

Chapter 13

Nonlinear Vibration

13.1 $\ddot{\theta}_1 + \omega_0^2 \theta_1 = f$, $\ddot{\theta}_2 + \omega_0^2 \theta_2 = \frac{\omega_0^2}{6} \theta_2^3$

Use $\theta = \theta_1 + \theta_2$ in $E_1 \cdot (E_2)$:

$$\ddot{\theta}_1 + \ddot{\theta}_2 + \omega_0^2 \theta_1 + \omega_0^2 \theta_2 = f + \frac{\omega_0^2}{6} \theta_1^3 + \frac{\omega_0^2}{6} \theta_2^3 + \frac{\omega_0^2}{2} \theta_1^2 \theta_2 + \frac{\omega_0^2}{2} \theta_1 \theta_2^2$$

Left hand side is not equal to the right hand side.

Thus superposition principle is not valid.

13.2 $T = \text{kinetic energy at time zero} = \frac{1}{2} m (\dot{x}_0)^2$

Let $x_2 = \text{maximum displacement on right side.}$

$V = \text{potential energy in spring at displacement}$

$$x_2 = \frac{1}{2} k_2 x_2^2 \quad (\dot{x} \text{ is zero at } x_2)$$

Since $T = V$, $x_2 = \sqrt{\frac{m(\dot{x}_0)^2}{k_2}} = \sqrt{\frac{m}{k_2}} \dot{x}_0$

Let $x_1 = \text{maximum displacement to left side.}$ $V = \frac{1}{2} k_1 x_1^2$

$T = V$ gives $x_1 = \sqrt{\frac{m}{k_1} (\dot{x}_0)^2} = \sqrt{\frac{m}{k_1}} \dot{x}_0$

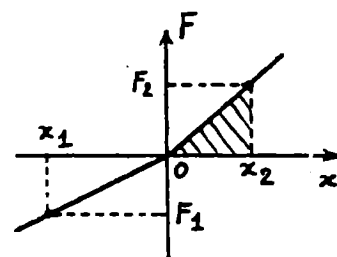
(a) Since $k_1 < k_2$, maximum deflection $= x_1 = \sqrt{\frac{m}{k_1}} \dot{x}_0$

(b) Period of vibration for a spring-mass system is $\tau_n = 2\pi \sqrt{\frac{m}{k}}$.

In the present case, $\tau_n = (\text{time for } m \text{ to go to } x = x_1 \text{ from } x = 0 \text{ and return to } x = 0) +$

$(\text{time for } m \text{ to go to } x = x_2 \text{ from } x = 0 \text{ and return to } x = 0)$

$$\therefore \tau_n = \pi \left(\sqrt{\frac{m}{k_1}} + \sqrt{\frac{m}{k_2}} \right)$$

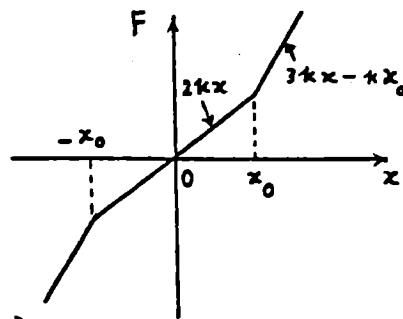


13.3 Restoring force = $F = \begin{cases} 2kx & ; 0 \leq x \leq x_0 \\ 2kx + k(x - x_0) & ; x > x_0 \end{cases}$

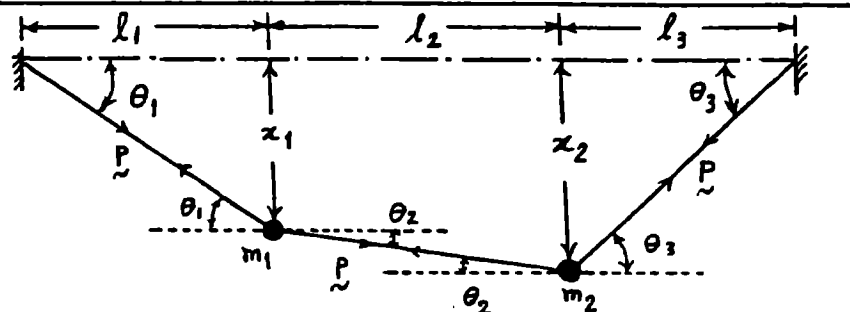
i.e. $F = \begin{cases} 2kx & ; 0 \leq x \leq x_0 \\ 3kx - kx_0 & ; x > x_0 \end{cases} \quad \dots (E_1)$

Equation of motion is

$m \ddot{x} + F = P \sin \omega t$ where F is given by (E_1) .



13.4



Let

$\tilde{P}$ = tension in wire after displacement of masses

P = initial tension in wire

$\tilde{l}_i$ = length of i^{th} segment of wire after displacement of masses

x_i = transverse displacement of mass i ($i = 1, 2$)

$$\tilde{l}_1 = \sqrt{l_1^2 + x_1^2}, \quad \tilde{l}_2 = \sqrt{l_2^2 + (x_2 - x_1)^2}, \quad \tilde{l}_3 = \sqrt{l_3^2 + x_2^2}$$

$$E = \text{strain in wire} = \left(\frac{\tilde{l}_1 + \tilde{l}_2 + \tilde{l}_3 - l_1 - l_2 - l_3}{l_1 + l_2 + l_3} \right) \quad (E_1)$$

$$\text{New tension in wire} = \tilde{P} = (P + AE E)$$

where A = cross-sectional area of wire and E = Young's modulus.

Equations of motion of masses:

$$m_1 \ddot{x}_1 + \tilde{P} \sin \theta_1 + \tilde{P} \sin \theta_2 = 0 \quad (E_2)$$

$$m_2 \ddot{x}_2 + \tilde{P} \sin \theta_2 + \tilde{P} \sin \theta_3 = 0 \quad (E_3)$$

$$\text{i.e., } m_1 \ddot{x}_1 + (P + AE E) \left(\frac{x_1}{\sqrt{l_1^2 + x_1^2}} + \frac{x_2 - x_1}{\sqrt{l_2^2 + (x_2 - x_1)^2}} \right) = 0 \quad (E_4)$$

$$m_2 \ddot{x}_2 + (P + AE E) \left(\frac{x_2 - x_1}{\sqrt{l_2^2 + (x_2 - x_1)^2}} + \frac{x_2}{\sqrt{l_3^2 + x_2^2}} \right) = 0 \quad (E_5)$$

$$\text{with } \frac{x_1}{\sqrt{l_1^2 + x_1^2}} = \frac{x_1}{l_1 \sqrt{1 + \left(\frac{x_1}{l_1}\right)^2}} = \frac{x_1}{l_1} \left[1 + \left(\frac{x_1}{l_1}\right)^2 \right]^{-\frac{1}{2}} \approx \frac{x_1}{l_1} \left[1 - \frac{1}{2} \left(\frac{x_1}{l_1}\right)^2 \right] \quad (E_6)$$

$$\frac{x_2 - x_1}{\sqrt{l_2^2 + (x_2 - x_1)^2}} \approx \frac{x_2 - x_1}{l_2} \left[1 - \frac{1}{2} \left(\frac{x_2 - x_1}{l_2} \right)^2 \right] \quad (E_7)$$

$$\text{and } \frac{x_2}{\sqrt{l_3^2 + x_2^2}} \approx \frac{x_2}{l_3} \left[1 - \frac{1}{2} \left(\frac{x_2}{l_3} \right)^2 \right] \quad (E_8)$$

$$\text{similarly, } l_1 - l_1 = \sqrt{l_1^2 + x_1^2} - l_1 \approx l_1 \left[1 + \frac{1}{2} \left(\frac{x_1}{l_1} \right)^2 \right] - l_1 \approx \frac{1}{2} \frac{x_1^2}{l_1}$$

$$l_2 - l_2 = \sqrt{l_2^2 + (x_2 - x_1)^2} - l_2 \approx \frac{1}{2} \frac{(x_2 - x_1)^2}{l_2}$$

$$l_3 - l_3 \approx \frac{1}{2} \frac{x_2^2}{l_3}$$

$$\text{and hence } \epsilon = \frac{\left\{ \frac{x_1^2}{l_1} + \frac{(x_2 - x_1)^2}{l_2} + \frac{x_2^2}{l_3} \right\}}{2(l_1 + l_2 + l_3)} \quad (E_9)$$

Substitution of Eqs. (E₆) to (E₉) into Eqs. (E₄) and (E₅) yields the nonlinear equations of motion of the masses m_1 and m_2 .

13.5

Using x and θ as the coordinates, the kinetic and potential energies of the system can be expressed as

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2 \quad (1)$$

where $J_0 = m(\ell + x)^2$ and

$$V = \frac{1}{2} k (x + \delta_{st})^2 - m g (\ell + x) \cos \theta \quad (2)$$

where $\delta_{st} = \frac{mg}{k}$. Equations (1) and (2) give

$$\frac{\partial T}{\partial \dot{x}} = m \dot{x} ; \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}} \right) = m \ddot{x}$$

$$\frac{\partial T}{\partial \dot{\theta}} = J_0 \dot{\theta} ; \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}} \right) = \dot{J}_0 \dot{\theta} + J_0 \ddot{\theta} = 2m(\ell + x) \dot{x} \dot{\theta} + J_0 \ddot{\theta}$$

$$\frac{\partial T}{\partial x} = m(\ell + x) \dot{\theta}^2 ; \quad \frac{\partial T}{\partial \theta} = 0$$

$$\frac{\partial V}{\partial x} = k(x + \delta_{st}) - m g \cos \theta ; \quad \frac{\partial V}{\partial \theta} = m g (\ell + x) \sin \theta$$

The equations of motion can be derived using Lagrange's equations, Eq. (6.44), as:

$$m \ddot{x} - m(\ell + x) \dot{\theta}^2 + kx + mg - mg \cos \theta = 0 \quad (3)$$

$$m(\ell + x)^2 \ddot{\theta} + 2m(\ell + x) \dot{x} \dot{\theta} + mg(\ell + x) \sin \theta = 0 \quad (4)$$

Using $\sin \theta \approx \theta$, $\cos \theta \approx 1$, and neglecting nonlinear terms involving $x^2 \ddot{\theta}$, $\dot{\theta}^2$, $x \theta$, and $\dot{x} \dot{\theta}$, Eqs. (3) and (4) can be reduced (linearized) to obtain:

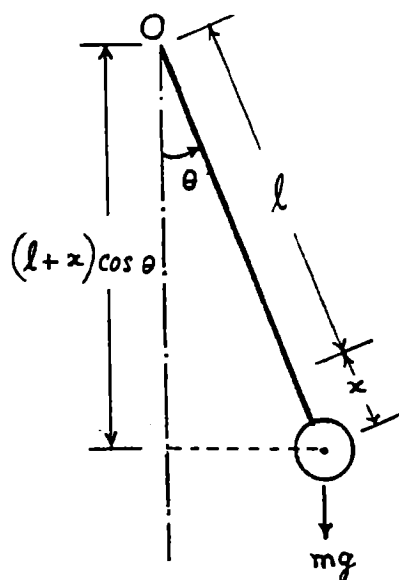
$$m \ddot{x} + k x = 0 \quad (5)$$

$$m \ell^2 \ddot{\theta} + m g \ell \theta = 0 \quad (6)$$

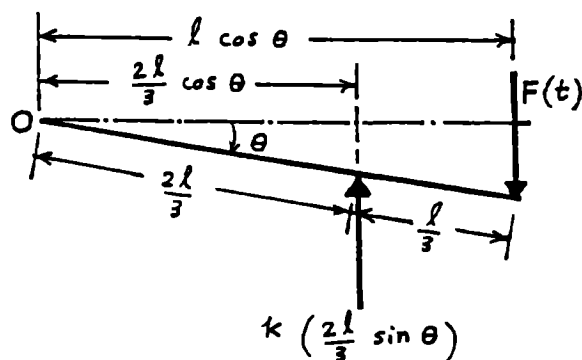
Equations (5) and (6) correspond to the natural frequencies:

$$\omega_{n1} = \sqrt{\frac{k}{m}} \quad (7)$$

$$\omega_{n2} = \sqrt{\frac{g}{\ell}} \quad (8)$$



13.6 $J_0 = \frac{1}{3} m \ell^2$



$$\Sigma M_0 = J_0 \ddot{\theta}$$

$$\text{or } J_0 \ddot{\theta} = -k \left(\frac{2\ell}{3} \sin \theta \right) \left(\frac{2\ell}{3} \cos \theta \right) + (\ell \cos \theta) F(t)$$

$$\text{or } \left(\frac{1}{3} m \ell^2 \right) \ddot{\theta} + \frac{4}{9} k \ell^2 \sin \theta \cos \theta = \ell \cos \theta F(t) \quad (1)$$

Eq. (1) can be approximated using

$$\sin \theta \approx \theta - \frac{\theta^3}{6} \quad \text{and} \quad \cos \theta \approx 1 - \frac{\theta^2}{2}$$

which yields

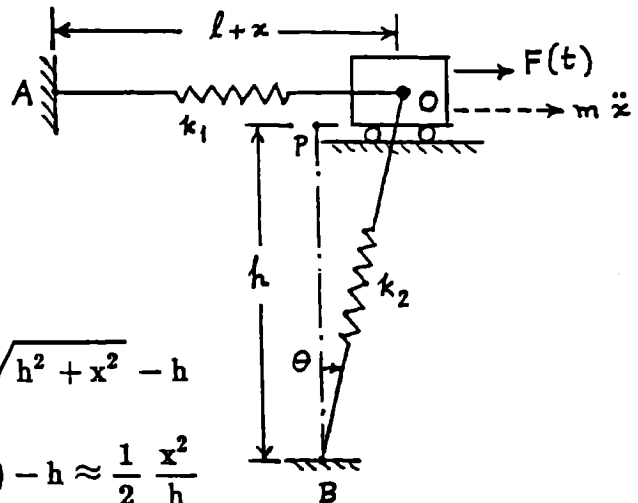
$$\left(\frac{1}{3} m \ell^2 \right) \ddot{\theta} + \frac{4}{9} k \ell^2 \left(\theta - \frac{\theta^3}{6} \right) \left(1 - \frac{\theta^2}{2} \right) = \ell \left(1 - \frac{\theta^2}{2} \right) F(t)$$

$$\text{or } \frac{1}{3} m \ell^2 \ddot{\theta} + \frac{4}{9} k \ell^2 \left(\theta - \frac{2}{3} \theta^3 \right) = \left(\ell - \frac{\ell \theta^2}{2} \right) F(t) \quad (2)$$

Neglecting the nonlinear term, Eq. (2) can be reduced to

$$\frac{1}{3} m \ell^2 \ddot{\theta} + \frac{4}{9} k \ell^2 \theta = \ell F(t) \quad (3)$$

13.7



Extension of k_2 :

$$\begin{aligned} OB - PB &= \sqrt{BP^2 + PO^2} - PB = \sqrt{h^2 + x^2} - h \\ &= h \left(1 + \frac{x^2}{h^2} \right)^{\frac{1}{2}} - h \approx h \left(1 + \frac{1}{2} \frac{x^2}{h^2} \right) - h \approx \frac{1}{2} \frac{x^2}{h} \end{aligned}$$

$$\sin \theta = \frac{x}{BO} = \frac{x}{h + \frac{1}{2} \frac{x^2}{h}}$$

$$\sum F = m \ddot{x}$$

$$\text{or } m \ddot{x} = -k_1 x - \left(\frac{1}{2} \frac{x^2}{h} \right) k_2 \sin \theta + F(t)$$

$$\text{or } m \ddot{x} + k_1 x + \left(\frac{\frac{1}{2} \frac{x^3 k_2}{h}}{h + \frac{1}{2} \frac{x^2}{h}} \right) = F(t) \quad (1)$$

Note that

$$\frac{x^3 k_2}{2 h \left(h + \frac{1}{2} \frac{x^2}{h} \right)} \approx \frac{x^3 k_2}{2 h^2} - \frac{x^5 k_2}{4 h^4} \quad (2)$$

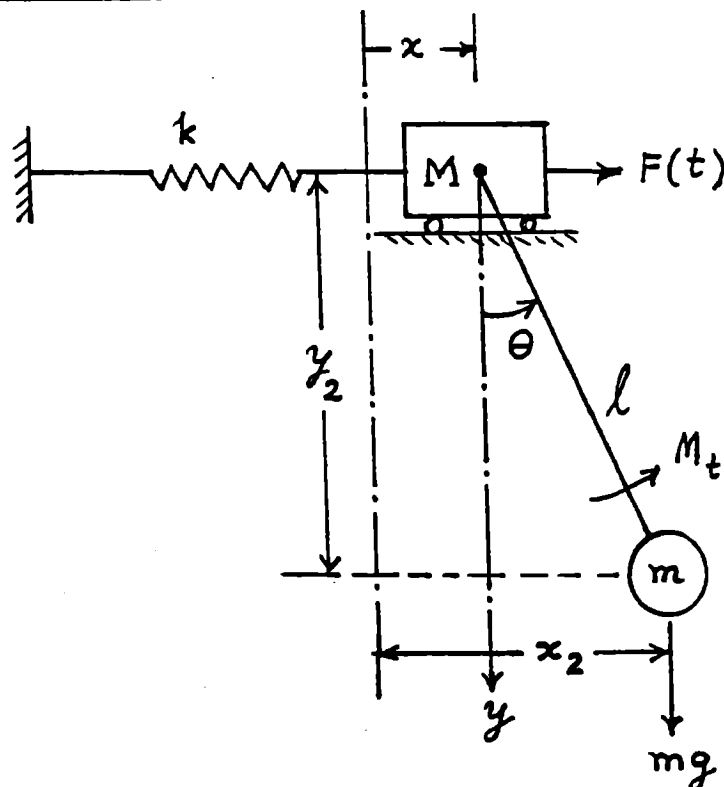
Substituting Eq. (2) into (1), we obtain

$$m \ddot{x} + k_1 x + \frac{x^3 k_2}{2 h^2} - \frac{x^5 k_2}{4 h^4} = F(t) \quad (3)$$

Neglecting the terms involving x^5 , Eq. (3) can be reduced to

$$m \ddot{x} + k_1 x + \frac{k_2}{2 h^2} x^3 = F(t) \quad (4)$$

13.8



$$x_2 = x + \ell \sin \theta ; \quad \dot{x}_2 = \dot{x} + \ell \dot{\theta} \cos \theta$$

$$y_2 = \ell \cos \theta ; \quad \dot{y}_2 = -\ell \dot{\theta} \sin \theta$$

$$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m (\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \left[(\dot{x} + \ell \dot{\theta} \cos \theta)^2 + (-\ell \dot{\theta} \sin \theta)^2 \right]$$

$$= \frac{1}{2} (M + m) \dot{x}^2 + \frac{1}{2} m \ell^2 \dot{\theta}^2 + m \ell \dot{x} \dot{\theta} \cos \theta$$

$$V = \frac{1}{2} k x^2 + m g \ell (1 - \cos \theta)$$

$$Q_x = F(t) ; \quad Q_\theta = M_t(t)$$

Equations of motion:

$$(M + m) \ddot{x} + m \ell \ddot{\theta} \cos \theta - m \ell \dot{\theta}^2 \sin \theta + k x = F(t) \quad (1)$$

$$m \ell^2 \ddot{\theta} + m \ell \ddot{x} \cos \theta - m \ell \dot{x} \dot{\theta} \sin \theta + m g \ell \sin \theta = M_t(t) \quad (2)$$

Using the approximations

$$\cos \theta \approx 1 - \frac{\theta^2}{2} ; \sin \theta \approx \theta - \frac{\theta^3}{6}$$

Eqs. (1) and (2) can be expressed as

$$(M + m) \ddot{x} + m \ell \ddot{\theta} - \frac{1}{2} m \ell \theta^2 \ddot{\theta} - m \ell \theta \dot{\theta}^2 + \frac{1}{6} m \ell \theta^3 \dot{\theta}^2 + k x = F(t) \quad (3)$$

$$\begin{aligned} m \ell^2 \ddot{\theta} + m \ell \ddot{x} - \frac{1}{2} m \ell \theta^2 \ddot{x} - m \ell \theta \dot{\theta} \dot{x} + \frac{1}{6} m \ell \theta^3 \dot{\theta} \dot{x} \\ + m g \ell \theta - \frac{1}{6} m g \ell \theta^3 = M_t(t) \end{aligned} \quad (4)$$

By neglecting the nonlinear terms, the linearized equations of motion can be written as

$$(M + m) \ddot{x} + m \ell \ddot{\theta} + k x = F(t) \quad (5)$$

$$m \ell^2 \ddot{\theta} + m \ell \ddot{x} + m g \ell \theta = M_t(t) \quad (6)$$

13.9

Eq. (13.1) is $\ddot{\theta} + \frac{g}{l} \sin \theta = 0 \quad \dots (E_1)$

But $\frac{d^2\theta}{dt^2} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta} \frac{d\theta}{dt} = \dot{\theta} \frac{d\dot{\theta}}{d\theta}$

Eq. (E₁) can be rewritten as $\dot{\theta} \frac{d\dot{\theta}}{d\theta} + \frac{g}{l} \sin \theta = 0$

Integrating this, we get $\frac{(\dot{\theta})^2}{2} - \frac{g}{l} \cos \theta = c \quad \dots (E_2)$

At $t=0$, $\dot{\theta}=0$ and $\theta=\theta_0 \Rightarrow c = -\left(\frac{g}{l}\right) \cos \theta_0$

Eq. (E₂) gives $\left(\frac{d\theta}{dt}\right)^2 = (\dot{\theta})^2 = 2 \left(c + \frac{g}{l} \cos \theta\right)$

$$dt = \frac{d\theta}{\sqrt{2 \left(c + \frac{g}{l} \cos \theta\right)}} = \frac{d\theta}{\sqrt{2 \frac{g}{l} (\cos \theta - \cos \theta_0)}}$$

$$t = \int \frac{d\theta}{\sqrt{2 \frac{g}{l} (\cos \theta - \cos \theta_0)}} \quad \dots (E_3)$$

But $\cos \theta - \cos \theta_0 = \left(1 - 2 \sin^2 \frac{\theta}{2}\right) - \left(1 - 2 \sin^2 \frac{\theta_0}{2}\right)$
 $= 2 \sin^2 \frac{\theta_0}{2} \left(1 - \frac{\sin^2 \frac{\theta}{2}}{\sin^2 \frac{\theta_0}{2}}\right) \quad \dots (E_4)$

If $\frac{\sin \frac{\theta}{2}}{\sin \frac{\theta_0}{2}} = \frac{\sin \frac{\theta}{2}}{k} = \sin \phi$ where $k = \sin \frac{\theta_0}{2}$,

differentiation gives

$$\frac{1}{k} \cos \frac{\theta}{2} \cdot \frac{d\theta}{2} = \cos \phi \cdot d\phi \Rightarrow d\theta = \frac{2k \cos \phi d\phi}{\cos \frac{\theta}{2}} \dots (E_5)$$

Eq. (E₃) to (E₅) give

$$t = \int \left(\frac{2\kappa \cos \phi \, d\phi}{\cos \frac{\theta}{2}} \right) / \left(2\sqrt{\frac{g}{l}} \sin \frac{\theta_0}{2} \cos \phi \right) = \sqrt{\frac{l}{g}} \int \frac{d\phi}{\cos \frac{\theta}{2}}$$

$$= \sqrt{\frac{l}{g}} \int \frac{d\phi}{\sqrt{1 - \sin^2 \frac{\theta}{2}}} = \sqrt{\frac{l}{g}} \int \frac{d\phi}{\sqrt{1 - \kappa^2 \sin^2 \phi}}$$

$$\tau = \text{period} = 4 \sqrt{\frac{l}{g}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \kappa^2 \sin^2 \phi}} \quad \text{where } \kappa = \sin \frac{\theta_0}{2}.$$

The integral in τ is the complete elliptic integral of the first kind. Hence the numerical value of τ corresponding to any θ_0 can be obtained from the elliptic integral tables.

