

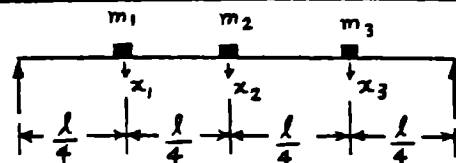
Chapter 7

Determination of Natural Frequencies and Mode Shapes

7.1 From Example 6.6,

$$\alpha_{11} = \alpha_{33} = \frac{3}{256} \frac{l^3}{EI}$$

$$\alpha_{22} = \frac{1}{48} \frac{l^3}{EI}$$



(a) Eq. (7.6) gives

$$\frac{1}{\omega_1^2} \approx \frac{ml^3}{EI} \left(\frac{3 \times 5}{256} + \frac{1}{48} \times 1 + \frac{3 \times 5}{256} \right) = \frac{106}{768} \frac{ml^3}{EI} = 0.13802 \frac{ml^3}{EI}$$

$$\omega_1 \approx 2.6917 \sqrt{\frac{EI}{ml^3}}$$

(b) $\frac{1}{\omega_1^2} \approx \frac{ml^3}{EI} \left(\frac{3}{256} + \frac{1 \times 5}{48} + \frac{3}{256} \right) = \frac{98}{758} \frac{ml^3}{EI} = 0.12760 \frac{ml^3}{EI}$

$$\omega_1 \approx 2.7994 \sqrt{\frac{EI}{ml^3}}$$

7.2 Flexibility influence coefficients:

α_{11} = rotation of J_1 when a unit torque is applied to $J_1 = 1/k_{t1}$

α_{22} = rotation of J_2 when a unit torque is applied to J_2

$$= \frac{1}{k_{teq}} = \frac{1}{k_{t1}} + \frac{1}{k_{t2}}$$

α_{33} = rotation of J_3 when a unit torque is applied to J_3

$$= \frac{1}{k_{teq}} = \frac{1}{k_{t1}} + \frac{1}{k_{t2}} + \frac{1}{k_{t3}}$$

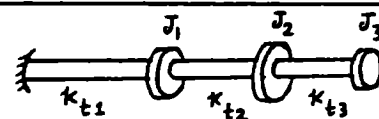
(a) Eq. (7.6) gives $\frac{1}{\omega_1^2} \approx \alpha_{11} J_1 + \alpha_{22} J_2 + \alpha_{33} J_3 = \frac{J_0}{k_t} (1+2+3)$

$$\omega_1 \approx 0.4082 \sqrt{k_t/J_0}$$

(b) $\frac{1}{\omega_1^2} \approx \frac{J_1}{k_{t1}} + J_2 \left(\frac{1}{k_{t1}} + \frac{1}{k_{t2}} \right) + J_3 \left(\frac{1}{k_{t1}} + \frac{1}{k_{t2}} + \frac{1}{k_{t3}} \right)$

$$\approx \frac{J_0}{k_t} + 2 J_0 \left(\frac{1}{k_t} + \frac{1}{2k_t} \right) + 3 J_0 \left(\frac{1}{k_t} + \frac{1}{2k_t} + \frac{1}{3k_t} \right) = \frac{9.5 J_0}{k_t}$$

$$\omega_1 \approx 0.3244 \sqrt{k_t/J_0}$$



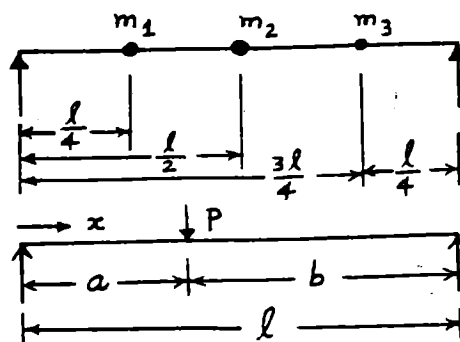
7.3

$$m_1 = m, m_2 = 2m, m_3 = 3m;$$

$$l_1 = l_2 = l_3 = l_4 = \frac{l}{4}$$

$$w(x) = \frac{Pbx}{6EI} (l^2 - b^2 - x^2); 0 \leq x \leq a$$

$$= -\frac{Pa(l-x)}{6EI} (a^2 + x^2 - 2lx); a \leq x \leq l$$



Deflection due to weight of m_1 : ($P = mg$)

At location of m_1 ($x = \frac{l}{4}$, $b = \frac{3l}{4}$, $l = l$)

$$w_1' = \frac{(mg)(\frac{3l}{4})(\frac{l}{4})}{6EI} \left\{ l^2 - \frac{9}{16}l^2 - \frac{1}{16}l^2 \right\} = \frac{3mg l^3}{256 EI}$$

At location of m_2 ($a = \frac{l}{4}$, $x = \frac{l}{2}$, $l = l$)

$$w_2' = -\frac{(mg)(\frac{l}{4})(l - \frac{l}{2})}{6EI} \left(\frac{l^2}{16} + \frac{l^2}{4} - l^2 \right) = \frac{11mg l^3}{768 EI}$$

At location of m_3 ($a = \frac{l}{4}$, $x = \frac{3l}{4}$, $l = l$)

$$w_3' = -\frac{(mg)(\frac{l}{4})(l - \frac{3l}{4})}{6EI} \left(\frac{l^2}{16} + \frac{9}{16}l^2 - \frac{6}{4}l^2 \right) = \frac{7mg l^3}{768 EI}$$

Deflection due to weight of m_2 : ($P = 2mg$)

At location of m_1 ($x = \frac{l}{4}$, $b = \frac{l}{2}$, $l = l$)

$$w_1'' = \frac{(2mg)(\frac{l}{2})(\frac{l}{4})}{6EI} \left(l^2 - \frac{l^2}{4} - \frac{1}{16}l^2 \right) = \frac{11mg l^3}{384 EI}$$

At location of m_2 ($x = \frac{l}{2}$, $b = \frac{l}{2}$, $l = l$)

$$w_2'' = \frac{(2mg)(\frac{l}{2})(\frac{l}{2})}{6EI} \left(l^2 - \frac{l^2}{4} - \frac{l^2}{4} \right) = \frac{mg l^3}{24 EI}$$

At location of m_3 ($x = \frac{3l}{4}$, $a = \frac{l}{2}$, $b = \frac{l}{2}$, $l = l$)

$$w_3'' = -\frac{(2mg)(\frac{l}{2})(l - \frac{3l}{4})}{6EI} \left(\frac{l^2}{4} + \frac{9}{16}l^2 - \frac{6}{4}l^2 \right) = \frac{11mg l^3}{1024 EI}$$

Deflection due to weight of m_3 : ($P = 3mg$)

At location of m_1 ($x = \frac{l}{4}$, $b = \frac{l}{4}$, $l = l$)

$$w_1''' = \frac{(3mg)(\frac{l}{4})(\frac{l}{4})}{6EI} \left(l^2 - \frac{l^2}{16} - \frac{l^2}{16} \right) = \frac{7mg l^3}{256 EI}$$

At location of m_2 ($x = \frac{l}{2}$, $b = \frac{l}{4}$, $l = l$)

$$w_2''' = \frac{(3mg)(\frac{l}{4})(\frac{l}{2})}{6EI l} \left(l^2 - \frac{l^2}{16} - \frac{l^2}{4} \right) = \frac{11mg l^3}{256 EI}$$

At location of m_3 ($x = \frac{3l}{4}$, $b = \frac{l}{4}$, $l = l$)

$$w_3''' = \frac{(3mg)(\frac{l}{4})(\frac{3l}{4})}{6EI l} \left(l^2 - \frac{1}{16} l^2 - \frac{9}{16} l^2 \right) = \frac{9mg l^3}{256 EI}$$

Total deflection of masses:

$$w_1 = w_1' + w_1'' + w_1''' = \frac{mg l^3}{EI} \left(\frac{3}{256} + \frac{11}{384} + \frac{7}{256} \right) = \frac{13}{192} \frac{mg l^3}{EI}$$

$$w_2 = w_2' + w_2'' + w_2''' = \frac{mg l^3}{EI} \left(\frac{11}{768} + \frac{1}{24} + \frac{11}{256} \right) = \frac{19}{192} \frac{mg l^3}{EI}$$

$$w_3 = w_3' + w_3'' + w_3''' = \frac{mg l^3}{EI} \left(\frac{7}{768} + \frac{11}{1024} + \frac{9}{256} \right) = \frac{169}{3072} \frac{mg l^3}{EI}$$

$$\omega = \left\{ \frac{g \frac{m^2 g l^3}{EI} \left(\frac{13}{192} + 2 \times \frac{19}{192} + 3 \times \frac{169}{3072} \right)}{\frac{m^2 g^2 l^6 m}{E^2 I^2} \left\{ \left(\frac{13}{192} \right)^2 + 2 \left(\frac{19}{192} \right)^2 + 3 \left(\frac{169}{3072} \right)^2 \right\}} \right\}^{1/2}$$

$$= 3.5987 \sqrt{\frac{EI}{m l^3}}$$

7.4

Weapon's mass 900 kg, stiffness of the wing 700 kN/m.

Solution:

ω_{11} = natural frequency of wing itself = 20 Hz = 125.664 rad/sec

ω_{22} = natural frequency of weapon attached at the tip of the wing
(neglecting the effect of mass of wing)

$$= \sqrt{\frac{k}{m}} = \sqrt{\frac{700 \times 10^3}{900}} = 27.8887 \text{ rad/sec}$$

New frequency of vibration of the wing with weapon is given by

$$\frac{1}{\omega_1^2} = \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{22}^2} = \frac{1}{125.664^2} + \frac{1}{27.8887^2}$$

$$= 134.9040 \times 10^{-5}$$

$$\therefore \omega_1 = 27.2262 \text{ rad/sec} = 4.3332 \text{ Hz}$$

- 7.5 For a simply supported beam, natural frequency is given by (assuming its mass to be concentrated at the middle)

$$\omega_{11} = \sqrt{\frac{k}{m}} \quad \text{where } k = \frac{48EI}{l^3}$$

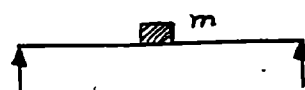
Neglecting the mass of beam, if trolley is placed at the middle of girder, its natural frequency is given by

$$\omega_{22} = \sqrt{\frac{k}{10m}}$$

Fundamental natural frequency of the combined system is given by

$$\frac{1}{\omega_1^2} = \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{22}^2} = \frac{m}{k} + \frac{10m}{k} = \frac{11m}{k}$$

$$\therefore \omega_1 = 0.3015 \sqrt{\frac{k}{m}} = 30.15\% \text{ of the natural frequency of the girder (without the trolley)}$$



- 7.6 Flexibility coefficients:

Let T = tension in string

a_{11} = deflection of m_1 due to unit force applied to m_1

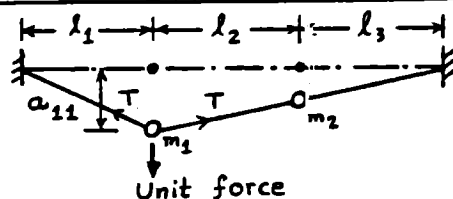
Unit force = sum of components of tension in vertical direction

$$\text{i.e. } 1 = T \left(\frac{a_{11}}{l_1} \right) + T \left(\frac{a_{11}}{l_2 + l_3} \right) = T a_{11} \left(\frac{1}{l} + \frac{1}{2l} \right) = \frac{3T a_{11}}{2l}$$

$$a_{11} = \frac{2l}{3T} = a_{22} \text{ (by symmetry)}$$

$$\frac{1}{\omega_1^2} \approx a_{11} m_1 + a_{22} m_2 = \frac{2l}{3T} (m + m) = \frac{4ml}{3T}$$

$$\omega_1 \approx 0.866 \sqrt{\frac{T}{ml}}; \text{ Exact solution is } \omega_1 = \sqrt{\frac{T}{ml}} \text{ (Problem 5.18).}$$



- 7.7 From Example 6.3,

$$[k] = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} = k \begin{bmatrix} 3 & -2 & 0 \\ -2 & 5 & -3 \\ 0 & -3 & 3 \end{bmatrix}$$

$$[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Assuming the mode shape as $\vec{x} = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$, Rayleigh's quotient becomes

$$R(\vec{x}) = \omega^2 = \frac{\vec{x}^T [k] \vec{x}}{\vec{x}^T [m] \vec{x}}$$

$$= \frac{(1 \ 2 \ 3) k \begin{bmatrix} 3 & -2 & 0 \\ -2 & 5 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}}{(1 \ 2 \ 3) m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}} = \frac{1}{6} \frac{k}{m}$$

$$\omega_1 = 0.4082 \sqrt{\frac{k}{m}}$$

Exact value is $\omega_1 = 0.3376 \sqrt{\frac{k}{m}}$ (Problem 6.46).

