

Chapter 14

Random Vibration

14.1

Solution:

$$p(x) = \begin{cases} k \left(1 - \frac{x}{30}\right) & ; 20 \leq x \leq 30 \\ 0 & ; \text{elsewhere} \end{cases}$$

Normalization:

$$\int_{-\infty}^{\infty} p(x) dx = k \int_{20}^{30} \left(1 - \frac{x}{30}\right) dx = k \left(x - \frac{x^2}{60}\right)_{20}^{30} = 1$$

$$\Rightarrow k = 0.6$$

$$\begin{aligned} P(x \geq 28) &= \int_{28}^{\infty} p(x) dx = k \int_{28}^{30} \left(1 - \frac{x}{30}\right) dx = k \left(x - \frac{x^2}{60}\right)_{28}^{30} \\ &= k \left(\frac{4}{60}\right) = 0.6 \left(\frac{4}{60}\right) = 0.04 \end{aligned}$$

14.2

$$p(x, y) = \begin{cases} \frac{xy}{9} & ; 0 \leq x \leq 2, \quad 0 \leq y \leq 3 \\ 0 & ; \text{elsewhere} \end{cases}$$

Normalization:

$$\begin{aligned} \int_{x=0}^2 \int_{y=0}^3 p(x, y) dx dy &= \int_{x=0}^2 \frac{x}{9} dx \int_{y=0}^3 y dy = \int_{x=0}^2 \frac{x}{9} dx \left(\frac{y^2}{2}\right)_0^3 \\ &= \frac{1}{2} \int_{x=0}^2 x dx = \frac{1}{2} \left(\frac{x^2}{2}\right)_0^2 = 1 \quad (\text{satisfied}) \end{aligned}$$

(a) Marginal density functions

$$p_x(x) = \int_{y=0}^3 p(x,y) dy = \frac{x}{9} \int_{y=0}^3 y dy = \frac{x}{2}$$

$$p_y(y) = \int_{x=0}^2 p(x,y) dx = \frac{y}{9} \int_{x=0}^2 x dx = \frac{2}{9} y$$

$$(b) \mu_x = \bar{x} = \int_0^2 p(x) \cdot x dx = \frac{1}{2} \int_0^2 x^2 dx = \frac{1}{2} \left(\frac{x^3}{3} \right)_0^2 = \frac{4}{3}$$

$$\mu_y = \bar{y} = \int_0^3 p(y) \cdot y dy = \frac{2}{9} \int_0^3 y^2 dy = \frac{2}{9} \left(\frac{y^3}{3} \right)_0^3 = 2$$

$$\begin{aligned} \sigma_x^2 &= E[(x - \mu_x)^2] = \int_0^2 (x - \mu_x)^2 p(x) dx = \int_0^2 \left(x - \frac{4}{3}\right)^2 \frac{x}{2} dx \\ &= \frac{1}{2} \left(\frac{x^4}{4} + \frac{16}{9} \frac{x^2}{2} - \frac{8}{3} \frac{x^3}{3} \right)_0^2 = \frac{2}{9} \end{aligned}$$

$$\sigma_x = 0.4714$$

$$\begin{aligned} \sigma_y^2 &= E[(y - \mu_y)^2] = \int_0^3 (y - \mu_y)^2 p(y) dy = \int_0^3 (y - 2)^2 \frac{2y}{9} dy \\ &= \frac{2}{9} \left(\frac{y^4}{4} + 2y^2 - \frac{4}{3} y^3 \right)_0^3 = \frac{1}{2} \end{aligned}$$

$$\sigma_y = 0.7071$$

$$\begin{aligned} (c) \sigma_{x,y} &= E[(x - \mu_x)(y - \mu_y)] = \int_{x=0}^2 \int_{y=0}^3 \left(x - \frac{4}{3}\right)(y - 2) \frac{xy}{9} dx dy \\ &= \int_{x=0}^2 \left(x - \frac{4}{3}\right) \frac{x}{9} dx \left(\frac{y^3}{3} - 2 \cdot \frac{y^2}{2} \right)_0^3 = 0 \end{aligned}$$

$$\rho_{x,y} = 0$$

$$14.3 \quad E[x] = \bar{x} = \int_{-\infty}^{\infty} x \cdot p(x) dx = \int_0^2 0.5 x dx = 1.0$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 p(x) dx = \int_0^2 0.5 x^2 dx = 1.3333$$

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x - \bar{x})^2 p(x) dx = \int_0^2 (x-1)^2 (0.5) dx = 0.5 \int_0^2 (x^2 - 2x + 1) dx$$

$$= 0.3333$$

$$\therefore \sigma_x = 0.5773$$

$$14.4 \quad z = x + y, \quad E[z^2] = E[(x+y)^2] = E[x^2 + y^2 + 2xy]$$

$$= E[x^2] + E[y^2] + 2E[xy]$$

Since x and y are independent, $E[xy] = E[x] \cdot E[y]$

$$E[z^2] = E[x^2] + E[y^2] + 2E[x] \cdot E[y]$$

$$14.5 \quad R_x(\tau) = 20 + \frac{5}{1 + 3\tau^2}$$

$$E[x^2] = R(0) = 20 + 5 = 25$$

$$14.6 \quad R_x(\tau) = A \cos \omega \tau$$

$$S_x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$$

$$= \frac{A}{2\pi} \int_{-\pi/2\omega}^{\pi/2\omega} \cos \omega \tau (\cos \omega \tau - i \sin \omega \tau) d\tau$$

$$= \frac{A}{2\pi} \int_{-\pi/2\omega}^{\pi/2\omega} (\cos^2 \omega \tau - i \frac{1}{2} \sin 2\omega \tau) d\tau$$

$$= \frac{A}{2\pi} \left\{ \left[\frac{\tau}{2} + \frac{1}{4\omega} \sin 2\omega \tau \right]_{-\pi/2\omega}^{\pi/2\omega} - \frac{i}{2} \left(-\frac{\cos 2\omega \tau}{2\omega} \right)_{-\pi/2\omega}^{\pi/2\omega} \right\}$$

$$= \frac{A}{4\omega}$$

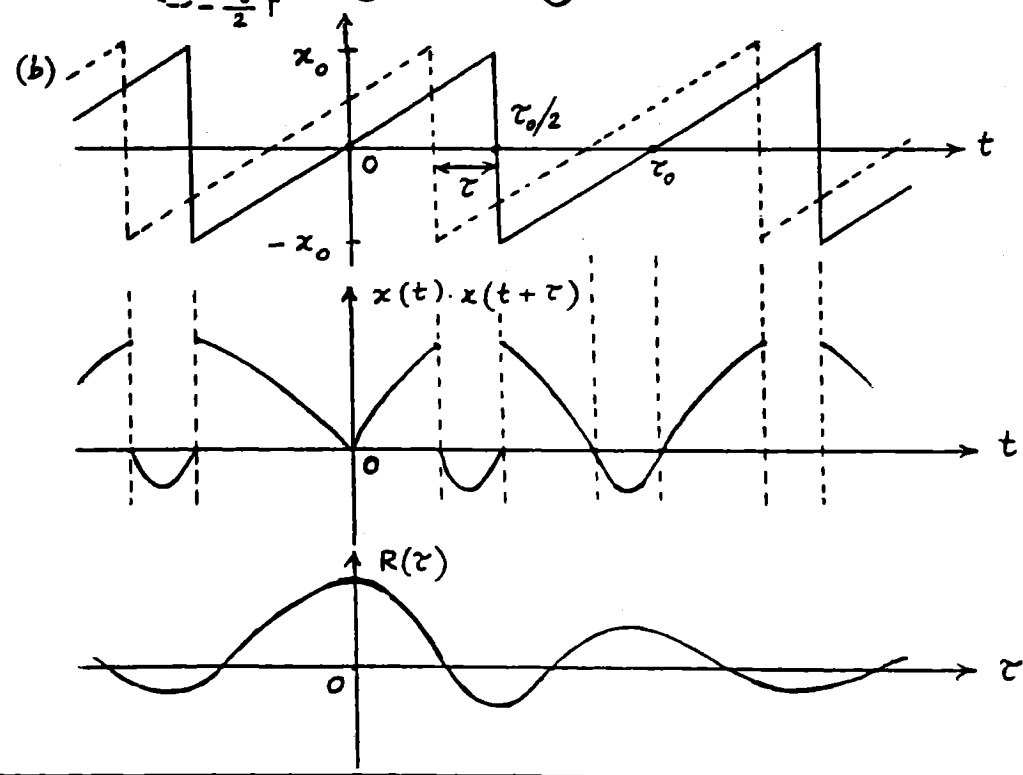
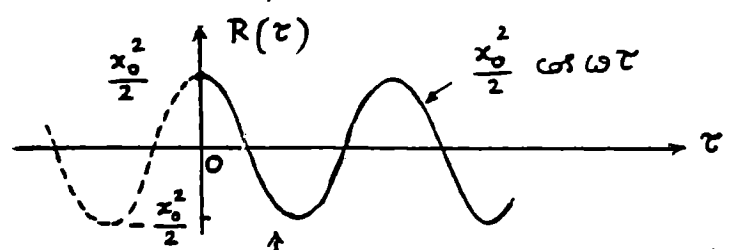
14.7