13.10

Pendulum length 0.8 m

$$\text{Equation of motion: } \ddot{\theta} + \frac{g}{l} \sin \theta = 0 \quad \text{or} \quad \frac{d}{d\theta} (\dot{\theta}^2) + \frac{2g}{l} \sin \theta = 0$$

$$\text{Let } \theta = \theta_0 \text{ at } \frac{d\theta}{dt} = 0, \quad \int_{\dot{\theta}=0}^{\dot{\theta}} \frac{d}{d\theta} (\dot{\theta}^2) d\theta + \int_{\theta=\theta_0}^{\theta} \frac{2g}{l} \sin \theta d\theta = 0$$

$$\text{i.e., } (\dot{\theta}^2) + \frac{2g}{l} (-\cos \theta) \Big|_{\theta=\theta_0}^{\theta} = 0$$

$$\text{i.e., } (\dot{\theta}^2) = \frac{2g}{l} (\cos \theta - \cos \theta_0)$$

$$\text{i.e., } \frac{d\theta}{dt} = \pm \sqrt{\frac{2g}{l}} \sqrt{\cos \theta - \cos \theta_0}$$

$$\text{i.e., } dt = \pm \sqrt{\frac{l}{2g}} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}}$$

Time taken by pendulum to reach vertical position can be found as

$$t = \int_0^t dt = \sqrt{\frac{l}{2g}} \int_{\theta=\theta_0}^0 \frac{d\theta}{(\cos \theta - \cos \theta_0)^{\frac{1}{2}}} \quad (E_1)$$

Using the relation $\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2}$, Eq. (E₁) can be rewritten as

$$t = \frac{1}{2} \sqrt{\frac{l}{g}} \int_{\theta_0}^0 \frac{d\theta}{\sqrt{\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2}}} \quad (E_2)$$

Let $a = \sin \frac{\theta_0}{2}$ and $\sin \frac{\theta}{2} = a \cdot \sin \phi$ so that when $\theta = 0$, $\phi = 0$ and when $\theta = \theta_0$, $\phi = \frac{\pi}{2}$.

$$\cos \frac{\theta}{2} \cdot \frac{d\theta}{2} = a \cos \phi \cdot d\phi; \quad d\theta = \left(\frac{2a \cos \phi}{\cos \frac{\theta}{2}} \right) d\phi$$

$$\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2} = a^2 - a^2 \sin^2 \phi = a^2 \cos^2 \phi$$

Eq. (E₂) becomes

$$t = \pm \frac{1}{2} \sqrt{\frac{\ell}{g}} \int_{\frac{\pi}{2}}^0 \frac{2a \cos \phi d\phi}{\cos \frac{\theta}{2} \cdot a \cos \phi} = \pm \sqrt{\frac{\ell}{g}} \int_{\frac{\pi}{2}}^0 \frac{1}{\sqrt{1 - \cos^2 \frac{\theta}{2}}} d\phi$$

$$= \sqrt{\frac{\ell}{g}} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - a^2 \sin^2 \phi}} = \sqrt{\frac{\ell}{g}} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - \sin^2 \frac{\theta_0}{2} \cdot \sin^2 \phi}}$$

$$= \sqrt{\frac{\ell}{g}} \cdot F\left(a, \frac{\pi}{2}\right) \quad (E_3)$$

where F is called the incomplete elliptic integral of the first kind.

Here $\sqrt{\frac{\ell}{g}} = \sqrt{\frac{0.8}{9.8}} = 0.2857$ and $a = \sin \frac{\theta_0}{2} = \sin 40^\circ = 0.6428$.

From CRC Standard Mathematical Tables, we find for $F\left(a, \frac{\pi}{2}\right) = F\left(\sin \frac{\theta_0}{2}, \frac{\pi}{2}\right)$ with $\theta_0 = 80^\circ$, $F = 1.7868$

$$\therefore t = 0.2857 (1.7868) = 0.5105 \text{ second}$$

Alternatively, $\ell = 0.8 \text{ m}$, $\theta_0 = 80^\circ = 1.3963 \text{ rad}$ and $g = 9.8 \text{ m/s}^2$.

Time taken by pendulum to reach vertical position (one-quarter of the time period) can be found from Eq. (E₁₅) in solution of problem 13.11 as

$$t = \frac{\tau}{4} = \frac{1}{\omega_0 \left(1 - \frac{\theta_0^2}{12}\right)} F\left(a, \frac{\pi}{2}\right) \quad (E_4)$$

Here $\omega_0 = \sqrt{\frac{g}{\ell}} = \sqrt{9.8/0.8} = 3.5$, $\theta_0 = 1.3963 \text{ rad}$,

$$a^2 = \frac{\theta_0^2}{12 \left(1 - \frac{1}{12} \theta_0^2\right)} = \frac{1.9496}{12 (0.8375)} = 0.1940 = (0.4405)^2 = (\sin^{-1} 26.1358^\circ)^2$$

$F\left(a, \frac{\pi}{2}\right) \approx 1.6490$ from CRC Standard Mathematical Tables.

$$t \approx \frac{1.6490}{(3.5)(0.8375)} = 0.5626 \text{ sec}$$

Note that this result is only approximately due to the approximation $\sin \theta \approx \theta - \frac{1}{6} \theta^3$ used in the solution of problem 13.11.

$$\ddot{\theta} + \omega_0^2 \left(\theta - \frac{1}{6} \theta^3 \right) = 0 \quad (E_1)$$

This equation is similar to Eq. (13.9) with

$$x = \theta, \quad \omega = \omega_0, \quad F(x) = F(\theta) = \theta - \frac{1}{6} \theta^3.$$

Eq. (E₁) can be rewritten as

$$\frac{d}{d\theta} (\dot{\theta}^2) + 2 \omega_0^2 \left(\theta - \frac{1}{6} \theta^3 \right) = 0 \quad (E_2)$$

which upon integration gives

$$\begin{aligned} \dot{\theta}^2 &= 2 \omega_0^2 \int_{\theta}^{\theta_0} F(\eta) \cdot d\eta = 2 \omega_0^2 \int_{\theta}^{\theta_0} \left(\eta - \frac{1}{6} \eta^3 \right) \cdot d\eta \\ &= 2 \omega_0^2 \left(\frac{1}{2} \eta^2 - \frac{1}{24} \eta^4 \right) \Big|_{\theta}^{\theta_0} = \omega_0^2 \left(\theta_0^2 - \frac{1}{12} \theta_0^4 - \theta^2 + \frac{1}{12} \theta^4 \right) \end{aligned} \quad (E_3)$$

$$= \omega_0^2 (\theta_0^2 - \theta^2) \left\{ 1 - \frac{1}{12} (\theta_0^2 + \theta^2) \right\} \quad (E_4)$$

Since the maximum value of θ is θ_0 , we assume

$$\theta(t) = \theta_0 \sin \beta \quad (E_5)$$

$$\text{Thus } \theta_0^2 - \theta^2 = \theta_0^2 - \theta_0^2 \sin^2 \beta = \theta_0^2 \cos^2 \beta \quad (E_6)$$

$$\theta_0^2 + \theta^2 = \theta_0^2 (1 + \sin^2 \beta) \quad (E_7)$$

$$\text{and } \dot{\theta} = A_0 \cos \beta \frac{d\beta}{dt} \quad (E_8)$$

Substitution of Eqs. (E₆) to (E₈) into (E₄) gives

$$\theta_0^2 \cos^2 \beta \left(\frac{d\beta}{dt} \right)^2 = \omega_0^2 \theta_0^2 \cos^2 \beta \left\{ 1 - \frac{1}{12} \theta_0^2 (1 + \sin^2 \beta) \right\}$$

i.e.,

$$\left(\frac{d\beta}{dt} \right)^2 = \omega_0^2 \left(1 - \frac{1}{12} \theta_0^2 \right) \left\{ 1 - \frac{\theta_0^2 \sin^2 \beta}{12 \left(1 - \frac{1}{12} \theta_0^2 \right)} \right\} \quad (E_9)$$

Defining

$$a^2 = \frac{\theta_0^2}{12 \left(1 - \frac{1}{12} \theta_0^2 \right)} \quad (E_{10})$$

Eq. (E₉) can be used to express (taking positive root):

$$\frac{d\beta}{dt} = \omega_0 \left(1 - \frac{1}{12} \theta_0^2 \right)^{\frac{1}{2}} \left(1 - a^2 \sin^2 \beta \right)^{\frac{1}{2}} \quad (E_{11})$$

i.e.,

$$\omega_0 \left(1 - \frac{1}{12} \theta_0^2 \right)^{\frac{1}{2}} dt = \int \frac{d\beta}{\sqrt{1 - a^2 \sin^2 \beta}} \quad (E_{12})$$

Integration of (E₁₂) yields

$$\omega_0 \left(1 - \frac{1}{12} \theta_0^2 \right)^{\frac{1}{2}} (t - t_0) = \int_{\beta_0}^{\beta} \frac{d\beta}{\sqrt{1 - a^2 \sin^2 \beta}} \quad (E_{13})$$

Using the initial conditions $\beta_0 = 0$ at $t_0 = 0$, E_8 (E_{13}) can be reduced to

$$\omega_0 \left(1 - \frac{1}{12} \theta_0^2\right)^{\frac{1}{2}} \cdot t = \int_0^{\beta} \frac{d\beta}{\sqrt{1 - a^2 \sin^2 \beta}} = F(a, \beta) \quad (E_{14})$$

where $F(a, \beta)$ is an incomplete elliptic integral of the first kind. Using $\beta = \frac{\pi}{2}$ when $\theta = \theta_0$ and $\beta = 0$ when $\theta = 0$, we get for one-quarter period,

$$\frac{\tau}{4} = t = \frac{1}{\omega_0 \left(1 - \frac{1}{12} \theta_0^2\right)^{\frac{1}{2}}} \cdot F\left(a, \frac{\pi}{2}\right) \quad (E_{15})$$

Thus the time period of the pendulum is given by

$$\tau = \frac{4}{\omega_0 \left(1 - \frac{1}{12} \theta_0^2\right)^{\frac{1}{2}}} \cdot F\left(a, \frac{\pi}{2}\right) \quad (E_{16})$$

13.12

$$\text{Equation: } E[x] = \ddot{x} + \omega_0^2 x - \frac{1}{6} \omega_0^2 x^3 = 0 \quad (E_1)$$

$$\text{Assumed solution: } \tilde{x}(t) = A_0 \sin \omega t + A_3 \sin 3\omega t \quad (E_2)$$

$$\ddot{\tilde{x}}(t) = -A_0 \omega^2 \sin \omega t - A_3 (3\omega)^2 \sin 3\omega t$$

$$\begin{aligned} \tilde{x}^3 &= A_0^3 \sin^3 \omega t + A_3^3 \sin^3 3\omega t \quad (\text{neglecting cross product terms}) \\ &= \sin \omega t \left(\frac{3}{4} A_0^3\right) + \sin 3\omega t \left(-\frac{1}{4} A_0^3 + \frac{3}{4} A_3^3\right) + \sin 9\omega t \left(-\frac{1}{4} A_3^3\right) \end{aligned}$$

This gives

$$\begin{aligned} \tilde{E} = E[\tilde{x}] &= -A_0 \omega^2 \sin \omega t - A_3 (3\omega)^2 \sin 3\omega t + A_0 \omega_0^2 \sin \omega t \\ &\quad + A_3 \omega_0^2 \sin 3\omega t - \frac{1}{8} \omega_0^2 A_0^3 \sin \omega t \\ &\quad + \frac{1}{24} \omega_0^2 (A_0^3 - 3A_3^3) \sin 3\omega t + \frac{1}{24} \omega_0^2 A_3^3 \sin 9\omega t \\ &= B_1 \sin \omega t + B_2 \sin 3\omega t + B_3 \sin 9\omega t \quad (E_3) \end{aligned}$$

$$\text{Where } B_1 = -A_0 \omega^2 + A_0 \omega_0^2 - \frac{1}{8} A_0^3 \omega_0^2$$

$$B_2 = -9A_3 \omega^2 + A_3 \omega_0^2 + \frac{1}{24} A_0^3 \omega_0^2 - \frac{1}{8} A_3^3 \omega_0^2$$

$$B_3 = \frac{1}{24} A_3^3 \omega_0^2$$

In Ritz - Galerkin method,

$$\int_0^{\tau} \tilde{E} \frac{\partial \tilde{E}}{\partial A_0} dt = 0 \quad (E_4)$$

$$\int_0^{\tau} \tilde{E} \frac{\partial \tilde{E}}{\partial A_3} dt = 0 \quad (E_5)$$

$$\text{i.e., } \int_0^{2\pi} (B_1 \sin \omega t + B_2 \sin 3\omega t + B_3 \sin 9\omega t)(B_4 \sin \omega t + B_5 \sin 3\omega t) dt = 0$$

$$\text{with } B_4 = -\omega^2 + \omega_0^2 - \frac{3}{8} A_0^2 \omega_0^2 \quad \text{and} \quad B_5 = \frac{1}{8} A_0^2 \omega_0^2.$$

$$\text{i.e., } \int_0^{2\pi} (B_1 B_4 \sin^2 \omega t + B_2 B_4 \sin \omega t \sin 3\omega t + B_3 B_4 \sin \omega t \sin 9\omega t + B_1 B_5 \sin \omega t \sin 3\omega t + B_2 B_5 \sin^2 3\omega t + B_3 B_5 \sin 3\omega t \sin 9\omega t) dt = 0$$

$$\text{i.e., } B_1 B_4 + B_2 B_5 = 0 \quad (E_6)$$

and Eq. (E5) leads to

$$\int_0^{2\pi} (B_1 \sin \omega t + B_2 \sin 3\omega t + B_3 \sin 9\omega t)(B_6 \sin 3\omega t + B_7 \sin 9\omega t) dt = 0$$

$$\text{with } B_6 = -9\omega^2 + \omega_0^2 - \frac{3}{8} A_3^2 \omega_0^2 \quad \text{and} \quad B_7 = \frac{1}{8} A_3^2 \omega_0^2.$$

$$\text{i.e., } B_2 B_6 + B_3 B_7 = 0 \quad (E_7)$$

Eqs. (E6) and (E7) can be rewritten as

$$(-A_0 \omega^2 + A_0 \omega_0^2 - \frac{1}{8} A_0^3 \omega_0^2)(-\omega^2 + \omega_0^2 - \frac{3}{8} A_0^2 \omega_0^2) + (-A_3 \cdot 9\omega^2 + A_3 \omega_0^2 + \frac{1}{24} A_0^3 \omega_0^2 - \frac{1}{8} A_3^3 \omega_0^2)(\frac{1}{8} A_0^2 \omega_0^2) = 0 \quad (E_8)$$

and

$$(-A_3 \cdot 9\omega^2 + A_3 \omega_0^2 + \frac{1}{24} A_0^3 \omega_0^2 - \frac{1}{8} A_3^3 \omega_0^2)(-9\omega^2 + \omega_0^2 - \frac{3}{8} A_3^2 \omega_0^2) + (\frac{1}{24} A_3^3 \omega_0^2)(\frac{1}{8} A_3^2 \omega_0^2) = 0 \quad (E_9)$$

The solution of Eqs. (E8) and (E9) gives the values of A_0 and A_3 .

Note that (E8) and (E9) are two simultaneous algebraic equations in A_0 and A_3 for which general closed-form solution is not possible.

Particular case :

If $A_3 = 0$, Eq. (E) yields

$$A_0 (-\omega^2 + \omega_0^2 - \frac{1}{8} A_0^2 \omega_0^2)(-\omega^2 + \omega_0^2 - \frac{3}{8} A_0^2 \omega_0^2) + \frac{1}{192} A_0^5 \omega_0^4 = 0$$

$$\text{i.e., } A_0 \left\{ \omega^4 + \omega^2 \left(-2\omega_0^2 + \frac{1}{2} A_0^2 \omega_0^2 \right) + \omega_0^4 \left(1 - \frac{1}{2} A_0^2 + \frac{5}{96} A_0^4 \right) \right\} = 0$$

Since $A_0 \neq 0$, we get

$$\omega^4 - \omega^2 \left(2\omega_0^2 - \frac{1}{2} A_0^2 \omega_0^2 \right) + \omega_0^4 \left(1 - \frac{1}{2} A_0^2 + \frac{5}{96} A_0^4 \right) = 0 \quad (E_{10})$$

which can be seen to be identical to Eq. (E.6) of Example 13.1.

13.13 Equation: $\ddot{x} + \omega_0^2 x + \alpha x^3 = 0$ (E₁)
we assume

$$x(t) = x_0(t) + \alpha x_1(t) + \alpha^2 x_2(t) \quad (E_2)$$

and $\omega^2 = \omega_0^2 + \alpha \omega_1(A_0) + \alpha^2 \omega_2(A_0)$ (E₃)

or $\omega_0^2 = \omega^2 - \alpha \omega_1(A_0) - \alpha^2 \omega_2(A_0)$ (E₄)

where A_0 = amplitude and ω = true fundamental frequency.

Substitution of (E₂) and (E₄) into (E₁) gives

$$\begin{aligned} \ddot{x}_0 + \alpha \ddot{x}_1 + \alpha^2 \ddot{x}_2 + \omega^2 x_0 - \alpha \omega_1 x_0 - \alpha^2 \omega_2 x_0 + \omega^2 \alpha x_1 \\ - \alpha^2 \omega_1 x_1 - \alpha^3 \omega_2 x_1 + \alpha^2 \omega^2 x_2 - \alpha^3 \omega_1 x_2 - \alpha^4 \omega_2 x_2 + \alpha x_0^3 \\ + \alpha^4 x_1^3 + \alpha^7 x_2^3 + 3\alpha^2 x_0^2 x_1 + 3\alpha^3 x_0 x_1^2 + 3\alpha^3 x_0^2 x_2 \\ + 3\alpha^5 x_0 x_2^2 + 3\alpha^5 x_1^2 x_2 + 3\alpha^6 x_1 x_2^2 + 6\alpha^4 x_0 x_1 x_2 = 0 \end{aligned} \quad (E_5)$$

Neglecting terms involving $\alpha^3, \alpha^4, \dots$, (E₅) can be rewritten as

$$\begin{aligned} \alpha^0 (\ddot{x}_0 + \omega^2 x_0) + \alpha^1 (\ddot{x}_1 - \omega_1 x_0 + \omega^2 x_1 + x_0^3) \\ + \alpha^2 (\ddot{x}_2 - \omega_2 x_0 - \omega_1 x_1 + \omega^2 x_2 + 3x_0^2 x_1) = 0 \end{aligned} \quad (E_6)$$

Setting the coefficient of α^0 to zero in (E₆), we get

$$\ddot{x}_0 + \omega^2 x_0 = 0 \quad (E_7)$$

whose solution is

$$x_0(t) = D_1 \cos \omega t + D_2 \sin \omega t \quad (E_8)$$

and hence $\dot{x}_0(t) = -D_1 \omega \sin \omega t + D_2 \omega \cos \omega t$ (E₉)

Let the initial conditions of the system be

$$x_0(t=0) = A_0 \quad \text{and} \quad \dot{x}_0(t=0) = 0 \quad (E_{10})$$

Eqs. (E₈) to (E₁₀) yield $D_1 = A_0$ and $D_2 = 0$ so that

$$x_0(t) = A_0 \cos \omega t \quad (E_{11})$$

Thus, if $\alpha = 0$, the solution becomes

$$x_0(t) = A_0 \cos \omega t, \quad \omega = \omega_0 \quad (E_{12})$$

By setting the coefficient of α^1 to zero in (E₆), we get

$$\ddot{x}_1 + \omega^2 x_1 = \omega_1 x_0 - x_0^3 \quad (E_{13})$$

Substituting Eq. (E₁₁) for x_0 , (E₁₃) leads to

$$\ddot{x}_1 + \omega^2 x_1 = (\omega_1 A_0 - \frac{3}{4} A_0^3) \cos \omega t - \frac{1}{4} A_0^3 \cos 3\omega t \quad (E_{14})$$

The complete solution of (E₁₄) can be expressed as (can be verified by substitution):

$$x_1(t) = D_1 \cos \omega t + D_2 \sin \omega t + (\omega_1 A_0 - \frac{3}{4} A_0^3) \frac{1}{\omega^2} \cos \omega t + \frac{1}{32} \cdot \frac{1}{\omega^2} \cdot A_0^3 \cos 3\omega t \quad (E_{15})$$

Note that the third term on the r.h.s. of (E₁₅) is a secular term which can be eliminated by setting

$$\omega_1 A_0 - \frac{3}{4} A_0^3 = 0 \quad \text{i.e.,} \quad A_0 = 0 \quad \text{or} \quad \omega_1 = \frac{3}{4} A_0^2 \quad (E_{16})$$

$$\text{Since } A_0 \neq 0, \text{ we have } \omega_1 = \frac{3}{4} A_0^2 \quad (E_{17})$$

Using the initial conditions $x_1(t=0) = \dot{x}_1(t=0) = 0$ in (E₁₅), we get

$$D_1 = -\frac{1}{32 \omega^2} A_0^3, \quad D_2 = 0 \quad (E_{18})$$

Hence the first-order correction is given by

$$x_1(t) = -\frac{A_0^3}{32 \omega^2} (\cos \omega t - \cos 3\omega t); \quad \omega_1 = \frac{3}{4} A_0^2 \quad (E_{19})$$

Finally, by setting the coefficient of α^2 in (E₆) to zero, we get

$$\ddot{x}_2 + \omega^2 x_2 = \omega_2 x_0 + x_1 \omega_1 - 3 x_0^2 x_1 \quad (E_{20})$$

