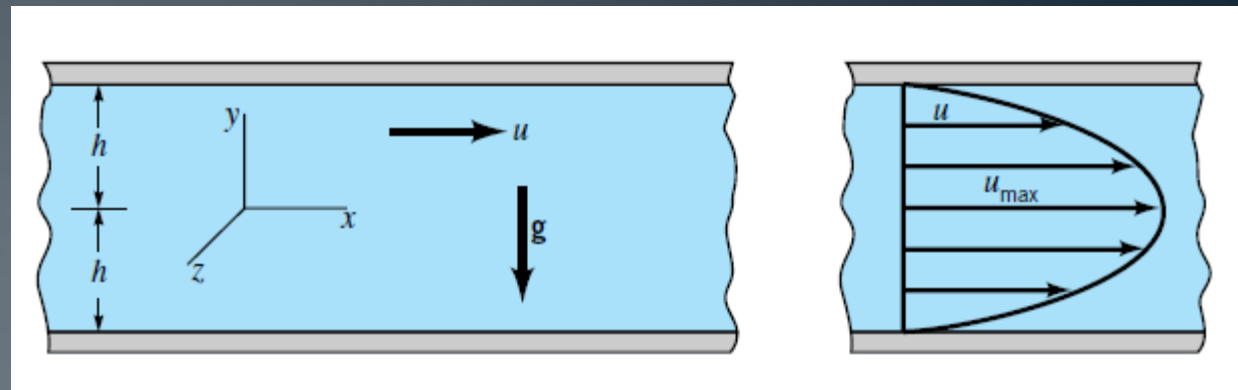
$$c_1 = 0$$

$$c_2 = -\frac{1}{2\mu} \left(\frac{\partial p}{\partial x} \right) h^2$$

$$u = \frac{1}{2\mu} \left(\frac{\partial p}{\partial x} \right) (y^2 - h^2)$$

- Steady, Laminar Flow Between Fixed Parallel Plates

Some Simple
Solutions for
Viscous,
Incompressible
Fluids



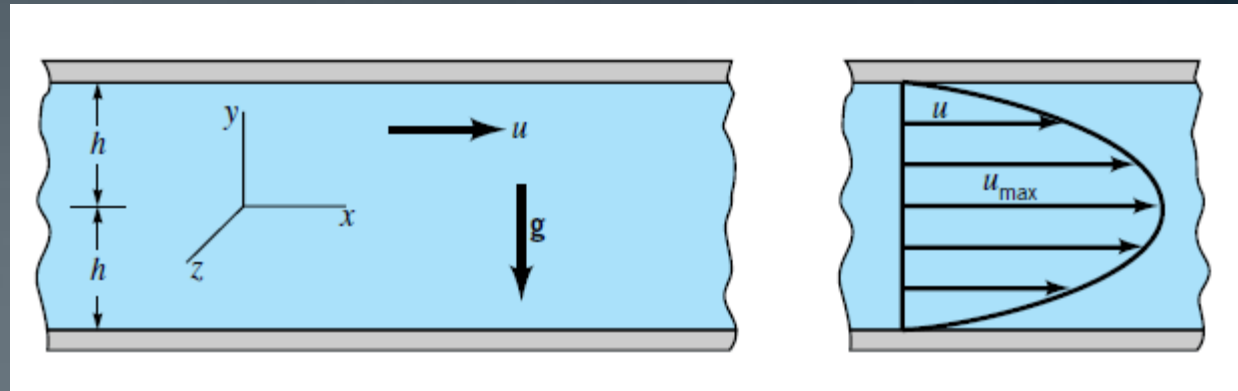
$$q = \int_{-h}^h u \, dy = \int_{-h}^h \frac{1}{2\mu} \left(\frac{\partial p}{\partial x} \right) (y^2 - h^2) \, dy$$

$$q = -\frac{2h^3}{3\mu} \left(\frac{\partial p}{\partial x} \right)$$

$$\frac{\Delta p}{\ell} = -\frac{\partial p}{\partial x}$$

$$q = \frac{2h^3 \Delta p}{3\mu \ell}$$

- Steady, Laminar Flow Between Fixed Parallel Plates



Some Simple
Solutions for
Viscous,
Incompressible
Fluids

$$V = \frac{h^2 \Delta p}{3\mu \ell}$$

$$u_{\max} = -\frac{h^2}{2\mu} \left(\frac{\partial p}{\partial x} \right)$$

$$u_{\max} = \frac{3}{2} V$$

$$f_1(x) = \left(\frac{\partial p}{\partial x} \right) x + p_0$$

$$p = -\rho g y + \left(\frac{\partial p}{\partial x} \right) x + p_0$$

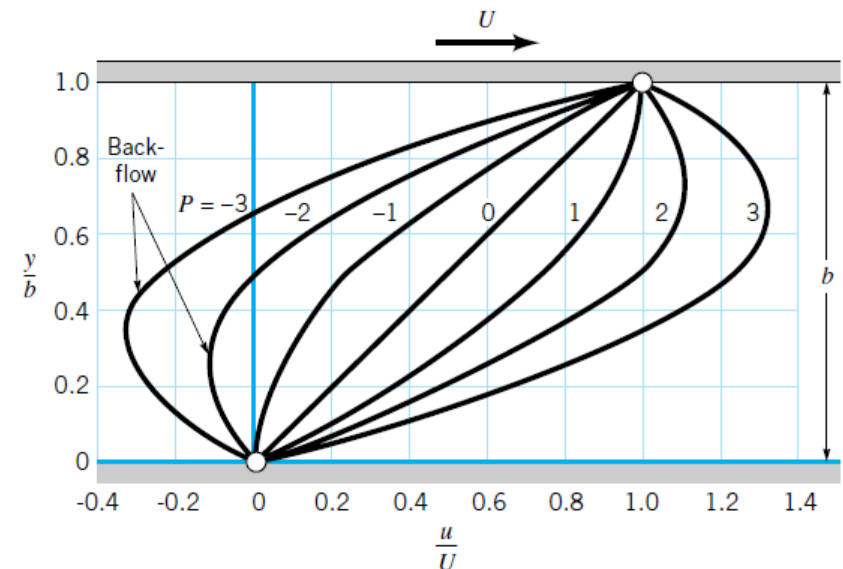
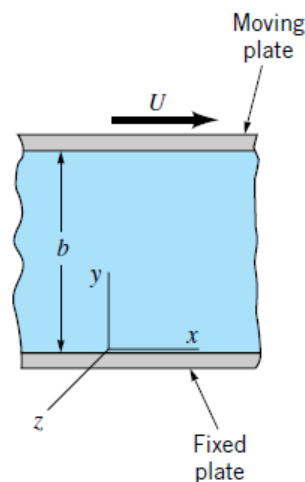
- Couette Flow**

Some Simple
Solutions for
Viscous,
Incompressible
Fluids

$$u = U \frac{y}{b} + \frac{1}{2\mu} \left(\frac{\partial p}{\partial x} \right) (y^2 - by)$$

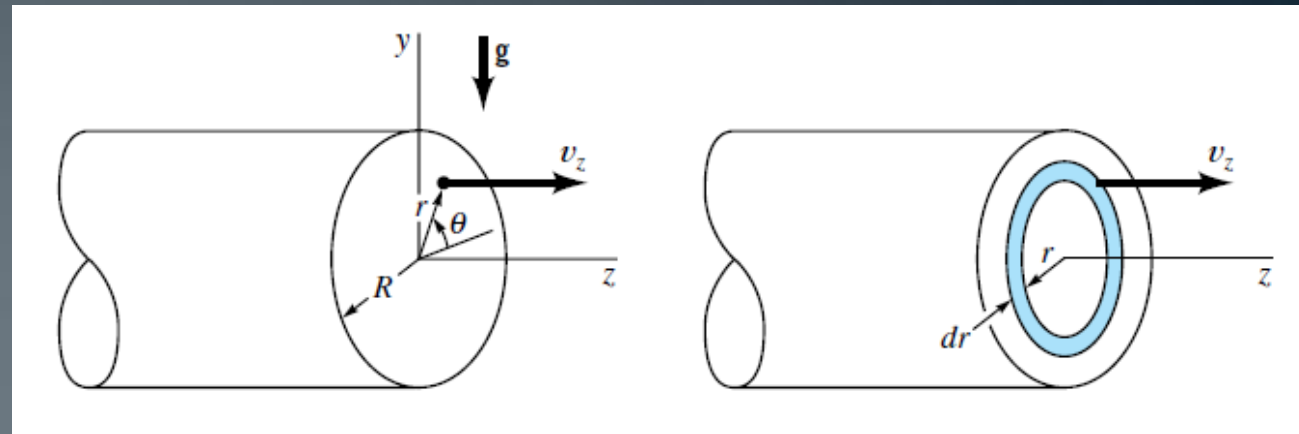
$$\frac{u}{U} = \frac{y}{b} - \frac{b^2}{2\mu U} \left(\frac{\partial p}{\partial x} \right) \left(\frac{y}{b} \right) \left(1 - \frac{y}{b} \right)$$

$$P = -\frac{b^2}{2\mu U} \left(\frac{\partial p}{\partial x} \right)$$



- Poiseuille flow

Steady, Laminar Flow in Circular Tubes



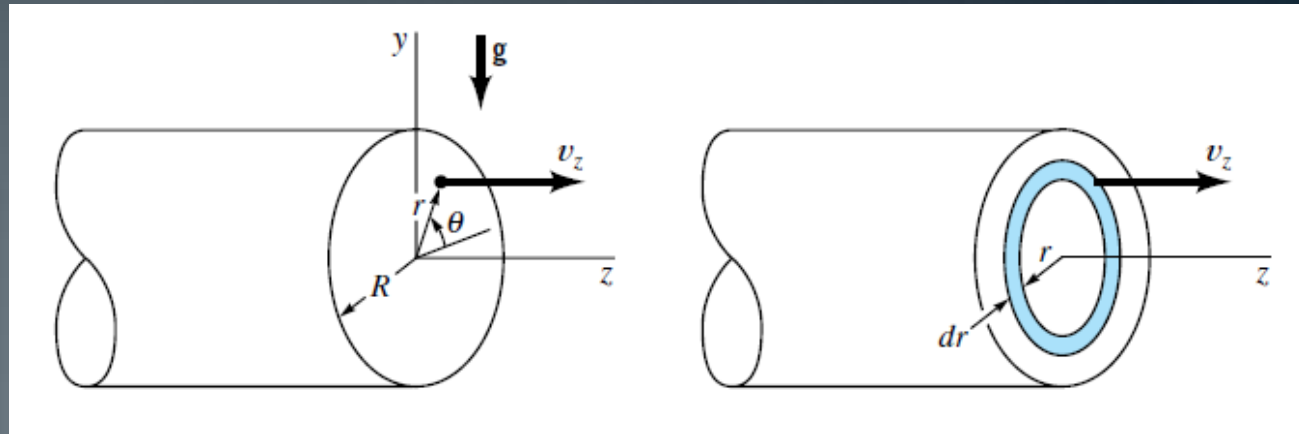
$$0 = -\rho g \sin \theta - \frac{\partial p}{\partial r}$$

$$0 = -\rho g \cos \theta - \frac{1}{r} \frac{\partial p}{\partial \theta}$$

$$0 = -\frac{\partial p}{\partial z} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) \right]$$

Steady, Laminar Flow in Circular Tubes

- **Poiseuille flow**



$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) = \frac{1}{\mu} \frac{\partial p}{\partial z}$$

$$r \frac{\partial v_z}{\partial r} = \frac{1}{2\mu} \left(\frac{\partial p}{\partial z} \right) r^2 + c_1$$

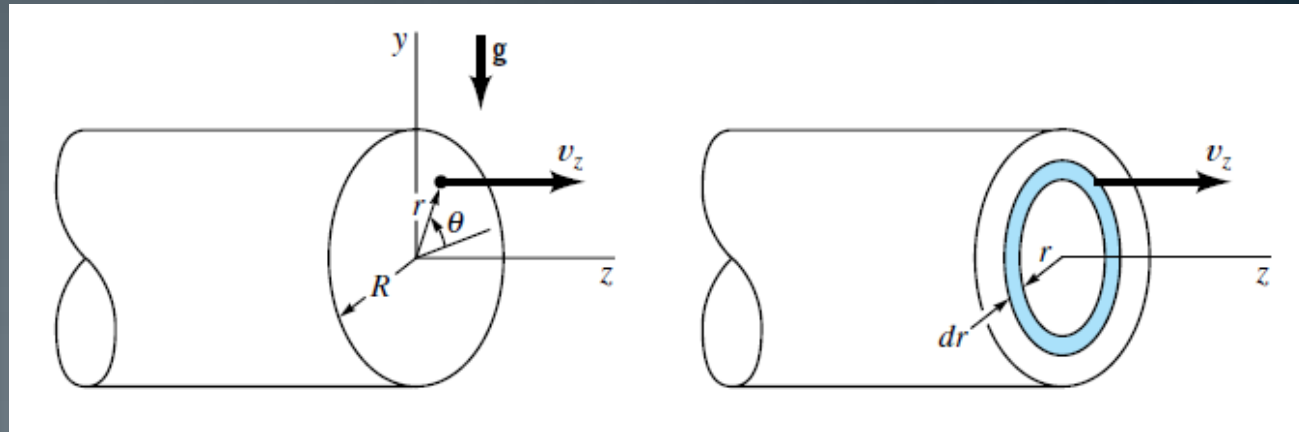
$$v_z = \frac{1}{4\mu} \left(\frac{\partial p}{\partial z} \right) r^2 + c_1 \ln r + c_2$$

$$c_2 = -\frac{1}{4\mu} \left(\frac{\partial p}{\partial z} \right) R^2$$

$$v_z = \frac{1}{4\mu} \left(\frac{\partial p}{\partial z} \right) (r^2 - R^2)$$

- Poiseuille flow

Steady, Laminar Flow in Circular Tubes



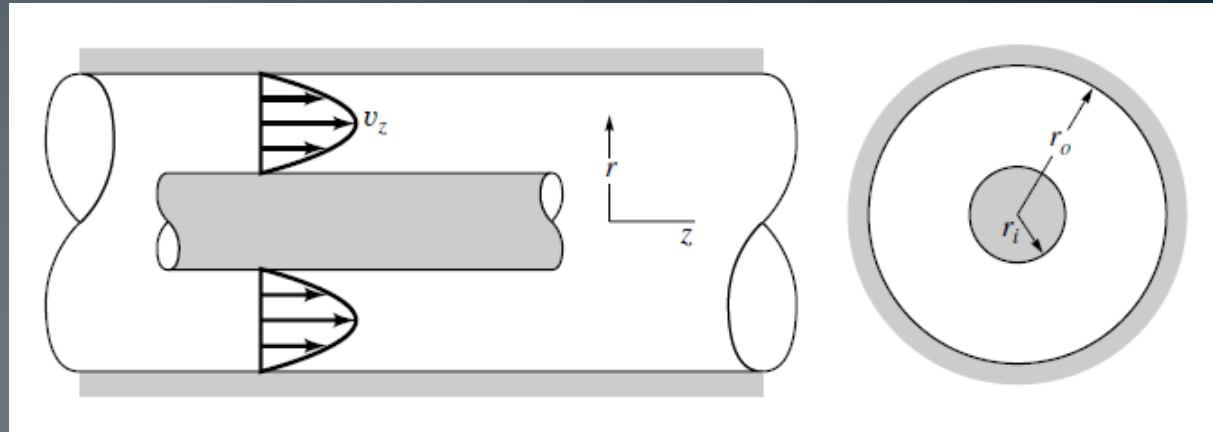
$$v_{\max} = -\frac{R^2}{4\mu} \left(\frac{\partial p}{\partial z} \right) = \frac{R^2 \Delta p}{4\mu \ell}$$

$$v_{\max} = 2V$$

$$\frac{v_z}{v_{\max}} = 1 - \left(\frac{r}{R} \right)^2$$

Steady, Laminar Flow in Circular Tubes

- Steady, Axial, Laminar Flow in an Annulus



$$v_z = \frac{1}{4\mu} \left(\frac{\partial p}{\partial z} \right) \left[r^2 - r_o^2 + \frac{r_i^2 - r_o^2}{\ln(r_o/r_i)} \ln \frac{r}{r_o} \right]$$

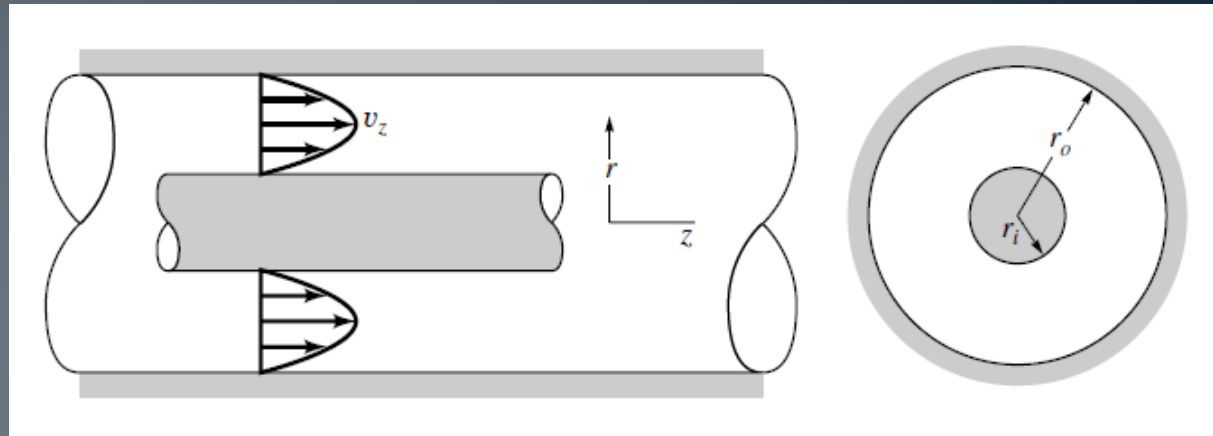
$$Q = \int_{r_i}^{r_o} v_z (2\pi r) dr = -\frac{\pi}{8\mu} \left(\frac{\partial p}{\partial z} \right) \left[r_o^4 - r_i^4 - \frac{(r_o^2 - r_i^2)^2}{\ln(r_o/r_i)} \right]$$

$$Q = \frac{\pi \Delta p}{8\mu \ell} \left[r_o^4 - r_i^4 - \frac{(r_o^2 - r_i^2)^2}{\ln(r_o/r_i)} \right]$$

$$r_m = \left[\frac{r_o^2 - r_i^2}{2 \ln(r_o/r_i)} \right]^{1/2}$$

- Steady, Axial, Laminar Flow in an Annulus

Steady, Laminar Flow in Circular Tubes

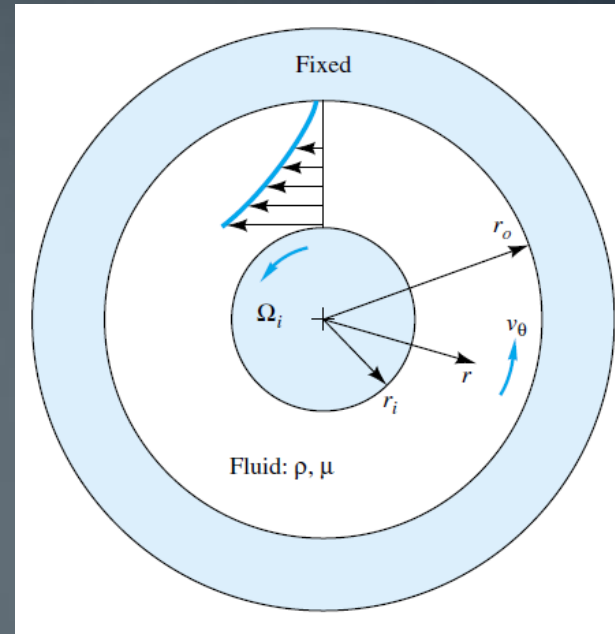


$$D_h = \frac{4 \times \text{cross-sectional area}}{\text{wetted perimeter}}$$

$$D_h = \frac{4\pi(r_o^2 - r_i^2)}{2\pi(r_o + r_i)} = 2(r_o - r_i)$$

Steady, Laminar Flow in Circular Tubes

- Flow between Long Concentric Cylinders



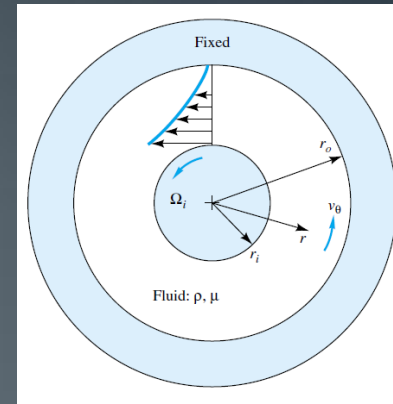
$$\frac{1}{r} \frac{\partial}{\partial r} (rv_r) + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} = 0 = \frac{1}{r} \frac{d}{dr} (rv_r) \quad \text{or} \quad rv_r = \text{const}$$

$$\rho(\mathbf{V} \cdot \nabla)v_\theta + \frac{\rho v_r v_\theta}{r} = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \rho g_\theta + \mu \left(\nabla^2 v_\theta - \frac{v_\theta}{r^2} \right)$$

$$\nabla^2 v_\theta = \frac{1}{r} \frac{d}{dr} \left(r \frac{dv_\theta}{dr} \right) = \frac{v_\theta}{r^2}$$

Steady, Laminar Flow in Circular Tubes

- Flow between Long Concentric Cylinders



$$\nabla^2 v_\theta = \frac{1}{r} \frac{d}{dr} \left(r \frac{dv_\theta}{dr} \right) = \frac{v_\theta}{r^2}$$

$$v_\theta = C_1 r + \frac{C_2}{r}$$

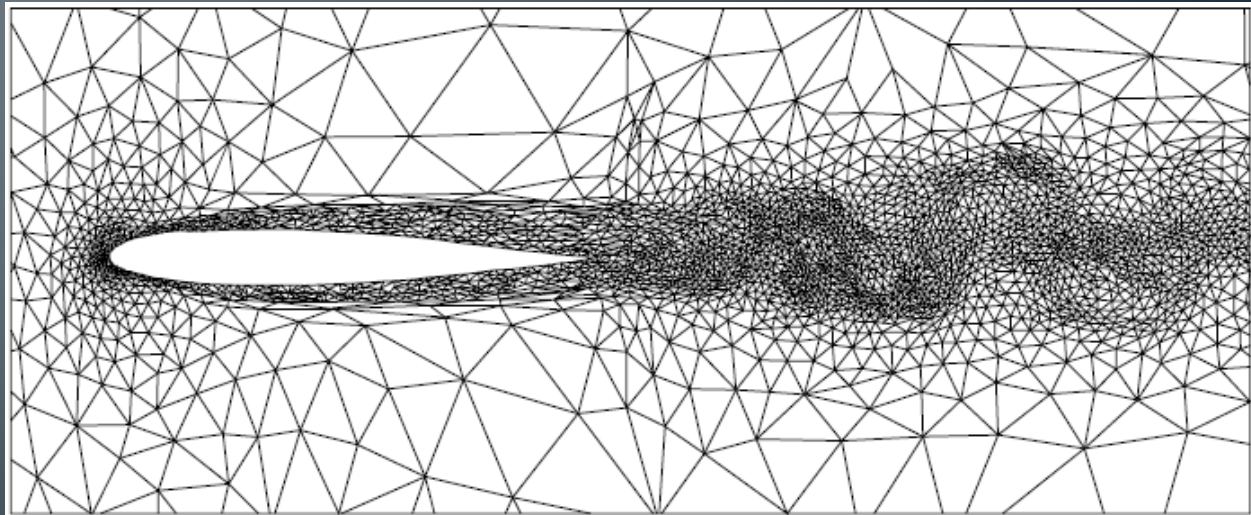
$$v_\theta = 0 = C_1 r_o + \frac{C_2}{r_o}$$

$$v_\theta = \Omega_i r_i = C_1 r_i + \frac{C_2}{r_i}$$

$$v_\theta = \Omega_i r_i \frac{r_o/r - r/r_o}{r_o/r_i - r_i/r_o}$$

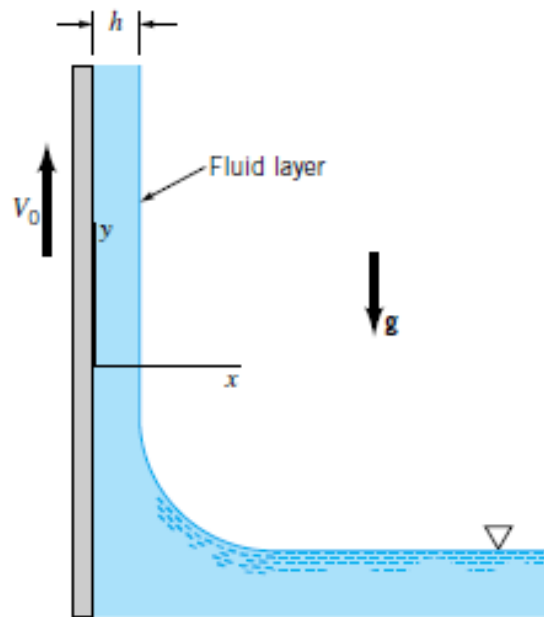
Numerical Methods

- Of the various techniques available for the numerical solution of the governing differential equations of fluid flow, the following three types are most common: (1) the **finite difference** method, (2) the **finite element** (or **finite volume**) method, and (3) the **boundary element** method.
- In each of these methods the continuous flow field (i.e., velocity or pressure as a function of space and time) is described in terms of **discrete** (rather than continuous) values at **prescribed locations**.



EXAMPLE 6.9

A wide moving belt passes through a container of a viscous liquid. The belt moves vertically upward with a constant velocity, V_0 , as illustrated in Fig. E6.9. Because of viscous forces the belt picks up a film of fluid of thickness h . Gravity tends to make the fluid drain down the belt. Use the Navier–Stokes equations to determine an expression for the average velocity of the fluid film as it is dragged up the belt. Assume that the flow is laminar, steady, and fully developed.



■ FIGURE E6.9

SOLUTION

Since the flow is assumed to be fully developed, the only velocity component is in the y direction (the v component) so that $u = w = 0$. It follows from the continuity equation that $\partial v / \partial y = 0$, and for steady flow $\partial v / \partial t = 0$, so that $v = v(x)$. Under these conditions the Navier–Stokes equations for the x direction (Eq. 6.127a) and the z direction (perpendicular to the paper) (Eq. 6.127c) simply reduce to

$$\frac{\partial p}{\partial x} = 0 \quad \frac{\partial p}{\partial z} = 0$$

This result indicates that the pressure does not vary over a horizontal plane, and since the pressure on the surface of the film ($x = h$) is atmospheric, the pressure throughout the film must be atmospheric (or zero gage pressure). The equation of motion in the y direction (Eq. 6.127b) thus reduces to

$$0 = -\rho g + \mu \frac{d^2 v}{dx^2}$$

or

$$\frac{d^2 v}{dx^2} = \frac{\gamma}{\mu} \quad (1)$$

Integration of Eq. 1 yields

$$\frac{dv}{dx} = \frac{\gamma}{\mu} x + c_1 \quad (2)$$

On the film surface ($x = h$) we assume the shearing stress is zero—that is, the drag of the air on the film is negligible. The shearing stress at the free surface (or any interior parallel surface) is designated as τ_{xy} , where from Eq. 6.125d

$$\tau_{xy} = \mu \left(\frac{dv}{dx} \right)$$

Thus, if $\tau_{xy} = 0$ at $x = h$, it follows from Eq. 2 that

$$c_1 = -\frac{\gamma h}{\mu}$$

A second integration of Eq. 2 gives the velocity distribution in the film as

$$v = \frac{\gamma}{2\mu}x^2 - \frac{\gamma h}{\mu}x + c_2$$

At the belt ($x = 0$) the fluid velocity must match the belt velocity, V_0 , so that

$$c_2 = V_0$$

and the velocity distribution is therefore

$$v = \frac{\gamma}{2\mu}x^2 - \frac{\gamma h}{\mu}x + V_0$$

With the velocity distribution known we can determine the flowrate per unit width, q , from the relationship

$$q = \int_0^h v \, dx = \int_0^h \left(\frac{\gamma}{2\mu}x^2 - \frac{\gamma h}{\mu}x + V_0 \right) dx$$

and thus

$$q = V_0 h - \frac{\gamma h^3}{3\mu}$$

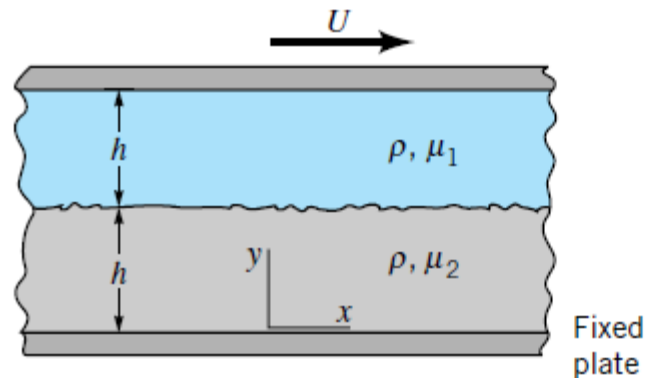
The average film velocity, V (where $q = Vh$), is therefore

$$V = V_0 - \frac{\gamma h^2}{3\mu} \quad (\text{Ans})$$

It is interesting to note from this result that there will be a net upward flow of liquid (positive V) only if $V_0 > \gamma h^2/3\mu$. It takes a relatively large belt speed to lift a small viscosity fluid.

Problems

6.81 Two immiscible, incompressible, viscous fluids having the same densities but different viscosities are contained between two infinite, horizontal, parallel plates (Fig. P6.81). The bottom plate is fixed and the upper plate moves with a constant velocity U . Determine the velocity at the interface. Express your answer in terms of U , μ_1 , and μ_2 . The motion of the fluid is caused entirely by the movement of the upper plate; that is, there is no pressure gradient in the x direction. The fluid velocity and shearing stress are continuous across the interface between the two fluids. Assume laminar flow.



Problems

For the specified conditions, $v=0$, $w=0$, $\frac{\partial P}{\partial x}=0$, and $g_x=0$, so that the x-component of the Navier-Stokes equations (Eq. 6.127a) for either the upper or lower layer reduces to

$$\frac{d^2 u}{dy^2} = 0 \quad (1)$$

Integration of Eq. (1) yields

$$u = Ay + B$$

which gives the velocity distribution in either layer.

In the upper layer at $y=2h$, $u=U$ so that

$$B_1 = U - A_1(2h)$$

where the subscript 1 refers to the upper layer.

For the lower layer at $y=0$, $u=0$ so that

$$B_2 = 0$$

where the subscript 2 refers to the lower layer. Thus,

$$u_1 = A_1(y - 2h) + U$$

and

$$u_2 = A_2 y$$

At $y=h$, $u_1 = u_2$ so that

$$A_1(h - 2h) + U = A_2 h$$

or

$$A_2 = -A_1 + \frac{U}{h} \quad (\text{con't}) \quad (2)$$

Problems

Since the velocity distribution is linear in each layer the shearing stress

$$\tau_{yx} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \mu \frac{du}{dy}$$

is constant throughout each layer. For the upper layer

$$\tau_1 = \mu_1 A_1$$

and for the lower layer

$$\tau_2 = \mu_2 A_2$$

At the interface $\tau_1 = \tau_2$ so that

$$\mu_1 A_1 = \mu_2 A_2$$

or

$$\frac{A_1}{A_2} = \frac{\mu_2}{\mu_1}$$

(3)

Substitution of Eq. (3) into Eq. (2) yields

$$A_2 = -\frac{\mu_2}{\mu_1} A_2 + \frac{U}{h}$$

or

$$A_2 = \frac{U/h}{1 + \mu_2/\mu_1}$$

Thus, velocity at the interface is

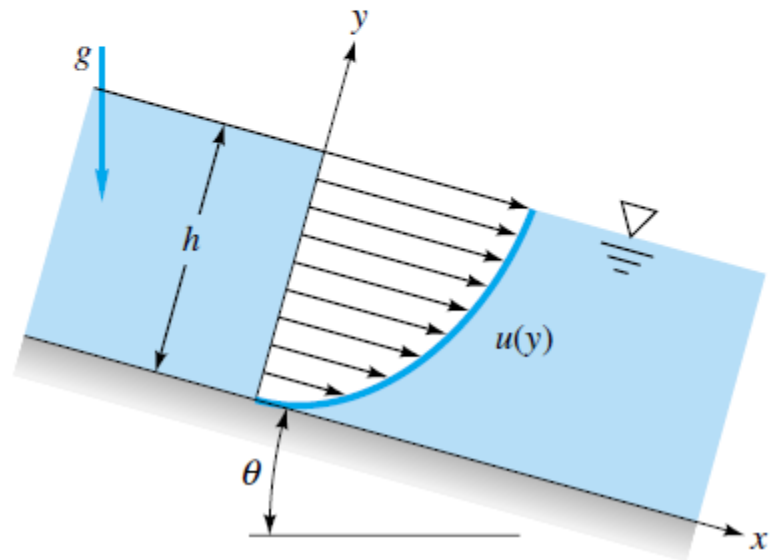
$$u_2(y=h) = A_2 h = \frac{U}{1 + \frac{\mu_2}{\mu_1}}$$

Problems

P4.36 A constant-thickness film of viscous liquid flows in laminar motion down a plate inclined at angle θ , as in Fig. P4.36. The velocity profile is

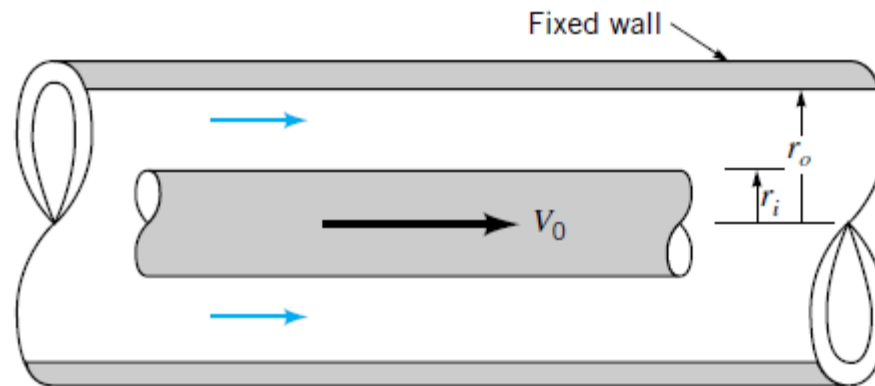
$$u = Cy(2h - y) \quad v = w = 0$$

Find the constant C in terms of the specific weight and viscosity and the angle θ . Find the volume flux Q per unit width in terms of these parameters.



Problems

6.93 An incompressible Newtonian fluid flows steadily between two infinitely long, concentric cylinders as shown in Fig. P6.93. The outer cylinder is fixed, but the inner cylinder moves with a longitudinal velocity V_0 as shown. For what value of V_0 will the drag on the inner cylinder be zero? Assume that the flow is laminar, axisymmetric, and fully developed.



Problems

Equation 6.147, which was developed for flow in circular tubes, applies in the annular region. Thus,

$$v_z = \frac{1}{4\mu} \left(\frac{\partial p}{\partial z} \right) r^2 + c_1 \ln r + c_2 \quad (1)$$

With boundary conditions, $r=r_o$, $v_z=0$, and $r=r_i$, $v_z=V_o$, it follows that:

$$0 = \frac{1}{4\mu} \left(\frac{\partial p}{\partial z} \right) r_o^2 + c_1 \ln r_o + c_2 \quad (2)$$

$$V_o = \frac{1}{4\mu} \left(\frac{\partial p}{\partial z} \right) r_i^2 + c_1 \ln r_i + c_2 \quad (3)$$

Subtract Eq. (2) from Eq. (3) to obtain

$$V_o = \frac{1}{4\mu} \left(\frac{\partial p}{\partial z} \right) (r_i^2 - r_o^2) + c_1 \ln \frac{r_i}{r_o}$$

so that

$$c_1 = \frac{V_o - \frac{1}{4\mu} \left(\frac{\partial p}{\partial z} \right) (r_i^2 - r_o^2)}{\ln \frac{r_i}{r_o}}$$

The drag on the inner cylinder will be zero if

$$(\tau_{rz})_{r=r_i} = 0$$

$$\text{Since,} \quad \tau_{rz} = \mu \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right) \quad (\text{Eq. 6.126f})$$

and with $v_r=0$, it follows that

$$\tau_{rz} = \mu \frac{\partial v_z}{\partial r} \quad (\text{con't})$$

Problems

(cont)

Differentiate Eq. (1) with respect to r to obtain

$$\frac{\partial v_z}{\partial r} = \frac{1}{2\mu} \left(\frac{\partial P}{\partial z} \right) r + \frac{C_1}{r}$$

so that at $r = r_i$

$$\left(\tau_{rz} \right)_{r=r_i} = \mu \left[\frac{1}{2\mu} \left(\frac{\partial P}{\partial z} \right) r_i + \frac{V_0 - \frac{1}{4\mu} \left(\frac{\partial P}{\partial z} \right) (r_i^2 - r_0^2)}{r_i \ln \frac{r_i}{r_0}} \right]$$

Thus, in order for the drag to be zero,

$$\frac{1}{2\mu} \left(\frac{\partial P}{\partial z} \right) r_i + \frac{V_0 - \frac{1}{4\mu} \left(\frac{\partial P}{\partial z} \right) (r_i^2 - r_0^2)}{r_i \ln \frac{r_i}{r_0}} = 0$$

or

$$\underline{\underline{V_0 = - \frac{1}{4\mu} \left(\frac{\partial P}{\partial z} \right) \left[2 r_i^2 \ln \frac{r_i}{r_0} - (r_i^2 - r_0^2) \right]}}$$

Problems

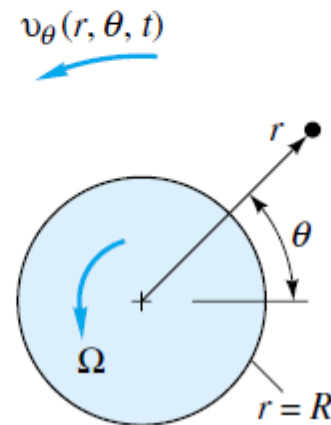
6.71 For a two-dimensional incompressible flow in the x - y plane show that the z component of the vorticity, ζ_z , varies in accordance with the equation

$$\frac{D\zeta_z}{Dt} = \nu \nabla^2 \zeta_z$$

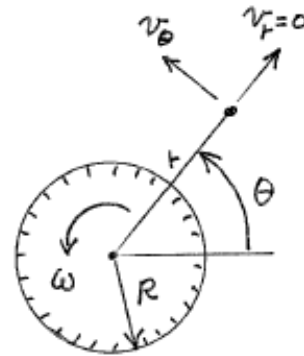
What is the physical interpretation of this equation for a non-viscous fluid? *Hint:* This *vorticity transport equation* can be derived from the Navier–Stokes equations by differentiating and eliminating the pressure between Eqs. 6.127a and 6.127b.

Problems

P4.82 A solid circular cylinder of radius R rotates at angular velocity Ω in a viscous incompressible fluid which is at rest far from the cylinder, as in Fig. P4.82. Make simplifying assumptions and derive the governing differential equation and boundary conditions for the velocity field v_θ in the fluid. Do not solve unless you are obsessed with this problem. What is the steady-state flow field for this problem?



Problems



For this flow field, $v_r=0$, $v_z=0$, and from the continuity equation,

$$\frac{1}{r} \frac{\partial(r v_r)}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \quad (\text{Eq. 6.35})$$

it follows that

$$\frac{\partial v_\theta}{\partial \theta} = 0 \quad (\text{See figure for notation.})$$

Thus, the Navier-Stokes equation in the θ -direction (Eq. 6.128b) for steady flow reduces to

$$0 = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_\theta}{\partial r} \right) - \frac{v_\theta}{r^2} \right]$$

Due to the symmetry of the flow,

$$\frac{\partial p}{\partial \theta} = 0$$

so that

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_\theta}{\partial r} \right) - \frac{v_\theta}{r^2} = 0$$

Problems

or

$$\frac{\partial^2 \vec{v}_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial \vec{v}_\theta}{\partial r} - \frac{\vec{v}_\theta}{r^2} = 0 \quad (1)$$

Since $\vec{v}_\theta$ is a function of only r , Eq. (1) can be expressed as an ordinary differential equation, and re-written as

$$\frac{d^2 \vec{v}_\theta}{dr^2} + \frac{d}{dr} \left(\frac{\vec{v}_\theta}{r} \right) = 0 \quad (2)$$

Equation (2) can be integrated to yield

$$\frac{d\vec{v}_\theta}{dr} + \frac{\vec{v}_\theta}{r} = c_1$$

or

$$r \frac{d\vec{v}_\theta}{dr} + \vec{v}_\theta = c_1 r \quad (3)$$

(con't)

Problems

Equation (3) can be expressed as

$$\frac{d(rv_{\theta})}{dr} = c_1 r$$

and a second integration yields

$$rv_{\theta} = \frac{c_1 r^2}{2} + c_2$$

or

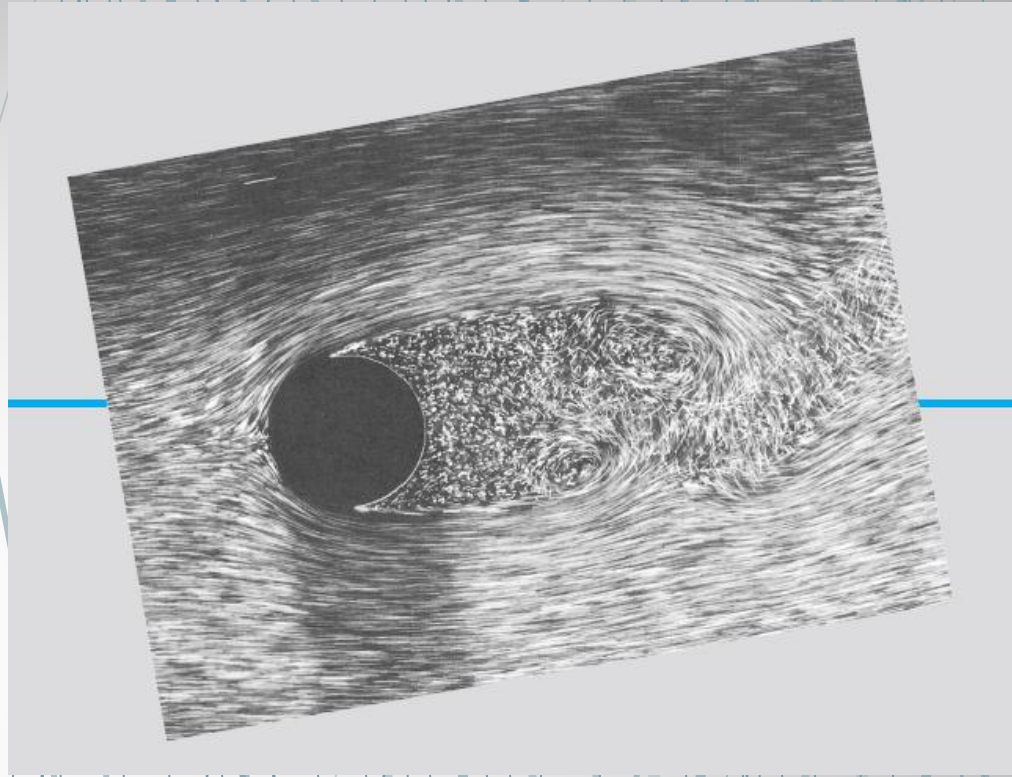
$$v_{\theta} = \frac{c_1 r}{2} + \frac{c_2}{r}$$

As $r \rightarrow \infty$, $v_{\theta} \rightarrow 0$, (since fluid is at rest at infinity)
so that $c_1 = 0$. Thus,

$$v_{\theta} = \frac{c_2}{r}$$

and since at $r = R$, $v_{\theta} = R\omega$, it follows that $c_2 = R^2\omega$
and

$$\underline{\underline{v_{\theta} = \frac{R^2\omega}{r}}}$$



Similitude, Dimensional Analysis, and Modeling

Similitude

- Very few problems involving real fluids can be solved by analysis alone. The solution to many problems is achieved through the use of a combination of **analysis** and **experimental data**.
- An obvious goal of any experiment is to make the results as widely applicable as possible.
- The concept of **similitude** is often used so that measurements made on one system (for example, in the laboratory) can be used to describe the behavior of other similar systems (outside the laboratory).
- The laboratory systems are usually thought of as **models** and are used to study the phenomenon of interest under carefully controlled conditions.
- From these model studies, **empirical formulations** can be developed, or specific predictions of one or more **characteristics** of some other similar system can be made.

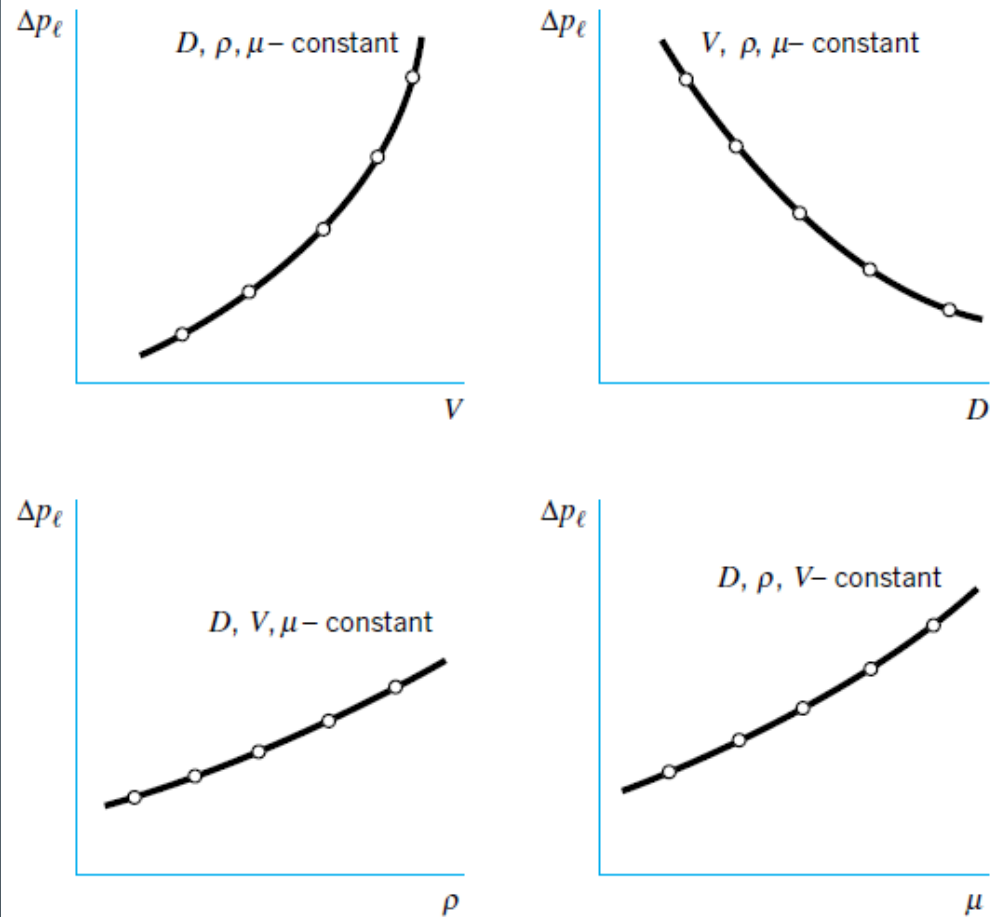
Dimensional Analysis

- To illustrate a typical fluid mechanics problem in which **experimentation** is required, consider the steady flow of an incompressible Newtonian fluid through a long, smooth-walled, horizontal, circular pipe.
- The first step in the planning of an experiment to study this problem would be to decide on the factors, or variables, that will have an effect on the pressure drop per unit length.
- We expect the list to include the **pipe diameter**, the **fluid density**, **fluid viscosity**, and the **mean velocity**, at which the fluid is flowing through the pipe. Thus, we can express this relationship as

$$\Delta p_\ell = f(D, \rho, \mu, V)$$

Dimensional Analysis

- To perform the experiments in a meaningful and systematic manner, it would be necessary to change one of the variables, such as the velocity, while holding all others constant, and measure the corresponding pressure drop.

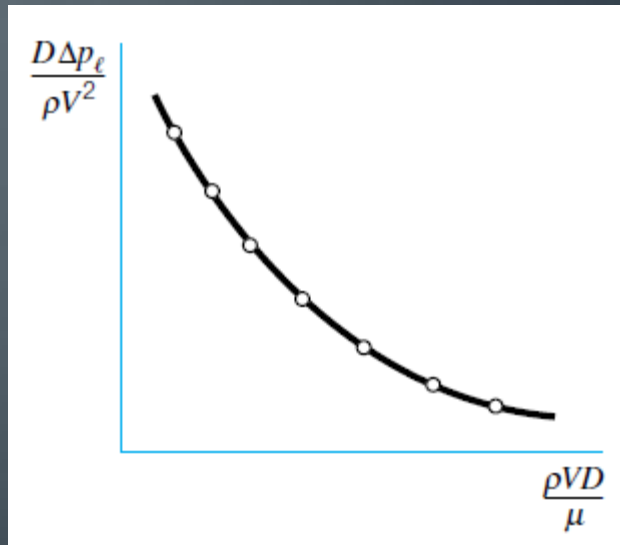


Dimensional Analysis

- Rather than working with the original list of variables, we can collect these into two **nondimensional** combinations of variables (called **dimensionless products** or **dimensionless groups**) so that

$$\frac{D \Delta p_\ell}{\rho V^2} = \phi \left(\frac{\rho V D}{\mu} \right)$$

- Thus, instead of having to work with five variables, we now have only two.



Dimensional Analysis

- A quick check of the dimensions of the two groups shows that they are in fact **dimensionless products**; that is,

$$\frac{D \Delta p_\ell}{\rho V^2} \doteq \frac{L(F/L^3)}{(FL^{-4}T^2)(LT^{-1})^2} \doteq F^0 L^0 T^0$$

$$\frac{\rho V D}{\mu} \doteq \frac{(FL^{-4}T^2)(LT^{-1})(L)}{(FL^{-2}T)} \doteq F^0 L^0 T^0$$

- Not only have we **reduced** the numbers of variables from five to two, but the new groups are dimensionless combinations of variables.
- This type of analysis is called **dimensional analysis**.

Buckingham Pi Theorem

- A fundamental question we must answer is how many dimensionless products are required to replace the original list of variables?
- If an equation involving k variables is dimensionally homogeneous, it can be reduced to a relationship among $k - r$ independent dimensionless products, where r is the minimum number of reference dimensions required to describe the variables.

$$u_1 = f(u_2, u_3, \dots, u_k)$$

$$\Pi_1 = \phi(\Pi_2, \Pi_3, \dots, \Pi_{k-r})$$

Determination of Pi Terms

- Several methods can be used to form the **dimensionless products**, or **pi terms**, that arise in a dimensional analysis.
- Essentially we are looking for a method that will allow us to systematically form the pi terms so that we are sure that they are **dimensionless** and **independent**, and that we have the right number.
- The method we will describe in detail in this section is called the **method of repeating variables**.

Method of Repeating Variables

Step 1. List all the variables that are involved in the problem.

Determination of Pi Terms

- The determination of the variables must be accomplished by the **experimenter's knowledge of the problem** and the **physical laws that govern the phenomenon**.
- Typically the variables will include those that are necessary to describe the **geometry** of the system (such as a pipe diameter), to define any **fluid properties** (such as a fluid viscosity), and to indicate **external effects** that influence the system (such as a driving pressure drop per unit length).
- Since we wish to keep the number of variables to a **minimum**, so that we can minimize the amount of laboratory work, it is important that all variables be **independent**.

Method of Repeating Variables

Step 2. Express each of the variables in terms of basic dimensions.

Step 3. Determine the required number of pi terms.

Determination of Pi Terms

- This can be accomplished by means of the Buckingham pi theorem, which indicates that the number of pi terms is equal to $k - r$ where k is the number of variables in the problem (which is determined from Step 1) and r is the number of reference dimensions required to describe these variables (which is determined from Step 2).
- There may be occasions (usually rare) in which the basic dimensions appear in combinations so that the number of reference dimensions is less than the number of basic dimensions.

Method of Repeating Variables

Determination of Pi Terms

Step 4. Select a number of repeating variables, where the number required is equal to the number of reference dimensions.

- All of the required reference dimensions must be included within the group of repeating variables.
- Each repeating variable must be dimensionally independent of the others. This means that the repeating variables cannot themselves be combined to form a dimensionless product.
- Do not choose the dependent variable as one of the repeating variables, since the repeating variables will generally appear in more than one pi term.

Method of Repeating Variables

Determination of Pi Terms

Step 5. Form a pi term by multiplying one of the nonrepeating variables by the product of the repeating variables, each raised to an exponent that will make the combination dimensionless.

Step 6. Repeat Step 5 for each of the remaining nonrepeating variables.

Step 7. Check all the resulting pi terms to make sure they are dimensionless.

Step 8. Express the final form as a relationship among the pi terms, and think about what it means.

$$\Pi_1 = \phi(\Pi_2, \Pi_3, \dots, \Pi_{k-r})$$

Determination of Pi Terms

Method of Repeating Variables

- To illustrate these various steps we will again consider the steady flow of an incompressible Newtonian fluid through a long, smooth-walled, horizontal circular pipe.