7.8

Stiffness matrix :

Give each disc a unit angular displacement holding other discs with zero rotation. Torque required will give stiffness coefficients.

$$\theta_1 = 1, \theta_2 = \theta_3 = 0 : M_{t1} = k_{t1} + k_{t2}, M_{t2} = -k_{t2}, M_{t3} = 0$$

$$\theta_2 = 1, \theta_1 = \theta_3 = 0 : M_{t2} = k_{t2} + k_{t3}, M_{t1} = -k_{t2}, M_{t3} = -k_{t3}$$

$$\theta_3 = 1, \theta_1 = \theta_2 = 0 : M_{t3} = k_{t3}, M_{t2} = -k_{t3}, M_{t1} = 0$$

$$[k] = \begin{bmatrix} k_{t1} + k_{t2} & -k_{t2} & 0 \\ -k_{t2} & k_{t2} + k_{t3} & -k_{t3} \\ 0 & -k_{t3} & k_{t3} \end{bmatrix} = k_t \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$[m] = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix} = J_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Assume the mode shape as $\vec{\theta} = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$

$$R(\vec{\theta}) = \omega^2 = \frac{(1 \ 2 \ 3) k_t \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}}{(1 \ 2 \ 3) J_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}} = \frac{1}{12} \frac{k_t}{J_0}$$

$$\omega_1 \approx 0.2887 \sqrt{\frac{k_t}{J_0}}$$

7.9 Stiffness matrix:

When $x_1 = 1, x_2 = 0$:

$$F_1 = k_{11} = \frac{T}{l_1} + \frac{T}{l_2}$$

$$F_2 = k_{21} = -\frac{T}{l_2}$$

When $x_1 = 0, x_2 = 1$:

$$F_2 = \frac{T}{l_2} + \frac{T}{l_3}$$

$$F_1 = k_{12} = -\frac{T}{l_2}$$

$$[k] = T \begin{bmatrix} \left(\frac{1}{l_1} + \frac{1}{l_2}\right) & -\frac{1}{l_2} \\ -\frac{1}{l_2} & \left(\frac{1}{l_2} + \frac{1}{l_3}\right) \end{bmatrix} = \frac{T}{l} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Assume the mode shape as $\vec{x} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$

$$R(\vec{x}) = \omega^2 = \frac{(1 \ 2) \frac{T}{l} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}}{(1 \ 2) m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}} = \frac{6}{5} \frac{T}{l_m}$$

$$\omega_1 \approx 1.0954 \sqrt{\frac{T}{l_m}}$$

$$\text{Exact value is } \omega_1 = \sqrt{\frac{T}{l_m}} \quad (\text{Problem 5.18}).$$

7.10

From problem 7.9, for $l_1 = l_2 = l_3 = l$,

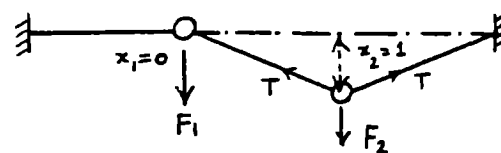
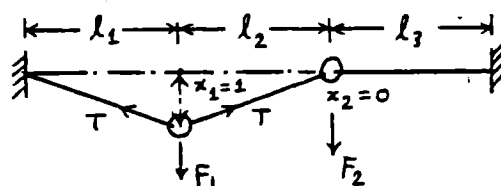
$$[k] = \frac{T}{l} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\text{and for } m_1 = m, m_2 = 5m, \quad [m] = m \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\text{Let } \vec{x} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

$$R(\vec{x}) = \omega^2 = \frac{(1 \ 2) \frac{T}{l} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}}{(1 \ 2) m \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}} = \frac{2}{7} \frac{T}{ml}$$

$$\omega_1 \approx 0.5345 \sqrt{\frac{T}{ml}}$$



7.11 Apply a unit load to masses m_1 and m_2 along x_1 and x_2 , respectively:

$$a_{11} = \frac{1}{2k_1} ; a_{22} = \frac{1}{2k_1} + \frac{1}{2k_2} = \frac{k_2 + k_1}{2k_1 k_2}$$

Dunkerley's equation gives

$$\frac{1}{\omega_1^2} = a_{11} m_1 + a_{22} m_2 = \frac{m_1}{2k_1} + m_2 \left(\frac{k_2 + k_1}{2k_1 k_2} \right) \quad (E_1)$$

Since $k_1 = k_2 = 3EI/h^3 \equiv k$, $m_1 = 2m$, $m_2 = m$ and hence (E₁) gives

$$\omega_1 = \sqrt{\frac{3EI}{2mh^3}}$$

7.12 Eq. (7.21) gives, for $r = n$,

$$R(\vec{x}) = \frac{c_n^2 \omega_n^2 + c_n^2 \sum_{i=1}^{n-1} \left(\frac{c_i}{c_n} \right)^2 \omega_i^2}{c_n^2 + c_n^2 \sum_{i=1}^{n-1} \left(\frac{c_i}{c_n} \right)^2} \quad (E_1)$$

Let $\left| \frac{c_i}{c_n} \right| = \epsilon_i \ll 1$. Then Eq. (E₁) becomes

$$\begin{aligned} R(\vec{x}) &= \frac{\omega_n^2 + \sum_{i=1}^{n-1} \epsilon_i^2 \omega_i^2}{1 + \sum_{i=1}^{n-1} \epsilon_i^2} \approx \left(\omega_n^2 + \sum_{i=1}^{n-1} \epsilon_i^2 \omega_i^2 \right) \left(1 - \sum_{i=1}^{n-1} \epsilon_i^2 \right) \\ &\approx \omega_n^2 + \sum_{i=1}^{n-1} \epsilon_i^2 \omega_i^2 - \omega_n^2 \sum_{i=1}^{n-1} \epsilon_i^2 \\ &\approx \omega_n^2 + \sum_{i=1}^{n-1} (\omega_i^2 - \omega_n^2) \epsilon_i^2 \quad (E_2) \end{aligned}$$

Since $\omega_i^2 < \omega_n^2$, in general, (E₂) shows that

$$R(\vec{x}) \leq \omega_n^2$$

7.13 Equations of motion

$$m_1 \ddot{x}_1 + k_1 (x_1 - x_2) = 0 \quad \dots (E.1)$$

$$m_2 \ddot{x}_2 + k_1 (x_2 - x_1) + k_2 (x_2 - x_3) = 0 \quad \dots (E.2)$$

$$m_3 \ddot{x}_3 + k_2 (x_3 - x_2) = 0 \quad \dots (E.3)$$

With $x_i(t) = X_i \cos \omega t$, Eqs. (E.1) and (E.2) give

$$m_1 \omega^2 X_1 = k_1 (X_1 - X_2) ; \quad m_2 \omega^2 X_2 = -\omega^2 m_1 X_1 + k_2 (X_2 - X_3)$$

$$\text{or } X_2 = X_1 - \frac{\omega^2 m_1 X_1}{k_1} ; \quad X_3 = X_2 - \frac{\omega^2}{k_2} (m_1 X_1 + m_2 X_2)$$

since the system is free-free, the resultant force applied to mass 3,

$$F = \sum_{i=1}^3 \omega^2 m_i X_i, \text{ must be zero.}$$

The computer program, with the subroutine FUN and the output, are given below.



```

C =====
C
C HOLZER METHOD
C THIS PROGRAM REQUIRES USER SUPPLIED SUBROUTINE FUN
C
C =====
C NT = NUMBER OF TIMES INCREMENT OF OM CHANGED
C FOLLOWING LINE CONTAINS PROBLEM-DEPENDENT DATA
C NR = NUMBER OF ROOTS REQUIRED
      NR=2  <-----
C END OF PROBLEM-DEPENDENT DATA
      NFUN=1
      OM=0.1 <-----
      CALL FUN (OM,F)
      PRINT 60,NFUN,OM,F
      FB=F
      INR=0
      OM=0.0
100  DEL=10.0 <-----
      NT=0
200  CONTINUE
      F1=FB
10   OM=OM+DEL
      CALL FUN (OM,F)
      NFUN=NFUN+1
      PRINT 60,NFUN,OM,F
      F2=F1+F
      IF (F2 .LT. 0.0) GO TO 20
      F1=F
      GO TO 10
20   NT=NT+1
      PRINT 30, OM,F,DEL,NFUN
30   FORMAT (//,31H CHANGE OF SIGN DETECTED AT OM=,E15.8,/,3H F=,
2 E15.8,/,5H DEL=,E15.8,/,6H NFUN=,I5,/)
      IF (NT .EQ. 6) GO TO 40
      IF (NT .EQ. 1) OMB=OM
      OM=OM-DEL
      DEL=DEL/10.0
      GO TO 200
40   INR=INR+1
      IF (INR .EQ. NR) GO TO 50
      OM=OMB
      CALL FUN (OM,F)
      NFUN=NFUN+1
      PRINT 60,NFUN,OM,F
      FB=F
      GO TO 100
50   CONTINUE
60   FORMAT (2X,6H NFUN=,I4,2X,4H OM=,E15.8,2X,3H F=,E15.8)
      STOP
      END
C =====
C
C SUBROUTINE FUN
C
C =====

```

```

SUBROUTINE FUN (OM,F)
  XM1=100.0
  XM2=20.0
  XM3=200.0
  AK1=8000.0
  AK2=4000.0
  OMS=OM**2
  X1=1.0
  X2=(1.0-(OMS*XM1/AK1))*X1
  X3=X2-(OMS/AK2)*(XM1*X1+XM2*X2)
  F=OMS*(XM1*X1+XM2*X2+XM3*X3)
  RETURN
END

```

$\omega_1 = 0$

```

-----
NFUN= 1   OM= 0.10000000E+00   F= 0.31991253E+01
NFUN= 2   OM= 0.10000000E+02   F=-0.43000000E+05

```

```

CHANGE OF SIGN DETECTED AT OM= 0.10000000E+02
F=-0.43000000E+05
DEL= 0.10000000E+02
NFUN= 2

```

```

NFUN= 3   OM= 0.10000000E+01   F= 0.31126251E+03
NFUN= 4   OM= 0.20000000E+01   F= 0.11408000E+04
NFUN= 5   OM= 0.30000000E+01   F= 0.21803625E+04
NFUN= 6   OM= 0.40000000E+01   F= 0.29312000E+04
NFUN= 7   OM= 0.50000000E+01   F= 0.27265625E+04
NFUN= 8   OM= 0.60000000E+01   F= 0.76320044E+03
NFUN= 9   OM= 0.70000000E+01   F=-0.38581377E+04

```

```

CHANGE OF SIGN DETECTED AT OM= 0.70000000E+01
F=-0.38581377E+04
DEL= 0.10000000E+01
NFUN= 9

```

```

CHANGE OF SIGN DETECTED AT OM= 0.62220016E+01
F=-0.28649828E+00
DEL= 0.99999999E-01
NFUN= 27

```

← $\omega_2 = 6.2220016$

```

NFUN= 28   OM= 0.10000000E+02   F=-0.43000000E+05
NFUN= 29   OM= 0.20000000E+02   F=-0.47200000E+06
NFUN= 30   OM= 0.30000000E+02   F= 0.23130000E+07

```

```

CHANGE OF SIGN DETECTED AT OM= 0.30000000E+02
F= 0.23130000E+07
DEL= 0.10000000E+02
NFUN= 30

```

```

NFUN= 31   OM= 0.21000000E+02   F=-0.48851225E+06
NFUN= 32   OM= 0.22000000E+02   F=-0.47761113E+06
NFUN= 33   OM= 0.23000000E+02   F=-0.42888006E+06

```

CHANGE OF SIGN DETECTED AT OM= 0.25715595E+02
 F= 0.21467506E+02
 DEL= 0.99999990E-04
 NFUN= 58

$$\omega_3 = 25.715595$$

7.14 Eigenvalue problem is

$$\begin{bmatrix} (\lambda-2) & 1 & 0 \\ 1 & (\lambda-2) & 1 \\ 0 & 1 & (2\lambda-3) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (E_1)$$

where $\lambda = \frac{m\omega^2}{k}$.

If $X_1 = 1$, (E_1) gives $X_2 = -(\lambda-2)X_1$, $X_3 = -X_1 - (\lambda-2)X_2$,

$E = X_2 + (2\lambda-3)X_3$ (should be zero)

The computer program and results are given.

```

C =====
C
C HOLZER METHOD
C THIS PROGRAM REQUIRES USER SUPPLIED SUBROUTINE FUN
C
C =====
C NT = NUMBER OF TIMES INCREMENT OF OM CHANGED
C FOLLOWING LINE CONTAINS PROBLEM-DEPENDENT DATA
C NR = NUMBER OF ROOTS REQUIRED
NR=3<-----
C END OF PROBLEM-DEPENDENT DATA
NFUN=1
OM=0.01<-----
CALL FUN (OM,F)
WRITE(13,60) NFUN,OM,F
FB=F
INR=0
OM=0.0
100 DEL=0.25<-----
    NT=0
200 CONTINUE
    F1=FB
    10 OM=OM+DEL
    CALL FUN (OM,F)
    NFUN=NFUN+1
    WRITE(13,60) NFUN,OM,F
    F2=F1*F
    IF (F2 .LT. 0.0) GO TO 20
    F1=F
    GO TO 10
20 NT=NT+1
  WRITE(13,30) OM,F,DEL,NFUN
30 FORMAT (//,31H CHANGE OF SIGN DETECTED AT OM=,E15.8,/,3H F=,
2 E15.8,/,5H DEL=,E15.8,/,6H NFUN=,I5,/)
  IF (NT .EQ. 6) GO TO 40
  IF (NT .EQ. 1) OMB=OM
  OM=OM-DEL
  DEL=DEL/10.0
  GO TO 200
40 INR=INR+1
  
```

IF (INR .EQ. NR) GO TO 50

OM=OMB

CALL FUN (OM,F)

NFUN=NFUN+1

WRITE(13,60) NFUN,OM,F

FB=F

GO TO 100

50 CONTINUE

60 FORMAT (2X,6H NFUN=,I4,2X,4H OM=,E15.8,2X,3H F=,E15.8)

STOP

END

C =====

C

C SUBROUTINE FUN

C

C =====

SUBROUTINE FUN (OM,F)