Substituting for x_0, x_1 and ω_1 , Eq. (E₂₀) leads to

$$\ddot{x}_2 + \omega^2 x_2 = (\omega_2 A_0 - \frac{3}{128} \frac{A_0^5}{\omega^2}) \cos \omega t + (\frac{3}{128} \frac{A_0^5}{\omega^2}) \cos 3\omega t + \frac{3}{32} \frac{A_0^5}{\omega^2} (\cos^3 \omega t - \cos^2 \omega t \cos 3\omega t) \quad (E_{21})$$

$$\text{Here } \cos^3 \omega t = \frac{3}{4} \cos \omega t + \frac{1}{4} \cos 3\omega t \quad (E_{22})$$

$$\text{and } \cos^2 \omega t \cdot \cos 3\omega t = \frac{1}{4} \cos \omega t + \frac{1}{2} \cos 3\omega t + \frac{1}{4} \cos 5\omega t \quad (E_{23})$$

Eqs. (E₂₁) to (E₂₃) lead to

$$\ddot{x}_2 + \omega^2 x_2 = (\omega_2 A_0 + \frac{3}{128} \frac{A_0^5}{\omega^2}) \cos \omega t - \frac{3}{128} \frac{A_0^5}{\omega^2} \cos 5\omega t \quad (E_{24})$$

Since the first term on the r.h.s. of (E₂₄) leads to the secular term, we eliminate it by setting

$$\omega_2 A_0 + \frac{3}{128} \frac{A_0^5}{\omega^2} = 0 \quad \text{or} \quad \omega_2 = -\frac{3}{128} \frac{A_0^4}{\omega^2} \quad (E_{25})$$

With this, the total solution of (E₂₄), after applying the initial conditions $x_2(t=0) = 0$ and $\dot{x}_2(t=0) = 0$, can be obtained as

$$x_2(t) = -\frac{A_0^5}{1024 \omega^4} (\cos \omega t - \cos 5\omega t); \quad \omega_2 = -\frac{3}{128} \frac{A_0^4}{\omega^2} \quad (E_{26})$$

Thus the total solution becomes

$$x(t) = A_0 \cos \omega t - \frac{1}{32} \frac{A_0^3 \alpha}{\omega^2} (\cos \omega t - \cos 3\omega t) \\ - \frac{1}{1024} \cdot \frac{A_0^5 \alpha^2}{\omega^4} (\cos \omega t - \cos 5\omega t)$$

and

$$\omega^2 = \omega_0^2 + \frac{3}{4} A_0^2 \alpha - \frac{3}{128} \frac{A_0^4}{\omega^2} \alpha^2 \quad (E_{27})$$

13.14

Equation: $\ddot{x} + c \dot{x} + k_1 x + k_2 x^3 = a_1 \cos 3\omega t - a_2 \sin 3\omega t \quad (E_1)$

Assume

$$x(t) = A \cos \omega t + B \cos 3\omega t + C \sin 3\omega t \quad (E_2)$$

so that $\dot{x}(t) = -\omega A \sin \omega t - 3\omega B \sin 3\omega t + 3\omega C \cos 3\omega t \quad (E_3)$

$$\ddot{x}(t) = -\omega^2 A \cos \omega t - 9\omega^2 B \cos 3\omega t - 9\omega^2 C \sin 3\omega t \quad (E_4)$$

$$\begin{aligned} x^3 &= A^3 \cos^3 \omega t + B^3 \cos^3 3\omega t + C^3 \sin^3 3\omega t \\ &+ 3A^2 B \cos^2 \omega t \cos 3\omega t + 3AB^2 \cos \omega t \cos^2 3\omega t \\ &+ 3A^2 C \cos^2 \omega t \sin 3\omega t + 3AC^2 \cos \omega t \sin^2 3\omega t \\ &+ 3B^2 C \cos^2 3\omega t \sin 3\omega t + 3BC^2 \cos 3\omega t \sin^2 3\omega t \\ &+ 6ABC \cos \omega t \cos 3\omega t \sin 3\omega t \\ &= \cos \omega t \left(\frac{3}{4} A^3 + \frac{3}{4} A^2 B + \frac{3}{2} AB^2 + \frac{3}{2} AC^2 \right) + \sin \omega t \left(\frac{3}{4} A^2 C \right) \\ &+ \cos 3\omega t \left(\frac{1}{4} A^3 + \frac{3}{4} B^3 + \frac{3}{2} A^2 B + \frac{3}{2} BC^2 \right) \\ &+ \sin 3\omega t \left(\frac{3}{4} C^3 + \frac{3}{2} A^2 C + \frac{3}{2} B^2 C - \frac{3}{4} B^2 C - \frac{3}{4} BC^2 \right) \\ &+ \cos 9\omega t \left(\frac{1}{4} B^3 \right) + \sin 9\omega t \left(-\frac{1}{4} C^3 + \frac{3}{4} B^2 C - \frac{3}{4} BC^2 \right) \\ &+ \cos 5\omega t \left(\frac{3}{4} A^2 B + \frac{3}{4} AB^2 - \frac{3}{4} AC^2 \right) \\ &+ \cos 7\omega t \left(\frac{3}{4} AB^2 - \frac{3}{4} AC^2 \right) + \sin 5\omega t \left(\frac{3}{4} A^2 C + \frac{3}{2} ABC \right) \\ &+ \sin 7\omega t \left(\frac{3}{2} ABC \right) \end{aligned} \quad (E_5)$$

For convenience, we use the notation

$$\theta = \omega t \quad \text{so that} \quad \dot{x} = \frac{dx}{dt} = \frac{dx}{d\theta} \frac{d\theta}{dt} = \omega \frac{dx}{d\theta} = \omega x'$$

$$\text{and} \quad \ddot{x} = \frac{d}{dt}(\omega x') = \omega^2 x''$$

With this, Eqs. (E₁) to (E₅) become

$$\omega^2 x'' + c\omega x' + k_1 x + k_2 x^3 = a_1 \cos 3\theta - a_2 \sin 3\theta \quad (E_6)$$

$$x(\theta) = A \cos \theta + B \cos 3\theta + C \sin 3\theta \quad (E_7)$$

$$x'(\theta) = \dot{x}/\omega = -A \sin \theta - 3B \sin 3\theta + 3C \cos 3\theta \quad (E_8)$$

$$x''(\theta) = \ddot{x}/\omega^2 = -A \cos \theta - 9B \cos 3\theta - 9C \sin 3\theta \quad (E_9)$$

$$x^3(\theta) = \left(\frac{3}{4} A^3 + \frac{3}{4} A^2 B + \frac{3}{2} AB^2 + \frac{3}{2} AC^2 \right) \cos \theta \\ + \left(\frac{3}{4} A^2 C \right) \sin \theta + \dots + \left(\frac{3}{2} ABC \right) \sin 7\theta \quad (E_{10})$$

Terms containing $\cos 5\theta$, $\sin 5\theta$, $\cos 7\theta$, $\sin 7\theta$, $\cos 9\theta$ and $\sin 9\theta$ in Eq. (E₁₀) can be neglected for the present analysis.

Then the substitution of Eqs. (E₇) to (E₁₀) into (E₆) and equating the coefficients of $\cos \theta$, $\sin \theta$, $\cos 3\theta$ and $\sin 3\theta$ on both sides gives:

$$(k_1 - \omega^2) + \frac{3}{4} k_2 (A^2 + AB + 2B^2 + 2C^2) = 0 \quad (E_{11})$$

$$-c\omega + \frac{3}{4} k_2 AC = 0 \quad (E_{12})$$

$$(k_1 - 9\omega^2)B + 3c\omega C + \frac{1}{4} k_2 (A^3 + 6A^2 B + 3B^3 + 3BC^2) = a_1 \quad (E_{13})$$

$$(k_1 - 9\omega^2)C - 3c\omega B + \frac{3}{4} k_2 C (C^2 + 2A^2 + B^2) = -a_2 \quad (E_{14})$$

Eqs. (E₁₁) to (E₁₄) represent four simultaneous nonlinear algebraic equations in A , B , C and ω . For any specific case, these equations should yield non-zero real values for A , B and C for the existence of subharmonic oscillations of order 3.

More specifically, for A :

Eqs. (E₁₁) and (E₁₂) can be solved to find

$$\omega^2 = k_1 + \frac{3}{4} k_2 (A^2 + AB + 2B^2 + 2C^2) \quad (E_{15})$$

$$C = \frac{4c\omega}{3k_2 A} \quad (E_{16})$$

Eq. (E₁₅) can be solved to obtain

$$A = -\frac{B}{2} \pm \frac{1}{2} \left\{ \frac{16}{3k_2} (\omega^2 - k_1) - 7B^2 - 8C^2 \right\}^{\frac{1}{2}} \quad (E_{17})$$

This equation gives the condition for the existence of subharmonic of order 3 (nonzero A) as

$$\frac{16}{3k_2} (\omega^2 - k_1) - 7B^2 - 8C^2 > 0$$

i.e., $\omega^2 > k_1 + \frac{3k_2}{16} (7B^2 + 8C^2)$ (E₁₈)

13.15

Equation: $\ddot{x} + c\dot{x} + k_1x + k_2x^2 = a \cos 2\omega t$ (E₁)

As in the solution of problem 13.14, we introduce

$$\theta = \omega t, \quad \dot{x} = \omega x' \quad \text{and} \quad \ddot{x} = \omega^2 x'' \quad \text{with} \quad x' = \frac{dx}{d\theta}.$$

Eq. (E₁) becomes

$$\omega^2 x'' + c\omega x' + k_1x + k_2x^2 = a \cos 2\theta \quad (E_2)$$

The solution of (E₂) must contain terms involving $\cos \theta$ and/or $\sin \theta$ in order to include subharmonics of order 2. If the nonlinear term k_2x^2 is assumed to be small with $k_2 > 0$, we can express c and a as $c = \epsilon k_2^2$ and $a = a_0 k_2$. Thus (E₂) becomes

$$\omega^2 x'' + \epsilon \omega k_2^2 x' + k_1x + k_2x^2 = k_2 a_0 \cos 2\theta \quad (E_3)$$

Using perturbation method, we assume

$$x(t) = x_0(t) + k_2 x_1(t) + k_2^2 x_2(t) + \dots \quad (E_4)$$

$$\omega = \omega_0 + k_2 \omega_1 + k_2^2 \omega_2 + \dots \quad (E_5)$$

Substituting (E₄) and (E₅) into (E₃), we get

$$\begin{aligned} & (\omega_0 + k_2 \omega_1 + k_2^2 \omega_2)^2 (x_0'' + k_2 x_1'' + k_2^2 x_2'') + \epsilon k_2^2 (\omega_0 + k_2 \omega_1 + k_2^2 \omega_2) (x_0' + k_2 x_1' + k_2^2 x_2') \\ & + k_1 (x_0 + k_2 x_1 + k_2^2 x_2) + k_2 (x_0 + k_2 x_1 + k_2^2 x_2)^2 = k_2 a_0 \cos 2\theta \end{aligned}$$

i.e.,

$$\begin{aligned} & k_2^0 (x_0'' \omega_0^2 + k_1 x_0) + k_2^1 (x_1'' \omega_0^2 + 2\omega_0 \omega_1 x_0'' + k_1 x_1 + x_0^2 - a_0 \cos 2\theta) \\ & + k_2^2 (x_2'' \omega_0^2 + x_0'' \omega_1^2 + 2\omega_0 \omega_1 x_1'' + 2\omega_0 \omega_2 x_0'' + \epsilon \omega_0 x_0' + k_1 x_2 \\ & + 2x_0 x_1) + k_2^3 (\dots) + \dots = 0 \end{aligned} \quad (E_6)$$

By setting the coefficients of various powers of k_2 equal to zero in (E₆), we get the following.

coefficient of κ_2^0 : $x_0'' \omega_0^2 + \kappa_1 x_0 = 0$ (E7)

Solution of (E7) is $x_0(t) = A_1 \cos \theta + A_2 \sin \theta$ (E8)

and $\omega_0^2 = \kappa_1$ (E9)

coefficient of κ_2^1 :

$x_1'' \omega_0^2 + \kappa_1 x_1 = -2\omega_0 \omega_1 x_0'' - x_0'^2 + a_0 \cos 2\theta$ (E10)

Substituting (E8) and (E9), Eq. (E10) becomes

$$x_1'' \omega_0^2 + \kappa_1 x_1 = -2\omega_0 \omega_1 (-A_1 \cos \theta - A_2 \sin \theta) - A_1^2 \cos^2 \theta - A_2^2 \sin^2 \theta - A_1 A_2 \sin 2\theta + a_0 \cos 2\theta$$

or,

$$x_1'' + x_1 = \left(\frac{2\omega_1 A_1}{\omega_0} \right) \cos \theta + \left(\frac{2\omega_1 A_2}{\omega_0} \right) \sin \theta + \frac{1}{\kappa_1} \left(-\frac{A_1^2}{2} + \frac{A_2^2}{2} + a_0 \right) \cos 2\theta - \left(\frac{A_1 A_2}{\kappa_1} \right) \sin 2\theta - \left(\frac{A_1^2 + A_2^2}{2\kappa_1} \right) \quad (E11)$$

Since the first two terms on the r.h.s. of (E11) lead to secular terms, we should have

$$\omega_1 = 0 \quad (E12)$$

to avoid those terms. Thus (E11) reduces to

$$x_1'' + x_1 = \frac{1}{\kappa_1} \left(-\frac{A_1^2}{2} + \frac{A_2^2}{2} + a_0 \right) \cos 2\theta - \frac{A_1 A_2}{\kappa_1} \sin 2\theta - \frac{A_1^2 + A_2^2}{2\kappa_1} \quad (E13)$$

The particular integral of (E13) can be written as

$$x_1(\theta) = -\left(\frac{A_1^2 + A_2^2}{2\kappa_1} \right) + A_3 \cos 2\theta + A_4 \sin 2\theta \quad (E14)$$

where A_3 and A_4 can be found by substituting (E14) into (E13):

$$A_3 = \frac{1}{3\kappa_1} \left(\frac{A_1^2 - A_2^2}{2} - a_0 \right), \quad A_4 = \frac{A_1 A_2}{3\kappa_1} \quad (E15)$$

coefficient of κ_2^2 :

$$\omega_0^2 x_2'' + \kappa_1 x_2 = -\omega_1^2 x_0'' - 2\omega_0 \omega_1 x_1'' - 2\omega_0 \omega_2 x_0'' - \epsilon \omega_0 x_0' - 2x_0 x_1$$

or, $x_2'' + x_2 = -2 \frac{\omega_2}{\omega_0} x_0'' - \frac{\epsilon}{\omega_0} x_0' - \frac{2}{\kappa_1} x_0 x_1$ (E16)

since $\omega_1 = 0$ and $\omega_0^2 = \kappa_1$. By substituting (E8) and (E14) on the r.h.s. of (E16), we get

$$x_2'' + x_2 = \left(\frac{2\omega_2}{\omega_0} A_1 - \frac{\epsilon}{\omega_0} A_2 + \frac{A_1}{\kappa_1^2} (A_1^2 + A_2^2) - \frac{A_1 A_3}{\kappa_1} - \frac{A_2 A_4}{\kappa_1} \right) \cos \theta + \left(\frac{2\omega_2}{\omega_0} A_2 + \frac{\epsilon}{\omega_0} A_1 + \frac{A_2}{\kappa_1^2} (A_1^2 + A_2^2) + \frac{A_2 A_3}{\kappa_1} - \frac{A_1 A_4}{\kappa_1} \right) \sin \theta$$

$$+ \left(-\frac{A_1 A_3}{k_1} + \frac{A_2 A_4}{k_1} \right) \cos 3\theta + \left(-\frac{A_2 A_3}{k_1} - \frac{A_1 A_4}{k_1} \right) \sin 3\theta \quad (E_{17})$$

To avoid secular terms, the coefficients of $\cos \theta$ and $\sin \theta$ in (E₁₇) must be zero. This gives

$$\frac{2\omega_2 A_1}{\omega_0} - \frac{\epsilon A_2}{\omega_0} + \frac{A_1}{k_1^2} (A_1^2 + A_2^2) - \frac{A_1 A_3}{k_1} - \frac{A_2 A_4}{k_1} = 0 \quad (E_{18})$$

$$\frac{2\omega_2 A_2}{\omega_0} + \frac{\epsilon A_1}{\omega_0} + \frac{A_2}{k_1^2} (A_1^2 + A_2^2) + \frac{A_2 A_3}{k_1} - \frac{A_1 A_4}{k_1} = 0 \quad (E_{19})$$

Thus (E₁₇) reduces to

$$x_2'' + x_2 = (-A_1 A_3 + A_2 A_4) \frac{1}{k_1} \cos 3\theta - (A_2 A_3 + A_1 A_4) \frac{1}{k_1} \sin 3\theta \quad (E_{20})$$

Substituting (E₁₅) into (E₁₈) and (E₁₉), we get

$$\frac{2\omega_2 A_1}{\omega_0} - \frac{\epsilon A_2}{\omega_0} + \frac{A_1}{6k_1^2} (6A_1^2 + 6A_2^2 - A_1^2 + A_2^2 - 2A_2^2) + \frac{A_1 \omega_0}{3k_1^2} = 0 \quad (E_{21})$$

$$\frac{2\omega_2 A_2}{\omega_0} + \frac{\epsilon A_1}{\omega_0} + \frac{A_2}{6k_1^2} (6A_1^2 + 6A_2^2 + A_1^2 - A_2^2 - 2A_1^2) - \frac{A_2 \omega_0}{3k_1^2} = 0 \quad (E_{22})$$

Dividing (E₂₁) and (E₂₂) by A_1 and A_2 , respectively, we get

$$\frac{2\omega_2}{\omega_0} - \frac{\epsilon}{\omega_0} \frac{A_2}{A_1} + \frac{5}{6k_1^2} (A_1^2 + A_2^2) + \frac{\omega_0}{3k_1^2} = 0 \quad (E_{23})$$

$$\frac{2\omega_2}{\omega_0} + \frac{\epsilon}{\omega_0} \frac{A_1}{A_2} + \frac{5}{6k_1^2} (A_1^2 + A_2^2) - \frac{\omega_0}{3k_1^2} = 0 \quad (E_{24})$$

Addition and subtraction of (E₂₃) and (E₂₄) give

$$\omega_2 = \frac{\epsilon}{4} \left(\frac{A_2}{A_1} - \frac{A_1}{A_2} \right) - \frac{5\omega_0}{12k_1^2} (A_1^2 + A_2^2) \quad (E_{25})$$

$$\text{and } r + \frac{1}{r} = p \quad (E_{26})$$

$$\text{where } r = A_2/A_1 \text{ and } p = \left(\frac{2\omega_0}{3\epsilon k_1 \omega_0} \right) \quad (E_{27})$$

$$\text{Solution of (E}_{26}\text{) is: } r = \frac{A_2}{A_1} = \frac{p \pm \sqrt{p^2 - 4}}{2} \quad (E_{28})$$

Thus ω_2 becomes (E₂₅):

$$\omega_2 = \pm \frac{(\omega_0^2 - 9\epsilon^2 k_1^3)^{1/2}}{6k_1 \omega_0} - \frac{5(A_1^2 + A_2^2)}{12k_1 \omega_0} \quad (E_{29})$$

The solution of E₂₀ can be determined as

$$x_2(\theta) = \left(\frac{A_1 A_3 - A_2 A_4}{8k_1} \right) \cos 3\theta + \left(\frac{A_2 A_3 + A_1 A_4}{8k_1} \right) \sin 3\theta \quad (E_{30})$$

Thus Eqs. (E₄) and (E₅) become

$$x(t) = x_0(t) + k_2 x_1(t) + k_2^2 x_2(t) ; \quad \omega = \omega_0 + k_2 \omega_1 + k_2^2 \omega_2$$

along with Eqs. (E₈), (E₉), (E₁₂), (E₁₄), (E₂₉) and (E₃₀).