- Step 1.

$$\Delta p_\ell = f(D, \rho, \mu, V)$$

- Step 2.

$$\Delta p_\ell \doteq FL^{-3}$$

$$D \doteq L$$

$$\rho \doteq FL^{-4}T^2$$

$$\mu \doteq FL^{-2}T$$

$$V \doteq LT^{-1}$$

Method of Repeating Variables

Determination of Pi Terms

- **Step 3.** According to the pi theorem there will be $(5 - 3)$ or two pi terms required.
- **Step 4.** The repeating variables to be used to form the pi terms need to be selected from the list.
- **Step 5.** We would start with the dependent variable and combine it with the repeating variables to form the first pi term; that is,

$$\Pi_1 = \Delta p_\ell D^a V^b \rho^c$$

Method of Repeating Variables

Determination of Pi Terms

$$\Pi_1 = \Delta p_\ell D^a V^b \rho^c$$

$$(FL^{-3})(L)^a(LT^{-1})^b(FL^{-4}T^2)^c \doteq F^0L^0T^0$$

$$1 + c = 0 \quad (\text{for } F)$$

$$-3 + a + b - 4c = 0 \quad (\text{for } L)$$

$$-b + 2c = 0 \quad (\text{for } T)$$

$$\Pi_1 = \frac{\Delta p_\ell D}{\rho V^2}$$

Method of Repeating Variables

- **Step 6.** The process is now repeated for the remaining nonrepeating variables

$$\Pi_2 = \mu D^a V^b \rho^c$$

$$(FL^{-2}T)(L)^a(LT^{-1})^b(FL^{-4}T^2)^c \doteq F^0L^0T^0$$

$$1 + c = 0 \quad (\text{for } F)$$

$$-2 + a + b - 4c = 0 \quad (\text{for } L)$$

$$1 - b + 2c = 0 \quad (\text{for } T)$$

$$\Pi_2 = \frac{\mu}{DV\rho}$$

Determination
of Pi Terms

Method of Repeating Variables

- **Step 7.** At this point stop and check to make sure the pi terms are actually dimensionless

Determination of Pi Terms

$$\Pi_1 = \frac{\Delta p_\ell D}{\rho V^2} \doteq \frac{(FL^{-3})(L)}{(FL^{-4}T^2)(LT^{-1})^2} \doteq F^0 L^0 T^0$$

$$\Pi_2 = \frac{\mu}{DV\rho} \doteq \frac{(FL^{-2}T)}{(L)(LT^{-1})(FL^{-4}T^2)} \doteq F^0 L^0 T^0$$

$$\Pi_1 = \frac{\Delta p_\ell D}{\rho V^2} \doteq \frac{(ML^{-2}T^{-2})(L)}{(ML^{-3})(LT^{-1})^2} \doteq M^0 L^0 T^0$$

$$\Pi_2 = \frac{\mu}{DV\rho} \doteq \frac{(ML^{-1}T^{-1})}{(L)(LT^{-1})(ML^{-3})} \doteq M^0 L^0 T^0$$

Determination of Pi Terms

Method of Repeating Variables

- **Step 8.** Finally, we can express the result of the dimensional analysis as

$$\frac{\Delta p_{\ell} D}{\rho V^2} = \tilde{\phi} \left(\frac{\mu}{DV\rho} \right)$$

- Dimensional analysis will **not** provide the form of the function. This can only be obtained from a suitable set of **experiments**.
- If desired, the pi terms can be **rearranged**.

$$\Pi_2 = \frac{\rho V D}{\mu}$$

$$\frac{D \Delta p_{\ell}}{\rho V^2} = \phi \left(\frac{\rho V D}{\mu} \right)$$

Some Additional Comments About Dimensional Analysis

Selection of Variables

- For most engineering problems, pertinent variables can be classified into three general groups—**geometry**, **material properties**, and **external effects**.
- **Geometry.** The geometric characteristics can usually be described by a series of lengths and angles.
- **Material Properties.** The material properties that relate the external effects and the responses must be included as variables.
- **External Effects.** This terminology is used to denote any variable that produces, or tends to produce, a change in the system.
- Since we wish to keep the number of variables to a minimum, it is important that all variables are **independent**.

Determination of Reference Dimensions

Some Additional Comments About Dimensional Analysis

- It is also important to know how many **reference dimensions** are required to describe the variables.
- **F**, **L**, and **T** appear to be a convenient set of **basic dimensions** for characterizing fluid-mechanical quantities. There is, however, really nothing “fundamental” about this set, and as previously noted **M**, **L**, and **T** would also be suitable.
- We occasionally find that the number of reference dimensions needed to describe all variables is smaller than the number of basic dimensions. This point is illustrated in Example 7.2.

EXAMPLE 7.2

An open, cylindrical tank having a diameter D is supported around its bottom circumference and is filled to a depth h with a liquid having a specific weight γ . The vertical deflection, δ , of the center of the bottom is a function of D , h , d , γ , and E , where d is the thickness of the bottom and E is the modulus of elasticity of the bottom material. Perform a dimensional analysis of this problem.

SOLUTION

From the statement of the problem

$$\delta = f(D, h, d, \gamma, E)$$

and the dimensions of the variables are

$$\delta \doteq L$$

$$D \doteq L$$

$$h \doteq L$$

$$d \doteq L$$

$$\gamma \doteq FL^{-3} \doteq ML^{-2}T^{-2}$$

$$E \doteq FL^{-2} \doteq ML^{-1}T^{-2}$$

where the dimensions have been expressed in terms of both the *FLT* and *MLT* systems.

We now apply the pi theorem to determine the required number of pi terms. First, let us use F , L , and T as our system of basic dimensions. There are six variables and two reference dimensions (F and L) required so that four pi terms are needed. For repeating variables, we can select D and γ so that

$$\Pi_1 = \delta D^a \gamma^b$$
$$(L)(L)^a (FL^{-3})^b \doteq F^0 L^0$$

and

$$1 + a - 3b = 0 \quad (\text{for } L)$$
$$b = 0 \quad (\text{for } F)$$

Therefore, $a = -1$, $b = 0$, and

$$\Pi_1 = \frac{\delta}{D}$$

Similarly,

$$\Pi_2 = h D^a \gamma^b$$

The remaining two pi terms can be found using the same procedure, with the result

$$\Pi_3 = \frac{d}{D} \quad \Pi_4 = \frac{E}{D\gamma}$$

Thus, this problem can be studied by using the relationship

$$\frac{\delta}{D} = \phi\left(\frac{h}{D}, \frac{d}{D}, \frac{E}{D\gamma}\right) \quad (\text{Ans})$$

Let us now solve the same problem using the *MLT* system. Although the number of variables is obviously the same, it would seem that there are three reference dimensions required, rather than two. If this were indeed true it would certainly be fortuitous, since we would reduce the number of required pi terms from four to three. Does this seem right? How can we reduce the number of required pi terms by simply using the *MLT* system of basic dimensions? The answer is that we cannot, and a closer look at the dimensions of the variables listed above reveals that actually only two reference dimensions, MT^{-2} and L , are required.

This is an example of the situation in which the number of reference dimensions differs from the number of basic dimensions. It does not happen very often and can be detected by looking at the dimensions of the variables (regardless of the systems used) and making sure how many reference dimensions are actually required to describe the variables. Once the number of reference dimensions has been determined, we can proceed as before. Since the number of repeating variables must equal the number of reference dimensions, it follows that two reference dimensions are still required and we could again use D and γ as repeating variables. The pi terms would be determined in the same manner. For example, the pi term containing E would be developed as

$$\begin{aligned}\Pi_4 &= ED^a\gamma^b \\ (ML^{-1}T^{-2})(L)^a(ML^{-2}T^{-2})^b &\doteq (MT^{-2})^0L^0 \\ 1 + b &= 0 \quad (\text{for } MT^{-2}) \\ -1 + a - 2b &= 0 \quad (\text{for } L)\end{aligned}$$

and, therefore, $a = -1$, $b = -1$ so that

$$\Pi_4 = \frac{E}{D\gamma}$$

which is the same as Π_4 obtained using the FLT system. The other pi terms would be the same, and the final result is the same; that is,

$$\frac{\delta}{D} = \phi\left(\frac{h}{D}, \frac{d}{D}, \frac{E}{D\gamma}\right) \quad (\text{Ans})$$

This will always be true—you cannot affect the required number of pi terms by using M , L , and T instead of F , L , and T , or vice versa.

Uniqueness of Pi Terms

- A little reflection on the process used to determine pi terms by the method of repeating variables reveals that the specific pi terms obtained depend on the somewhat **arbitrary** selection of repeating variables.

$$\frac{\Delta p_{\ell} D}{\rho V^2} = \phi \left(\frac{\rho V D}{\mu} \right)$$

$$\frac{\Delta p_{\ell} D^2}{V \mu} = \phi_1 \left(\frac{\rho V D}{\mu} \right)$$

- There is **not** a **unique set of pi terms** which arises from a dimensional analysis. However, the **required number of pi terms** is **fixed**.
- Once a correct set is determined all other possible sets can be developed from this set by combinations of products of powers of the original set.

Uniqueness of Pi Terms

- Thus, if we have a problem involving, say, three pi terms,

$$\Pi_1 = \phi(\Pi_2, \Pi_3)$$

$$\Pi'_2 = \Pi_2^a \Pi_3^b$$

- where a and b are arbitrary exponents. Then the relationship could be expressed as

$$\Pi_1 = \phi_1(\Pi'_2, \Pi_3)$$

$$\Pi_1 = \phi_2(\Pi_2, \Pi'_2)$$

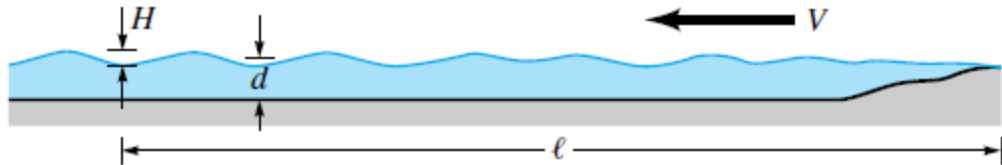
$$\left(\frac{\Delta p_\ell D}{\rho V^2}\right)\left(\frac{\rho V D}{\mu}\right) = \frac{\Delta p_\ell D^2}{V \mu}$$

Common Dimensionless Groups in Fluid Mechanics

Dimensionless Groups	Name	Interpretation (Index of Force Ratio Indicated)	Types of Applications
$\frac{\rho V \ell}{\mu}$	Reynolds number, Re	$\frac{\text{inertia force}}{\text{viscous force}}$	Generally of importance in all types of fluid dynamics problems
$\frac{V}{\sqrt{g \ell}}$	Froude number, Fr	$\frac{\text{inertia force}}{\text{gravitational force}}$	Flow with a free surface
$\frac{p}{\rho V^2}$	Euler number, Eu	$\frac{\text{pressure force}}{\text{inertia force}}$	Problems in which pressure, or pressure differences, are of interest
$\frac{\rho V^2}{E_v}$	Cauchy number, ^a Ca	$\frac{\text{inertia force}}{\text{compressibility force}}$	Flows in which the compressibility of the fluid is important
$\frac{V}{c}$	Mach number, ^a Ma	$\frac{\text{inertia force}}{\text{compressibility force}}$	Flows in which the compressibility of the fluid is important
$\frac{\omega \ell}{V}$	Strouhal number, St	$\frac{\text{inertia (local) force}}{\text{inertia (convective) force}}$	Unsteady flow with a characteristic frequency of oscillation
$\frac{\rho V^2 \ell}{\sigma}$	Weber number, We	$\frac{\text{inertia force}}{\text{surface tension force}}$	Problems in which surface tension is important

Problems

7.7 It is desired to determine the wave height when wind blows across a lake. The wave height, H , is assumed to be a function of the wind speed, V , the water density, ρ , the air density, ρ_a , the water depth, d , the distance from the shore, ℓ , and the acceleration of gravity, g , as shown in Fig. P7.7. Use d , V , and ρ as repeating variables to determine a suitable set of pi terms that could be used to describe this problem.



Problems

$$H = f(V, \rho, \rho_a, d, l, g)$$

$$H \doteq L \quad V \doteq LT^{-1} \quad \rho \doteq FL^{-3}T^2 \quad \rho_a \doteq FL^{-3}T^2 \quad d \doteq L \quad l \doteq L \quad g \doteq LT^{-2}$$

From the pi theorem, $7-3=4$ pi terms required. Use d, V , and ρ as repeating variables. Thus,

$$\pi_1 = H d^a V^b \rho^c$$

and

$$(L)(L)^a (LT^{-1})^b (FL^{-3}T^2)^c = F^0 L^0 T^0$$

so that

$$c = 0 \quad (\text{for } F)$$

$$1 + a + b - 4c = 0 \quad (\text{for } L)$$

$$-b + 2c = 0 \quad (\text{for } T)$$

It follows that $a = -1, b = 0, c = 0$, and therefore

$$\pi_1 = \frac{H}{d}$$

which is obviously dimensionless.

For π_2 :

$$\pi_2 = \rho_a d^a V^b \rho^c$$

and

$$(FL^{-3}T^2)(L)^a (LT^{-1})^b (FL^{-3}T^2)^c = F^0 L^0 T^0$$

so that

$$1 + c = 0 \quad (\text{for } F)$$

$$-4 + a + b - 4c = 0 \quad (\text{for } L)$$

$$2 - b + 2c = 0 \quad (\text{for } T)$$

It follows that $a = 0, b = 0, c = -1$ so that

$$\pi_2 = \frac{\rho_a}{\rho}$$

which is obviously dimensionless.

(con't)

Problems

For π_3 :

$$\pi_3 = l d^a v^b \rho^c$$

and as for π_1 , $a = -1$, $b = 0$, $c = 0$ so that

$$\pi_3 = \frac{l}{d}$$

For π_4 :

$$\pi_4 = g d^a v^b \rho^c$$

$$(L T^{-2})(L)^a (L T^{-1})^b (F L^{-4} T^2)^c = F^0 L^0 T^0$$

$$c = 0$$

(for F)

$$1 + a + b - 4c = 0$$

(for L)

$$-2 - b + 2c = 0$$

(for T)

It follows that $a =$, $b = -2$, $c = 0$, and therefore

$$\pi_4 = \frac{g d}{v^2}$$

Check dimensions:

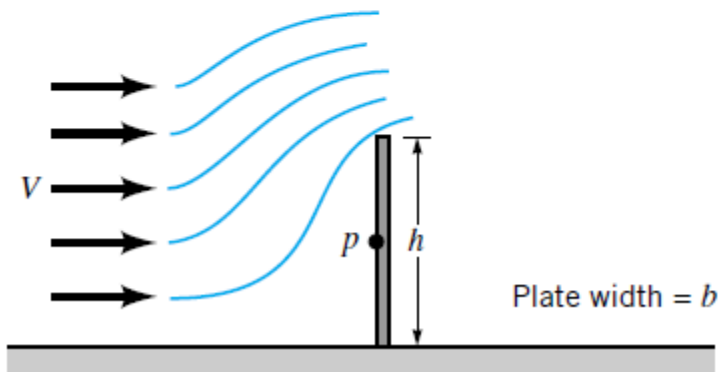
$$\frac{g d}{v^2} = \frac{(L T^{-2})(L)}{(L T^{-1})^2} = L^0 T^0 \quad \therefore \text{OK}$$

Thus,

$$\underline{\underline{\frac{H}{d} = \phi \left(\frac{\rho_a}{\rho}, \frac{l}{d}, \frac{g d}{v^2} \right)}}$$

Problems

7.17 When a fluid flows slowly past a vertical plate of height h and width b (see Fig. P7.17), pressure develops on the face of the plate. Assume that the pressure, p , at the midpoint of the plate is a function of plate height and width, the approach velocity, V , and the fluid viscosity, μ . Make use of dimensional analysis to determine how the pressure, p , will change when the fluid velocity, V , is doubled.

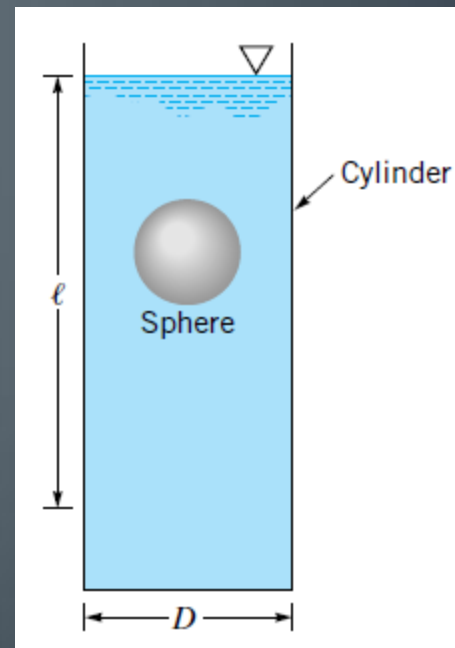


Problems

7.19 The viscosity, μ , of a liquid can be measured by determining the time, t , it takes for a sphere of diameter, d , to settle slowly through a distance, ℓ , in a vertical cylinder of diameter, D , containing the liquid (see Fig. P7.19). Assume that

$$t = f(\ell, d, D, \mu, \Delta\gamma)$$

where $\Delta\gamma$ is the difference in specific weights between the sphere and the liquid. Use dimensional analysis to show how t is related to μ , and describe how such an apparatus might be used to measure viscosity.



Problems

$$t \doteq T \quad l \doteq L \quad d \doteq L \quad D \doteq L \quad \Delta \gamma \doteq FL^{-3} \quad \mu \doteq FL^{-2}T$$

From the pi Theorem, $6-3 = 3$ pi terms required. By inspection for π_1 (containing t):

$$\pi_1 = \frac{t \Delta \gamma d}{\mu} \doteq \frac{(T)(FL^{-3})(L)}{(FL^{-2}T)} = F^0 L^0 T^0$$

Check using MLT:

$$\frac{t \Delta \gamma d}{\mu} \doteq \frac{(T)(ML^{-2}T^{-2})(L)}{(ML^{-1}T^{-1})} \doteq M^0 L^0 T^0 \quad \therefore \text{OK}$$

For π_2 (containing D):

$$\pi_2 = \frac{D}{d}$$

Which is obviously dimensionless.

For π_3 (containing l):

$$\pi_3 = \frac{l}{d}$$

Which is obviously dimensionless. Thus,

$$\frac{t \Delta \gamma d}{\mu} = \phi\left(\frac{D}{d}, \frac{l}{d}\right)$$

and for a fixed geometry

$$\frac{t \Delta \gamma d}{\mu} = C$$

where C is a constant, or

$$\mu = \frac{d}{C} \Delta \gamma t = C_1 \Delta \gamma t$$

where $C_1 = \frac{d}{C}$ a constant for a fixed geometry. Thus, for this device

$$\underline{\mu = C_1 \Delta \gamma t} \quad (1)$$

The constant C_1 can be determined by calibration with a fluid of known viscosity. With C_1 known the viscosity of other fluids can be determined through a measurement of the time t in conjunction with Eq. (1).

Correlation of Experimental Data

- **Problems with One Pi Term**
- Application of the pi theorem indicates that if the number of variables minus the number of reference dimensions is equal to unity, then only one pi term is required to describe the phenomenon.
- The functional relationship that must exist for one pi term is

$$\Pi_1 = C$$

- where C is a constant.

EXAMPLE 7.3

Assume that the drag, $\mathcal{D}$, acting on a spherical particle that falls very slowly through a viscous fluid is a function of the particle diameter, d , the particle velocity, V , and the fluid viscosity, μ . Determine, with the aid of dimensional analysis, how the drag depends on the particle velocity.

SOLUTION

From the information given, it follows that

$$\mathcal{D} = f(d, V, \mu)$$

and the dimensions of the variables are

$$\mathcal{D} \doteq F$$

$$d \doteq L$$

$$V \doteq LT^{-1}$$

$$\mu \doteq FL^{-2}T$$

We see that there are four variables and three reference dimensions (F , L , and T) required to describe the variables. Thus, according to the pi theorem, one pi term is required. This pi term can be easily formed by inspection and can be expressed as

$$\Pi_1 = \frac{\mathcal{D}}{\mu V d}$$

Since there is only one pi term, it follows that

$$\frac{\mathcal{D}}{\mu V d} = C$$

or

$$\mathcal{D} = C \mu V d$$

Thus, for a given particle and fluid, the drag varies directly with the velocity so that

$$\mathcal{D} \propto V \quad (\text{Ans})$$

Actually, the dimensional analysis reveals that the drag not only varies directly with the velocity, but it also varies directly with the particle diameter and the fluid viscosity. We could not, however, predict the value of the drag, since the constant, C , is unknown. An experiment would have to be performed in which the drag and the corresponding velocity are measured for a given particle and fluid. Although in principle we would only have to run a single test, we would certainly want to repeat it several times to obtain a reliable value for C . It should be emphasized that once the value of C is determined it is not necessary to run similar tests by using different spherical particles and fluids; that is, C is a universal constant so long as the drag is a function only of particle diameter, velocity, and fluid viscosity.

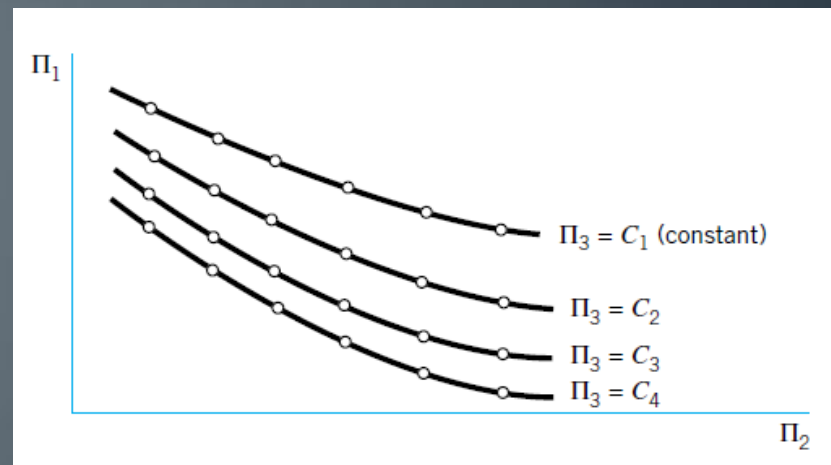
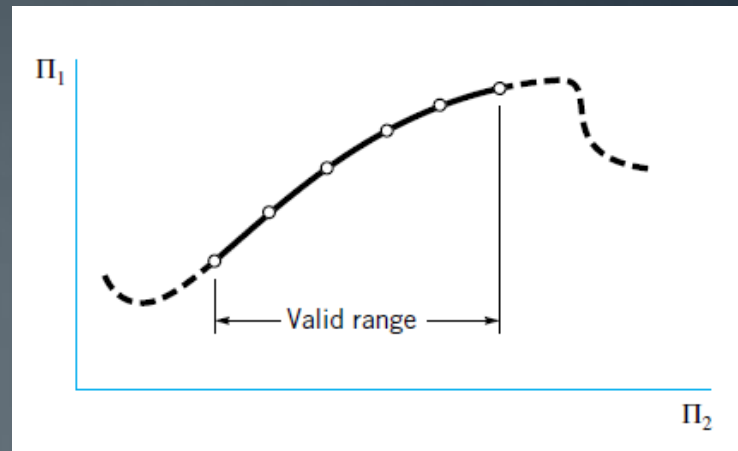
An approximate solution to this problem can also be obtained theoretically, from which it is found that $C = 3\pi$ so that

$$\mathcal{D} = 3\pi\mu Vd$$

This equation is commonly called *Stokes law* and is used in the study of the settling of particles. Our experiments would reveal that this result is only valid for small Reynolds numbers ($\rho Vd/\mu \ll 1$). This follows, since in the original list of variables, we have neglected inertial effects (fluid density is not included as a variable). The inclusion of an additional variable would lead to another pi term so that there would be two pi terms rather than one.

Correlation of Experimental Data

- Problems with Two or More Pi Terms



- For more complicated problems, it is often more feasible to use **models** to predict specific characteristics of the system rather than to try to develop general correlations.

Modeling and Similitude

- A **model** is a representation of a physical system that may be used to **predict** the behavior of the system in some desired respect.
- The physical system for which the predictions are to be made is called the **prototype**.
- Usually a model is **smaller** than the prototype.
- With the successful development of a valid model, it is possible to **predict** the behavior of the prototype under a certain set of conditions.
- We may also wish to examine a priori the effect of **possible design changes** that are proposed for a hydraulic structure or fluid-flow system.

Theory of Models

- The theory of models can be readily developed by using the principles of dimensional analysis.

$$\Pi_1 = \phi(\Pi_2, \Pi_3, \dots, \Pi_n)$$

- If the above equation describes the behavior of a particular prototype, a similar relationship can be written for a model of this prototype; that is,

$$\Pi_{1m} = \phi(\Pi_{2m}, \Pi_{3m}, \dots, \Pi_{nm})$$

$$\Pi_{2m} = \Pi_2$$

$$\Pi_{3m} = \Pi_3$$

$$\vdots$$

$$\Pi_{nm} = \Pi_n$$

$$\Pi_1 = \Pi_{1m}$$

Theory of Models

- Consider the problem of determining the drag, on
- a thin rectangular plate placed normal to a fluid with velocity.

$$\mathcal{D} = f(w, h, \mu, \rho, V)$$

- Application of the pi theorem yielded

$$\frac{\mathcal{D}}{w^2 \rho V^2} = \phi \left(\frac{w}{h}, \frac{\rho V w}{\mu} \right)$$

$$\frac{\mathcal{D}_m}{w_m^2 \rho_m V_m^2} = \phi \left(\frac{w_m}{h_m}, \frac{\rho_m V_m w_m}{\mu_m} \right)$$

$$\frac{w_m}{h_m} = \frac{w}{h} \quad \frac{\rho_m V_m w_m}{\mu_m} = \frac{\rho V w}{\mu}$$

$$w_m = \frac{h_m}{h} w$$

Theory of Models

$$V_m = \frac{\mu_m}{\mu} \frac{\rho}{\rho_m} \frac{w}{w_m} V$$

- With the foregoing similarity requirements satisfied, the prediction equation for the drag is

$$\frac{\mathcal{D}}{w^2 \rho V^2} = \frac{\mathcal{D}_m}{w_m^2 \rho_m V_m^2}$$

$$\mathcal{D} = \left(\frac{w}{w_m} \right)^2 \left(\frac{\rho}{\rho_m} \right) \left(\frac{V}{V_m} \right)^2 \mathcal{D}_m$$

- Thus, a measured drag on the model, must be multiplied by the ratio of the square of the plate widths, the ratio of the fluid densities, and the ratio of the square of the velocities to obtain the predicted value of the prototype drag.

Theory of Models

- **Distorted Models**
- Models for which one or more of the similarity requirements are not satisfied are called distorted models.
- The classic example of a distorted model occurs in the study of open channel or free-surface flows.
- Typically in these problems both the Reynolds number, and the Froude number, are involved.
- Froude number similarity requires

$$\frac{V_m}{\sqrt{g_m \ell_m}} = \frac{V}{\sqrt{g \ell}}$$

Theory of Models

- Distorted Models

$$\frac{V_m}{V} = \sqrt{\frac{\ell_m}{\ell}} = \sqrt{\lambda_\ell}$$

- Reynolds number similarity requires

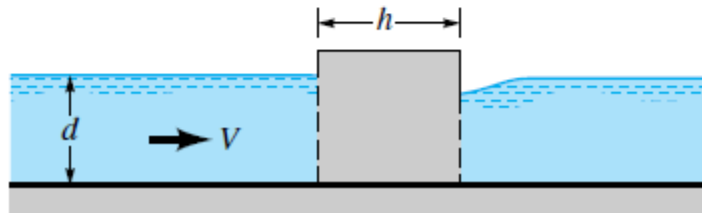
$$\frac{\rho_m V_m \ell_m}{\mu_m} = \frac{\rho V \ell}{\mu}$$

$$\frac{V_m}{V} = \frac{\mu_m}{\mu} \frac{\rho}{\rho_m} \frac{\ell}{\ell_m}$$

$$\frac{\mu_m / \rho_m}{\mu / \rho} = \frac{v_m}{v} = (\lambda_\ell)^{3/2}$$

Problems

7.43 A solid block in the shape of a cube rests partially submerged on the bottom of a river as shown in Fig. P7.43. The drag, $\mathcal{D}$, on the block depends on the river depth, d , the block dimension, h , the stream velocity, V , the fluid density, ρ , and the acceleration of gravity, g . (a) Perform a dimensional analysis for this problem. (b) The drag is to be determined from a model study using a length scale of $1/5$. What model velocity should be used to predict the drag on the prototype located in a river with a velocity of 9 ft/s? Water is to be used for the model fluid. Determine the expected prototype drag in terms of the model drag.



Problems

$$(a) \quad D = f(d, h, V, \rho, g)$$

$$D \doteq f \quad d \doteq L \quad h \doteq L \quad V \doteq LT^{-1} \quad \rho \doteq FL^{-3}T^{-2} \quad g \doteq LT^{-2}$$

From the pi theorem, $6-3=3$ pi terms required, and a dimensional analysis yields

$$\underline{\underline{\frac{D}{\rho V^2 h^2} = \phi\left(\frac{d}{h}, \frac{V}{\sqrt{g d}}\right)}}$$

(b) For similarity between model and prototype

$$\frac{d_m}{h_m} = \frac{d}{h} \quad \text{and} \quad \frac{V_m}{\sqrt{g_m d_m}} = \frac{V}{\sqrt{g d}}$$

$$\text{Thus, } \frac{d_m}{d} = \frac{h_m}{h} = \frac{1}{5}$$

$$\text{and } V_m = \sqrt{\frac{g_m}{g}} \sqrt{\frac{d_m}{d}} V = \sqrt{\frac{1}{5}} \left(9 \frac{\text{ft}}{\text{s}}\right) = \underline{\underline{4.02 \frac{\text{ft}}{\text{s}}}}$$

The prediction equation is

$$\frac{D}{\rho V^2 h^2} = \frac{D_m}{\rho_m V_m^2 h_m^2}$$

so that

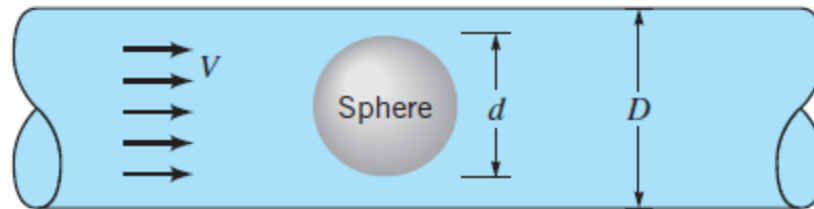
$$D = \frac{\rho}{\rho_m} \left(\frac{V}{V_m}\right)^2 \left(\frac{h}{h_m}\right)^2 D_m$$

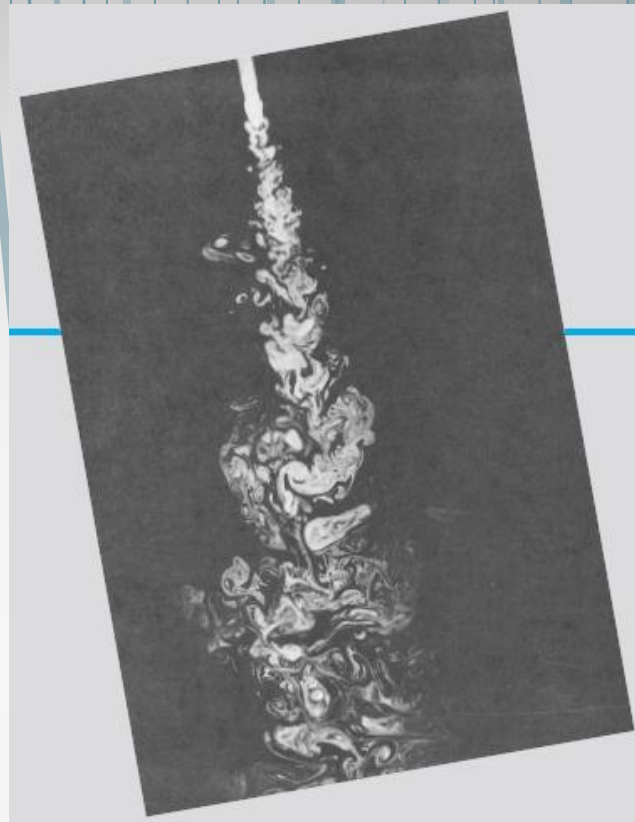
and with $\rho = \rho_m$

$$D = \left(\frac{9 \frac{\text{ft}}{\text{s}}}{4.02 \frac{\text{ft}}{\text{s}}}\right)^2 (5)^2 D_m = \underline{\underline{125 D_m}}$$

Problems

7.50 The drag, $\mathcal{D}$, on a sphere located in a pipe through which a fluid is flowing is to be determined experimentally (see Fig. P7.50). Assume that the drag is a function of the sphere diameter, d , the pipe diameter, D , the fluid velocity, V , and the fluid density, ρ . (a) What dimensionless parameters would you use for this problem? (b) Some experiments using water indicate that for $d = 0.2$ in., $D = 0.5$ in., and $V = 2$ ft/s, the drag is 1.5×10^{-3} lb. If possible, estimate the drag on a sphere located in a 2-ft-diameter pipe through which water is flowing with a velocity of 6 ft/s. The sphere diameter is such that geometric similarity is maintained. If it is not possible, explain why not.

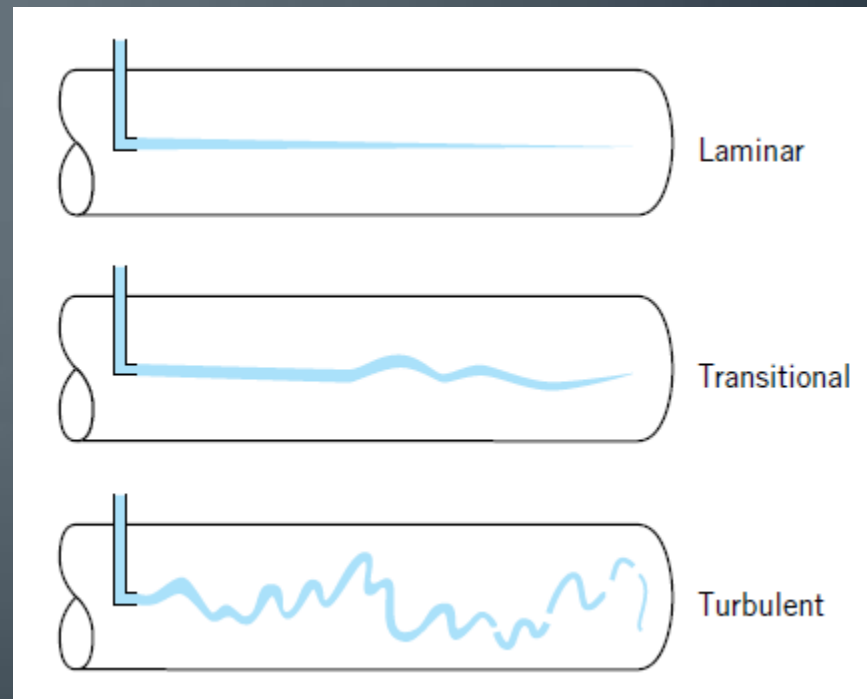
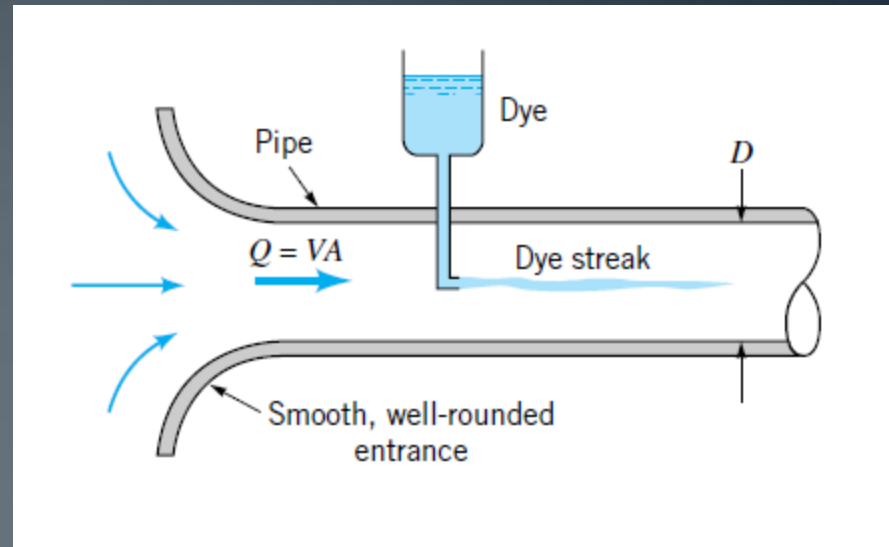




Viscous Flow in Pipes

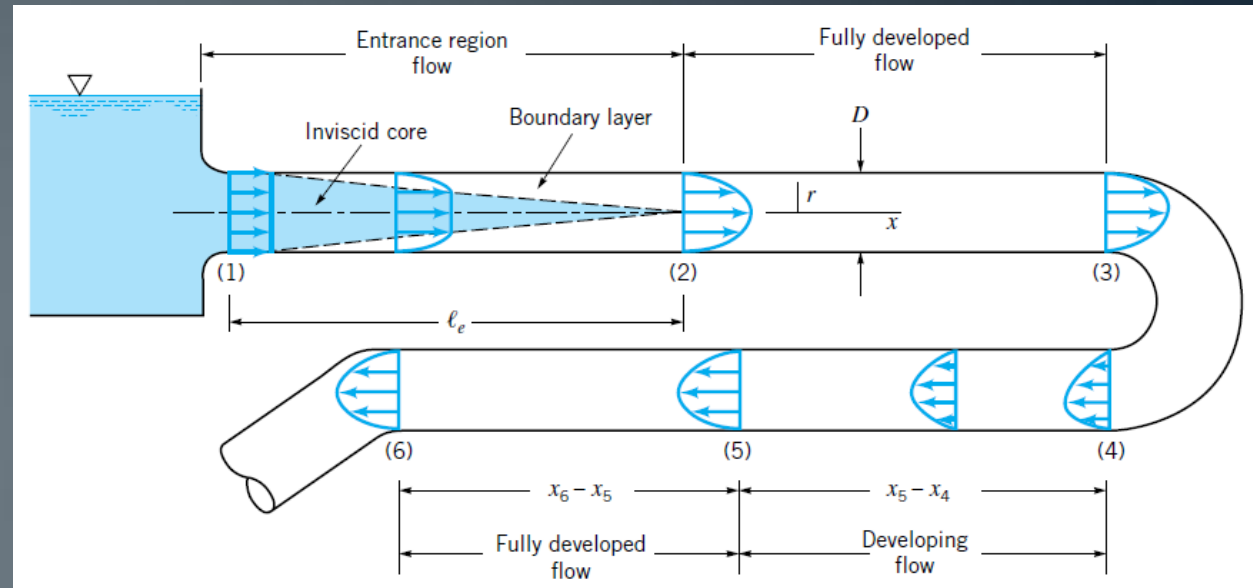
General Characteristics of Pipe Flow

- Laminar or Turbulent Flow



- Entrance Region and Fully Developed Flow

General Characteristics of Pipe Flow

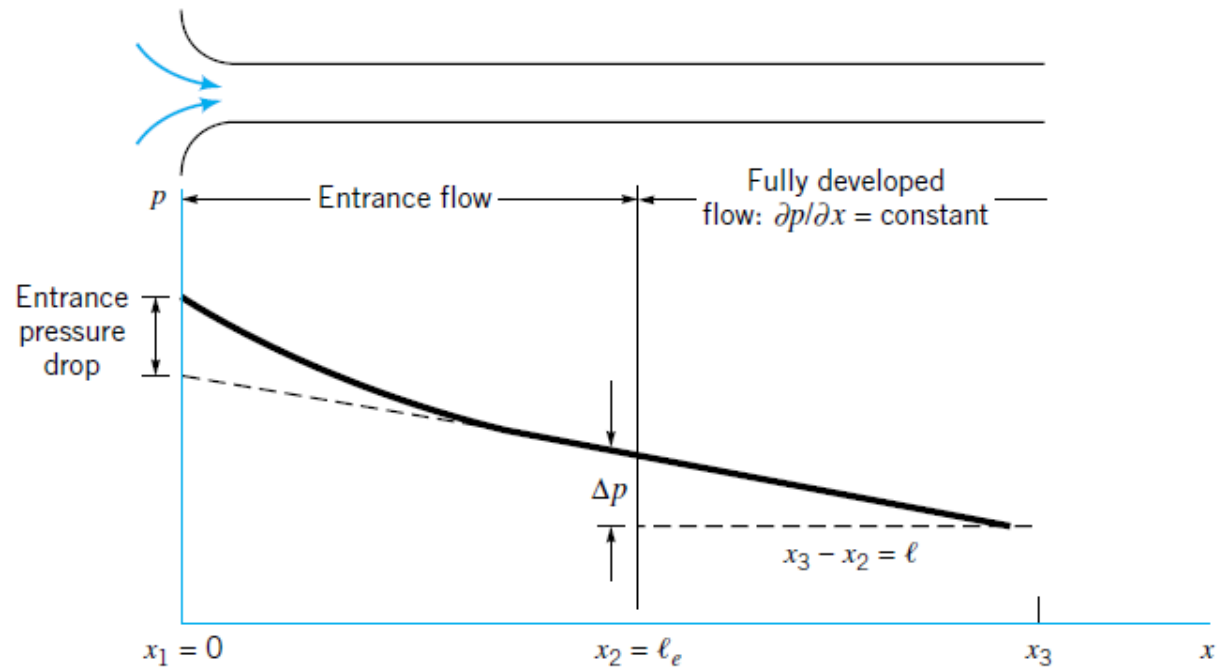


$$\frac{\ell_e}{D} = 0.06 \text{ Re for laminar flow}$$

$$\frac{\ell_e}{D} = 4.4 (\text{Re})^{1/6} \text{ for turbulent flow}$$

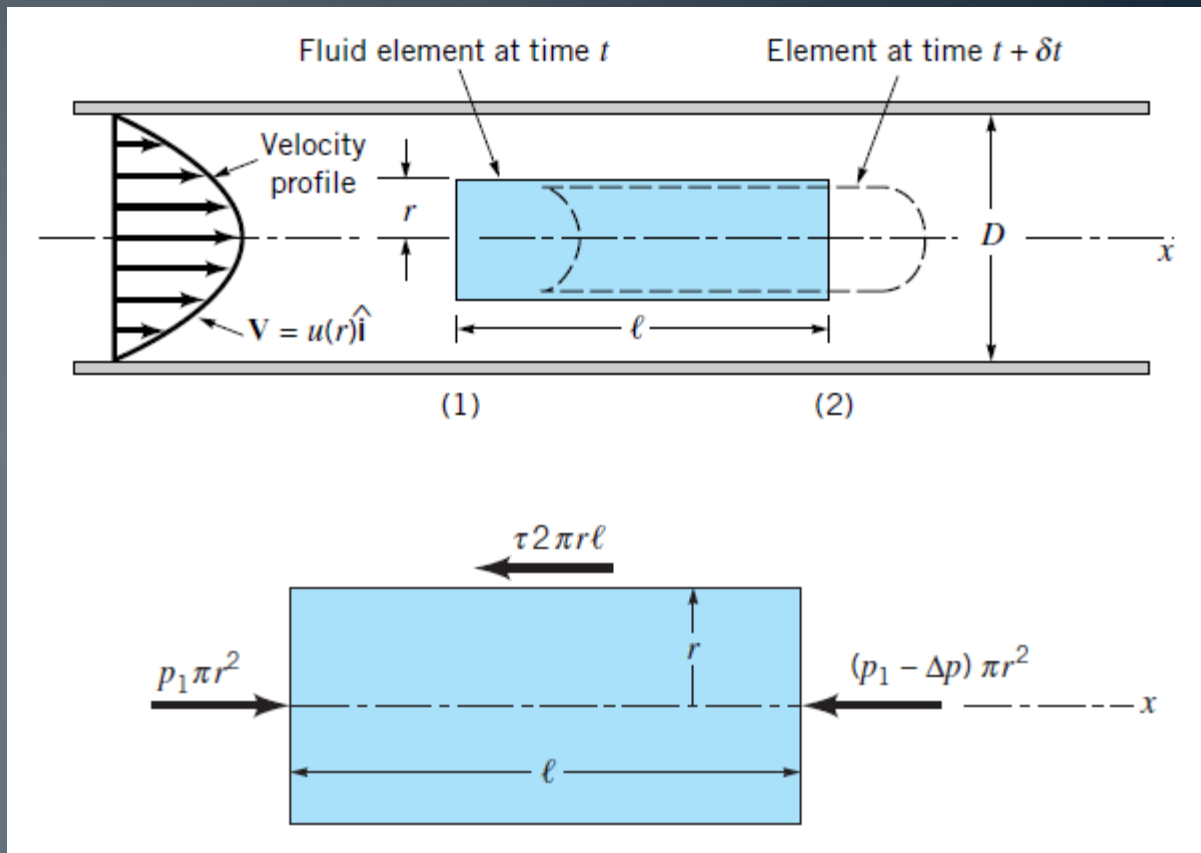
- Pressure and Shear Stress

General Characteristics of Pipe Flow



Fully Developed Laminar Flow

- From $F = ma$ Applied Directly to a Fluid Element



$$(p_1)\pi r^2 - (p_1 - \Delta p)\pi r^2 - (\tau)2\pi r \ell = 0$$

$$\frac{\Delta p}{\ell} = \frac{2\tau}{r}$$

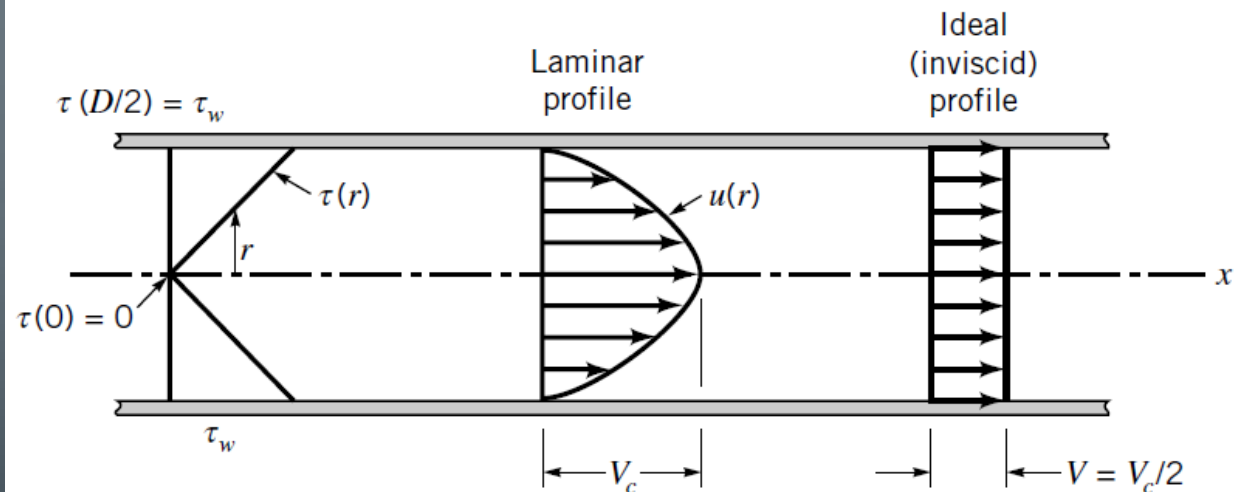
Fully Developed Laminar Flow

- From $F = ma$ Applied Directly to a Fluid Element

$$\frac{\Delta p}{\ell} = \frac{2\tau}{r}$$

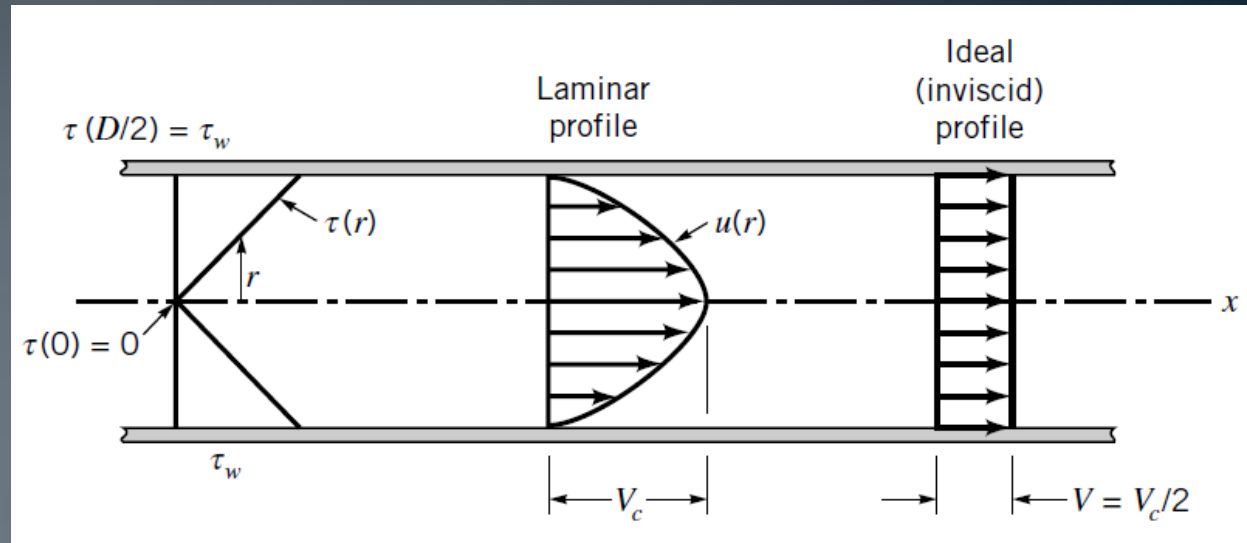
$$\tau = \frac{2\tau_w r}{D}$$

$$\Delta p = \frac{4\ell\tau_w}{D}$$



Fully Developed Laminar Flow

- From $F = ma$ Applied Directly to a Fluid Element



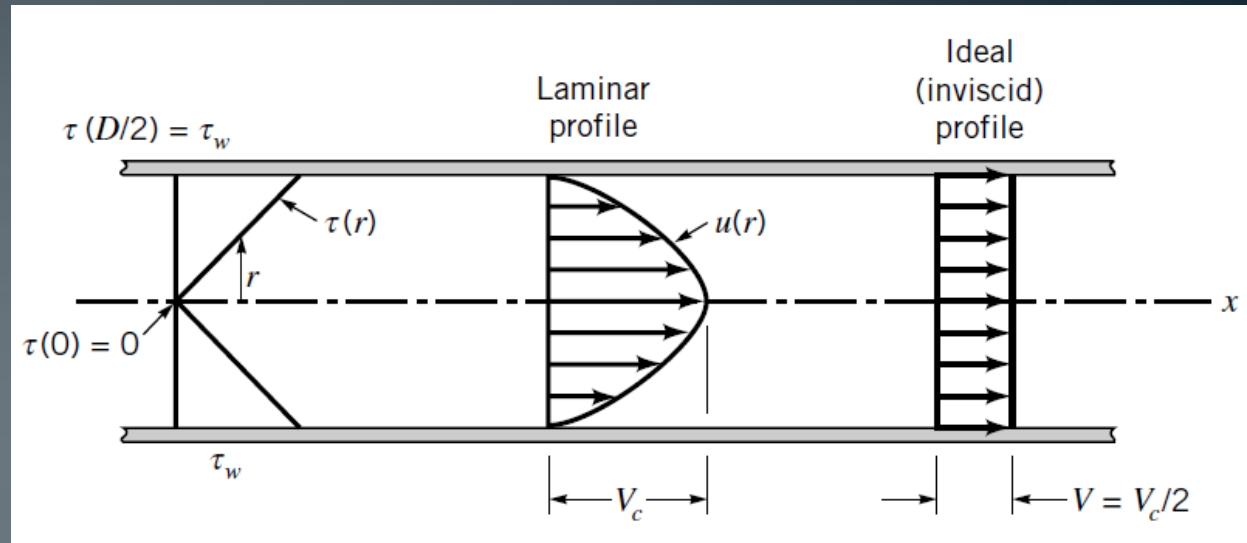
$$\frac{du}{dr} = -\left(\frac{\Delta p}{2\mu\ell}\right)r$$

$$u = -\left(\frac{\Delta p}{4\mu\ell}\right)r^2 + C_1$$

$$u(r) = \left(\frac{\Delta p D^2}{16\mu\ell}\right)\left[1 - \left(\frac{2r}{D}\right)^2\right] = V_c\left[1 - \left(\frac{2r}{D}\right)^2\right]$$

Fully Developed Laminar Flow

- From $F = ma$ Applied Directly to a Fluid Element



$$\frac{du}{dr} = -\left(\frac{\Delta p}{2\mu\ell}\right)r$$

$$u = -\left(\frac{\Delta p}{4\mu\ell}\right)r^2 + C_1$$

$$u(r) = \left(\frac{\Delta p D^2}{16\mu\ell}\right)\left[1 - \left(\frac{2r}{D}\right)^2\right] = V_c\left[1 - \left(\frac{2r}{D}\right)^2\right]$$

- From $F = ma$ Applied Directly to a Fluid Element

$$u(r) = \frac{\tau_w D}{4\mu} \left[1 - \left(\frac{r}{R} \right)^2 \right]$$

$$Q = \int u \, dA = \int_{r=0}^{r=R} u(r) 2\pi r \, dr$$

$$= 2\pi V_c \int_0^R \left[1 - \left(\frac{r}{R} \right)^2 \right] r \, dr$$

$$Q = \frac{\pi R^2 V_c}{2}$$

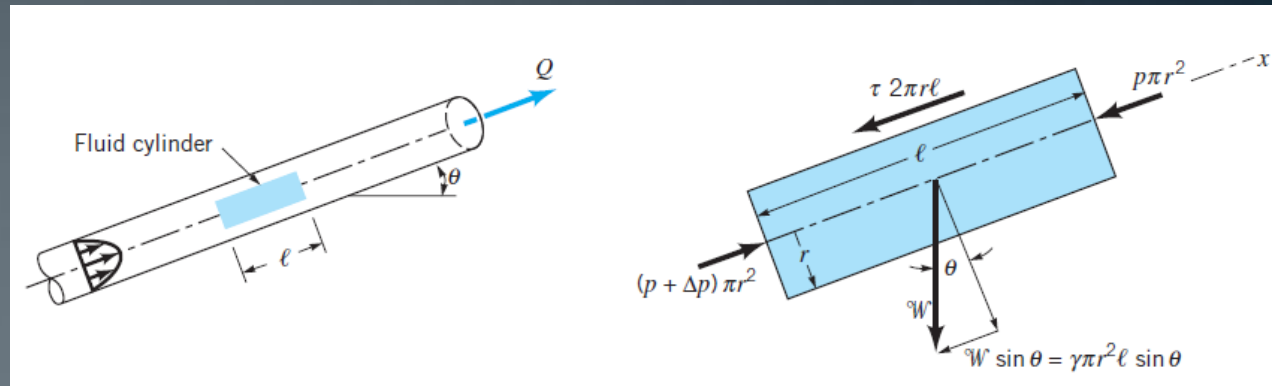
$$V = \frac{\pi R^2 V_c}{2\pi R^2} = \frac{V_c}{2} = \frac{\Delta p D^2}{32\mu \ell}$$

$$Q = \frac{\pi D^4 \Delta p}{128\mu \ell}$$

Fully
Developed
Laminar Flow

Fully Developed Laminar Flow

- From $F = ma$ Applied Directly to a Fluid Element



$$\frac{\Delta p - \gamma \ell \sin \theta}{\ell} = \frac{2\tau}{r}$$

$$V = \frac{(\Delta p - \gamma \ell \sin \theta) D^2}{32 \mu \ell}$$

$$Q = \frac{\pi (\Delta p - \gamma \ell \sin \theta) D^4}{128 \mu \ell}$$

- From the Navier–Stokes Equations

Fully
Developed
Laminar Flow

$$\nabla \cdot \mathbf{V} = 0$$

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = -\frac{\nabla p}{\rho} + \mathbf{g} + \nu \nabla^2 \mathbf{V}$$

$$\nabla \cdot \mathbf{V} = 0$$

$$\nabla p + \rho g \hat{\mathbf{k}} = \mu \nabla^2 \mathbf{V}$$

$$\frac{\partial p}{\partial x} + \rho g \sin \theta = \mu \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right)$$

Fully Developed Laminar Flow

- From Dimensional Analysis

$$\Delta p = F(V, \ell, D, \mu)$$

$$\frac{D \Delta p}{\mu V} = \phi\left(\frac{\ell}{D}\right)$$

- It seems reasonable to impose a further assumption that the pressure drop is directly proportional to the pipe length.

$$\frac{D \Delta p}{\mu V} = \frac{C \ell}{D}$$

$$\frac{\Delta p}{\ell} = \frac{C \mu V}{D^2}$$

$$Q = AV = \frac{(\pi/4C) \Delta p D^4}{\mu \ell}$$

Fully Developed Laminar Flow

- The value of C must be determined by theory or experiment. For a round pipe, $C = 32$.

$$\Delta p = 32\mu\ell V/D^2$$

- It is usually advantageous to describe a process in terms of dimensionless quantities.

$$\frac{\Delta p}{\frac{1}{2}\rho V^2} = \frac{(32\mu\ell V/D^2)}{\frac{1}{2}\rho V^2} = 64 \left(\frac{\mu}{\rho V D} \right) \left(\frac{\ell}{D} \right) = \frac{64}{\text{Re}} \left(\frac{\ell}{D} \right)$$

$$\Delta p = f \frac{\ell}{D} \frac{\rho V^2}{2}$$

$$f = \Delta p(D/\ell)/(\rho V^2/2)$$

- is termed the **friction factor**, or sometimes the **Darcy friction factor**.