(a) $x(t) = x_0 \sin \omega t$, $x(t+\tau) = x_0 \sin \omega(t+\tau)$

$x(t) x(t+\tau) = x_0^2 (\sin^2 \omega t \cos \omega \tau + \sin \omega t \cdot \cos \omega t \cdot \sin \omega \tau)$

$R(\tau) = \lim_{T \rightarrow \infty} \frac{x_0^2}{T} \int_{-T/2}^{T/2} \left[\left(\frac{1 - \cos 2\omega t}{2} \right) \cos \omega \tau + \frac{\sin 2\omega t}{2} \cdot \sin \omega \tau \right] dt$

$= \lim_{T \rightarrow \infty} \frac{x_0^2}{T} \left[\frac{T}{2} \cos \omega \tau + (0) \sin \omega \tau \right] = \frac{x_0^2}{2} \cos \omega \tau$



$$14.8 \quad x(t) = \begin{cases} (2x_0 t/\tau) & ; \quad 0 \leq t \leq \frac{\tau}{2} \\ (2x_0 t/\tau) - 2x_0 & ; \quad \frac{\tau}{2} \leq t \leq \tau \end{cases}$$

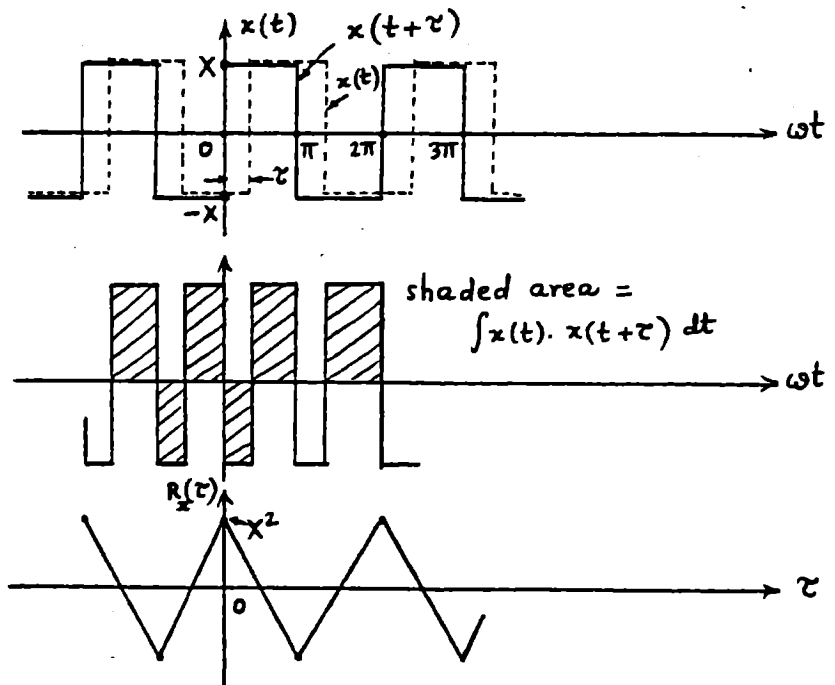
$$\begin{aligned} c_n &= \frac{1}{\tau} \int_0^{\tau} x(t) e^{-in\omega t} dt = \frac{1}{\tau} \left[\int_0^{\tau/2} \frac{2x_0 t}{\tau} e^{-in\omega t} dt \right. \\ &\quad \left. + \int_{\tau/2}^{\tau} \left(\frac{2x_0 t}{\tau} - 2x_0 \right) e^{-in\omega t} dt \right] \\ &= \frac{2x_0}{\tau^2} \int_0^{\tau/2} t e^{-in\omega t} dt + \frac{2x_0}{\tau^2} \int_{\tau/2}^{\tau} t e^{-in\omega t} dt - \frac{2x_0}{\tau} \int_{\tau/2}^{\tau} e^{-in\omega t} dt \\ &= \frac{2x_0}{\tau^2} \left(\frac{1}{n^2 \omega^2} \right) \left[e^{-in\omega t} (in\omega t) + e^{-in\omega t} \right]_0^{\tau/2} \\ &\quad + \frac{2x_0}{\tau^2} \left(\frac{1}{n^2 \omega^2} \right) \left[e^{-in\omega t} (in\omega t) + e^{-in\omega t} \right]_{\tau/2}^{\tau} \\ &\quad - \frac{2x_0}{\tau} \left(\frac{1}{-in\omega} \right) \left(e^{-in\omega t} \right)_{\tau/2}^{\tau} \\ &= \frac{2x_0}{n^2 \pi^2} \left[in\pi e^{-in\pi} + e^{-in\pi} - 1 \right] - \frac{2x_0 i}{n\pi} \left[e^{-in\pi} - e^{-in\frac{\pi}{2}} \right] \\ &= \frac{2x_0}{n^2 \pi^2} \cos n\pi + i \frac{2x_0}{n\pi} \cos \frac{n\pi}{2} + \frac{2x_0}{n\pi} \sin \frac{n\pi}{2} - \frac{2x_0}{n^2 \pi^2} \\ &= \frac{2x_0}{n^2 \pi^2} \left[(-1)^n - 1 \right] + i \underbrace{\frac{2x_0}{n\pi} (-1)^{\frac{n}{2}}}_{n = \text{even}} + \underbrace{\frac{2x_0}{n\pi} (-1)^{\frac{n-1}{2}}}_{n = \text{odd}} \end{aligned}$$

$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} c_n e^{in\omega t} \\ &= \sum_{n=-\infty}^{\infty} \frac{2x_0}{n^2 \pi^2} \left[(-1)^n - 1 \right] e^{in\omega t} \\ &\quad + i \frac{2x_0}{\pi} \sum_{\substack{n=-\infty \\ (n = \text{even})}}^{\infty} (-1)^{\frac{n}{2}} \frac{1}{n} e^{in\omega t} \\ &\quad + \frac{2x_0}{\pi} \sum_{\substack{n=-\infty \\ (n = \text{odd})}}^{\infty} (-1)^{\frac{n-1}{2}} \frac{1}{n} e^{in\omega t} \end{aligned}$$

14.9

For $x(t) = X \sin \omega t$, $R_x(\tau) = \frac{X^2}{2} \cos \omega \tau$ [from Problem 14.7]

For square wave:



14.10

$$p(t) = \begin{cases} \lambda e^{-\lambda t} & ; t \geq 0 \\ 0 & ; t < 0 \end{cases}$$

$$(i) P(t) = \int_{-\infty}^t p(t') dt' = \lambda \int_0^t e^{-\lambda t'} dt' = 1 - e^{-\lambda t}$$

$$(ii) \bar{T} = \int_{-\infty}^{\infty} t p(t) dt = \lambda \int_0^{\infty} t e^{-\lambda t} dt = \frac{\lambda}{(-\lambda)^2} [e^{-\lambda t} (-\lambda t - 1)]_0^{\infty} = \frac{1}{\lambda}$$

$$(iii) \sigma_T^2 = \int_0^{\infty} (t - \frac{1}{\lambda})^2 p(t) dt$$

$$= \lambda \int_0^{\infty} t^2 e^{-\lambda t} dt + \frac{1}{\lambda} \int_0^{\infty} e^{-\lambda t} dt - 2 \int_0^{\infty} t e^{-\lambda t} dt$$

$$= \lambda \left[-\frac{t^2}{\lambda} e^{-\lambda t} - \frac{2t}{\lambda^2} e^{-\lambda t} - \frac{2}{\lambda^3} e^{-\lambda t} \right]_0^{\infty}$$

$$- \frac{1}{\lambda^2} (e^{-\lambda t})_0^{\infty} + \frac{2}{\lambda^2} (\lambda t e^{-\lambda t} + e^{-\lambda t})_0^{\infty} = \frac{1}{\lambda^2}$$

$$\sigma_T = \frac{1}{\lambda}$$

14.11 $x(t) = x_0 \sin \frac{\pi t}{2}$

$$\langle x(t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x_0 \sin \frac{\pi t}{2} dt = \lim_{T \rightarrow \infty} \frac{1}{T} \left(\frac{-x_0 \cos \frac{\pi t}{2}}{\frac{\pi}{2}} \right) \Big|_{-T/2}^{T/2}$$

$$= \lim_{T \rightarrow \infty} -\frac{x_0}{T} \left(\frac{2}{\pi} \right) \left[\cos \frac{\pi T}{4} - \cos \left(-\frac{\pi T}{4} \right) \right] = 0$$

$$\langle x^2(t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x_0^2 \sin^2 \frac{\pi t}{2} dt$$