13.16 Eq. (13.82) can be rewritten as

$$(\omega^2)^3 - \omega_0^2 (\omega^2)^2 - \frac{3}{4} \alpha A^2 (\omega^2)^2 + \frac{3}{32} \alpha F A (\omega^2) - \frac{3}{128} \alpha F^2 = 0 \quad (E_1)$$

Differentiation of (E₁) gives

$$d\omega^2 \left(3\omega^4 - 2\omega_0^2 \omega^2 - \frac{3}{2} \alpha A^2 \omega^2 + \frac{3}{32} \alpha F A \right) + dA \left(-\frac{3}{2} \alpha \omega^4 A + \frac{3}{32} \alpha F \omega^2 \right) = 0 \quad (E_2)$$

$$\text{Setting } \frac{d\omega^2}{dA} = 0, \text{ Eq. (E}_2\text{) gives } A = \frac{1}{16} \frac{F}{\omega^2} \quad (E_3)$$

With this Eq. (E₃), Eq. (E₁) becomes

$$\omega^6 - \omega_0^2 \omega^4 = \frac{3}{128} \alpha F^2 + \frac{3}{1024} \alpha F^2 - \frac{3}{512} \alpha F^2 = \frac{21}{1024} \alpha F^2 \quad (E_4)$$

Dividing (E₄) by ω_0^6 , we get

$$\left(\frac{\omega}{\omega_0} \right)^6 - \left(\frac{\omega}{\omega_0} \right)^4 = \frac{21}{1024} \frac{\alpha F^2}{\omega_0^6} \quad (E_5)$$

Letting $\frac{\omega}{\omega_0} = 1 + \delta$ with $\delta \ll 1$, Eq. (E₅) can be expressed as

$$(1 + \delta)^6 - (1 + \delta)^4 = \frac{21}{1024} \frac{\alpha F^2}{\omega_0^6} \quad (E_6)$$

Since $(1 + \delta)^6 \approx 1 + 6\delta$ and $(1 + \delta)^4 \approx 1 + 4\delta$,

Eq. (E₆) becomes

$$2\delta = \frac{21}{1024} \frac{\alpha F^2}{\omega_0^6}$$

$$\text{or } \omega_{\min} = \omega_0 + \frac{21}{2048} \frac{\alpha F^2}{\omega_0^5} \quad (E_7)$$

13.17

Derivation of Eq. (13.113b):

Using $a_0 = \frac{1}{4}$ and $y_0 = \sin \frac{t}{2}$, Eq. (13.103) gives

$$\ddot{y}_1 + a_0 y_1 = -a_1 y_0 - y_0 \cos t$$

$$\text{or } \ddot{y}_1 + \frac{1}{4} y_1 = \left(-a_1 + \frac{1}{2} \right) \sin \frac{t}{2} - \frac{1}{2} \sin \frac{3t}{2} \quad (E_1)$$

Homogeneous solution of (E₁) is:

$$y_1(t) = A_1 \cos \frac{t}{2} + A_2 \sin \frac{t}{2} \quad (E_2)$$

Where A_1 and A_2 are constants of integration. To avoid secular term, the coefficient of $\sin \frac{t}{2}$ must be set equal to zero in (E₁). This gives $a_1 = \frac{1}{2}$ and

$$\ddot{y}_1 + \frac{1}{4} y_1 = -\frac{1}{2} \sin \frac{3t}{2} \quad (E_3)$$

Using the particular solution

$$y_1(t) = A_2 \sin \frac{3t}{2} \quad (E_4)$$

in (E₃), we get $A_2 = \frac{1}{4}$ and $y_1(t) = \frac{1}{4} \sin \frac{3t}{2}$ (E₅)

With $\alpha_0 = \frac{1}{4}$, $\alpha_1 = \frac{1}{2}$, $y_0 = \sin \frac{t}{2}$ and $y_1 = \frac{1}{4} \sin \frac{3t}{2}$, E_g. (13.104) becomes

$$\ddot{y}_2 + \frac{1}{4} y_2 = (-\alpha_2 - \frac{1}{8}) \sin \frac{t}{2} - \frac{1}{8} \sin \frac{3t}{2} - \frac{1}{8} \sin \frac{5t}{2} \quad (E_6)$$

Again the coefficient of $\sin \frac{t}{2}$ in (E₆) must be set equal to zero to avoid the secular term. This gives $\alpha_2 = -\frac{1}{8}$. Thus E_g. (13.100) becomes

$$\omega = \alpha_0 + \epsilon \alpha_1 + \epsilon^2 \alpha_2 = \frac{1}{4} + \frac{\epsilon}{2} - \frac{\epsilon^2}{8} \quad (13.113b)$$

Derivation of E_g. (13.116b):

with $\alpha_0 = 1$ and $y_0 = \sin t$, E_g. (13.103) becomes

$$\ddot{y}_1 + y_1 = -\alpha_1 y_0 - y_0 \cos t = -\alpha_1 \sin t - \frac{1}{2} \sin 2t \quad (E_7)$$

Coefficient of $\sin t$ in (E₇) must be zero to avoid the secular term. This gives $\alpha_1 = 0$ and hence (E₇) reduces to

$$\ddot{y}_1 + y_1 = -\frac{1}{2} \sin 2t \quad (E_8)$$

By substituting the particular solution

$$y_1(t) = A_2 \sin 2t \quad (E_9)$$

into (E₈), we get $A_2 = \frac{1}{6}$. Using $\alpha_0 = 1$, $y_0 = \sin t$, $\alpha_1 = 0$ and $y_1 = \frac{1}{6} \sin 2t$, E_g. (13.104) becomes

$$\begin{aligned} \ddot{y}_2 + \alpha_0 y_2 &= -\alpha_2 y_0 - \alpha_1 y_1 - y_1 \cos t \\ &= (-\alpha_2 - \frac{1}{12}) \sin t - \frac{1}{12} \sin 3t \end{aligned} \quad (E_{10})$$

To avoid secular terms, the coefficient of $\sin t$ on the r.h.s. of E_g. (E₁₀) must be zero. This gives $\alpha_2 = -\frac{1}{12}$. Thus E_g. (13.100) becomes

$$\omega = \alpha_0 + \epsilon \alpha_1 + \epsilon^2 \alpha_2 = 1 - \frac{1}{12} \epsilon^2 \quad (13.116b)$$

13.18

$\ddot{x} + 0.4 \dot{x} + 0.8 x = 0$. This is in the form $\ddot{x} + 2\gamma\omega_n \dot{x} + \omega_n^2 x = 0$
with $\omega_n = \sqrt{0.8} = 0.89443$ and $\gamma = 0.22361$.

$$\omega_d = \omega_n \sqrt{1 - \gamma^2} = 0.87178.$$

$$\text{Solution is } x(t) = e^{-\gamma\omega_n t} (A_1 \cos \omega_d t + A_2 \sin \omega_d t)$$

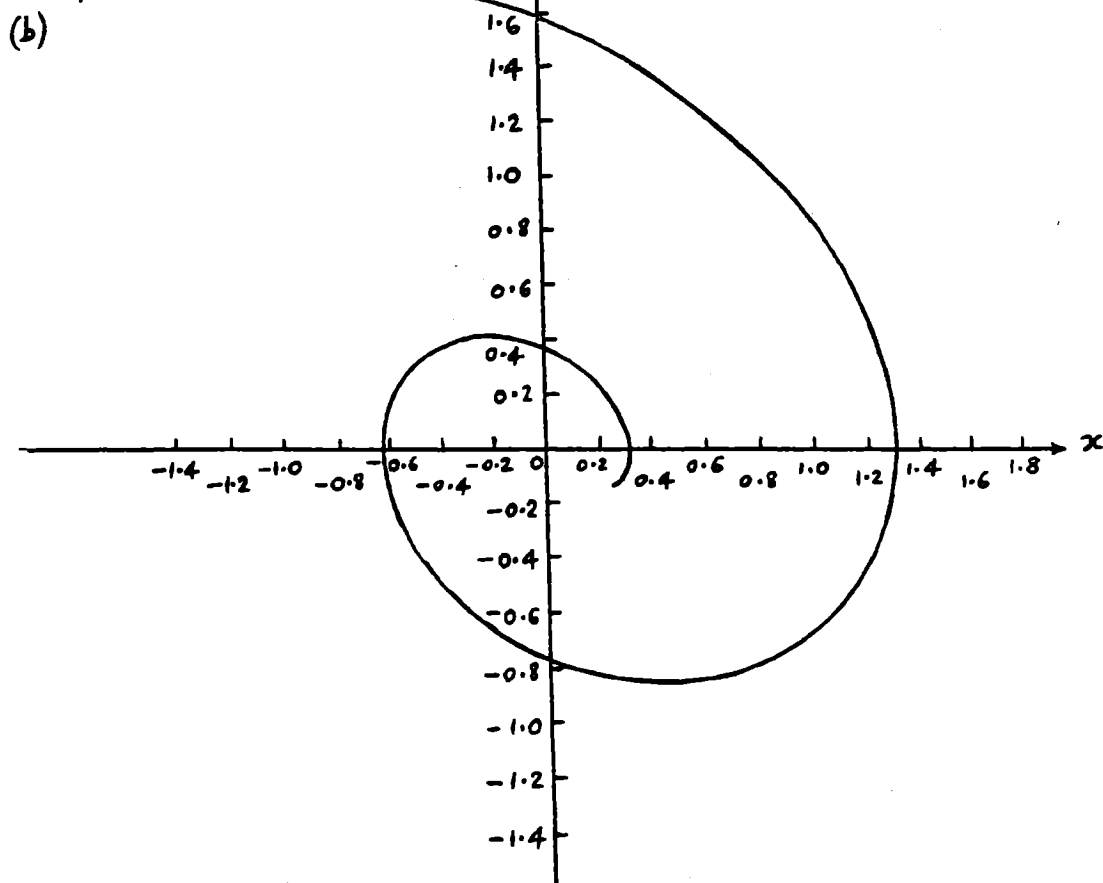
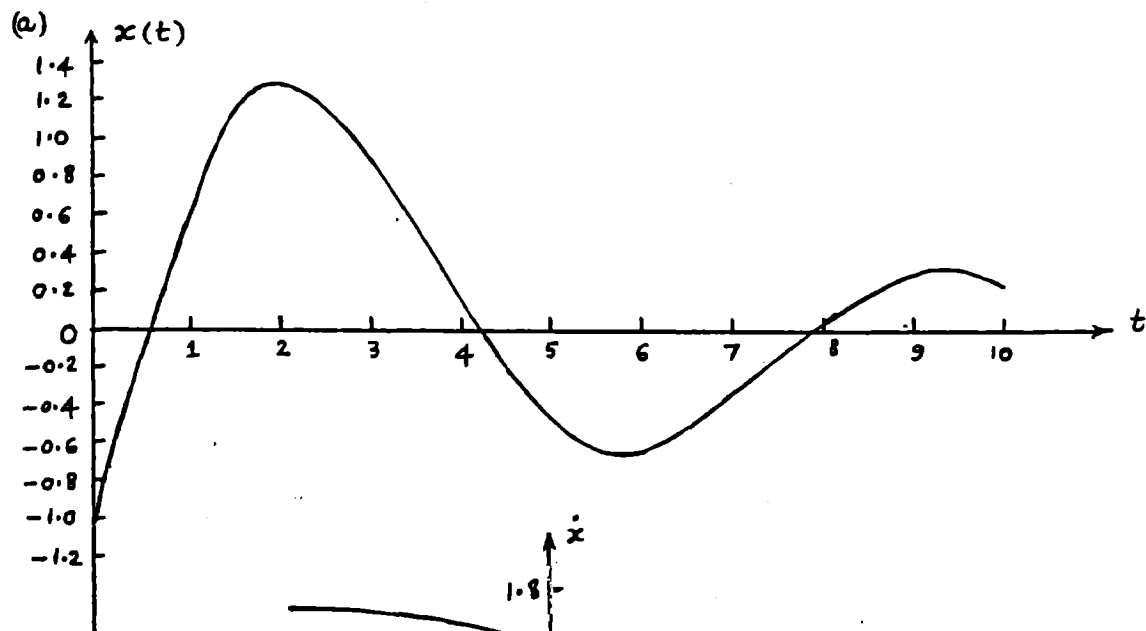
$$\dot{x}(t) = e^{-\gamma\omega_n t} [(-\gamma\omega_n A_1 + \omega_d A_2) \cos \omega_d t - (\omega_d A_1 + \gamma\omega_n A_2) \sin \omega_d t]$$

Using $x(0) = A_1 = -1$ and $\dot{x}(0) = -\gamma\omega_n A_1 + \omega_d A_2 = 2$, we obtain

$$A_1 = -1 \text{ and } A_2 = (2 + \gamma\omega_n A_1) / \omega_d = 1.77058$$

$$\therefore x(t) = e^{-0.2t} (-\cos 0.87178t + 1.77058 \sin 0.87178t)$$

and $\dot{x}(t) = e^{-0.2t} (1.74356 \cos 0.87178t + 0.51766 \sin 0.87178t)$



13.19

$$\ddot{x} + 0.1(x^2 - 1)\dot{x} + x = 0 \quad \text{or} \quad \ddot{x} = -[0.1(x^2 - 1)\dot{x} + x]$$

$$\text{Let } x = x_1, \quad \dot{x}_1 = x_2 = f_1(x_1, x_2)$$

$$\dot{x}_2 = -[0.1(x_1^2 - 1)x_2 + x_1] = f_2(x_1, x_2)$$

For equilibrium, from Eq. (13.129),

$$f_1 = 0 \Rightarrow x_2 = 0; \quad f_2 = 0 \Rightarrow x_1 = 0$$

Eq. (13.132) becomes

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

where

$$a_{11} = \left. \frac{\partial f_1}{\partial x_1} \right|_{(0,0)} = 0, \quad a_{12} = \left. \frac{\partial f_1}{\partial x_2} \right|_{(0,0)} = 1,$$

$$a_{21} = \left. \frac{\partial f_2}{\partial x_1} \right|_{(0,0)} = -[0.2x_1x_2 + 1]_{(0,0)} = -1,$$

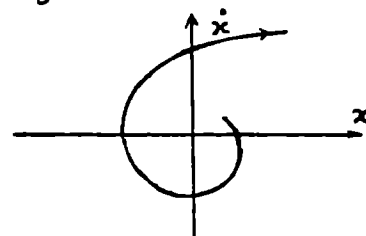
$$a_{22} = \left. \frac{\partial f_2}{\partial x_2} \right|_{(0,0)} = -[0.1(x_1^2 - 1)]_{(0,0)} = 0.1$$

From Eq. (13.135), we find $p = 0.1$, $q = 1$

$$\lambda_1, \lambda_2 = \frac{1}{2}(0.1 \pm \sqrt{0.01 - 4}) = \text{complex with positive real parts}$$

Since $p > 0$, the system is unstable at the equilibrium point $(x, \dot{x}) = (0, 0)$.

Hence the phase-plane trajectory in the neighborhood of the equilibrium position appears as shown in the figure.



13.20

$$\ddot{x} + 0.4\dot{x} + 0.8x = 0$$

$$\text{Let } x = x, \quad \dot{x} = y, \quad \dot{y} = -0.4y - 0.8x$$

Initial conditions:

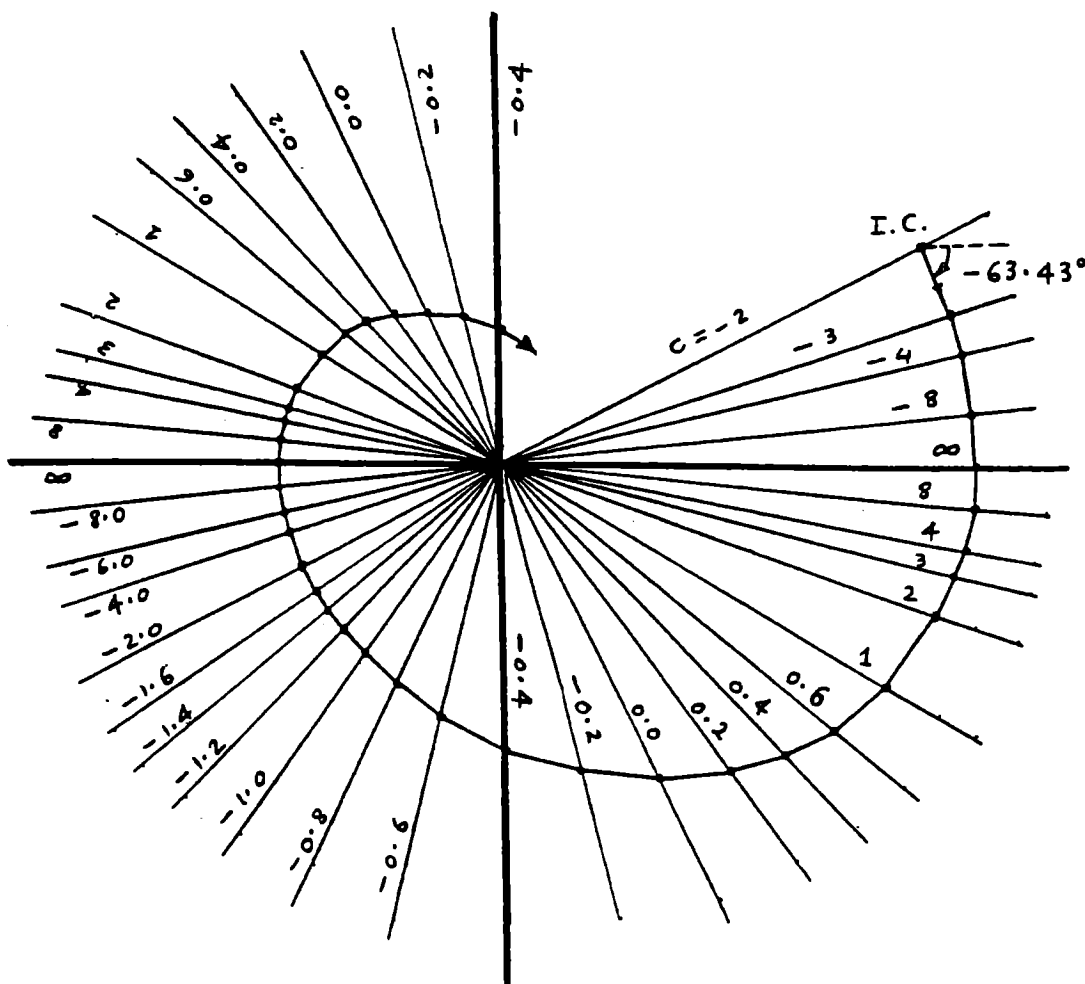
$$x(0) = 2, \quad y(0) = 1$$

$$\frac{dy}{dx} = \frac{-0.4y - 0.8x}{y} = C = \text{slope of the trajectory}$$

Equation of isocline is $-0.4y - 0.8x = cy$ or $\frac{y}{x} = \frac{-0.8}{c + 0.4}$
This equation is plotted for different values of c . Then small line segments are drawn and slopes are used for extrapolation.

c	-2	-3	-4	-8	∞	8	4
value of y/x	0.5	0.3077	0.2222	0.1053	0	-0.0952	-0.1818
Angle	26.5651°	17.10°	12.53°	6.01°	0°	-5.44°	-10.30°
Slope of trajectory	-63.43°	-71.56°	-75.96°	-82.87°	-90°	82.87°	75.96°

3	2	1	0.6	0.4	0.2	0	-0.2	-0.4
-0.2353	-0.3333	-0.5714	-0.8	-1.0	-1.3333	-2.0	-4.0	-∞
-13.24°	-18.43°	-29.74°	-38.65°	-45°	-53.13°	-63.43°	-75.96°	-90°
71.56°	63.43°	45°	30.96°	21.80°	11.31°	0°	-11.31°	-21.80°
-0.6	-0.8	-1.0	-1.2	-1.6	-2.0	-4	-8	∞
4.0	2.0	1.3333	1.0	0.6667	0.5	0.2222	0.1053	0
75.96°	63.43°	53.13°	45°	33.69°	26.57°	12.53°	6.01°	0°
-30.96°	-38.66°	-45.00°	-50.19°	-57.99°	-63.43°	-75.96°	-82.87°	-90°



13.21 Equation of motion: $\ddot{x} + 0.1 \dot{x} + x = 5$ (E₁)

Letting $\frac{dx}{dt} = y$, $\frac{dy}{dt} = 5 - x - 0.1 y$ (E₂)

we obtain $\frac{dy}{dx} = \frac{5 - x - 0.1 y}{y}$ (E₃)

Instead of integrating (E₃) directly, we find the solution of (E₁) using the procedures of chapter 4.

Noting that $m=1$, $c=0.1$, $k=1$, $F_0=5$, we find

$$\omega_n = \sqrt{\frac{k}{m}} = 1 \text{ and } \zeta = \frac{c}{2\sqrt{km}} = \frac{0.1}{2(1)} = 0.05.$$

Hence the solution of (E₁), with the initial conditions

$x(0) = \dot{x}(0) = 0$, is given by [see Example 4.6] :

$$x(t) = \frac{F_0}{k} \left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos(\omega_d t - \phi) \right]$$

$$\text{where } \omega_d = \omega_n \sqrt{1-\zeta^2} = 1 \sqrt{1-0.05^2} = 0.9987$$

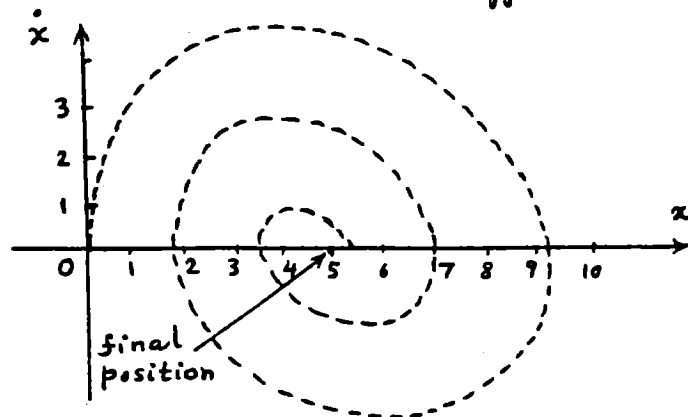
$$\text{and } \phi = \tan^{-1} \left(\frac{\zeta}{\sqrt{1-\zeta^2}} \right) = \tan^{-1} \left(\frac{0.05}{0.9987} \right) = 2.8681^\circ$$

$$\therefore x(t) = 5 \left[1 - 1.0013 e^{-0.05t} \cos(0.9987t - 2.8681^\circ) \right] \quad (E_4)$$

Velocity is given by (differentiating E₄·(E₄)) :

$$\begin{aligned} \dot{x}(t) = & 0.2503 e^{-0.05t} \cos(0.9987t - 2.8681^\circ) \\ & + 5.0 e^{-0.05t} \sin(0.9987t - 2.8681^\circ) \end{aligned} \quad (E_5)$$

The trajectory in the phase-plane ($x-\dot{x}$ plane) appears as shown below.



13.22 Equation of motion: $\ddot{x} + f \frac{\dot{x}}{|\dot{x}|} + \omega_n^2 x = 0 \quad (E_1)$

i.e. $\ddot{x} + \omega_n^2 (x+a) = 0 \quad \text{for } \dot{x} > 0 \quad (E_2)$

and $\ddot{x} + \omega_n^2 (x-a) = 0 \quad \text{for } \dot{x} < 0 \quad (E_3)$

where $a = f/\omega_n^2 \quad (E_4)$

Multiplying by $2\dot{x}$ and integrating, (E₂) and (E₃) yield

$$\dot{x}^2 + \omega_n^2 (x+a)^2 = R_j^2 \quad \text{for } \dot{x} > 0 \quad (E_5)$$

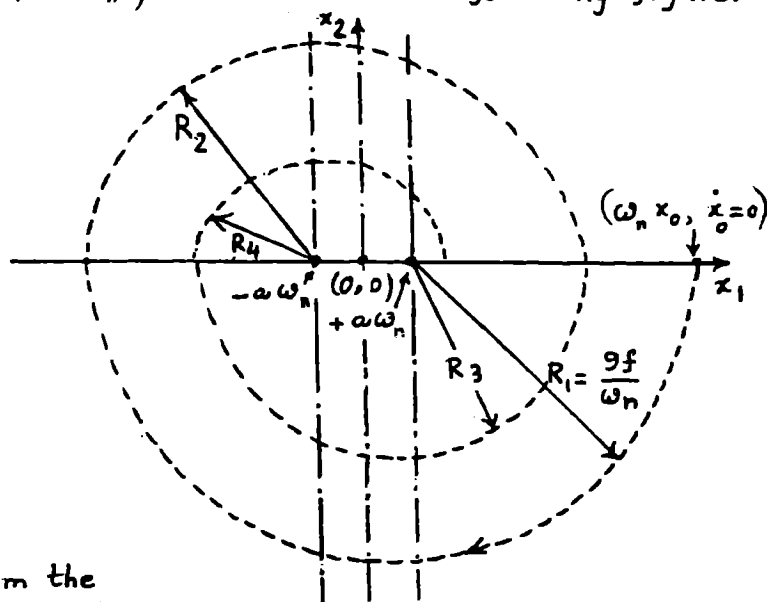
$$\dot{x}^2 + \omega_n^2 (x-a)^2 = R_{j+1}^2 \quad \text{for } \dot{x} < 0 \quad (E_6)$$

where R_j^2 and R_{j+1}^2 are integration constants which are to be computed at each switching of sign of $\dot{x}$.

We can plot the trajectories of a representative point whose coordinates are

$$x_1 = \omega_n x, \quad x_2 = \dot{x} \quad (E_7)$$

Eqs. (E₅) and (E₆) show that the trajectory is made of semicircles whose centers are located at $x = -a$ (or $x_1 = -a\omega_n$) and $x = +a$ (or $x_1 = +a\omega_n$) as shown in the following figure.