Fully Developed Laminar Flow

$$f = \Delta p(D/\ell)/(\rho V^2/2)$$

- The friction factor for laminar fully developed pipe flow is simply

$$f = \frac{64}{\text{Re}}$$

- By substituting the pressure drop in terms of the wall shear stress, we obtain an alternate expression for the friction factor as a dimensionless wall shear stress

$$f = \frac{8\tau_w}{\rho V^2}$$

Fully Developed Laminar Flow

- **Energy Considerations**
- consider the energy equation for incompressible, steady flow between two locations

$$\frac{p_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \alpha_2 \frac{V_2^2}{2g} + z_2 + h_L$$

- Even though the velocity profile in viscous pipe flow is not uniform, for fully developed flow it does not change from section (1) to section (2)

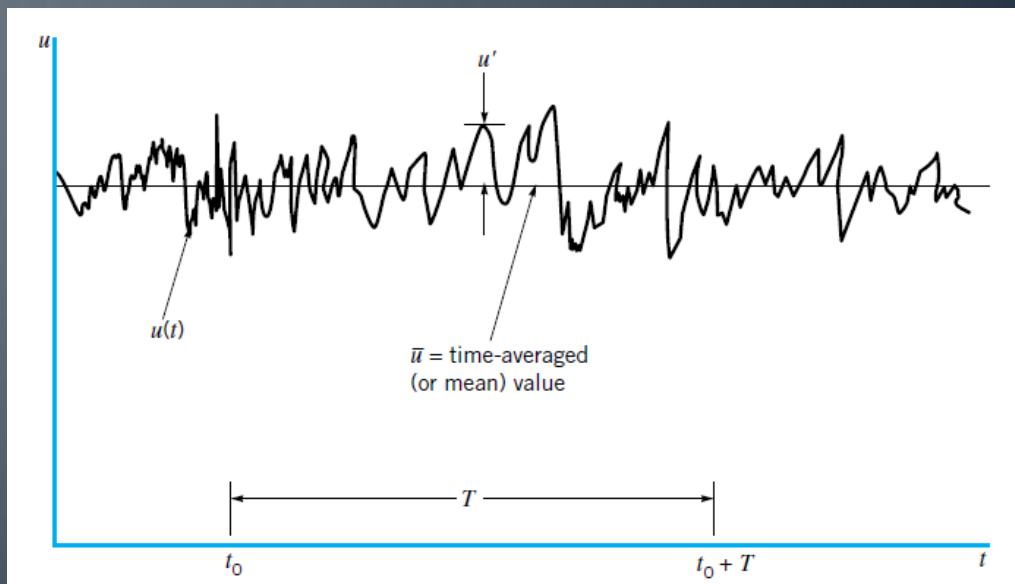
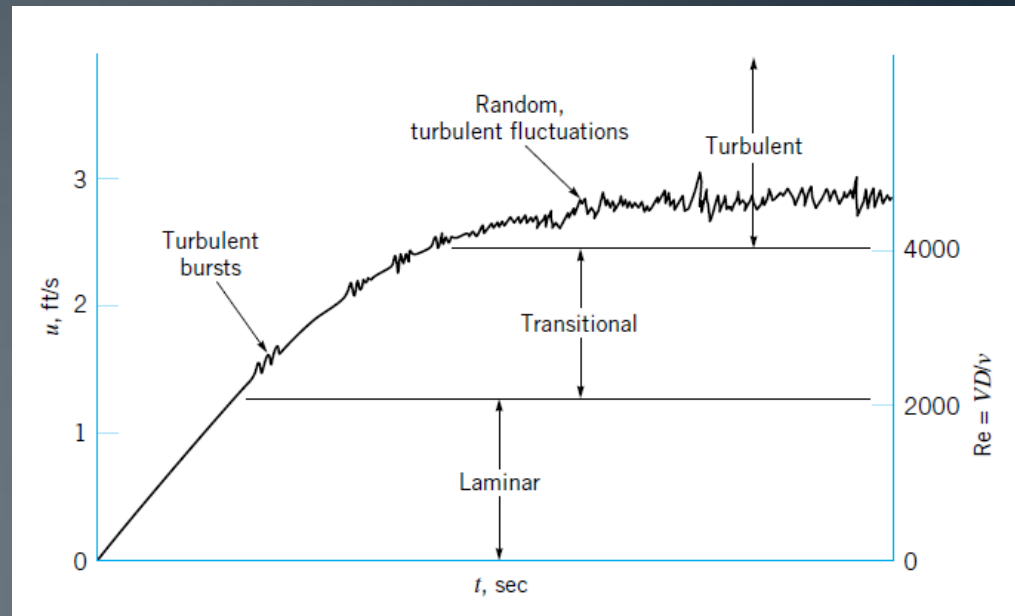
$$\left(\frac{p_1}{\gamma} + z_1 \right) - \left(\frac{p_2}{\gamma} + z_2 \right) = h_L$$

$$h_L = \frac{2\tau\ell}{\gamma r}$$

$$h_L = \frac{4\ell\tau_w}{\gamma D}$$

Fully Developed Turbulent Flow

- Transition from Laminar to Turbulent Flow



Fully Developed Turbulent Flow

- **Turbulent Shear Stress**
- If $u=u(x, y, z)$ is the x component of instantaneous velocity, then its **time mean** (or **time average**) value is

$$\bar{u} = \frac{1}{T} \int_{t_0}^{t_0+T} u(x, y, z, t) dt$$

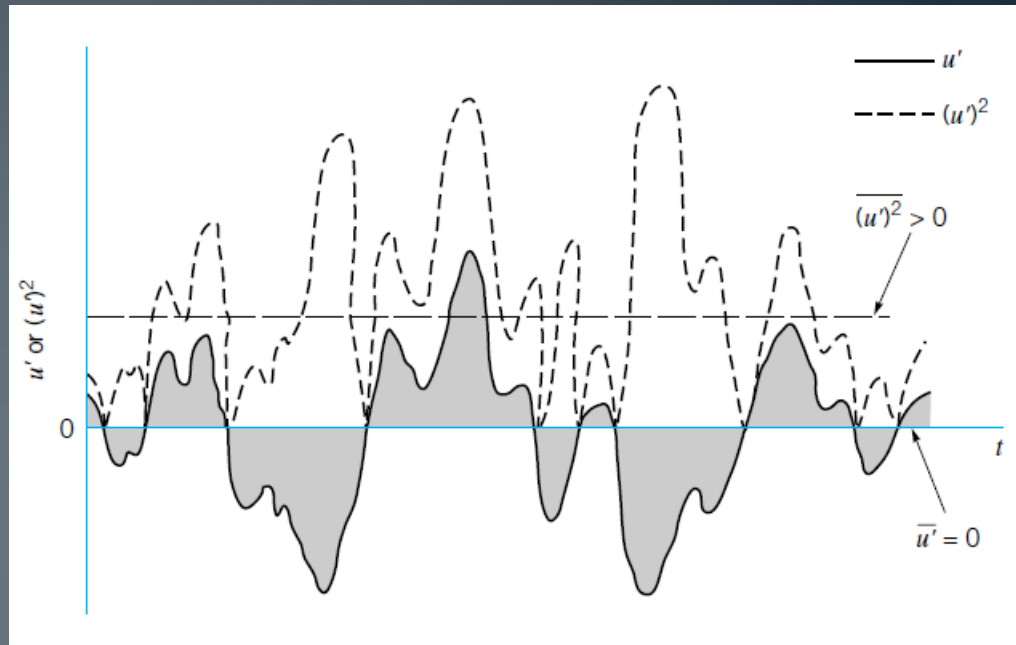
- The **fluctuating part** of the velocity, is that time-varying portion that differs from the average value

$$u = \bar{u} + u' \quad \text{or} \quad u' = u - \bar{u}$$

$$\begin{aligned} \overline{u'} &= \frac{1}{T} \int_{t_0}^{t_0+T} (u - \bar{u}) dt = \frac{1}{T} \left(\int_{t_0}^{t_0+T} u dt - \bar{u} \int_{t_0}^{t_0+T} dt \right) \\ &= \frac{1}{T} (T\bar{u} - T\bar{u}) = 0 \end{aligned}$$

Fully Developed Turbulent Flow

- Turbulent Shear Stress

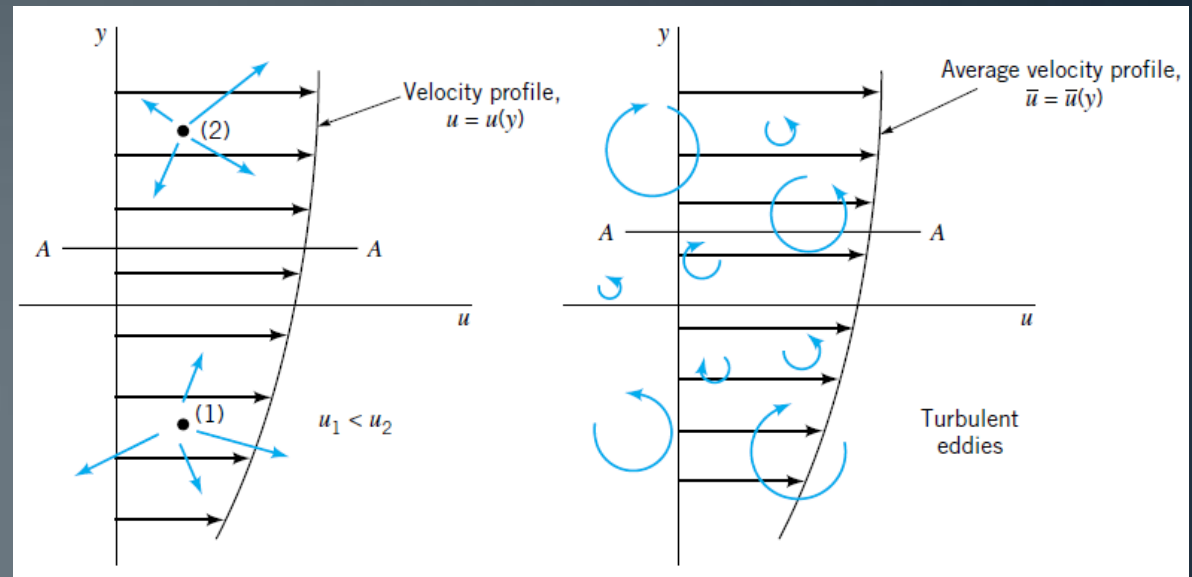


- The turbulence intensity, is often defined as the square root of the mean square of the fluctuating velocity divided by the time-averaged velocity, or

$$\mathcal{J} = \frac{\sqrt{\overline{(u')^2}}}{\bar{u}} = \frac{\left[\frac{1}{T} \int_{t_0}^{t_0+T} (u')^2 dt \right]^{1/2}}{\bar{u}}$$

Fully Developed Turbulent Flow

- Turbulent Shear Stress

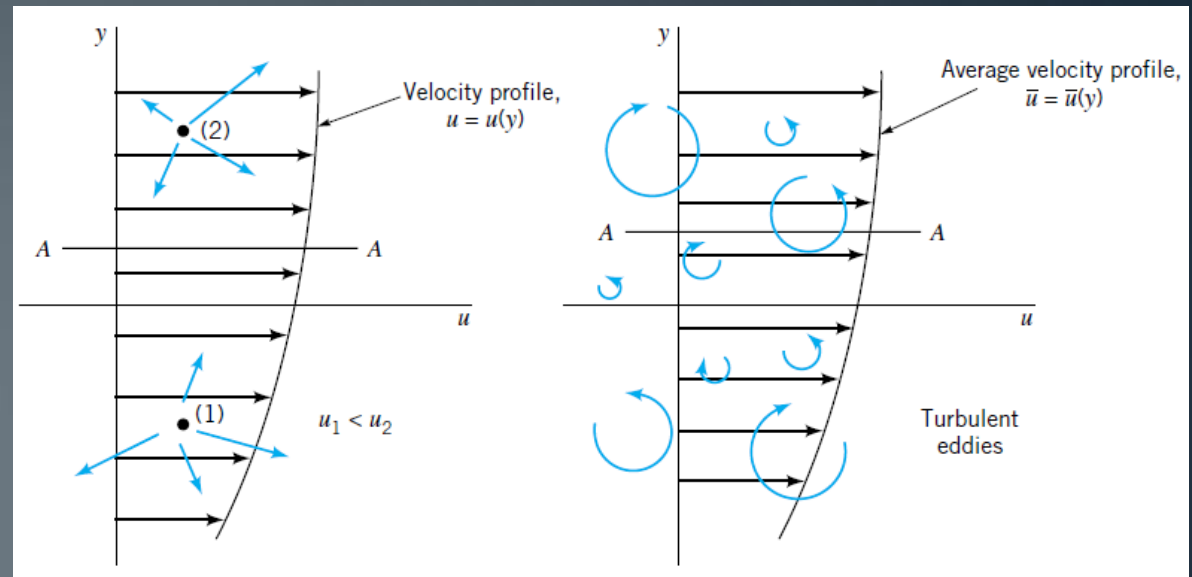


- Finite parcels of fluid (not merely individual molecules as in laminar flow) are randomly transported across the A-A plane, resulting in a relatively large (when compared with laminar flow) shear force.

$$\tau = \mu \frac{d\bar{u}}{dy} - \overline{\rho u'v'} = \tau_{\text{lam}} + \tau_{\text{turb}}$$

Fully Developed Turbulent Flow

- Turbulent Shear Stress

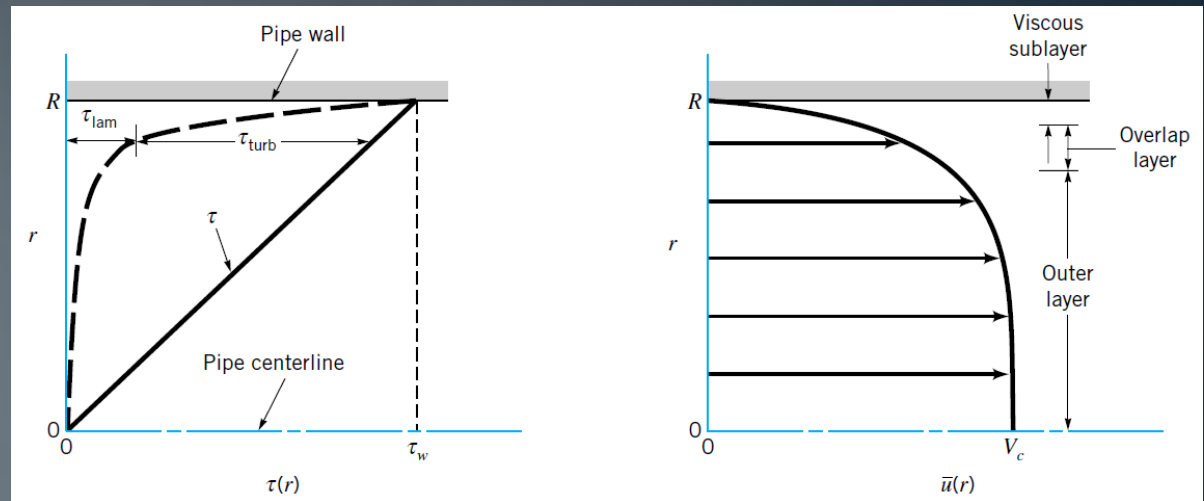


- Finite parcels of fluid (not merely individual molecules as in laminar flow) are randomly transported across the A-A plane, resulting in a relatively large (when compared with laminar flow) shear force.

$$\tau = \mu \frac{d\bar{u}}{dy} - \overline{\rho u'v'} = \tau_{\text{lam}} + \tau_{\text{turb}}$$

Fully Developed Turbulent Flow

- Turbulent Shear Stress



- In a very narrow region near the wall (the **viscous sublayer**), the laminar shear stress is dominant.
- Away from the wall (in the **outer layer**) the turbulent portion of the shear stress is dominant.
- The transition between these two regions occurs in the **overlap layer**.

$$\tau = \mu \frac{d\bar{u}}{dy} - \rho \overline{u'v'} = \tau_{\text{lam}} + \tau_{\text{turb}}$$

Fully Developed Turbulent Flow

- **Turbulent Shear Stress**
- An alternate form for the shear stress for turbulent flow is given in terms of the **eddy viscosity**, where

$$\tau = \eta \frac{d\bar{u}}{dy}$$

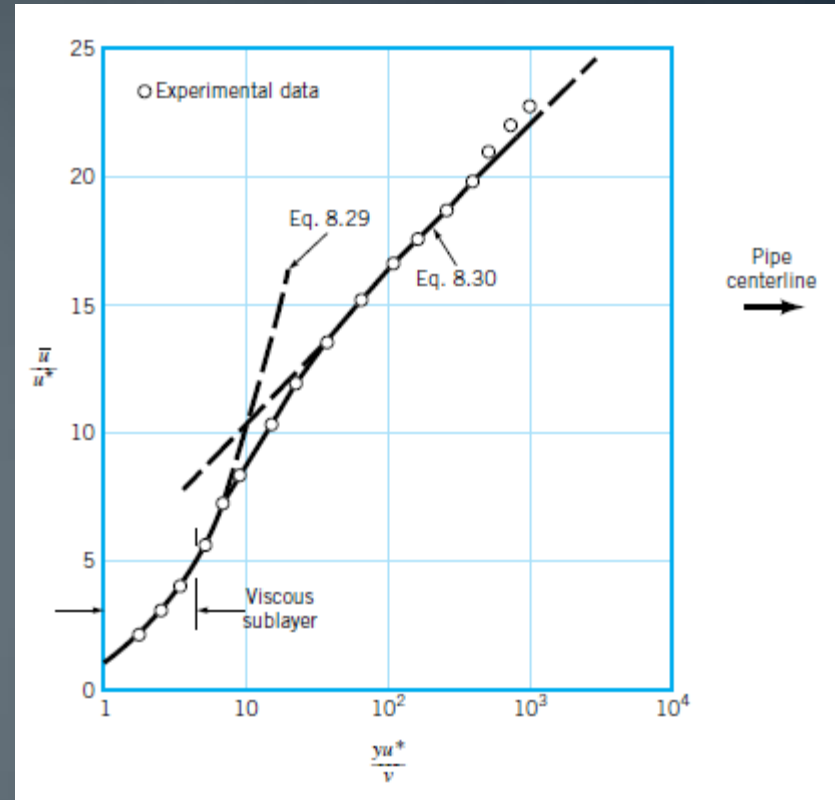
- **Prandtl** proposed that the turbulent process could be viewed as the random transport of bundles of fluid particles over a certain distance, the **mixing length**, from a region of one velocity to another region of a different velocity.

$$\eta = \rho \ell_m^2 \left| \frac{d\bar{u}}{dy} \right|$$

$$\tau_{\text{turb}} = \rho \ell_m^2 \left(\frac{d\bar{u}}{dy} \right)^2$$

Fully Developed Turbulent Flow

- Turbulent Velocity Profile



- Fully developed turbulent flow in a pipe can be broken into three regions which are characterized by their distances from the wall: the **viscous sublayer** very near the pipe wall, the **overlap region**, and the outer **turbulent layer** throughout the center portion of the flow.

Fully Developed Turbulent Flow

- **Turbulent Velocity Profile**
- In the **viscous sublayer** the velocity profile can be written in dimensionless form as (**law of the wall**)

$$\frac{\bar{u}}{u^*} = \frac{yu^*}{\nu}$$

Friction velocity

$$u^* = (\tau_w/\rho)^{1/2}$$

- The law of the wall is valid very near the smooth wall.

$$0 \leq yu^*/\nu \lesssim 5$$

Fully Developed Turbulent Flow

- **Turbulent Velocity Profile**
- Dimensional analysis arguments indicate that in the **overlap region** the velocity should vary as the logarithm of y .

$$\frac{\bar{u}}{u^*} = 2.5 \ln \left(\frac{yu^*}{\nu} \right) + 5.0$$

- where the constants 2.5 and 5.0 have been determined experimentally.
- In the central region (the **outer turbulent layer**) often-used correlation is the empirical **power-law velocity profile**

$$\frac{\bar{u}}{V_c} = \left(1 - \frac{r}{R} \right)^{1/n}$$

Fully Developed Turbulent Flow

- **Turbulent Velocity Profile**
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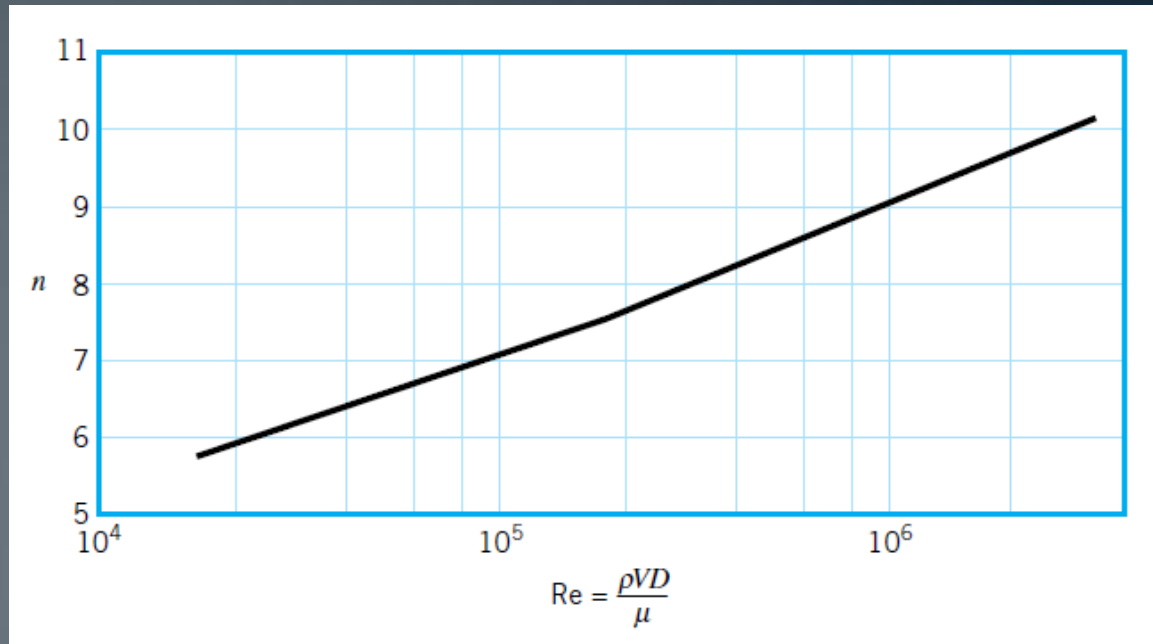
- where the constants 2.5 and 5.0 have been determined experimentally.
- In the **central region** (the **outer turbulent layer**) often-used correlation is the empirical power-law velocity profile

$$\frac{\bar{u}}{V_c} = \left(1 - \frac{r}{R} \right)^{1/n}$$

- The one-seventh **power-law velocity profile** is often used as a reasonable approximation for many practical flows.

Fully Developed Turbulent Flow

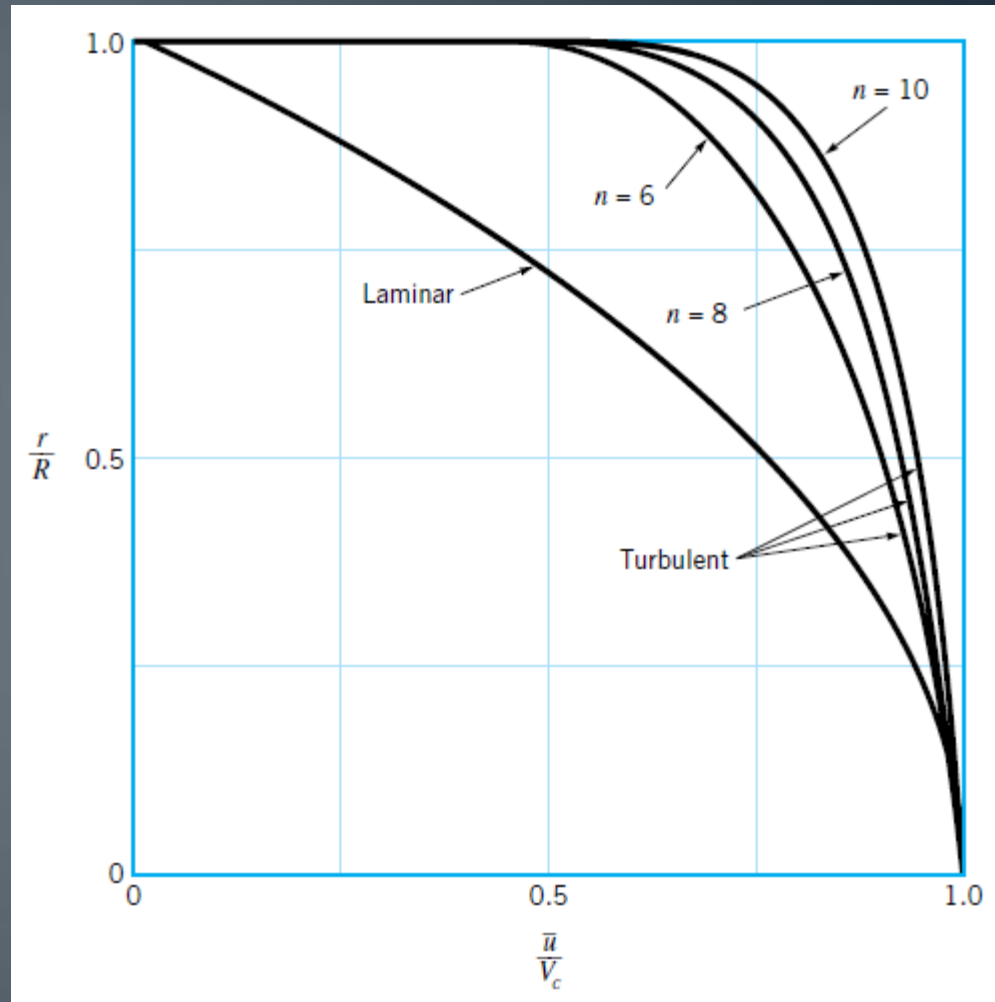
- Turbulent Velocity Profile



- Closer examination of the above equation shows that the power-law profile cannot be valid **near the wall**, since according to this equation the velocity gradient is infinite there. In addition, the equation cannot be precisely valid **near the centerline**.

- Turbulent Velocity Profile

Fully
Developed
Turbulent Flow



EXAMPLE 8.4

Water at 20 °C ($\rho = 998 \text{ kg/m}^3$ and $\nu = 1.004 \times 10^{-6} \text{ m}^2/\text{s}$) flows through a horizontal pipe of 0.1-m diameter with a flowrate of $Q = 4 \times 10^{-2} \text{ m}^3/\text{s}$ and a pressure gradient of 2.59 kPa/m. (a) Determine the approximate thickness of the viscous sublayer. (b) Determine the approximate centerline velocity, V_c . (c) Determine the ratio of the turbulent to laminar shear stress, $\tau_{\text{turb}}/\tau_{\text{lam}}$, at a point midway between the centerline and the pipe wall (i.e., at $r = 0.025 \text{ m}$).

SOLUTION

- (a) According to Fig. 8.16, the thickness of the viscous sublayer, δ_s , is approximately

$$\frac{\delta_s u^*}{\nu} = 5$$

or

$$\delta_s = 5 \frac{\nu}{u^*}$$

where

$$u^* = \left(\frac{\tau_w}{\rho} \right)^{1/2} \quad (1)$$

The wall shear stress can be obtained from the pressure drop data and Eq. 8.5, which is valid for either laminar or turbulent flow. Thus,

$$\tau_w = \frac{D \Delta p}{4\ell} = \frac{(0.1 \text{ m})(2.59 \times 10^3 \text{ N/m}^2)}{4(1 \text{ m})} = 64.8 \text{ N/m}^2$$

Hence, from Eq. 1 we obtain

$$u^* = \left(\frac{64.8 \text{ N/m}^2}{998 \text{ kg/m}^3} \right)^{1/2} = 0.255 \text{ m/s}$$

$$\delta_s = \frac{5(1.004 \times 10^{-6} \text{ m}^2/\text{s})}{0.255 \text{ m/s}} = 1.97 \times 10^{-5} \text{ m} \approx 0.02 \text{ mm} \quad (\text{Ans})$$

As stated previously, the viscous sublayer is very thin. Minute imperfections on the pipe wall will protrude into this sublayer and affect some of the characteristics of the flow (i.e., wall shear stress and pressure drop).

- (b) The centerline velocity can be obtained from the average velocity and the assumption of a power-law velocity profile as follows. For this flow with

$$V = \frac{Q}{A} = \frac{0.04 \text{ m}^3/\text{s}}{\pi(0.1 \text{ m})^2/4} = 5.09 \text{ m/s}$$

the Reynolds number is

$$\text{Re} = \frac{VD}{\nu} = \frac{(5.09 \text{ m/s})(0.1 \text{ m})}{(1.004 \times 10^{-6} \text{ m}^2/\text{s})} = 5.07 \times 10^5$$

Thus, from Fig. 8.17, $n = 8.4$ so that

$$\frac{\bar{u}}{V_c} \approx \left(1 - \frac{r}{R}\right)^{1/8.4}$$

To determine the centerline velocity, V_c , we must know the relationship between V (the average velocity) and V_c . This can be obtained by integration of the power-law velocity profile as follows. Since the flow is axisymmetric,

$$Q = AV = \int \bar{u} dA = V_c \int_{r=0}^{r=R} \left(1 - \frac{r}{R}\right)^{1/n} (2\pi r) dr$$

which can be integrated to give

$$Q = 2\pi R^2 V_c \frac{n^2}{(n+1)(2n+1)}$$

Thus, since $Q = \pi R^2 V$, we obtain

$$\frac{V}{V_c} = \frac{2n^2}{(n+1)(2n+1)}$$

With $n = 8.4$ in the present case, this gives

$$\begin{aligned} V_c &= \frac{(n+1)(2n+1)}{2n^2} V = 1.186V = 1.186 (5.09 \text{ m/s}) \\ &= 6.04 \text{ m/s} \end{aligned} \quad (\text{Ans})$$

Recall that $V_c = 2V$ for laminar pipe flow.

- (c) From Eq. 8.4, which is valid for laminar or turbulent flow, the shear stress at $r = 0.025 \text{ m}$ is

$$\tau = \frac{2\tau_w r}{D} = \frac{2(64.8 \text{ N/m}^2)(0.025 \text{ m})}{(0.1 \text{ m})}$$

or

$$\tau = \tau_{\text{lam}} + \tau_{\text{turb}} = 32.4 \text{ N/m}^2$$

where $\tau_{\text{lam}} = -\mu d\bar{u}/dr$. From the power-law velocity profile (Eq. 8.31) we obtain the gradient of the average velocity as

$$\frac{d\bar{u}}{dr} = -\frac{V_c}{nR} \left(1 - \frac{r}{R}\right)^{(1-n)/n}$$

which gives

$$\frac{d\bar{u}}{dr} = -\frac{(6.04 \text{ m/s})}{8.4(0.05\text{m})} \left(1 - \frac{0.025 \text{ m}}{0.05 \text{ m}}\right)^{(1-8.4)/8.4} = -26.5/\text{s}$$

Thus,

$$\begin{aligned}\tau_{\text{lam}} &= -\mu \frac{d\bar{u}}{dr} = -(\nu\rho) \frac{d\bar{u}}{dr} \\ &= -(1.004 \times 10^{-6} \text{ m}^2/\text{s})(998 \text{ kg/m}^3)(-26.5/\text{s}) \\ &= 0.0266 \text{ N/m}^2\end{aligned}$$

Thus, the ratio of turbulent to laminar shear stress is given by

$$\frac{\tau_{\text{turb}}}{\tau_{\text{lam}}} = \frac{\tau - \tau_{\text{lam}}}{\tau_{\text{lam}}} = \frac{32.4 - 0.0266}{0.0266} = 1220 \quad (\text{Ans})$$

As expected, most of the shear stress at this location in the turbulent flow is due to the turbulent shear stress.

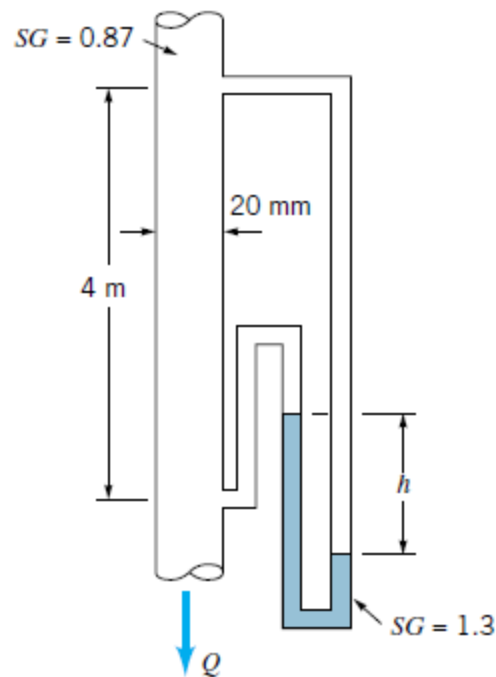
Problems

8.13 Some fluids behave as a non-Newtonian power-law fluid characterized by $\tau = -C(du/dr)^n$, where $n = 1, 3, 5$, and so on, and C is a constant. (If $n = 1$, the fluid is the customary Newtonian fluid.) For flow in a round pipe of a diameter D , integrate the force balance equation (Eq. 8.3) to obtain the velocity profile

$$u(r) = \frac{-n}{(n+1)} \left(\frac{\Delta p}{2\ell C} \right)^{1/n} \left[r^{(n+1)/n} - \left(\frac{D}{2} \right)^{(n+1)/n} \right]$$

Problems

8.22 Oil of $SG = 0.87$ and a kinematic viscosity $\nu = 2.2 \times 10^{-4} \text{ m}^2/\text{s}$ flows through the vertical pipe shown in Fig. P8.22 at a rate of $4 \times 10^{-4} \text{ m}^3/\text{s}$. Determine the manometer reading, h .



Problems

$$V = \frac{Q}{A} = \frac{4 \times 10^{-4} \frac{m^3}{s}}{\frac{\pi}{4} (0.02m)^2} = 1.27 \frac{m}{s} \text{ so that}$$

$$Re = \frac{\rho V D}{\mu} = \frac{V D}{\nu} = \frac{(1.27 \frac{m}{s})(0.02m)}{2.2 \times 10^{-4} \frac{m^2}{s}} = 115 < 2100$$

The flow is laminar with

$$Q = \frac{\pi(\Delta p - \gamma L) D^4}{128 \mu L}, \text{ or } \Delta p = p_1 - p_2 = \frac{128 \mu L Q}{\pi D^4} + \gamma L$$

Hence, with $\gamma = SG \gamma_{H_2O} = 0.87(9.81 \frac{kN}{m^3}) = 8.53 \frac{kN}{m^3}$ and

$$\mu = \nu \rho = \nu SG \rho_{H_2O} = (2.2 \times 10^{-4} \frac{m^2}{s})(0.87)(1000 \frac{kg}{m^3}) = 0.191 \frac{N \cdot s}{m^2}$$

Eq. (1) gives

$$\Delta p = \frac{128 (0.191 \frac{N \cdot s}{m^2})(4m)(4 \times 10^{-4} \frac{m^3}{s})}{\pi (0.020m)^4} + (8.53 \frac{kN}{m^3})(4m)(10^{-3} \frac{N}{kN})$$

or

$$\Delta p = 1.119 \times 10^5 \frac{N}{m^2} = 111.9 \frac{kN}{m^2}$$

Problems

From manometer considerations

$$p_1 - \delta h_1 + \delta_m h - \delta h_2 = p_2, \text{ where } \delta_m = SG_m \delta_{H_2O} = 1.3(9.81 \frac{kN}{m^3}) = 12.74 \frac{kN}{m^3}$$

$$\text{and } h_2 = \ell + h - h_1 \text{ or } h_2 + h_1 = \ell + h$$

Thus,

$$p_1 - p_2 = \Delta p = \delta(h_2 + h_1) - \delta_m h = -(\delta_m - \delta)h + \delta \ell \quad (3)$$

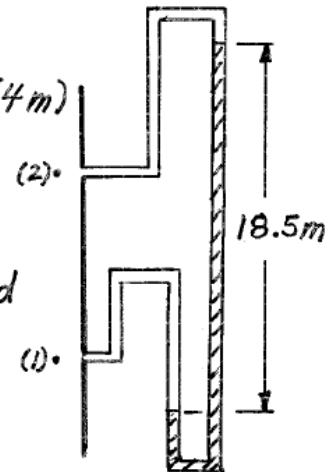
Combine Eqs. (2) and (3) to give

$$111.9 \frac{kN}{m^2} = -(12.74 - 8.53) \frac{kN}{m^3} h + 8.53 \frac{kN}{m^3} (4m)$$

or

$$h = \underline{\underline{-18.5m}}$$

Note: Since $h < 0$ the manometer is displaced in the direction opposite that shown in the (1). original figure.



Dimensional Analysis of Pipe Flow

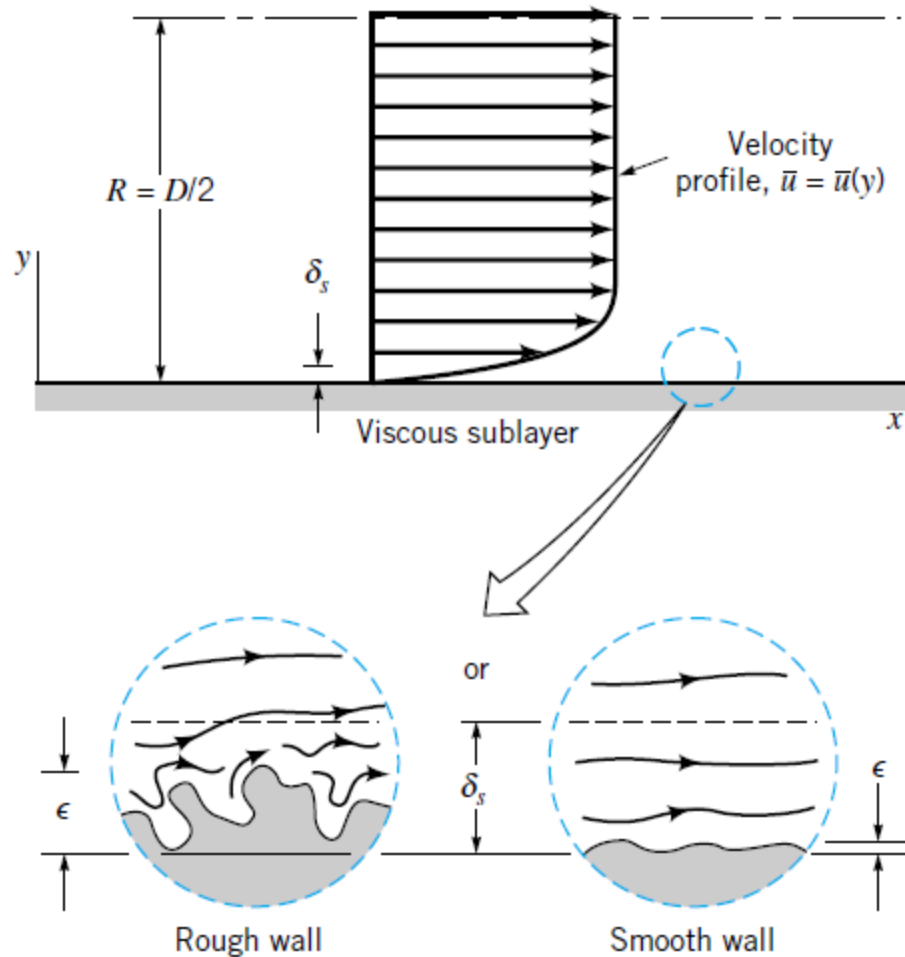
- The Moody Chart
- The pressure drop, for steady, incompressible turbulent flow in a horizontal round pipe of diameter D can be written in functional form as

$$\Delta p = F(V, D, \ell, \varepsilon, \mu, \rho)$$

- Although the pressure drop for laminar pipe flow is found to be independent of the roughness of the pipe, it is necessary to include this parameter when considering turbulent flow.

- The Moody Chart

Dimensional Analysis of Pipe Flow



Dimensional Analysis of Pipe Flow

- The Moody Chart

$$\Delta p = F(V, D, \ell, \varepsilon, \mu, \rho)$$

$$\frac{\Delta p}{\frac{1}{2}\rho V^2} = \tilde{\phi}\left(\frac{\rho V D}{\mu}, \frac{\ell}{D}, \frac{\varepsilon}{D}\right)$$

- As was done for laminar flow, the functional representation can be simplified by imposing the reasonable assumption that the pressure drop should be proportional to the pipe length.

$$\frac{\Delta p}{\frac{1}{2}\rho V^2} = \frac{\ell}{D} \phi\left(\text{Re}, \frac{\varepsilon}{D}\right)$$

Dimensional Analysis of Pipe Flow

- The Moody Chart

$$\frac{\Delta p}{\frac{1}{2}\rho V^2} = \frac{\ell}{D} \phi \left(\text{Re}, \frac{\varepsilon}{D} \right)$$

$$\Delta p = f \frac{\ell}{D} \frac{\rho V^2}{2}$$

$$f = \phi \left(\text{Re}, \frac{\varepsilon}{D} \right)$$

- For laminar fully developed flow, the value of f is simply

$$f = 64/\text{Re}$$

Dimensional Analysis of Pipe Flow

- The **Moody Chart**
- The energy equation for steady incompressible flow is

$$\frac{p_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \alpha_2 \frac{V_2^2}{2g} + z_2 + h_L$$

- With the assumption of a constant diameter so that horizontal pipe with fully developed flow this becomes

$$\Delta p = p_1 - p_2 = \gamma h_L$$

$$h_L = f \frac{\ell}{D} \frac{V^2}{2g}$$

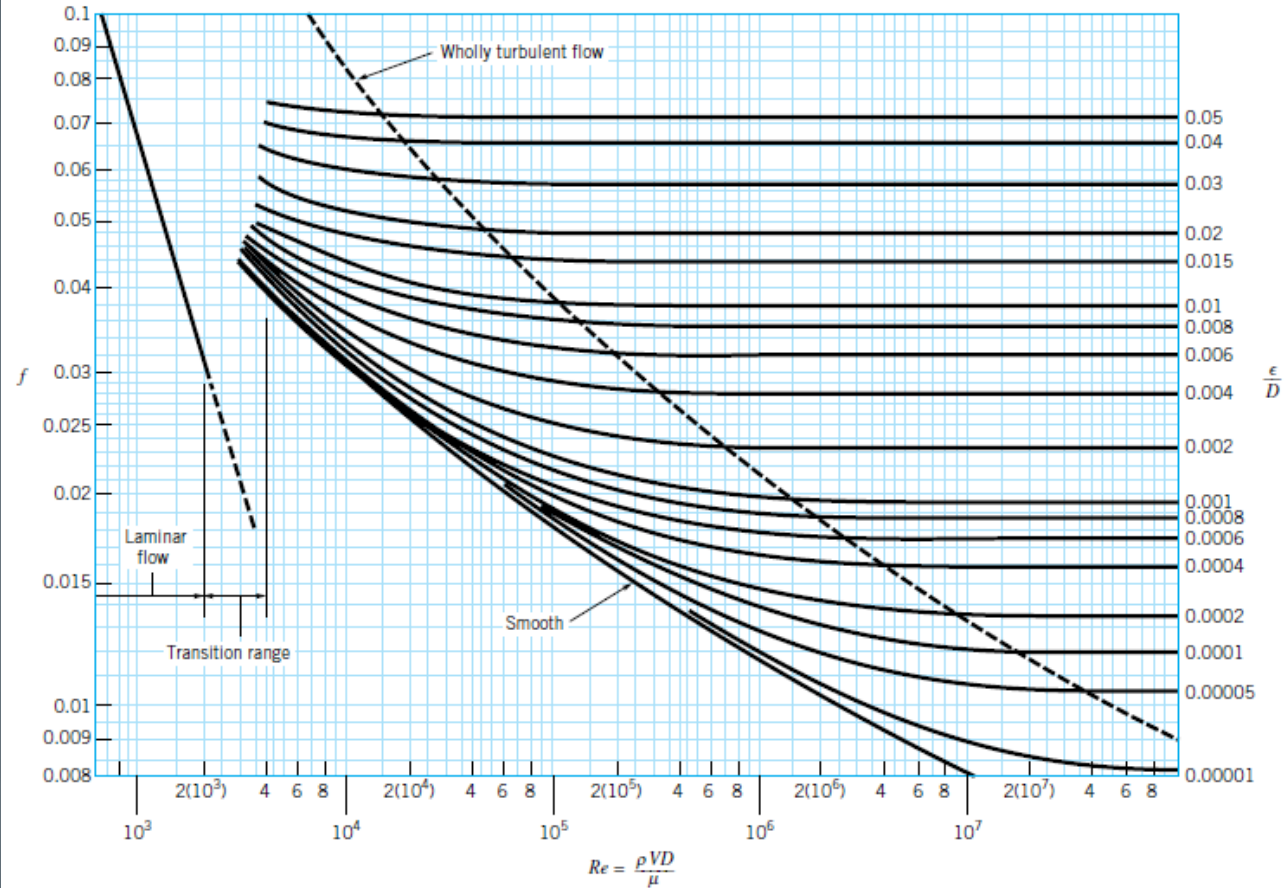
- The above equation called the **Darcy–Weisbach equation**, is valid for any fully developed, steady, incompressible pipe flow—whether the pipe is horizontal or on a hill.

Dimensional Analysis of Pipe Flow

- The Moody Chart
- It is not easy to determine the functional dependence of the friction factor on the Reynolds number and relative roughness.
- Much of this information is a result of experiments conducted by J. Nikuradse in 1933 and amplified by many others since then.
- L. F. Moody, who, along with C. F. Colebrook, correlated the original data of Nikuradse in terms of the relative roughness of commercially available pipe materials.

- The Moody Chart

Dimensional Analysis of Pipe Flow



Dimensional Analysis of Pipe Flow

- The Moody Chart
- For very large Reynolds numbers, f is independent of the Reynolds number.
- For such flows, commonly termed **completely turbulent flow** (or **wholly turbulent flow**), the laminar sublayer is so thin (its thickness decreases with increasing Re) that the surface roughness completely dominates the character of the flow near the wall.
- For flows with moderate values of Re , the friction factor is indeed dependent on both the Reynolds number and relative roughness.
- The gap in the figure for which no values of f are given is a result of the fact that the flow in this **transition range** may be laminar or turbulent.
- Note that even for smooth pipes (called **hydraulically smooth**) the friction factor is not zero.

Dimensional Analysis of Pipe Flow

- The Moody Chart
- Various investigators have attempted to obtain an analytical expression for f .
- The following equation from Colebrook is valid for the entire nonlaminar range of the Moody chart

$$\frac{1}{\sqrt{f}} = -2.0 \log \left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{\text{Re} \sqrt{f}} \right)$$

EXAMPLE 8.5

Air under standard conditions flows through a 4.0-mm-diameter drawn tubing with an average velocity of $V = 50$ m/s. For such conditions the flow would normally be turbulent. However, if precautions are taken to eliminate disturbances to the flow (the entrance to the tube is very smooth, the air is dust free, the tube does not vibrate, etc.), it may be possible to maintain laminar flow. (a) Determine the pressure drop in a 0.1-m section of the tube if the flow is laminar. (b) Repeat the calculations if the flow is turbulent.

SOLUTION

Under standard temperature and pressure conditions the density and viscosity are $\rho = 1.23 \text{ kg/m}^3$ and $\mu = 1.79 \times 10^{-5} \text{ N} \cdot \text{s/m}^2$. Thus, the Reynolds number is

$$\text{Re} = \frac{\rho V D}{\mu} = \frac{(1.23 \text{ kg/m}^3)(50 \text{ m/s})(0.004 \text{ m})}{1.79 \times 10^{-5} \text{ N} \cdot \text{s/m}^2} = 13,700$$

which would normally indicate turbulent flow.

- (a) If the flow were laminar, then $f = 64/\text{Re} = 64/13,700 = 0.00467$ and the pressure drop in a 0.1-m-long horizontal section of the pipe would be

$$\Delta p = f \frac{\ell}{D} \frac{1}{2} \rho V^2 = (0.00467) \frac{(0.1 \text{ m})}{(0.004 \text{ m})} \frac{1}{2} (1.23 \text{ kg/m}^3)(50 \text{ m/s})^2$$

or

$$\Delta p = 0.179 \text{ kPa} \quad (\text{Ans})$$

Note that the same result is obtained from Eq. 8.8.

$$\Delta p = \frac{32\mu\ell}{D^2} V = \frac{32(1.79 \times 10^{-5} \text{ N} \cdot \text{s/m}^2)(0.1 \text{ m})(50 \text{ m/s})}{(0.004 \text{ m})^2} = 179 \text{ N/m}^2$$

- (b) If the flow were turbulent, then $f = \phi(\text{Re}, \varepsilon/D)$, where from Table 8.1, $\varepsilon = 0.0015 \text{ mm}$ so that $\varepsilon/D = 0.0015 \text{ mm}/4.0 \text{ mm} = 0.000375$. From the Moody chart with $\text{Re} = 1.37 \times 10^4$ and $\varepsilon/D = 0.000375$ we obtain $f = 0.028$. Thus, the pressure drop in this case would be approximately

$$\Delta p = f \frac{\ell}{D} \frac{1}{2} \rho V^2 = (0.028) \frac{(0.1 \text{ m})}{(0.004 \text{ m})} \frac{1}{2} (1.23 \text{ kg/m}^3)(50 \text{ m/s})^2$$

or

$$\Delta p = 1.076 \text{ kPa} \quad (\text{Ans})$$

A considerable savings in effort to force the fluid through the pipe could be realized (0.179 kPa rather than 1.076 kPa) if the flow could be maintained as laminar flow at this Reynolds number. In general this is very difficult to do, although laminar flow in pipes has been maintained up to $\text{Re} \approx 100,000$ in rare instances.

An alternate method to determine the friction factor for the turbulent flow would be to use the Colebrook formula, Eq. 8.35. Thus,

$$\frac{1}{\sqrt{f}} = -2.0 \log \left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{\text{Re} \sqrt{f}} \right) = -2.0 \log \left(\frac{0.000375}{3.7} + \frac{2.51}{1.37 \times 10^4 \sqrt{f}} \right)$$

or

$$\frac{1}{\sqrt{f}} = -2.0 \log \left(1.01 \times 10^{-4} + \frac{1.83 \times 10^{-4}}{\sqrt{f}} \right) \quad (1)$$

An iterative procedure to obtain f can be done as follows. We assume a value of f ($f = 0.02$, for example), substitute it into the right-hand side of Eq. 1, and calculate a new f ($f = 0.0307$ in this case). Since the two values do not agree, the assumed value is not the solution. Hence, we try again. This time we assume $f = 0.0307$ (the last value calculated) and calculate the new value as $f = 0.0289$. Again this is still not the solution. Two more iterations show that the assumed and calculated values converge to the solution $f = 0.0291$, in agreement (within the accuracy of reading the graph) with the Moody chart method of $f = 0.028$.

Numerous other empirical formulas can be found in the literature (Ref. 5) for portions of the Moody chart. For example, an often-used equation, commonly referred to as the Blasius formula, for turbulent flow in smooth pipes ($\varepsilon/D = 0$) with $\text{Re} < 10^5$ is

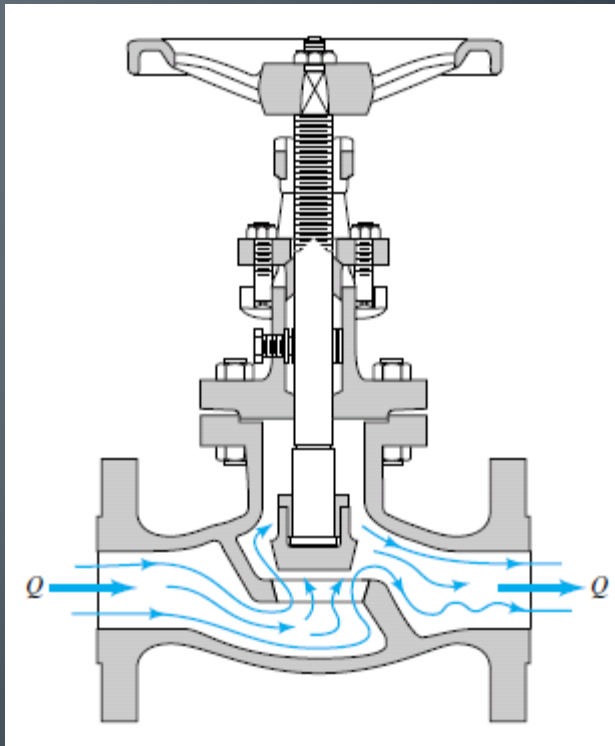
$$f = \frac{0.316}{\text{Re}^{1/4}}$$

For our case this gives

$$f = 0.316(13,700)^{-0.25} = 0.0292$$

Dimensional Analysis of Pipe Flow

- **Minor Losses**
- The flow pattern through a typical component such as a valve is shown below.
- the head loss information for essentially all components is given in dimensionless form and based on experimental data.