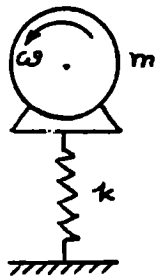
$$= \lim_{T \rightarrow \infty} \frac{x_0^2}{T} \int_{-T/2}^{T/2} \left\{ \frac{1}{2} - \frac{1}{2} \cos \pi t \right\} dt = \lim_{T \rightarrow \infty} \frac{x_0^2}{2T} \left[T - \frac{2}{\pi} \sin \frac{\pi T}{2} \right]$$

$$= \lim_{T \rightarrow \infty} \left(\frac{x_0^2}{2} - \frac{x_0^2}{2} \left\{ \frac{\sin \frac{\pi T}{2}}{\frac{\pi T}{2}} \right\} \right) = \frac{x_0^2}{2}$$

14.12 $\omega = 2\pi(1800)/60 = 188.496 \text{ rad/sec}$

$k = (\bar{k}, \sigma_k) = (2.25 \times 10^6, 0.225 \times 10^6) \text{ N/m}$
(normally distributed)

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{k}{100}} = \frac{\sqrt{k}}{10} \text{ rad/sec}$



$P(\omega_n \geq \omega) = P[\omega_n^2 \geq \omega^2]$

$= P[k \geq 100(188.496)^2] = P[k \geq 3553074.202] \quad (E_1)$

Defining standard normal variate z as $z = \left(\frac{k - \bar{k}}{\sigma_k} \right)$,
Eq. (E₁) can be rewritten as

$$P[\omega_n > \omega] = P\left[\frac{k - \bar{k}}{\sigma_k} \geq \frac{3.5531 \times 10^6 - 2.25 \times 10^6}{0.225 \times 10^6} \right]$$

$$= P[z \geq 5.7916] = 0.3316 \times 10^{-8}$$

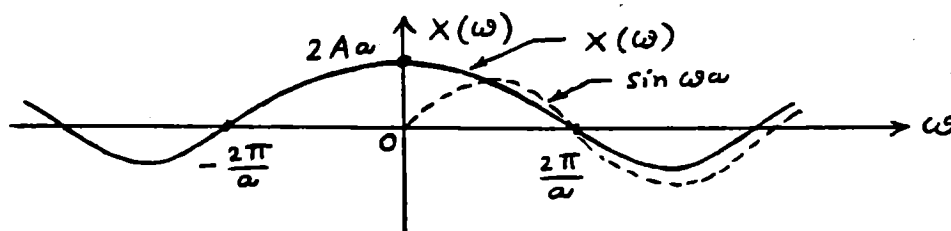
from standard normal distribution tables
[see, for example, Ref. 14.5]

14.13 $x(t) = \begin{cases} A & ; -a \leq t \leq a \\ 0 & ; \text{elsewhere} \end{cases}$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt = A \int_{-a}^a e^{-i\omega t} dt = \frac{A}{-i\omega} \left(e^{-i\omega t} \right)_{-a}^a$$

$$\begin{aligned}
 &= \frac{Ai}{\omega} (e^{-i\omega a} - e^{i\omega a}) \\
 &= \frac{Ai}{\omega} (\cos \omega a - i \sin \omega a - \cos \omega a - i \sin \omega a) \\
 &= \frac{2Aa}{\omega a} \sin \omega a \quad (E_1)
 \end{aligned}$$

Eq. (E₁) shows that $X(\omega) = 2Aa$ as $\omega \rightarrow 0$.

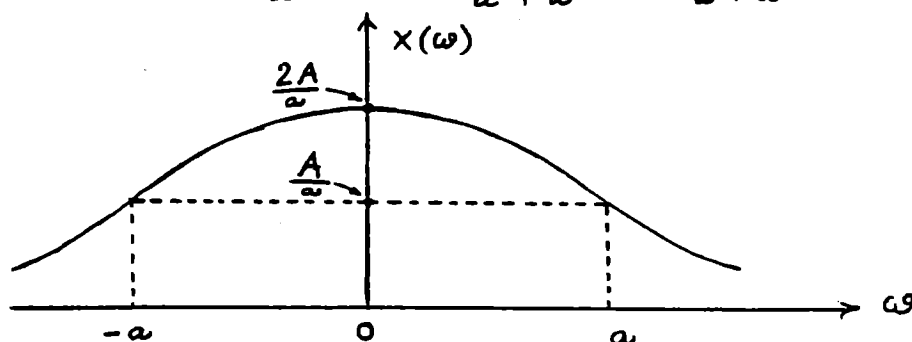


14.14 $x(t) = \begin{cases} A e^{-at} & ; t \geq 0 \\ 0 & ; \text{elsewhere} \end{cases}$

$$\begin{aligned}
 X(\omega) &= \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt = A \int_0^{\infty} e^{-(a+i\omega)t} dt \\
 &= \frac{-A}{(a+i\omega)} \left[e^{-(a+i\omega)t} \right]_0^{\infty} = \frac{A}{a+i\omega} = \frac{Aa}{a^2+\omega^2} - i \frac{A\omega}{a^2+\omega^2}
 \end{aligned}$$

14.15 $x(t) = A e^{-a|t|}$

$$\begin{aligned}
 X(\omega) &= A \int_{-\infty}^0 e^{at} e^{-i\omega t} dt + A \int_0^{\infty} e^{-at} e^{-i\omega t} dt \\
 &= \frac{A}{-(i\omega-a)} \left[e^{-(i\omega-a)t} \right]_{-\infty}^0 - \frac{A}{(a+i\omega)} \left[e^{-(a+i\omega)t} \right]_0^{\infty} \\
 &= \frac{A}{a-i\omega} + \frac{A}{a+i\omega} = \frac{A(a+i\omega)}{a^2+\omega^2} + \frac{A(a-i\omega)}{a^2+\omega^2} = \frac{2Aa}{a^2+\omega^2}
 \end{aligned}$$

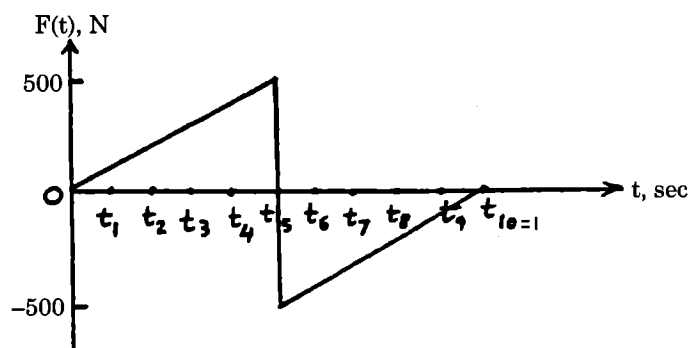


14.16 $x(t) = \delta(t-a)$

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t-a) e^{-i\omega t} dt = e^{-i\omega a} = \cos \omega a - i \sin \omega a$$

since, by definition, the Dirac delta function is zero every where except at $t=a$. At $t=a$, $e^{-i\omega t} = e^{-i\omega a}$

14.17



j	0	1	2	3	4	5	6	7	8	9	10
t_j (sec)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$F_j = F(t_j)$, N	0	100	200	300	400	500	-400	-300	-200	-100	0

(i) Spectrum of $F(t)$:

$$\text{From Eq. (14.46), } c_n = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} F(t) e^{-in\omega_0 t} dt \quad (E_1)$$

where τ = time period = $\frac{2\pi}{\omega_0}$ and ω_0 = fundamental frequency, $\tau = 1.0$ sec,

$\omega_0 = 2\pi$ rad/sec.