R_1 can be obtained from the initial conditions using Eq. (E₆) as:

$$R_1^2 = 0^2 + \omega_n^2 \left(\frac{10f}{\omega_n^2} - \frac{f}{\omega_n^2} \right)^2 = \left(\frac{9f}{\omega_n} \right)^2 ; \quad R_1 = \frac{9f}{\omega_n} \quad (E_8)$$

Notice that the radii of the circles $R_1, R_2, \dots$ decrease according to the relation

$$R_j = R_{j-1} - 2a\omega_n ; \quad j = 1, 2, \dots$$

and the system will stop when

$$R_k \leq 2a\omega_n$$

$$\text{Here } R_1 = \frac{9f}{\omega_n}, \quad R_2 = R_1 - \frac{2f}{\omega_n} = \frac{7f}{\omega_n}, \quad R_3 = R_2 - \frac{2f}{\omega_n} = \frac{5f}{\omega_n},$$

$$R_4 = R_3 - \frac{2f}{\omega_n} = \frac{3f}{\omega_n}, \quad R_5 = R_4 - \frac{2f}{\omega_n} = \frac{f}{\omega_n},$$

and the motion stops at this point (after five half-cycles) since $R_6 < 2a\omega_n = \frac{2f}{\omega_n}$.

13.23

$$\ddot{\theta} + c \dot{\theta} + \sin \theta = 0 \quad \text{or} \quad \ddot{\theta} = -c \dot{\theta} - \sin \theta$$

$$\text{Let } x = \theta \text{ and } y = \frac{dx}{dt} = \dot{\theta}$$

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -cy - \sin x \quad (E_1)$$

Equilibrium or critical point (where $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$) of this system is $(x=0, y=0)$. Linearization of Eqs. (E₁) about the equilibrium point (origin) leads to

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -cy - x$$

$$\text{or} \quad \begin{Bmatrix} dx/dt \\ dy/dt \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} \quad (E_2)$$

The eigenvalues of this system are given by

$$\left| \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\text{i.e.} \quad \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda - c \end{vmatrix} = 0 \quad \text{i.e.,} \quad \lambda^2 + \lambda c + 1 = \lambda^2 + p\lambda + q = 0$$

$$\text{i.e.,} \quad \lambda_{1,2} = -\frac{c}{2} \pm \sqrt{\left(\frac{c}{2}\right)^2 - 1} \quad (E_3)$$

$$\text{If } c=0: \quad p=0; q=1; \lambda_{1,2} = \pm \sqrt{-1}$$

The origin will be a center.

$$\text{If } 0 < c < 2: \quad p > 0; q > 0; \lambda_{1,2} = \text{complex conjugates}$$

The origin will be a stable focal point (spiral point).

$$\text{If } c=2: \quad p > 0; q > 0; \lambda_{1,2} = \text{negative and equal.}$$

The origin will be a stable nodal point.

$$\text{If } c > 2: \quad p > 0; q > 0; \text{If } \lambda_{1,2} = \text{negative real, the origin will be a stable nodal point.}$$

$$\text{If } -2 < c < 0: \quad p < 0; q > 0; \lambda_{1,2} = \text{complex conjugates.}$$

The origin will be an unstable focal point (spiral point).

13.24

$$\text{Equation of motion:} \quad \ddot{\theta} + 0.5 \dot{\theta} + \sin \theta = 0.8 \quad (E_1)$$

$$\text{Let } x = \theta \text{ and } y = dx/dt$$

$$\therefore \frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\sin x - 0.5 y + 0.8 \quad (E_2)$$

$$\frac{dy}{dx} = \frac{-\sin x - 0.5y + 0.8}{y} \quad (E_3)$$

At $(x = \sin^{-1} 0.8, y = 0)$, $\frac{dy}{dx} = \frac{0}{0}$ and hence it will be an equilibrium point. To investigate the nature of singularity, we rewrite Eqs. (E₂) in linearized form as

$$\left. \begin{aligned} \frac{dx}{dt} &= (0)x + (1)y \\ \frac{dy}{dt} &= (0)x - 0.5y \end{aligned} \right\} \quad (E_4)$$

Thus the eigenvalues of the system are given by

$$\left| \begin{bmatrix} 0 & 1 \\ 0 & -0.5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0 \quad \text{or} \quad \begin{vmatrix} -\lambda & 1 \\ 0 & -0.5 - \lambda \end{vmatrix} = 0$$

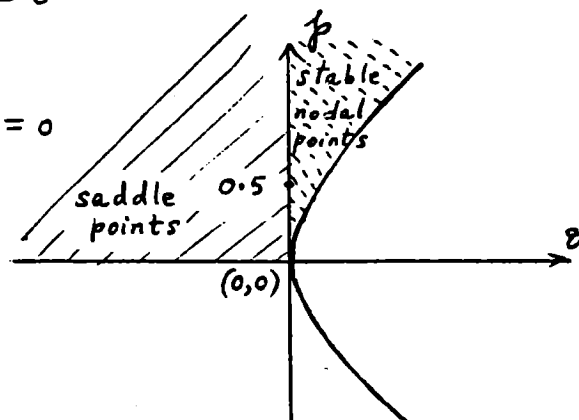
$$\text{i.e., } \lambda^2 + 0.5\lambda \equiv \lambda^2 + p\lambda + q = 0$$

$$\therefore \lambda_1 = 0, \lambda_2 = \text{negative}$$

Here $p = \text{positive}$, $q = 0$, $\lambda_1 = 0$

and $\lambda_2 = \text{negative}$.

Thus the equilibrium point falls on the border of saddle points and stable nodal points as shown in the adjacent figure.



13.25

$$\frac{dx}{dt} = (0)x + (1)y \quad (E_1)$$

$$\frac{dy}{dt} = -1.x - c.y + (0.1)x^3 \quad (E_2)$$

Eqs. (E₁) and (E₂) are zero at $(x=0, y=0)$. Hence the origin $(0,0)$ will be equilibrium point (singularity). The eigenvalues are given by

$$\left| \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} -\lambda & 1 \\ -1 & -c-\lambda \end{vmatrix} = 0$$

$$\text{i.e., } \lambda^2 + \lambda c + 1 \equiv \lambda^2 + p\lambda + q = 0$$

$$\text{i.e., } \lambda_{1,2} = \left\{ \frac{-c \pm \sqrt{c^2 - 4}}{2} \right\}$$

For $c > 0$ and $c < 2$:

$p > 0$, $q > 0$ and $\lambda_{1,2} = \text{complex conjugates}$.

Hence the origin will be a stable focus (or spiral point).

For $c \geq 2$:

$p > 0$, $q > 0$; $\lambda_{1,2} = \text{negative real}$.

Hence the origin will be a stable nodal point.

13.26

$$\ddot{x} - \alpha \dot{x} (1 - x^2) + x = 0, \quad \alpha > 0 \quad (E_1)$$

This equation can be rewritten as

$$\left. \begin{aligned} dx/dt &= y \\ dy/dt &= -x + y \alpha (1 - x^2) \end{aligned} \right\} \quad (E_2)$$

$$\therefore \frac{dy}{dx} = \frac{-x + y \alpha (1 - x^2)}{y} \quad (E_3)$$

Thus the system has singularity at $(x=0, y=0)$. Near the origin, the nature of singularity can be investigated by considering the linearized equations:

$$\left. \begin{aligned} dx/dt &= (0)x + (1)y \\ dy/dt &= (-1)x + (\alpha)y \end{aligned} \right\} \quad (E_4)$$

The eigenvalues corresponding to Eqs. (E₄) are given by

$$\left| \begin{bmatrix} 0 & 1 \\ -1 & \alpha \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} -\lambda & 1 \\ -1 & \alpha - \lambda \end{vmatrix} = 0$$

$$\text{i.e., } \lambda^2 - \alpha \lambda + 1 \equiv \lambda^2 + p \lambda + q = 0$$

$$\text{i.e., } \lambda_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 - 4}}{2}$$

Here $p = -\alpha$, $q = 1$ and $\lambda_{1,2} = \text{complex conjugates}$ for $\alpha > 0$ and $\alpha = \text{small}$. Hence the origin will be an unstable focus (spiral point).

13.27

$$\ddot{x} + \omega_n^2 x (1 + \kappa^2 x^2) = 0 \quad (E_1)$$

$$\text{Let } dx/dt = y, \quad dy/dt = -\omega_n^2 x - \omega_n^2 \kappa^2 x^3 \quad (E_2)$$

singularity is at $(x=0, y=0)$. Eigenvalues are given by

$$\left| \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} -\lambda & 1 \\ -\omega_n^2 & -\lambda \end{vmatrix} = 0$$

$$\text{i.e., } \lambda^2 + \omega_n^2 \equiv \lambda^2 + p\lambda + q = 0$$

$$\text{i.e., } \lambda_{1,2} = \pm i\omega_n$$

Here $p=0$; $q = \omega_n^2 = \text{positive}$; $\lambda_{1,2} = \text{imaginary (conjugates)}$

$\therefore$ Equilibrium point is a center.

$$\text{13.28 } \ddot{x} + \omega_n^2 x (1 - \kappa^2 x^2) = 0 \quad (E_1)$$

$$\text{with } dx/dt = y, \quad dy/dt = -\omega_n^2 x + \omega_n^2 \kappa^2 x^3 \quad (E_2)$$

Singularity at $(x=0, y=0)$.

Linearized equations about origin are

$$dx/dt = y, \quad dy/dt = -\omega_n^2 x \quad (E_3)$$

Eigenvalues are given by

$$\left| \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\text{i.e., } \lambda^2 + \omega_n^2 \equiv \lambda^2 + p\lambda + q = 0$$

$$\text{i.e., } \lambda_{1,2} = \pm i\omega_n$$

$\therefore p=0$; $q = \text{positive} = \omega_n^2$; $\lambda_{1,2} = \text{conjugates (imaginary)}$

Hence equilibrium point is a center.

$$\text{13.29 } \ddot{\theta} + \omega_n^2 \sin \theta = 0 \quad (E_1)$$

Using $x = \theta$, E_1 can be expressed as

$$dx/dt = y, \quad dy/dt = -\omega_n^2 \sin x \quad (E_2)$$

Singularity is at $(x=0, y=0)$.

Linearized equations about origin yield

$$dx/dt = y, \quad dy/dt = -\omega_n^2 x$$

$$\text{Eigenvalues are given by } \left| \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\text{i.e., } \lambda^2 + \omega_n^2 \equiv \lambda^2 + p\lambda + q = 0; \quad \lambda_{1,2} = \pm i\omega_n$$

Here $p=0$; $q = \omega_n^2 = \text{positive}$; $\lambda_{1,2} = \text{imaginary (conjugates)}$

$\therefore$ Equilibrium point is a center.

13.30 (a) $\dot{x} = x - y$
 $\dot{y} = x + 3y$

Assuming the solution as $\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} X \\ Y \end{Bmatrix} e^{\lambda t}$,

we get the following eigenvalue problem

$$\begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix} \begin{Bmatrix} X \\ Y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (E_1)$$

The eigenvalues are given by

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 4 = 0$$

i.e., $\lambda_{1,2} = \frac{4 \pm \sqrt{16-4}}{2} = 2, 2$

Both eigenvalues are same.

Using $\lambda_1 = 2$ in the first equation of (E_1) , we get

$$Y = -X.$$

Letting $X=1$ arbitrarily, we get $\begin{Bmatrix} X \\ Y \end{Bmatrix}^{(1)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$ for $\lambda_1 = 2$ (E₂)

Since the substitution of $\lambda_2 = 2$ gives the same eigenvector as in (E_2) , we seek a

second solution in the form $\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} X \\ Y \end{Bmatrix} t e^{2t}$ (E₃)

Substitution of (E_3) into the original equations gives

$$\begin{Bmatrix} X \\ Y \end{Bmatrix} e^{2t} + 2 \begin{Bmatrix} X \\ Y \end{Bmatrix} t e^{2t} - \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{Bmatrix} X \\ Y \end{Bmatrix} t e^{2t} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (E_4)$$

It can be seen that the only way (E_4) can be satisfied for all values of t is to have $\begin{Bmatrix} X \\ Y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$. This implies that solution of the form of E_3 cannot exist. Hence we seek a solution of the form

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} e^{2t} + \begin{Bmatrix} x_2 \\ y_2 \end{Bmatrix} t e^{2t} \quad (E_5)$$

where $\begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix}$ and $\begin{Bmatrix} x_2 \\ y_2 \end{Bmatrix}$ are to be determined. Substitution of

(E_5) into original equations gives

$$2 \begin{Bmatrix} x_2 \\ y_2 \end{Bmatrix} t e^{2t} + \left(\begin{Bmatrix} x_2 \\ y_2 \end{Bmatrix} + 2 \begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} \right) e^{2t} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \left(\begin{Bmatrix} x_2 \\ y_2 \end{Bmatrix} t e^{2t} + \begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} e^{2t} \right) \quad (E_6)$$

By equating the coefficients of $t e^{2t}$ and e^{2t} on both sides of (E_6) , we get

$$2 \begin{Bmatrix} x_2 \\ y_2 \end{Bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{Bmatrix} x_2 \\ y_2 \end{Bmatrix} \quad (E_7)$$

$$\text{and } \begin{Bmatrix} x_2 \\ y_2 \end{Bmatrix} + 2 \begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} \quad (E_8)$$

It can be seen that (E_7) will be satisfied if

$$\begin{Bmatrix} x_2 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = \text{eigenvector corresponding to } \lambda_1 = 2.$$

Eq. (E_8) can be rewritten in scalar form as

$$\left. \begin{aligned} x_2 + 2x_1 &= x_1 - y_1 \\ y_2 + 2y_1 &= x_1 + 3y_1 \end{aligned} \right\} \text{ or, } \left. \begin{aligned} x_1 + y_1 &= -x_2 \\ -x_1 - y_1 &= -y_2 \end{aligned} \right\} \quad (E_9)$$

Thus for $x_2 = 1$ and $y_2 = -1$, (E_9) give $x_1 = c = \text{any constant}$ and $y_1 = -1 - c$.

$$\begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} = \begin{Bmatrix} c \\ -1-c \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} + c \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \quad (E_{10})$$

$\therefore$ The solution of (E_5) becomes

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} e^{2t} + c \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} e^{2t} + \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} t e^{2t} \quad (E_{11})$$

We can ignore the middle term on the r.h.s. of (E_{11}) since it is same as $\begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix}^{(1)}$. Thus the second solution of original equations is

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} e^{2t} + \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} t e^{2t} \quad (E_{12})$$

$$(b) \quad \begin{cases} \dot{x} = x + y \\ \dot{y} = 4x + y \end{cases} \quad (E_1)$$

$$\text{Assuming the solution as } \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} x \\ y \end{Bmatrix} e^{\lambda t} \quad (E_2)$$

Eq. (E_1) give

$$\begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (E_3)$$

$$\text{Eigenvalues are given by } \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = 0 \quad (E_4)$$

$$\text{i.e., } \lambda_1 = -1, \quad \lambda_2 = 3$$

$$\text{Substitution of } \lambda_1 = -1 \text{ in } (E_3) \text{ gives } \left. \begin{aligned} 2x^{(1)} + y^{(1)} &= 0 \\ 4x^{(1)} + 2y^{(1)} &= 0 \end{aligned} \right\}$$

$$\text{or } y^{(1)} = -2x^{(1)}$$

choosing $x^{(1)} = 1$, arbitrarily, the first eigenvector becomes

$$\begin{Bmatrix} x \\ y \end{Bmatrix}^{(1)} = \begin{Bmatrix} 1 \\ -2 \end{Bmatrix}$$

Next, by substituting $\lambda_2 = 3$ in (E_3) , we get $-2x^{(2)} + y^{(2)} = 0$

or $y^{(2)} = 2x^{(2)}$. By choosing $x^{(2)} = 1$ arbitrarily, the second eigenvector becomes $\begin{Bmatrix} x \\ y \end{Bmatrix}^{(2)} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$.

$$(13.31) \quad \begin{cases} \dot{x} = x - 2y \\ \dot{y} = 4x - 5y \end{cases}, \quad \frac{dy}{dx} = \frac{4x - 5y}{x - 2y} \quad (E_1)$$

Eigenvalues are defined by

$$\left[\begin{bmatrix} 1 & -2 \\ 4 & -5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (E_2)$$

$$\text{This gives } \begin{vmatrix} 1-\lambda & -2 \\ 4 & -5-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 3 = 0$$

$$\lambda_1 = -3, \quad \lambda_2 = -1.$$

Eigenvectors: For $\lambda_1 = -3$, $E_2(E_2)$ gives $4x = 2y$

For $\lambda_2 = -1$, $E_2(E_2)$ gives $2x = 2y$

$$\vec{x}^{(1)} = \begin{Bmatrix} x \\ y \end{Bmatrix}^{(1)} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \quad \text{and} \quad \vec{x}^{(2)} = \begin{Bmatrix} x \\ y \end{Bmatrix}^{(2)} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Thus the solutions are: $\vec{x}^{(1)} = \begin{Bmatrix} x \\ y \end{Bmatrix}^{(1)} e^{\lambda_1 t}$, $\vec{x}^{(2)} = \begin{Bmatrix} x \\ y \end{Bmatrix}^{(2)} e^{\lambda_2 t}$

General solution is

$$\vec{x} = \begin{Bmatrix} x(t) \\ y(t) \end{Bmatrix} = c_1 \vec{x}^{(1)}(t) + c_2 \vec{x}^{(2)}(t) = c_1 \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} e^{-3t} + c_2 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} e^{-t} \quad (E_3)$$

where c_1 and c_2 are arbitrary constants.

The solution can be represented in the xy plane for various values of c_1 and c_2 .

First, we consider $c_2 = 0$ so that

$$x = c_1 e^{-3t}, \quad y = 2c_1 e^{-3t}$$

Eliminating t between these,

we find that the solution lies on the straight line $y = 2x$.

The solution $\vec{x}^{(1)}(t)$ approaches origin along the line $y = 2x$.

Similarly, $\vec{x}^{(2)}(t)$ approaches origin along the line $y = x$.

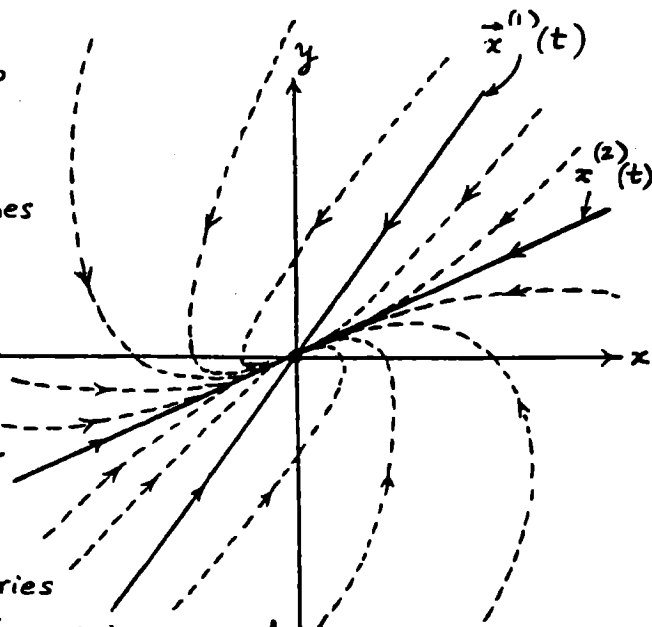
As $t \rightarrow \infty$, $\vec{x}^{(1)}(t)$ is negligible compared to $\vec{x}^{(2)}(t)$.

Hence $\vec{x}(t)$ of (E_3)

approaches origin tangential

to line $y = x$. The trajectories

are shown in the figure. Here the origin is a node.



$$\begin{aligned} \dot{x} &= x - y, \quad \dot{y} = x + 3y & (E_1) \\ \frac{dy}{dx} &= \frac{x + 3y}{x - y} & (E_2) \end{aligned}$$

The eigenvalues and eigenvectors of this system were found in Problem 13.30(a) as

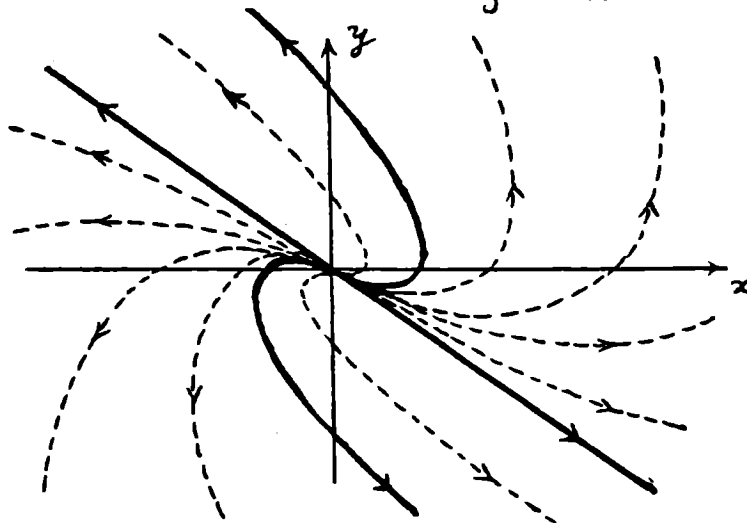
$$\lambda_1 = 2, \quad \lambda_2 = 2, \quad \vec{X}^{(1)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = \text{only one independent eigenvector}$$

$$\vec{x}^{(1)}(t) = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} e^{2t}, \quad \vec{x}^{(2)}(t) = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} t e^{2t} + \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} e^{2t} \quad (E_3)$$

The general solution of (E₁) is

$$\vec{x}(t) = c_1 \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} e^{2t} + c_2 \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} t e^{2t} + c_2 \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} e^{2t} \quad (E_4)$$

To draw the trajectories given by (E₄), we first observe the following behavior. As $t \rightarrow \infty$, $\vec{x}(t)$ becomes unbounded and as $t \rightarrow -\infty$, $\vec{x}(t) \rightarrow \vec{0}$. Also, as $t \rightarrow -\infty$, the solutions approach the origin $\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ tangentially to the line $y = -x$ (which denotes the eigenvector, $\vec{X}^{(1)}$). Further, as $t \rightarrow \infty$, the trajectories will lie asymptotic to the line $y = -x$. The origin will be a node in this case. The trajectories are shown below.



$$\begin{aligned} \dot{x} &= 2x + y, \quad \dot{y} = -3x - 2y & (E_1) \\ \frac{dy}{dx} &= \frac{-3x - 2y}{2x + y} & (E_2) \end{aligned}$$

$$\text{Eigenvalue problem is: } \left[\begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (E_3)$$

$$\begin{aligned} \text{Eigenvalues are given by } \begin{vmatrix} 2-\lambda & 1 \\ -3 & -2-\lambda \end{vmatrix} &= \lambda^2 - 1 = 0 \\ \lambda_1 &= -1, \quad \lambda_2 = 1 & (E_4) \end{aligned}$$

Eigenvectors: For $\lambda_1 = -1$; $(2 - \lambda_1)X + Y = 0 \Rightarrow Y = -3X$.