Dimensional Analysis of Pipe Flow

- **Minor Losses**
- The most common method used to determine these head losses or pressure drops is to specify the **loss coefficient**, which is defined as

$$K_L = \frac{h_L}{(V^2/2g)} = \frac{\Delta p}{\frac{1}{2}\rho V^2}$$

$$\Delta p = K_L \frac{1}{2}\rho V^2$$

$$h_L = K_L \frac{V^2}{2g}$$

Dimensional Analysis of Pipe Flow

- **Minor Losses**
- The actual value of loss coefficient is strongly dependent on the geometry of the component considered. It may also be dependent on the fluid properties. That is,

$$K_L = \phi(\text{geometry}, \text{Re})$$

- In most cases of practical interest the loss coefficients for components are a function of geometry only,

$$K_L = \phi(\text{geometry})$$

Dimensional Analysis of Pipe Flow

- **Minor Losses**
- Minor losses are sometimes given in terms of an **equivalent length**,

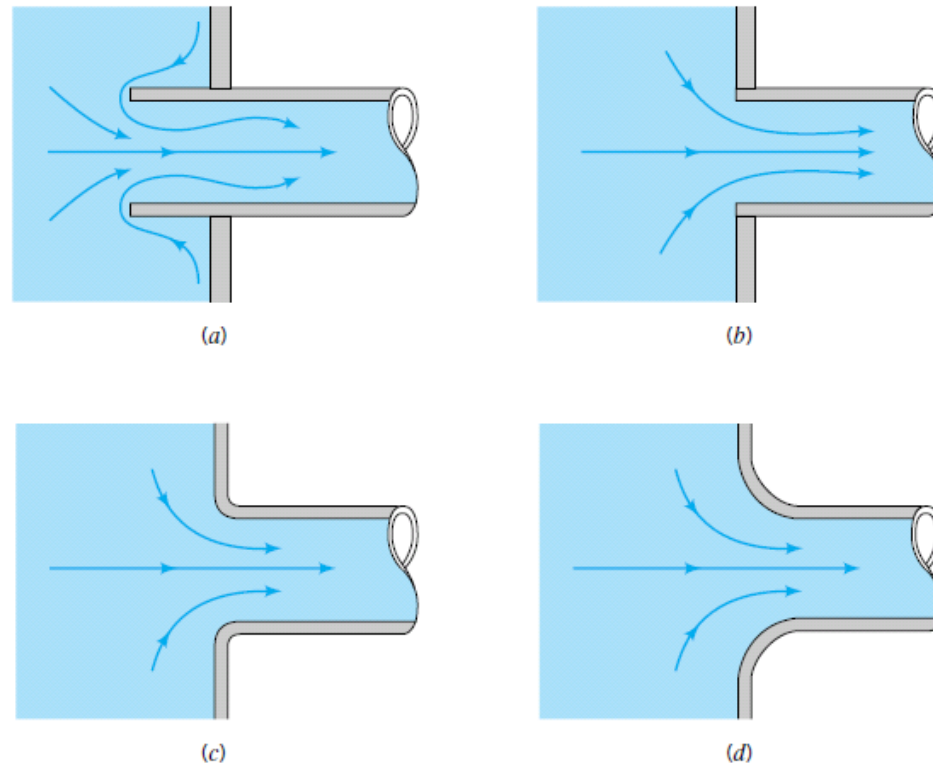
$$h_L = K_L \frac{V^2}{2g} = f \frac{\ell_{eq}}{D} \frac{V^2}{2g}$$

$$\ell_{eq} = \frac{K_L D}{f}$$

- Many pipe systems contain various transition sections in which the pipe diameter changes from one size to another.
- The extreme cases involve flow into a pipe from a reservoir (**an entrance**) or out of a pipe into a reservoir (**an exit**).

- Minor Losses

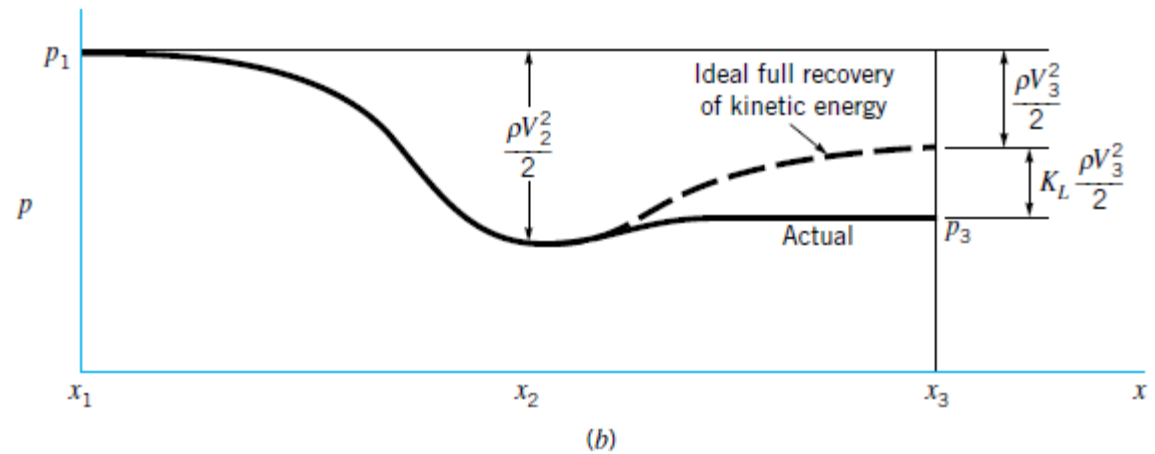
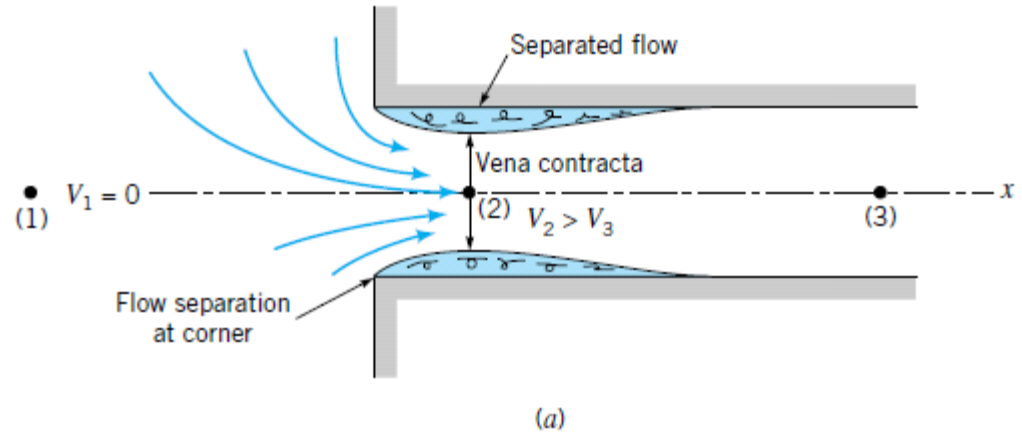
Dimensional Analysis of Pipe Flow



■ **FIGURE 8.22** Entrance flow conditions and loss coefficient (Refs. 28, 29). (a) Reentrant, $K_L = 0.8$, (b) sharp-edged, $K_L = 0.5$, (c) slightly rounded, $K_L = 0.2$ (see Fig. 8.24), (d) well-rounded, $K_L = 0.04$ (see Fig. 8.24).

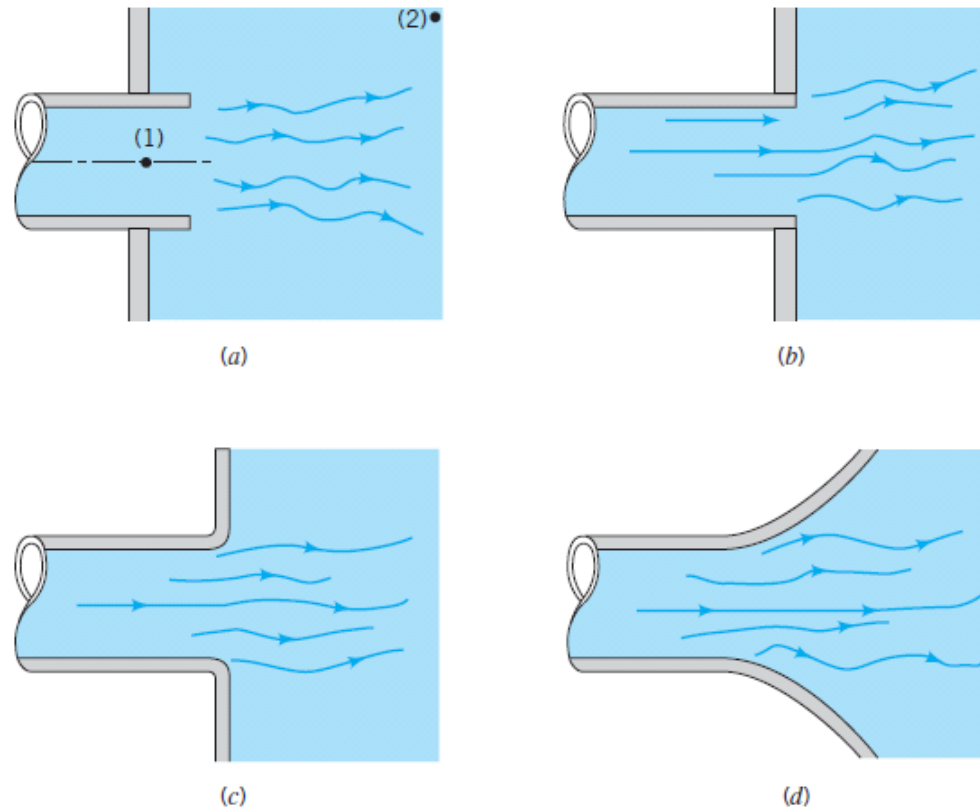
Dimensional Analysis of Pipe Flow

- Minor Losses



Dimensional Analysis of Pipe Flow

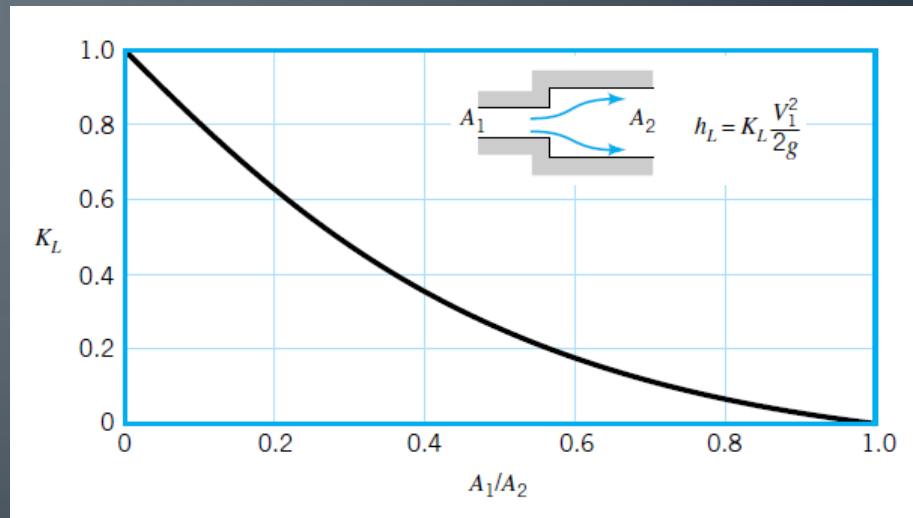
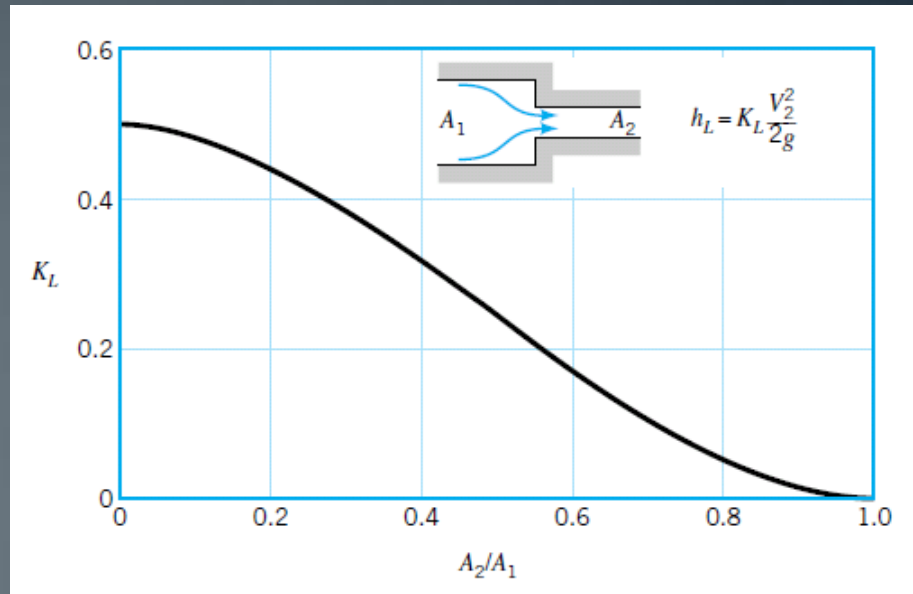
- Minor Losses



■ **FIGURE 8.25** Exit flow conditions and loss coefficient. (a) Reentrant, $K_L = 1.0$, (b) sharp-edged, $K_L = 1.0$, (c) slightly rounded, $K_L = 1.0$, (d) well-rounded, $K_L = 1.0$.

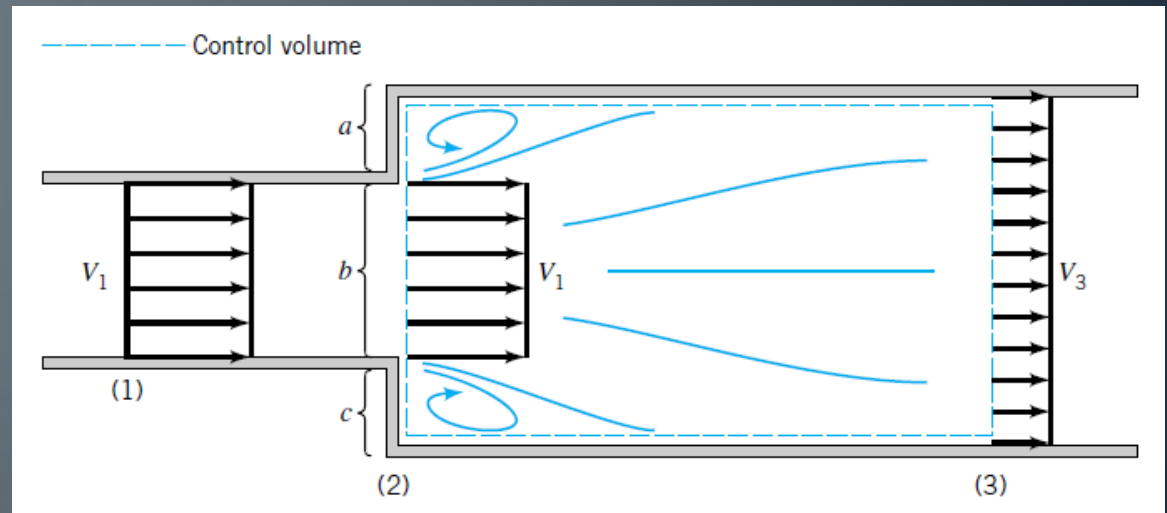
Dimensional Analysis of Pipe Flow

- Minor Losses

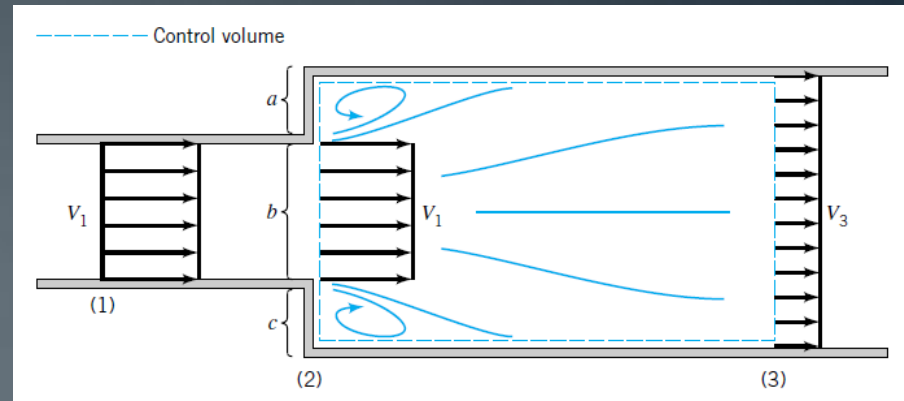


Dimensional Analysis of Pipe Flow

- **Minor Losses**
- A sudden expansion is one of the few components (perhaps the only one) for which the loss coefficient can be obtained by means of a simple analysis.



- Minor Losses



Dimensional Analysis of Pipe Flow

$$A_1 V_1 = A_3 V_3$$

$$p_1 A_3 - p_3 A_3 = \rho A_3 V_3 (V_3 - V_1)$$

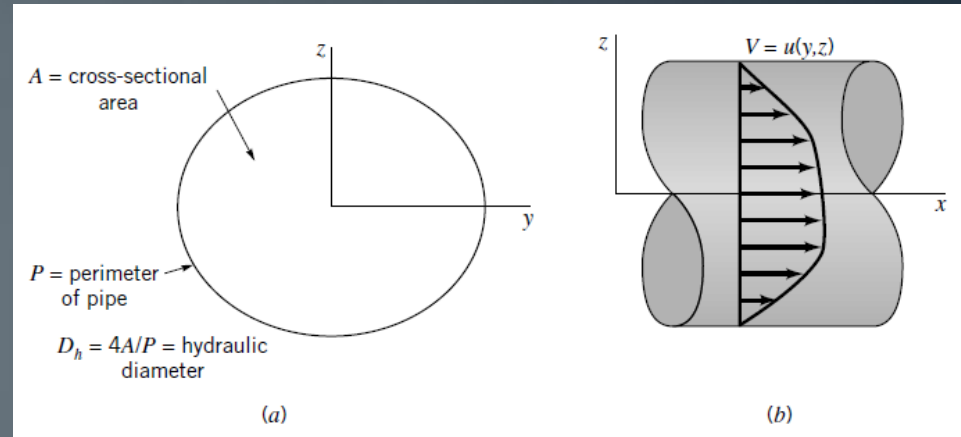
$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} = \frac{p_3}{\gamma} + \frac{V_3^2}{2g} + h_L$$

$$K_L = h_L / (V_1^2 / 2g)$$

$$K_L = \left(1 - \frac{A_1}{A_2} \right)^2$$

Dimensional Analysis of Pipe Flow

- **Noncircular Conduits**
- The hydraulic diameter defined as is four times the ratio of the cross sectional flow area divided by the wetted perimeter of the pipe as is illustrated below.

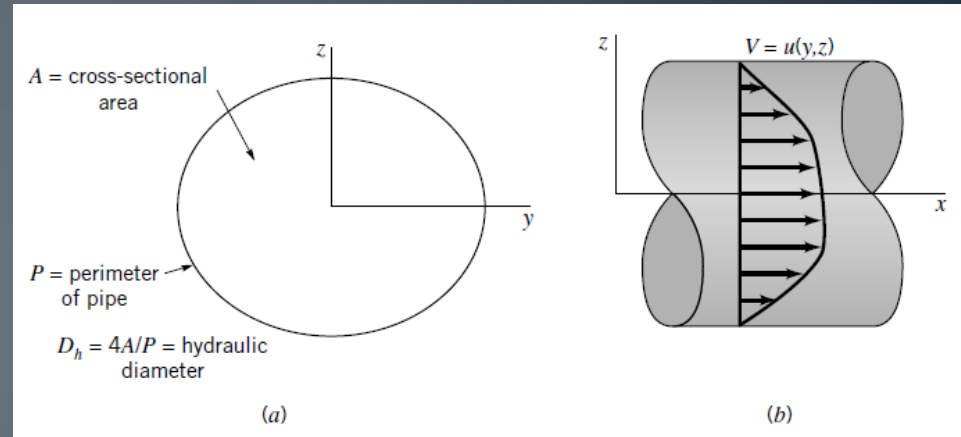


Friction Factors for Laminar Flow in Noncircular Ducts (Data from Ref. 18)

Shape	Parameter	$C = f Re_h$
I. Concentric Annulus $D_h = D_2 - D_1$	D_1/D_2	
	0.0001	71.8
	0.01	80.1
	0.1	89.4
	0.6	95.6
	1.00	96.0
II. Rectangle $D_h = \frac{2ab}{a+b}$	a/b	
	0	96.0
	0.05	89.9
	0.10	84.7
	0.25	72.9
	0.50	62.2
	0.75	57.9
	1.00	56.9

Pipe Flow Examples

- **Noncircular Conduits**
- The hydraulic diameter defined as is four times the ratio of the cross sectional flow area divided by the wetted perimeter of the pipe as is illustrated below.

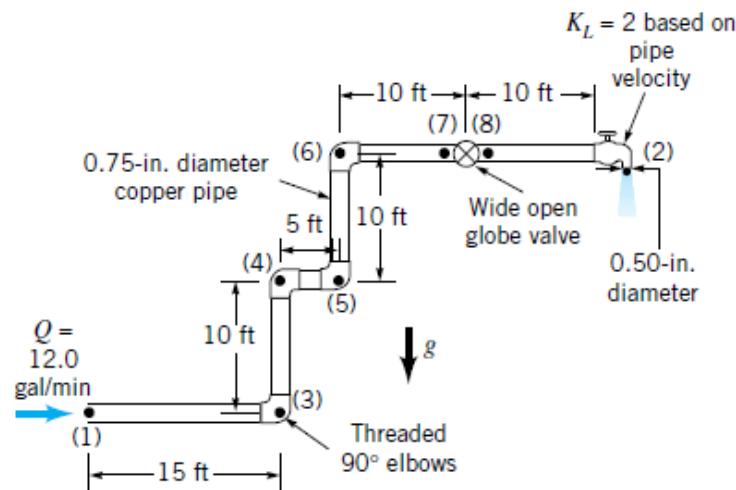


Friction Factors for Laminar Flow in Noncircular Ducts (Data from Ref. 18)

Shape	Parameter	$C = f Re_h$
I. Concentric Annulus $D_h = D_2 - D_1$	D_1/D_2	
	0.0001	71.8
	0.01	80.1
	0.1	89.4
	0.6	95.6
	1.00	96.0
II. Rectangle $D_h = \frac{2ab}{a+b}$	a/b	
	0	96.0
	0.05	89.9
	0.10	84.7
	0.25	72.9
	0.50	62.2
	0.75	57.9
	1.00	56.9

EXAMPLE 8.8 (TYPE I, DETERMINE PRESSURE DROP)

Water at 60 °F flows from the basement to the second floor through the 0.75-in. (0.0625-ft)-diameter copper pipe (a drawn tubing) at a rate of $Q = 12.0 \text{ gal/min} = 0.0267 \text{ ft}^3/\text{s}$ and exits through a faucet of diameter 0.50 in. as shown in Fig. E8.8a. Determine the pressure at point (1) if: (a) all losses are neglected, (b) the only losses included are major losses, or (c) all losses are included.



■ FIGURE E8.8a

SOLUTION

Since the fluid velocity in the pipe is given by $V_1 = Q/A_1 = Q/(\pi D^2/4) = (0.0267 \text{ ft}^3/\text{s})/[\pi(0.0625 \text{ ft})^2/4] = 8.70 \text{ ft/s}$, and the fluid properties are $\rho = 1.94 \text{ slugs/ft}^3$ and $\mu = 2.34 \times 10^{-5} \text{ lb} \cdot \text{s/ft}^2$ (see Table B.1), it follows that $\text{Re} = \rho V D / \mu = (1.94 \text{ slugs/ft}^3)(8.70 \text{ ft/s})(0.0625 \text{ ft})/(2.34 \times 10^{-5} \text{ lb} \cdot \text{s/ft}^2) = 45,000$. Thus, the flow is turbulent. The governing equation for either case (a), (b), or (c) is Eq. 8.21,

$$\frac{p_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \alpha_2 \frac{V_2^2}{2g} + z_2 + h_L$$

where $z_1 = 0$, $z_2 = 20 \text{ ft}$, $p_2 = 0$ (free jet), $\gamma = \rho g = 62.4 \text{ lb/ft}^3$, and the outlet velocity is $V_2 = Q/A_2 = (0.0267 \text{ ft}^3/\text{s})/[\pi(0.50/12)^2 \text{ ft}^2/4] = 19.6 \text{ ft/s}$. We assume that the kinetic energy coefficients α_1 and α_2 are unity. This is reasonable because turbulent velocity profiles are nearly uniform across the pipe. Thus,

$$p_1 = \gamma z_2 + \frac{1}{2} \rho (V_2^2 - V_1^2) + \gamma h_L \quad (1)$$

where the head loss is different for each of the three cases.

(a) If all losses are neglected ($h_L = 0$), Eq. 1 gives

$$\begin{aligned} p_1 &= (62.4 \text{ lb/ft}^3)(20 \text{ ft}) \\ &\quad + \frac{1.94 \text{ slugs/ft}^3}{2} \left[\left(19.6 \frac{\text{ft}}{\text{s}} \right)^2 - \left(8.70 \frac{\text{ft}}{\text{s}} \right)^2 \right] \\ &= (1248 + 299) \text{ lb/ft}^2 = 1547 \text{ lb/ft}^2 \end{aligned}$$

or

$$p_1 = 10.7 \text{ psi} \quad (\text{Ans})$$

Note that for this pressure drop, the amount due to elevation change (the hydrostatic effect) is $\gamma(z_2 - z_1) = 8.67 \text{ psi}$ and the amount due to the increase in kinetic energy is $\rho(V_2^2 - V_1^2)/2 = 2.07 \text{ psi}$.

- (b) If the only losses included are the major losses, the head loss is

$$h_L = f \frac{\ell}{D} \frac{V_1^2}{2g}$$

From [Table 8.1](#) the roughness for a 0.75-in.-diameter copper pipe (drawn tubing) is $\varepsilon = 0.00005$ ft so that $\varepsilon/D = 8 \times 10^{-5}$. With this ε/D and the calculated Reynolds number ($Re = 45,000$), the value of f is obtained from the Moody chart as $f = 0.0215$. Note that the Colebrook equation ([Eq. 8.35](#)) would give the same value of f . Hence, with the total length of the pipe as $\ell = (15 + 5 + 10 + 10 + 20)$ ft = 60 ft and the elevation and kinetic energy portions the same as for part (a), [Eq. 1](#) gives

$$\begin{aligned} p_1 &= \gamma z_2 + \frac{1}{2} \rho (V_2^2 - V_1^2) + \rho f \frac{\ell}{D} \frac{V_1^2}{2} \\ &= (1248 + 299) \text{ lb/ft}^2 \\ &\quad + (1.94 \text{ slugs/ft}^3)(0.0215) \left(\frac{60 \text{ ft}}{0.0625 \text{ ft}} \right) \frac{(8.70 \text{ ft/s})^2}{2} \\ &= (1248 + 299 + 1515) \text{ lb/ft}^2 = 3062 \text{ lb/ft}^2 \end{aligned}$$

or

$$p_1 = 21.3 \text{ psi} \quad (\text{Ans})$$

Of this pressure drop, the amount due to pipe friction is approximately $(21.3 - 10.7)$ psi = 10.6 psi.

- (c) If major and minor losses are included, [Eq. 1](#) becomes

$$p_1 = \gamma z_2 + \frac{1}{2} \rho (V_2^2 - V_1^2) + f \gamma \frac{\ell}{D} \frac{V_1^2}{2g} + \sum \rho K_L \frac{V^2}{2}$$

or

$$p_1 = 21.3 \text{ psi} + \sum \rho K_L \frac{V^2}{2} \quad (2)$$

where the 21.3 psi contribution is due to elevation change, kinetic energy change, and major losses [part (b)], and the last term represents the sum of all of the minor losses. The loss coefficients of the components ($K_L = 1.5$ for each elbow and $K_L = 10$ for the wide-open globe valve) are given in [Table 8.2](#) (except for the loss coefficient of the faucet, which is given in Fig. E8.8a as $K_L = 2$). Thus,

$$\begin{aligned}\sum \rho K_L \frac{V^2}{2} &= (1.94 \text{ slugs/ft}^3) \frac{(8.70 \text{ ft})^2}{2} [10 + 4(1.5) + 2] \\ &= 1321 \text{ lb/ft}^2\end{aligned}$$

or

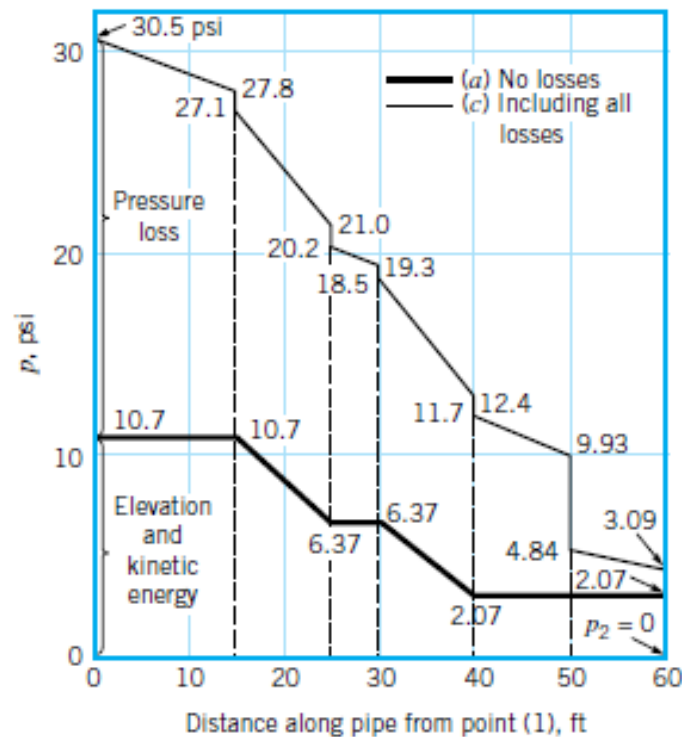
$$\sum \rho K_L \frac{V^2}{2} = 9.17 \text{ psi} \quad (3)$$

Note that we did not include an entrance or exit loss because points (1) and (2) are located within the fluid streams, not within an attaching reservoir where the kinetic energy is zero. Thus, by combining Eqs. 2 and 3 we obtain the entire pressure drop as

$$p_1 = (21.3 + 9.17) \text{ psi} = 30.5 \text{ psi} \quad (\text{Ans})$$

This pressure drop calculated by including all losses should be the most realistic answer of the three cases considered.

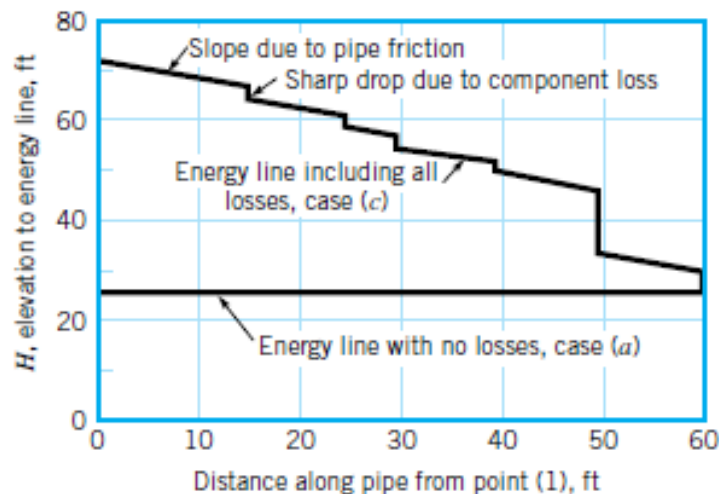
More detailed calculations will show that the pressure distribution along the pipe is as illustrated in Fig. E8.8b for cases (a) and (c)—neglecting all losses or including all losses. Note that not all of the pressure drop, $p_1 - p_2$, is a “pressure loss.” The pressure change due to the elevation and velocity changes is completely reversible. The portion due to the major and minor losses is irreversible.



Location: (1) (3) (4) (5) (6) (7) (8) (2)

■ FIGURE E8.8b

This flow can be illustrated in terms of the energy line and hydraulic grade line concepts introduced in [Section 3.7](#). As is shown in Fig. E8.8c, for case (a) there are no losses and the energy line (EL) is horizontal, one velocity head ($V^2/2g$) above the hydraulic grade line (HGL), which is one pressure head (γz) above the pipe itself. For case (c) the energy line is not horizontal. Each bit of friction in the pipe or loss in a component reduces the available energy, thereby lowering the energy line. Thus, for case



■ FIGURE E8.8c

(a) the total head remains constant throughout the flow with a value of

$$\begin{aligned} H &= \frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{(1547 \text{ lb/ft}^2)}{(62.4 \text{ lb/ft}^3)} + \frac{(8.70 \text{ ft/s})^2}{2(32.2 \text{ ft/s}^2)} + 0 \\ &= 26.0 \text{ ft.} \\ &= \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 = \frac{p_3}{\gamma} + \frac{V_3^2}{2g} + z_3 = \dots \end{aligned}$$

For case (c) the energy line starts at

$$H_1 = \frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{(30.5 \times 144) \text{ lb/ft}^2}{(62.4 \text{ lb/ft}^3)} + \frac{(8.70 \text{ ft/s})^2}{2(32.2 \text{ ft/s}^2)} + 0 = 71.6 \text{ ft}$$

and falls to a final value of

$$H_2 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 = 0 + \frac{(19.6 \text{ ft/s})^2}{2(32.2 \text{ ft/s}^2)} + 20 \text{ ft} = 26.0 \text{ ft}$$

The elevation of the energy line can be calculated at any point along the pipe. For example, at point (7), 50 ft from point (1),

$$H_7 = \frac{p_7}{\gamma} + \frac{V_7^2}{2g} + z_7 = \frac{(9.93 \times 144) \text{ lb/ft}^2}{(62.4 \text{ lb/ft}^3)} + \frac{(8.70 \text{ ft/s})^2}{2(32.2 \text{ ft/s}^2)} + 20 \text{ ft} = 44.1 \text{ ft}$$

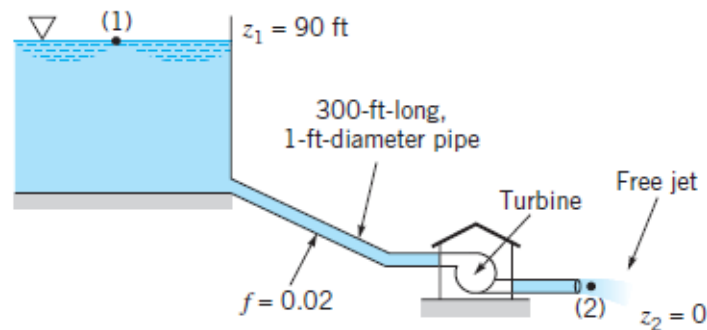
The head loss per foot of pipe is the same all along the pipe. That is,

$$\frac{h_L}{\ell} = f \frac{V^2}{2gD} = \frac{0.0215(8.70 \text{ ft/s})^2}{2(32.2 \text{ ft/s}^2)(0.0625 \text{ ft})} = 0.404 \text{ ft/ft}$$

Thus, the energy line is a set of straight line segments of the same slope separated by steps whose height equals the head loss of the minor component at that location. As is seen from Fig. E8.8c, the globe valve produces the largest of all the minor losses.

EXAMPLE 8.11 (TYPE II, DETERMINE FLOWRATE)

The turbine shown in Fig. E8.11 extracts 50 hp from the water flowing through it. The 1-ft-diameter, 300-ft-long pipe is assumed to have a friction factor of 0.02. Minor losses are negligible. Determine the flowrate through the pipe and turbine.



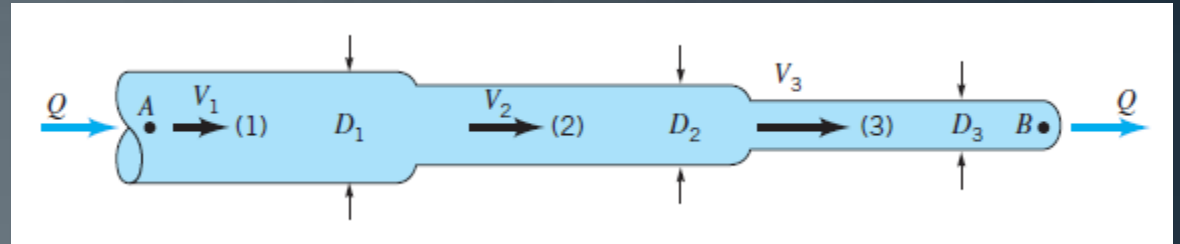
■ FIGURE E8.11

Either of these two flowrates gives the same power, $\mathcal{P}_a = \gamma Q h_T$. The reason for two possible solutions can be seen from the following. With the low flowrate ($Q = 5.17 \text{ ft}^3/\text{s}$), we obtain the head loss and turbine head as $h_L = 4.04 \text{ ft}$ and $h_T = 85.3 \text{ ft}$. Because of the relatively low velocity there is a relatively small head loss and, therefore, a large head available for the turbine. With the large flowrate ($Q = 19.6 \text{ ft}^3/\text{s}$), we find $h_L = 57.8 \text{ ft}$ and $h_T = 22.5 \text{ ft}$. The high-speed flow in the pipe produces a relatively large loss due to friction, leaving a relatively small head for the turbine. However, in either case the product of the turbine head times the flowrate is the same. That is, the power extracted ($\mathcal{P}_a = \gamma Q h_T$) is identical for each case. Although either flowrate will allow the extraction of 50 hp from the water, the details of the design of the turbine itself will depend strongly on which flowrate is to be used. Such information can be found in [Chapter 12](#) and various references about turbomachines (Refs. 14, 19, 20).

If the friction factor were not given, the solution to the problem would be much more lengthy. A trial-and-error solution similar to that in Example 8.10 would be required along with the solution of a cubic equation.

- Multiple Pipe Systems

Pipe Flow Examples

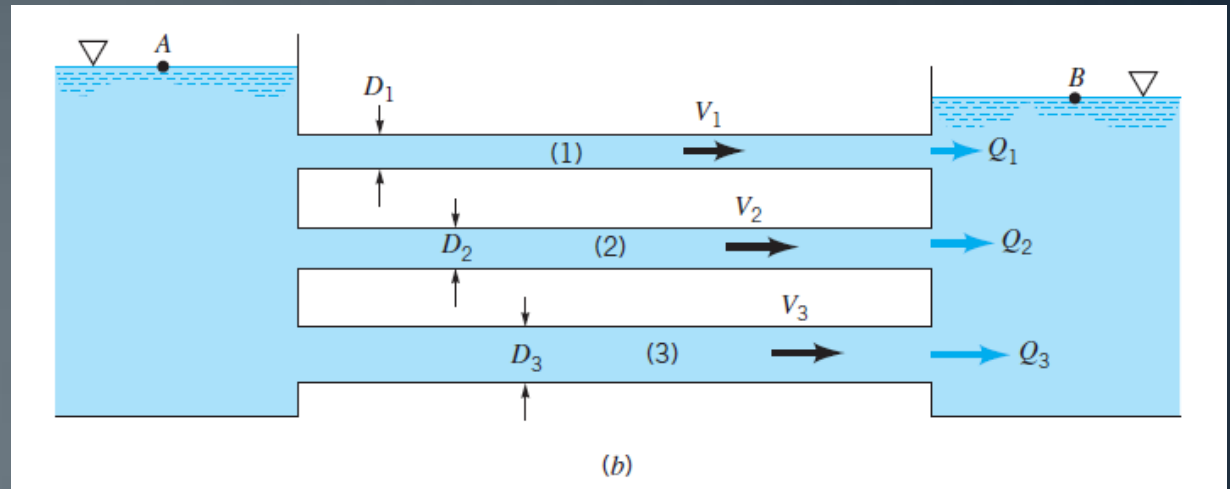


$$Q_1 = Q_2 = Q_3$$

$$h_{L_{A-B}} = h_{L_1} + h_{L_2} + h_{L_3}$$

- Multiple Pipe Systems

Pipe Flow Examples

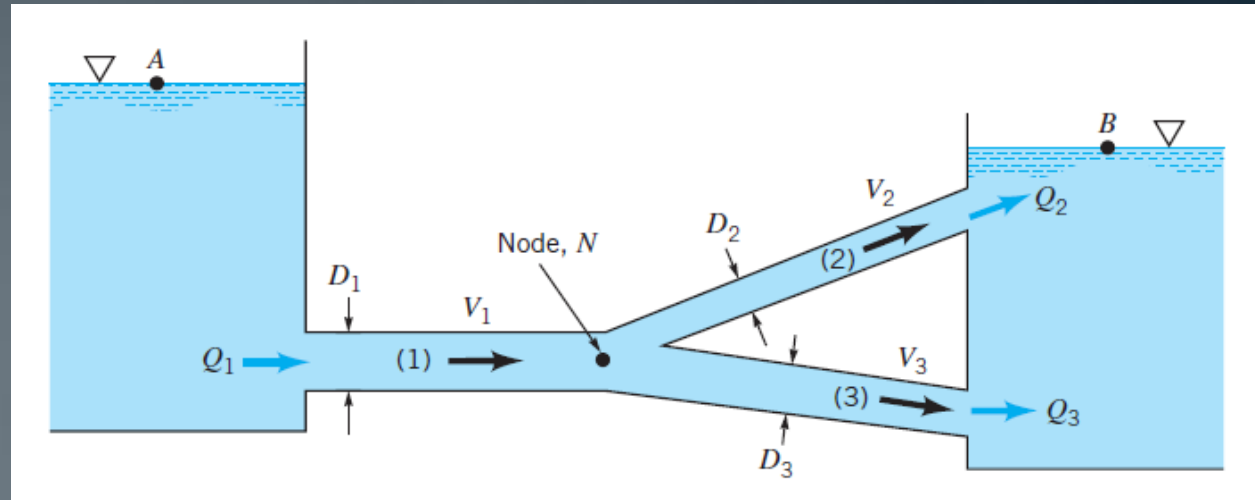


$$Q = Q_1 + Q_2 + Q_3$$

$$h_{L_1} = h_{L_2} = h_{L_3}$$

- Multiple Pipe Systems

Pipe Flow Examples

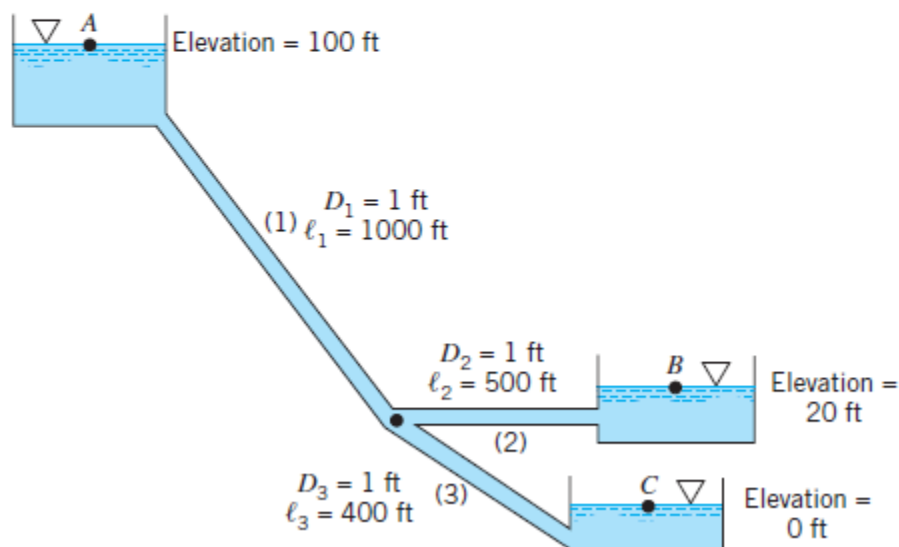


$$\frac{p_A}{\gamma} + \frac{V_A^2}{2g} + z_A = \frac{p_B}{\gamma} + \frac{V_B^2}{2g} + z_B + h_{L_1} + h_{L_2}$$

$$\frac{p_A}{\gamma} + \frac{V_A^2}{2g} + z_A = \frac{p_B}{\gamma} + \frac{V_B^2}{2g} + z_B + h_{L_1} + h_{L_3}$$

EXAMPLE 8.14

Three reservoirs are connected by three pipes as are shown in Fig. E8.14. For simplicity we assume that the diameter of each pipe is 1 ft, the friction factor for each is 0.02, and because of the large length-to-diameter ratio, minor losses are negligible. Determine the flowrate into or out of each reservoir.



■ FIGURE E8.14

SOLUTION

It is not obvious which direction the fluid flows in pipe (2). However, we assume that it flows out of reservoir *B*, write the governing equations for this case, and check our assumption. The continuity equation requires that $Q_1 + Q_2 = Q_3$, which, since the diameters are the same for each pipe, becomes simply

$$V_1 + V_2 = V_3 \quad (1)$$

The energy equation for the fluid that flows from *A* to *C* in pipes (1) and (3) can be written as

$$\frac{p_A}{\gamma} + \frac{V_A^2}{2g} + z_A = \frac{p_C}{\gamma} + \frac{V_C^2}{2g} + z_C + f_1 \frac{\ell_1}{D_1} \frac{V_1^2}{2g} + f_3 \frac{\ell_3}{D_3} \frac{V_3^2}{2g}$$

By using the fact that $p_A = p_C = V_A = V_C = z_C = 0$, this becomes

$$z_A = f_1 \frac{\ell_1}{D_1} \frac{V_1^2}{2g} + f_3 \frac{\ell_3}{D_3} \frac{V_3^2}{2g}$$

For the given conditions of this problem we obtain

$$100 \text{ ft} = \frac{0.02}{2(32.2 \text{ ft/s}^2)} \frac{1}{(1 \text{ ft})} [(1000 \text{ ft})V_1^2 + (400 \text{ ft})V_3^2]$$

or

$$322 = V_1^2 + 0.4V_3^2 \quad (2)$$

where V_1 and V_3 are in ft/s. Similarly the energy equation for fluid flowing from B and C is

$$\frac{p_B}{\gamma} + \frac{V_B^2}{2g} + z_B = \frac{p_C}{\gamma} + \frac{V_C^2}{2g} + z_C + f_2 \frac{\ell_2}{D_2} \frac{V_2^2}{2g} + f_3 \frac{\ell_3}{D_3} \frac{V_3^2}{2g}$$

or

$$z_B = f_2 \frac{\ell_2}{D_2} \frac{V_2^2}{2g} + f_3 \frac{\ell_3}{D_3} \frac{V_3^2}{2g}$$

For the given conditions this can be written as

$$64.4 = 0.5V_2^2 + 0.4V_3^2 \quad (3)$$

Equations 1, 2, and 3 (in terms of the three unknowns V_1 , V_2 , and V_3) are the governing equations for this flow, provided the fluid flows from reservoir B . It turns out, however, that there is no solution for these equations with positive, real values of the velocities. Although these equations do not appear to be complicated, there is no simple way to solve them directly. Thus, a trial-and-error solution is suggested. This can be accomplished as follows. Assume a value of $V_1 > 0$, calculate V_3 from Eq. 2, and then V_2 from Eq. 3. It is found that the resulting V_1 , V_2 , V_3 trio does not satisfy Eq. 1 for any value of V_1 assumed. There is no solution to Eqs. 1, 2, and 3 with real, positive values of V_1 , V_2 , and V_3 . Thus, our original assumption of flow out of reservoir B must be incorrect.

To obtain the solution, assume the fluid flows into reservoirs B and C and out of A . For this case the continuity equation becomes

$$Q_1 = Q_2 + Q_3$$

or

$$V_1 = V_2 + V_3 \quad (4)$$

Application of the energy equation between points A and B and A and C gives

$$z_A = z_B + f_1 \frac{\ell_1}{D_1} \frac{V_1^2}{2g} + f_2 \frac{\ell_2}{D_2} \frac{V_2^2}{2g}$$

and

$$z_A = z_C + f_1 \frac{\ell_1}{D_1} \frac{V_1^2}{2g} + f_3 \frac{\ell_3}{D_3} \frac{V_3^2}{2g}$$

which, with the given data, become

$$258 = V_1^2 + 0.5 V_2^2 \quad (5)$$

and

$$322 = V_1^2 + 0.4 V_3^2 \quad (6)$$

Equations 4, 5, and 6 can be solved as follows. By subtracting Eq. 5 from 6 we obtain

$$V_3 = \sqrt{160 + 1.25V_2^2}$$

Thus, Eq. 5 can be written as

$$258 = (V_2 + V_3)^2 + 0.5V_2^2 = (V_2 + \sqrt{160 + 1.25V_2^2})^2 + 0.5V_2^2$$

or

$$2V_2\sqrt{160 + 1.25V_2^2} = 98 - 2.75V_2^2 \quad (7)$$

which, upon squaring both sides, can be written as

$$V_2^4 - 460 V_2^2 + 3748 = 0$$

By using the quadratic formula we can solve for V_2^2 to obtain either $V_2^2 = 452$ or $V_2^2 = 8.30$. Thus, either $V_2 = 21.3$ ft/s or $V_2 = 2.88$ ft/s. The value $V_2 = 21.3$ ft/s is not a root of the original equations. It is an extra root introduced by squaring Eq. 7, which with $V_2 = 21.3$ becomes “1140 = -1140.” Thus, $V_2 = 2.88$ ft/s and from Eq. 5, $V_1 = 15.9$ ft/s. The corresponding flowrates are

$$\begin{aligned} Q_1 &= A_1 V_1 = \frac{\pi}{4} D_1^2 V_1 = \frac{\pi}{4} (1 \text{ ft})^2 (15.9 \text{ ft/s}) \\ &= 12.5 \text{ ft}^3/\text{s from A} \end{aligned} \quad (\text{Ans})$$

$$\begin{aligned} Q_2 &= A_2 V_2 = \frac{\pi}{4} D_2^2 V_2 = \frac{\pi}{4} (1 \text{ ft})^2 (2.88 \text{ ft/s}) \\ &= 2.26 \text{ ft}^3/\text{s into B} \end{aligned} \quad (\text{Ans})$$

and

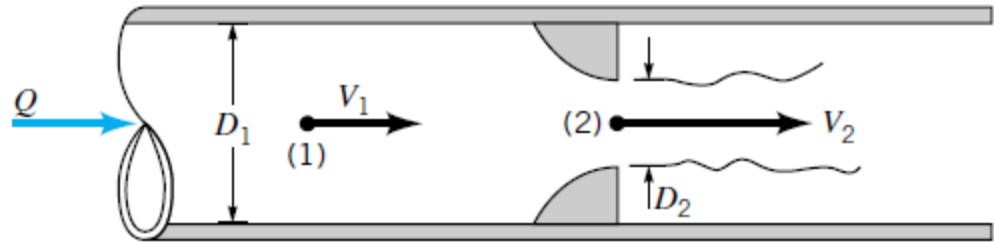
$$Q_3 = Q_1 - Q_2 = (12.5 - 2.26) \text{ ft}^3/\text{s} = 10.2 \text{ ft}^3/\text{s into C} \quad (\text{Ans})$$

Note the slight differences in the governing equations depending on the direction of the flow in pipe (2)—compare Eqs. 1, 2, and 3 with Eqs. 4, 5, and 6.

If the friction factors were not given, a trial-and-error procedure similar to that needed for Type II problems (see [Section 8.5.1](#)) would be required.