Since the integration in Eq. (E_1) is over one-time period, c_n can be expressed as

$$c_n = \frac{1}{\Delta t N} \sum_{j=1}^N F(t_j) \cdot e^{-in\left(\frac{2\pi}{\tau}\right)t_j} \cdot \Delta t \quad \text{where } \frac{t_j}{\tau} = \frac{j}{N}$$

$$\begin{aligned} \therefore c_n &= \frac{1}{N} \sum_{j=1}^N F_j e^{-i\left(\frac{2\pi n j}{N}\right)} = \frac{1}{N} \sum_{j=1}^N F_j \left\{ \cos \frac{2\pi n j}{N} - i \sin \frac{2\pi n j}{N} \right\} \\ &= \text{real}(c_n) + i \text{imag}(c_n) \end{aligned} \quad (E_2)$$

Eq. (E₂) gives the following result for N = 10:

n	Real (c _n)	Imag (c _n)	Spectrum of F (t) = (c _n) ² = Real (c _n) ² + Imag (c _n) ²
0	50	0	0.25 × 10 ⁴
1	- 50	- 153.8842	2.6180 × 10 ⁴
2	50	68.8191	0.7236 × 10 ⁴
3	- 50	- 36.3271	0.3820 × 10 ⁴
4	50	16.2460	0.2764 × 10 ⁴
5	- 50	0.0000	0.2500 × 10 ⁴
6	50	- 16.2460	0.2764 × 10 ⁴
7	- 50	36.3271	0.3820 × 10 ⁴
8	50	- 68.8191	0.7236 × 10 ⁴
9	- 50	153.8842	2.6180 × 10 ⁴

(ii) Mean square value of F (t):

Using Eq. 14.49,

$$\overline{F^2(t)} = \sum_{n=0}^{N-1} |C_n|^2 = \sum_{n=0}^9 |C_n|^2 = 85000.00$$

14.18

$$R_X(\tau) = a e^{-b/|\tau|}$$

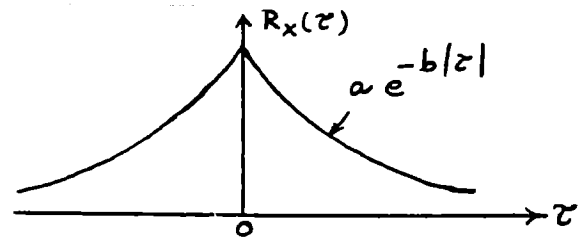
$$S_X(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$$

$$= \frac{1}{2\pi} \int_{-\infty}^0 a e^{b\tau} e^{-i\omega\tau} d\tau$$

$$+ \frac{1}{2\pi} \int_0^{\infty} a e^{-b\tau} e^{-i\omega\tau} d\tau$$

$$= \left(\frac{a}{2\pi}\right) \frac{1}{-(i\omega - b)} \left[e^{-(i\omega - b)\tau} \right]_{-\infty}^0 + \left(\frac{a}{2\pi}\right) \frac{1}{-(i\omega + b)} \left[e^{-(i\omega + b)\tau} \right]_0^{\infty}$$

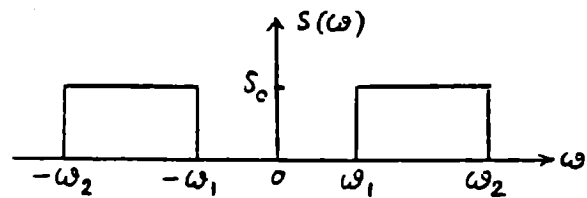
$$= \frac{-a}{2\pi(i\omega - b)} + \frac{a}{2\pi(i\omega + b)} = \frac{ab}{\pi(b^2 + \omega^2)}$$



$$(14.19) \quad R(\tau) = \int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} d\omega \quad \dots (E_1)$$

Since $S(\omega) = S_0$ is symmetric and real, we can neglect the imaginary component in (E_1) and write $R(\tau)$ as

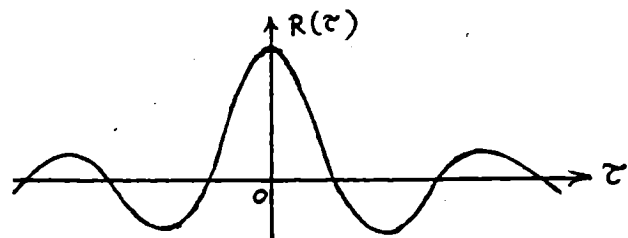
$$R(\tau) = 2 S_0 \int_{\omega_1}^{\omega_2} \cos \omega\tau \cdot d\omega = \frac{2 S_0}{\tau} \left(\sin \omega_2 \tau - \sin \omega_1 \tau \right) \quad \dots (E_2)$$



$R(\tau)$ as $\tau \rightarrow 0$ is given by

$$\lim_{\tau \rightarrow 0} \left[2 S_0 \left(\frac{\omega_2 \sin \omega_2 \tau}{\omega_2 \tau} \right) - 2 S_0 \left(\frac{\omega_1 \sin \omega_1 \tau}{\omega_1 \tau} \right) \right] = 2 S_0 (\omega_2 - \omega_1) \quad \dots (E_3)$$

The variation of $R(\tau)$ is shown in the figure.



14.20 From Eq. (14.60),

$$S_x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_x(\tau) e^{-i\omega\tau} d\tau = \frac{1}{2\pi} \sigma_x^2 \int_{-\infty}^{\infty} e^{-\alpha|v\tau|} \cos \beta v \tau e^{-i\omega\tau} d\tau$$

$$\text{But } \cos \beta v \tau = \frac{1}{2} (e^{i\beta v \tau} + e^{-i\beta v \tau})$$

$$\begin{aligned} S_x(\omega) &= \frac{\sigma_x^2}{4\pi} \int_{-\infty}^{\infty} \left\{ e^{-\alpha|v\tau|} e^{-i\omega\tau} (e^{i\beta v \tau} + e^{-i\beta v \tau}) d\tau \right\} \\ &= \frac{\sigma_x^2}{4\pi} \left[\int_0^{\infty} e^{(-\alpha v - i\omega + i\beta v)\tau} d\tau + \int_0^{\infty} e^{(-\alpha v - i\omega - i\beta v)\tau} d\tau \right. \\ &\quad \left. + \int_{-\infty}^0 e^{(\alpha v - i\omega + i\beta v)\tau} d\tau + \int_{-\infty}^0 e^{(\alpha v - i\omega - i\beta v)\tau} d\tau \right] \\ &= \frac{\sigma_x^2}{2\pi} \left[\frac{\alpha v}{\alpha^2 v^2 + (\omega - \beta v)^2} + \frac{\alpha v}{\alpha^2 v^2 + (\omega + \beta v)^2} \right] \\ &= \frac{\alpha v \sigma_x^2}{\pi} \left[\frac{\omega^2 + \beta^2 v^2 + \alpha^2 v^2}{\{\alpha^2 v^2 + (\omega - \beta v)^2\} \{\alpha^2 v^2 + (\omega + \beta v)^2\}} \right] \end{aligned}$$

For asphalt surface:

$$\sigma_x = 1.1, \quad \alpha = 0.2, \quad \beta = 0.4$$

$$\begin{aligned} S_x(\omega) &= \frac{(0.2) v (1.21)}{\pi} \left[\frac{\omega^2 + 0.16 v^2 + 0.04 v^2}{\{0.04 v^2 + (\omega - 0.4 v)^2\} \{0.04 v^2 + (\omega + 0.4 v)^2\}} \right] \\ &= 0.07703 v \left[\frac{\omega^2 + 0.2 v^2}{\{0.04 v^2 + (\omega - 0.4 v)^2\} \{0.04 v^2 + (\omega + 0.4 v)^2\}} \right] \end{aligned}$$

For paved surface:

$$\sigma_x = 1.6, \quad \alpha = 0.3, \quad \beta = 0.6$$

$$S_x(\omega) = \frac{(0.3) v (2.56)}{\pi} \left[\frac{\omega^2 + 0.36 v^2 + 0.09 v^2}{\{0.09 v^2 + (\omega - 0.6 v)^2\} \{0.09 v^2 + (\omega + 0.6 v)^2\}} \right]$$

$$= 0.24446 \, v \left[\frac{\omega^2 + 0.45 \, v^2}{\{0.09 \, v^2 + (\omega - 0.6 \, v)^2\} \{0.09 \, v^2 + (\omega + 0.6 \, v)^2\}} \right]$$

For gravel surface:

$$\sigma_x = 1.8, \quad \alpha = 0.5, \quad \beta = 0.9$$

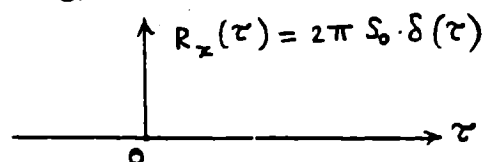
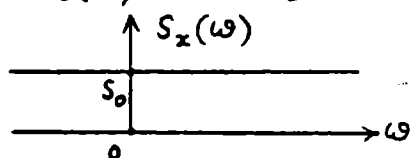
$$S_x(\omega) = \frac{(0.5) v (3.24)}{\pi} \left[\frac{\omega^2 + 0.81 \, v^2 + 0.25 \, v^2}{\{0.25 \, v^2 + (\omega - 0.9 \, v)^2\} \{0.25 \, v^2 + (\omega + 0.9 \, v)^2\}} \right]$$

$$= 0.51566 \, v \left[\frac{\omega^2 + 1.06 \, v^2}{\{0.25 \, v^2 + (\omega - 0.9 \, v)^2\} \{0.25 \, v^2 + (\omega + 0.9 \, v)^2\}} \right]$$

14.21 From Eq. (14.61), $R_x(\tau) = \int_{-\infty}^{\infty} S_x(\omega) e^{i\omega\tau} d\omega = S_0 \int_{-\infty}^{\infty} e^{i\omega\tau} d\omega$

$$= 2\pi S_0 \delta(\tau)$$

Where $\delta(\tau)$ is the Dirac delta function.