For $\lambda_2 = 1$; $(2 - \lambda_2)X + Y = 0 \Rightarrow Y = -X$

$$\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ -3 \end{Bmatrix}, \quad \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

Hence the general solution is: $\vec{x}(t) = c_1 \begin{Bmatrix} 1 \\ -3 \end{Bmatrix} e^{-t} + c_2 \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} e^t$ (E₅)
where c_1 and c_2 are arbitrary constants.

To plot the trajectories, first consider the case of $c_1 = 0$.

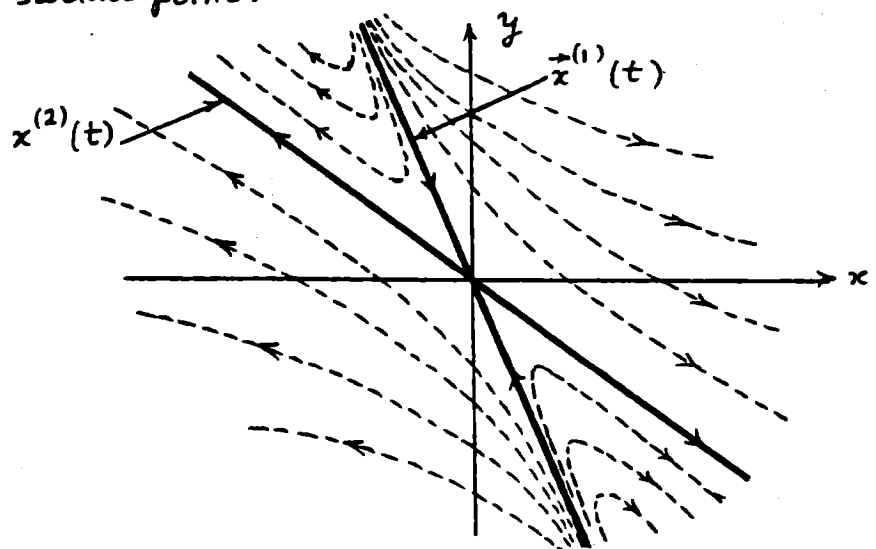
$$\vec{x}(t) = c_2 \vec{x}^{(2)}(t) \quad \text{i.e.,} \quad x(t) = c_2 e^t, \quad y(t) = -c_2 e^t \quad (E_6)$$

Eliminating t from (E₆), we get $y = -x$. This line denotes the eigenvector $\vec{X}^{(2)}$. If we consider the trajectory of a representative point, the particle moves away from origin as t increases. Next, we consider $c_2 = 0$. Eq. (E₅) gives

$$x(t) = c_1 e^{-t}, \quad y(t) = -3c_1 e^{-t} \quad (E_7)$$

This leads to the equation $y = -3x$, which represents the eigenvector, $\vec{X}^{(1)}$. A representative point in this case goes to origin as $t \rightarrow \infty$.

In the general solution of (E₅), the term $c_2 \vec{x}^{(2)}(t)$ dominates for large values of t . Thus the solutions with $c_2 \neq 0$ will be asymptotic to the line $y = -x$ as $t \rightarrow \infty$. Similarly, all solutions with $c_1 \neq 0$ will be asymptotic to the line $y = -3x$ as $t \rightarrow -\infty$. The trajectories are shown below, with origin denoting a saddle point.



13.34

Van der Pol's equation: $\ddot{x} - \alpha(1-x^2)\dot{x} + x = 0, \quad \alpha > 0$ (E₁)

Assume $x(t) = x_0(t) + \alpha x_1(t) + \alpha^2 x_2(t)$ (E₂)

$$\omega_0^2 = 1 = \omega^2 - \alpha \omega_1 - \alpha^2 \omega_2 \quad (E_3)$$

where $\omega_0^2 = 1 = \text{coefficient of } x \text{ in } E_0. (E_1).$

Substitution of (E_2) and (E_3) into (E_1) gives

$$\begin{aligned} \alpha^0 [\ddot{x}_0 + \omega^2 x_0] + \alpha^1 [\ddot{x}_1 - \dot{x}_0 + \dot{x}_0 x_0^2 - \omega_1 x_0 + \omega^2 x_1] \\ + \alpha^2 [\ddot{x}_2 - \dot{x}_1 + \dot{x}_1 x_0^2 + 2 x_0 \dot{x}_0 x_1 - \omega_2 x_0 - \omega_1 x_1 + \omega^2 x_2] \\ + \alpha^3 [\dots] + \dots = 0 \end{aligned} \quad (E_4)$$

Setting coefficient of α^0 in (E_4) to zero, we obtain

$$\ddot{x}_0 + \omega^2 x_0 = 0 \quad , \quad \text{i.e.,} \quad x_0(t) = A_1 \cos \omega t + A_2 \sin \omega t \quad (E_5)$$

Assuming the initial conditions $x(0) = A$ and $\dot{x}(0) = 0$, we get $A_1 = A$ and $A_2 = 0$. Thus (E_5) reduces to

$$x_0(t) = A \cos \omega t \quad (E_6)$$

setting coefficient of α^1 to zero, in $E_0. (E_4)$,

$$\begin{aligned} \ddot{x}_1 + \omega^2 x_1 &= \dot{x}_0 - \dot{x}_0 x_0^2 + \omega_1 x_0 \\ &= -A\omega \sin \omega t + A^3 \omega \sin \omega t \cos^2 \omega t + \omega_1 A \cos \omega t \\ &= \left(-A\omega + \frac{1}{4} A^3 \omega\right) \sin \omega t + \omega_1 A \cos \omega t + \frac{A^3 \omega}{4} \sin 3\omega t \end{aligned} \quad (E_7)$$

The coefficients of $\sin \omega t$ and $\cos \omega t$ must be zero in $E_0. (E_7)$ to avoid secular terms. This gives

$$A = \pm 2 \quad , \quad \omega_1 = 0 \quad (E_8)$$

Thus the particular solution of (E_7) can be expressed as

$$x_1(t) = A_3 \sin 3\omega t + A_4 \cos 3\omega t \quad (E_9)$$

substitution of (E_8) and (E_9) into E_7 gives

$$A_3 = \frac{1}{32} \frac{A^3}{\omega} \quad \text{and} \quad A_4 = 0 \quad (E_{10})$$

$$\text{Thus} \quad x_1(t) = \frac{1}{32} \frac{A^3}{\omega} \sin 3\omega t \quad (E_{11})$$

Finally, setting coefficient of α^2 in (E_4) to zero, we get

$$\ddot{x}_2 + \omega^2 x_2 = \dot{x}_1 - \dot{x}_1 x_0^2 - 2 x_0 \dot{x}_0 x_1 + \omega_2 x_0 + \omega_1 x_1 \quad (E_{12})$$

substitution of (E_{11}) , (E_6) and (E_8) into (E_{12}) leads to

$$\begin{aligned} \ddot{x}_2 + \omega^2 x_2 &= \frac{3}{32} A^3 \cos 3\omega t - \left(\frac{3}{32} A^3 \cos 3\omega t\right) A^2 \cos^2 \omega t \\ &\quad - 2(A \cos \omega t)(-A\omega \sin \omega t) \left(\frac{A^3}{32\omega} \sin 3\omega t\right) + \omega_2 A \cos \omega t \\ &= \left(-\frac{3}{128} A^5 + \frac{1}{64} A^5 + A\omega_2\right) \cos \omega t + \left(\frac{3}{32} A^3 - \frac{3}{64} A^5\right) \cos 3\omega t \\ &\quad + \left(-\frac{3}{128} A^5 - \frac{1}{64} A^5\right) \cos 5\omega t \end{aligned} \quad (E_{13})$$

To avoid secular terms, the coefficient of $\cos \omega t$ in (E_{13}) must be zero. This gives $\omega_2 = \frac{1}{128} A^4$ (E_{14})

With this, and using $A=2$, E_{13} reduces to

$$\ddot{x}_2 + \omega^2 x_2 = -\frac{3}{4} \cos 3\omega t - \frac{5}{4} \cos 5\omega t \quad (E_{15})$$

$$\text{Assuming } x_2(t) = A_5 \cos 3\omega t + A_6 \cos 5\omega t \quad (E_{16})$$

we find, from E_{15} ,

$$A_5 = \frac{3}{32} \cdot \frac{1}{\omega^2}, \quad A_6 = \frac{5}{96} \cdot \frac{1}{\omega^2} \quad (E_{17})$$

$$\therefore x_2(t) = \frac{3}{32\omega^2} \cos 3\omega t + \frac{5}{96\omega^2} \cos 5\omega t \quad (E_{18})$$

Thus the complete solution, E_{12} , (E_2) and (E_3) , become

$$x(t) = 2 \cos \omega t + \frac{\alpha}{4\omega} \sin 3\omega t + \frac{3\alpha^2}{32\omega^2} \cos 3\omega t + \frac{5\alpha^2}{96\omega^2} \cos 5\omega t$$

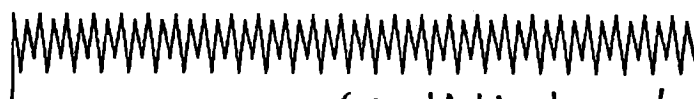
$$\text{and } \omega^2 = 1 + \frac{\alpha^2}{8}.$$

13.35

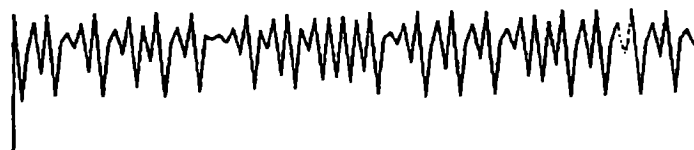
With $k = 3.25$, the sequence of values generated oscillate between two distinct values. With $k = 3.5$, the sequence of values generated oscillate between four distinct values. With $k = 3.75$, the sequence of values generated do not show any apparent periodicity (i.e., exhibit chaotic behavior). The resulting plots are shown below.



$k = 3.25$ (2 distinct values)



$k = 3.50$ (4 distinct values)



$k = 3.75$ (chaotic)

13.36

By taking two different initial values of x_1 which apparently do not differ much from one another, completely different sequences of values are generated. The sequences are shown plotted in the following figure.



$$x_1 = 1.002$$



$$x_1 = 1.003$$

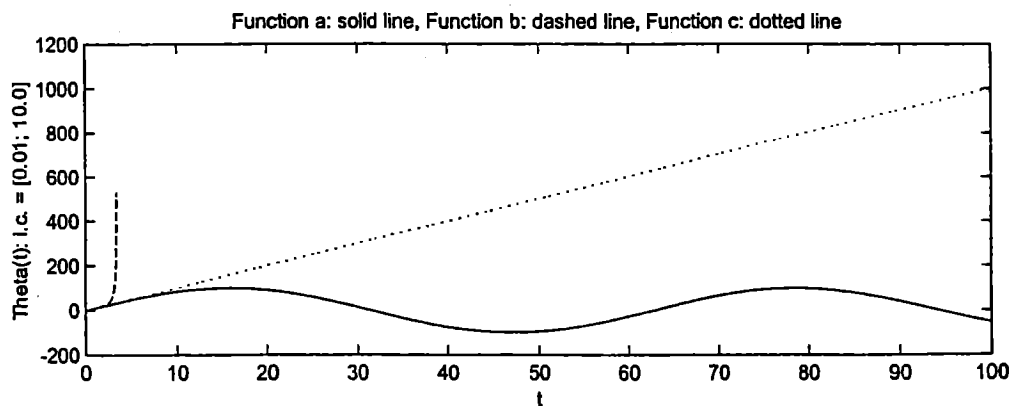
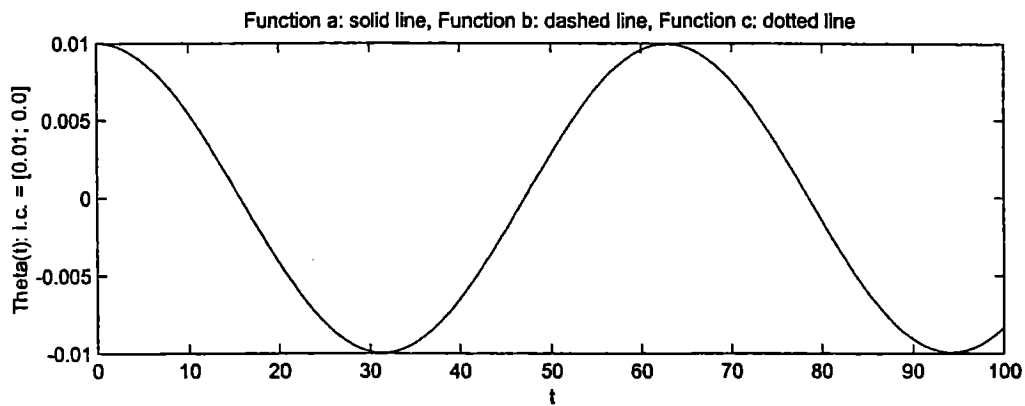
13.37

```
% Ex13_37.m
% This program will use the function dfunc1_a.m, dfunc1_b.m and dfunc1_c.m
% they should be in the same folder, two different initial conditions
tspan = [0: 0.1: 100];
x0 = [0.01; 0.0];
x0_1 = [0.01; 10.0];
[t, xa] = ode23('dfunc1_a', tspan, x0);
[t, xb] = ode23('dfunc1_b', tspan, x0);
[t, xc] = ode23('dfunc1_c', tspan, x0);
[t1, xa1] = ode23('dfunc1_a', tspan, x0_1);
[t2, xb1] = ode23('dfunc1_b', tspan, x0_1);
[t3, xc1] = ode23('dfunc1_c', tspan, x0_1);
subplot(211);
plot(t, xa(:,1));
ylabel('Theta(t): i.c. = [0.01; 0.0]');
xlabel('t');
title...
('Function a: solid line, Function b: dashed line, Function c: dotted line');
hold on;
plot(t, xb(:,1), '--');
hold on;
plot(t, xc(:,1), ':');
subplot(212);
hold off;
plot(t1, xa1(:,1));
ylabel('Theta(t): i.c. = [0.01; 10.0]');
xlabel('t');
title...
('Function a: solid line, Function b: dashed line, Function c: dotted line');
hold on;
plot(t2, xb1(:,1), '--');
hold on;
plot(t3, xc1(:,1), ':');
```

```
% dfunc1_a.m
function f = dfunc1_a(t,x);
w0 = 0.1;
f = zeros(2,1);
f(1) = x(2);
f(2) = -w0^2 * x(1);

% dfunc1_b.m
function f = dfunc1_b(t,x);
w0 = 0.1;
f = zeros(2,1);
f(1) = x(2);
f(2) = w0^2 * ((x(1))^3) / 6.0 - w0^2 * x(1);

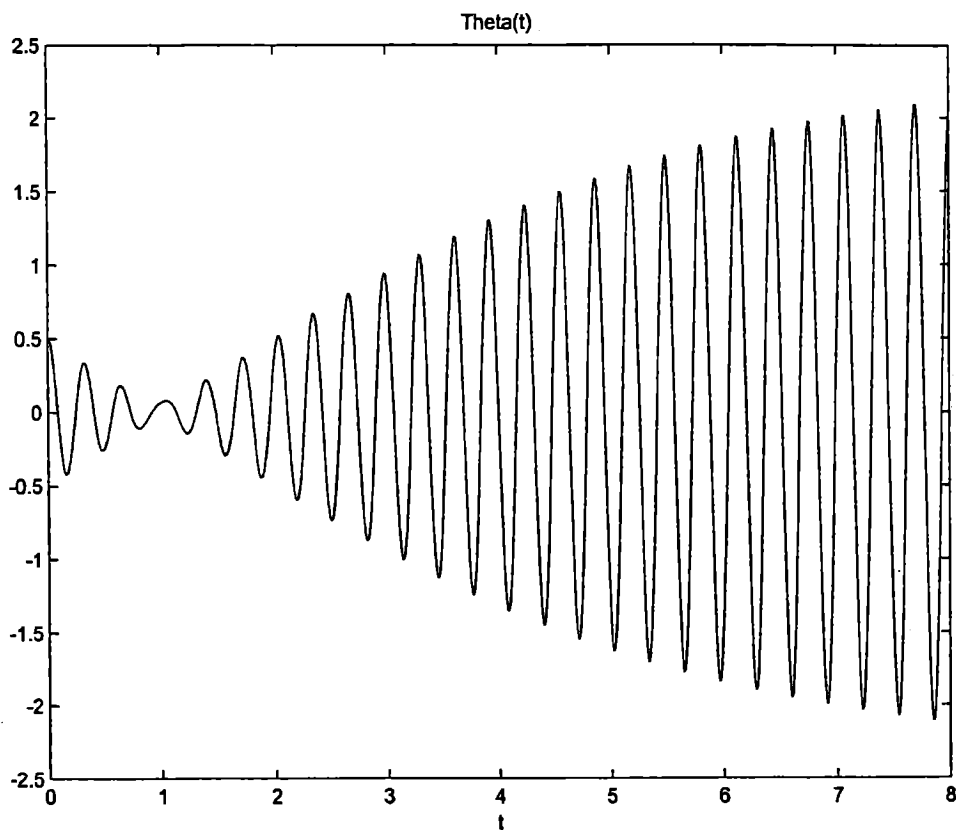
% dfunc1_c.m
function f = dfunc1_c(t,x);
w0 = 0.1;
f = zeros(2,1);
f(1) = x(2);
f(2) = -w0^2 * sin(x(1));
```



13.38

```
% Ex13_38.m
% This program will use the function dfunc13_38.m,
% they should be in the same folder
tspan = [0: 0.001: 8];
x0 = [0.5; 1.0];
[t,x] = ode23('dfunc13_38', tspan, x0);
plot(t,x(:,1));
title('Theta(t)');
xlabel('t');

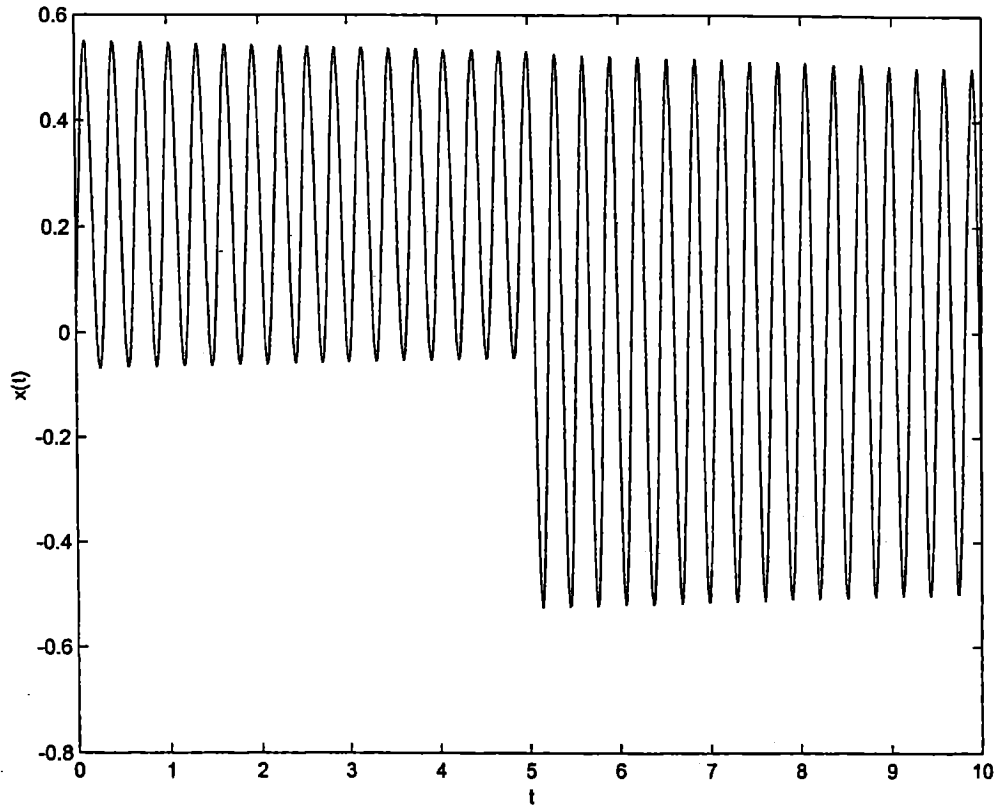
% dfunc13_38.m
function f = dfunc13_38(t,x);
f0 = 200;
m = 10;
c = 0.1;
k = 4000;
w = 20;
f = zeros(2,1);
f(1) = x(2);
f(2) = f0*sin(w*t)/m - c* x(2)^2 * sign(x(2))/m - k*x(1)/m;
```



13.39

```
% Ex13_39.m
% This program will use the function dfunc13_39.m,
% they should be in the same folder
tspan = [0: 0.01: 10];
x0 = [0.05; 5.0];
[t,x] = ode23('dfunc13_39', tspan, x0);
plot(t,x(:,1));
ylabel('x(t)');
xlabel('t');
```

```
% dfunc13_39.m
function f = dfunc13_39(t,x)
f0 = 1000;
m = 10;
k1 = 4000;
k2 = 1000;
FF = f0* ( stepfun(t, 0.0)-stepfun(t, 5.0) );
f = zeros(2,1);
f(1) = x(2);
f(2) = FF/m - k1*x(1)/m - k2*x(1)^3/m;
```