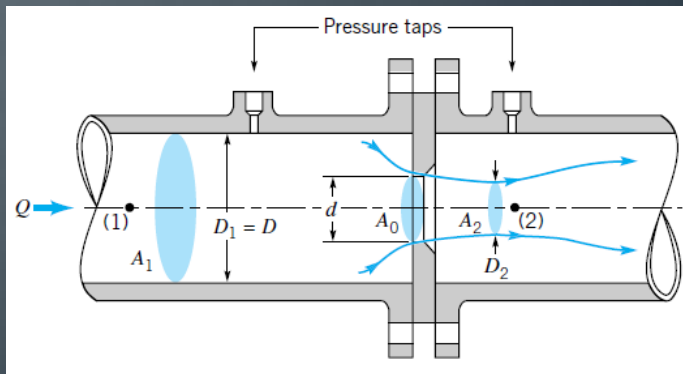
Flowrate Measurement

- Pipe Flowrate Meters



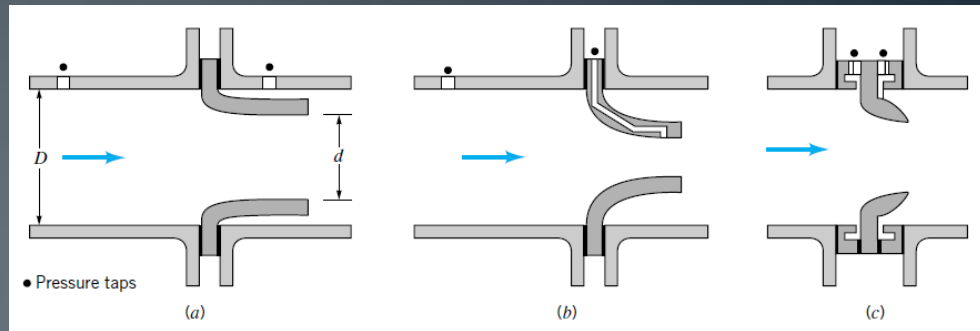
$$Q_{\text{ideal}} = A_2 V_2 = A_2 \sqrt{\frac{2(p_1 - p_2)}{\rho(1 - \beta^4)}}$$

$$Q = C_o Q_{\text{ideal}} = C_o A_o \sqrt{\frac{2(p_1 - p_2)}{\rho(1 - \beta^4)}}$$

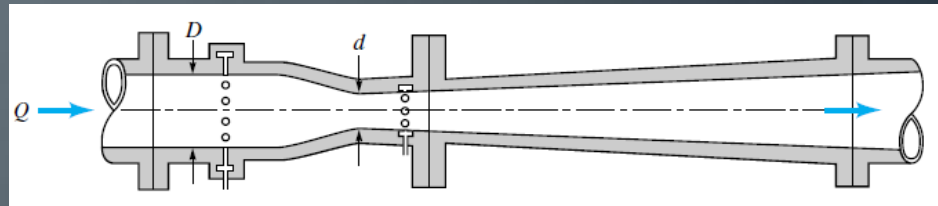


- Pipe Flowrate Meters

Flowrate Measurement



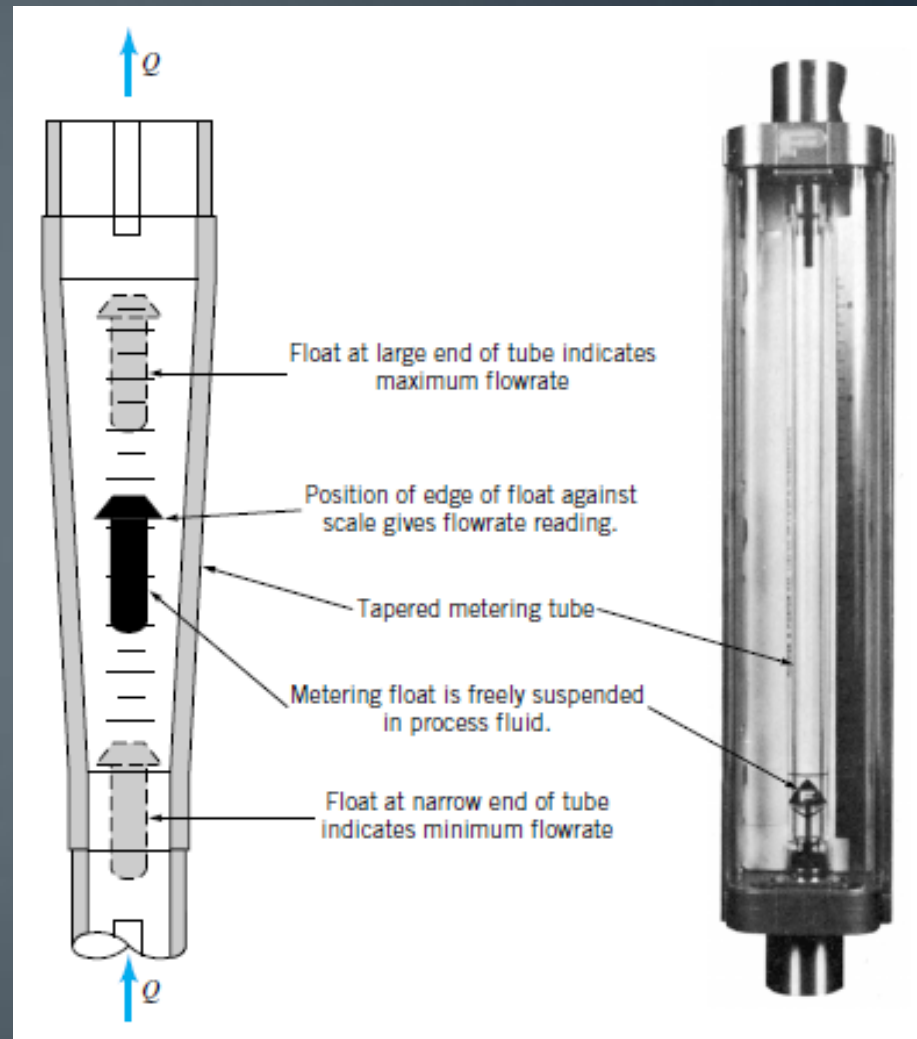
$$Q = C_n Q_{\text{ideal}} = C_n A_n \sqrt{\frac{2(p_1 - p_2)}{\rho(1 - \beta^4)}}$$



$$Q = C_v Q_{\text{ideal}} = C_v A_T \sqrt{\frac{2(p_1 - p_2)}{\rho(1 - \beta^4)}}$$

- Pipe Flowrate Meters

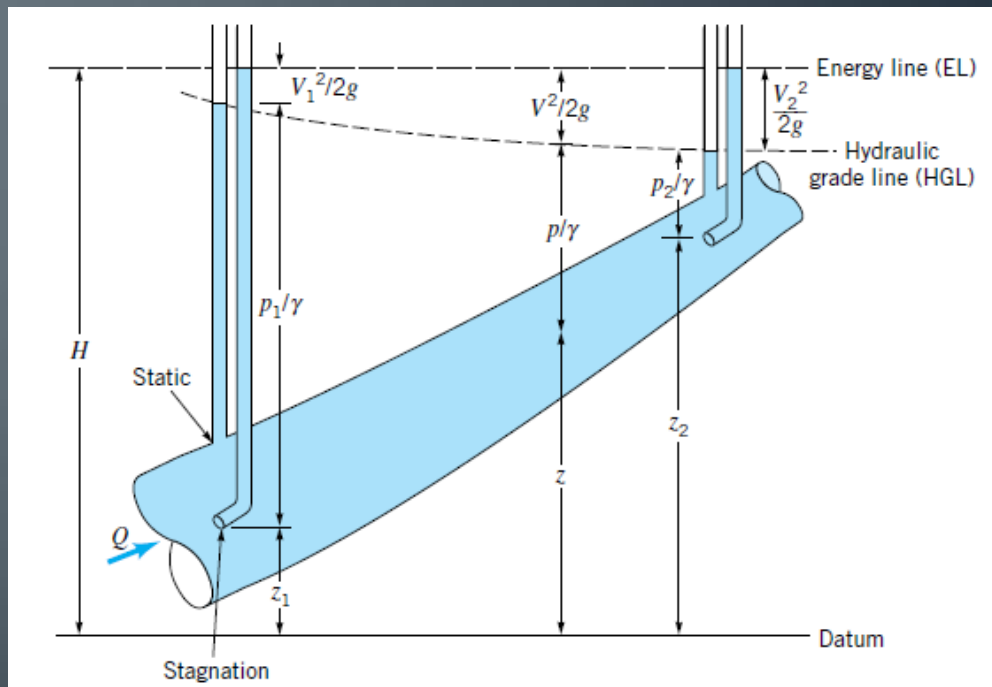
Flowrate Measurement



Review

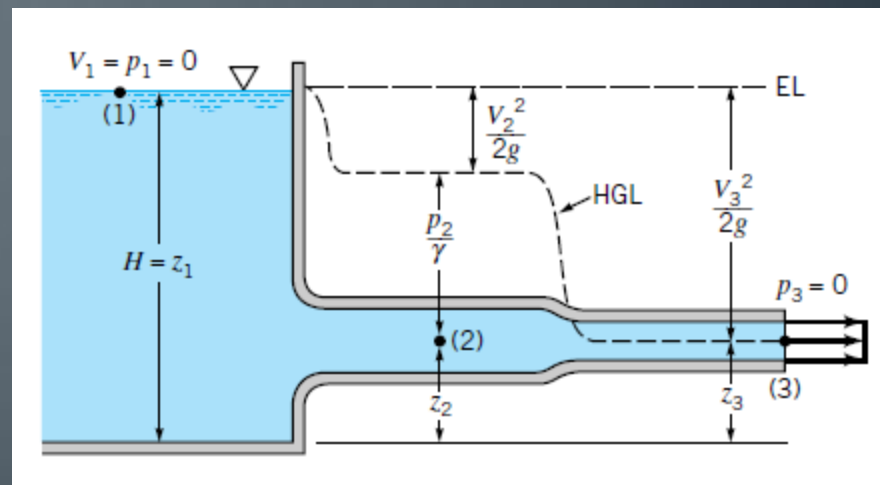
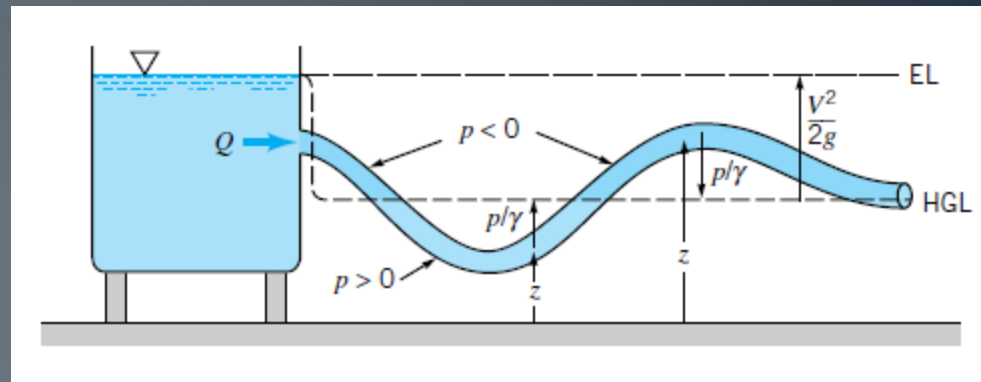
- The Energy Line and the Hydraulic Grade Line
- The Bernoulli equation states that the sum of the pressure head, the velocity head, and the elevation head is constant along a streamline. This constant is called the **total head, H** .

$$\frac{p}{\gamma} + \frac{V^2}{2g} + z = \text{constant on a streamline} = H$$



- The Energy Line and the Hydraulic Grade Line

Review



Review

- Comparison of the Energy Equation with the Bernoulli Equation

$$\dot{m} \left[\check{u}_{\text{out}} - \check{u}_{\text{in}} + \frac{p_{\text{out}}}{\rho} - \frac{p_{\text{in}}}{\rho} + \frac{V_{\text{out}}^2 - V_{\text{in}}^2}{2} + g(z_{\text{out}} - z_{\text{in}}) \right] = \dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft net in}}$$

$$\frac{p_{\text{out}}}{\rho} + \frac{V_{\text{out}}^2}{2} + gz_{\text{out}} = \frac{p_{\text{in}}}{\rho} + \frac{V_{\text{in}}^2}{2} + gz_{\text{in}} + w_{\text{shaft net in}} - (\check{u}_{\text{out}} - \check{u}_{\text{in}} - q_{\text{net in}})$$

$$\frac{p_{\text{out}}}{\rho} + \frac{V_{\text{out}}^2}{2} + gz_{\text{out}} = \frac{p_{\text{in}}}{\rho} + \frac{V_{\text{in}}^2}{2} + gz_{\text{in}} + w_{\text{shaft net in}} - \text{loss}$$

- It is sometimes called the **mechanical energy equation** or the **extended Bernoulli equation**.

$$\frac{p_{\text{out}}}{\gamma} + \frac{V_{\text{out}}^2}{2g} + z_{\text{out}} = \frac{p_{\text{in}}}{\gamma} + \frac{V_{\text{in}}^2}{2g} + z_{\text{in}} + h_s - h_L$$

$$h_s = w_{\text{shaft net in}}/g = \frac{\dot{W}_{\text{shaft net in}}}{\dot{m}g} = \frac{\dot{W}_{\text{shaft net in}}}{\gamma Q}$$

- Application of the Energy Equation to Nonuniform Flows

$$\frac{\partial}{\partial t} \int_{cv} e \rho dV + \int_{cs} \left(\check{u} + \frac{p}{\rho} + \frac{V^2}{2} + gz \right) \rho \mathbf{V} \cdot \hat{\mathbf{n}} dA = \dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft net in}}$$

$$\int_{cs} \frac{V^2}{2} \rho \mathbf{V} \cdot \hat{\mathbf{n}} dA = \dot{m} \left(\frac{\alpha_{\text{out}} \bar{V}_{\text{out}}^2}{2} - \frac{\alpha_{\text{in}} \bar{V}_{\text{in}}^2}{2} \right)$$

Review

- Where α is the kinetic energy coefficient.

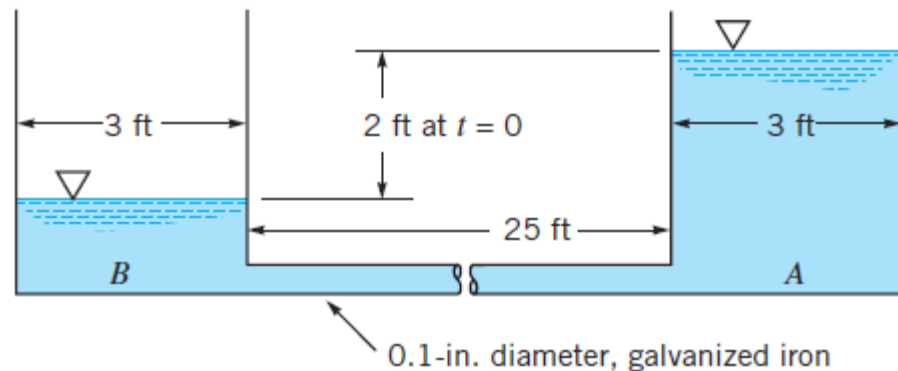
$$\frac{\dot{m} \alpha \bar{V}^2}{2} = \int_A \frac{V^2}{2} \rho \mathbf{V} \cdot \hat{\mathbf{n}} dA$$

$$\alpha = \frac{\int_A (V^2/2) \rho \mathbf{V} \cdot \hat{\mathbf{n}} dA}{\dot{m} \bar{V}^2/2}$$

$$\frac{p_{\text{out}}}{\gamma} + \frac{\alpha_{\text{out}} \bar{V}_{\text{out}}^2}{2g} + z_{\text{out}} = \frac{p_{\text{in}}}{\gamma} + \frac{\alpha_{\text{in}} \bar{V}_{\text{in}}^2}{2g} + z_{\text{in}} + \frac{w_{\text{shaft net in}}}{g} - h_L$$

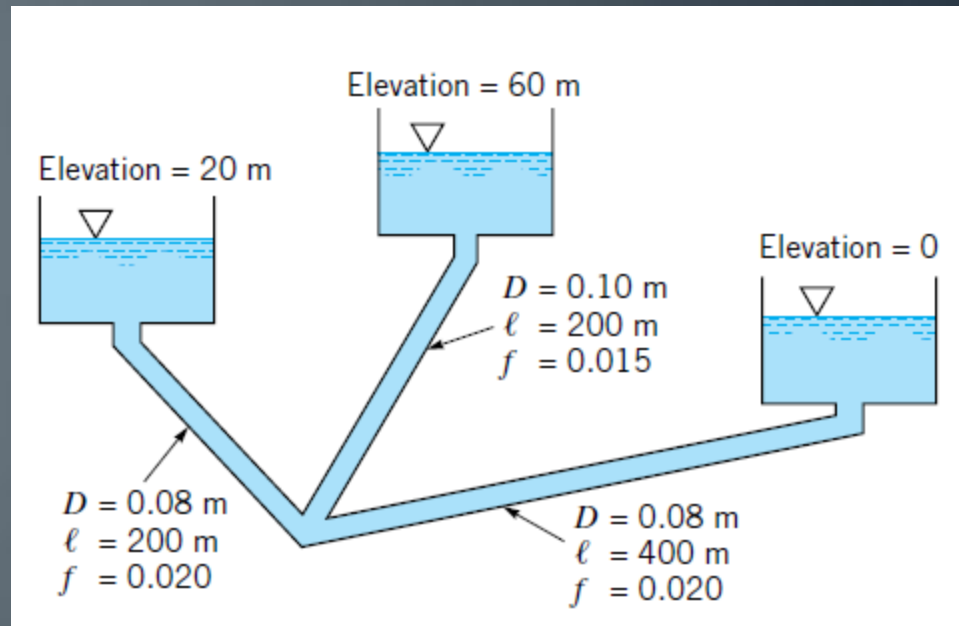
Problems

8.49 At time $t = 0$ the level of water in tank A shown in Fig. P8.49 is 2 ft above that in tank B . Plot the elevation of the water in tank A as a function of time until the free surfaces in both tanks are at the same elevation. Assume quasisteady conditions—that is, the steady pipe flow equations are assumed valid at any time, even though the flowrate does change (slowly) in time. Neglect minor losses. *Note:* Verify and use the fact that the flow is laminar.



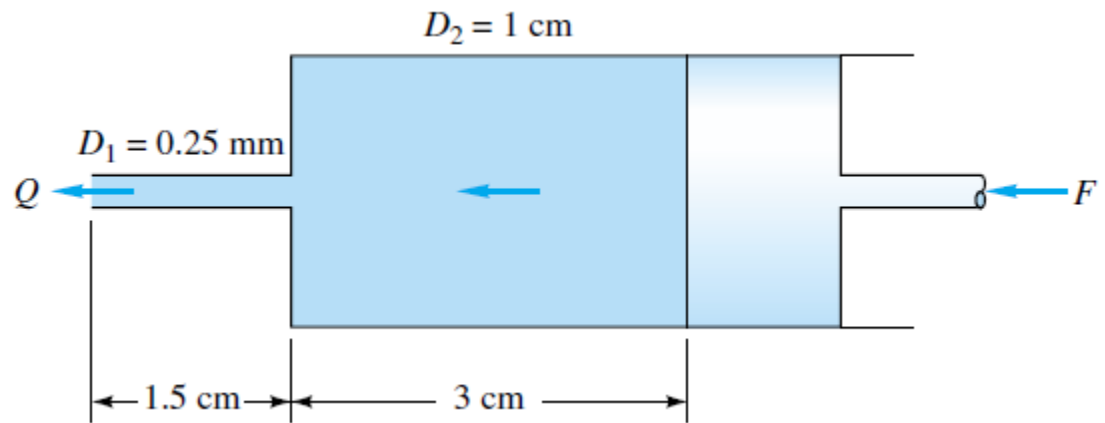
Problems

8.102 The three water-filled tanks shown in Fig. P8.102 are connected by pipes as indicated. If minor losses are neglected, determine the flowrate in each pipe.



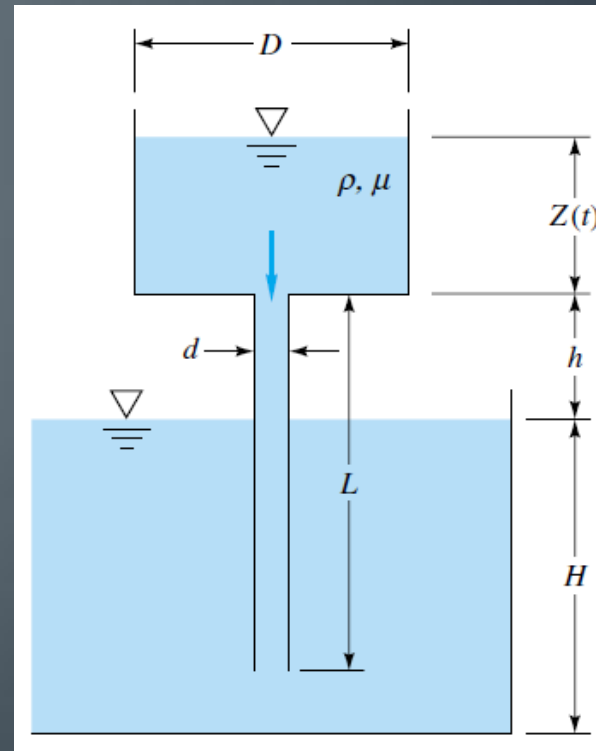
Problems

P6.30 A steady push on the piston in Fig. P6.30 causes a flow rate $Q = 0.15 \text{ cm}^3/\text{s}$ through the needle. The fluid has $\rho = 900 \text{ kg/m}^3$ and $\mu = 0.002 \text{ kg/(m} \cdot \text{s)}$. What force F is required to maintain the flow?



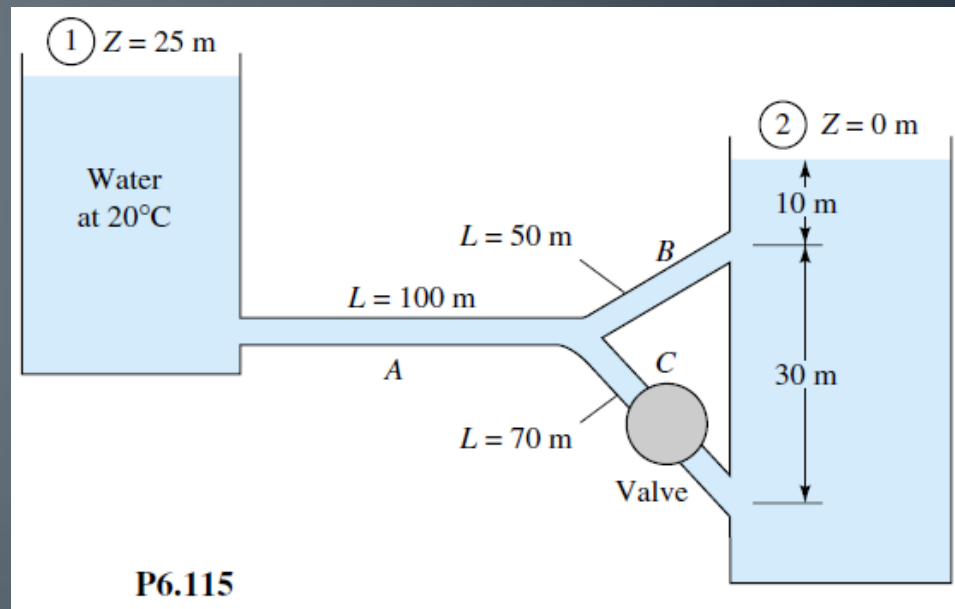
Problems

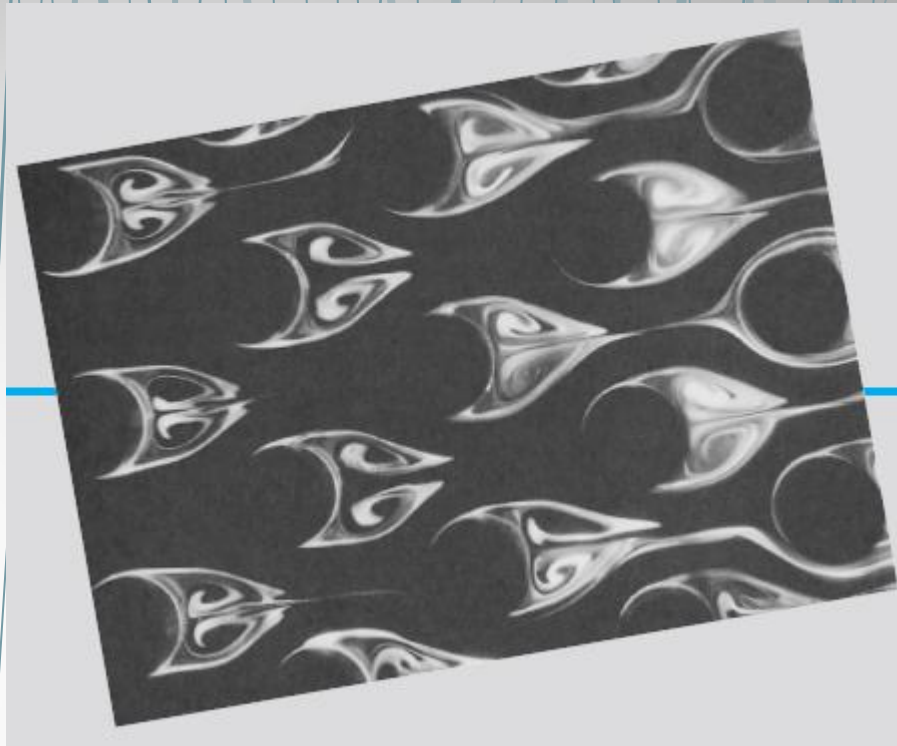
***P6.35** Let us attack Prob. 6.33 in symbolic fashion, using Fig. P6.35. All parameters are constant except the upper tank depth $Z(t)$. Find an expression for the flow rate $Q(t)$ as a function of $Z(t)$. Set up a differential equation, and solve for the time t_0 to drain the upper tank completely. Assume quasi-steady laminar flow.



Problems

P6.115 In Fig. P6.115 all pipes are 8-cm-diameter cast iron. Determine the flow rate from reservoir 1 if valve C is (a) closed and (b) open, $K = 0.5$.

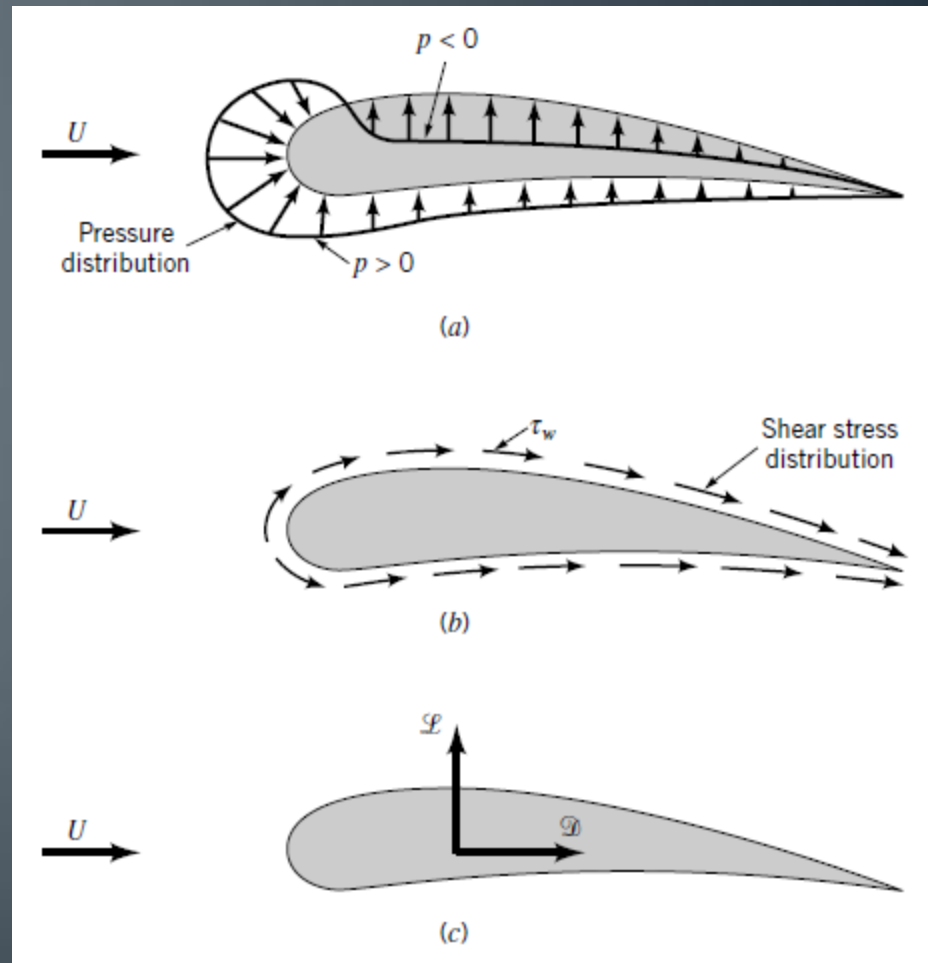




Flow Over Immersed Bodies

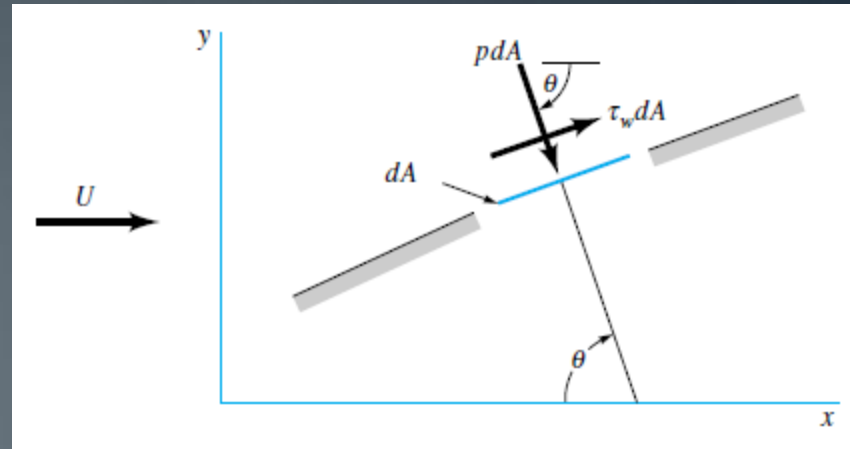
General External Flow Characteristics

- **Lift and Drag Concepts**
- The resultant force in the direction of the upstream velocity is termed the **drag**, and the resultant force normal to the upstream velocity is termed the **lift**.



- Lift and Drag Concepts

General External Flow Characteristics



$$dF_x = (p dA) \cos \theta + (\tau_w dA) \sin \theta$$

$$dF_y = -(p dA) \sin \theta + (\tau_w dA) \cos \theta$$

$$\mathcal{D} = \int dF_x = \int p \cos \theta dA + \int \tau_w \sin \theta dA$$

$$\mathcal{L} = \int dF_y = - \int p \sin \theta dA + \int \tau_w \cos \theta dA$$

General External Flow Characteristics

- Lift and Drag Concepts
- The lift coefficient, and the drag coefficient, are defined as

$$C_L = \frac{\mathcal{L}}{\frac{1}{2}\rho U^2 A}$$

$$C_D = \frac{\mathcal{D}}{\frac{1}{2}\rho U^2 A}$$

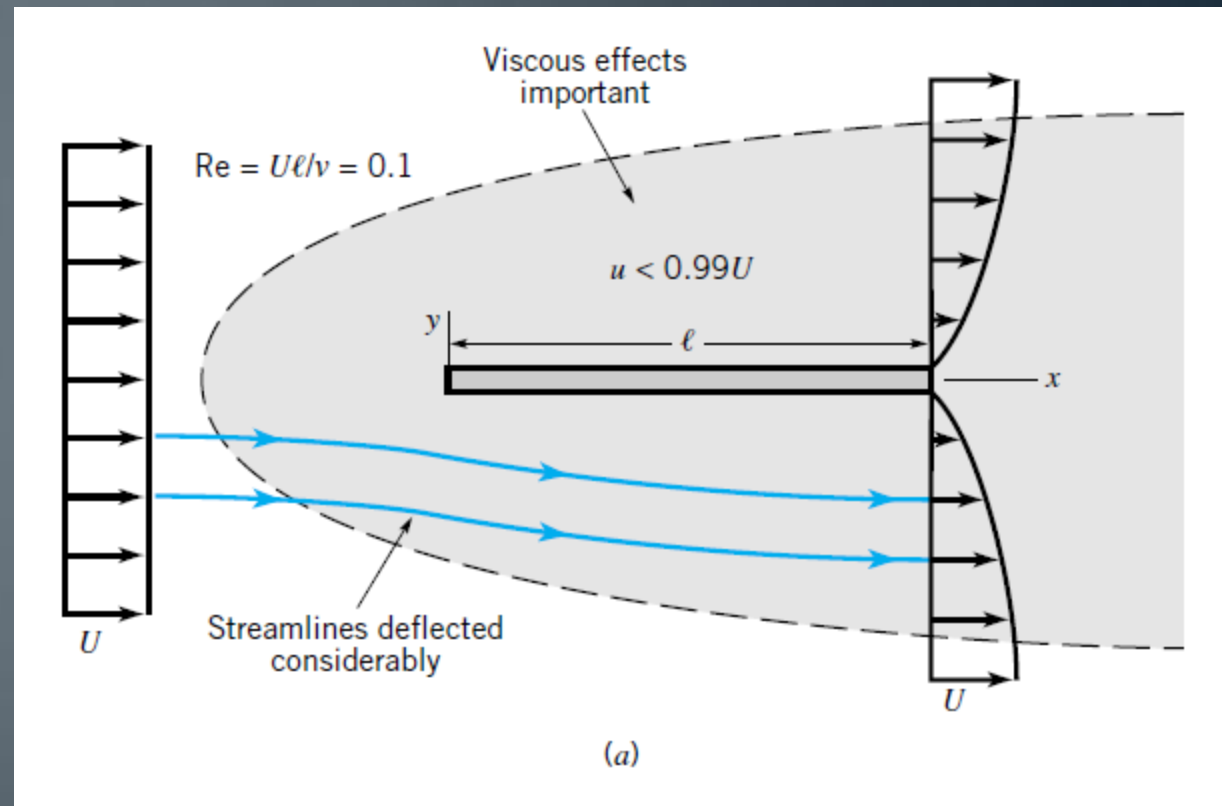
- Where A is a characteristic area of the object. Typically, A is taken to be **frontal area**, in other situations A is taken to be the **planform area**.

General External Flow Characteristics

- **Characteristics of Flow Past an Object**
- One of the great advancements in fluid mechanics occurred in 1904 as a result of theinsight of **Ludwig Prandtl**, a German physicist and aerodynamicist.
- He conceived of the idea of the boundary layer—a thin region on the surface of a body in which viscous effects are very important and outside of which the fluid behaves essentially as if it were inviscid.

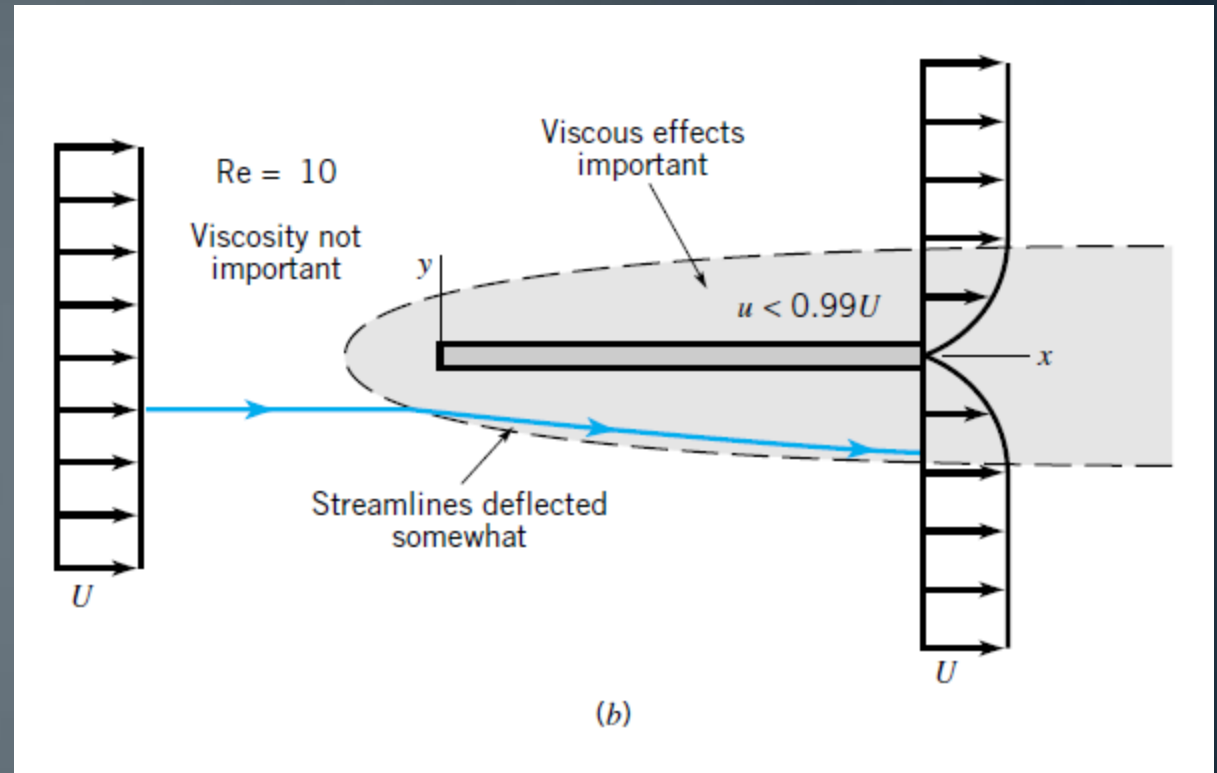
General External Flow Characteristics

- Characteristics of Flow Past an Object
- Flow past the flat plate



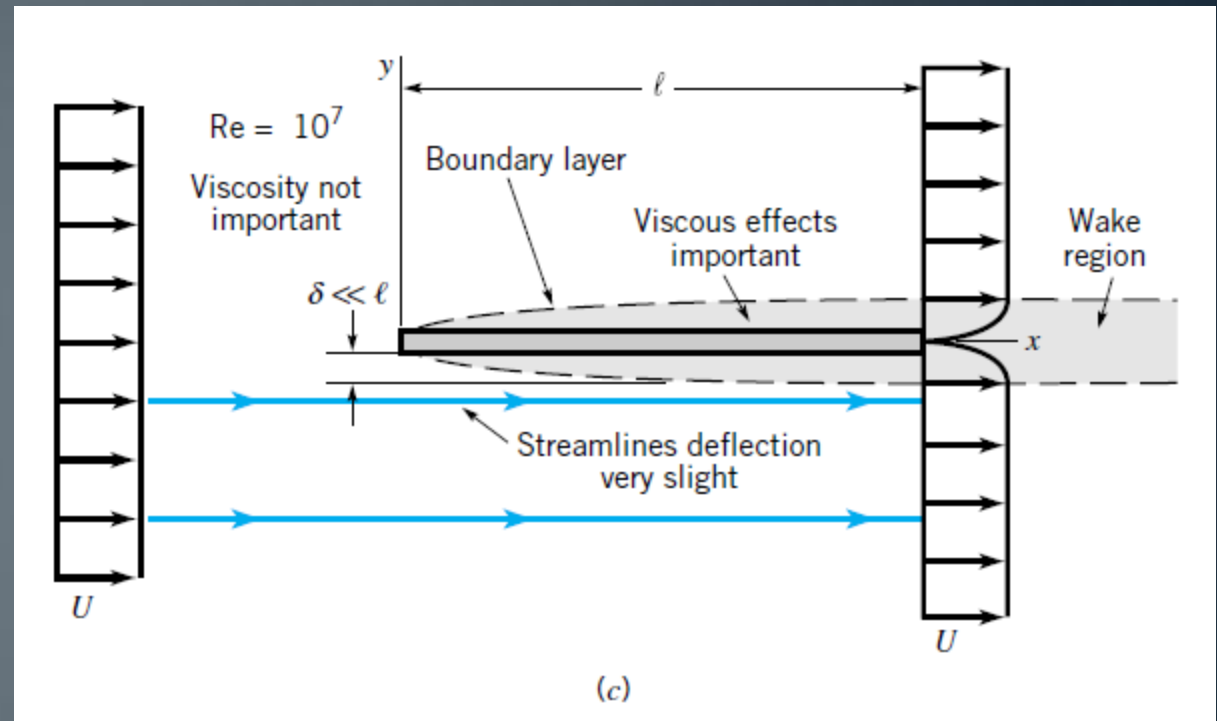
- Characteristics of Flow Past an Object

General External Flow Characteristics



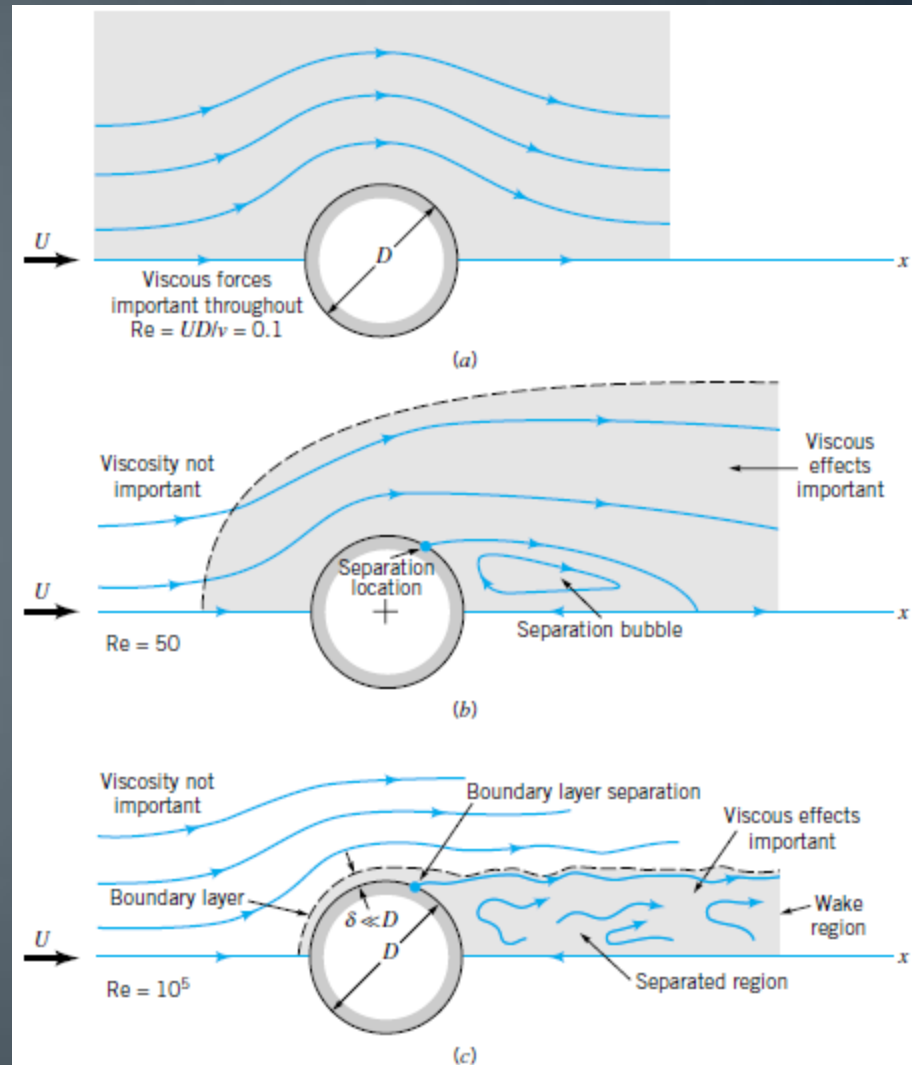
- Characteristics of Flow Past an Object

General External Flow Characteristics



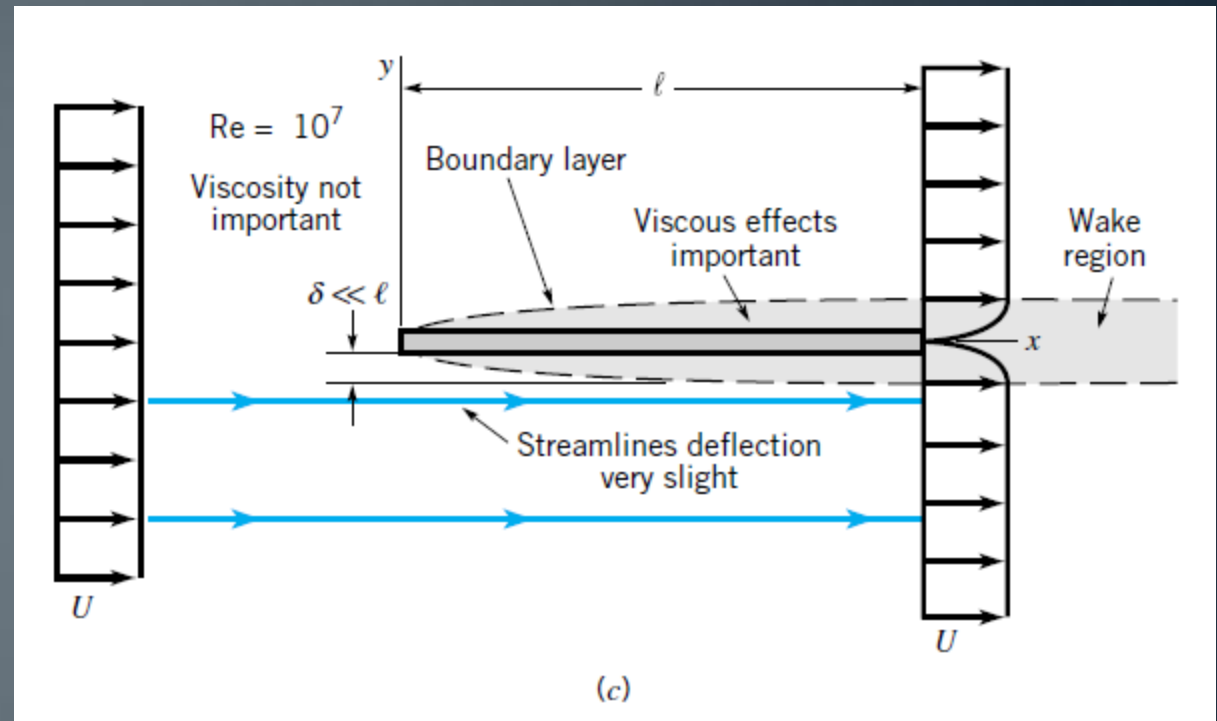
General External Flow Characteristics

- Characteristics of Flow Past an Object
- Flow past a blunt object



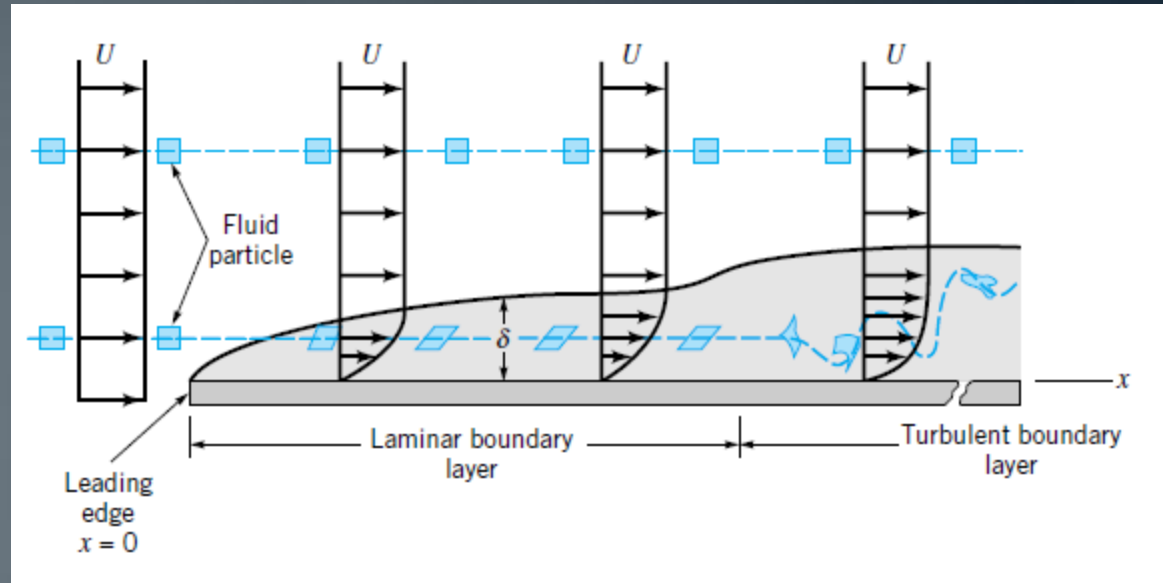
- Characteristics of Flow Past an Object

General External Flow Characteristics



General External Flow Characteristics

- Boundary Layer Structure and Thickness on a Flat Plate

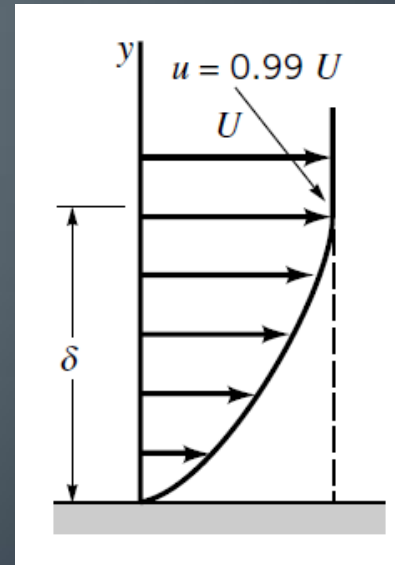


- The presence of the plate is felt only in the relatively thin boundary layer and wake regions.
- The flow is said to be irrotational outside the boundary layer and rotational within the boundary layer.

General External Flow Characteristics

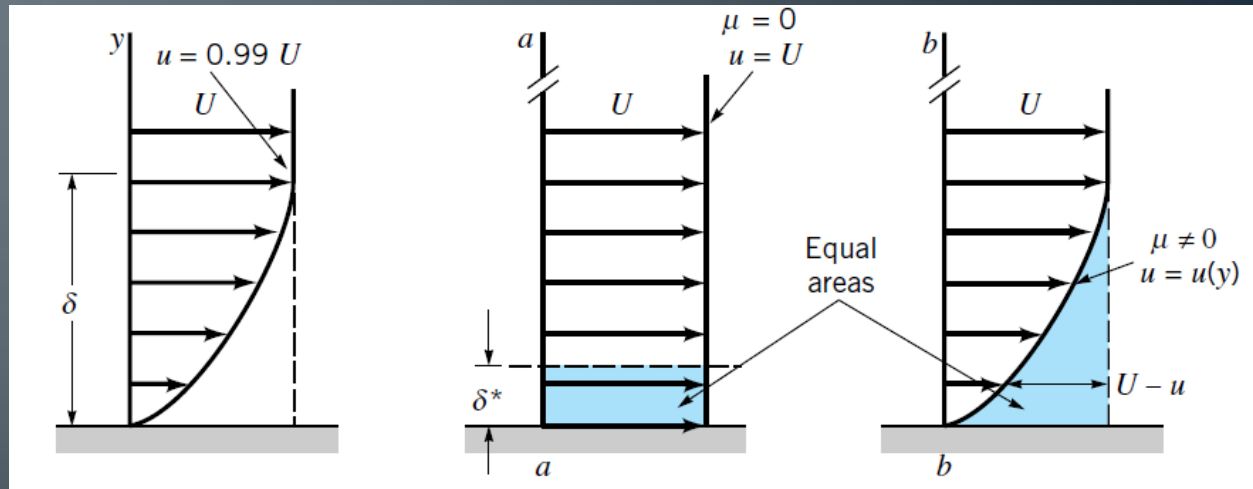
- **Boundary Layer Structure and Thickness on a Flat Plate**
- We define **the boundary layer thickness**, as that distance from the plate at which the fluid velocity is within some arbitrary value of the upstream velocity.

$$\delta = y \quad \text{where} \quad u = 0.99 U$$



General External Flow Characteristics

- **Boundary Layer Structure and Thickness on a Flat Plate**
- Because of the velocity deficit, within the boundary layer, the flowrate across section b–b is less than that across section a–a. However, if we displace the plate at section a–a by an appropriate amount **the boundary layer displacement thickness**, the flowrates across each section will be identical.



General External Flow Characteristics

- Boundary Layer Structure and Thickness on a Flat Plate

$$\delta^* b U = \int_0^{\infty} (U - u) b \, dy$$

$$\delta^* = \int_0^{\infty} \left(1 - \frac{u}{U} \right) dy$$

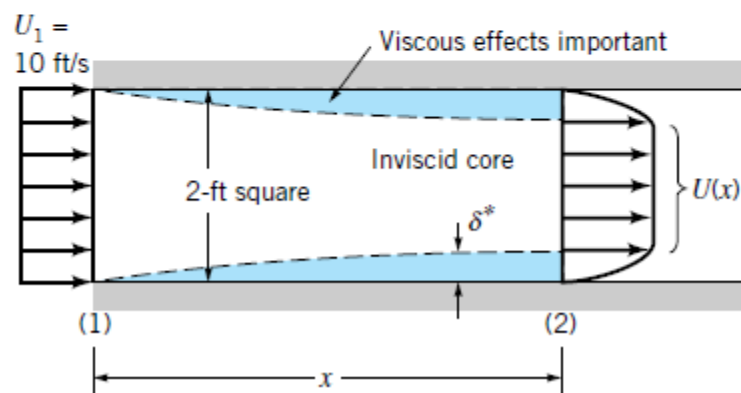
- The displacement thickness represents the amount that the thickness of the body must be increased so that the fictitious uniform inviscid flow has the same mass flowrate properties as the actual viscous flow.
- It represents the outward displacement of the streamlines caused by the viscous effects on the plate.

EXAMPLE 9.3

Air flowing into a 2-ft-square duct with a uniform velocity of 10 ft/s forms a boundary layer on the walls as shown in Fig. E9.3. The fluid within the core region (outside the boundary layers) flows as if it were inviscid. From advanced calculations it is determined that for this flow the boundary layer displacement thickness is given by

$$\delta^* = 0.0070(x)^{1/2} \quad (1)$$

where δ^* and x are in feet. Determine the velocity $U = U(x)$ of the air within the duct but outside of the boundary layer.



■ FIGURE E9.3

SOLUTION

If we assume incompressible flow (a reasonable assumption because of the low velocities involved), it follows that the volume flowrate across any section of the duct is equal to that at the entrance (i.e., $Q_1 = Q_2$). That is,

$$U_1 A_1 = 10 \text{ ft/s} (2 \text{ ft})^2 = 40 \text{ ft}^3/\text{s} = \int_{(2)} u \, dA$$

According to the definition of the displacement thickness, δ^* , the flowrate across section (2) is the same as that for a uniform flow with velocity U through a duct whose walls have been moved inward by δ^* . That is,

$$40 \text{ ft}^3/\text{s} = \int_{(2)} u \, dA = U(2 \text{ ft} - 2\delta^*)^2 \quad (2)$$

By combining Eqs. 1 and 2 we obtain

$$40 \text{ ft}^3/\text{s} = 4U(1 - 0.0070x^{1/2})^2$$

or

$$U = \frac{10}{(1 - 0.0070x^{1/2})^2} \text{ ft/s} \quad (\text{Ans})$$

SOLUTION

If we assume incompressible flow (a reasonable assumption because of the low velocities involved), it follows that the volume flowrate across any section of the duct is equal to that at the entrance (i.e., $Q_1 = Q_2$). That is,

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By combining Eqs. 1 and 2 we obtain

$$40 \text{ ft}^3/\text{s} = 4U(1 - 0.0070x^{1/2})^2$$

or

$$U = \frac{10}{(1 - 0.0070x^{1/2})^2} \text{ ft/s} \quad (\text{Ans})$$

Note that U increases in the downstream direction. For example, $U = 11.6$ ft/s at $x = 100$ ft. The viscous effects that cause the fluid to stick to the walls of the duct reduce the effective size of the duct, thereby (from conservation of mass principles) causing the fluid to accelerate. The pressure drop necessary to do this can be obtained by using the Bernoulli equation (Eq. 3.7) along the inviscid streamlines from section (1) to (2). (Recall that this equation is not valid for viscous flows within the boundary layer. It is, however, valid for the inviscid flow outside the boundary layer.) Thus,

$$p_1 + \frac{1}{2}\rho U_1^2 = p + \frac{1}{2}\rho U^2$$

Hence, with $\rho = 2.38 \times 10^{-3}$ slugs/ft³ and $p_1 = 0$ we obtain

$$\begin{aligned} p &= \frac{1}{2}\rho (U_1^2 - U^2) \\ &= \frac{1}{2}(2.38 \times 10^{-3} \text{ slugs/ft}^3) \left[(10 \text{ ft/s})^2 - \frac{10^2}{(1 - 0.0079x^{1/2})^4} \text{ ft}^2/\text{s}^2 \right] \end{aligned}$$

or

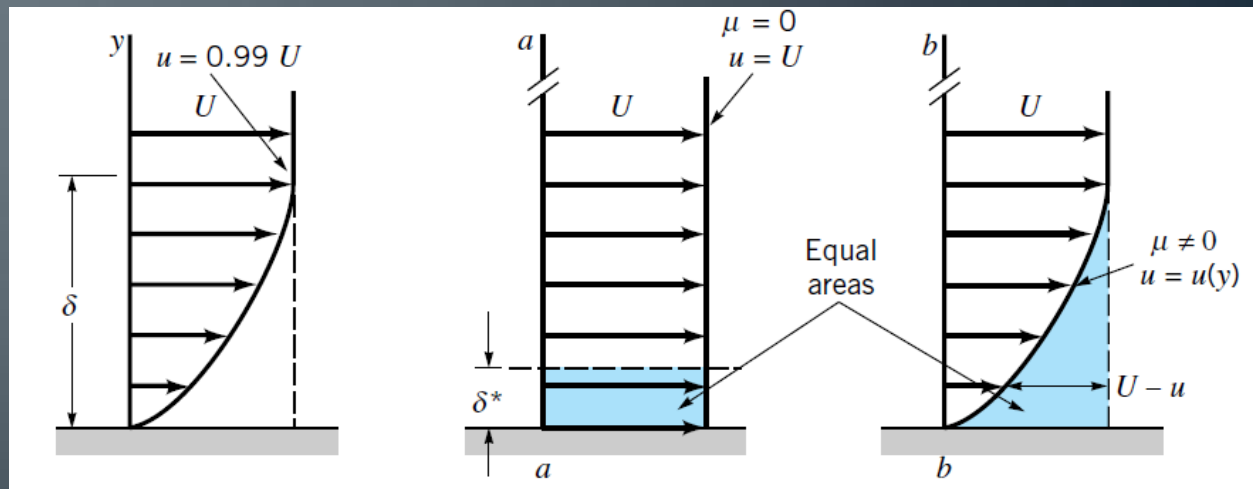
$$p = 0.119 \left[1 - \frac{1}{(1 - 0.0079x^{1/2})^4} \right] \text{ lb/ft}^2$$

For example, $p = -0.0401$ lb/ft² at $x = 100$ ft.