X1=1.0

X2=-(OM-2.0)*X1

X3=-X1-(OM-2.0)*X2

F=X2+(2.0*OM-3.0)*X3

RETURN

END

NFUN=	1	OM=	0.99999998E-02	F=-0.68310986E+01
NFUN=	2	OM=	0.25000000E+00	F=-0.34062500E+01
NFUN=	3	OM=	0.50000000E+00	F=-0.10000000E+01
NFUN=	4	OM=	0.75000000E+00	F= 0.40625000E+00

CHANGE OF SIGN DETECTED AT OM= 0.75000000E+00

F= 0.40625000E+00

DEL= 0.25000000E+00

NFUN= 4

NFUN=	5	OM=	0.52499998E+00	F=-0.81746900E+00
NFUN=	6	OM=	0.54999995E+00	F=-0.64475036E+00
NFUN=	7	OM=	0.57499993E+00	F=-0.48165655E+00
NFUN=	8	OM=	0.59999990E+00	F=-0.32800055E+00
NFUN=	9	OM=	0.62499988E+00	F=-0.18359447E+00
NFUN=	10	OM=	0.64999986E+00	F=-0.48250675E-01
NFUN=	11	OM=	0.67499983E+00	F= 0.78217864E-01
	:			
NFUN=	28	OM=	0.65932715E+00	F=-0.38385391E-04
NFUN=	29	OM=	0.65932965E+00	F=-0.25749207E-04
NFUN=	30	OM=	0.65933216E+00	F=-0.12755394E-04
NFUN=	31	OM=	0.65933466E+00	F=-0.11920929E-06
NFUN=	32	OM=	0.65933716E+00	F= 0.12636185E-04

CHANGE OF SIGN DETECTED AT OM= 0.65933716E+00

F= 0.12636185E-04

DEL= 0.24999997E-05

NFUN= 32

← $\lambda_1 = 0.65933716$

NFUN= 33 OM= 0.75000000E+00 F= 0.40625000E+00

NFUN= 34	OM= 0.10000000E+01	F= 0.10000000E+01
NFUN= 35	OM= 0.12500000E+01	F= 0.96875000E+00
NFUN= 36	OM= 0.15000000E+01	F= 0.50000000E+00
NFUN= 37	OM= 0.17500000E+01	F=-0.21875000E+00
:		
NFUN= 63	OM= 0.16789526E+01	F= 0.32067299E-04
NFUN= 64	OM= 0.16789551E+01	F= 0.24497509E-04
NFUN= 65	OM= 0.16789576E+01	F= 0.16927719E-04
NFUN= 66	OM= 0.16789601E+01	F= 0.93579292E-05
NFUN= 67	OM= 0.16789626E+01	F= 0.17881393E-05
NFUN= 68	OM= 0.16789651E+01	F=-0.57816505E-05

CHANGE OF SIGN DETECTED AT OM= 0.16789651E+01

← $\lambda_2 = 1.6789651$

F=-0.57816505E-05

DEL= 0.24999997E-05

NFUN= 68

NFUN= 69	OM= 0.17500000E+01	F=-0.21875000E+00
NFUN= 70	OM= 0.20000000E+01	F=-0.10000000E+01
NFUN= 71	OM= 0.22500000E+01	F=-0.16562500E+01
NFUN= 72	OM= 0.25000000E+01	F=-0.20000000E+01
NFUN= 73	OM= 0.27500000E+01	F=-0.18437500E+01
NFUN= 74	OM= 0.30000000E+01	F=-0.10000000E+01
NFUN= 75	OM= 0.32500000E+01	F= 0.71875000E+00
:		
NFUN= 95	OM= 0.31615264E+01	F=-0.13036728E-02
NFUN= 96	OM= 0.31615515E+01	F=-0.11179447E-02
NFUN= 97	OM= 0.31615765E+01	F=-0.93221664E-03
NFUN= 98	OM= 0.31616015E+01	F=-0.74636936E-03
NFUN= 99	OM= 0.31616266E+01	F=-0.56064129E-03
NFUN= 100	OM= 0.31616516E+01	F=-0.37479401E-03
NFUN= 101	OM= 0.31616766E+01	F=-0.18906593E-03
NFUN= 102	OM= 0.31617017E+01	F=-0.32186508E-05
NFUN= 103	OM= 0.31617267E+01	F= 0.18215179E-03

CHANGE OF SIGN DETECTED AT OM= 0.31617267E+01

← $\lambda_3 = 3.1617267$

F= 0.18215179E-03

DEL= 0.24999998E-04

NFUN= 103

7.15 The program listed in Problems 7.13 and 7.14 is used with NR= 1, initial value of OM= 0.01, DEL= 0.25 and

C =====

C

C SUBROUTINE FUN

C

C =====

SUBROUTINE FUN (OM,F)

$\Theta_1 = 1$

X1=1.0

$\Theta_2 = (-\lambda + 2) \Theta_1$; $\lambda = \left(\frac{J_0 \omega^2}{k_t} \right)$

X2=(2.0-OM)*X1

$\Theta_3 = -\Theta_1 + (-\lambda + 2) \Theta_2$

X3=-X1+(2.0-OM)*X2

$F = -\Theta_2 + (-\lambda + 1) \Theta_3$

F=-X2+(1.0-OM)*X3

RETURN

END

The output of the program is given below.

NFUN=	1	OM=	0.99999998E-02	F=	0.94049907E+00
NFUN=	2	OM=	0.25000000E+00	F=	-0.20312500E+00

CHANGE OF SIGN DETECTED AT OM= 0.25000000E+00

F=-0.20312500E+00

DEL= 0.25000000E+00

NFUN= 2

NFUN=	3	OM=	0.25000000E-01	F=	0.85310948E+00
NFUN=	4	OM=	0.50000001E-01	F=	0.71237516E+00
NFUN=	5	OM=	0.75000003E-01	F=	0.57770300E+00
NFUN=	6	OM=	0.10000000E+00	F=	0.44899976E+00
NFUN=	7	OM=	0.12500000E+00	F=	0.32617188E+00
NFUN=	8	OM=	0.15000001E+00	F=	0.20912516E+00
NFUN=	9	OM=	0.17500001E+00	F=	0.97765565E-01
NFUN=	10	OM=	0.20000002E+00	F=	-0.80001354E-02

CHANGE OF SIGN DETECTED AT OM= 0.20000002E+00

F=-0.80001354E-02

DEL= 0.25000000E-01

NFUN= 10

NFUN=	11	OM=	0.17750001E+00	F=	0.86938858E-01
NFUN=	12	OM=	0.18000001E+00	F=	0.76168060E-01
NFUN=	13	OM=	0.18250000E+00	F=	0.65452933E-01
NFUN=	14	OM=	0.18500000E+00	F=	0.54793477E-01
NFUN=	15	OM=	0.18750000E+00	F=	0.44189453E-01
NFUN=	16	OM=	0.19000000E+00	F=	0.33641100E-01
NFUN=	17	OM=	0.19250000E+00	F=	0.23147941E-01
NFUN=	18	OM=	0.19499999E+00	F=	0.12710214E-01
NFUN=	19	OM=	0.19749999E+00	F=	0.23275614E-02
NFUN=	20	OM=	0.19999999E+00	F=	-0.79998970E-02

CHANGE OF SIGN DETECTED AT OM= 0.19999999E+00

F=-0.79998970E-02

DEL= 0.24999999E-02

NFUN= 20

NFUN=	21	OM=	0.19774999E+00	F=	0.12923479E-02
NFUN=	22	OM=	0.19799998E+00	F=	0.25773048E-03
NFUN=	23	OM=	0.19824998E+00	F=	-0.77641010E-03

CHANGE OF SIGN DETECTED AT OM= 0.19824998E+00

F=-0.77641010E-03

DEL= 0.24999998E-03

NFUN= 23

NFUN= 24	DM= 0.19802499E+00	F= 0.15425682E-03
NFUN= 25	DM= 0.19804999E+00	F= 0.50902367E-04
NFUN= 26	DM= 0.19807500E+00	F=-0.52571297E-04

CHANGE OF SIGN DETECTED AT DM= 0.19807500E+00

F=-0.52571297E-04

DEL= 0.24999998E-04

NFUN= 26

NFUN= 27	DM= 0.19805250E+00	F= 0.40531158E-04
NFUN= 28	DM= 0.19805500E+00	F= 0.30040741E-04
NFUN= 29	DM= 0.19805750E+00	F= 0.19669533E-04
NFUN= 30	DM= 0.19806001E+00	F= 0.92983246E-05
NFUN= 31	DM= 0.19806251E+00	F=-0.10728836E-05

CHANGE OF SIGN DETECTED AT DM= 0.19806251E+00

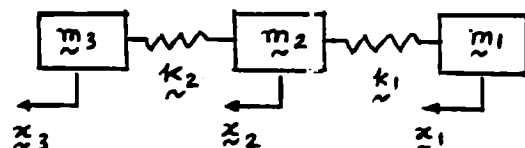
F=-0.10728836E-05

DEL= 0.24999997E-05

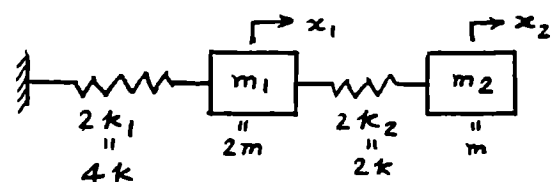
NFUN= 31

← $\lambda_1 = 0.19806251$

- 7.16 The system can be modeled as shown.
The system can be redrawn as follows:



Here $m_1 = m$, $k_1 = 2k$, $m_2 = 2m$,
 $k_2 = 4k$, $x_1 = x_2$, $x_2 = x_1$, $x_3 = 0$,
 $m_3 = \text{any value}$.



Let $x_1 = 1$

$$x_2 = x_1 - \frac{\omega^2 m_1 x_1}{k_1}$$

$$= x_1 \left(1 - \frac{m \omega^2}{2k} \right)$$

$$x_3 = x_2 - \frac{\omega^2}{k_2} (m_1 x_1 + m_2 x_2)$$

$$= x_2 \left(1 - \frac{m \omega^2}{2k} \right) - x_1 \left(\frac{m \omega^2}{4k} \right)$$

$$= x_1 \left(1 + \frac{m^2 \omega^4}{4k^2} - \frac{5m \omega^2}{4k} \right)$$

Holzer's procedure involves assuming different values for ω^2 and finding, out of those values, the correct frequency ω as the one which gives $x_3 = 0$ (boundary condition to be satisfied).

From the expression for x_3 , we can find the correct

frequency (without trial and error) by setting

$$\omega^4 \left(\frac{m^2}{4k^2} \right) - \omega^2 \left(\frac{5m}{4k} \right) + 1 = 0$$

$$\text{or } \omega^2 = \frac{\left(\frac{5m}{4k} \right) \pm \sqrt{\frac{25m^2}{16k^2} - \frac{m^2}{k^2}}}{\left(\frac{m^2}{4k^2} \right)} = \frac{\frac{5m}{4k} \pm \frac{3m}{4k}}{\left(\frac{m^2}{4k^2} \right)}$$

Thus the first natural frequency is given by

$$\omega_1^2 = \left(\frac{5m}{4k} - \frac{3m}{4k} \right) / \left(\frac{m^2}{4k^2} \right) \quad \text{or} \quad \omega_1 = \sqrt{\frac{k}{m}}$$

7.17 Eqs. (7.32) to (7.35) give

$$\Theta_2 = \Theta_1 - \frac{\omega^2 J_1}{k_{t1}} \Theta_1$$

$$\Theta_3 = \Theta_2 - \frac{\omega^2}{k_{t2}} (J_1 \Theta_1 + J_2 \Theta_2)$$

$$E = \sum_{i=1}^3 \omega^2 J_i \Theta_i = \text{sum of inertia torques (should be zero)}$$

The computer program of Problems 7.13 and 7.14 is used with

NR=3, OM=0.01 and DEL=10.0.

The subroutine FUN and results are given.

```

C =====
C
C SUBROUTINE FUN
C
C =====
      SUBROUTINE FUN (OM,F)
      XJ1=10.0
      XJ2=5.0
      XJ3=1.0
      XKT1=1.0E+06
      XKT2=1.0E+06
      X1=1.0
      OMS=OM**2
      X2=X1-(OMS*XJ1/XKT1)*X1
      X3=X2-(OMS/XKT2)*(XJ1*X1+XJ2*X2)
      F=OMS*(XJ1*X1+XJ2*X2+XJ3*X3)
      RETURN
      END
      END
      NFUN= 1  OM= 0.99999998E-02  F= 0.16000000E-02
      NFUN= 2  OM= 0.10000000E+02  F= 0.15992500E+04
      NFUN= 3  OM= 0.20000000E+02  F= 0.63880034E+04
      NFUN= 4  OM= 0.30000000E+02  F= 0.14339287E+05
      NFUN= 5  OM= 0.40000000E+02  F= 0.25408205E+05
      NFUN= 6  OM= 0.50000000E+02  F= 0.39532031E+05
      NFUN= 7  OM= 0.60000000E+02  F= 0.56630336E+05
      NFUN= 8  OM= 0.70000000E+02  F= 0.76605133E+05
      NFUN= 9  OM= 0.80000000E+02  F= 0.99341109E+05
      NFUN= 10 OM= 0.90000000E+02  F= 0.12470582E+06
      ...