13.40

Results of Ex13_40

» program18

Solution of nonlinear vibration problem
by fourth order Runge-kutta method

Data:

ym = 1.000000e+000

yc = 5.000000e-001

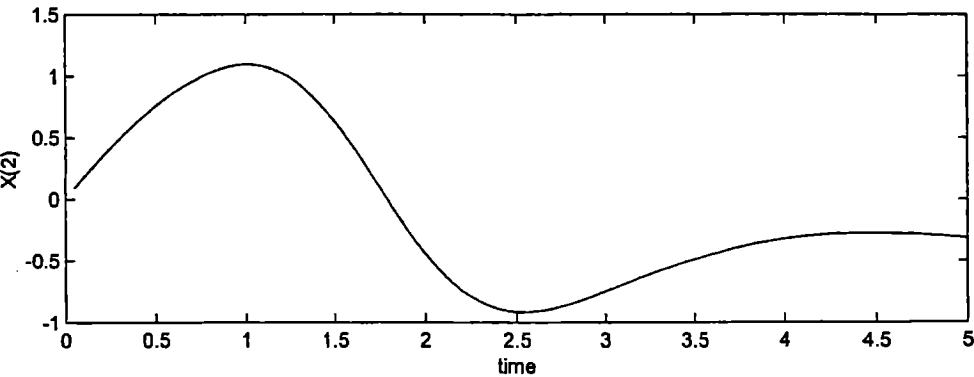
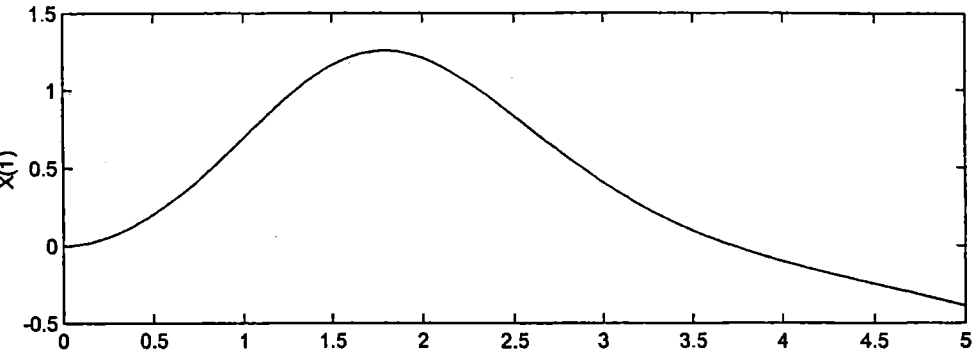
yk = 1.00000000e+000

yks = 1.20000000e+000

Results

i	time(i)	x(i,1)	x(i,2)
1	5.000000e-002	2.230824e-003	8.884132e-002
2	1.000000e-001	8.843311e-003	1.752342e-001
3	1.500000e-001	1.971028e-002	2.589902e-001
4	2.000000e-001	3.469540e-002	3.399312e-001
5	2.500000e-001	5.365366e-002	4.178885e-001
6	3.000000e-001	7.643182e-002	4.927007e-001

7	3.500000e-001	1.028687e-001	5.642099e-001
8	4.000000e-001	1.327951e-001	6.322582e-001
9	4.500000e-001	1.660341e-001	6.966810e-001
10	5.000000e-001	2.023999e-001	7.573016e-001
11	5.500000e-001	2.416976e-001	8.139235e-001
12	6.000000e-001	2.837219e-001	8.663231e-001
13	6.500000e-001	3.282553e-001	9.142430e-001
14	7.000000e-001	3.750665e-001	9.573844e-001
15	7.500000e-001	4.239083e-001	9.954025e-001
...			
81	4.050000e+000	-1.175452e-001	-3.112257e-001
82	4.100000e+000	-1.328833e-001	-3.025009e-001
83	4.150000e+000	-1.478152e-001	-2.949702e-001
84	4.200000e+000	-1.623999e-001	-2.886102e-001
85	4.250000e+000	-1.766953e-001	-2.833937e-001
86	4.300000e+000	-1.907578e-001	-2.792902e-001
87	4.350000e+000	-2.046423e-001	-2.762657e-001
88	4.400000e+000	-2.184017e-001	-2.742829e-001
89	4.450000e+000	-2.320873e-001	-2.733010e-001
90	4.500000e+000	-2.457478e-001	-2.732763e-001
91	4.550000e+000	-2.594301e-001	-2.741616e-001
92	4.600000e+000	-2.731783e-001	-2.759064e-001
93	4.650000e+000	-2.870341e-001	-2.784572e-001
94	4.700000e+000	-3.010365e-001	-2.817569e-001
95	4.750000e+000	-3.152213e-001	-2.857454e-001
96	4.800000e+000	-3.296215e-001	-2.903593e-001
97	4.850000e+000	-3.442666e-001	-2.955318e-001
98	4.900000e+000	-3.591828e-001	-3.011930e-001
99	4.950000e+000	-3.743928e-001	-3.072695e-001
100	5.000000e+000	-3.899154e-001	-3.136849e-001



13.41

% Results of Ex13_41

>> program18

Solution of nonlinear vibration problem
by fourth order Runge-kutta method

Data:

ym = 1.000000e+000

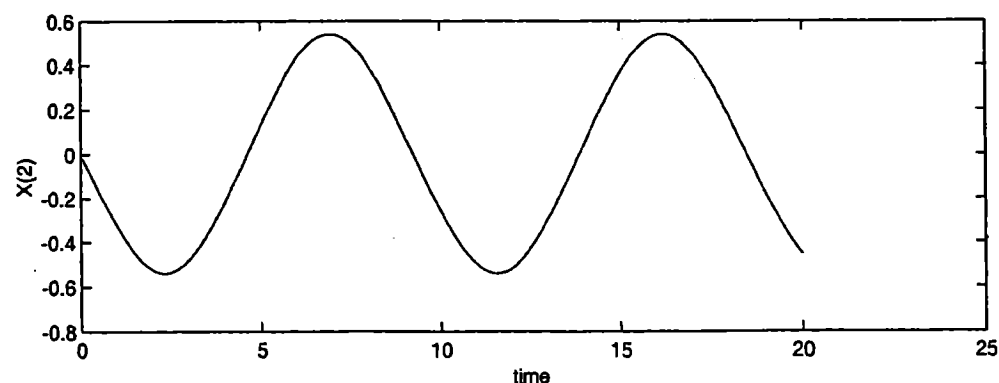
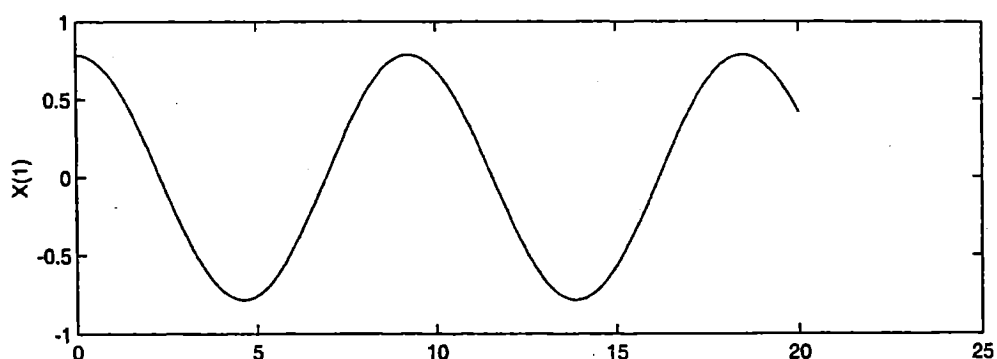
yc = 0.000000e+000

yk = 5.00000000e-001

yks = -8.33333333e-002

Results

i	time(i)	x(i,1)	x(i,2)
1	5.000000e-002	7.849578e-001	-1.761378e-002
6	3.000000e-001	7.695847e-001	-1.051476e-001
11	5.500000e-001	7.325772e-001	-1.903594e-001
16	8.000000e-001	6.747684e-001	-2.712057e-001
21	1.050000e+000	5.975204e-001	-3.454950e-001
26	1.300000e+000	5.027601e-001	-4.109125e-001
31	1.550000e+000	3.929984e-001	-4.651246e-001
36	1.800000e+000	2.713140e-001	-5.059555e-001
41	2.050000e+000	1.412872e-001	-5.316073e-001
46	2.300000e+000	6.877705e-003	-5.408763e-001
51	2.550000e+000	-1.277459e-001	-5.333127e-001
...	...	...	...
371	1.855000e+001	7.849876e-001	-1.700658e-002
376	1.880000e+001	7.697655e-001	-1.045496e-001
381	1.905000e+001	7.329049e-001	-1.897843e-001
386	1.930000e+001	6.752355e-001	-2.706680e-001
391	1.955000e+001	5.981157e-001	-3.450103e-001
396	1.980000e+001	5.034682e-001	-4.104971e-001



13.42

Results of Ex13_42

>> program18

Solution of nonlinear vibration problem
by fourth order Runge-kutta method

Data:

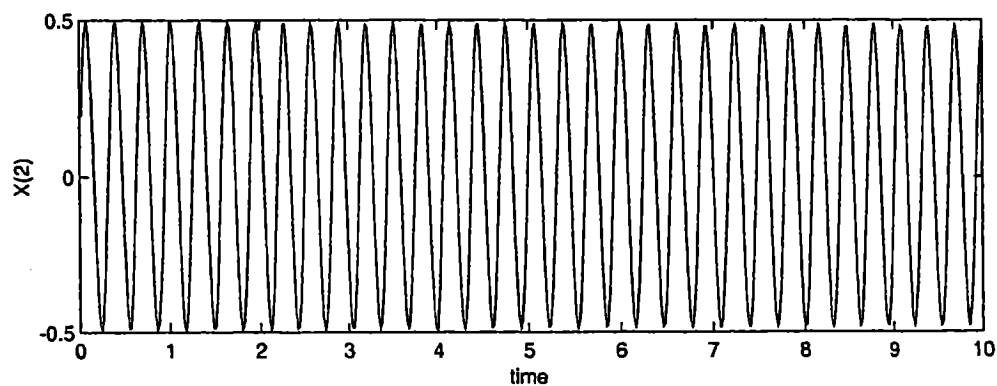
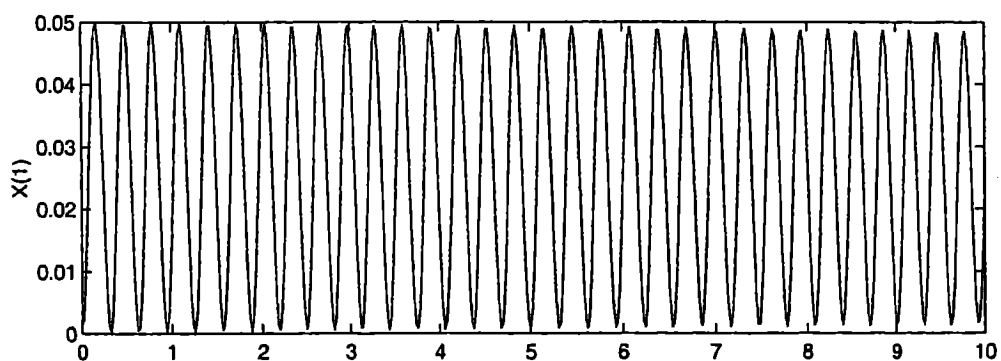
yc = 1.000000e+001

yk = 8.00000000e+005

yks = 6.00000000e+003

Results

i	time(i)	x(i,1)	x(i,2)
1	2.000000e-002	1.973319e-003	1.946639e-001
6	1.200000e-001	4.341183e-002	3.375669e-001
11	2.200000e-001	3.266651e-002	-4.754773e-001
16	3.200000e-001	2.145722e-004	5.888314e-002
21	4.200000e-001	3.800441e-002	4.259926e-001
26	5.200000e-001	3.888710e-002	-4.145664e-001
31	6.200000e-001	3.953964e-004	-7.910512e-002
36	7.200000e-001	3.170805e-002	4.804600e-001
41	8.200000e-001	4.394467e-002	-3.234886e-001
.	.	.	.
461	9.220000e+000	3.397870e-002	-4.512661e-001
466	9.320000e+000	1.308378e-003	4.378072e-002
471	9.420000e+000	3.776651e-002	4.106881e-001
476	9.520000e+000	3.692116e-002	-4.211170e-001
481	9.620000e+000	1.278773e-003	-2.297464e-002
486	9.720000e+000	3.492314e-002	4.419094e-001
491	9.820000e+000	3.953430e-002	-3.843744e-001
496	9.920000e+000	1.673648e-003	-8.686608e-002



```

%=====
%
% Program18.m
% Main program for solving a nonlinear vibration problem using the
% subroutine RK4
%
%=====
% Run "Program18" in MATLAB command window. Program18.m, fun.m, and
% rk4.m should be in the same folder, and set the Matlab path to this folder
% following 8 lines contain problem-dependent data
xx=[0 0];
n=2;
nstep=500;
dt=0.02;
yc=10;
yk=8e5;
yks=6000;
%end of problem-dependent data
t=0;
fprintf('Solution of nonlinear vibration problem\n');
fprintf('by fourth order Runge-kutta method\n\n');
fprintf('Data:\n');
fprintf('yc = %8.6e\n',yc);
fprintf('yk = %10.8e\n',yk);
fprintf('yks = %10.8e\n\n',yks);
fprintf('Results\n\n');
fprintf('   i           time(i)           x(i,1)           x(i,2)\n\n');
for i=1:nstep
    [xx,f,t]=rk4(t,dt,n,xx);
    time(i)=t;
    for j=1:n
        x(i,j)=xx(j);
    end
end
for i=1:5:nstep
    fprintf('%3.0f   %8.6e   %8.6e   %8.6e\n',i,time(i),x(i,1),x(i,2))
end
subplot(211);
plot(time,x(1:nstep,1));
ylabel('X(1)');
subplot(212);
plot(time,x(1:nstep,2));
xlabel('time');
ylabel('X(2)');

%=====
%
% Function rk4.m
%
%=====
function [xx,f,t]=rk4(t,dt,n,xx)
[xi]=fun(xx,n,t);
for i=1:n
    uu(i)=xx(i)+.5*dt*xi(i);
end
tn=t+0.5*dt;
[xj]=fun(uu,n,tn);

```

```

for i=1:n
    uu(i)=xx(i)+.5*dt*xj(i);
end
[xk]=fun(uu,n,tn);
for i=1:n
    uu(i)=xx(i)+dt*xk(i);
end
tn=t+dt;
[xl]=fun(uu,n,tn);
for i=1:n
    f(i)=xl(i);
    xx(i)=xx(i)+(xi(i)+2*xj(i)+2*xk(i)+xl(i))*dt/6;
end
t=t+dt;

%=====
%
% Function fun.m
%
%=====
function [f]=fun(x,n,t)
ym=(2000 - 10*t)*( stepfun(t,0) - stepfun(t,100) );
yc=10;
yk=8e5;
yks=6000;
f(1)=x(2);
f(2)=10*2000*( stepfun(t,0) - stepfun(t,100) )/ym...
    - (yc/ym)*x(2) - (20/ym)* x(2)^2 - (yk/ym)*x(1) - (yks/ym)*(x(1)^3);

```

Results of Ex13_43

SOLUTION OF NONLINEAR VIBRATION PROBLEM
BY FOURTH ORDER RUNGE-KUTTA METHOD

DATA:

YM = 1.000000
YC = 0.500000
YK = 1.000000
YKS = 1.200000

RESULTS:

I	TIME(I)	X(I,1)	X(I,2)
1	0.05000000	0.00223082	0.08884132
2	0.10000000	0.00884331	0.17523421
3	0.15000000	0.01971028	0.25899017
4	0.20000000	0.03469540	0.33993117
5	0.25000000	0.05365366	0.41788851
6	0.30000000	0.07643182	0.49270065
7	0.35000000	0.10286867	0.56420995
8	0.40000000	0.13279513	0.63225817
9	0.45000000	0.16603407	0.69668101
10	0.50000000	0.20239986	0.75730159
11	0.55000000	0.24169759	0.81392345
12	0.60000000	0.28372185	0.86632314
13	0.65000000	0.32825526	0.91424296
14	0.70000000	0.37506653	0.95738436
15	0.75000000	0.42390833	0.99540253

.
.
.

91	4.55000000	-0.25943007	-0.27416160
92	4.60000000	-0.27317830	-0.27590643
93	4.65000000	-0.28703415	-0.27845716
94	4.70000000	-0.30103650	-0.28175687
95	4.75000000	-0.31522132	-0.28574539
96	4.80000000	-0.32962147	-0.29035928
97	4.85000000	-0.34426656	-0.29553181
98	4.90000000	-0.35918279	-0.30119299
99	4.95000000	-0.37439279	-0.30726955
100	5.00000000	-0.38991540	-0.31368495

13.44

Results of Ex13_44

SOLUTION OF NONLINEAR VIBRATION PROBLEM
BY FOURTH ORDER RUNGE-KUTTA METHOD

DATA:

YM = 1.000000

YC = 0.000000

YK = 0.500000

YKS = -0.083333

RESULTS:

I	TIME(I)	X(I,1)	X(I,2)
1	0.05000000	0.78495777	-0.01761378
6	0.30000000	0.76958469	-0.10514758
11	0.55000000	0.73257718	-0.19035941
16	0.80000000	0.67476835	-0.27120566
21	1.05000000	0.59752039	-0.34549495
26	1.30000000	0.50276011	-0.41091248
31	1.55000000	0.39299837	-0.46512455
36	1.80000000	0.27131399	-0.50595545
41	2.05000000	0.14128723	-0.53160732
46	2.30000000	0.00687770	-0.54087626
51	2.55000000	-0.12774587	-0.53331264
56	2.80000000	-0.25840436	-0.50928396
61	3.05000000	-0.38110846	-0.46992334
66	3.30000000	-0.49223282	-0.41697757
71	3.55000000	-0.58864363	-0.35259431
76	3.80000000	-0.66777078	-0.27910032
81	4.05000000	-0.72762849	-0.19881842
86	4.30000000	-0.76679863	-0.11395498
91	4.55000000	-0.78439597	-0.02656762
96	4.80000000	-0.78003441	0.06139708
101	5.05000000	-0.75380896	0.14802117
106	5.30000000	-0.70630037	0.23132208
111	5.55000000	-0.63860072	0.30918427
.			
.			
.			
331	16.55000000	0.19315002	0.52342651
336	16.80000000	0.32035201	0.49162680
341	17.05000000	0.43775099	0.44529523
346	17.30000000	0.54195230	0.38640534
351	17.55000000	0.63009547	0.31721971
356	17.80000000	0.69989971	0.24008210
361	18.05000000	0.74966380	0.15726679
366	18.30000000	0.77823802	0.07090574
371	18.55000000	0.78498761	-0.01700659
376	18.80000000	0.76976545	-0.10454959
381	19.05000000	0.73290488	-0.18978427
386	19.30000000	0.67523547	-0.27066796
391	19.55000000	0.59811564	-0.34501030
396	19.80000000	0.50346820	-0.41049707

13.45

Results of Ex13_45

SOLUTION OF NONLINEAR VIBRATION PROBLEM
BY FOURTH ORDER RUNGE-KUTTA METHOD

DATA:

YC = 10.000000

YK = 800000.000000

YKS = 6000.000000

RESULTS:

I	TIME(I)	X(I,1)	X(I,2)
1	0.02000000	0.00197333	0.19466640
6	0.12000000	0.04341811	0.33766449
11	0.22000000	0.03267006	-0.47558080
16	0.32000000	0.00021457	0.05888324
21	0.42000000	0.03800922	0.42610043
26	0.52000000	0.03889213	-0.41467399
31	0.62000000	0.00039540	-0.07910520
36	0.72000000	0.03171139	0.48056153
:			
471	9.42000000	0.03777123	0.41079244
476	9.52000000	0.03692556	-0.42122285
481	9.62000000	0.00127876	-0.02297349
486	9.72000000	0.03492716	0.44201314
491	9.82000000	0.03953938	-0.38447832
496	9.92000000	0.00167363	-0.08686515

13.46

Equations of motion:

$$\ddot{\vec{x}}(t) = \begin{Bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{Bmatrix} = \begin{Bmatrix} x_2(t) \\ [1.8 \cos 0.4t - 0.5 x_2 - x_1 - 1.2 x_1^3] \end{Bmatrix} = \vec{F}$$

The main program which calls RK4 and subroutine FUN
alongwith the output are given below:

```

C =====
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE RK4
C
C =====
C THE FOLLOWING 7 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
  DIMENSION TIME(100), X(100,2), XX(2), F(2), YI(2), YJ(2), YK(2), YL(2),
2  UU(2)
  XX(1)=0.0
  XX(2)=0.0
  NEG=2
  NSTEP=100
  DT=0.05
  T=0.0
  PRINT 10
10  FORMAT (//,3X,5H I ,10H TIME(I),7X,5H X(1),12X,5H X(2),/)
  DO 40 I=1, NSTEP
    CALL RK4 (T, DT, NEG, XX, F, YI, YJ, YK, YL, UU)
    TIME(I)=T
    DO 20 J=1, NEG
20  X(I,J)=XX(J)
    PRINT 30, I, TIME(I), (X(I,J), J=1, NEG)
30  FORMAT (2X, I5, F10.4, 2X, E15.8, 2X, E15.8)
40  CONTINUE
    STOP
  END

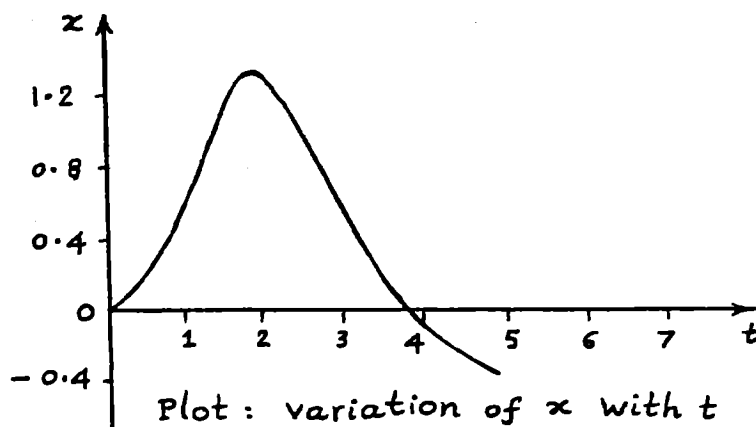
```

```

C =====
C SUBROUTINE FUN FOR USE IN THE SUBROUTINE RK4
C THIS SUBROUTINE CHANGES FROM PROBLEM TO PROBLEM
C =====
      SUBROUTINE FUN (X,F,N,T)
      DIMENSION X(N),F(N)
      F(1)=X(2)
      F(2)=1.8*COS(0.4*T)-0.5*X(2)-X(1)-1.2*(X(1)**3)
      RETURN
      END