If it were desired to maintain a constant velocity along the centerline of this entrance region of the duct, the walls could be displaced outward by an amount equal to the boundary layer displacement thickness, δ^* .

General External Flow Characteristics

- **Boundary Layer Structure and Thickness on a Flat Plate**
- Another boundary layer thickness definition, the **boundary layer momentum thickness**, is often used when determining the drag on an object.
- Again because of the velocity deficit, in the boundary layer, the momentum flux across section b-b is less than that across section a-a.
- This deficit in momentum flux for the actual boundary layer flow is given by



General External Flow Characteristics

- **Boundary Layer Structure and Thickness on a Flat Plate**
- This deficit in momentum flux for the actual boundary layer flow is given by

$$\int \rho u(U - u) dA = \rho b \int_0^{\infty} u(U - u) dy$$

$$\rho b U^2 \Theta = \rho b \int_0^{\infty} u(U - u) dy$$

$$\Theta = \int_0^{\infty} \frac{u}{U} \left(1 - \frac{u}{U} \right) dy$$

General External Flow Characteristics

- **Prandtl/Blasius Boundary Layer Solution**
- For steady, two dimensional laminar flows with negligible gravitational effects, the governing Navier-Stokes equations reduce to the following:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$
$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$$

- In addition, the conservation of mass equation, for incompressible flow is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

General External Flow Characteristics

- **Prandtl/Blasius Boundary Layer Solution**
- By using boundary layer concepts introduced in the previous sections, **Prandtl** was able to impose certain approximations, and thereby to simplify the governing equations.
- In 1908, **H. Blasius**, one of Prandtl's students, was able to solve these simplified equations for the boundary layer flow past a flat plate parallel to the flow.

General External Flow Characteristics

- Prandtl/Blasius Boundary Layer Solution
- Since the boundary layer is thin, it is expected that the component of velocity normal to the plate is much smaller than that parallel to the plate and that the rate of change of any parameter across the boundary layer should be much greater than that along the flow direction. That is,

$$v \ll u \quad \text{and} \quad \frac{\partial}{\partial x} \ll \frac{\partial}{\partial y}$$

General External Flow Characteristics

- Prandtl/Blasius Boundary Layer Solution
- With these assumptions it can be shown that the governing equations reduce to the following boundary layer equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$
$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}$$

- The flow represents a balance between **viscous and inertial effects**, with pressure playing no role.
- The boundary conditions

$$u = v = 0 \quad \text{on} \quad y = 0$$

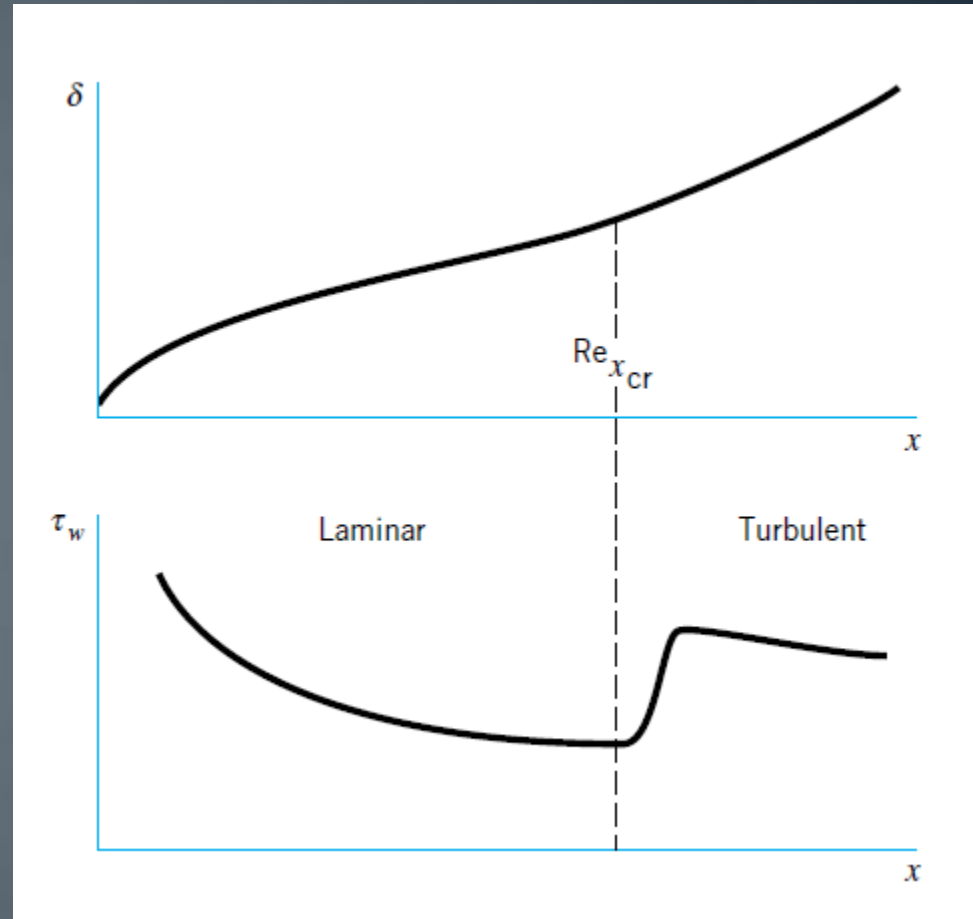
$$u \rightarrow U \quad \text{as} \quad y \rightarrow \infty$$

General External Flow Characteristics

- Prandtl/Blasius Boundary Layer Solution
- In mathematical terms, the Navier–Stokes equations and the continuity equation are **elliptic equations**, whereas the equations for boundary layer flow are **parabolic equations**.
- The nature of the solutions to these two sets of equations, therefore, is different. Physically, this fact translates to the idea that what happens downstream of a given location in a boundary layer cannot affect what happens upstream of that point.
- In general, the solutions of nonlinear partial differential equations are extremely difficult to obtain. However, by applying a clever coordinate transformation and change of variables, Blasius reduced the **partial differential equations** to an **ordinary differential equation** that he was able to solve.

General External Flow Characteristics

- Prandtl/Blasius Boundary Layer Solution



General External Flow Characteristics

- **Prandtl/Blasius Boundary Layer Solution**
- It can be argued that in dimensionless form the boundary layer velocity profiles on a flat plate should be similar regardless of the location along the plate. That is,

$$\frac{u}{U} = g\left(\frac{y}{\delta}\right)$$

- In addition, by applying an order of magnitude analysis of the forces acting on fluid within the boundary layer, it can be shown that the boundary layer thickness grows as the square root of x and inversely proportional to the square root of U . That is,

$$\delta \sim \left(\frac{\nu x}{U}\right)^{1/2}$$

General External Flow Characteristics

- Prandtl/Blasius Boundary Layer Solution
- Thus, we introduce the dimensionless **similarity variable**

$$\eta = (U/\nu x)^{1/2} y$$

- and the stream function

$$\psi = (\nu x U)^{1/2} f(\eta)$$

- By substitution

$$u = U f'(\eta)$$

$$v = \left(\frac{\nu U}{4x} \right)^{1/2} (\eta f' - f)$$

General External Flow Characteristics

- Prandtl/Blasius Boundary Layer Solution
- We substitute the above equations into the governing equations, to obtain the following nonlinear, third-order ordinary differential equation:

$$2f''' + ff'' = 0$$

- The boundary conditions can be written as

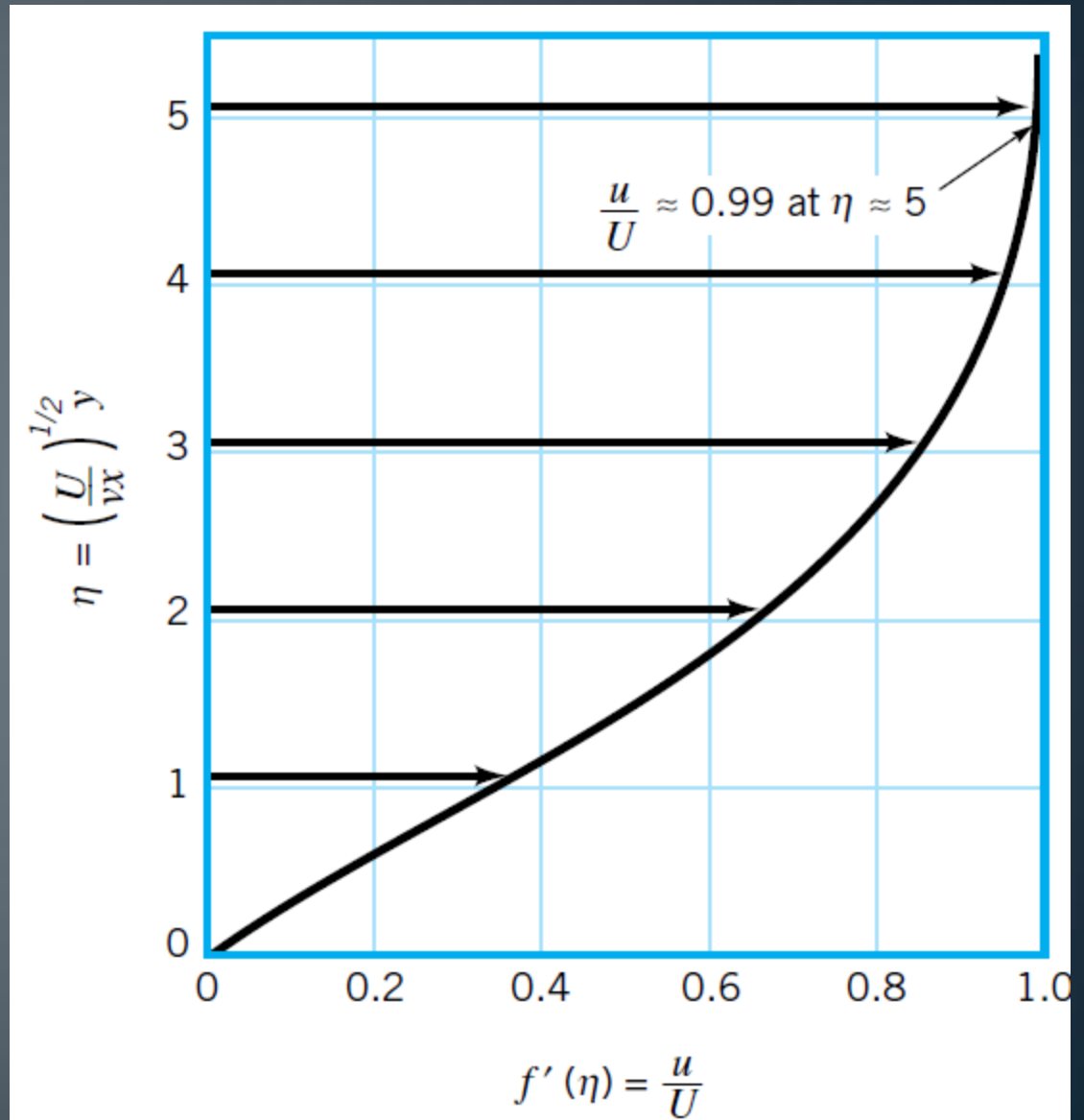
$$f = f' = 0 \text{ at } \eta = 0 \quad \text{and} \quad f' \rightarrow 1 \text{ as } \eta \rightarrow \infty$$

- From the solution it is found that

$$u/U \approx 0.99 \text{ when } \eta = 5.0$$

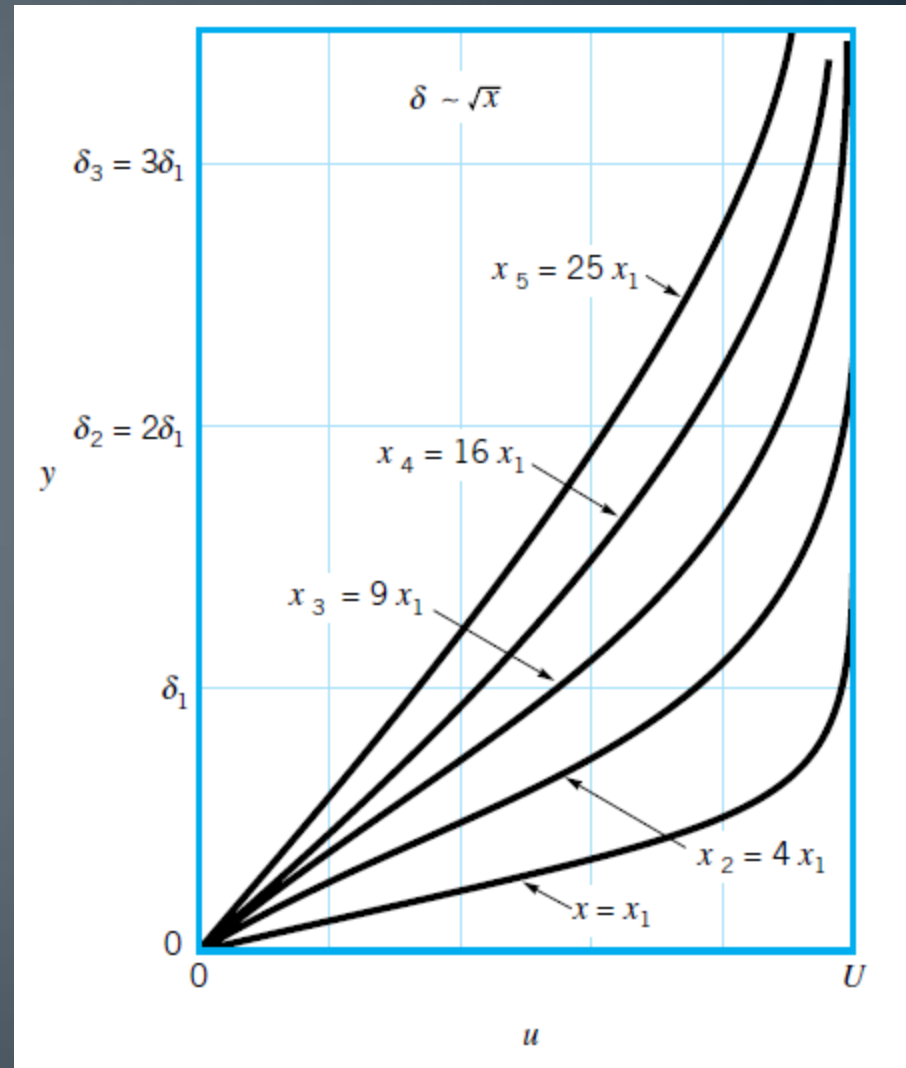
- Prandtl/Blasius Boundary Layer Solution

General External Flow Characteristics



- Prandtl/Blasius Boundary Layer Solution

General External Flow Characteristics



- Prandtl/Blasius Boundary Layer Solution

General External Flow Characteristics

Laminar Flow Along a Flat Plate (the Blasius Solution)

$\eta = y(U/\nu x)^{1/2}$	$f'(\eta) = u/U$	η	$f'(\eta)$
0	0	3.6	0.9233
0.4	0.1328	4.0	0.9555
0.8	0.2647	4.4	0.9759
1.2	0.3938	4.8	0.9878
1.6	0.5168	5.0	0.9916
2.0	0.6298	5.2	0.9943
2.4	0.7290	5.6	0.9975
2.8	0.8115	6.0	0.9990
3.2	0.8761	∞	1.0000

General External Flow Characteristics

- Prandtl/Blasius Boundary Layer Solution
- It can also be shown that

$$\frac{\delta}{x} = \frac{5}{\sqrt{\text{Re}_x}}$$

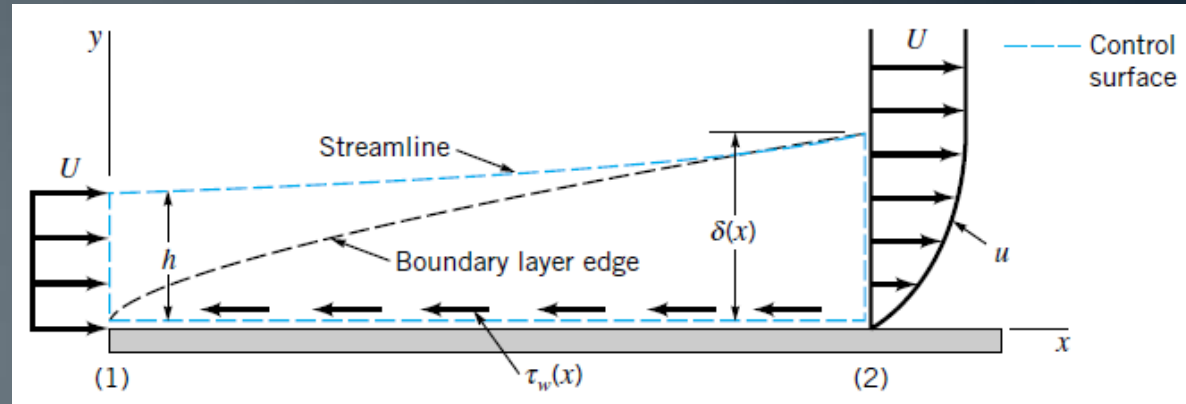
$$\frac{\delta^*}{x} = \frac{1.721}{\sqrt{\text{Re}_x}}$$

$$\frac{\Theta}{x} = \frac{0.664}{\sqrt{\text{Re}_x}}$$

$$\tau_w = 0.332 U^{3/2} \sqrt{\frac{\rho \mu}{x}}$$

General External Flow Characteristics

- Momentum-Integral Boundary Layer Equation for a Flat Plate



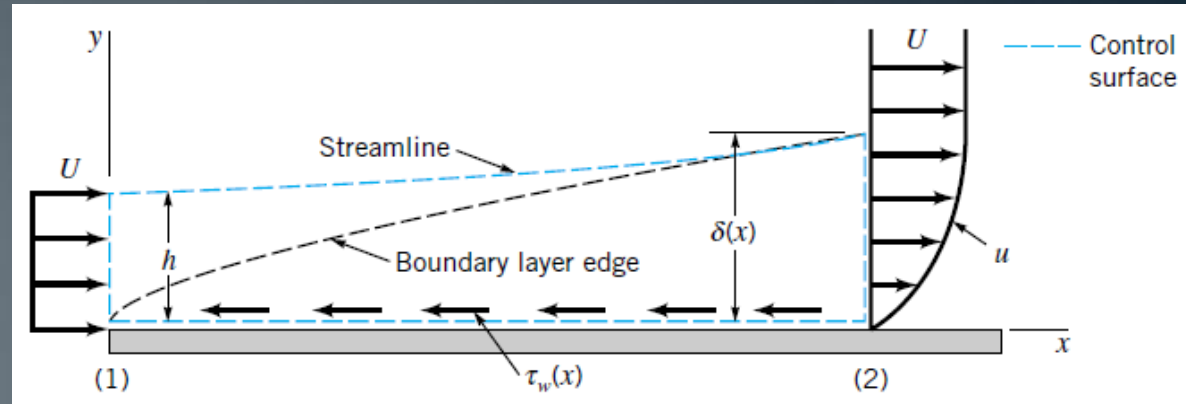
$$\sum F_x = \rho \int_{(1)} u \mathbf{V} \cdot \hat{\mathbf{n}} dA + \rho \int_{(2)} u \mathbf{V} \cdot \hat{\mathbf{n}} dA$$

$$\sum F_x = -\mathcal{D} = - \int_{\text{plate}} \tau_w dA = -b \int_{\text{plate}} \tau_w dx$$

$$-\mathcal{D} = \rho \int_{(1)} U(-U) dA + \rho \int_{(2)} u^2 dA$$

General External Flow Characteristics

- Momentum-Integral Boundary Layer Equation for a Flat Plate



$$\mathcal{D} = \rho U^2 b h - \rho b \int_0^{\delta} u^2 dy$$

$$U h = \int_0^{\delta} u dy$$

$$\rho U^2 b h = \rho b \int_0^{\delta} U u dy$$

General External Flow Characteristics

- Momentum-Integral Boundary Layer Equation for a Flat Plate

$$\mathcal{D} = \rho b \int_0^{\delta} u(U - u) dy$$

- The development of the above equation and its use was first put forth in 1921 by **T. von Karman** (1881–1963), a Hungarian/German aerodynamicist.

$$\mathcal{D} = \rho b U^2 \Theta$$

$$\frac{d\mathcal{D}}{dx} = \rho b U^2 \frac{d\Theta}{dx}$$

$$\frac{d\mathcal{D}}{dx} = b \tau_w$$

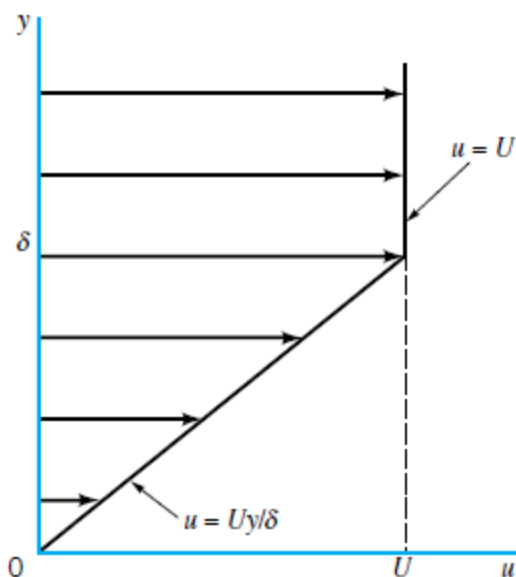
General External Flow Characteristics

- Momentum-Integral Boundary Layer Equation for a Flat Plate
- Hence, by combining the previous equations we obtain the **momentum integral equation** for the boundary layer flow on a flat plate

$$\tau_w = \rho U^2 \frac{d\Theta}{dx}$$

EXAMPLE 9.4

Consider the laminar flow of an incompressible fluid past a flat plate at $y = 0$. The boundary layer velocity profile is approximated as $u = Uy/\delta$ for $0 \leq y \leq \delta$ and $u = U$ for $y > \delta$, as is shown in Fig. E9.4. Determine the shear stress by using the momentum integral equation. Compare these results with the Blasius results given by Eq. 9.18.



■ FIGURE E9.4

SOLUTION

From Eq. 9.26 the shear stress is given by

$$\tau_w = \rho U^2 \frac{d\Theta}{dx} \quad (1)$$

while for laminar flow we know that $\tau_w = \mu(\partial u/\partial y)_{y=0}$. For the assumed profile we have

$$\tau_w = \mu \frac{U}{\delta} \quad (2)$$

and from Eq. 9.4

$$\Theta = \int_0^\infty \frac{u}{U} \left(1 - \frac{u}{U}\right) dy = \int_0^\delta \frac{u}{U} \left(1 - \frac{u}{U}\right) dy = \int_0^\delta \left(\frac{y}{\delta}\right) \left(1 - \frac{y}{\delta}\right) dy$$

or

$$\Theta = \frac{\delta}{6} \quad (3)$$

Note that as yet we do not know the value of δ (but suspect that it should be a function of x).

By combining Eqs. 1, 2, and 3 we obtain the following differential equation for δ :

$$\frac{\mu U}{\delta} = \frac{\rho U^2}{6} \frac{d\delta}{dx}$$

or

$$\delta d\delta = \frac{6\mu}{\rho U} dx$$

This can be integrated from the leading edge of the plate, $x = 0$ (where $\delta = 0$) to an arbitrary location x where the boundary layer thickness is δ . The result is

$$\frac{\delta^2}{2} = \frac{6\mu}{\rho U} x$$

or

$$\delta = 3.46 \sqrt{\frac{\nu x}{U}} \quad (4)$$

Note that this approximate result (i.e., the velocity profile is not actually the simple straight line we assumed) compares favorably with the (much more laborious to obtain) Blasius result given by Eq. 9.15.

The wall shear stress can also be obtained by combining Eqs. 1, 3, and 4 to give

$$\tau_w = 0.289 U^{3/2} \sqrt{\frac{\rho \mu}{x}} \quad (\text{Ans})$$

Again this approximate result is close (within 13%) to the Blasius value of τ_w given by Eq. 9.18.

General External Flow Characteristics

- **Momentum-Integral Boundary Layer Equation for a Flat Plate**
- The momentum integral equation can be used along with an assumed velocity profile to obtain reasonable, approximate boundary layer results. Thus, we consider a general velocity profile

$$\frac{u}{U} = g(Y) \quad \text{for} \quad 0 \leq Y \leq 1$$

$$\frac{u}{U} = 1 \quad \text{for} \quad Y > 1$$

$$g(0) = 0 \quad \text{and} \quad g(1) = 1$$

- Momentum-Integral Boundary Layer Equation for a Flat Plate

General External Flow Characteristics

$$\mathcal{D} = \rho b \int_0^\delta u(U - u) dy = \rho b U^2 \delta \int_0^1 g(Y)[1 - g(Y)] dY$$

$$\mathcal{D} = \rho b U^2 \delta C_1$$

$$C_1 = \int_0^1 g(Y)[1 - g(Y)] dY$$

$$\tau_w = \mu \left. \frac{\partial u}{\partial y} \right|_{y=0} = \frac{\mu U}{\delta} \left. \frac{dg}{dY} \right|_{Y=0} = \frac{\mu U}{\delta} C_2$$

$$C_2 = \left. \frac{dg}{dY} \right|_{Y=0}$$

General External Flow Characteristics

- Momentum-Integral Boundary Layer Equation for a Flat Plate
- By combining the above equations we obtain

$$\delta \, d\delta = \frac{\mu C_2}{\rho U C_1} dx$$

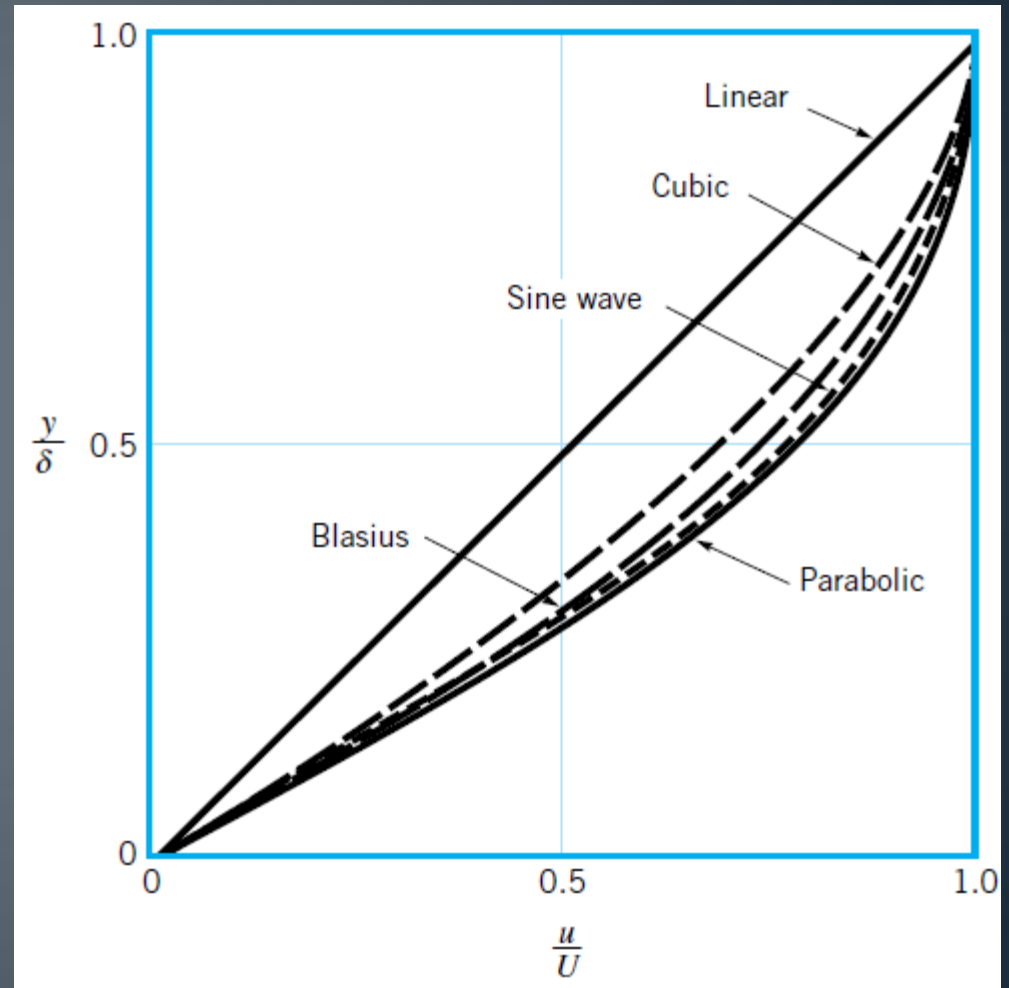
$$\delta = \sqrt{\frac{2\nu C_2 x}{U C_1}}$$

$$\frac{\delta}{x} = \frac{\sqrt{2C_2/C_1}}{\sqrt{\text{Re}_x}}$$

$$\tau_w = \sqrt{\frac{C_1 C_2}{2}} U^{3/2} \sqrt{\frac{\rho \mu}{x}}$$

General External Flow Characteristics

- Momentum-Integral Boundary Layer Equation for a Flat Plate



General External Flow Characteristics

- Momentum-Integral Boundary Layer Equation for a Flat Plate
- It is often convenient to use the dimensionless local friction coefficient, defined as

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho U^2}$$

$$c_f = \sqrt{2C_1C_2} \sqrt{\frac{\mu}{\rho Ux}} = \frac{\sqrt{2C_1C_2}}{\sqrt{\text{Re}_x}}$$

- while the Blasius solution result is given by

$$c_f = \frac{0.664}{\sqrt{\text{Re}_x}}$$

- Momentum-Integral Boundary Layer Equation for a Flat Plate

General External Flow Characteristics

Flat Plate Momentum-Integral Results for Various Assumed Laminar Flow Velocity Profiles

Profile Character	$\delta \text{Re}_x^{1/2}/x$	$c_f \text{Re}_x^{1/2}$	$C_{Df} \text{Re}_x^{1/2}$
a. Blasius solution	5.00	0.664	1.328
b. Linear $u/U = y/\delta$	3.46	0.578	1.156
c. Parabolic $u/U = 2y/\delta - (y/\delta)^2$	5.48	0.730	1.460
d. Cubic $u/U = 3(y/\delta)/2 - (y/\delta)^3/2$	4.64	0.646	1.292
e. Sine wave $u/U = \sin[\pi(y/\delta)/2]$	4.79	0.655	1.310

General External Flow Characteristics

- Momentum-Integral Boundary Layer Equation for a Flat Plate
- It is often convenient to use the dimensionless local friction coefficient, defined as

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho U^2}$$

$$c_f = \sqrt{2C_1C_2} \sqrt{\frac{\mu}{\rho Ux}} = \frac{\sqrt{2C_1C_2}}{\sqrt{\text{Re}_x}}$$

- while the Blasius solution result is given by

$$c_f = \frac{0.664}{\sqrt{\text{Re}_x}}$$

General External Flow Characteristics

- **Momentum-Integral Boundary Layer Equation for a Flat Plate**
- For a flat plate of length and width b , the net friction drag, can be expressed in terms of the friction drag coefficient, as

$$C_{Df} = \frac{\mathcal{D}_f}{\frac{1}{2}\rho U^2 b \ell} = \frac{b \int_0^\ell \tau_w dx}{\frac{1}{2}\rho U^2 b \ell}$$

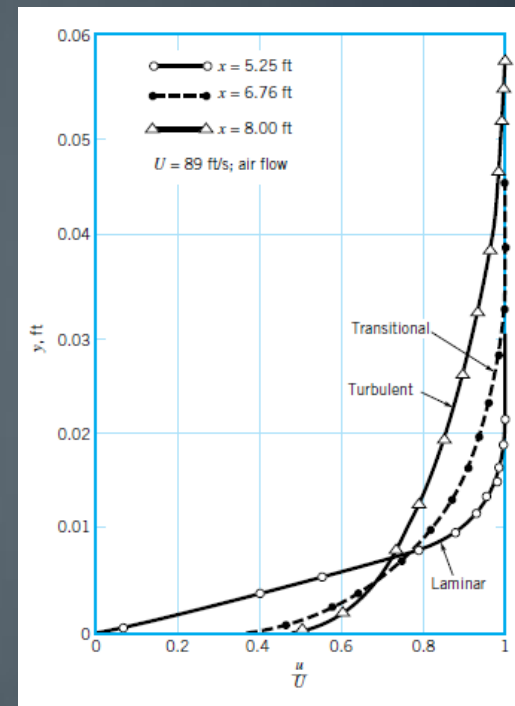
$$C_{Df} = \frac{1}{\ell} \int_0^\ell c_f dx$$

$$C_{Df} = \frac{\sqrt{8C_1C_2}}{\sqrt{\text{Re}_\ell}}$$

$$C_{Df} = \frac{1.328}{\sqrt{\text{Re}_\ell}}$$

General External Flow Characteristics

- **Transition from Laminar to Turbulent Flow**
- Transition from laminar to turbulent flow involves a noticeable change in the shape of the boundary layer velocity profile.



- The turbulent profiles are **flatter**, have a **larger velocity gradient at the wall**, and produce a **larger boundary layer thickness** than do the laminar profiles.

General External Flow Characteristics

- **Turbulent Boundary Layer Flow**
- The structure of turbulent boundary layer flow is very complex, random, and irregular.
- There are no “**exact**” solutions for turbulent boundary layer flow.
- It is possible to solve the Prandtl boundary layer equations for laminar flow past a flat plate to obtain the Blasius solution. Which is “exact” within the framework of the assumptions involved in the boundary layer equations.
- Since there is no precise expression for the shear stress in turbulent flow, solutions are not available for turbulent flow.
- Approximate turbulent boundary layer results can also be obtained by use of **the momentum integral equation** which is valid for either laminar or turbulent flow.

EXAMPLE 9.6

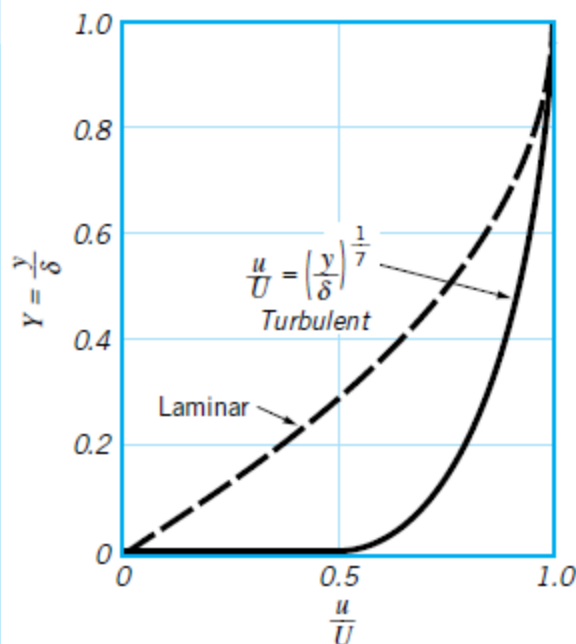
Consider turbulent flow of an incompressible fluid past a flat plate. The boundary layer velocity profile is assumed to be $u/U = (y/\delta)^{1/7} = Y^{1/7}$ for $Y = y/\delta \leq 1$ and $u = U$ for $Y > 1$ as shown in Fig. E9.6. This is a reasonable approximation of experimentally observed profiles, except very near the plate where this formula gives $\partial u/\partial y = \infty$ at $y = 0$. Note the differences between the assumed turbulent profile and the laminar profile. Also assume that the shear stress agrees with the experimentally determined formula:

$$\tau_w = 0.0225\rho U^2 \left(\frac{\nu}{U\delta} \right)^{1/4} \quad (1)$$

Determine the boundary layer thicknesses δ , δ^* , and Θ and the wall shear stress, τ_w , as a function of x . Determine the friction drag coefficient, C_{Df} .

SOLUTION

Whether the flow is laminar or turbulent, it is true that the drag force is accounted for by a reduction in the momentum of the fluid flowing past the plate. The shear is obtained from



■ FIGURE E9.6

Eq. 9.26 in terms of the rate at which the momentum boundary layer thickness, Θ , increases with distance along the plate as

$$\tau_w = \rho U^2 \frac{d\Theta}{dx}$$

For the assumed velocity profile, the boundary layer momentum thickness is obtained from Eq. 9.4 as

$$\Theta = \int_0^\infty \frac{u}{U} \left(1 - \frac{u}{U}\right) dy = \delta \int_0^1 \frac{u}{U} \left(1 - \frac{u}{U}\right) dY$$

or by integration

$$\Theta = \delta \int_0^1 Y^{1/7} (1 - Y^{1/7}) dY = \frac{7}{72} \delta \quad (2)$$

where δ is an unknown function of x . By combining the assumed shear force dependence (Eq. 1) with Eq. 2, we obtain the following differential equation for δ :

$$0.0225 \rho U^2 \left(\frac{\nu}{U\delta}\right)^{1/4} = \frac{7}{72} \rho U^2 \frac{d\delta}{dx}$$

or

$$\delta^{1/4} d\delta = 0.231 \left(\frac{\nu}{U}\right)^{1/4} dx$$

This can be integrated from $\delta = 0$ at $x = 0$ to obtain

$$\delta = 0.370 \left(\frac{\nu}{U}\right)^{1/5} x^{4/5} \quad (3) \quad (\text{Ans})$$

or in dimensionless form

$$\frac{\delta}{x} = \frac{0.370}{\text{Re}_x^{1/5}}$$

Strictly speaking, the boundary layer near the leading edge of the plate is laminar, not turbulent, and the precise boundary condition should be the matching of the initial turbulent boundary layer thickness (at the transition location) with the thickness of the laminar boundary layer at that point. In practice, however, the laminar boundary layer often exists over a relatively short portion of the plate, and the error associated with starting the turbulent boundary layer with $\delta = 0$ at $x = 0$ can be negligible.

The displacement thickness, δ^* , and the momentum thickness, Θ , can be obtained from Eqs. 9.3 and 9.4 by integrating as follows:

$$\begin{aligned}\delta^* &= \int_0^\infty \left(1 - \frac{u}{U}\right) dy = \delta \int_0^1 \left(1 - \frac{u}{U}\right) dY \\ &= \delta \int_0^1 (1 - Y^{1/7}) dY = \frac{\delta}{8}\end{aligned}$$

Thus, by combining this with Eq. 3 we obtain

$$\delta^* = 0.0463 \left(\frac{\nu}{U}\right)^{1/5} x^{4/5} \quad (\text{Ans})$$

Similarly, from Eq. 2,

$$\Theta = \frac{7}{72} \delta = 0.0360 \left(\frac{\nu}{U}\right)^{1/5} x^{4/5} \quad (4) \quad (\text{Ans})$$

The functional dependence for δ , δ^* , and Θ is the same; only the constants of proportionality are different. Typically, $\Theta < \delta^* < \delta$.

By combining Eqs. 1 and 3, we obtain the following result for the wall shear stress

$$\tau_w = 0.0225\rho U^2 \left[\frac{\nu}{U(0.370)(\nu/U)^{1/5}x^{4/5}} \right]^{1/4} = \frac{0.0288\rho U^2}{\text{Re}_x^{1/5}} \quad (\text{Ans})$$

This can be integrated over the length of the plate to obtain the friction drag on one side of the plate, $\mathcal{D}_f$, as

$$\mathcal{D}_f = \int_0^\ell b\tau_w dx = b(0.0288\rho U^2) \int_0^\ell \left(\frac{\nu}{Ux} \right)^{1/5} dx$$

or

$$\mathcal{D}_f = 0.0360\rho U^2 \frac{A}{\text{Re}_\ell^{1/5}}$$

where $A = b\ell$ is the area of the plate. (This result can also be obtained by combining Eq. 9.23 and the expression for the momentum thickness given in Eq. 4.) The corresponding friction drag coefficient, C_{Df} , is

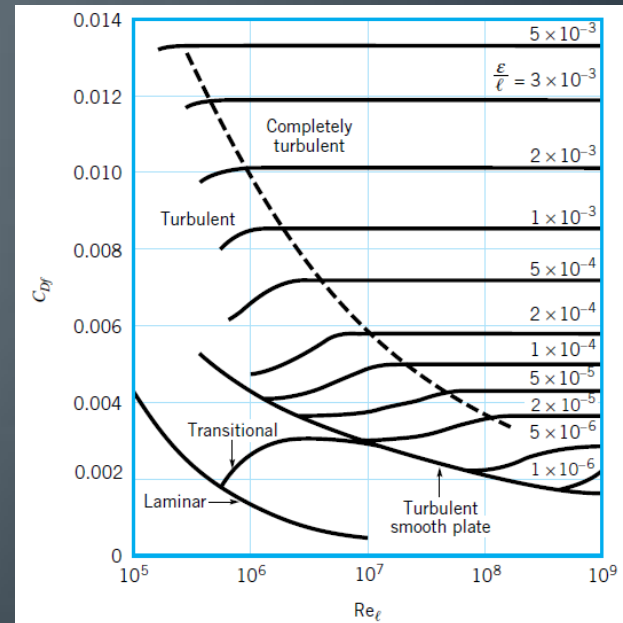
$$C_{Df} = \frac{\mathcal{D}_f}{\frac{1}{2}\rho U^2 A} = \frac{0.0720}{\text{Re}_\ell^{1/5}} \quad (\text{Ans})$$

Note that for the turbulent boundary layer flow the boundary layer thickness increases with x as $\delta \sim x^{4/5}$ and the shear stress decreases as $\tau_w \sim x^{-1/5}$. For laminar flow these dependencies are $x^{1/2}$ and $x^{-1/2}$, respectively. The random character of the turbulent flow causes a different structure of the flow.

Obviously the results presented in this example are valid only in the range of validity of the original data—the assumed velocity profile and shear stress. This range covers smooth flat plates with $5 \times 10^5 < \text{Re}_\ell < 10^7$.

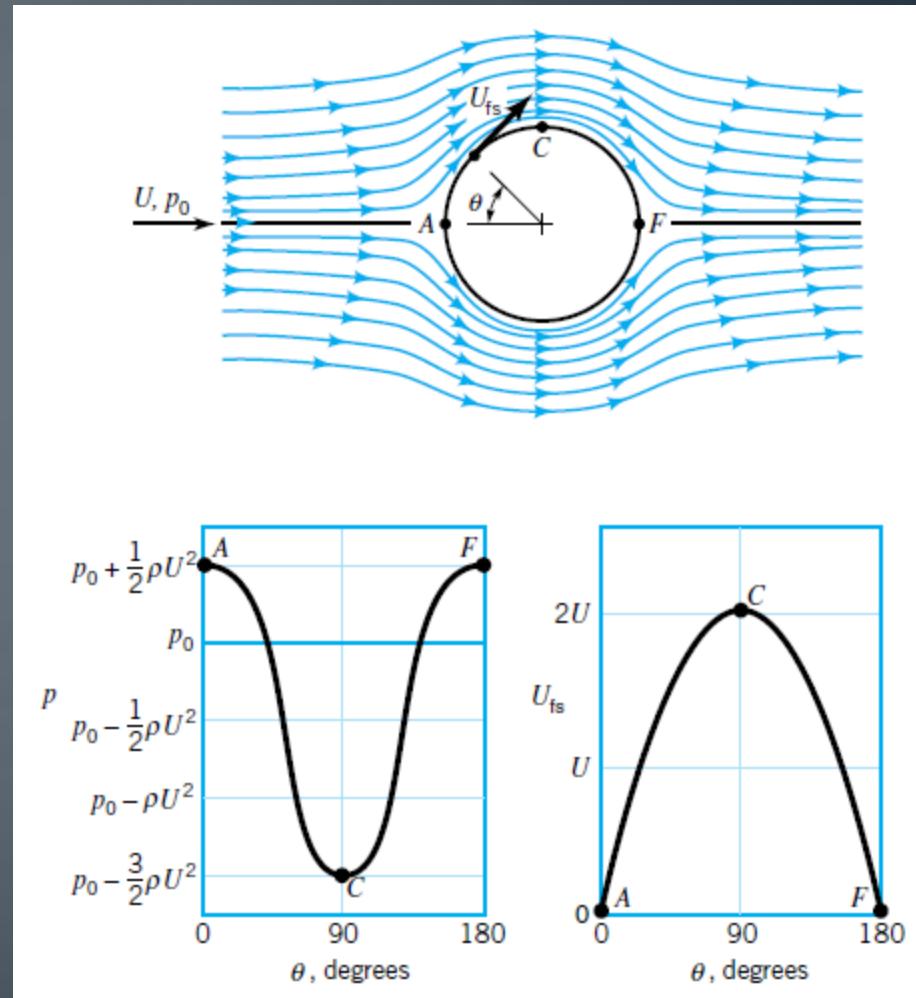
General External Flow Characteristics

- **Turbulent Boundary Layer Flow**
- In general, the drag coefficient for a flat plate of length is a function of the **Reynolds number**, and the **relative roughness**.
- Fully developed horizontal pipe flow is governed by a balance between **pressure forces** and **viscous forces**. The fluid inertia remains constant throughout the flow. Boundary layer flow on a horizontal flat plate is governed by a balance between **inertia effects** and **viscous forces**. The pressure remains constant throughout the flow.



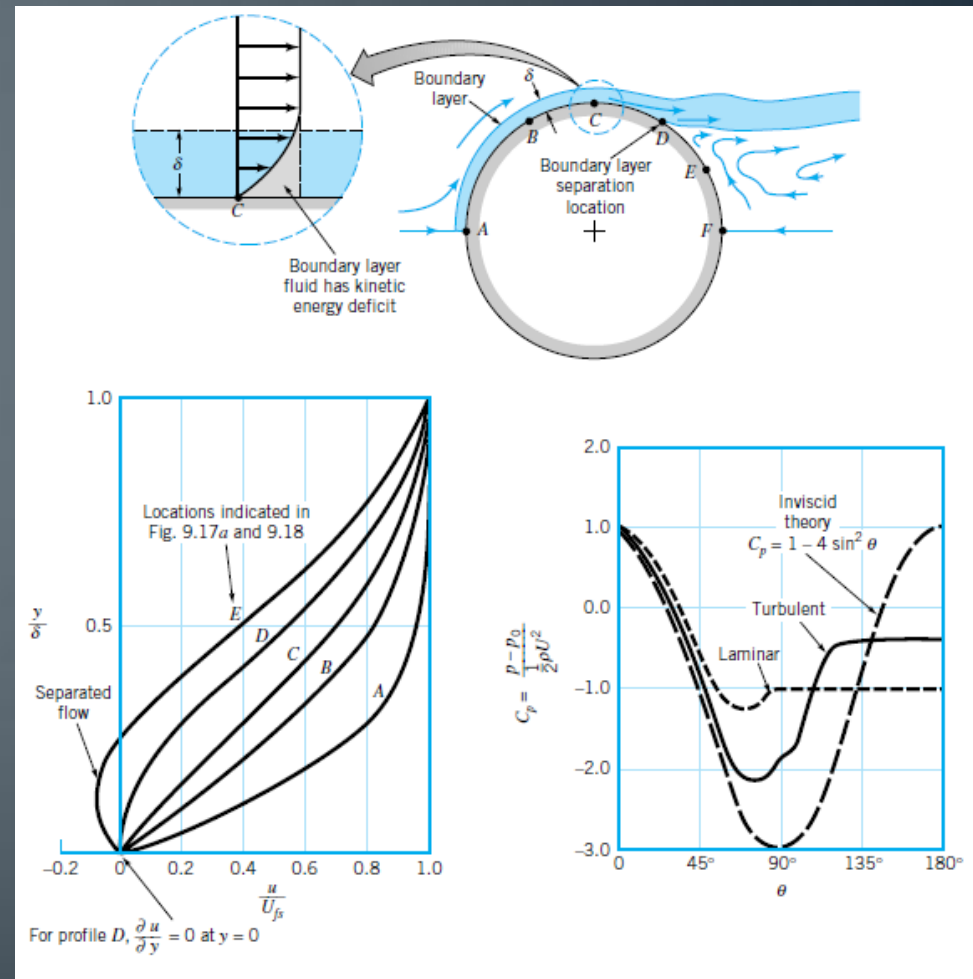
General External Flow Characteristics

- Effects of Pressure Gradient
- The pressure and free-stream velocity distributions in the flow past a circular cylinder are shown below.



General External Flow Characteristics

- **Effects of Pressure Gradient**
- The decrease in pressure in the direction of flow along the front half of the cylinder is termed a **favorable pressure gradient**. The increase in pressure in the direction of flow along the rear half of the cylinder is termed an **adverse pressure gradient**.



General External Flow Characteristics

- **Momentum-Integral Boundary Layer Equation with Nonzero Pressure Gradient**
- If the freestream velocity is not constant, the pressure will not be constant.

$$\frac{dp}{dx} = -\rho U_{fs} \frac{dU_{fs}}{dx}$$

- For a given body the free-stream velocity and the corresponding pressure gradient on the surface can be obtained from inviscid flow techniques.
- By using the x component of the momentum equation with the appropriate shear forces and pressure forces the following integral momentum equation for boundary layer flows is obtained:

$$\tau_w = \rho \frac{d}{dx} (U_{fs}^2 \Theta) + \rho \delta^* U_{fs} \frac{dU_{fs}}{dx}$$

- The general momentum integral equation for two-dimensional boundary layer flow, represents a balance between **viscous forces**, **pressure forces**, and the **fluid momentum**.

Drag

- Most of the information pertaining to drag on objects is a result of numerous **experiments** with wind tunnels, water tunnels, towing tanks, and other ingenious devices that are used to measure the drag on scale models.
- Typically, the result for a given-shaped object is a **drag coefficient**, where

$$C_D = \frac{\mathcal{D}}{\frac{1}{2}\rho U^2 A}$$

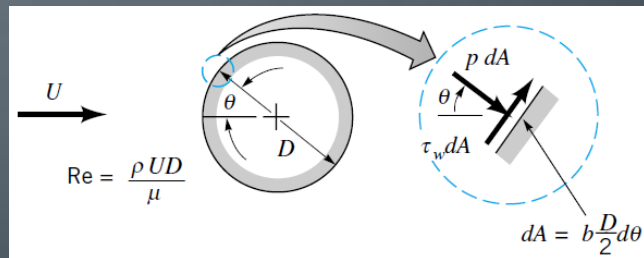
- and drag coefficient is a function of other dimensionless parameters such as Reynolds number, Mach number, Froude number, and relative roughness of the surface, That is,

$$C_D = \phi(\text{shape}, \text{Re}, \text{Ma}, \text{Fr}, \varepsilon/\ell)$$

Drag

- **Friction Drag**
 - Friction drag, is that part of the drag that is due directly to the shear stress, on the object. It is a function of not only the **magnitude of the wall shear stress**, but also of the **orientation of the surface** on which it acts.
 - Because the viscosity of most common fluids is small, the contribution of the shear force to the overall drag on a body is often quite **small**.
 - For highly streamlined bodies or for low Reynolds number flow, however, most of the drag may be due to friction drag.

$$\mathcal{D}_f = \frac{1}{2} \rho U^2 b \ell C_{Df}$$



$$\mathcal{D}_f = \int \tau_w \sin \theta dA$$

Drag

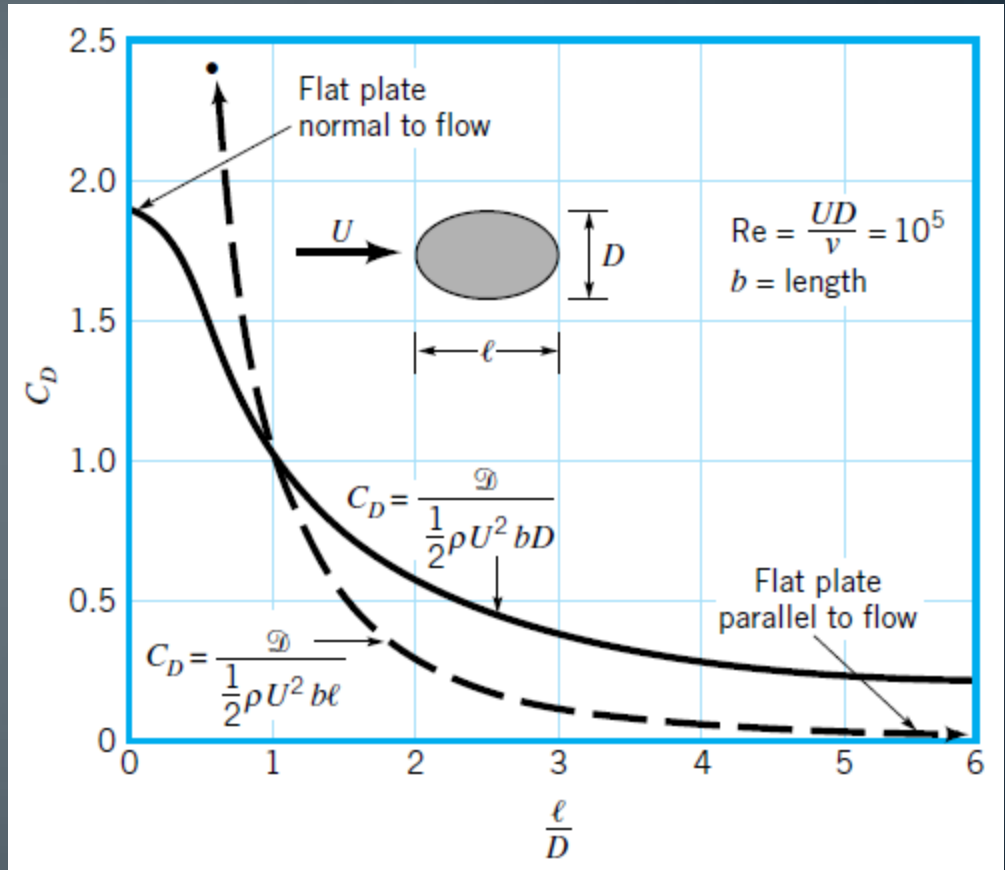
- **Pressure Drag**
- Pressure drag, is that part of the drag that is due directly to the pressure, on an object.
- It is often referred to as **form drag** because of its strong dependency on the shape or form of the object.
- Pressure drag is a function of the **magnitude of the pressure** and the **orientation of the surface element** on which the pressure force acts.

$$\mathcal{D}_p = \int p \cos \theta dA$$

$$C_{Dp} = \frac{\mathcal{D}_p}{\frac{1}{2}\rho U^2 A} = \frac{\int p \cos \theta dA}{\frac{1}{2}\rho U^2 A} = \frac{\int C_p \cos \theta dA}{A}$$

Drag

- Drag Coefficient Data and Examples
- Shape Dependence.