```

$\omega_1 = 0$

```

NFUN= 50   OM= 0.49000000E+03   F= 0.21006358E+06
NFUN= 51   OM= 0.50000000E+03   F= 0.93750000E+05
NFUN= 52   OM= 0.51000000E+03   F=-0.32486393E+05

```

CHANGE OF SIGN DETECTED AT OM= 0.51000000E+03

F=-0.32486393E+05

DEL= 0.10000000E+02

NFUN= 52

⋮

```

NFUN= 69   OM= 0.50750012E+03   F= 0.93337460E+01
NFUN= 70   OM= 0.50750021E+03   F= 0.79214058E+01
NFUN= 71   OM= 0.50750031E+03   F= 0.66932836E+01
NFUN= 72   OM= 0.50750040E+03   F= 0.57721915E+01
NFUN= 73   OM= 0.50750049E+03   F= 0.45440674E+01
NFUN= 74   OM= 0.50750058E+03   F= 0.33159423E+01
NFUN= 75   OM= 0.50750067E+03   F= 0.20878162E+01
NFUN= 76   OM= 0.50750076E+03   F= 0.92109573E+00
NFUN= 77   OM= 0.50750085E+03   F=-0.30703202E+00

```

CHANGE OF SIGN DETECTED AT OM= 0.50750085E+03

← $\omega_2 = 507.50085$

F=-0.30703202E+00

DEL= 0.99999990E-04

NFUN= 77

```

NFUN= 78   OM= 0.51000000E+03   F=-0.32486393E+05
NFUN= 79   OM= 0.52000000E+03   F=-0.16878158E+06
NFUN= 80   OM= 0.53000000E+03   F=-0.31524272E+06

```

⋮

```

NFUN= 137  OM= 0.11000000E+04   F=-0.18694431E+07
NFUN= 138  OM= 0.11100000E+04   F=-0.62095219E+06
NFUN= 139  OM= 0.11200000E+04   F= 0.74758494E+06

```

CHANGE OF SIGN DETECTED AT OM= 0.11200000E+04

F= 0.74758494E+06

DEL= 0.10000000E+02

NFUN= 139

⋮

```

NFUN= 168  OM= 0.11146489E+04   F=-0.37916328E+02
NFUN= 169  OM= 0.11146490E+04   F=-0.23697710E+02
NFUN= 170  OM= 0.11146492E+04   F=-0.94790859E+01
NFUN= 171  OM= 0.11146493E+04   F= 0.47395439E+01

```

CHANGE OF SIGN DETECTED AT OM= 0.11146493E+04

← $\omega_3 = 1114.6493$

F= 0.47395439E+01

DEL= 0.99999990E-04

NFUN= 171

7.18 Eigenvector $\vec{x}^{(1)}$ corresponding to $\lambda_1 = 10.38068 (= \frac{1}{\omega_1^2})$ is given by

$$\begin{bmatrix} (2.5 - \lambda_1) & -1 & 0 \\ -1 & (5 - \lambda_1) & -\sqrt{2} \\ 0 & -\sqrt{2} & (10 - \lambda_1) \end{bmatrix} \begin{Bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{i.e. } x_2^{(1)} = -7.88068 x_1^{(1)} \quad \text{and} \quad x_3^{(1)} = -3.71497 x_2^{(1)} = 29.27649 x_1^{(1)}$$

$$\vec{x}^{(1)} = \alpha \begin{Bmatrix} 1 \\ -7.88068 \\ 29.27649 \end{Bmatrix} \quad \text{where } \alpha = \text{value of } x_1^{(1)}$$

$$\text{When } \vec{x}^{(1)} \text{ is normalized as } \vec{x}^{(1)T} [m] \vec{x}^{(1)} = 1, \quad \alpha = 0.03296$$

$$\therefore \vec{x}^{(1)} = \begin{Bmatrix} 0.03296 \\ -0.25975 \\ 0.96495 \end{Bmatrix}$$

$$[D_2] = [D_1] - \lambda_1 \vec{x}^{(1)} \vec{x}^{(1)T} [m]$$

$$= \begin{bmatrix} 2.5 & -1.0 & 0.0 \\ -1.0 & 5.0 & -1.4142 \\ 0.0 & -1.4142 & 10.0 \end{bmatrix} - (10.38068) \begin{Bmatrix} 0.03296 \\ -0.25975 \\ 0.96495 \end{Bmatrix} \begin{Bmatrix} 0.03296 & -0.25975 & 0.96495 \end{Bmatrix}$$

$$= \begin{bmatrix} 2.4887228 & -0.9111273 & -0.3301549 \\ -0.9111273 & 4.2996149 & 1.1876738 \\ -0.3301549 & 1.1876738 & 0.3342530 \end{bmatrix}$$

$$\text{If } \vec{x}_0 = \begin{Bmatrix} 1.0 \\ -7.88068 \\ 29.27649 \end{Bmatrix} \text{ is used, the iterative procedure } \vec{x}_{i+1} = [D_2] \vec{x}_i$$

gives the following results (with $x_{1,i+1} = 1$):

Iter. No. (i)	$\vec{x}_{i+1}$ as a row vector (with $x_{1,i+1} = 1$)
ITER= 0	0.10000000E+01-0.78806801E+01 0.29276489E+02
ITER= 1	0.10000000E+01-0.10666667E+02 0.29333334E+02
ITER= 2	0.10000000E+01-0.47272644E+01-0.13066854E+01
ITER= 3	0.10000000E+01-0.31531227E+01-0.88291872E+00
ITER= 4	0.10000000E+01-0.27448514E+01-0.77301961E+00
ITER= 5	0.10000000E+01-0.25989382E+01-0.73374254E+00
ITER= 6	0.10000000E+01-0.25411220E+01-0.71817952E+00
ITER= 7	0.10000000E+01-0.25172873E+01-0.71176374E+00
ITER= 8	0.10000000E+01-0.25073018E+01-0.70907575E+00
ITER= 9	0.10000000E+01-0.25030899E+01-0.70794201E+00
ITER= 10	0.10000000E+01-0.25013082E+01-0.70746231E+00
ITER= 11	0.10000000E+01-0.25005538E+01-0.70725930E+00
ITER= 12	0.10000000E+01-0.25002341E+01-0.70717323E+00
ITER= 13	0.10000000E+01-0.25000987E+01-0.70713681E+00
ITER= 14	0.10000000E+01-0.25000415E+01-0.70712131E+00
ITER= 15	0.10000000E+01-0.25000172E+01-0.70711482E+00

converged value of $\lambda_2 = 5.00004216$ (or $\omega_2 = 0.44721171$)

By using a similar procedure, $[D_3]$ is found. with the same $\vec{x}_0$, the results obtained from $\vec{x}_{i+1} = [D_3] \vec{x}_i$ are given below.

Iter. no. (i)	$\vec{X}_{i+1}$ as a row vector (with $X_{1,i+1} = 1$)
ITER= 0	0.10000000E+01 -0.78806801E+01 0.29276489E+02
ITER= 1	0.10000000E+01 0.19600000E+02 -0.72824997E+02
ITER= 2	0.10000000E+01 0.38067123E+00 0.68348452E-01
ITER= 3	0.10000000E+01 0.38067168E+00 0.68312615E-01
ITER= 4	0.10000000E+01 0.38067159E+00 0.68312593E-01

Converged value of $\lambda_3 = 2.11932243$ ($\omega_3 = 0.68691260$).

7.19 $[K]^{-1} = \frac{1}{k} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2.5 \end{bmatrix}$, $[D] = [K]^{-1} [m] = \frac{m}{k} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \\ 1 & 2 & 5 \end{bmatrix}$

Using $\vec{X}_0 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$, the following results are obtained:

Iteration number (i)	$\vec{X}_{i+1} = [D] \vec{X}_i$ with $X_{1,i} = 1$ (given as row vector)
ITER= 0	0.10000000E+01 0.10000000E+01 0.10000000E+01
ITER= 1	0.10000000E+01 0.17500000E+01 0.20000000E+01
ITER= 2	0.10000000E+01 0.18518518E+01 0.21481481E+01
ITER= 3	0.10000000E+01 0.18601036E+01 0.21606219E+01
ITER= 4	0.10000000E+01 0.18607503E+01 0.21616161E+01
ITER= 5	0.10000000E+01 0.18608015E+01 0.21616952E+01
ITER= 6	0.10000000E+01 0.18608055E+01 0.21617017E+01 $\leftarrow \vec{X}^{(1)}$

Converged frequency = $0.373088 = \omega_1 \sqrt{\frac{m}{k}}$

Repetition of the procedure with $[D_2]$ and $[D_3]$ gives the following results:

$$\omega_2 \sqrt{\frac{m}{k}} = 1.321324$$

$$\vec{X}^{(2)} = \{ 1.000000 \quad 0.254098 \quad -0.340706 \}$$

$$\omega_3 \sqrt{\frac{m}{k}} = 2.028523$$

$$\vec{X}^{(3)} = \{ 1.000000 \quad -2.114801 \quad 0.678847 \}$$

$$7.20 \quad [k]^{-1} = \frac{1}{k} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.5 & 1.5 \\ 1 & 1.5 & 1.833 \end{bmatrix}, \quad [k]^{-1} [m] = \frac{m}{k} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.5 & 1.5 \\ 1 & 1.5 & 1.833 \end{bmatrix}$$

$$[D] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.5 & 1.5 \\ 1 & 1.5 & 1.833 \end{bmatrix}$$

With $\vec{x}_0 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$, the following results are obtained (Program 10.F):

Iteration number (i)	$\vec{x}_{i+1} = [D] \vec{x}_i$ with $x_{1,i} = 1$ (given as row vector)		
0	0.10000000E+01	0.10000000E+01	0.10000000E+01
1	0.10000000E+01	0.13333334E+01	0.14443334E+01
2	0.10000000E+01	0.13676430E+01	0.14949605E+01
3	0.10000000E+01	0.13705537E+01	0.14994360E+01
4	0.10000000E+01	0.13708007E+01	0.14998223E+01
5	0.10000000E+01	0.13708220E+01	0.14998555E+01
6	0.10000000E+01	0.13708237E+01	0.14998584E+01 ← $\vec{x}^{(1)}$

converged frequency = 0.50828409 = ω_1

0	0.10000000E+01	0.10000000E+01	0.10000000E+01
1	0.10000000E+01	-0.56849640E-01	-0.61459929E+00
2	0.10000000E+01	-0.22656711E-01	-0.64602280E+00
3	0.10000000E+01	-0.91092410E-02	-0.65840477E+00
4	0.10000000E+01	-0.38191630E-02	-0.66323972E+00
5	0.10000000E+01	-0.17635337E-02	-0.66511846E+00
6	0.10000000E+01	-0.96627331E-03	-0.66584712E+00
7	0.10000000E+01	-0.65728393E-03	-0.66612953E+00
8	0.10000000E+01	-0.53756207E-03	-0.66623896E+00
9	0.10000000E+01	-0.49118389E-03	-0.66628140E+00
10	0.10000000E+01	-0.47320157E-03	-0.66629785E+00
11	0.10000000E+01	-0.46624581E-03	-0.66630417E+00
12	0.10000000E+01	-0.46355074E-03	-0.66630661E+00
13	0.10000000E+01	-0.46252197E-03	-0.66630769E+00
14	0.10000000E+01	-0.46209697E-03	-0.66630799E+00
15	0.10000000E+01	-0.46194042E-03	-0.66630810E+00
16	0.10000000E+01	-0.46188448E-03	-0.66630810E+00
17	0.10000000E+01	-0.46187337E-03	-0.66630816E+00
18	0.10000000E+01	-0.46186216E-03	-0.66630822E+00
19	0.10000000E+01	-0.46185096E-03	-0.66630816E+00
20	0.10000000E+01	-0.46185098E-03	-0.66630822E+00 ← $\vec{x}^{(2)}$

converged frequency = 1.7323176 = ω_2

0	0.10000000E+01	0.10000000E+01	0.10000000E+01
1	0.10000000E+01	-0.23654659E+01	0.15024464E+01
2	0.10000000E+01	-0.23733659E+01	0.15024524E+01
3	0.10000000E+01	-0.23733664E+01	0.15024524E+01 ← $\vec{x}^{(3)}$

converged frequency = 2.783294 = ω_3

$$7.21 \quad \text{From problem 6.18, } [k] = \frac{GJ}{l} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad [J_d] = J_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[D] = \begin{bmatrix} 0.75 & 0.50 & 0.25 \\ 0.50 & 1.00 & 0.50 \\ 0.25 & 0.50 & 0.75 \end{bmatrix}$$

With $\vec{X}_0 = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$, the following results are obtained (Program 10.F):

Iteration number (i)	$\vec{X}_{i+1} = [D] \vec{X}_i$ with $X_{1,i} = 1$ (given as row vector)		
0	0.10000000E+01	0.20000000E+01	0.30000000E+01
1	0.10000000E+01	0.16000000E+01	0.14000000E+01
2	0.10000000E+01	0.14736842E+01	0.11052632E+01
3	0.10000000E+01	0.14328358E+01	0.10298507E+01
4	0.10000000E+01	0.14199134E+01	0.10086581E+01
5	0.10000000E+01	0.14159292E+01	0.10025285E+01
6	0.10000000E+01	0.14147242E+01	0.10007399E+01
7	0.10000000E+01	0.14143645E+01	0.10002166E+01
8	0.10000000E+01	0.14142580E+01	0.10000634E+01
9	0.10000000E+01	0.14142267E+01	0.10000186E+01
10	0.10000000E+01	0.14142175E+01	0.10000055E+01
11	0.10000000E+01	0.14142147E+01	0.10000015E+01

converged frequency = 0.76536608 = ω_1

0	0.10000000E+01	0.20000000E+01	0.30000000E+01
1	0.10000000E+01	0.29291141E+00	-0.14141805E+01
2	0.10000000E+01	0.15801974E+00	-0.12234681E+01
3	0.10000000E+01	0.88471927E-01	-0.11251128E+01
4	0.10000000E+01	0.50517395E-01	-0.10714371E+01
5	0.10000000E+01	0.29161688E-01	-0.10412357E+01

...