14.22 $S_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau) e^{-i\omega\tau} d\tau$ or $S_x(f) = \int_{-\infty}^{\infty} R_x(\tau) e^{-i2\pi f\tau} d\tau$ --- (E₁)

and

$$R_x(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega) e^{i\omega\tau} d\omega$$
 or $R_x(\tau) = \int_{-\infty}^{\infty} S_x(f) e^{i2\pi f\tau} df$ --- (E₂)

where $S_x(f)$ is the two-sided power spectral density function for $-\infty < f < \infty$. Noting that

$S_x(f) = S_x(-f)$ and $R_x(\tau) = R_x(-\tau)$, (E₁) and (E₂) can be written as

$$S_x(f) = 2 \int_0^{\infty} R_x(\tau) \cdot \{\cos 2\pi f\tau - i \sin 2\pi f\tau\} d\tau \quad \text{for } -\infty < f < \infty$$
 --- (E₃)

and

$$R_x(\tau) = 2 \int_0^{\infty} S_x(f) \cdot \{\cos 2\pi f\tau + i \sin 2\pi f\tau\} d\tau \quad \text{for } -\infty < f < \infty$$
 --- (E₄)

For a real random function $x(t)$, $S_x(f)$ and $R_x(\tau)$ are real-valued functions and hence, by neglecting the imaginary parts in (E₃) and (E₄), we obtain

$$S_x(f) = 2 \int_0^{\infty} R_x(\tau) \cdot \cos 2\pi f\tau \cdot d\tau \quad \text{for } -\infty < f < \infty$$
 --- (E₅)

and

$$R_x(\tau) = 2 \int_0^{\infty} S_x(f) \cdot \cos 2\pi f \tau \cdot df \quad \text{for } -\infty < f < \infty \dots (E_6)$$

If $S(f)$ denotes the one-sided power spectral density function,

$$S(f) = 2 S_x(f) \quad \text{and} \quad R(\tau) = R_x(\tau), \quad \text{Eqs. (E5) and (E6) become}$$

$$S(f) = 4 \int_0^{\infty} R(\tau) \cos 2\pi f \tau \cdot d\tau \quad \text{for } 0 \leq f < \infty \dots (E_7)$$

and

$$R(\tau) = \int_0^{\infty} S(f) \cos 2\pi f \tau \cdot df \quad \text{for } 0 \leq f < \infty \dots (E_8)$$

14.23

Mean square value of a single d.o.f. system is given by Eq. (14.95):

$$E[y^2] = \int_{-\infty}^{\infty} |H(\omega)|^2 S_x(\omega) d\omega \equiv \int_{-\infty}^{\infty} A(\omega) d\omega \quad (E_1)$$

where

$$|H(\omega)| = \frac{1}{(k - m\omega^2)^2 + c^2\omega^2} \quad (E_2)$$

Let $-\omega^{(1)}$ and $+\omega^{(2)}$ denote finite lower and upper bounds to be used for integration in Eq. (E1) [$\omega^{(1)}$ and $\omega^{(2)}$ should be sufficiently large].

We can use trapezoidal rule of integration for simplicity. In this method, we divide the interval $(\omega^{(2)} + \omega^{(1)})$ into $(n-1)$ equal divisions so that

$$\left. \begin{aligned} \omega_1 &= -\omega^{(1)} \\ \omega_2 &= -\omega_1 + \Delta\omega \\ \omega_3 &= -\omega_1 + 2\Delta\omega \\ \vdots \\ \omega_n &= -\omega_1 + (n-1)\Delta\omega = \omega^{(2)} \end{aligned} \right\} \quad (E_3)$$

Then

$$\begin{aligned} E[y^2] &= \int_{-\omega^{(1)}}^{\omega^{(2)}} A(\omega) d\omega = \sum_{p=1}^{n-1} [A(\omega_p) + A(\omega_{p+1})] \frac{\Delta\omega}{2} \\ &= [A(-\omega^{(1)}) + A(\omega^{(2)})] \frac{\Delta\omega}{2} + \sum_{p=2}^{n-1} A(\omega_p) \cdot \Delta\omega \dots (E_4) \end{aligned}$$

The computer program listing is given below. In this program, the following notation is used:

$$OM1 = \omega^{(1)}, \quad OM2 = \omega^{(2)}, \quad OMP = \omega_p, \quad SX = S_x(\omega),$$

$$HOM = |H(\omega)|^2, \quad SUM = E[y^2].$$

```

REAL  M, K, C
DATA  M, K, C /... /
DATA  OM1, OM2 /... /
N = ...
N1 = N - 1
DOM = (OM2 + OM1) / REAL (N1)
OM11 = -OM1
CALL PSD (OM11, SX1, HOM1)
CALL PSD (OM2, SX2, HOM2)
SUM = [(SX1/HOM1) + (SX2/HOM2)] * DOM / 2.0
DO 11 I = 2, N1
  OMP = -OM1 + REAL (I-1) * DOM
  CALL PSD (OMP, SXP, HOMP)
11 SUM = SUM + (SXP/HOMP) * DOM
PRINT 12, SUM
12 FORMAT (...)
STOP
END
C SUBROUTINE TO EVALUATE POWER SPECTRAL DENSITY AND HOM
SUBROUTINE PSD (OM, SX, HOM)
  SX = ...
  HOM = ...
  RETURN
END

```

14.24 $m = 900 \text{ kg}$, $k = 7 \times 10^6 \text{ N/m}$, $c = 135 \text{ N} \cdot \text{m/sec}$
 Mean square response of the machine is given by Eq. (14.95):

$$E[y^2] = \int_{-\infty}^{\infty} |H(\omega)|^2 S_x(\omega) d\omega = \sum_{n=0}^{N-1} \left| \frac{1}{-m\omega_n^2 + i c \omega_n + k} \right|^2 \cdot |c_n|^2$$

where $\omega_n = n \omega_0 = n \left(\frac{2\pi}{\tau} \right) = 2\pi n \text{ rad/sec}$ (E₁)

and $|c_n|^2$ are given in the solution of problem 14.17

Eq. (E₁) can be rewritten as $E[y^2] = \sum_{n=0}^{N-1} \frac{|c_n|^2}{(k - m\omega_n^2)^2 + c^2 \omega_n^2}$ (E₂)

Computations

n	ω_n	$"(k - m \omega_n^2)^2 + c^2 \omega_n^2" (\times 10^{13})$	$\left\{ \frac{ c_n ^2}{(k - m \omega_n^2)^2 + c^2 \omega_n^2} \right\} (\times 10^{-8} \text{ m}^2)$
0	0	4.9	0.0051
1	2π	4.8504	0.0540
2	4π	4.7030	0.0154
3	6π	4.4625	0.0086
4	8π	4.1364	0.0067
5	10π	3.7353	0.0067
6	12π	3.2729	0.0084
7	14π	2.7657	0.0138
8	16π	2.2336	0.0324
9	18π	1.6991	0.1541