```

I	TIME(I)	X(1)	X(2)
1	0.0500	0.22308242E-02	0.88841319E-01
2	0.1000	0.88433102E-02	0.17523426E+00
3	0.1500	0.19710280E-01	0.25899023E+00
4	0.2000	0.34695394E-01	0.33993125E+00
5	0.2500	0.53653665E-01	0.41788858E+00
6	0.3000	0.76431811E-01	0.49270076E+00
7	0.3500	0.10286868E+00	0.56421006E+00
8	0.4000	0.13279516E+00	0.63225830E+00
9	0.4500	0.16603410E+00	0.69668114E+00
10	0.5000	0.20239991E+00	0.75730175E+00
11	0.5500	0.24169767E+00	0.81392360E+00
12	0.6000	0.28372198E+00	0.86632329E+00
13	0.6500	0.32825541E+00	0.91424310E+00
14	0.7000	0.37506670E+00	0.95738453E+00
15	0.7500	0.42390853E+00	0.99540269E+00
16	0.8000	0.47451508E+00	0.10279036E+01
17	0.8500	0.52659953E+00	0.10544405E+01
18	0.9000	0.57985157E+00	0.10745211E+01
19	0.9500	0.63393515E+00	0.10876102E+01
20	1.0000	0.68848670E+00	0.10931435E+01
.			
.			
.			
.			
86	4.2999	-0.19073796E+00	-0.27927017E+00
87	4.3499	-0.20462143E+00	-0.27624547E+00
88	4.3999	-0.21837986E+00	-0.27426249E+00
89	4.4499	-0.23206437E+00	-0.27328062E+00
90	4.4999	-0.24572390E+00	-0.27325594E+00
91	4.5499	-0.25940514E+00	-0.27414125E+00
92	4.5999	-0.27315235E+00	-0.27588624E+00
93	4.6499	-0.28700721E+00	-0.27843720E+00
94	4.6999	-0.30100858E+00	-0.28173721E+00
95	4.7499	-0.31519240E+00	-0.28572613E+00
96	4.7999	-0.32959157E+00	-0.29034048E+00
97	4.8499	-0.34423572E+00	-0.29551363E+00
98	4.8999	-0.35915107E+00	-0.30117553E+00
99	4.9499	-0.37436020E+00	-0.30725288E+00
100	4.9999	-0.38988203E+00	-0.31366915E+00



13.47 $m l^2 \ddot{\theta} + m g l (\theta - \frac{1}{6} \theta^3) = 0$ i.e. $\ddot{\theta} + \frac{g}{l} \theta - \frac{1}{12} \theta^3 = 0$ for $\frac{g}{l} = 0.5$

$$\vec{F} = \begin{Bmatrix} x_2 \\ -\frac{x_1}{2} + \frac{1}{12} x_1^3 \end{Bmatrix} \quad \text{where } x_1 = \theta \text{ and } x_2 = \dot{\theta}$$

Since the natural time period of the linear system is $\frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{0.5}} = 8.8858$, Δt is taken as 0.25 in the present case. Using

$\theta(t=0) = 0.7854$ rad and $\dot{\theta}(t=0) = 0$, the Runge-Kutta method is used to find the solution. The main program which calls RK4 and the subroutine FUN along with the output are given below.

```

C =====
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE RK4
C
C =====
C THE FOLLOWING 7 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
  DIMENSION TIME(100),X(100,2),XX(2),F(2),YI(2),YJ(2),YK(2),YL(2),
2  UU(2)
  XX(1)=0.7854
  XX(2)=0.0
  NEQ=2
  NSTEP=100
  DT=0.25
  T=0.0
  WRITE (15,10)
10  FORMAT (//,3X,5H I ,10H TIME(I),7X,5H X(1),12X,5H X(2),/)
  DO 40 I=1,NSTEP
    CALL RK4 (T,DT,NEQ,XX,F,YI,YJ,YK,YL,UU)
    TIME(I)=T
    DO 20 J=1,NEQ
20  X(I,J)=XX(J)
  WRITE (15,30) I,TIME(I),(X(I,J),J=1,NEQ)
30  FORMAT (2X,15,F10.4,2X,E15.8,2X,E15.8)
40  CONTINUE
  STOP
  END

```

```

C =====
C SUBROUTINE FUN FOR USE IN THE SUBROUTINE RK4
C THIS SUBROUTINE CHANGES FROM PROBLEM TO PROBLEM
C =====
      SUBROUTINE FUN (X,F,N,T)
      DIMENSION X(N),F(N)
      F(1)=X(2)
      F(2)=-X(1)/2.0+(X(1)**3)/12.0
      RETURN
      END

```

I	TIME(I)	X(1)	X(2)
1	0.2500	0.77440965E+00	-0.87762989E-01
2	0.5000	0.74167883E+00	-0.17359795E+00
3	0.7500	0.68793911E+00	-0.25548738E+00
4	1.0000	0.61444390E+00	-0.33127007E+00
5	1.2500	0.52300471E+00	-0.39865014E+00
6	1.5000	0.41601548E+00	-0.45528558E+00
7	1.7500	0.29644495E+00	-0.49895233E+00
8	2.0000	0.16778104E+00	-0.52775872E+00
9	2.2500	0.33919662E-01	-0.54036677E+00
10	2.5000	-0.10099708E+00	-0.53616732E+00
11	2.7500	-0.23277587E+00	-0.51536494E+00
12	3.0000	-0.35737443E+00	-0.47894892E+00
13	3.2500	-0.47108278E+00	-0.42855906E+00
14	3.5000	-0.57066125E+00	-0.36628109E+00
15	3.7500	-0.65342385E+00	-0.29442233E+00
16	4.0000	-0.71726823E+00	-0.21531790E+00
17	4.2500	-0.76066524E+00	-0.13120258E+00
18	4.5000	-0.78262591E+00	-0.44162579E-01
19	4.7500	-0.78266639E+00	0.43838933E-01
20	5.0000	-0.76078576E+00	0.13088597E+00
...			
94	23.5000	-0.76101488E+00	0.13025595E+00
95	23.7500	-0.71784961E+00	0.21441360E+00
96	24.0000	-0.65422338E+00	0.29358405E+00
97	24.2500	-0.57165933E+00	0.36553413E+00
98	24.5000	-0.47225332E+00	0.42792925E+00
99	24.7500	-0.35868505E+00	0.47846088E+00
100	25.0000	-0.23418844E+00	0.51503974E+00

13.48 (a) Equation of motion

$$(M - m_0 t) \ddot{x} + 10 \dot{x} + 20 x^2 + 8 \times 10^5 x + 6000 x^3 = m_0 v$$

i.e., $(2000 - 10 t) \ddot{x} + 10 \dot{x} + 20 x^2 + 8 \times 10^5 x + 6000 x^3 = (10)(2000) = 2 \times 10^4 \quad (E_1)$

(b) let $x_1 = x, \quad x_2 = \frac{dx_1}{dt} = \frac{dx}{dt}$

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \vec{F} = \begin{Bmatrix} x_2 \\ (2 \times 10^4 - 10 x_2 - 20 x_2^2 - 8 \times 10^5 x_1 - 6000 x_2^3) / (2000 - 10 t) \end{Bmatrix}$$

Initial conditions: assume $x_1(0) = x_2(0) = 0$.

The main program, which calls RK4 and subroutine FUN, alongwith the output are given below.

```

C =====
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE RK4
C
C =====
C THE FOLLOWING 7 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
  DIMENSION TIME(200), X(200, 2), XX(2), F(2), YI(2), YJ(2), YK(2), YL(2),
2  UU(2)
  XX(1)=0.0
  XX(2)=0.0
  NEG=2
  NSTEP=200
  DT=0.1
  T=0.0
  PRINT 10
10  FORMAT (//, 3X, 5H I , 10H TIME(I), 7X, 5H X(1), 12X, 5H X(2), /)
  DO 40 I=1, NSTEP
    CALL RK4 (T, DT, NEG, XX, F, YI, YJ, YK, YL, UU)
    TIME(I)=T
    DO 20 J=1, NEG
20    X(I, J)=XX(J)

    PRINT 30, I, TIME(I), (X(I, J), J=1, NEG)
30    FORMAT (2X, I5, F10.4, 2X, E15.8, 2X, E15.8)
40    CONTINUE
    STOP
    END

C =====
C SUBROUTINE FUN FOR USE IN THE SUBROUTINE RK4
C THIS SUBROUTINE CHANGES FROM PROBLEM TO PROBLEM
C =====
  SUBROUTINE FUN (X, F, N, T)
    DIMENSION X(N), F(N)
    F(1)=X(2)
    F(2)=(20000.0-10.0*X(2)-20.0*(X(2)**2)
2  -800000.0*X(1)-6000.0*(X(1)**3))/(2000.0-10.0*T)
    RETURN
    END

  I      TIME(I)      X(1)      X(2)

  1      0.1000      0.33320766E-01      0.33307534E+00
  2      0.2000      0.33306178E-01      -0.22198296E+00
  3      0.3000      0.14845021E-01      -0.36448002E-01
  4      0.4000      0.27181853E-01      0.14729625E+00
  5      0.5000      0.29156089E-01      -0.78316689E-01
  6      0.6000      0.21012224E-01      -0.29013485E-01
  7      0.7000      0.25375746E-01      0.62707484E-01
  8      0.8000      0.26947644E-01      -0.26040435E-01
  9      0.9000      0.23485284E-01      -0.17102312E-01
  10     1.0000      0.24944551E-01      0.25843836E-01
  :
  :
  195    19.5003      0.24999883E-01      0.47695581E-09
  196    19.6003      0.24999883E-01      0.54962435E-09
  197    19.7003      0.24999883E-01      0.62233330E-09

```

198	19. 8003	0. 24999883E-01	0. 69508244E-09
199	19. 9003	0. 24999883E-01	0. 76787199E-09
200	20. 0003	0. 24999883E-01	0. 84070195E-09

13.49 For $\ddot{x} + a^2 F(x) = 0$,
$$\tau = \frac{2\sqrt{2}}{a} \int_0^{x_0} \frac{dF}{\left\{ \int_{\xi}^{x_0} F(\eta) \cdot d\eta \right\}^{1/2}} \quad (E.1)$$

Here $\ddot{\theta} + \frac{g}{l} \left(\theta - \frac{\theta^3}{6} \right) = 0$; $\frac{g}{l} = 0.5 \Rightarrow a = \sqrt{0.5}$; $F(\theta) = \theta - \frac{\theta^3}{6}$;
 $\theta_0 = 0.7854$ rad.

Simpson's rule is used for integration; note that one integral is embedded inside a second integral in (E.1). According to Simpson's rule,

$$\int_a^b f(x) \cdot dx = \frac{h}{3} \left\{ f(a) + 2 \sum_{i=1}^{\left(\frac{N}{2}-1\right)} f_{2i+1} + 4 \sum_{i=1}^{\left(\frac{N}{2}\right)} f_{2i} + f(b) \right\}$$

where $h = \frac{b-a}{N}$, $N =$ even number of equal intervals and

$$f_i = f(x = x_i = a + \{i-1\} \left\{ \frac{b-a}{N} \right\}).$$

The program listing and the result are given below.

```

C =====
C
C PROBLEM 13.49
C
C =====
      AA=0.0
      BB=0.7854
      NN=40
      HH=(BB-AA)/FLOAT(NN)
      CALL FFINT (AA,FA)
      BB1=BB-(BB-AA)/100.0
      CALL FFINT (BB1,FB)
      CC=BB-HH
      CALL FFINT (CC,FC)
      TAU=HH*(FA+FB+4.0*FC)/3.0
      KK=NN/2
      KKM1=KK-1
      DO 10 JJ=1,KKM1
      XX=AA+HH*FLOAT(2*JJ-1)
      CALL FFINT (XX,FX)
      XXHH=XX+HH
      CALL FFINT (XXHH,FXH)
      TAU=TAU+HH*(4.0*FX+2.0*FXH)/3.0
10  CONTINUE
      TAU=4.0*TAU
      PRINT 20, TAU
20  FORMAT (2X,E15.8)
      STOP
      END

```

```

C =====
C SUBROUTINE FFINT
C =====
      SUBROUTINE FFINT (X,Y)
      A=X
      B=0.7854
      N=40
      CALL SIM (A,B,N,FINT)
      Y=1.0/SQRT(FINT)
      RETURN
      END
C =====
C SUBROUTINE SIM
C =====
      SUBROUTINE SIM (A,B,N,FINT)
      H=(B-A)/FLOAT(N)
      FINT=H*(F(A)+F(B)+4.0*F(B-H))/3.0
      K=N/2
      KM1=K-1
      DO 10 J=1,KM1
      X=A+H*FLOAT(2*J-1)
      FINT=FINT+H*(4.0*F(X)+2.0*F(X+H))/3.0
10    CONTINUE
      RETURN
      END
C =====
C FUNCTION F
C =====
      FUNCTION F(X)
      F=X-(X**3)/6.0
      RETURN
      END

```

RESULT: $\tau = 8.7627392$

This can be compared with the value of τ of a linear pendulum
 which is equal to $\frac{2\pi}{\omega} = 2\pi / \sqrt{\frac{g}{l}} = 2\pi / \sqrt{9.8} = 8.857867$.

13.50 van der Pol's equation: $\ddot{x} - \alpha(1-x^2)\dot{x} + x = 0$ (E₁)

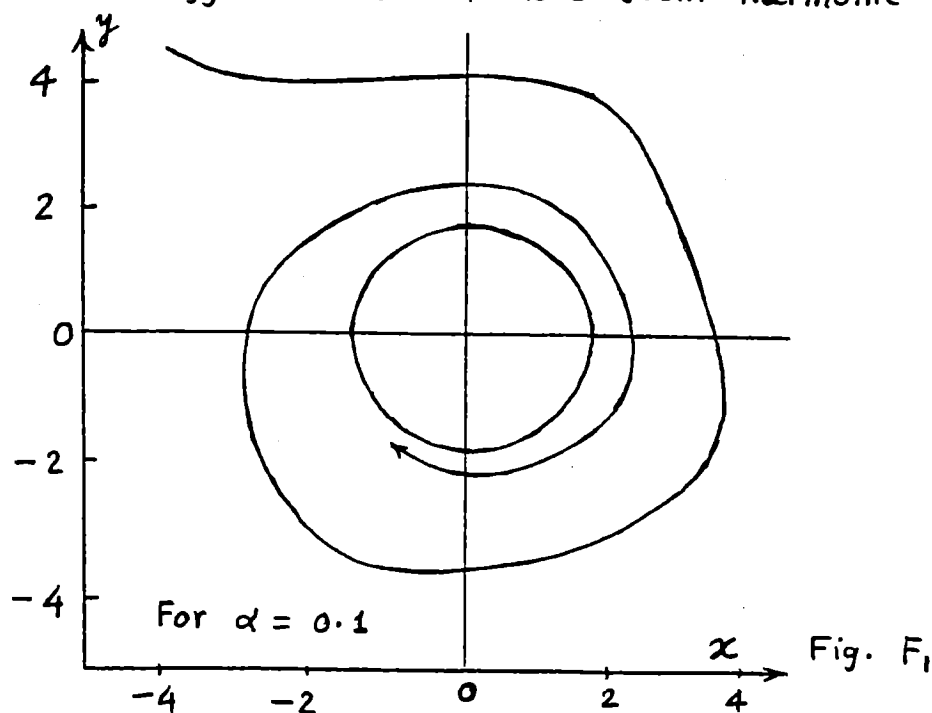
i.e.,
$$\begin{cases} \frac{dx}{dt} \\ \frac{dy}{dt} \end{cases} = \begin{cases} y \\ \alpha(1-x^2)y - x \end{cases}$$
 (E₂)

(a) Equation for the phase plane trajectory is

$$\frac{dy}{dx} = \frac{\alpha(1-x^2)y - x}{y} \quad (E_3)$$

The trajectories corresponding to $\alpha = 0.1$, $\alpha = 1.0$ and $\alpha = 10$ are shown in Figs. F₁, F₂ and F₃, respectively. It can be seen, from these figures, that irrespective of the initial point, the trajectories converge to closed curves. These closed curves denote periodic motions with constant amplitude. In addition, as the value of α increases, the closed curve deviates more and more from a circle.

(b) If (E₁) is solved numerically, the variation of the solution $x(t)$ can be plotted against the t -axis. The result will be as shown in Figs. F₄, F₅ and F₆ for $\alpha = 0.1$, 1.0 and 10 , respectively. It can be seen from these figures, that as the value of α is increased, the variation of x with respect to t differs more and more from harmonic motion.



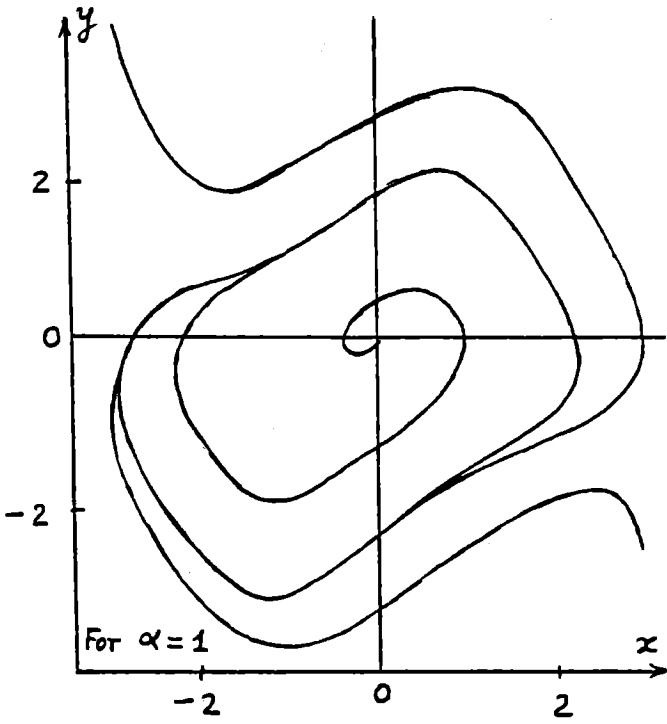


Fig. F₂

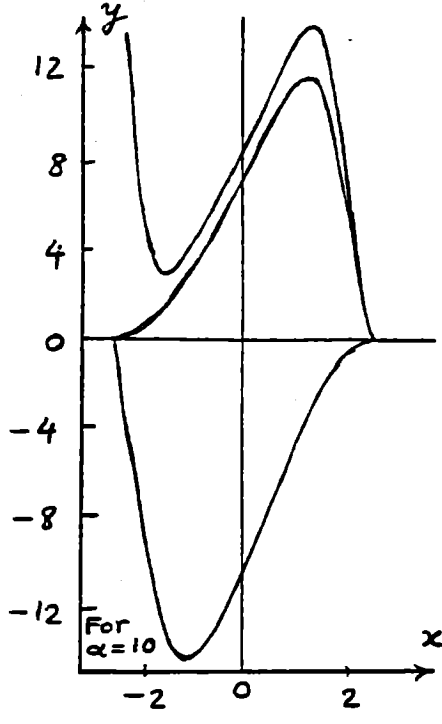


Fig. F₃

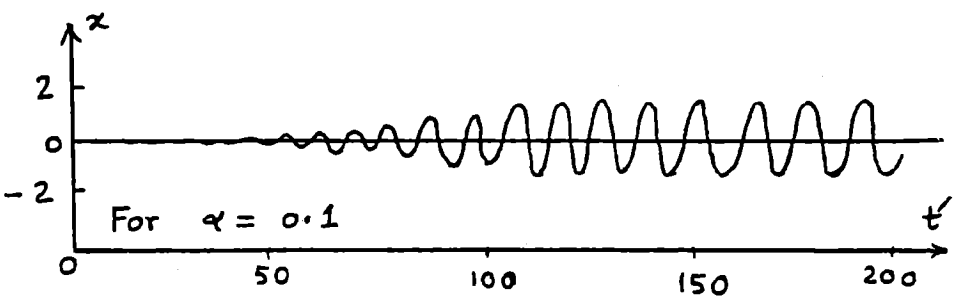


Fig. F₄

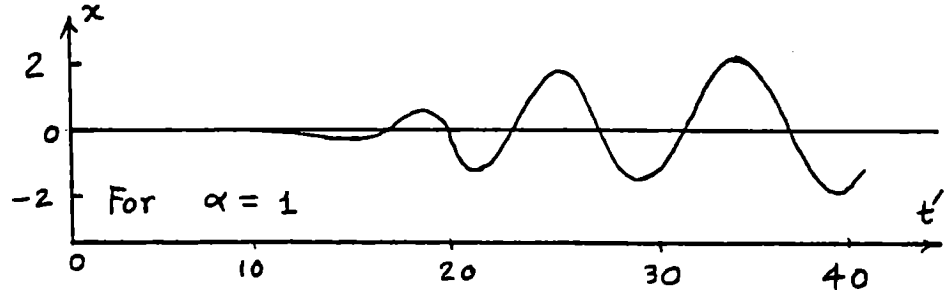


Fig. F₅

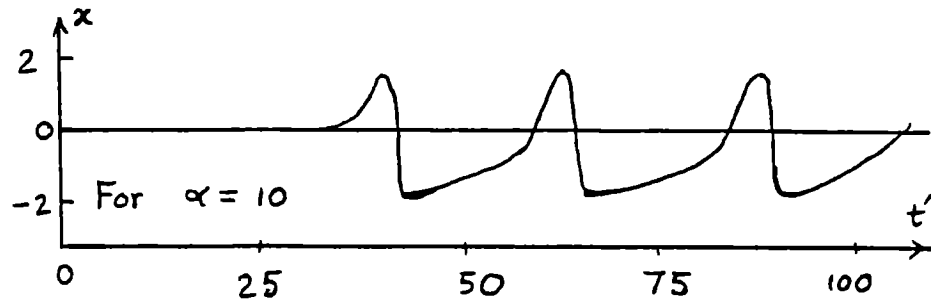


Fig. F₆

13.51 Equations of motion:

$$m \ddot{x} + k_{11}(x - l_1 \theta) + k_{12}(x - l_1 \theta)^3 + k_{21}(x + l_2 \theta) + k_{22}(x + l_2 \theta)^3 = 0$$

$$J_0 \ddot{\theta} - k_{11}(x - l_1 \theta)l_1 - k_{12}(x - l_1 \theta)^3 l_1 + k_{21}(x + l_2 \theta)l_2 + k_{22}(x + l_2 \theta)^3 l_2 = 0$$

Using $x_1 = x$, $x_2 = \theta$, $x_3 = \frac{dx}{dt} = \dot{x}$, and $x_4 = \frac{d\theta}{dt} = \dot{\theta}$, the equations of motion can be expressed as

$$\dot{\mathbf{x}} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{Bmatrix} = \mathbf{F} \equiv \begin{Bmatrix} x_3 \\ x_4 \\ -\frac{1}{m} [k_{11}(x_1 - l_1 x_2) + k_{12}(x_1 - l_1 x_2)^3 + k_{21}(x_1 + l_2 x_2) + k_{22}(x_1 + l_2 x_2)^3] \\ \frac{1}{J_0} [k_{11}(x_1 - l_1 x_2)l_1 + k_{12}(x_1 - l_1 x_2)^3 l_1 - k_{21}(x_1 + l_2 x_2)l_2 - k_{22}(x_1 + l_2 x_2)^3 l_2] \end{Bmatrix}$$

Data:

$$m = 1000 \text{ kg}, \quad J_0 = 2500 \text{ kg-m}^2, \quad l_1 = 1 \text{ m}, \quad l_2 = 1.5 \text{ m},$$

$$k_{11} = 40000, \quad k_{12} = 10000, \quad k_{21} = 50000, \quad k_{22} = 5000.$$

Runge-Kutta method can be used to solve the equations of motion using the initial conditions $x(0) = \dot{x}(0) = \theta(0) = \dot{\theta}(0) = 0$.