27	0.10000000E+01	0.19669531E-05	-0.99999768E+00
28	0.10000000E+01	0.18924472E-05	-0.99999750E+00
29	0.10000000E+01	0.18179417E-05	-0.99999750E+00
30	0.10000000E+01	0.18030403E-05	-0.99999738E+00
31	0.10000000E+01	0.17732382E-05	-0.99999744E+00
32	0.10000000E+01	0.17732380E-05	-0.99999726E+00

converged frequency = 1.4142134 = ω_2

0	0.10000000E+01	0.20000000E+01	0.30000000E+01
1	0.10000000E+01	-0.14143940E+01	0.10000018E+01
2	0.10000000E+01	-0.14142135E+01	0.10000004E+01
3	0.10000000E+01	-0.14142135E+01	0.10000004E+01

converged frequency = 1.8477590 = ω_3

7.22 From Problem 7.9, $[k] = \frac{T}{l} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $[m] = m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $[k]^{-1} = \frac{l}{3T} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$$[D] = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$$

With $\vec{X}_0 = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$, the following results are obtained (Program 10.F):

Iteration number (i)	$\vec{X}_{i+1} = [D] \vec{X}_i$ with $X_{1,i} = 1$ (given as row vector)	
0	0.10000000E+01	0.20000000E+01
1	0.10000000E+01	0.12500000E+01
2	0.10000000E+01	0.10769231E+01
3	0.10000000E+01	0.10250001E+01
4	0.10000000E+01	0.10082645E+01
5	0.10000000E+01	0.10027474E+01
6	0.10000000E+01	0.10009151E+01
7	0.10000000E+01	0.10003049E+01

8	0.10000000E+01	0.10001017E+01	
9	0.10000000E+01	0.10000340E+01	
10	0.10000000E+01	0.10000113E+01	
11	0.10000000E+01	0.10000038E+01	$\vec{x}^{(1)}$
Converged frequency = 0.99999809 = $\omega_1 \sqrt{m/l/T}$			
0	0.10000000E+01	0.20000000E+01	
1	0.10000000E+01	-0.99991989E+00	
2	0.10000000E+01	-0.99998856E+00	
3	0.10000000E+01	-0.99998856E+00	$\vec{x}^{(2)}$
Converged frequency = 1.7320508 = $\omega_2 \sqrt{m/l/T}$			

7.23 $[k]^{-1} = \frac{1}{k} \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 1.0 & 1.0 & 1.0 \\ 0.5 & 1.0 & 2.0 & 2.0 \\ 0.5 & 1.0 & 2.0 & 3.0 \end{bmatrix}$, $[D] = \frac{m}{k} \begin{bmatrix} 1.5 & 1.0 & 0.5 & 0.5 \\ 1.5 & 2.0 & 1.0 & 1.0 \\ 1.5 & 2.0 & 2.0 & 2.0 \\ 1.5 & 2.0 & 2.0 & 3.0 \end{bmatrix}$

With $\vec{x}_0 = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$, the following results are obtained for $\vec{x}^{(i)}$ and ω_1 .

Iteration number (i)	$\vec{x}_{i+1} = [D] \vec{x}_i$ with $x_{1,i} = 1$ (given as row vector)			
0	0.10000000E+01	0.10000000E+01	0.10000000E+01	0.10000000E+01
1	0.10000000E+01	0.15714285E+01	0.21428571E+01	0.24285715E+01
2	0.10000000E+01	0.17200000E+01	0.25733335E+01	0.30266669E+01
3	0.10000000E+01	0.17508308E+01	0.26810634E+01	0.31838319E+01
4	0.10000000E+01	0.17574103E+01	0.27059193E+01	0.32208292E+01
5	0.10000000E+01	0.17588729E+01	0.27116063E+01	0.32293591E+01
6	0.10000000E+01	0.17592046E+01	0.27129092E+01	0.32313194E+01
7	0.10000000E+01	0.17592804E+01	0.27132087E+01	0.32317696E+01
8	0.10000000E+01	0.17592978E+01	0.27132771E+01	0.32318730E+01
9	0.10000000E+01	0.17593019E+01	0.27132931E+01	0.32318966E+01

Converged frequency = 0.40058133 = $\omega_1 \sqrt{m/k}$

7.24 The problem-dependent data for Program 9.F and output are given.

C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA

DIMENSION D(3,3),E(3,3)

DATA N,ITMAX,EPS/3,200,1.0E-05/

DATA D/3.0,-2.0,0.0,-2.0,5.0,-3.0,0.0,-3.0,3.0/

C END OF PROBLEM-DEPENDENT DATA

EIGENVALUE SOLUTION BY JACOBI METHOD

GIVEN MATRIX

0.300000E+01	-0.200000E+01	0.000000E+00
-0.200000E+01	0.500000E+01	-0.300000E+01
0.000000E+00	-0.300000E+01	0.300000E+01

EIGEN VALUES ARE

0.774166E+01	0.300000E+01	0.258343E+00
--------------	--------------	--------------

EIGEN VECTORS

FIRST	SECOND	THIRD
0.335734E+00	0.832048E+00	0.441564E+00
-0.796032E+00	-0.343903E-05	0.605254E+00
0.503602E+00	-0.554704E+00	0.662336E+00

7.25

The problem-dependent data for Program 9-F and output are given.

```

C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
  DIMENSION D(3,3),E(3,3)
  DATA N,ITMAX,EPS/3,200,1.0E-05/
  DATA D/3.0,2.0,1.0,2.0,2.0,1.0,1.0,1.0,1.0/
C END OF PROBLEM-DEPENDENT DATA

```

EIGENVALUE SOLUTION BY JACOBI METHOD

GIVEN MATRIX

```

0.300000E+01  0.200000E+01  0.100000E+01
0.200000E+01  0.200000E+01  0.100000E+01
0.100000E+01  0.100000E+01  0.100000E+01

```

EIGEN VALUES ARE

```

0.504892E+01  0.643104E+00  0.307979E+00

```

EIGEN VECTORS

```

      FIRST      SECOND      THIRD
0.736978E+00 -0.591004E+00  0.327991E+00
0.591007E+00  0.327977E+00 -0.736982E+00
0.327985E+00  0.736984E+00  0.590999E+00

```

7.26

The main program which calls DECOMP and results are given.

```

C =====
C
C PROGRAM
C MAIN PROGRAM WHICH CALLS DECOMP
C
C =====
  DIMENSION A(4,4),U(4,4)
  DATA A/4.0,-2.0,6.0,4.0,-2.0,2.0,-1.0,3.0,6.0,-1.0,22.0,13.0,
2  4.0,3.0,13.0,46.0/
  N=4
  CALL DECOMP (A,U,N)
  WRITE (17,10)
10  FORMAT (/,25H UPPER TRIANGULAR MATRIX:,:)
  DO 30 I=1,N
  WRITE (17,20) (U(I,J),J=1,N)
20  FORMAT (3E15.8)
30  CONTINUE
  STOP
  END
-----
UPPER TRIANGULAR MATRIX:

0.20000000E+01-0.10000000E+01 0.30000000E+01
0.20000000E+01
0.00000000E+00 0.10000000E+01 0.20000000E+01
0.00000000E+00
0.00000000E+00 0.00000000E+00 0.30000000E+01
0.23333333E+01
0.00000000E+00 0.00000000E+00 0.00000000E+00
0.60461192E+01

```

7.27 From Eq. (7.84), $u_{11} = \sqrt{5} = 2.236068$, $u_{12} = \frac{a_{12}}{u_{11}} = \frac{-1}{u_{11}} = -0.44721359$,

$$u_{13} = \frac{a_{13}}{u_{11}} = \frac{1}{u_{11}} = 0.44721359$$

$$u_{22} = (a_{22} - u_{12}^2)^{1/2} = 2.408319, \quad u_{23} = \frac{1}{u_{22}} (a_{23} - u_{12} u_{13}) = -1.5778641$$

$$u_{33} = (a_{33} - u_{13}^2 - u_{23}^2)^{1/2} = 0.55708611$$

$$[U] = \begin{bmatrix} 2.236068 & -0.44721359 & 0.44721359 \\ 0 & 2.408319 & -1.5778641 \\ 0 & 0 & 0.55708611 \end{bmatrix}$$

$$\alpha_{11} = \frac{1}{u_{11}} = 0.44721359$$

$$\alpha_{12} = -\frac{1}{u_{11}} (u_{12} \alpha_{22}) = 0.083045475$$

$$\alpha_{22} = \frac{1}{u_{22}} = 0.41522738$$

$$\alpha_{23} = -\frac{1}{u_{22}} (u_{23} \alpha_{33}) = 1.1760702$$

$$\alpha_{33} = \frac{1}{u_{33}} = 1.7950547$$

$$\alpha_{13} = -\frac{1}{u_{11}} (u_{12} \alpha_{23} + u_{13} \alpha_{33}) = -0.12379687$$

$$[U]^{-1} = [\alpha_{ij}] = \begin{bmatrix} 0.44721359 & 0.083045475 & -0.12379687 \\ 0 & 0.41522738 & 1.1760702 \\ 0 & 0 & 1.7950547 \end{bmatrix}$$

7.28 From Eqs. (7.84) and (7.86),

$$u_{11} = \sqrt{2} = 1.4142135, \quad u_{12} = \frac{a_{12}}{u_{11}} = \frac{5}{u_{11}} = 3.5355339, \quad u_{13} = \frac{a_{13}}{u_{11}} = 5.6568542$$

$$u_{22} = (a_{22} - u_{12}^2)^{1/2} = 1.8708287, \quad u_{23} = \frac{1}{u_{22}} (a_{23} - u_{12} u_{13}) = 4.2761798$$

$$u_{33} = (a_{33} - u_{13}^2 - u_{23}^2)^{1/2} = 1.9272485$$

$$[U] = \begin{bmatrix} 1.4142135 & 3.5355339 & 5.6568542 \\ 0 & 1.8708287 & 4.2761798 \\ 0 & 0 & 1.9272485 \end{bmatrix}$$

$$\alpha_{11} = \frac{1}{u_{11}} = 0.70710677$$

$$\alpha_{12} = -\frac{1}{u_{11}} (u_{12} \alpha_{22}) = -1.3363062$$

$$\alpha_{22} = \frac{1}{u_{22}} = 0.53452247$$

$$\alpha_{23} = -\frac{1}{u_{22}} (u_{23} \alpha_{33}) = -1.1859988$$

$$\alpha_{33} = \frac{1}{u_{33}} = 0.51887447$$

$$\alpha_{13} = -\frac{1}{u_{11}} (u_{12} \alpha_{23} + u_{13} \alpha_{33}) = 0.88949931$$

$$[U]^{-1} = [\alpha_{ij}] = \begin{bmatrix} 0.70710677 & -1.3363062 & 0.88949931 \\ 0 & 0.53452247 & -1.1859988 \\ 0 & 0 & 0.51887447 \end{bmatrix}$$

7.29 The problem-dependent data to be used in Program 9.F and results are given.

~~C~~ FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA

DIMENSION D(4,4), E(4,4)

DATA N, ITMAX, EPS/4, 200, 1.0E-05/

DATA D/4., -2., 6., 4., -2., 2., -1., 3., 6., -1., 22., 13., 4., 3., 13., 46./

C END OF PROBLEM-DEPENDENT DATA

EIGENVALUE SOLUTION BY JACOBI METHOD

GIVEN MATRIX

```

0.400000E+01  -0.200000E+01  0.600000E+01
0.400000E+01
-0.200000E+01  0.200000E+01  -0.100000E+01
0.300000E+01
0.600000E+01  -0.100000E+01  0.220000E+02
0.130000E+02
0.400000E+01  0.300000E+01  0.130000E+02
0.460000E+02

```

EIGEN VALUES ARE

```

0.525424E+02  0.178109E+02  0.346931E+01  0.177413E+00

```

EIGEN VECTORS

```

FIRST          SECOND          THIRD
0.123182E+00  -0.274316E+00  0.718788E+00
0.626834E+00  0.407039E-01  0.167114E+00
-0.614583E+00  0.769873E+00  0.407591E+00
-0.851753E+00  -0.316805E+00  -0.895647E-01
0.903902E+00  0.413933E+00  0.725754E-01
-0.797051E-01

```

$$7.30 \quad [k] = k \begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 3 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad [m] = m \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Eq. (7.84) gives, for $[A] = [k]$,

$$[U] = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 1.41421 & -0.707107 & 0 \\ 0 & 0 & 1.22474 & -0.816497 \\ 0 & 0 & 0 & 0.577350 \end{bmatrix} \sqrt{k}$$