Thus Eq. (E₂) gives the mean square of the response as

$$E[y^2] = 3.0513 \times 10^{-9} \text{ m}^2$$

14.25

Equation of motion is

$$m\ddot{x} + c\dot{x} = F(t) \quad (E_1)$$

Let $y = \dot{x} = \text{velocity}$. Then (E₁) becomes

$$m\dot{y} + cy = F(t) \quad (E_2)$$

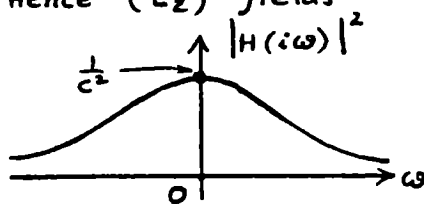
Let the excitation be

$$F(t) = F_0 e^{i\omega t}$$

and the velocity response be

$$y(t) = Y e^{i\omega t}$$

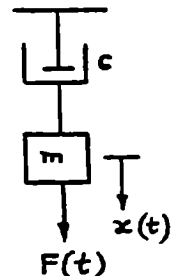
Hence (E₂) yields



$$m i \omega Y e^{i\omega t} + c Y e^{i\omega t} = F_0 e^{i\omega t}$$

$$\text{or} \quad \frac{Y}{Y_0} = \frac{1}{i \omega m + c} \equiv H(i\omega)$$

$$|H(i\omega)|^2 = \frac{1}{m^2 \omega^2 + c^2}$$

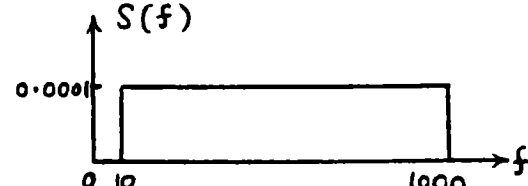


14.26 $S(f) = 0.0001 \text{ m}^2/\text{cycle per second}$ for $10 \text{ Hz} \leq f \leq 1000 \text{ Hz}$

$$\bar{x}^2 = (0.0001)(1000 - 10) = 0.099 \text{ m}^2$$

$$\text{RMS value} = \sqrt{\bar{x}^2} = 0.3146 \text{ m}$$

$$\sigma = \sqrt{\bar{x}^2 - (\bar{x})^2} = \sqrt{0.099 - 0.0025}$$

$$= 0.3106 \text{ m}$$


14.27 $\int_{t=-\tau/2}^{\tau/2} \cos(n-m)\omega_0 t \, dt = \frac{1}{(n-m)\omega_0} \left[\sin(n-m)\omega_0 t \right]_{t=-\tau/2}^{t=\tau/2}$

$$= \frac{1}{(n-m)\omega_0} \left[\sin(n-m)\pi + \sin(n-m)\pi \right] = 0 \text{ for all } m \neq n$$

since $\tau = \frac{2\pi}{\omega_0}$.

$$\int_{-\tau/2}^{\tau/2} \sin(n-m)\omega_0 t \, dt = \frac{1}{-(n-m)\omega_0} \left[\cos(n-m)\omega_0 t \right]_{t=-\tau/2}^{t=\tau/2}$$

$$= \frac{-1}{(n-m)\omega_0} \left[\cos(n-m)\pi - \cos(n-m)\pi \right] = 0 \text{ for all } m \neq n$$

$$\int_{-\tau/2}^{\tau/2} \cos 0 \, dt = \left(t \right)_{-\tau/2}^{\tau/2} = \tau \text{ for } m=n$$

$$\int_{-\tau/2}^{\tau/2} \sin 0 \, dt = 0 \text{ for } m=n$$

$\therefore$ Eq. (14.45) becomes

$$\int_{-\tau/2}^{\tau/2} x(t) e^{-in\omega_0 t} \, dt = c_n \tau \Rightarrow c_n = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} x(t) \cdot e^{-in\omega_0 t} \, dt$$

-----(14.46)

14.28

This system can be modeled as a single d.o.f. system with random base excitation. The equations of motion are given by:

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{x} \quad (E_1)$$

where $z = y - x$.

The frequency response function of the system can be derived as follows:

$$\left. \begin{aligned} \text{Let } x(t) &= e^{i\omega t} \\ z(t) &= H(\omega) e^{i\omega t} \end{aligned} \right\} \quad (E_2)$$

Substitution of (E_2) into (E_1) gives

$$(-\omega^2 m + ic\omega + k) H(\omega) e^{i\omega t} = m\omega^2 e^{i\omega t}$$

$$\text{i.e., } H(\omega) = \frac{m\omega^2}{-m\omega^2 + k + ic\omega} = \frac{\omega^2}{(\omega_n^2 - \omega^2) + i2\zeta\omega\omega_n}$$

$$\text{i.e., } |H(\omega)|^2 = \frac{\omega^4}{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega^2\omega_n^2} \quad (E_3)$$

The power spectral density of the response $z(t)$ is given by

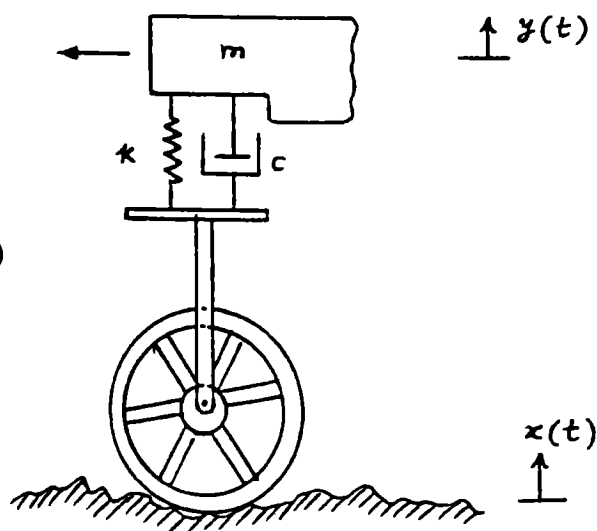
$$S_z(\omega) = |H(\omega)|^2 S_x(\omega) \quad (E_4)$$

In the present case, $S_x(\omega) = S_0$ and Eq. (E_4) reduces to

$$S_z(\omega) = S_0 |H(\omega)|^2 \quad (E_5)$$

The mean square value of the relative displacement of the mass can be found as

$$\begin{aligned} E[z^2] &= \int_{-\infty}^{\infty} S_z(\omega) \cdot d\omega = S_0 \int_{-\infty}^{\infty} \left| \frac{\omega^2}{-\omega^2 + \omega_n^2 + \frac{i\omega c}{m}} \right|^2 \cdot d\omega \\ &= S_0 \omega^4 \pi \left\{ \frac{\left(\frac{1}{\omega_n^2} \right)}{\left(\frac{c}{m} \right)} \right\} = \frac{\pi S_0 \omega^4}{2\zeta \omega_n^3} \quad (E_6) \end{aligned}$$



$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \frac{1}{m} F \quad \text{--- (E}_1\text{)}$$

This gives $S_x(\omega) = \frac{1}{m^2} \frac{S_F(\omega)}{\{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2\}}$ --- (E₂)

$$E[x^2] = \frac{1}{m^2} \int_{-\infty}^{\infty} \frac{S_F(\omega) \cdot d\omega}{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2} \quad \text{--- (E}_3\text{)}$$

For small damping, (E₃) becomes

$$E[x^2] \simeq \frac{1}{m^2} S_F(\omega_n) \cdot \frac{\pi}{2\zeta\omega_n^3} \quad \text{--- (E}_4\text{)}$$

Here $S_F(\omega) = A^2 \frac{1 + \left(\frac{L\omega}{v}\right)^2}{\left\{1 + \left(\frac{L\omega}{v}\right)^2\right\}^2 \left(1 + \frac{\pi\omega c}{v}\right)}$ --- (E₅)

$$\therefore E[x^2] = \frac{\pi A^2}{2\zeta\omega_n^3 m^2} \left\{ \frac{1 + \left(\frac{L\omega_n}{v}\right)^2}{\left(1 + \frac{L^2\omega_n^2}{v^2}\right)^2 \left(1 + \frac{\pi\omega_n c}{v}\right)} \right\} \quad \text{--- (E}_6\text{)}$$

14.30 $\omega_1 = \text{undamped natural frequency} = \sqrt{k_{eq}/m_{eq}}$
 or $k_{eq} = \omega_1^2 m_{eq}$ (E₁)

$$\begin{aligned} \omega_2 &= \text{damped natural frequency} = \omega_n \sqrt{1 - \zeta^2} \\ &= \omega_1 \sqrt{1 - \zeta^2} = \sqrt{\frac{k_{eq}}{m_{eq}}} \cdot \sqrt{1 - \zeta^2} = \sqrt{\frac{k_{eq}}{m_{eq}}} \cdot \sqrt{\left(1 - \frac{c_{eq}^2}{4 k_{eq} m_{eq}}\right)} \end{aligned}$$

$$\omega_2^2 = \frac{k_{eq}}{m_{eq}} \left(1 - \frac{c_{eq}^2}{4 k_{eq} m_{eq}}\right) = \omega_1^2 - \frac{c_{eq}^2}{4 m_{eq}^2}$$

or $\left(\frac{c_{eq}}{2 m_{eq}}\right)^2 = \omega_1^2 - \omega_2^2$

or $c_{eq} = 2 m_{eq} \sqrt{\omega_1^2 - \omega_2^2}$ (E₂)