Drag

- Drag Coefficient Data and Examples
- *Reynolds Number Dependence.*
- Low Reynolds number flows are governed by a balance between **viscous** and **pressure forces**. Inertia effects are negligibly small.

$$\mathcal{D} = f(U, \ell, \mu)$$

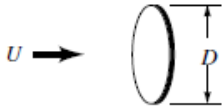
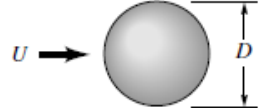
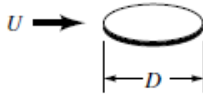
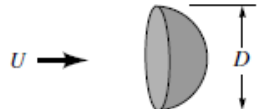
$$\mathcal{D} = C\mu\ell U$$

$$C_D = \frac{\mathcal{D}}{\frac{1}{2}\rho U^2 \ell^2} = \frac{2C\mu\ell U}{\rho U^2 \ell^2} = \frac{2C}{\text{Re}}$$

Drag

- Drag Coefficient Data and Examples
- Reynolds Number Dependence.

Low Reynolds Number Drag Coefficients (Ref. 7) ($Re = \rho U D / \mu, A = \pi D^2 / 4$)

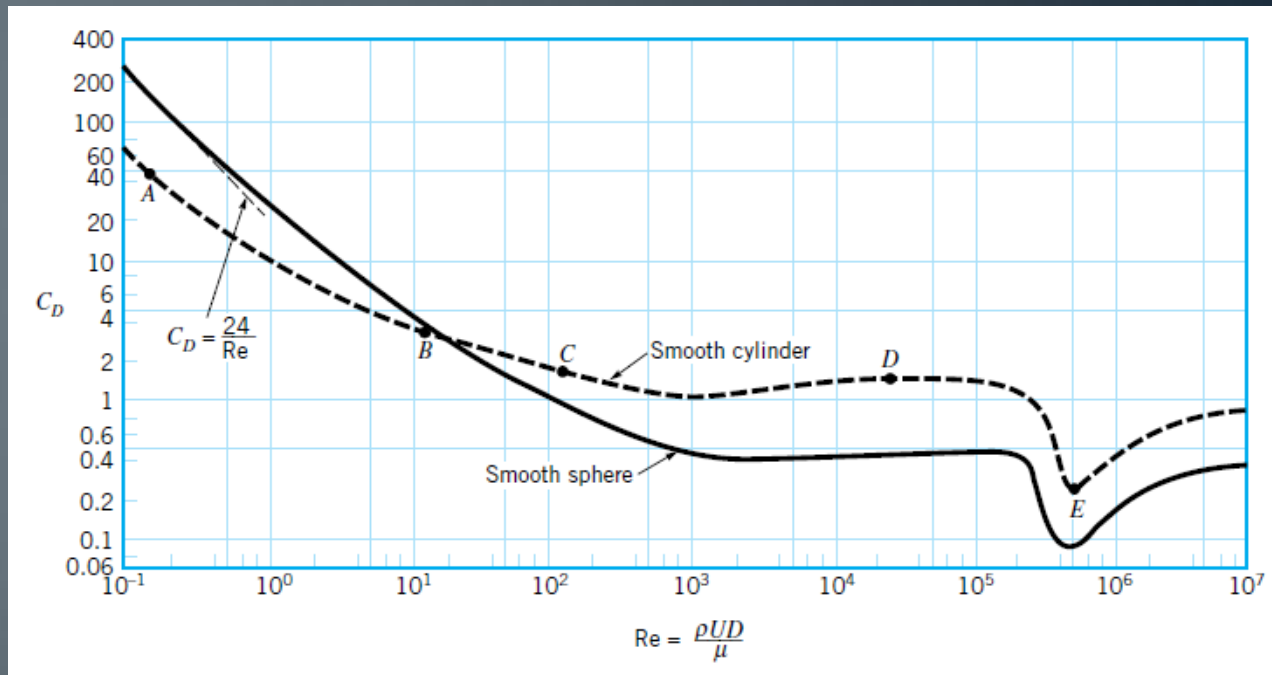
Object	$C_D = \mathcal{D} / (\rho U^2 A / 2)$ (for $Re \lesssim 1$)	Object	C_D
a. Circular disk normal to flow 	$20.4/Re$	c. Sphere 	$24.0/Re$
b. Circular disk parallel to flow 	$13.6/Re$	d. Hemisphere 	$22.2/Re$

Drag

- Drag Coefficient Data and Examples
- *Reynolds Number Dependence.*
- Moderate Reynolds number flows tend to take on a boundary layer flow structure.
- For such flows past **streamlined bodies**, the drag coefficient tends to **decrease** slightly with Reynolds number.
- Moderate Reynolds number flows past **blunt bodies** generally produce drag coefficients that are relatively **constant**.
- For many shapes there is a sudden change in the character of the drag coefficient when the boundary layer becomes turbulent.
- For streamlined bodies, the drag coefficient increases when the boundary layer becomes turbulent because most of the drag is due to the shear force, which is greater for turbulent flow than for laminar flow.
- On the other hand, the drag coefficient for a relatively blunt object, such as a cylinder or sphere, actually decreases when the boundary layer becomes turbulent.

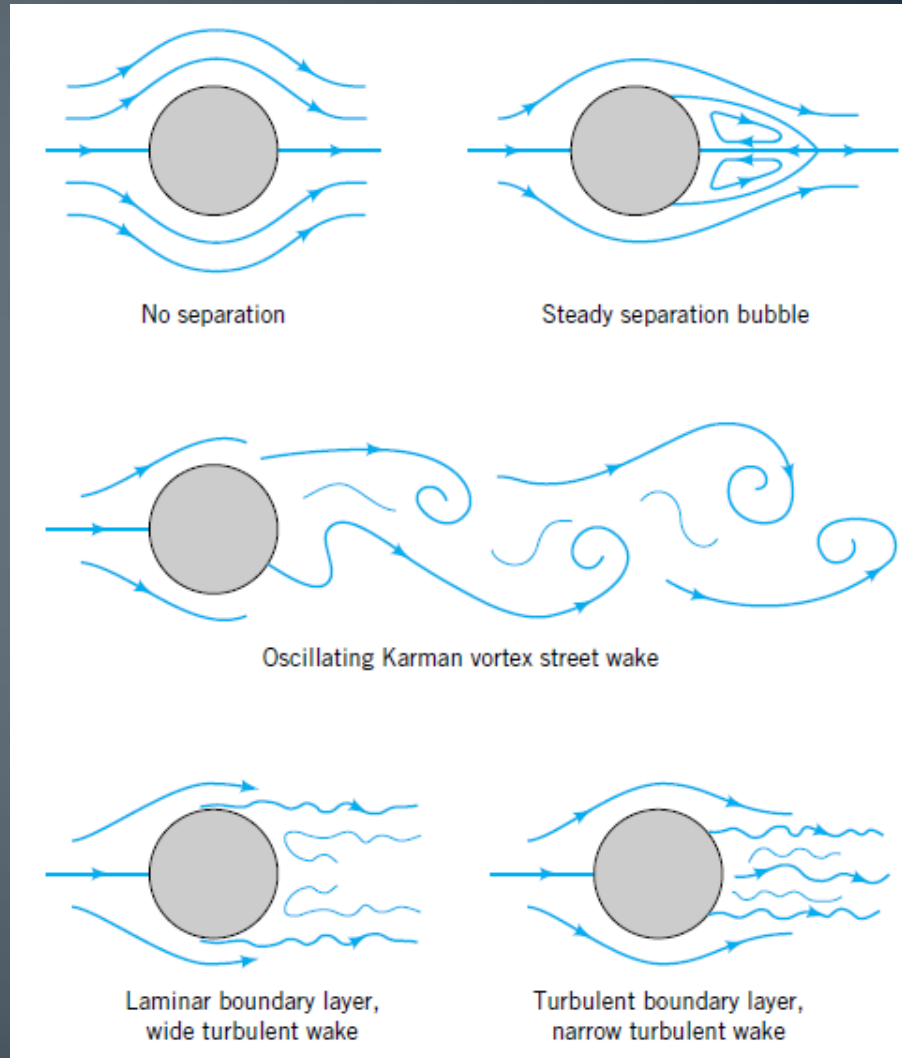
Drag

- Drag Coefficient Data and Examples
- *Reynolds Number Dependence.*



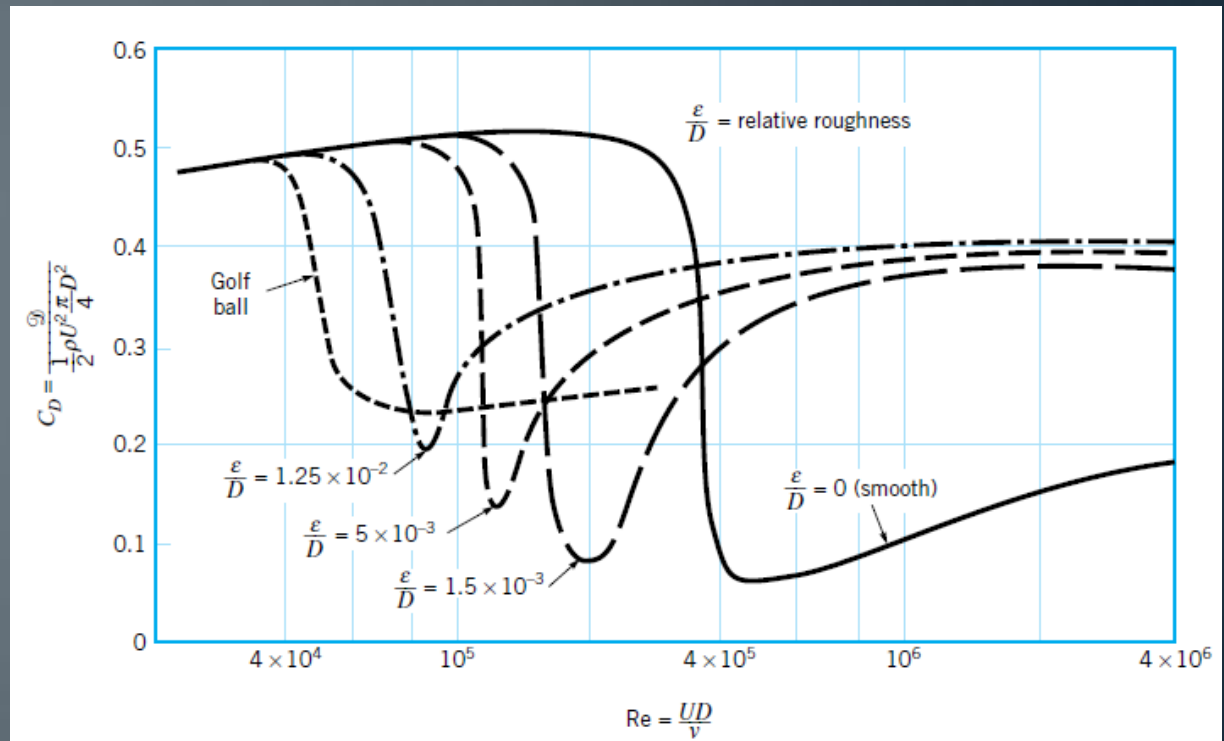
- Drag Coefficient Data and Examples
- *Reynolds Number Dependence.*

Drag



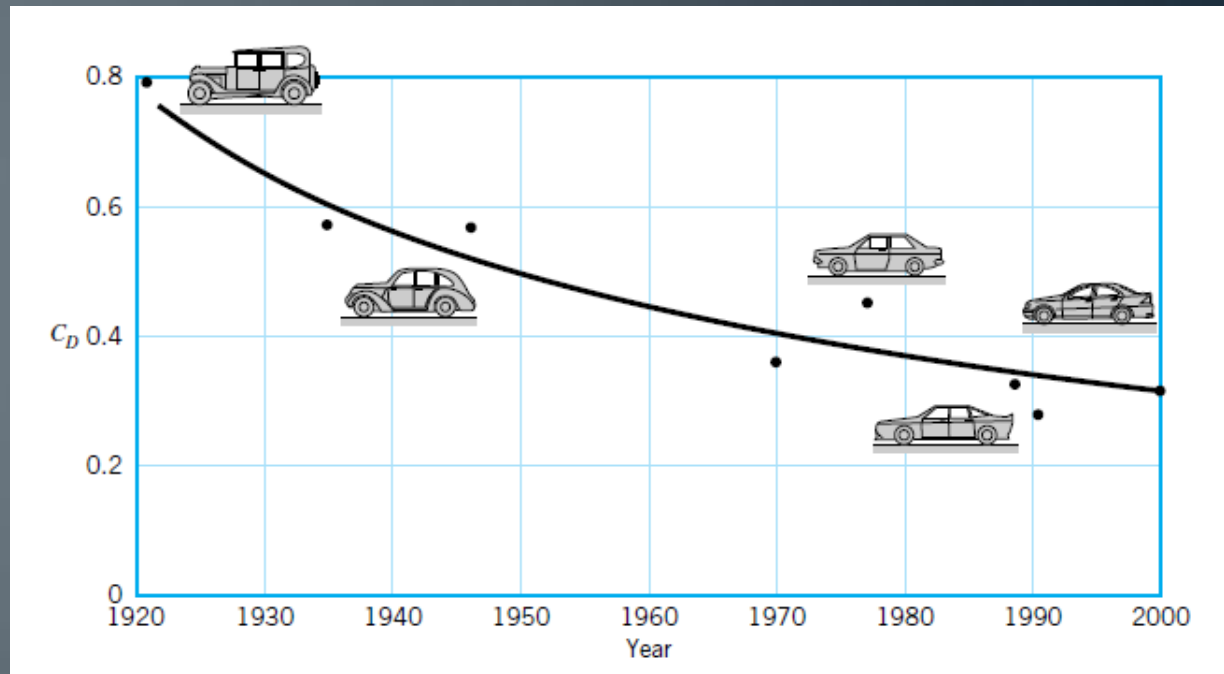
Drag

- Drag Coefficient Data and Examples
- Surface Roughness.



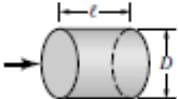
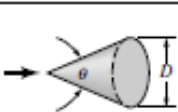
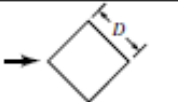
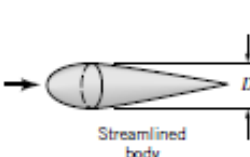
Drag

- Drag Coefficient Data and Examples
- *Composite Body Drag.*



Drag

- Drag Coefficient Data and Examples
- Composite Body Drag.

Shape	Reference area A	Drag coefficient C_D	Reynolds number $Re = \rho U D / \mu$										
 Solid hemisphere D	$A = \frac{\pi}{4} D^2$	$\rightarrow$ 1.17 $\leftarrow$ 0.42	$Re > 10^4$										
 Hollow hemisphere D	$A = \frac{\pi}{4} D^2$	$\rightarrow$ 1.42 $\leftarrow$ 0.38	$Re > 10^4$										
 Thin disk D	$A = \frac{\pi}{4} D^2$	1.1	$Re > 10^3$										
 Circular rod parallel to flow ℓ D	$A = \frac{\pi}{4} D^2$	<table><tr><th>ℓ/D</th><th>C_D</th></tr><tr><td>0.5</td><td>1.1</td></tr><tr><td>1.0</td><td>0.93</td></tr><tr><td>2.0</td><td>0.83</td></tr><tr><td>4.0</td><td>0.85</td></tr></table>	ℓ/D	C_D	0.5	1.1	1.0	0.93	2.0	0.83	4.0	0.85	$Re > 10^3$
ℓ/D	C_D												
0.5	1.1												
1.0	0.93												
2.0	0.83												
4.0	0.85												
 Cone θ D	$A = \frac{\pi}{4} D^2$	<table><tr><th>θ, degrees</th><th>C_D</th></tr><tr><td>10</td><td>0.30</td></tr><tr><td>30</td><td>0.55</td></tr><tr><td>60</td><td>0.80</td></tr><tr><td>90</td><td>1.15</td></tr></table>	θ , degrees	C_D	10	0.30	30	0.55	60	0.80	90	1.15	$Re > 10^4$
θ , degrees	C_D												
10	0.30												
30	0.55												
60	0.80												
90	1.15												
 Cube D	$A = D^2$	1.05	$Re > 10^4$										
 Cube D	$A = D^2$	0.80	$Re > 10^4$										
 Streamlined body D	$A = \frac{\pi}{4} D^2$	0.04	$Re > 10^3$										

Lift

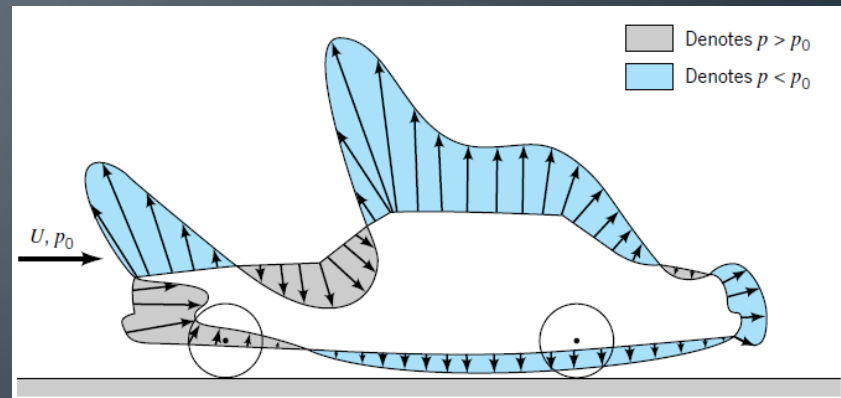
- Any object moving through a fluid will experience a net force of the fluid on the object. For symmetrical objects, this force will be in the direction of the free stream—a drag. If the object is not symmetrical or if it does not produce a symmetrical flow field, such as the flow around a rotating sphere, there may also be a force normal to the free stream—a lift.
- Some objects, such as an airfoil, are designed to generate lift. Other objects are designed to reduce the lift generated.
- For example, the lift on a car tends to reduce the contact force between the wheels and the ground, causing reduction in traction and cornering ability. It is desirable to reduce this lift.

Lift

- **Surface Pressure Distribution**
- Typically, the lift is given in terms of the lift coefficient.

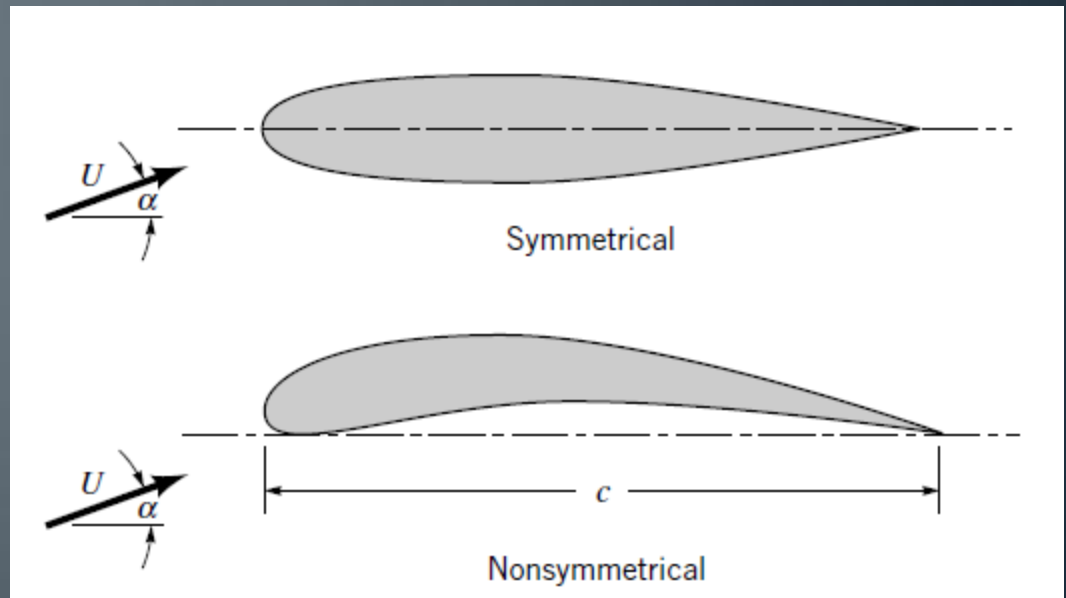
$$C_L = \frac{\mathcal{L}}{\frac{1}{2}\rho U^2 A}$$

$$C_L = \phi(\text{shape}, \text{Re}, \text{Ma}, \text{Fr}, \epsilon/\ell)$$



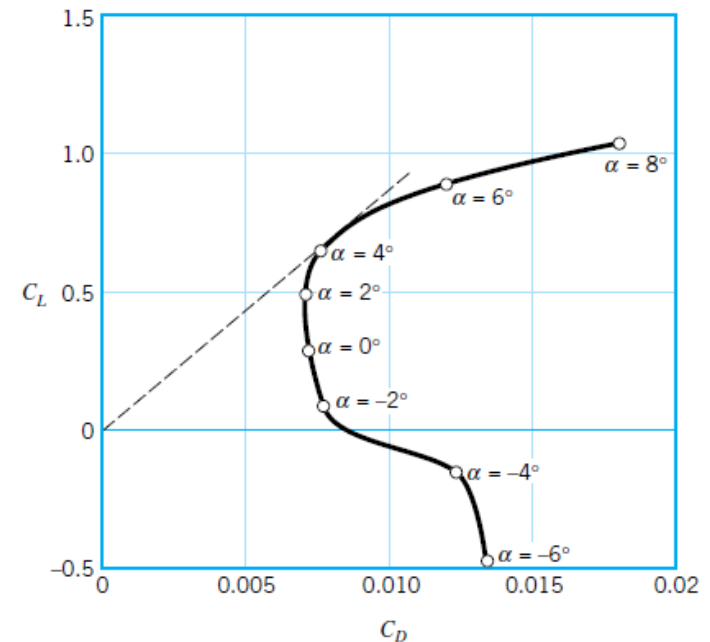
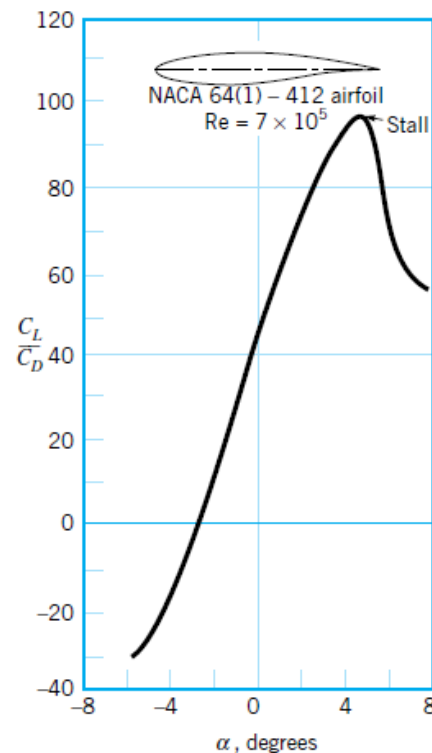
Lift

- **Surface Pressure Distribution**
- A typical device designed to produce lift does so by generating a pressure distribution that is different on the top and bottom surfaces.



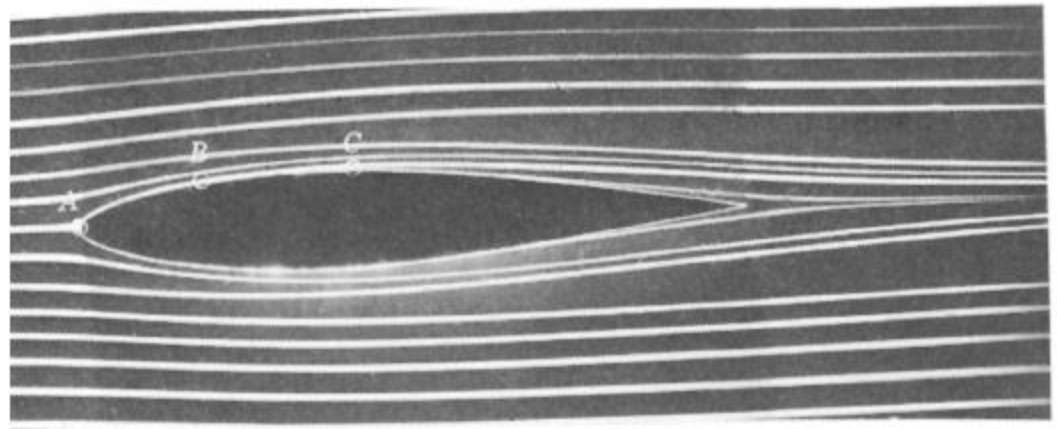
Lift

- Surface Pressure Distribution
- If angle of attack is too large, the boundary layer on the upper surface separates, the flow over the wing develops a wide, turbulent wake region, the lift decreases, and the drag increases. The airfoil **stalls**.

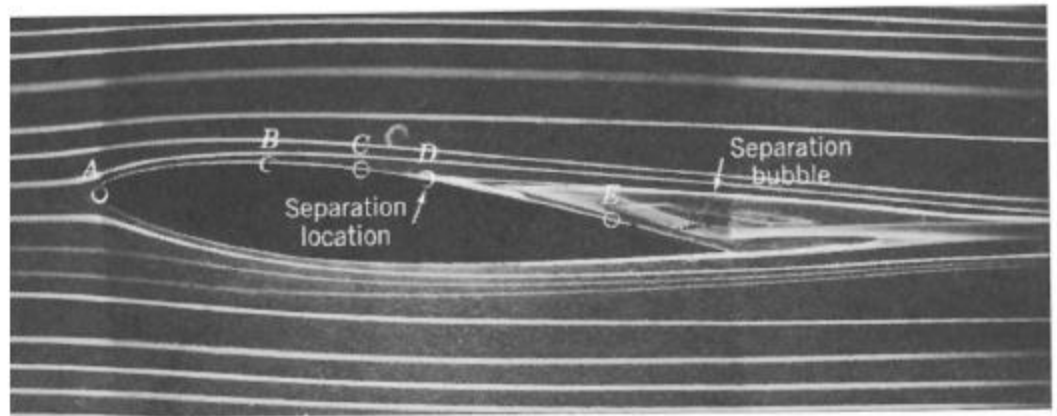


- Surface Pressure Distribution

Lift



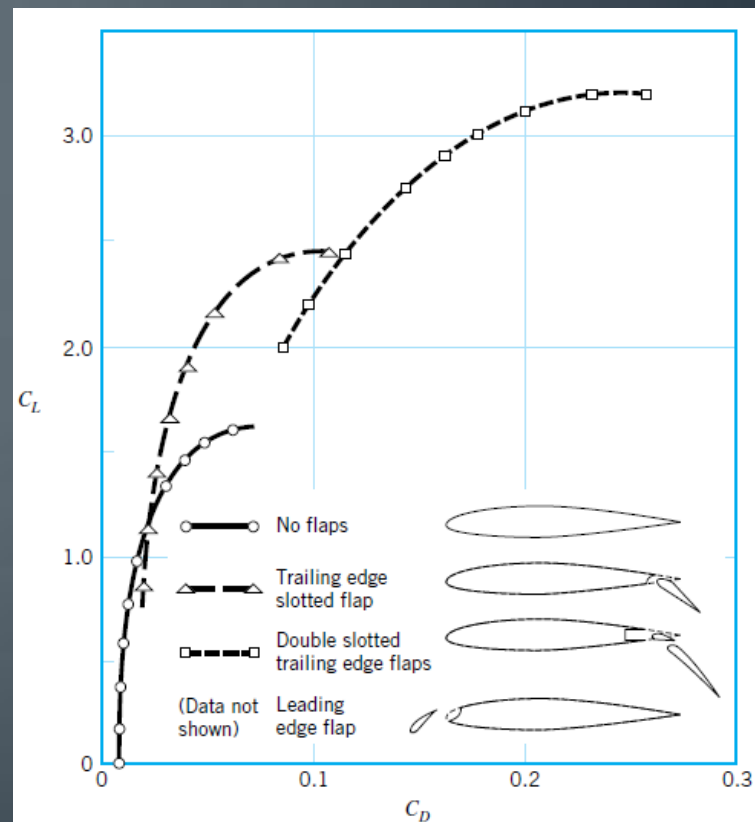
(a)



(b)

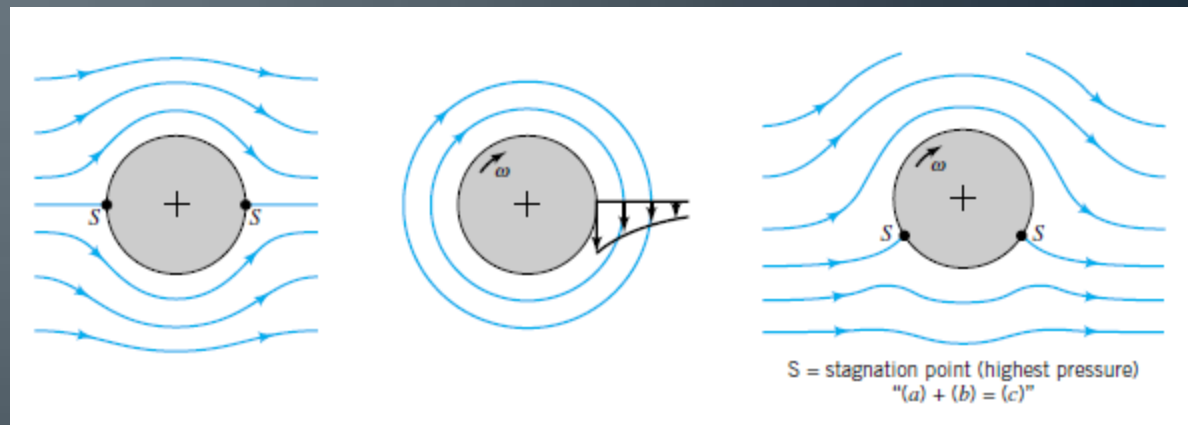
Lift

- **Surface Pressure Distribution**
- To generate the necessary lift during the relatively low-speed landing and takeoff procedures, the airfoil shape is altered by extending special flaps on the front and or rear portions of the wing.



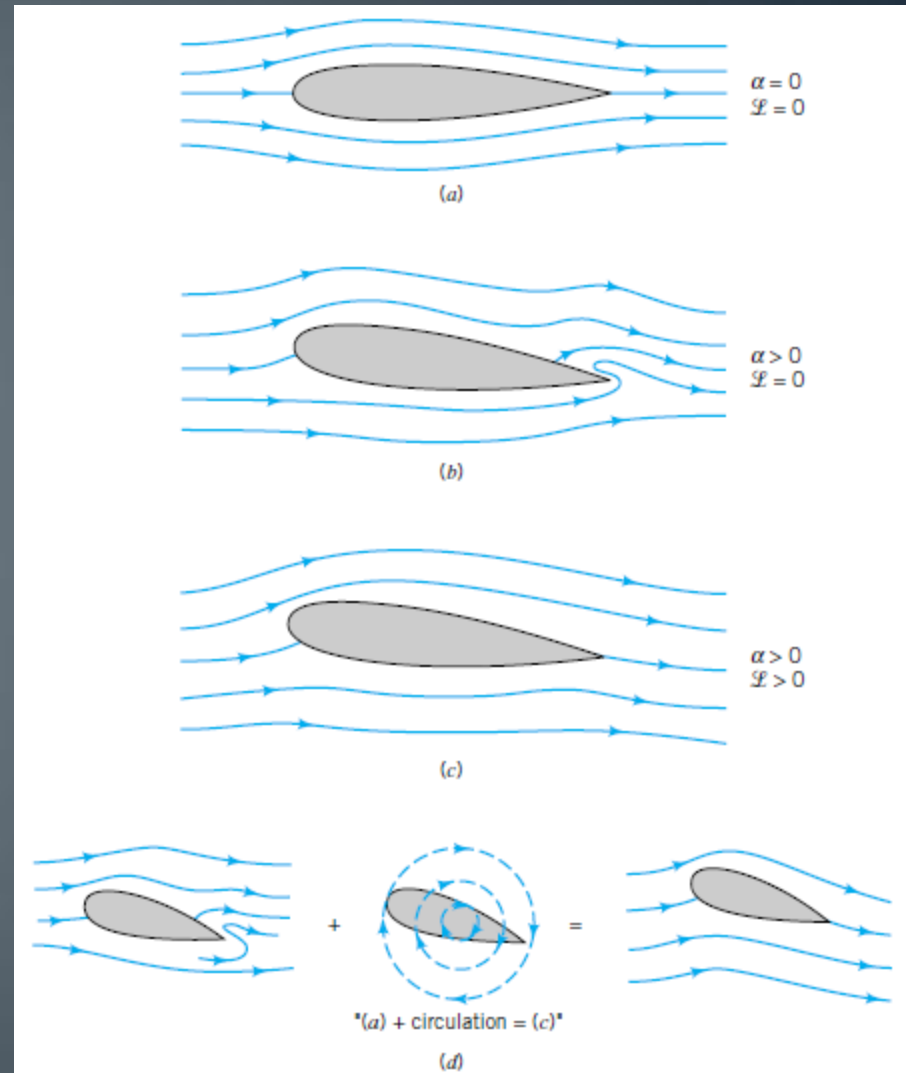
Lift

- Circulation
- Since viscous effects are of minor importance in the generation of lift, it should be possible to calculate the lift force on an airfoil by integrating the pressure distribution obtained from the equations governing **inviscid flow** past the airfoil.
- That is, the potential flow theory should provide a method to determine the lift.



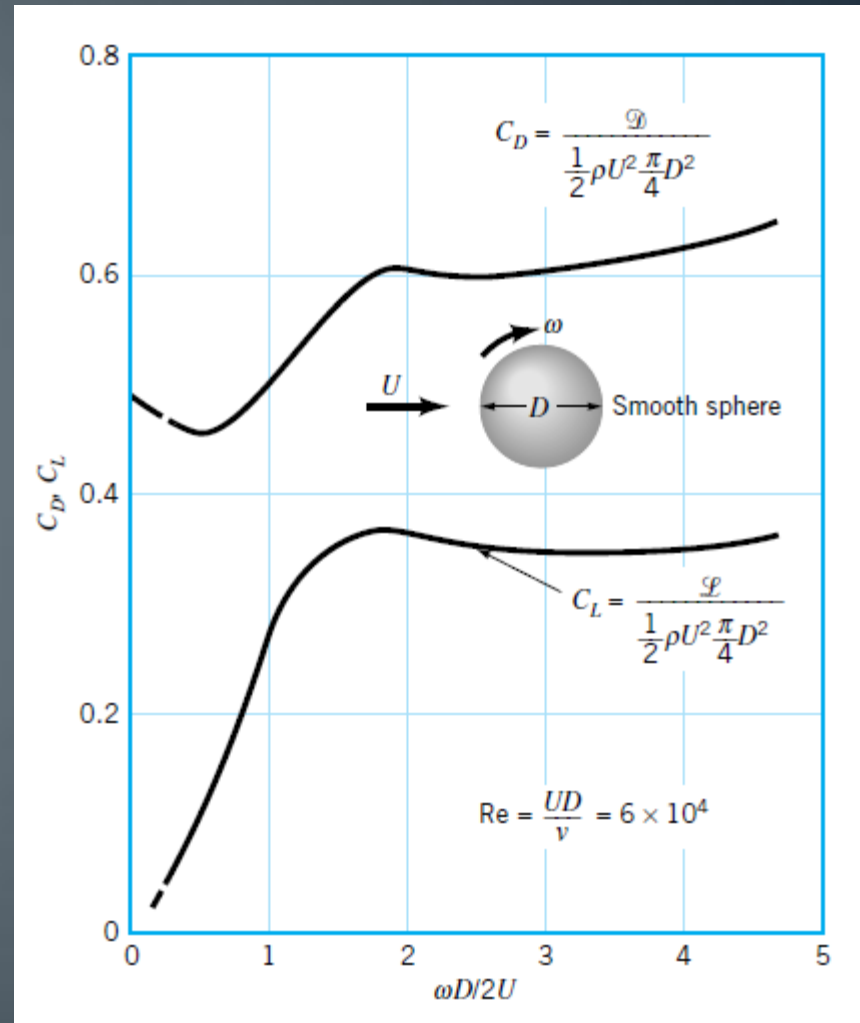
- Circulation

Lift



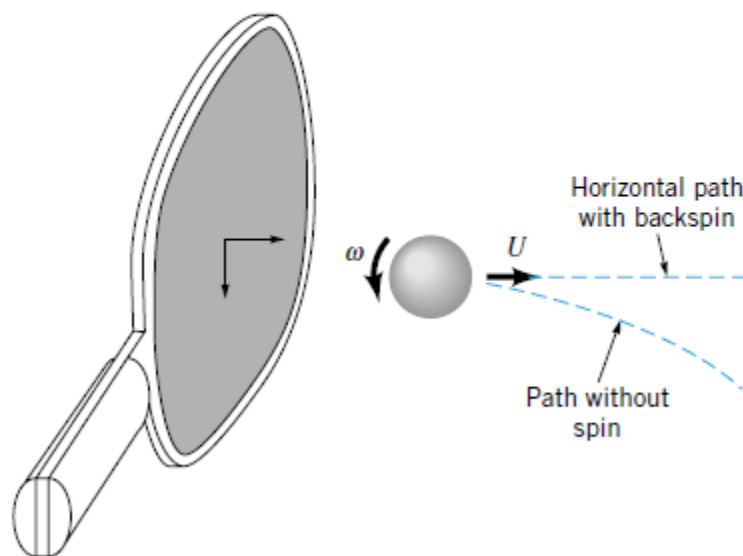
- Circulation

Lift



EXAMPLE 9.16

A table tennis ball weighing 2.45×10^{-2} N with diameter $D = 3.8 \times 10^{-2}$ m is hit at a velocity of $U = 12$ m/s with a back spin of angular velocity ω as is shown in Fig. E9.16. What is the value of ω if the ball is to travel on a horizontal path, not dropping due to the acceleration of gravity?



■ FIGURE E9.16

SOLUTION

For horizontal flight, the lift generated by the spinning of the ball must exactly balance the weight, $\mathcal{W}$, of the ball so that

$$\mathcal{W} = \mathcal{L} = \frac{1}{2}\rho U^2 A C_L$$

$$C_L = \frac{2\mathcal{W}}{\rho U^2 (\pi/4) D^2}$$

where the lift coefficient, C_L , can be obtained from Fig. 9.39. For standard atmospheric conditions with $\rho = 1.23 \text{ kg/m}^3$ we obtain

$$C_L = \frac{2(2.45 \times 10^{-2} \text{ N})}{(1.23 \text{ kg/m}^3)(12 \text{ m/s})^2 (\pi/4)(3.8 \times 10^{-2} \text{ m})^2} = 0.244$$

which, according to Fig. 9.39, can be achieved if

$$\frac{\omega D}{2U} = 0.9$$

or

$$\omega = \frac{2U(0.9)}{D} = \frac{2(12 \text{ m/s})(0.9)}{3.8 \times 10^{-2} \text{ m}} = 568 \text{ rad/s}$$

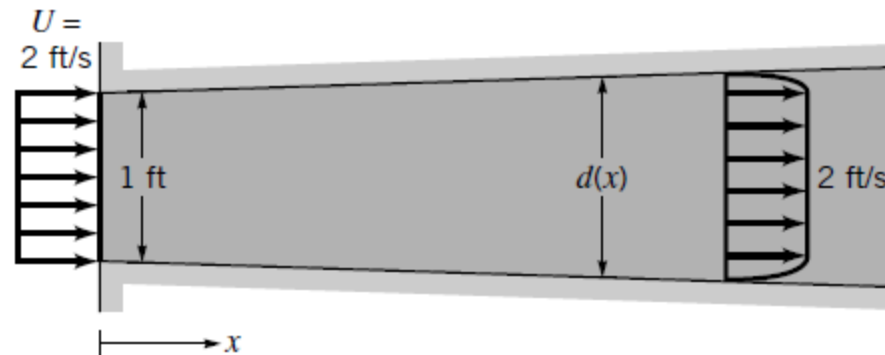
Thus,

$$\omega = (568 \text{ rad/s})(60 \text{ s/min})(1 \text{ rev}/2\pi \text{ rad}) = 5420 \text{ rpm} \quad (\text{Ans})$$

Is it possible to impart this angular velocity to the ball? With larger angular velocities the ball will rise and follow an upward curved path. Similar trajectories can be produced by a well-hit golf ball—rather than falling like a rock, the golf ball trajectory is actually curved up and the spinning ball travels a greater distance than one without spin. However, if top spin is imparted to the ball (as in an improper tee shot) the ball will curve downward more quickly than under the action of gravity alone—the ball is “topped” and a negative lift is generated. Similarly, rotation about a vertical axis will cause the ball to hook or slice to one side or the other.

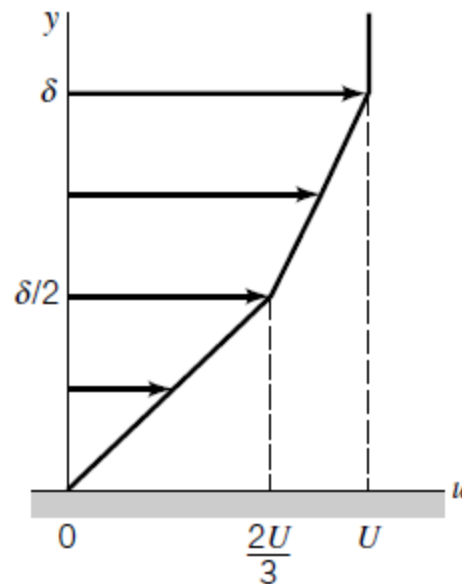
Problems

9.15 Air enters a square duct through a 1-ft opening as is shown in Fig. P9.15. Because the boundary layer displacement thickness increases in the direction of flow, it is necessary to increase the cross-sectional size of the duct if a constant $U = 2$ ft/s velocity is to be maintained outside the boundary layer. Plot a graph of the duct size, d , as a function of x for $0 \leq x \leq 10$ ft if U is to remain constant. Assume laminar flow.



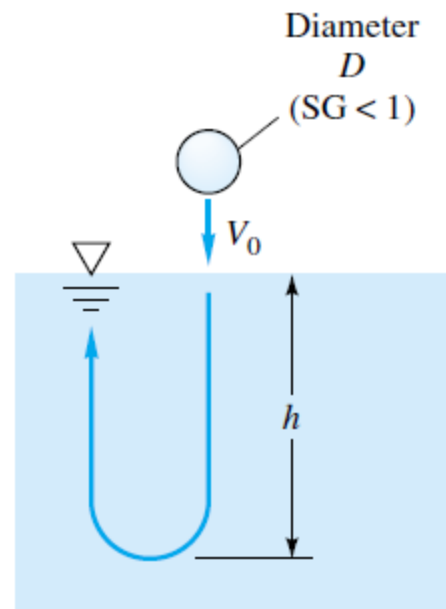
Problems

9.25 A laminar boundary layer velocity profile is approximated by the two straight-line segments indicated in Fig. P9.25. Use the momentum integral equation to determine the boundary layer thickness, $\delta = \delta(x)$, and wall shear stress, $\tau_w = \tau_w(x)$. Compare these results with those in Table 9.2.



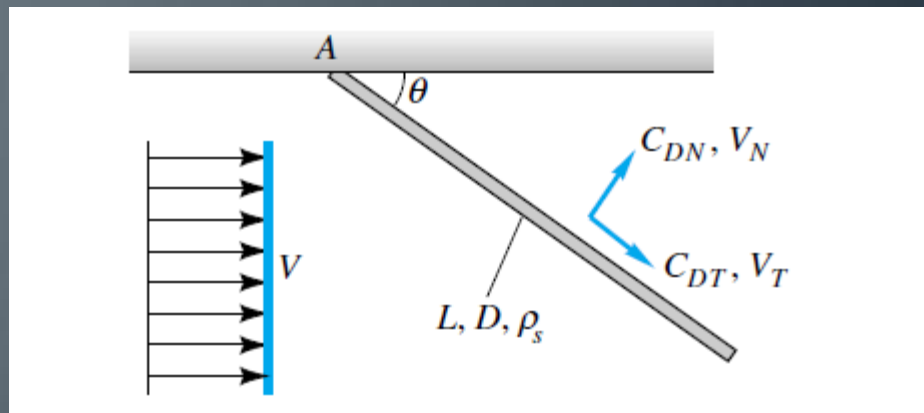
Problems

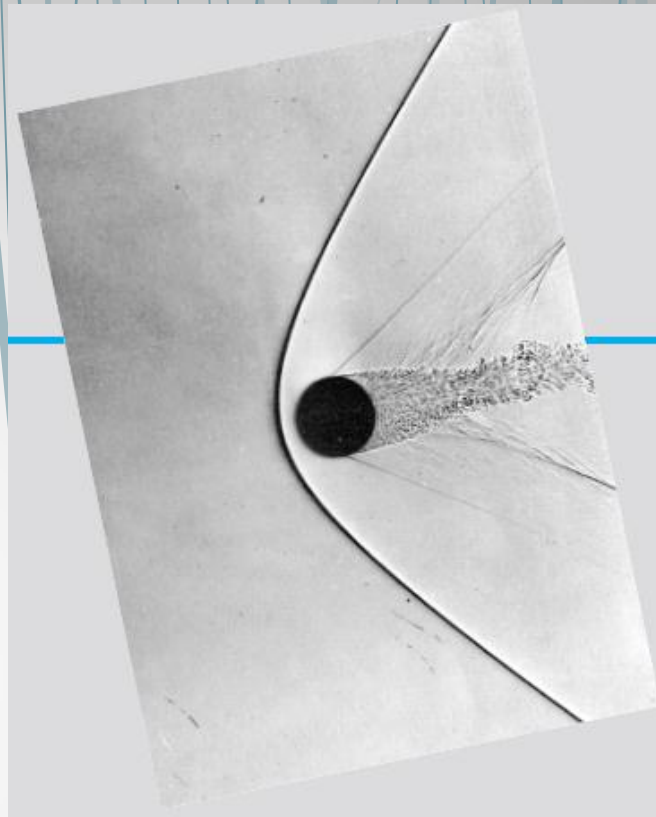
***P7.98** A buoyant ball of specific gravity $SG < 1$ dropped into water at inlet velocity V_0 will penetrate a distance h and then pop out again, as in Fig. P7.98. Make a dynamic analysis of this problem, assuming a constant drag coefficient, and derive an expression for h as a function of the system properties. How far will a 5-cm-diameter ball with $SG = 0.5$ and $C_D \approx 0.47$ penetrate if it enters at 10 m/s?



Problems

P7.104 Suppose that the body in Fig. P7.103 is a thin plate weighing 0.6 N, of length $L = 20$ cm and width $b = 4$ cm into the paper. For sea-level standard air, plot Pote's angle θ versus velocity V in the range $0 < V < 40$ m/s. Would this device make a good air-velocity meter (anemometer)?





Compressible Flow

Ideal Gas Relationships

$$p = \rho RT$$

$$\check{u}_2 - \check{u}_1 = c_v(T_2 - T_1)$$

$$\check{h}_2 - \check{h}_1 = c_p(T_2 - T_1)$$

$$c_p - c_v = R$$

$$c_p = \frac{Rk}{k - 1}$$

$$c_v = \frac{R}{k - 1}$$

Ideal Gas Relationships

$$s_2 - s_1 = c_v \ln \frac{T_2}{T_1} + R \ln \frac{\rho_1}{\rho_2}$$

$$s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}$$

$$\frac{p}{\rho^k} = \text{constant}$$

$$\left(\frac{T_2}{T_1}\right)^{k/(k-1)} = \left(\frac{\rho_2}{\rho_1}\right)^k = \left(\frac{p_2}{p_1}\right)$$

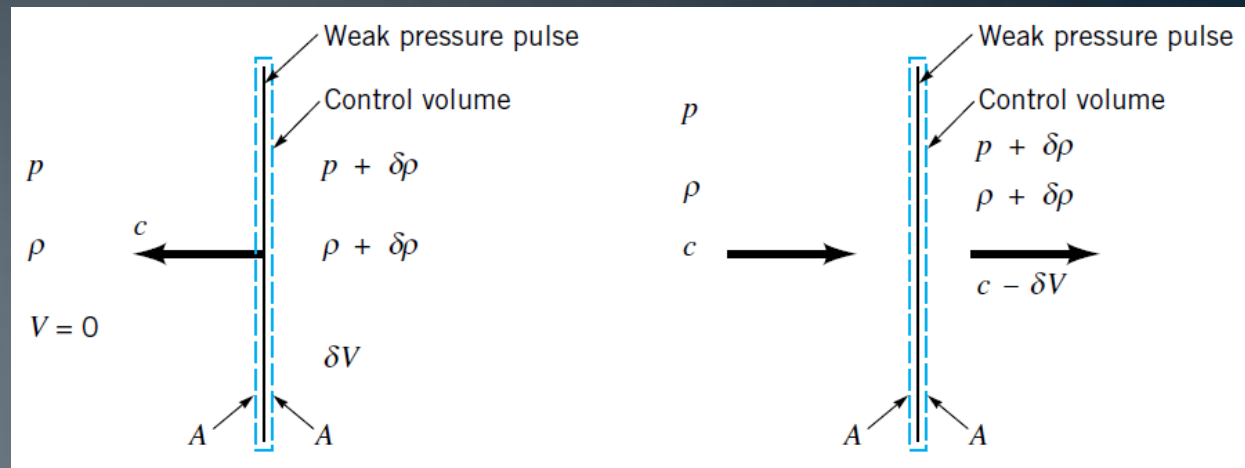
Mach Number and Speed of Sound

- The Mach number is defined as the ratio of the value of the local flow velocity, V , to the local speed of sound, c . In other words,

$$\text{Ma} = \frac{V}{c}$$

- What we perceive as sound generally consists of **weak pressure pulses** that move through air. When our ear drums respond to a succession of moving pressure pulses, we hear sounds.
- To better understand the notion of speed of sound, we analyze the one-dimensional fluid mechanics of an infinitesimally thin, weak pressure pulse moving at the speed of sound through a fluid at rest.