Eq. (7.86) gives

$$[U]^{-1} = \begin{bmatrix} 0.5 & 0.35355338 & 0.20412414 & 0.28867510 \\ 0 & 0.70710677 & 0.40824828 & 0.57735020 \\ 0 & 0 & 0.81649655 & 1.1547004 \\ 0 & 0 & 0 & 1.7320507 \end{bmatrix} \frac{1}{\sqrt{k}}$$

standard eigenproblem is $[D] \vec{Y} = \lambda \vec{Y}$

where

$$[D] = ([U]^{-1})^T [m] [U]^{-1}$$

$$= \frac{m}{k} \begin{bmatrix} 0.75 & 0.53033006 & 0.30618620 & 0.43301266 \\ 0.53033006 & 1.3749999 & 0.79385662 & 1.1226827 \\ 0.30618620 & 0.79385662 & 1.125 & 1.5909899 \\ 0.43301266 & 1.1226827 & 1.5909899 & 5.2499990 \end{bmatrix}$$

$$\lambda = \frac{1}{\omega^2}$$

$$\text{and } \vec{Y} = [U] \vec{X}$$

7.31 From Eq. (7.84),

$$u_{11} = \sqrt{a_{11}} = \sqrt{16} = 4, \quad u_{12} = \frac{a_{12}}{u_{11}} = -\frac{20}{4} = -5, \quad u_{13} = \frac{a_{13}}{u_{11}} = -\frac{24}{4} = -6$$

$$u_{22} = (a_{22} - u_{12}^2)^{1/2} = (89 - 25)^{1/2} = 8$$

$$u_{23} = \frac{1}{u_{22}} (a_{23} - u_{12} u_{13}) = \frac{1}{8} (-50 - (-5)(-6)) = -10$$

$$u_{33} = (a_{33} - u_{13}^2 - u_{23}^2)^{1/2} = (280 - 36 - 100)^{1/2} = 12$$

$$[A] = [U]^T [U]$$

with $[U] = \begin{bmatrix} 4 & -5 & -6 \\ 0 & 8 & -10 \\ 0 & 0 & 12 \end{bmatrix}$

7.32 From Example 7.3, $\omega = 0.028222 \sqrt{EI}$

Since $E = 2.07 \times 10^{11} \text{ N/m}^2$, $\omega = 12840.2346 \sqrt{I} \text{ rad/sec}$

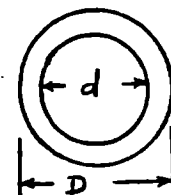
In order to have $\omega = 0.5 \text{ Hz} = 3.1416 \text{ rad/sec}$,

$$12840.2346 \sqrt{I} = 3.1416 \Rightarrow I = 59862.43022 \times 10^{-12} \text{ m}^4$$

For a tubular section,

$$I = \frac{\pi}{64} (D^4 - d^4) = 59862.43022 \times 10^{-12}$$

$$\text{i.e. } D^4 - d^4 = 1.219504 \times 10^{-6} \text{ m}^4$$



To minimize weight, we need to minimize $(D^2 - d^2)$.

Problem is: Find D and d to minimize $(D^2 - d^2)$
 subject to $D^4 - d^4 = 1.219504 \times 10^{-6}$

or Find d and $r = D/d$

to minimize $f = d^2(r^2 - 1)$ (E1)

subject to $d^4(r^4 - 1) = 121.9504 \times 10^{-8}$ (E2)

Eq. (E2) gives $d^2 = \frac{11.0431 \times 10^{-4}}{\sqrt{r^4 - 1}}$

Eq. (E1) becomes $f = \frac{11.0431 \times 10^{-4} (r^2 - 1)}{\sqrt{r^4 - 1}}$

For minimum of f , $\frac{df}{dr} = \frac{d}{dr} \left(\frac{r^2 - 1}{\sqrt{r^4 - 1}} \right) = 0$

i.e., $\frac{(r^2 - 1) \cdot \frac{1}{2} (r^4 - 1)^{-\frac{1}{2}} (4r^3) - \sqrt{r^4 - 1} (2r)}{r^4 - 1} = 0$

$$\text{i.e., } r^2(r^2-1) = (r^2+1)(r^2-1)$$

$$\text{i.e., } r^2=1 \text{ or } r^2=r^2+1$$

i.e., $r=1$ is the only feasible solution.

i.e., A solid circular section.

$$\text{Since } I = \frac{\pi D^4}{64} = 0.059862 \times 10^{-6}, D = 0.03323 \text{ m}$$

7.33

% Ex 7_33.m

```
>> A = [3 -2 0; -2 5 -3; 0 -1 1]
```

A =

```

3    -2    0
-2    5   -3
0    -1    1
```

```
>> [V, D] = eig(A)
```

V =

```

-0.4765  -0.8962  0.4154
 0.8656  -0.3444  0.5950
-0.1537   0.2797  0.6880
```

D =

```

6.6334    0    0
0    2.2315    0
0    0    0.1351
```

7.34

% Ex 7_34.m

```
>> A = [-5 2 1; 1 -9 -1; 2 -1 7]
```

A =

```

-5    2    1
 1   -9   -1
 2   -1    7
```

```
>> [V, D] = eig(A)
```

V =

```

-0.0723  -0.9572  0.4172
 0.0570  -0.2514  -0.9027
-0.9958   0.1431  -0.1048
```

D =

```

7.2024    0    0
0   -4.6241    0
0    0   -9.5783
```

7.35

% Results of Ex7_35

>> program9

Eigenvalue solution by Jacobi Method

Given matrix

2.50000000e+000	-1.00000000e+000	0.00000000e+000
-1.00000000e+000	5.00000000e+000	-1.41421356e+000
0.00000000e+000	-1.41421356e+000	1.00000000e+001

Eigen values are

1.03806779e+001	5.00000000e+000	2.11932209e+000
-----------------	-----------------	-----------------

Eigen vectors are

First	Second	Third
3.29649826e-002	3.59210604e-001	9.32674140e-001
-2.59786425e-001	-8.98026512e-001	3.55048443e-001
9.65103271e-001	-2.54000247e-001	6.37146104e-002

7.36

% Results of Ex7_36

>> program10

Solution of eigenvalue problem by
matrix iteration method

Natural frequencies:

4.450417e-001	1.246977e+000	1.801938e+000
---------------	---------------	---------------

Mode shapes (Columnwise):

1.000000e+000	1.000000e+000	1.000000e+000
8.019379e-001	-5.549503e-001	-2.246941e+000
4.450421e-001	-1.246987e+000	1.801867e+000

7.37

% Results of Ex7_37

>> program11

Upper triangular matrix [U]:

2.000000e+000	-1.000000e+000	0.000000e+000
0.000000e+000	1.414214e+000	-7.071068e-001
0.000000e+000	0.000000e+000	1.224745e+000
0.000000e+000	0.000000e+000	0.000000e+000

Inverse of the upper triangular matrix:

5.000000e-001	3.535534e-001	2.041241e-001
0.000000e+000	7.071068e-001	4.082483e-001
0.000000e+000	0.000000e+000	8.164966e-001
0.000000e+000	0.000000e+000	0.000000e+000

Matrix [UMU]=[UTI][M][UI]:

7.500000e-001	5.303301e-001	3.061862e-001
5.303301e-001	1.375000e+000	7.938566e-001
3.061862e-001	7.938566e-001	1.125000e+000
4.330127e-001	1.122683e+000	1.590990e+000

Eigenvalues:

6.231904e+000	1.431905e+000	5.000000e-001	3.361911e-001
---------------	---------------	---------------	---------------

Eigenvectors(Columnwise):

4.804506e-001	-4.370337e-001	2.672613e-001	-8.209847e-002
8.452583e-001	-4.162514e-001	-2.672612e-001	2.021007e-001
1.303605e+000	2.067115e-001	-2.672612e-001	-4.318039e-001
1.552770e+000	6.853100e-001	2.672613e-001	2.186930e-001

7.38

Results of Ex7_38

Please input N:
3

Please input matrix D row by row:
2 2 2
2 5 5
2 5 12

EIGENVALUE SOLUTION BY JACOBI METHOD

GIVEN MATRIX
2.000000 2.000000 2.000000
2.000000 5.000000 5.000000
2.000000 5.000000 12.000000

EIGEN VALUES ARE
15.15868265 2.87890731 0.96241005

EIGEN VECTORS
FIRST SECOND THIRD
0.20164232 -0.49011982 0.84801117
0.46419929 -0.71456420 -0.52337082
0.86247284 0.49917989 0.08342689

7.39

Results of Ex7_39

Please input N and NVEC:
3 3

Please input matrix D row by row
2 2 2
2 5 5
2 5 12

Please input matrix XM row by row
1 0 0
0 1 0
0 0 1

SOLUTION OF EIGENVALUE PROBLEM BY
MATRIX ITERATION METHOD

NATURAL FREQUENCIES

0.256844 0.589369 1.019342

MODE SHAPES (COLUMNWISE):

1.000000 1.000000 1.000000
2.302093 1.457934 -0.617178
4.277246 -1.018484 0.098382

7.40

Results of Ex7_40

Please input ND:

3

Please input BK matrix row by row:

10 -4 0

-4 6 -2

0 -2 2

Please input BM matrix row by row:

3 0 0

0 2 0

0 0 1

UPPER TRIANGULAR MATRIX [U]:

3.16227766	-1.26491106	0.00000000
0.00000000	2.09761770	-0.95346259
0.00000000	0.00000000	1.04446594

INVERSE OF THE UPPER TRIANGULAR MATRIX, [UI],

0.31622777	0.19069252	0.17407766
0.00000000	0.47673129	0.43519414
0.00000000	0.00000000	0.95742711

MATRIX [UMU] = [UTI][M][UI]:

0.30000000	0.18090681	0.16514456
0.18090681	0.56363636	0.51452725
0.16514456	0.51452725	1.38636364

EIGENVALUES:

1.67156962	0.38337151	0.19505887
------------	------------	------------

EIGENVECTORS (COLUMNWISE):

0.28923075	-0.23770397	0.16281604
0.59330466	-0.12923308	-0.21898643
0.84651373	0.42480537	0.14007703

7.41

```

C =====
C
C PROGRAM 9.F
C MAIN PROGRAM FOR CALLING JACOBI
C
C =====
C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
  DIMENSION D(3,3),E(3,3)
  DATA N,ITMAX,EPS/3,200,1.0E-05/
  DATA D/2.,2.,2.,2.,5.,5.,2.,5.,12./
C END OF PROBLEM-DEPENDENT DATA
  PRINT 50
50  FORMAT (//,37H EIGENVALUE SOLUTION BY JACOBI METHOD)
  PRINT 40
40  FORMAT (/,13H GIVEN MATRIX)
  DO 30 I=1,N
30  PRINT 20,(D(I,J),J=1,N)
20  FORMAT (3E15.6)
  CALL JACOBI (D,N,E,EPS,ITMAX)
  PRINT 10, (D(I,I),I=1,N), ((E(I,J),J=1,N),I=1,N)
10  FORMAT (/,1X,17H EIGEN VALUES ARE,/,3E15.6,/,1X,
2    14H EIGEN VECTORS,/,6X,5HFIRST,10X,6HSECOND,9X,5HTHIRD,/,
3    (1X,3E15.6))
  STOP
  END

```

```

C =====
C
C SUBROUTINE JACOBI
C
C =====
      SUBROUTINE JACOBI (D,N,E,EPS,ITMAX)
      DIMENSION D(N,N),E(N,N)
      ITER=0
      DO 110 I=1,N
      DO 110 J=1,N
      E(I,J)=0.0
110   E(I,I)=1.0
120   ZZ=0.0
      NM1=N-1
      DO 130 I=1,NM1
      IP1=I+1
      DO 130 J=IP1,N
      IF (ABS(D(I,J)) .LE. ZZ) GO TO 130
      ZZ=ABS(D(I,J))
      IR=I
      IC=J
130   CONTINUE
      IF (ITER .EQ. 0) YY=ZZ*EPS
      IF (ZZ .LE. YY) GO TO 210
      DIF=D(IR,IR) - D(IC,IC)
      TANZ=(-DIF+SQRT(DIF**2+4.0*ZZ**2))/(2.0*D(IR,IC))
      COSZ=1.0/SQRT(1.0+TANZ**2)
      SINZ=COSZ*TANZ
      DO 140 I=1,N
      ZZZ=E(I,IR)
      E(I,IR)=COSZ*ZZZ+SINZ*E(I,IC)
140   E(I,IC)=COSZ*E(I,IC)-SINZ*ZZZ
      I=1
150   IF (I .EQ. IR) GO TO 160
      YYY=D(I,IR)
      D(I,IR)=COSZ*YYY+SINZ*D(I,IC)
      D(I,IC)=COSZ*D(I,IC)-SINZ*YYY
      I=I+1
      GO TO 150
160   I=IR+1
170   IF (I .EQ. IC) GO TO 180
      YYY=D(IR,I)
      D(IR,I)=COSZ*YYY+SINZ*D(I,IC)
      D(I,IC)=COSZ*D(I,IC)-SINZ*YYY
      I=I+1
      GO TO 170
180   I=IC+1
190   IF (I .GT. N) GO TO 200
      ZZZ=D(IR,I)
      D(IR,I)=COSZ*ZZZ+SINZ*D(IC,I)
      D(IC,I)=COSZ*D(IC,I)-SINZ*ZZZ
      I=I+1
      GO TO 190
200   YYY=D(IR,IR)
      D(IR,IR)=YYY*COSZ**2+D(IR,IC)*2.0*COSZ*SINZ+D(IC,IC)
      2 *SINZ**2
      D(IC,IC)=D(IC,IC)*COSZ**2+YYY*SINZ**2-D(IR,IC)*2.0*COSZ*SINZ
      D(IR,IC)=0.0
      ITER=ITER+1
      IF (ITER .LT. ITMAX) GO TO 120
210   RETURN
      END