Mean square value of the displacement of the wing (m_{eq})

$$E[y^2] = \frac{\pi S_0}{k_{eq} c_{eq}} = \delta \quad \text{or} \quad c_{eq} = \frac{\pi S_0}{\delta k_{eq}} = \delta. \quad (E_3)$$

Eqs. (E₁) and (E₃) give

$$c_{eq} = \frac{\pi S_0}{\delta \omega_1^2 m_{eq}} \quad (E_4)$$

Equating (E₂) and (E₄),

$$c_{eq} = 2 m_{eq} \sqrt{\omega_1^2 - \omega_2^2} = \frac{\pi S_0}{\delta \omega_1^2 m_{eq}} \quad (E_5)$$

Eqs. (E₁) and (E₂), in view of (E₅), yield

$$k_{eq} = \left[\frac{\pi S_0 \omega_1^2}{2 \delta (\omega_1^2 - \omega_2^2)^{\frac{1}{2}}} \right]^{\frac{1}{2}} \quad (E_6)$$

and

$$\begin{aligned} c_{eq} &= 2 (\omega_1^2 - \omega_2^2)^{\frac{1}{2}} \left[\frac{\pi S_0}{2 \delta \omega_1^2 (\omega_1^2 - \omega_2^2)^{\frac{1}{2}}} \right]^{\frac{1}{2}} \\ &= \left[\frac{2 \pi S_0 (\omega_1^2 - \omega_2^2)^{\frac{1}{2}}}{\delta \omega_1^2} \right]^{\frac{1}{2}} \quad (E_7) \end{aligned}$$

14.31 In case of structural damping, the uncoupled equations of motion are given by

$$\ddot{q}_i(t) + (1 + i \beta) \omega_i^2 q_i(t) = Q_i(t) ; i = 1, 2, \dots, n \quad (1)$$

where β denotes the structural damping coefficient. The mean square values of $x_i(t)$ are given by Eq. (14.113):

$$\overline{x_i^2(t)} = \sum_{r=1}^n \left(X_i^{(r)} \right)^2 \frac{N_r^2}{\omega_r^4} \int_{-\infty}^{\infty} |H_r(\omega)|^2 S_r(\omega) d\omega \quad (2)$$

where, from Eq. (3.106),

$$|H_r(\omega)| = \frac{1}{\left\{1 - \frac{\omega^2}{\omega_n^2}\right\}^2 + \beta^2} \quad (3)$$

For $\beta \ll 1$, Eq. (2) can be approximated as

$$\begin{aligned} \overline{x_i^2(t)} &\approx \sum_{r=1}^n \left(X_i^{(r)}\right)^2 \frac{N_r^2}{\omega_r^4} S_r(\omega_r) \int_{-\infty}^{\infty} |H_r(\omega)|^2 d\omega \\ &\approx \sum_{r=1}^n \left(X_i^{(r)}\right)^2 \frac{N_r^2}{\omega_r^4} S_r(\omega_r) \frac{\pi \omega_r}{\beta} \end{aligned} \quad (4)$$

Using the computational details of Example 14.7, we can obtain (using $g = 0.01$ in place of $2 \zeta_r = 0.04$):

$$\overline{z_1^2(t)} = 0.0021253 \text{ m}^2 \quad (5)$$

$$\overline{z_2^2(t)} = 0.0055983 \text{ m}^2 \quad (6)$$

$$\overline{z_3^2(t)} = 0.0086582 \text{ m}^2 \quad (7)$$

14.32

$$S(\omega) = \frac{1}{4 + \omega^2} \frac{\text{m}^2/\text{s}^4}{\text{rad/sec}} ; \quad \zeta_i = 0.02$$

Since the natural frequencies (rad/sec) are given by $\omega_1 = 14.0734$, $\omega_2 = 39.4368$, and $\omega_3 = 57.0001$, we can approximate $S(\omega_r)$ for use in Eq. (14.115) as

$$S_r(\omega_1) = \frac{1}{4 + 14.0734^2} = 0.0049490 \quad (1)$$

$$S_r(\omega_2) = \frac{1}{4 + 39.4368^2} = 0.0006413 \quad (2)$$

$$S_r(\omega_3) = \frac{1}{4 + 57.0001^2} = 0.0003074 \quad (3)$$

and hence the mean square values of the relative displacements of the various floors can be computed as (using Eq. (E.18) of Example 14.7):

$$\begin{aligned} \overline{z_1^2(t)} &= \frac{\pi}{2 \zeta_r} \left[\left(Z_1^{(1)}\right)^2 \frac{N_1^2}{\omega_1^3} S(\omega_1) + \left(Z_1^{(2)}\right)^2 \frac{N_2^2}{\omega_2^3} S(\omega_2) + \left(Z_1^{(3)}\right)^2 \frac{N_3^2}{\omega_3^3} S(\omega_3) \right] \\ &= \frac{\pi}{0.04} \left[0.0001058 (0.004949) + 0.0000243 (0.0006413) + 0.0000052 (0.0003074) \right] \\ &= 42.4744 (10^{-6}) \text{ m}^2 \\ \overline{z_2^2(t)} &= \frac{\pi}{2 \zeta_r} \left[\left(Z_2^{(1)}\right)^2 \frac{N_1^2}{\omega_1^3} S(\omega_1) + \left(Z_2^{(2)}\right)^2 \frac{N_2^2}{\omega_2^3} S(\omega_2) + \left(Z_2^{(3)}\right)^2 \frac{N_3^2}{\omega_3^3} S(\omega_3) \right] \end{aligned} \quad (4)$$

$$= \frac{\pi}{0.04} [0.0003436 (0.004949) + 0.0000048 (0.0006413) + 0.0000080 (0.0003074)]$$

$$= 133.9971 (10^{-6}) \text{ m}^2 \quad (5)$$

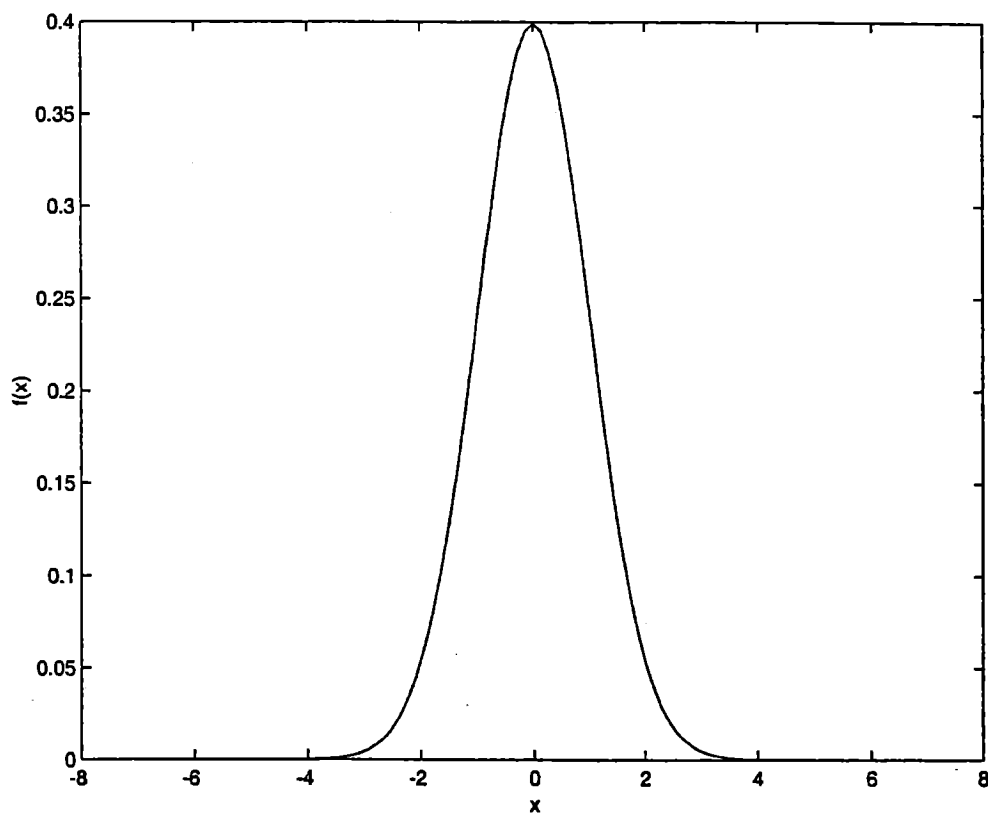
$$\overline{z_3^2(t)} = \frac{\pi}{2 \zeta_r} \left[\left(Z_3^{(1)} \right)^2 \frac{N_1^2}{\omega_1^3} S(\omega_1) + \left(Z_3^{(2)} \right)^2 \frac{N_2^2}{\omega_2^3} S(\omega_2) + \left(Z_3^{(3)} \right)^2 \frac{N_3^2}{\omega_3^3} S(\omega_3) \right]$$