Mach Number and Speed of Sound



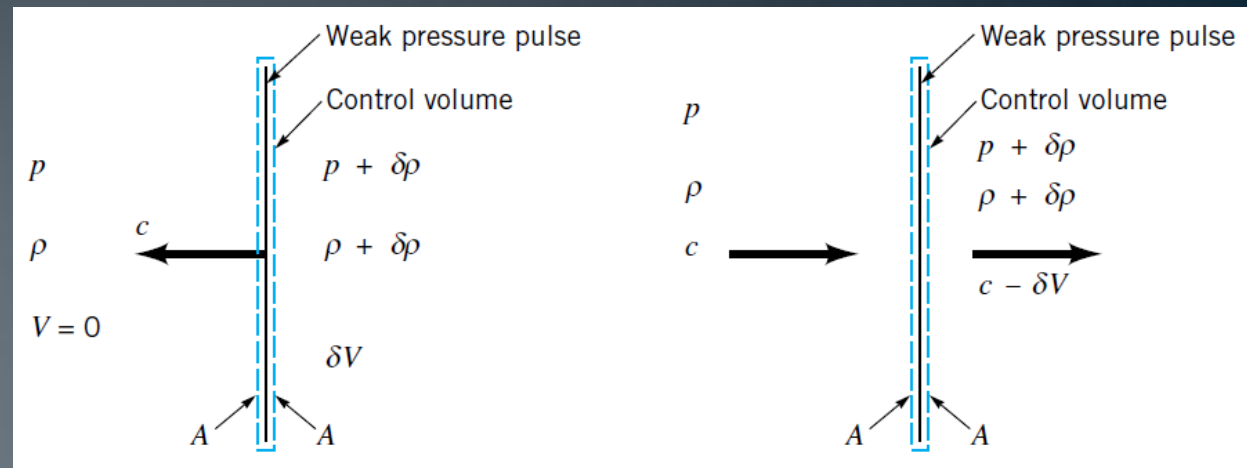
- When the continuity equation is applied to the flow through this control volume, the result is

$$\rho A c = (\rho + \delta \rho) A (c - \delta V)$$

$$\rho c = \rho c - \rho \delta V + c \delta \rho - (\delta \rho)(\delta V)$$

$$\rho \delta V = c \delta \rho$$

Mach Number and Speed of Sound



- The linear momentum equation can also be applied to the flow through the control volume. The result is

$$-c\rho cA + (c - \delta V)(\rho + \delta\rho)(c - \delta V)A = pA - (p + \delta p)A$$

$$-c\rho cA + (c - \delta V)\rho Ac = -\delta pA$$

$$\rho\delta V = \frac{\delta p}{c}$$

Mach Number and Speed of Sound

- From continuity and linear momentum we obtain

$$c^2 = \frac{\delta p}{\delta \rho}$$

$$c = \sqrt{\frac{\delta p}{\delta \rho}}$$

- If the energy equation is used for the flow through this control volume, the result is

$$\frac{\delta p}{\rho} + \delta \left(\frac{V^2}{2} \right) + g \delta z = \delta(\text{loss})$$

$$\frac{\delta p}{\rho} + \frac{(c - \delta V)^2}{2} - \frac{c^2}{2} = 0$$

Mach Number and Speed of Sound

$$\rho \delta V = \frac{\delta p}{c}$$

- By combining continuity and energy we again find that

$$c = \sqrt{\frac{\delta p}{\delta \rho}}$$

- Thus, the conservation of linear momentum and the conservation of energy principles lead to the same result. If we further assume that the **frictionless flow** through the control volume is **adiabatic** (no heat transfer), then the flow is **isentropic**.

$$c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}$$

Mach Number and Speed of Sound

- For the isentropic flow of an ideal gas, we have

$$p = (\text{constant})(\rho^k)$$

$$\left(\frac{\partial p}{\partial \rho}\right)_s = (\text{constant}) k \rho^{k-1} = \frac{p}{\rho^k} k \rho^{k-1} = \frac{p}{\rho} k = RTk$$

$$c = \sqrt{RTk}$$

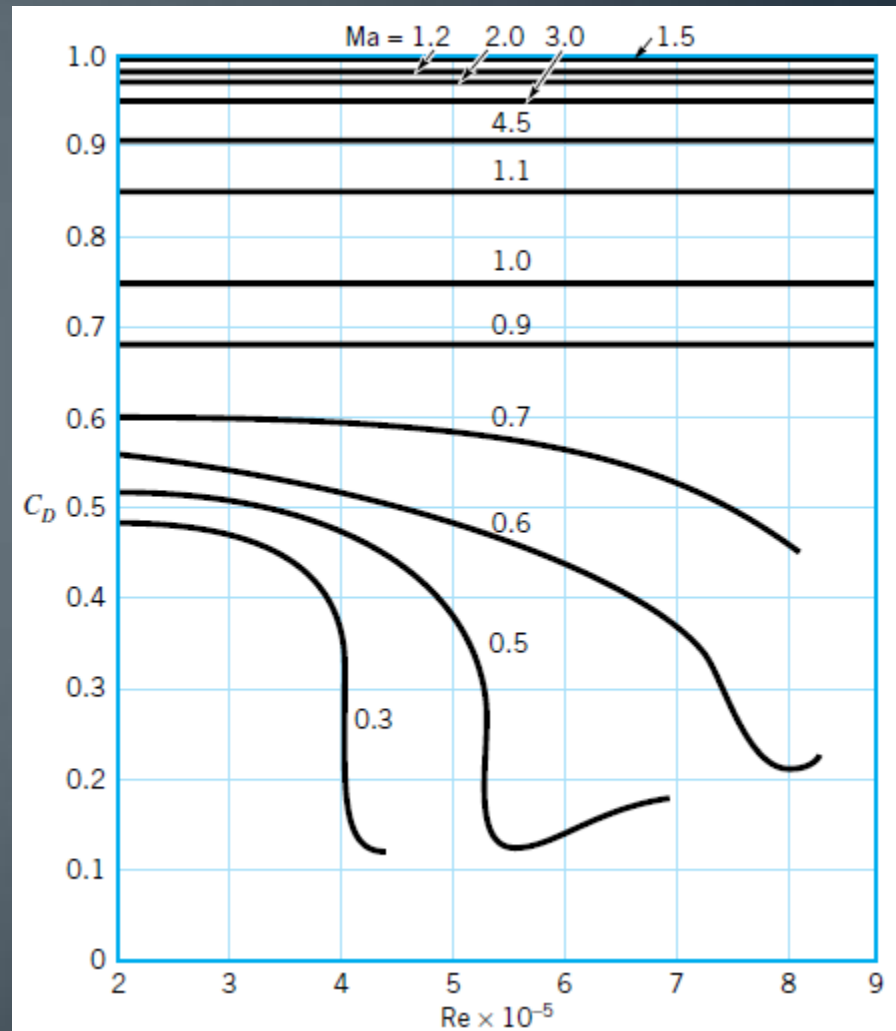
- More generally, the **bulk modulus of elasticity**, of any fluid including liquids is defined as

$$E_v = \frac{dp}{d\rho/\rho} = \rho \left(\frac{\partial p}{\partial \rho}\right)_s$$

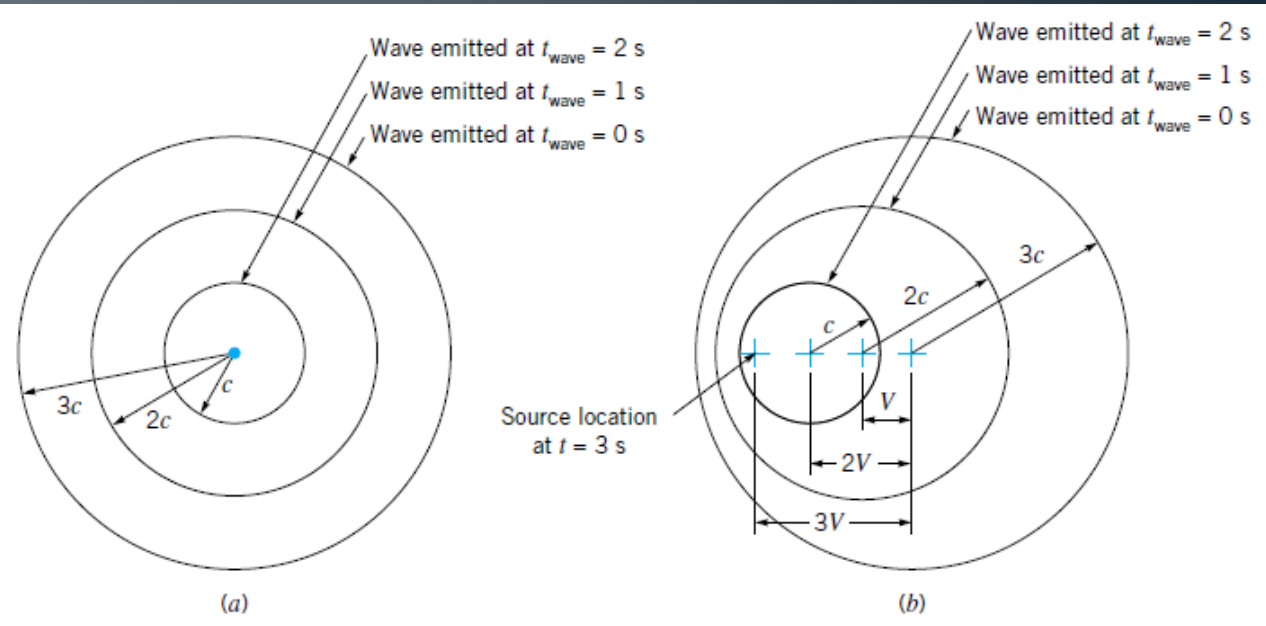
$$c = \sqrt{\frac{E_v}{\rho}}$$

Categories of Compressible Flow

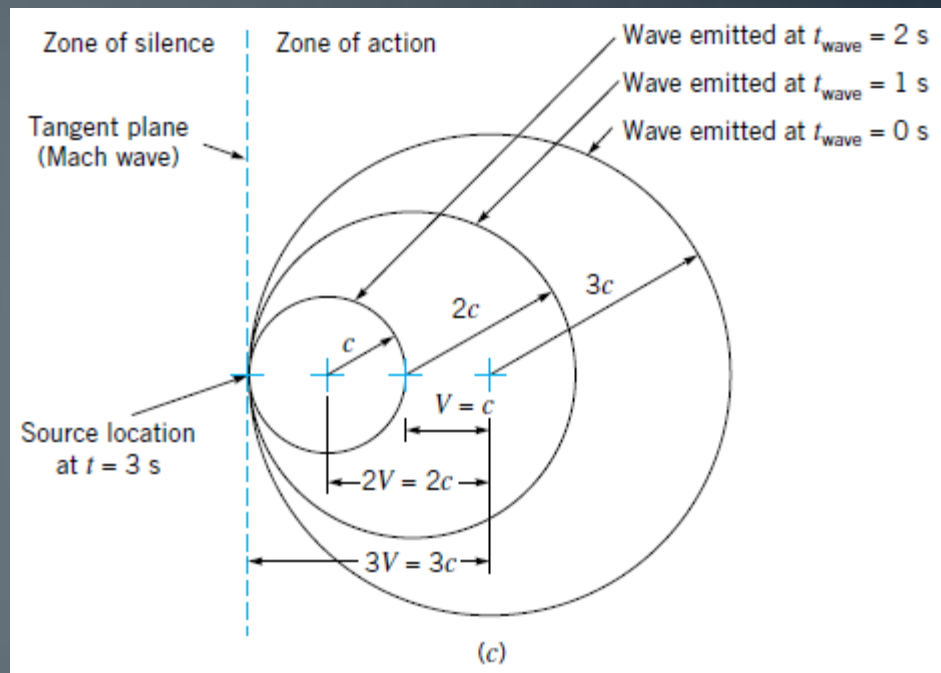
- The variation of drag coefficient with Reynolds number and Mach number is shown for air flow over a sphere. Compressibility effects can be of considerable importance.



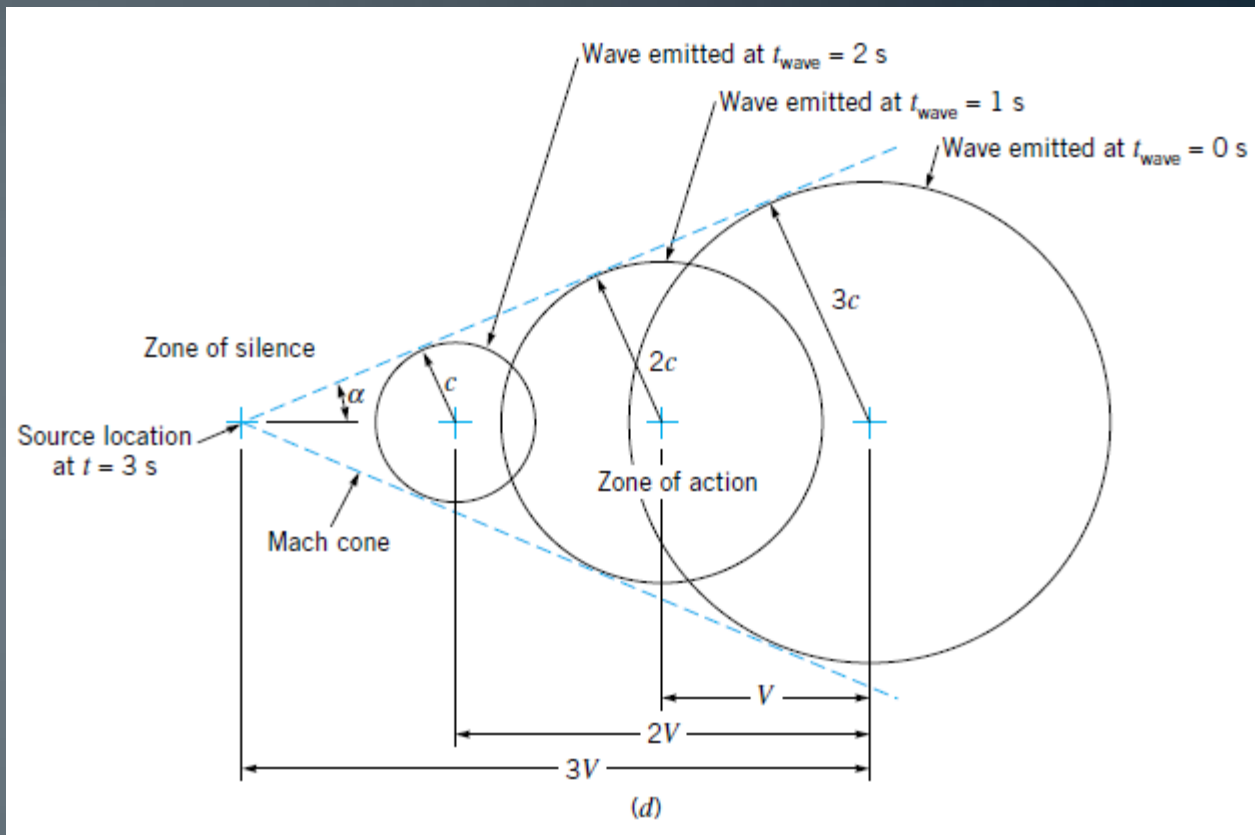
Categories of Compressible Flow



Categories of Compressible Flow



Categories of Compressible Flow



Categories of Compressible Flow

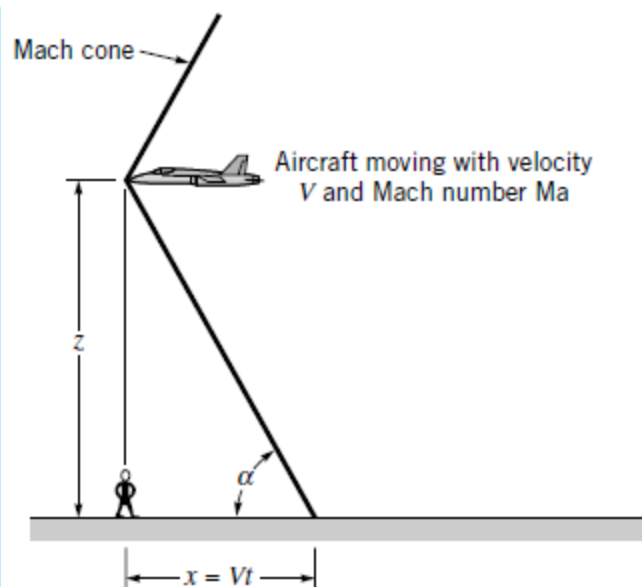
- This discussion about pressure wave patterns suggests the following categories of fluid flow:
- 1. **Incompressible flow**: $Ma \leq 0.3$. Unrestricted, nearly symmetrical and instantaneous pressure communication.
- 2. **Compressible subsonic flow**: $0.3 < Ma < 1$. Unrestricted but noticeably asymmetrical pressure communication.
- 3. **Compressible supersonic flow**: $Ma \geq 1$. Formation of Mach wave; pressure communication restricted to zone of action.
- In addition to the above-mentioned categories of flows, two other regimes are commonly referred to: namely, transonic flows ($0.9 \leq Ma \leq 1.2$) and hypersonic flows ($Ma \geq 5$).

EXAMPLE 11.4

An aircraft cruising at 1000-m elevation, z , above you moves past in a flyby. How many seconds after the plane passes overhead do you expect to wait before you hear the aircraft if it is moving with a Mach number equal to 1.5 and the ambient temperature is 20 °C?

SOLUTION

Since the aircraft is moving supersonically ($Ma > 1$), we can imagine a Mach cone originating from the forward tip of the craft as is illustrated in Fig. E11.4. When the surface of



■ FIGURE E11.4

the cone reaches the observer, the “sound” of the aircraft is perceived. The angle α in Fig. E11.4 is related to the elevation of the plane, z , and the ground distance, x , by

$$\alpha = \tan^{-1} \frac{z}{x} = \tan^{-1} \frac{1000}{Vt} \quad (1)$$

Also, assuming negligible change of Mach number with elevation, we can use Eq. 11.39 to relate Mach number to the angle α . Thus,

$$\text{Ma} = \frac{1}{\sin \alpha} \quad (2)$$

Combining Eqs. 1 and 2 we obtain

$$\text{Ma} = \frac{1}{\sin [\tan^{-1} (1000/Vt)]} \quad (3)$$

The speed of the aircraft can be related to the Mach number with

$$V = (\text{Ma})c \quad (4)$$

where c is the speed of sound. From [Table B.4](#), $c = 343.3 \text{ m/s}$. Using $\text{Ma} = 1.5$, we get from Eqs. 3 and 4

$$1.5 = \frac{1}{\sin \left\{ \tan^{-1} \left[\frac{1000 \text{ m}}{(1.5)(343.3 \text{ m/s})t} \right] \right\}}$$

or

$$t = 2.17 \text{ s} \quad (\text{Ans})$$

Isentropic Flow of an Ideal Gas

- Effect of Variations in Flow Cross-Sectional Area

$$\dot{m} = \rho AV = \text{constant}$$

- For the inviscid and steady flow of a fluid particle.

$$dp + \frac{1}{2}\rho d(V^2) + \gamma dz = 0$$

$$\frac{dp}{\rho V^2} = -\frac{dV}{V}$$

- If we form the logarithm of both sides of the continuity equation, the result is

$$\ln \rho + \ln A + \ln V = \text{constant}$$

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0$$

- Effect of Variations in Flow Cross-Sectional Area

Isentropic Flow of an Ideal Gas

$$-\frac{dV}{V} = \frac{d\rho}{\rho} + \frac{dA}{A}$$

$$\frac{dp}{\rho V^2} \left(1 - \frac{V^2}{dp/d\rho} \right) = \frac{dA}{A}$$

$$c = \sqrt{\left(\frac{\partial p}{\partial \rho} \right)_s}$$

$$\text{Ma} = \frac{V}{c}$$

$$\frac{dp}{\rho V^2} (1 - \text{Ma}^2) = \frac{dA}{A}$$

Isentropic Flow of an Ideal Gas

- Effect of Variations in Flow Cross-Sectional Area

$$\frac{dV}{V} = -\frac{dA}{A} \frac{1}{(1 - \text{Ma}^2)}$$

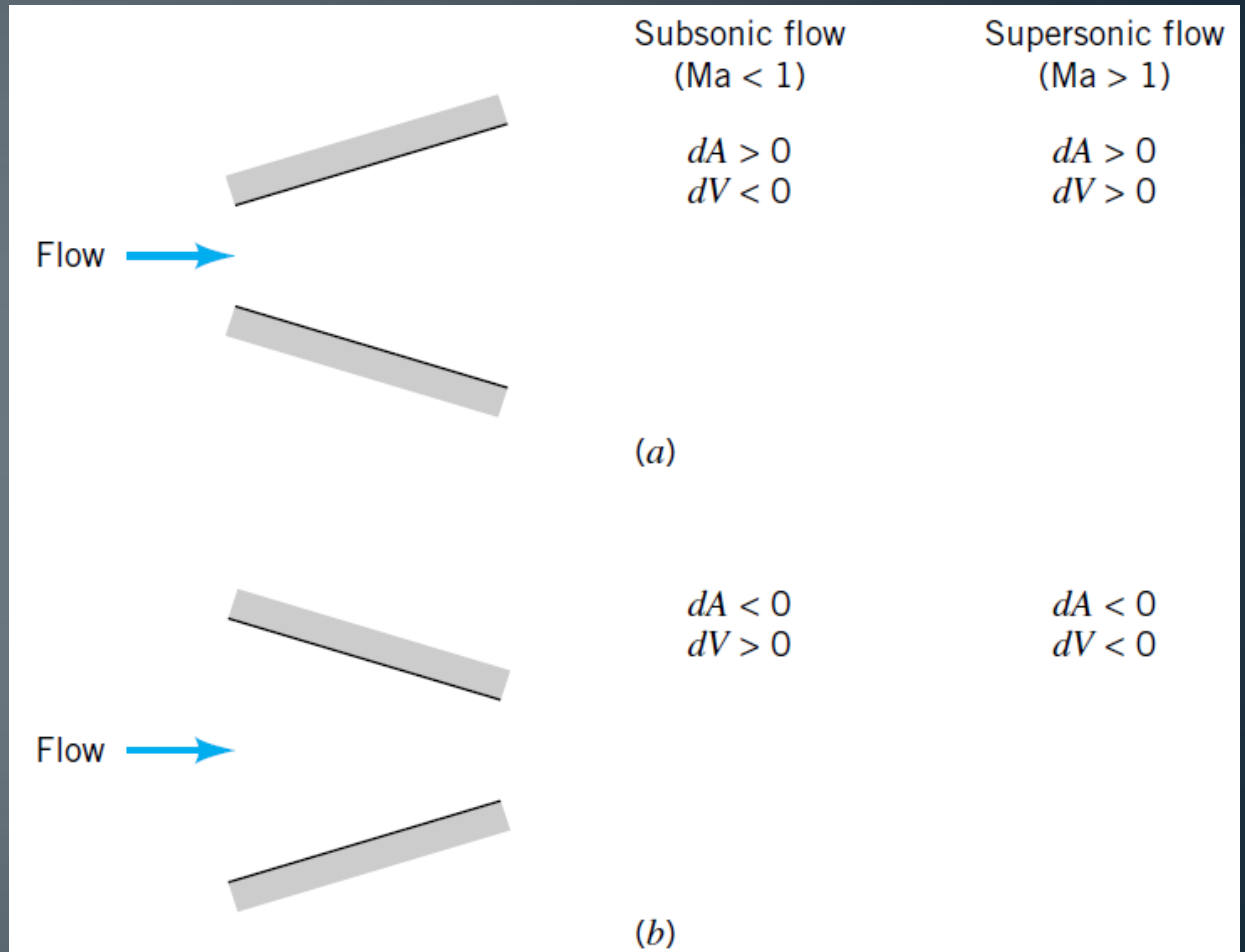
- We can use the above to conclude that when the flow is **subsonic** velocity and section area changes are in **opposite** directions, and when the flow is **supersonic** velocity and area changes are in the **same** direction.

$$\frac{d\rho}{\rho} = \frac{dA}{A} \frac{\text{Ma}^2}{(1 - \text{Ma}^2)}$$

- We can conclude that for **subsonic** flows density and area changes are in the **same** direction, whereas for **supersonic** flows density and area changes are in **opposite** directions.

- Effect of Variations in Flow Cross-Sectional Area

Isentropic Flow of an Ideal Gas



Isentropic Flow of an Ideal Gas

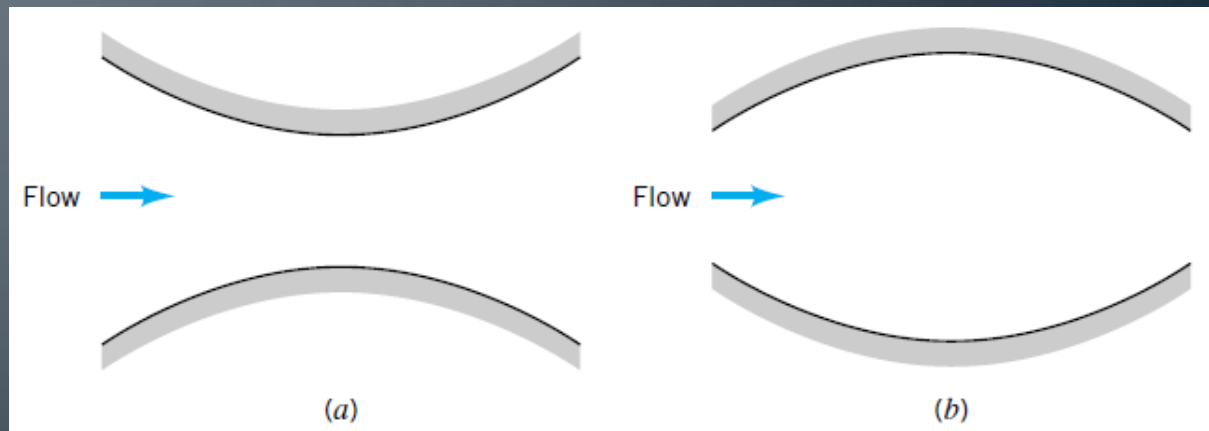
- Effect of Variations in Flow Cross-Sectional Area

$$\frac{dV}{V} = -\frac{dA}{A} \frac{1}{(1 - \text{Ma}^2)}$$

- By rearranging the above equation, we can obtain

$$\frac{dA}{dV} = -\frac{A}{V} (1 - \text{Ma}^2)$$

- This result suggests that the area associated with $\text{Ma} = 1$ is either a minimum or a maximum amount.



Converging-Diverging Duct Flow

- It is convenient to use the **stagnation state** of the fluid as a reference state for compressible flow calculations.
- The stagnation state is associated with **zero flow velocity** and an entropy value that corresponds to the **entropy of the flowing fluid**.
- The stagnation state can also be achieved by **isentropically decelerating** a flow to zero velocity.

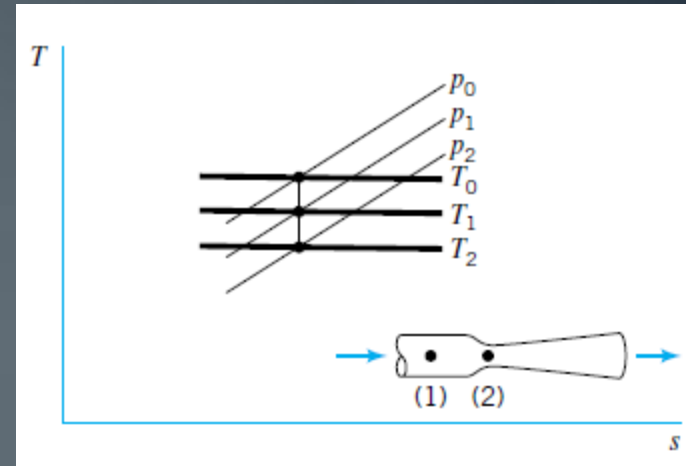
$$\frac{T}{T_0} = \frac{1}{1 + [(k - 1)/2]Ma^2}$$

$$\frac{p}{p_0} = \left\{ \frac{1}{1 + [(k - 1)/2]Ma^2} \right\}^{k/(k-1)}$$

$$\frac{\rho}{\rho_0} = \left\{ \frac{1}{1 + [(k - 1)/2]Ma^2} \right\}^{1/(k-1)}$$

Converging-Diverging Duct Flow

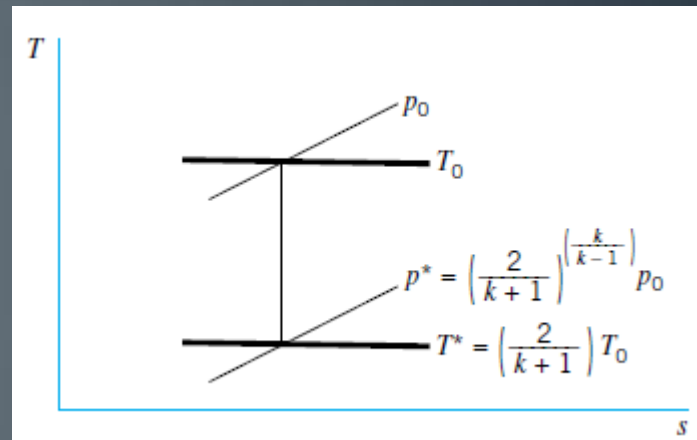
- A very useful means of keeping track of the states of an isentropic flow of an ideal gas involves a temperature-entropy diagram, as is shown below



- When the duct back pressure is lowered sufficiently, the Mach number at the throat of the duct will be 1.
- Any further decrease of the back pressure will not affect the flow in the converging portion of the duct because **information** about pressure cannot move upstream when $Ma=1$.
- When at the throat of the converging-diverging duct, we have a condition called **choked flow**.

Converging-Diverging Duct Flow

- We have already used the **stagnation state** for which $Ma=0$ as a reference condition.
- It will prove helpful to us to use the state associated with $Ma=1$ and the same entropy level as the flowing fluid as another reference condition we shall call the **critical state**.



$$\frac{p^*}{p_0} = \left(\frac{2}{k+1} \right)^{k/(k-1)}$$

$$\frac{T^*}{T_0} = \frac{2}{k+1}$$

Converging-Diverging Duct Flow

- For choked flow through the converging-diverging duct, the conservation of mass equation yields

$$\rho AV = \rho^* A^* V^*$$

$$\frac{A}{A^*} = \left(\frac{\rho^*}{\rho} \right) \left(\frac{V^*}{V} \right)$$

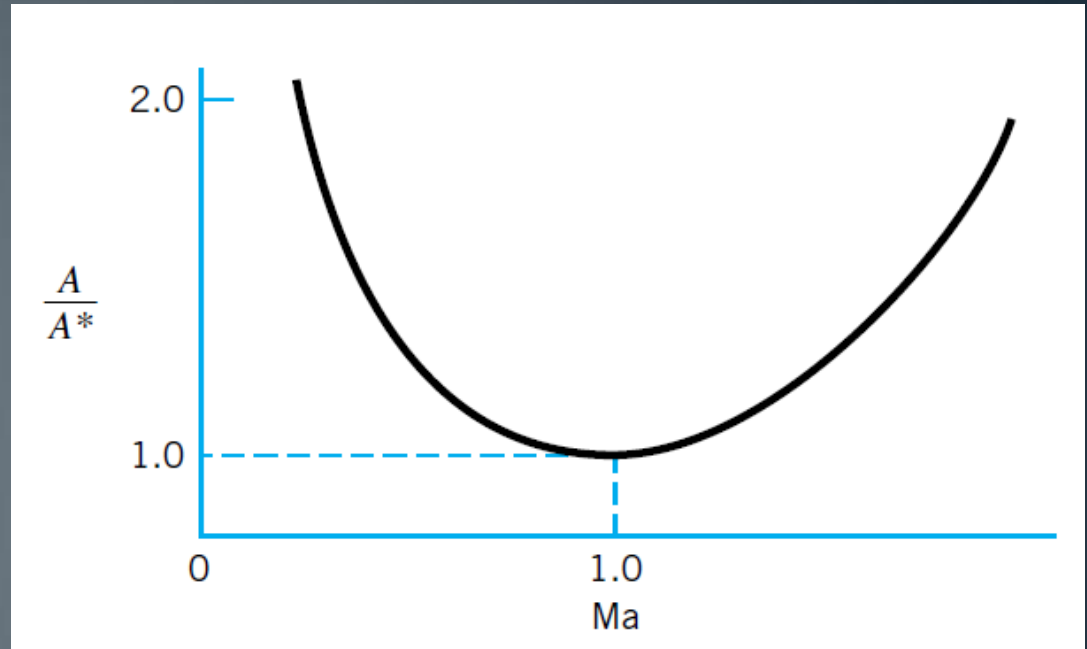
$$V^* = \sqrt{RT^*k}$$

$$V = \text{Ma} \sqrt{RTk}$$

$$\frac{A}{A^*} = \frac{1}{\text{Ma}} \left(\frac{\rho^*}{\rho_0} \right) \left(\frac{\rho_0}{\rho} \right) \sqrt{\frac{(T^*/T_0)}{(T/T_0)}}$$

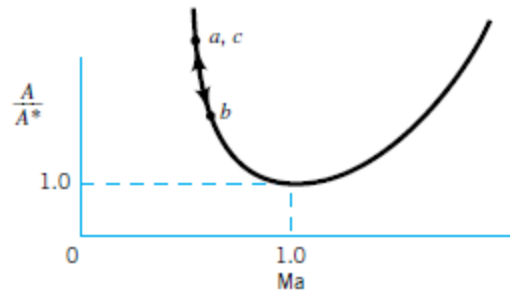
$$\frac{A}{A^*} = \frac{1}{\text{Ma}} \left\{ \frac{1 + [(k-1)/2]\text{Ma}^2}{1 + [(k-1)/2]} \right\}^{(k+1)/[2(k-1)]}$$

Converging-Diverging Duct Flow

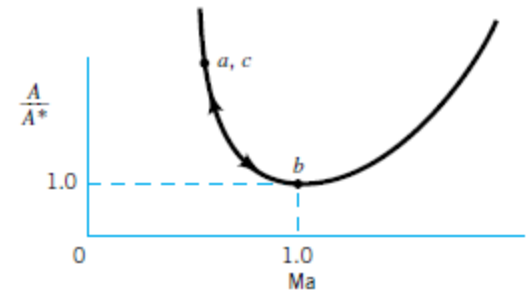


$$\frac{A}{A^*} = \frac{1}{Ma} \left\{ \frac{1 + [(k-1)/2]Ma^2}{1 + [(k-1)/2]} \right\}^{(k+1)/[2(k-1)]}$$

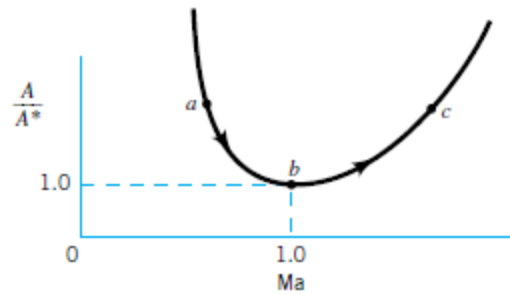
Converging-Diverging Duct Flow



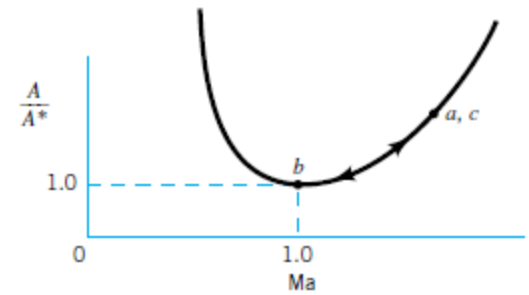
(a)



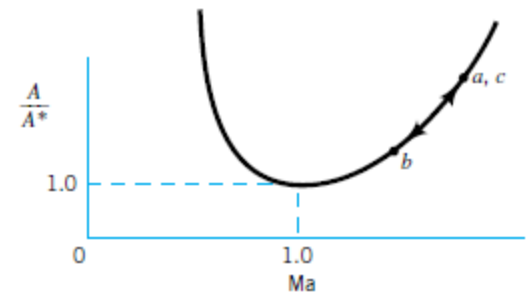
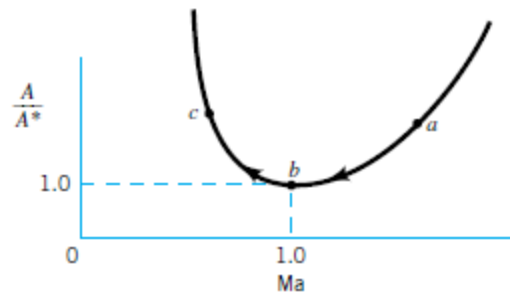
(b)



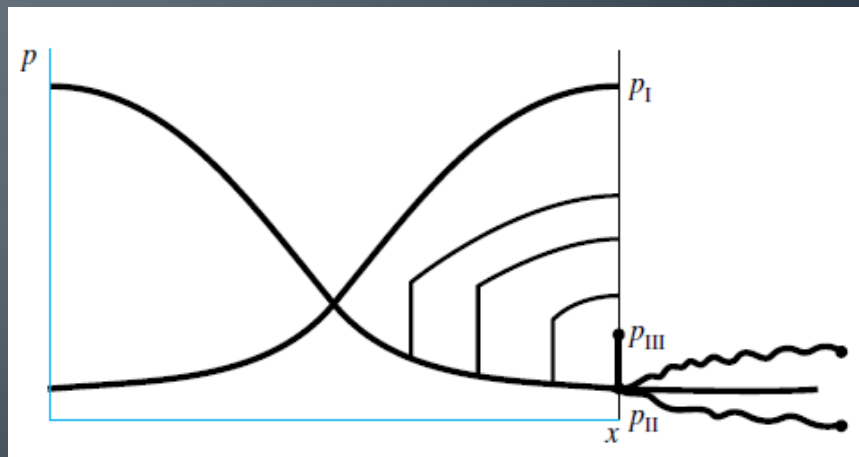
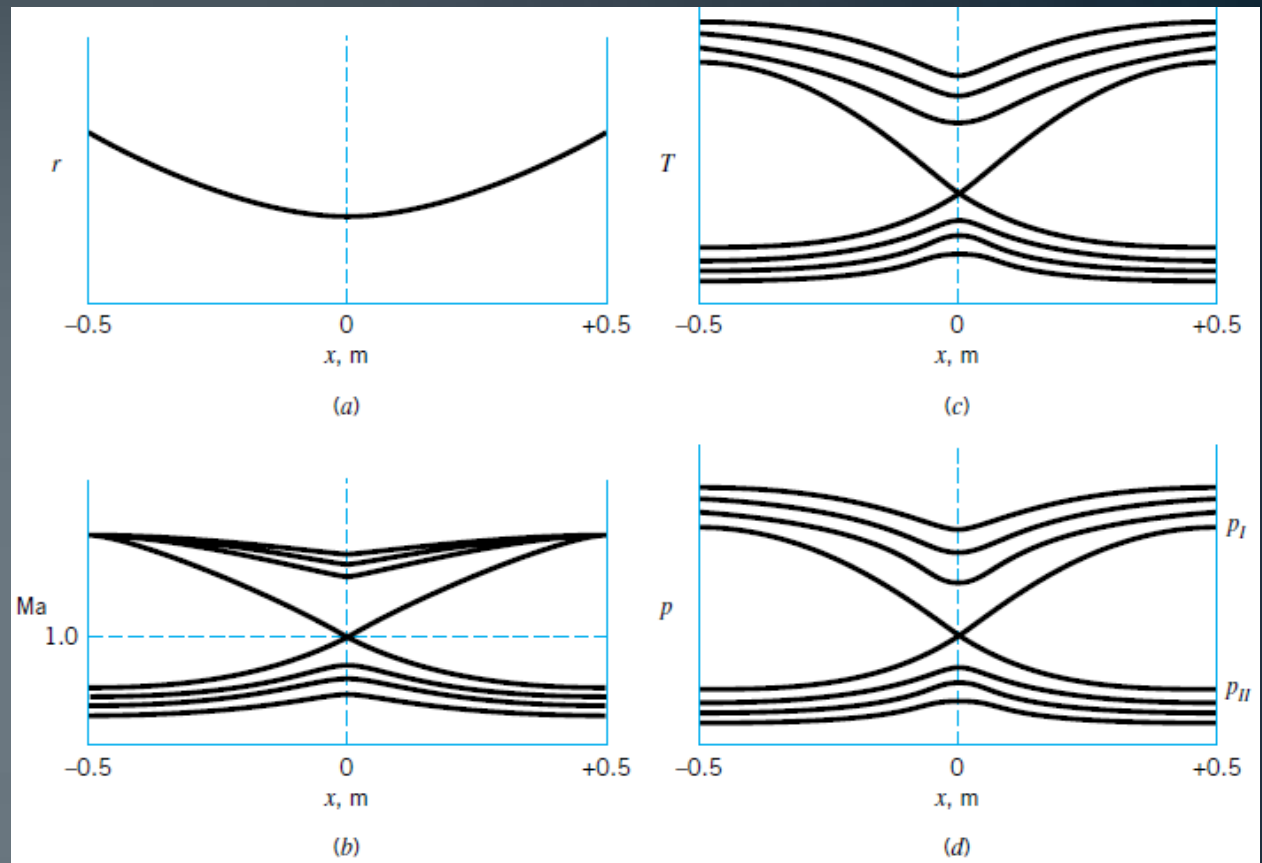
(c)



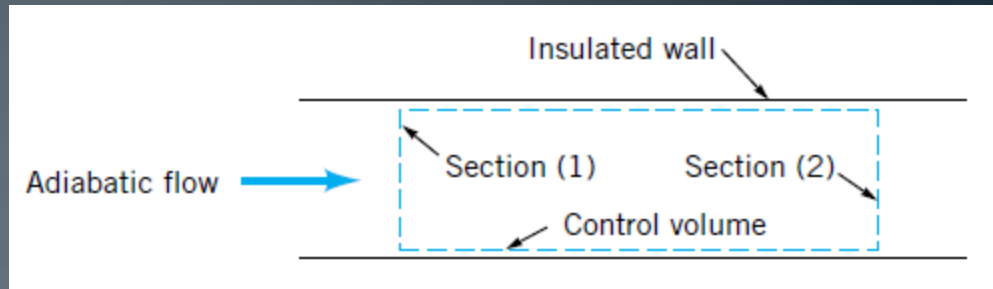
(d)



Converging-Diverging Duct Flow



- Adiabatic Constant-Area Duct Flow with Friction (Fanno Flow)



Nonisentropic Flow of an Ideal Gas

$$\dot{m} \left[\check{h}_2 - \check{h}_1 + \frac{V_2^2 - V_1^2}{2} + g(z_2 - z_1) \right] = \cancel{\dot{Q}_{\text{net in}}} + \cancel{\dot{W}_{\text{shaft net in}}}$$

Annotations for the equation above:

- 0 (negligibly small for gas flow) points to $g(z_2 - z_1)$
- 0 (flow is adiabatic) points to $\cancel{\dot{Q}_{\text{net in}}}$
- 0 (flow is steady throughout) points to $\cancel{\dot{W}_{\text{shaft net in}}}$

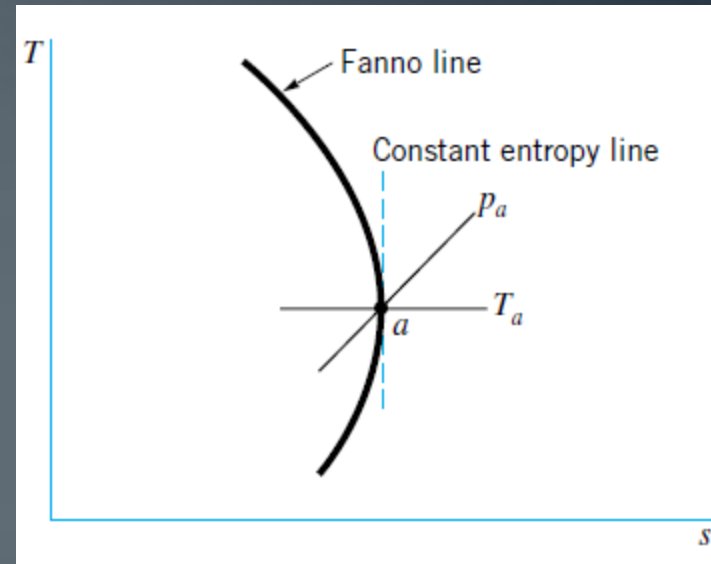
$$\check{h} + \frac{V^2}{2} = \check{h}_0 = \text{constant}$$

$$\check{h} - \check{h}_0 = c_p(T - T_0)$$

$$T + \frac{V^2}{2c_p} = T_0 = \text{constant}$$

- Adiabatic Constant-Area Duct Flow with Friction (Fanno Flow)

Nonisentropic Flow of an Ideal Gas

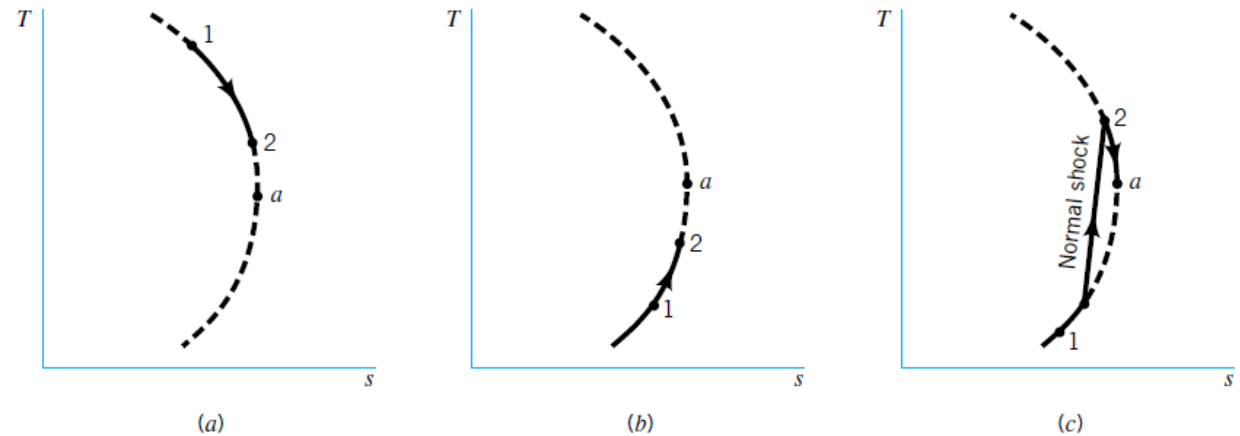


$$T + \frac{(\rho V)^2 T^2}{2c_p(p^2/R^2)} = T_0 = \text{constant}$$

$$s - s_1 = c_p \ln \frac{T}{T_1} - R \ln \frac{p}{p_1}$$

Nonisentropic Flow of an Ideal Gas

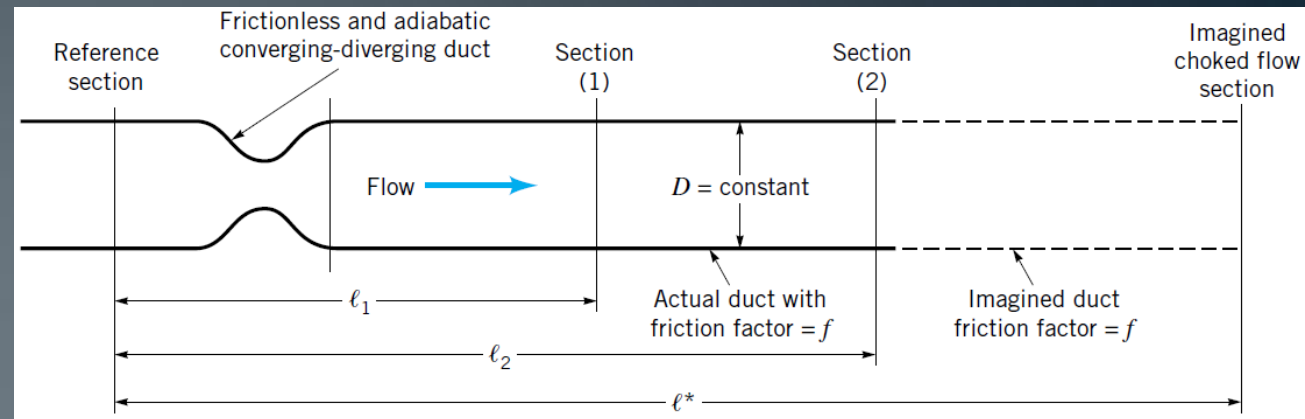
- Adiabatic Constant-Area Duct Flow with Friction (Fanno Flow)



Summary of Fanno Flow Behavior

Parameter	Flow	
	Subsonic Flow	Supersonic Flow
Stagnation temperature	Constant	Constant
Ma	Increases (maximum is 1)	Decreases (minimum is 1)
Friction	Accelerates flow	Decelerates flow
Pressure	Decreases	Increases
Temperature	Decreases	Increases

- Adiabatic Constant-Area Duct Flow with Friction (Fanno Flow)



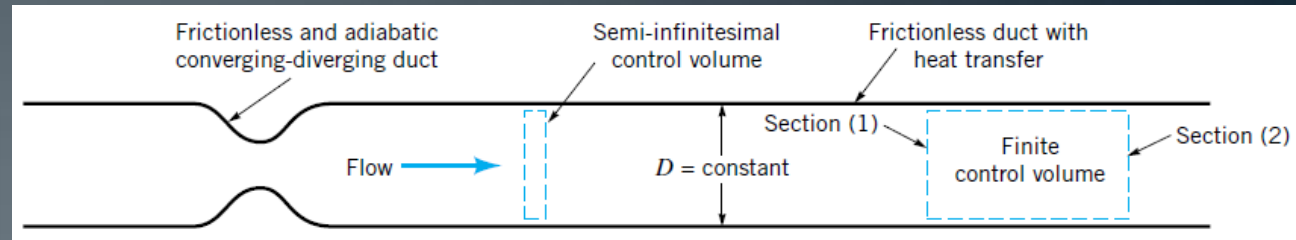
Nonisentropic Flow of an Ideal Gas

$$\frac{T}{T^*} = \frac{(k + 1)/2}{1 + [(k - 1)/2] \text{Ma}^2}$$

$$\frac{p}{p^*} = \frac{1}{\text{Ma}} \left\{ \frac{(k + 1)/2}{1 + [(k - 1)/2] \text{Ma}^2} \right\}^{1/2}$$

Nonisentropic Flow of an Ideal Gas

- Frictionless Constant-Area Duct Flow with Heat Transfer (Rayleigh Flow)



$$p_1 A_1 + \dot{m} V_1 = p_2 A_2 + \dot{m} V_2 + \overset{0(\text{frictionless flow})}{R_x}$$

$$p + \frac{(\rho V)^2}{\rho} = \text{constant}$$

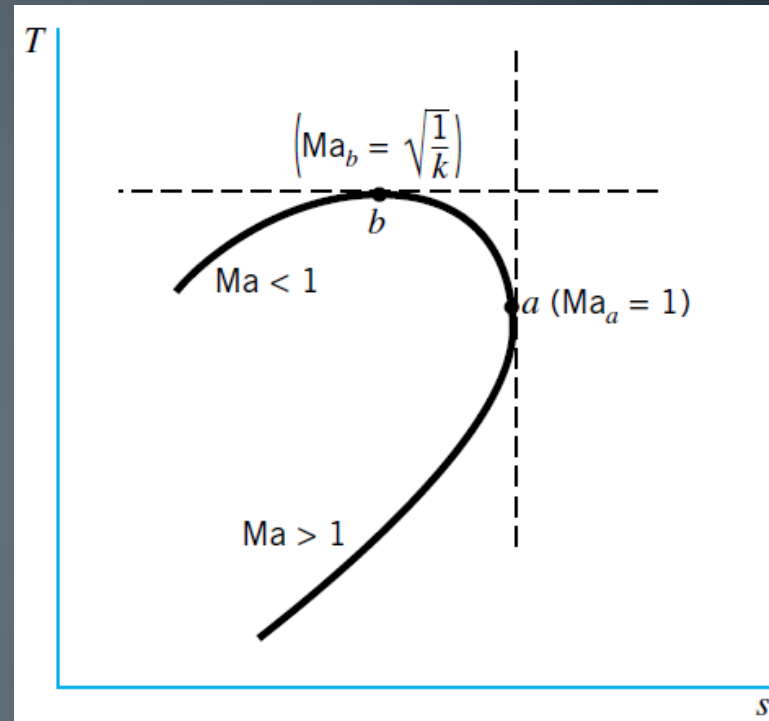
$$p + \frac{(\rho V)^2 RT}{p} = \text{constant}$$

- Since the flow cross-sectional area remains constant for Rayleigh flow, from the continuity equation we conclude that

$$\rho V = \text{constant}$$

Nonisentropic Flow of an Ideal Gas

- Frictionless Constant-Area Duct Flow with Heat Transfer (Rayleigh Flow)

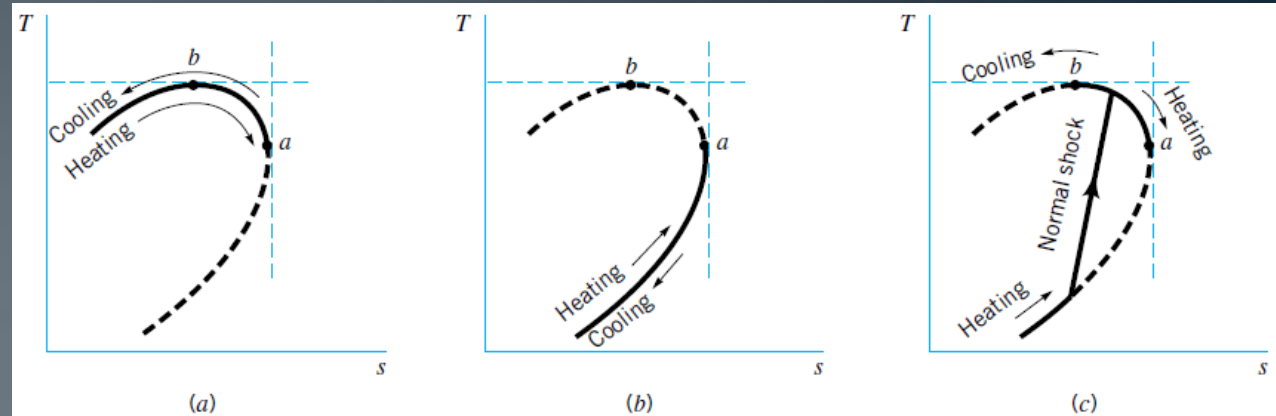


$$p + \frac{(\rho V)^2 RT}{p} = \text{constant}$$

$$s - s_1 = c_p \ln \frac{T}{T_1} - R \ln \frac{p}{p_1}$$

- Frictionless Constant-Area Duct Flow with Heat Transfer (Rayleigh Flow)

Nonisentropic Flow of an Ideal Gas



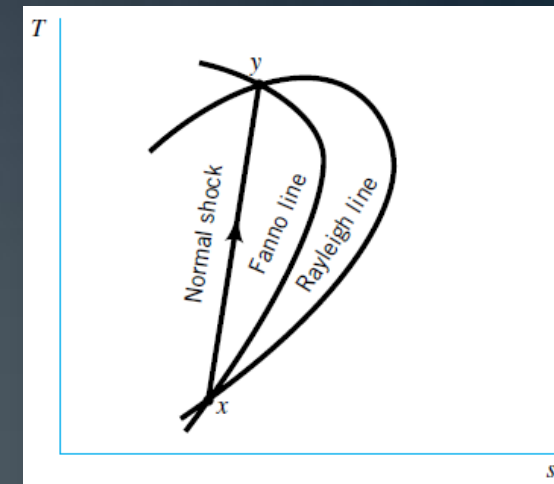
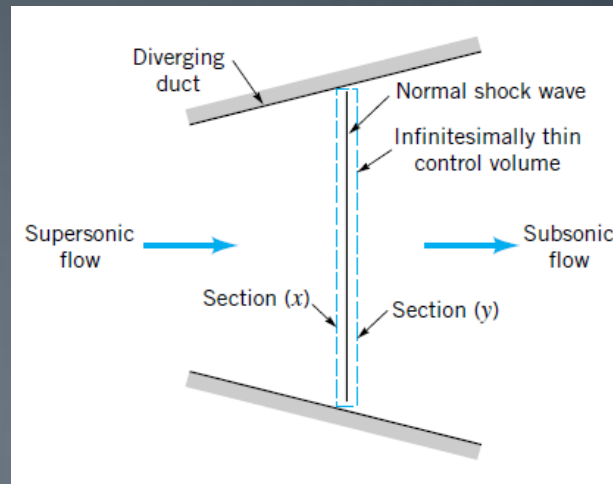
		Heating		Cooling	
		Subsonic	Supersonic	Subsonic	Supersonic
V	Increase	Increase	Decrease	Decrease	Increase
Ma	Increase	Increase	Decrease	Decrease	Increase
T	Increase for $0 \leq Ma \leq \sqrt{1/k} = 0.845$ Decrease for $\sqrt{1/k} \leq Ma \leq 1$	Increase	Increase	Decrease for $0 \leq Ma \leq \sqrt{1/k} = 0.845$ Increase for $\sqrt{1/k} \leq Ma \leq 1$	Decrease
T_0	Increase	Increase	Increase	Decrease	Decrease
p	Decrease	Increase	Increase	Increase	Decrease
p_0	Decrease	Decrease	Decrease	Increase	Increase

$$\frac{p}{p_a} = \frac{1 + k}{1 + kMa^2}$$

$$\frac{T}{T_a} = \left[\frac{(1 + k)Ma}{1 + kMa^2} \right]^2$$

Nonisentropic Flow of an Ideal Gas

- Normal Shock Waves



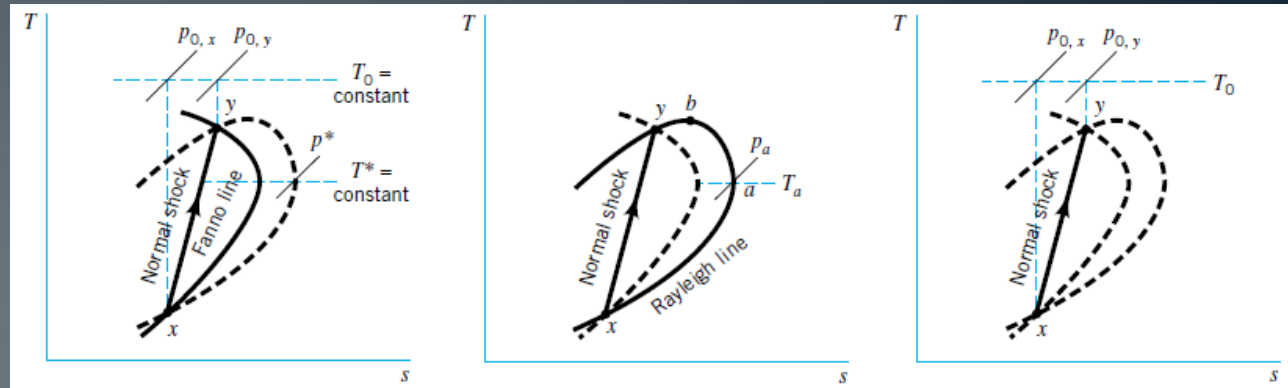
$$\rho V = \text{constant}$$

$$p + \frac{(\rho V)^2 R T}{p} = \text{constant}$$

$$T + \frac{(\rho V)^2 T^2}{2c_p(p^2/R^2)} = T_0 = \text{constant}$$

Nonisentropic Flow of an Ideal Gas

- Normal Shock Waves



Summary of Normal Shock Wave Characteristics

Variable	Change Across Normal Shock Wave
Mach number	Decrease
Static pressure	Increase
Stagnation pressure	Decrease
Static temperature	Increase
Stagnation temperature	Constant
Density	Increase
Velocity	Decrease

$$\frac{p_y}{p_x} = \frac{1 + k\text{Ma}_x^2}{1 + k\text{Ma}_y^2}$$

$$\frac{T_y}{T_x} = \frac{1 + [(k - 1)/2]\text{Ma}_x^2}{1 + [(k - 1)/2]\text{Ma}_y^2}$$