```

EIGENVALUE SOLUTION BY JACOBI METHOD

GIVEN MATRIX

0.200000E+01	0.200000E+01	0.200000E+01
0.200000E+01	0.500000E+01	0.500000E+01
0.200000E+01	0.500000E+01	0.120000E+02

EIGEN VALUES ARE

0.151587E+02 0.287891E+01 0.962410E+00

EIGEN VECTORS

FIRST	SECOND	THIRD
0.201639E+00	-0.490153E+00	0.847993E+00
0.464200E+00	-0.714545E+00	-0.523397E+00
0.862474E+00	0.499175E+00	0.834492E-01

7.42

```

C =====
C
C PROGRAM 10.F
C MAIN PROGRAM FOR CALLING THE SUBROUTINE MITER
C
C =====
C FOLLOWING 8 LINES CONTAIN PROBLEM-DEPENDENT DATA
  DIMENSION D(3,3),X(3),XS(3),B(3,3),C(3,3),XX(3),XM(3,3),FREQ(3),
2  EIG(3,3)
  N=3
  NVEC=3
  DATA D/2.,2.,2.,2.,5.,5.,2.,5.,12./
  DATA XM/1.0,0.0,0.0,0.0,1.0,0.0,0.0,0.0,1.0/
  EPS=0.00001
  DATA XS/1.0,1.0,1.0/
C END OF PROBLEM-DEPENDENT DATA
  CALL MITER (D,X,XS,N,NVEC,B,C,XX,XM,EPS,FREQ,EIG)
  PRINT 10
10  FORMAT (//,34H SOLUTION OF EIGENVALUE PROBLEM BY,/,
2    24H MATRIX ITERATION METHOD)
  PRINT 20,(FREQ(I),I=1,NVEC)
20  FORMAT (//,20H NATURAL FREQUENCIES,//,3(E15.8,1X))
  PRINT 30
30  FORMAT (//,26H MODE SHAPES (COLUMNWISE):,/)
  DO 40 I=1,NVEC
40  PRINT 50,(EIG(I,J),J=1,N)
50  FORMAT (3(E15.8,1X),/)
  STOP
  END
C=====
C
C SUBROUTINE MITER
C
C=====
  SUBROUTINE MITER (D,X,XS,N,NVEC,B,C,XX,XM,EPS,FREQ,EIG)
  DIMENSION D(N,N),X(N),XS(N),B(N,N),C(N,N),XX(N),XM(N,N),
2  FREQ(NVEC),EIG(N,NVEC)
  CON=XS(1)
  DO 10 I=1,N
10  X(I)=XS(I)/CON
50  ICON=0
  CALL MULT(D,X,N,XX)
  ALAM=XX(1)
  DO 20 I=1,N
20  XX(I)=XX(I)/ALAM
  DO 30 I=1,N
30  IF(ABS((XX(I)-X(I))/X(I)) .GT. EPS) ICON=1
  DO 40 I=1,N
40  X(I)=XX(I)
  IF (ICON .EQ. 0) GO TO 60
  GO TO 50
60  ICON=0
  FREQ(1)=SQRT(1.0/ALAM)
  DO 70 I=1,N
70  EIG(I,1)=X(I)
  II=1
100 II=II+1
  SUM=0.0
  DO 110 I=1,N
110 SUM=SUM+X(I)**2

```

```

      ALP=SQRT(1.0/SUM)
      DO 120 I=1,N
120     X(I)=X(I)*ALP
      DO 130 I=1,N
      DO 130 J=1,N
130     C(I,J)=X(I)*X(J)
      CALL MATMUL (B,C,XM,N,N,N)
      DO 140 I=1,N
      DO 140 J=1,N
140     D(I,J)=D(I,J)-ALAM*B(I,J)
      CON=XS(1)
      DO 150 I=1,N
150     X(I)=XS(I)/CON
250     ICON=0
      CALL MULT (D,X,N,XX)
      ALAM=XX(1)
      DO 220 I=1,N
220     XX(I)=XX(I)/ALAM
      DO 230 I=1,N
230     IF (ABS((XX(I)-X(I))/X(I)) .GT. EPS) ICON=1
      DO 240 I=1,N
240     X(I)=XX(I)
      IF (ICON .EQ. 0) GO TO 260
      GO TO 250
260     ICON=0
      FREQ(II)=SQRT(1.0/ALAM)
      DO 270 I=1,N
270     EIG(I,II)=X(I)
      IF (II .EQ. NVEC) GO TO 300
      GO TO 100
300     RETURN
      END
C =====
C
C SUBROUTINE MULT
C
C =====
      SUBROUTINE MULT (D,X,N,XX)
      DIMENSION D(N,N),X(N),XX(N)
      DO 10 I=1,N
      XX(I)=0.0
      DO 20 J=1,N
20     XX(I)=XX(I)+D(I,J)*X(J)
10     CONTINUE
      RETURN
      END
C =====
C
C SUBROUTINE MATMUL
C
C =====
C MATRIX MULTIPLICATION SUBROUTINE: A = B * C
C B(L,M) AND C(M,N) ARE INPUT MATRICES, A(L,N) IS OUTPUT MATRIX
      SUBROUTINE MATMUL (A,B,C,L,M,N)
      DIMENSION A(L,N),B(L,M),C(M,N)
      DO 10 I=1,L
      DO 10 J=1,N
      A(I,J)=0.0
      DO 10 K=1,M
10     A(I,J)=A(I,J)+B(I,K)*C(K,J)
      RETURN
      END

```

SOLUTION OF EIGENVALUE PROBLEM BY
MATRIX ITERATION METHOD

NATURAL FREQUENCIES

0.25684494E+00 0.58936858E+00 0.10193421E+01

MODE SHAPES (COLUMNWISE):

```
0.10000000E+01  0.10000000E+01  0.10000000E+01
0.23020921E+01  0.14579326E+01 -0.61717808E+00
0.42772374E+01 -0.10184917E+01  0.98382086E-01
```

7.43

```
C =====
C
C PROGRAM 11.F
C DETERMINATION OF EIGENVALUES AND EIGENVECTORS BY FINDING THE MATRIX
C [D] ACCORDING TO THE RELATION [D] = [UTI] [M] [UI]
C
C =====
C FOLLOWING 5 LINES CONTAIN PROBLEM-DEPENDENT DATA
  DIMENSION BK(3,3),BM(3,3),U(3,3),UI(3,3),UTI(3,3),BMU(3,3),
  2  UMU(3,3),XF(3,3),EV(3,3)
  DATA BK/10.,-4.,0.,-4.,6.,-2.,0.,-2.,2./
  DATA BM/3.,0.,0.,0.,2.,0.,0.,0.,1./
  ND=3
C END OF PROBLEM-DEPENDENT DATA
C DECOMPOSING THE MATRIX [BK] INTO TRIANGULAR MATRICES
  CALL DECOMP (BK,U,ND)
  PRINT 100
  FORMAT (//,29H UPPER TRIANGULAR MATRIX [U]:,/)
  DO 110 I=1,ND
  110  PRINT 120,(U(I,J),J=1,ND)
  120  FORMAT (4(1X,E15.6))
  DO 130 I=1,ND
  DO 130 J=1,ND
  130  UI(I,J)=0.0
  DO 140 I=1,ND
  140  UI(I,I)=1.0/U(I,I)
  DO 160 J=1,ND
  DO 160 II=1,ND
  I=ND-II+1
  IF (I .GE. J) GO TO 160
  IP=I+1
  SUM=0.0
  DO 150 K=IP,J
  150  SUM =SUM+U(I,K)*UI(K,J)
  UI(I,J)=-SUM/U(I,I)
  160  CONTINUE
  PRINT 170
  FORMAT (/,46H INVERSE OF THE UPPER TRIANGULAR MATRIX, [UI],/)
  DO 180 I=1,ND
  180  PRINT 120,(UI(I,J),J=1,ND)
  DO 190 I=1,ND
  DO 190 J=1,ND
  190  UTI(I,J)=UI(J,I)
  CALL MATMUL (BMU,BM,UI,ND,ND,ND)
  CALL MATMUL (UMU,UTI,BMU,ND,ND,ND)
  PRINT 200
  FORMAT (/,29H MATRIX [UMU] = [UTI] [M] [UI]:,/)
  DO 210 I=1,ND
  210  PRINT 120,(UMU(I,J),J=1,ND)
  CALL JACOBI (UMU,ND,EV,1.0E-05,200)
  PRINT 230
  230  FORMAT (/,13H EIGENVALUES: ,/)
  PRINT 120,(UMU(I,I),I=1,ND)
  PRINT 240
  240  FORMAT (/,27H EIGENVECTORS (COLUMNWISE):,/)
  CALL MATMUL (XF,UI,EV,ND,ND,ND)
  DO 250 I=1,ND
  250  PRINT 120,(XF(I,J),J=1,ND)
  STOP
  END
```

```

C =====
C
C SUBROUTINE DECOMP
C =====
      SUBROUTINE DECOMP (A,U,N)
      DIMENSION A(N,N),U(N,N)
      DO 10 I=1,N
      DO 10 J=1,N
10    U(I,J)=0.0
      U(1,1)=SQRT(A(1,1))
      DO 90 J=2,N
90    U(1,J)=A(1,J)/U(1,1)
      DO 40 I=2,N
      IM=I-1
      SUM=0.0
      DO 30 K=1,IM
30    SUM=SUM+U(K,I)**2
      U(I,I)=SQRT(A(I,I)-SUM)
      J=I+1
      SUM=0.0
      DO 50 K=1,IM
50    SUM=SUM+U(K,I)*U(K,J)
      U(I,J)=(A(I,J)-SUM)/U(I,I)
40    CONTINUE
      RETURN
      END

```

UPPER TRIANGULAR MATRIX [U]:

0.316228E+01	-0.126491E+01	0.000000E+00
0.000000E+00	0.209762E+01	-0.953463E+00
0.000000E+00	0.000000E+00	0.104447E+01

INVERSE OF THE UPPER TRIANGULAR MATRIX, [UI],

0.316228E+00	0.190693E+00	0.174078E+00
0.000000E+00	0.476731E+00	0.435194E+00
0.000000E+00	0.000000E+00	0.957427E+00

MATRIX [UMU] = [UTI] [M] [UI]:

0.300000E+00	0.180907E+00	0.165145E+00
0.180907E+00	0.563636E+00	0.514527E+00
0.165145E+00	0.514527E+00	0.138636E+01

EIGENVALUES:

0.167157E+01	0.383372E+00	0.195059E+00
--------------	--------------	--------------

EIGENVECTORS (COLUMNWISE):

0.289231E+00	-0.237705E+00	0.162815E+00
0.593305E+00	-0.129231E+00	-0.218987E+00
0.846513E+00	0.424805E+00	0.140081E+00

7.44

```

C =====
C
C PROGRAM 9.F
C MAIN PROGRAM FOR CALLING JACOBI
C
C =====
C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION D(3,3),E(3,3)
      DATA N,ITMAX,EPS/3,200,1.0E-05/
      DATA D/5.,-1.,1.,-1.,6.,-4.,1.,-4.,3./
C END OF PROBLEM-DEPENDENT DATA
      PRINT 50
50    FORMAT (//,37H EIGENVALUE SOLUTION BY JACOBI METHOD)
      PRINT 40

```

```

40  FORMAT (/,13H GIVEN MATRIX)
    DO 30 I=1,N
30  PRINT 20, (D(I,J),J=1,N)
20  FORMAT (3E15.6)
    CALL JACOBI (D,N,E,EPS,ITMAX)
    PRINT 10, (D(I,I),I=1,N), ((E(I,J),J=1,N),I=1,N)
10  FORMAT (/,1X,17H EIGEN VALUES ARE,/,3E15.6,/,1X,
      2  14H EIGEN VECTORS,/,6X,5HFIRST,10X,6HSECOND,9X,5HTHIRD,/,
      3  (1X,3E15.6))
    STOP
    END

```

EIGENVALUE SOLUTION BY JACOBI METHOD

GIVEN MATRIX

0.500000E+01	-0.100000E+01	0.100000E+01
-0.100000E+01	0.600000E+01	-0.400000E+01
0.100000E+01	-0.400000E+01	0.300000E+01

EIGEN VALUES ARE

0.923048E+01	0.455549E+01	0.214034E+00
--------------	--------------	--------------

EIGEN VECTORS

FIRST	SECOND	THIRD
0.312915E+00	0.948172E+00	-0.552614E-01
-0.775620E+00	0.288684E+00	0.561315E+00
0.548176E+00	-0.132782E+00	0.825755E+00

7.45

```

C =====
C
C PROGRAM 9.F
C MAIN PROGRAM FOR CALLING JACOBI
C
C =====
C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
  DIMENSION D(3,3),E(3,3)
  DATA N,ITMAX,EPS/3,200,1.0E-05/
  DATA D/16.,-20.,-24.,-20.,89.,-50.,-24.,-50.,280./
C END OF PROBLEM-DEPENDENT DATA
  PRINT 50
50  FORMAT (//,37H EIGENVALUE SOLUTION BY JACOBI METHOD)
  PRINT 40
40  FORMAT (/,13H GIVEN MATRIX)
    DO 30 I=1,N
30  PRINT 20, (D(I,J),J=1,N)
20  FORMAT (3E15.6)
    CALL JACOBI (D,N,E,EPS,ITMAX)
    PRINT 10, (D(I,I),I=1,N), ((E(I,J),J=1,N),I=1,N)
10  FORMAT (/,1X,17H EIGEN VALUES ARE,/,3E15.6,/,1X,
      2  14H EIGEN VECTORS,/,6X,5HFIRST,10X,6HSECOND,9X,5HTHIRD,/,
      3  (1X,3E15.6))
    STOP
    END