$$= \frac{\pi}{0.04} [0.0005340 (0.004949) + 0.0000156 (0.0006413) + 0.0000016 (0.0003074)]$$

$$= 208.3902 (10^{-6}) \text{ m}^2 \quad (6)$$

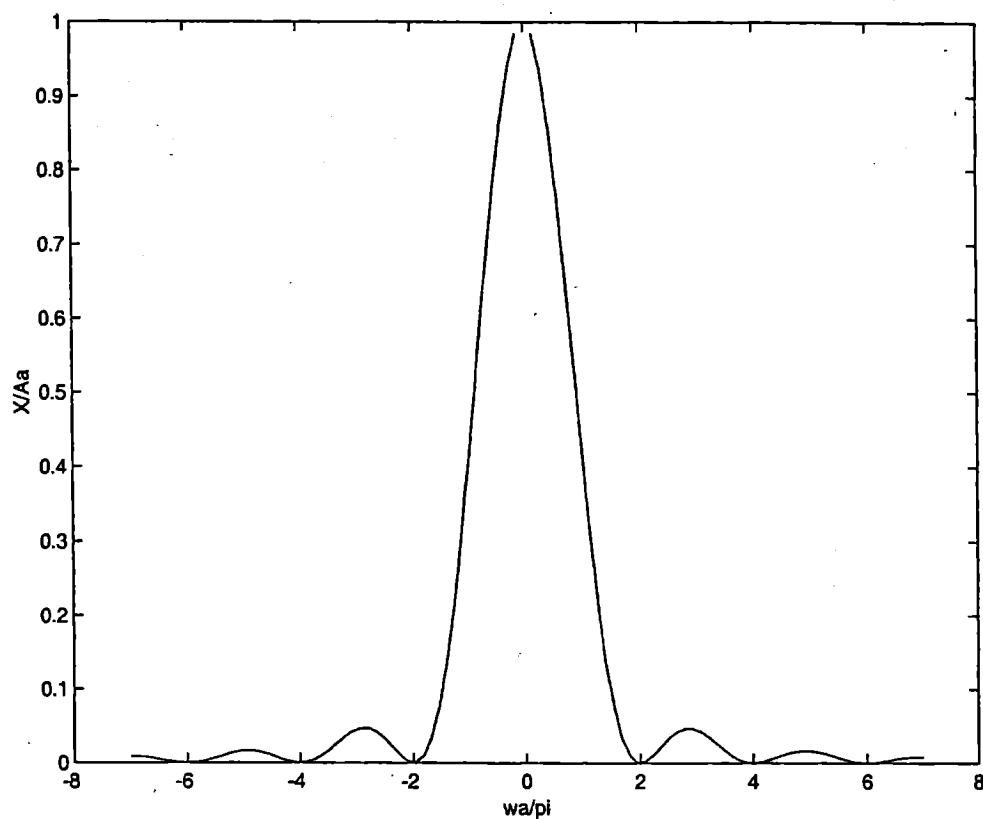
14.33

```
% Ex14_33.m
for i = 1: 101
    x(i) = -7 + 14 * (i-1)/100;
    f(i) = exp(-0.5 * x(i)^2)/sqrt(2*pi);
end
plot(x, f);
xlabel('x');
ylabel('f(x)');
```



14.34

```
% Ex14_34.m
for i = 1: 101
    wa_pi(i) = -7 + 14 * (i-1)/100;
    x_Aa(i) = (4/(pi^2)) * (1/wa_pi(i))^2 * (sin(wa_pi(i)*pi/2.0))^2;
end
plot(wa_pi, x_Aa);
xlabel('wa/pi');
ylabel('X/Aa');
```



14.35

```
% Ex14_35.m
f = [0 20 40 60 80 100 -80 -60 -40 -20 0];
k = 4e4;
c = 1200;
m = 5.1760;
N = 10;
sumE = 0.0;
for i = 1: N
    n = i - 1;
    wn = 2*pi*n;
    sumC = 0.0;
    for j = 1: N
        sumC = sumC + f(j+1)*complex(cos(2*pi*n*j/N), -sin(2*pi*n*j/N))/N;
    end
    cn = sumC;
    sumE = sumE + ( abs(cn)^2 )/( (k - m*wn^2)^2 + (c*wn)^2 );
end
sumE

% Results: sumE = 1.3760e-006
```

14.36

Let d = inner diameter of column

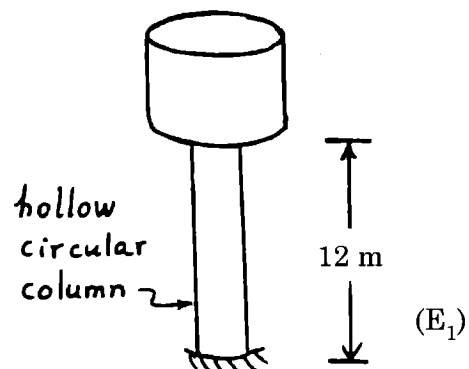
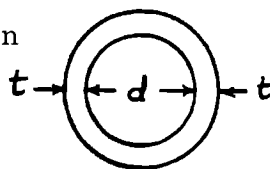
t = wall thickness

l = height = 12 m

$E = 200 \times 10^9 \text{ N/m}^2$

k = stiffness of column (cantilever)

$$= \frac{3 E I}{l^3} = \frac{3 (200 \times 10^9)}{12^3} \frac{\pi}{64} [(d + t)^4 - d^4]$$



Let tank be a thin-walled cylindrical vessel with

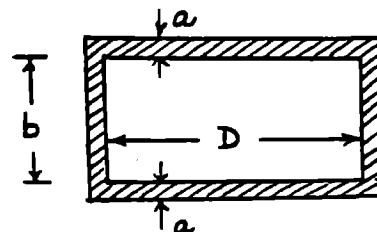
D = mean diameter

a = thickness of shell (top & bottom)

b = axial length of shell

water volume = 40 m³

$$\text{Volume of tank} = \frac{\pi D^2}{4} b = 40 \text{ m}^3$$



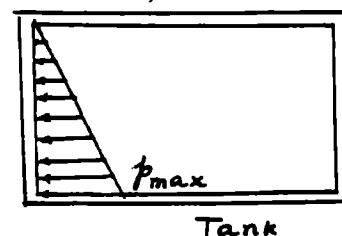
Max pressure in tank = $p_{\max} = \gamma h$; where γ = weight density of water = 9810 N/m³

$$\text{Max tangential stress in the tank} = \frac{p_{\max} D}{2 a}$$

$$\text{Let permissible stress in tank} = \sigma_p = \frac{\sigma_y}{2}$$

$$= \frac{200 \times 10^6}{2}$$

$$= 100 \times 10^6 \text{ N/m}^2$$



$$\text{Equating (E}_3\text{) and (E}_4\text{), } 100 \times 10^6 = \frac{9810 h D}{2 a}$$

Weight of empty steel tank = $\gamma_s \pi D a b$, where

γ_s = weight density of steel = 7800 g N/m³ = 76 518 N/m³ ($g = 9.81 \text{ m/s}^2$)

m_t = mass of empty steel tank = $7800 \pi D a b$

m_w = mass of water = (40) (1000) = 40 000 kg

$$\text{Natural frequency of empty tank} = \sqrt{\frac{m_t}{k}} > 2 \pi$$

$$\text{Natural frequency of full tank} = \sqrt{\frac{m_t + m_w}{k}} > 2 \pi$$

where m_t , m_w and k are given by (E₆), (E₇) and (E₁)

Due to ground acceleration with power spectral density $S(\omega)$, the mean square of relative displacement of tank is given by Eq. (E₁₂) of example (14.6):

$$E[z^2] = \frac{S_0 \pi m^2}{k c} \quad (E_{10})$$

where $c = 0.1$, $c_c = 0.2 \sqrt{k m}$ and $S_0 = 0.0002/\text{m}^2/\text{cycle}/\text{sec}$ when empty

$$E[z^2] = \frac{\pi S_0 m_t^2}{k \left(0.2 \sqrt{k m_t} \right)} \leq 100 \times 10^{-4} \text{ m}^2 \quad (E_{11})$$

and when full,

$$E[z^2] = \frac{\pi S_0 (m_t + m_w)^2}{k \left(0.2 \sqrt{k (m_t + m_w)} \right)} \leq 100 \times 10^{-4} \text{ m}^2 \quad (E_{12})$$

$\therefore$ We need to find the values of d , t , D , a and b to satisfy the inequalities (E₈), (E₉), (E₁₁) and (E₁₂).