```

EIGENVALUE SOLUTION BY JACOBI METHOD

GIVEN MATRIX

0.160000E+02	-0.200000E+02	-0.240000E+02
-0.200000E+02	0.890000E+02	-0.500000E+02
-0.240000E+02	-0.500000E+02	0.280000E+03

EIGEN VALUES ARE

0.293547E+03	0.855832E+02	0.586943E+01
--------------	--------------	--------------

EIGEN VECTORS

FIRST	SECOND	THIRD
0.673507E-01	-0.332766E+00	0.940601E+00
0.230709E+00	0.922386E+00	0.309802E+00
-0.970689E+00	0.196140E+00	0.138895E+00

7.46 The problem-dependent data and output are given (Program 11.F):

C FOLLOWING 5 LINES CONTAIN PROBLEM-DEPENDENT DATA

DIMENSION BK(3,3),BM(3,3),U(3,3),UI(3,3),UT1(3,3),BMU(3,3),

~~2~~ UMU(3,3),XF(3,3),EV(3,3)

DATA BK/2.,-1.,0.,-1.,2.,-1.,0.,-1.,3./

DATA BM/1.,0.,0.,0.,1.,0.,0.,0.,2./

ND=3

C END OF PROBLEM-DEPENDENT DATA

UPPER TRIANGULAR MATRIX [U]:

0.141421E+01 -0.707107E+00 0.000000E+00

0.000000E+00 0.122474E+01 -0.816497E+00

~~0.000000E+00 0.000000E+00 0.152753E+01~~

INVERSE OF THE UPPER TRIANGULAR MATRIX, [UI],

0.707107E+00 0.408248E+00 0.218218E+00

0.000000E+00 0.816497E+00 0.436436E+00

~~0.000000E+00 0.000000E+00 0.654654E+00~~

MATRIX [UMU] = [UT1][M][UI]:

0.500000E+00 0.288675E+00 0.154303E+00

0.288675E+00 0.833333E+00 0.445435E+00

~~0.154303E+00 0.445435E+00 0.109524E+01~~

EIGENVALUES:

0.151668E+01 0.595606E+00 0.316285E+00

EIGENVECTORS (COLUMNWISE):

0.610531E+00 -0.468617E+00 0.349193E+00

~~0.818509E+00 -0.150438E+00 -0.405654E+00~~

0.486814E+00 0.420340E+00 0.122055E+00

7.47 The problem-dependent data for Program 11.F and output are given.

C FOLLOWING 5 LINES CONTAIN PROBLEM-DEPENDENT DATA

DIMENSION BK(3,3),BM(3,3),U(3,3),UI(3,3),UT1(3,3),BMU(3,3),

~~2~~ UMU(3,3),XF(3,3),EV(3,3)

DATA BK/2.,-1.,0.,-1.,3.,-2.,0.,-2.,2./

DATA BM/1.,0.,0.,0.,1.,0.,0.,0.,2./

ND=3

C END OF PROBLEM-DEPENDENT DATA

UPPER TRIANGULAR MATRIX [U]:

0.141421E+01 -0.707107E+00 0.000000E+00

0.000000E+00 0.158114E+01 -0.126491E+01

0.000000E+00 0.000000E+00 0.632456E+00

INVERSE OF THE UPPER TRIANGULAR MATRIX, [UI],

0.707107E+00	0.316228E+00	0.632456E+00
0.000000E+00	0.632456E+00	0.126491E+01
0.000000E+00	0.000000E+00	0.158114E+01

MATRIX [UMU] = [UTI][M][UI]:

0.500000E+00	0.223607E+00	0.447214E+00
0.223607E+00	0.500000E+00	0.100000E+01
0.447214E+00	0.100000E+01	0.700000E+01

EIGENVALUES:

0.718421E+01	0.572771E+00	0.243019E+00
--------------	--------------	--------------

EIGENVECTORS (COLUMNWISE):

0.721309E+00	-0.664609E+00	0.194954E+00
0.134220E+01	-0.168876E+00	-0.412289E+00
0.155923E+01	0.226436E+00	0.132344E+00

7.48

The problem-dependent data and results are given (Program 11.F):

C FOLLOWING 5 LINES CONTAIN PROBLEM-DEPENDENT DATA

```

1  DIMENSION BK(4,4),BM(4,4),U(4,4),UI(4,4),UTI(4,4),BMU(4,4),
2  UMU(4,4),XF(4,4),EV(4,4)
3  DATA BK/4.,-2.,0.,0.,-2.,3.,-1.,0.,0.,-1.,2.,-1.,0.,0.,-1.,1./
4  DATA BM/3.,0.,0.,0.,0.,2.,0.,0.,0.,0.,1.,0.,0.,0.,0.,1./
5  ND=4

```

C END OF PROBLEM-DEPENDENT DATA

UPPER TRIANGULAR MATRIX [U]:

0.200000E+01	-0.100000E+01	0.000000E+00	0.000000E+00
0.000000E+00	0.141421E+01	-0.707107E+00	0.000000E+00
0.000000E+00	0.000000E+00	0.122474E+01	-0.816497E+00
0.000000E+00	0.000000E+00	0.000000E+00	0.577350E+00

INVERSE OF THE UPPER TRIANGULAR MATRIX, [UI],

0.500000E+00	0.353553E+00	0.204124E+00	0.288675E+00
0.000000E+00	0.707107E+00	0.408248E+00	0.577350E+00
0.000000E+00	0.000000E+00	0.816497E+00	0.115470E+01
0.000000E+00	0.000000E+00	0.000000E+00	0.173205E+01

MATRIX [UMU] = [UTI][M][UI]:

0.750000E+00	0.530330E+00	0.306186E+00	0.433013E+00
0.530330E+00	0.137500E+01	0.793857E+00	0.112268E+01
0.306186E+00	0.793857E+00	0.112500E+01	0.159099E+01
0.433013E+00	0.112268E+01	0.159099E+01	0.525000E+01

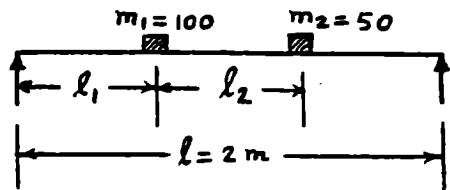
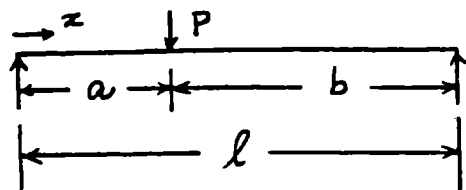
EIGENVALUES:

0.623190E+01	0.143190E+01	0.500000E+00	0.336191E+00
--------------	--------------	--------------	--------------

EIGENVECTORS (COLUMNWISE):

0.480456E+00	-0.437028E+00	0.267261E+00	-0.821001E-01
-0.845263E+00	-0.416241E+00	-0.267258E+00	0.202104E+00
0.130360E+01	0.206727E+00	-0.267263E+00	-0.431800E+00
0.155276E+01	0.685329E+00	0.267266E+00	0.218693E+00

7.49



Basic relationship:

$$w(x) = \begin{cases} \frac{Pbx}{6EI} (l^2 - b^2 - x^2); & 0 \leq x \leq a \\ -\frac{Pa(l-x)}{6EI} (a^2 + x^2 - 2lx); & a \leq x \leq l \end{cases} \quad (E_1)$$

Deflection of mass m_1 due to load $m_1 g$:

Using $x = l_1$, $b = 2 - l_1$ and $l = 2$ in (E_1) :

$$w_1' = \frac{(100 \times 9.81)(2 - l_1)l_1}{6EI(2)} \{4 - (2 - l_1)^2 - l_1^2\} = \frac{981 l_1^2 (2 - l_1)^2}{6EI} \quad (E_3)$$

Deflection of mass m_2 due to load $m_1 g$:

Using $x = l_1 + l_2$, $a = l_1$, $b = 2 - l_1$, $l = 2$ in (E_2) :

$$w_2' = -\frac{(100 \times 9.81)l_1(2 - l_1 - l_2)}{6EI(2)} \{l_1^2 + (l_1 + l_2)^2 - 2(2)(l_1 + l_2)\} \\ = -\frac{981 l_1(2 - l_1 - l_2)}{12EI} (2l_1^2 + l_2^2 + 2l_1 l_2 - 4l_1 - 4l_2) \quad (E_4)$$

Deflection of mass m_1 due to load $m_2 g$:

Using $x = l_1$, $l = 2$, $b = (2 - l_1 - l_2)$ in (E_1) :

$$w_1'' = \frac{(50 \times 9.81)(2 - l_1 - l_2)l_1}{6EI(2)} \{4 - (2 - l_1 - l_2)^2 - l_1^2\} \\ = \frac{490.5 l_1(2 - l_1 - l_2)(-2l_1^2 - l_2^2 + 4l_1 + 4l_2 - 2l_1 l_2)}{12EI} \quad (E_5)$$

Deflection of mass m_2 due to load $m_2 g$:

Using $x = l_1 + l_2$, $l = 2$ and $b = 2 - l_1 - l_2$ in (E₁):

$$\begin{aligned} w_2'' &= \frac{(50 \times 9.81)(2 - l_1 - l_2)(l_1 + l_2)}{6EI(2)} \{4 - (2 - l_1 - l_2)^2 - (l_1 + l_2)^2\} \\ &= \frac{490.5 (l_1 + l_2)(2 - l_1 - l_2)(-2l_1^2 - 2l_2^2 + 4l_1 + 4l_2 - 4l_1l_2)}{12EI} \end{aligned} \quad (E_6)$$

Total deflection of masses m_1 and m_2 are:

$$\begin{aligned} w_1 = w_1' + w_1'' &= \frac{981 l_1^2 (2 - l_1)^2}{6EI} + \\ &+ \frac{490.5 l_1 (2 - l_1 - l_2)(-2l_1^2 - 2l_2^2 + 4l_1 + 4l_2 - 2l_1l_2)}{12EI} \end{aligned} \quad (E_7)$$

$$\begin{aligned} w_2 = w_2' + w_2'' &= \frac{-981 l_1 (2 - l_1 - l_2)(2l_1^2 + l_2^2 + 2l_1l_2 - 4l_1 - 4l_2)}{12EI} \\ &+ \frac{490.5 (l_1 + l_2)(2 - l_1 - l_2)(-2l_1^2 - 2l_2^2 + 4l_1 + 4l_2 - 4l_1l_2)}{12EI} \end{aligned} \quad (E_8)$$

Fundamental natural frequency is given by

$$\omega = \left\{ \frac{g(m_1 w_1 + m_2 w_2)}{(m_1 w_1^2 + m_2 w_2^2)} \right\}^{\frac{1}{2}} = 3.1321 \left(\frac{2w_1 + w_2}{2w_1^2 + w_2^2} \right) \quad (E_9)$$

To maximize ω , we can maximize ω^2 .

Problem is: Find l_1 and l_2

$$\text{to maximize } f = \left(\frac{2w_1 + w_2}{2w_1^2 + w_2^2} \right)$$

where w_1 and w_2 are given by (E₇) and (E₈).

Problem can be solved as follows:

Treat f as a function of l_1 and l_2 .

$$\text{Set } \frac{\partial f}{\partial l_1} = 0 \quad \text{and} \quad \frac{\partial f}{\partial l_2} = 0 \quad (E_{10})$$

Solve Eqs. (E₁₀) for l_1 and l_2 .

7.50

Solution:

Stiffness of one girder (simply supported beam)

$$k_{g1} = \frac{48 EI}{\ell^3} = \frac{48 (200 \times 10^9) \left(\frac{1}{12} a^4 \right)}{9^3} = 1.097 \times 10^9 a^4$$

where a = width and depth of cross-section (m) of girder.

$$k_g = 2 k_{g1} = 2.195 \times 10^9 a^4 \text{ (N/m)}$$

$$m_t = \text{mass of trolley} = 20000 \text{ kg}$$

$$\text{Stiffness of rope} = k_r = \frac{AE}{\ell} = \frac{\frac{\pi d^2}{4} (200 \times 10^9)}{6} = 26.1799 d^2$$

$$m_\ell = \text{mass of lifted load} = 5000 \text{ kg}$$

From section 5.3, the natural frequencies of a 2 d.o.f. system (shown in adjacent figure) are given by

$$\omega_{1,2}^2 = \frac{1}{2} \left\{ \frac{(k_1 + k_2) m_2 + k_2 m_1}{m_1 m_2} \right\} \mp \left[\left\{ \frac{(k_1 + k_2) m_2 + k_2 m_1}{m_1 m_2} \right\}^2 - 4 \left\{ \frac{(k_1 + k_2) k_2 - k_2^2}{m_1 m_2} \right\} \right]^{\frac{1}{2}} \quad (E_1)$$

Here $k_1 = k_g$, $k_2 = k_r$, $m_1 = m_t$, $m_2 = m_\ell$ and hence ω_1^2 and ω_2^2 can be expressed as functions of a and d :

$$\omega_1^2 = \omega_1^2(a, d); \quad \omega_2^2 = \omega_2^2(a, d) \quad (E_2)$$

Requirement is

$$\omega_1^2(a, d) > (157.08)^2; \quad \omega_2^2(a, d) > (157.08)^2 \quad (E_3)$$

Since 1500 rpm = 157.08 rad/sec, choose a and d such that the inequalities (E_3) are satisfied. A trial and error procedure can be used for this purpose.
