

Vibration Damping

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Summary

Damping in vibrating systems occurs through the dissipation of mechanical energy. This chapter presents modeling, analysis, and measurement of mechanical damping. The types of damping covered include material internal damping (including viscoelastic damping and hysteretic damping), structural damping, fluid damping, interface damping, viscous damping, Coulomb friction, and Stribeck damping. Representations of various types of damping using equivalent viscous damping models are analyzed. Damping in rotating devices is also studied.

1.1 Introduction

Damping is the phenomenon by which mechanical energy is dissipated (usually by conversion into internal thermal energy) in dynamic systems. Knowledge of the level of damping in a dynamic system is important in the utilization, analysis, and testing of the system. For example, a device with natural frequencies within the seismic range (that is, less than 33 Hz) and which has relatively low damping, could produce damaging motions under resonance conditions when subjected to a seismic disturbance. This effect could be further magnified by low-frequency support structures and panels with low damping. This example shows that knowledge of damping in constituent devices, components, and support structures is important in the design and operation of complex mechanical systems. The nature and the level of component damping should be known in order to develop a dynamic model of the system and its peripherals. Knowledge of damping in a system is also important in imposing dynamic environmental limitations on the system (that is, the maximum dynamic excitation the system can withstand) under in-service conditions. Furthermore, knowledge of a system's damping can be useful in order to make design modifications in a system that has failed the acceptance test.

However, the significance of knowledge of damping levels in a test object for the development of test excitation (input) is often overemphasized. Specifically, if the response spectrum method is used to represent the required excitation in a vibration test, then there is no need for the damping value used in the development of the required response spectrum specification to be equal to the actual damping in the

test object. The only requirement is that the damping used in the specified response spectrum be equal to that used in the test response spectrum. The degree of dynamic interaction between the test object and the shaker table, however, will depend on the actual level of damping in these systems. Furthermore, when testing near the resonant frequency of a test object, it is desirable to know about the damping in the test object.

In characterizing damping in a dynamic system it is important, first, to understand the major mechanisms associated with mechanical energy dissipation in the system. Then a suitable damping model should be chosen to represent the associated energy dissipation. Finally, damping values (model parameters) should be determined, for example, by testing the system or a representative physical model, by monitoring system response under transient conditions during normal operation or by employing already available data.

1.2 Types of Damping

There is some form of mechanical energy dissipation in any dynamic system. In the modeling of systems, damping can be neglected if the mechanical energy that is dissipated during the time duration of interest is small in comparison to the initial total mechanical energy of excitation in the system. Even for highly damped systems, it is useful to perform an analysis with the damping terms neglected, in order to study several crucial dynamic characteristics, e.g., modal characteristics (undamped natural frequencies and mode shapes).

Several types of damping are inherently present in a mechanical system. If the level of damping that is available in this manner is not adequate for proper functioning of the system then external damping devices may be added either during the original design or during subsequent design modifications of the system. Three primary mechanisms of damping are important in the study of mechanical systems. They are:

1. Internal damping (of material)
2. Structural damping (at joints and interfaces)
3. Fluid damping (through fluid–structure interactions)

Internal (material) damping results from mechanical energy dissipation within the material due to various microscopic and macroscopic processes. Structural damping is caused by mechanical energy dissipation resulting from relative motions between components in a mechanical structure that has common points of contact, joints or supports. Fluid damping arises from the mechanical energy dissipation resulting from drag forces and associated dynamic interactions when a mechanical system or its components move in a fluid.

Two general types of external dampers may be added to a mechanical system in order to improve its energy dissipation characteristics. They are:

1. Passive dampers
2. Active dampers

Passive dampers are devices that dissipate energy through some kind of motion, without needing an external power source or actuators. Active dampers have actuators that need external sources of power. They operate by actively controlling the motion of the system that needs damping. Dampers may be considered as vibration controllers. In the present chapter, the emphasis will be on damping that is inherently present in a mechanical system.

1.2.1 Material (Internal) Damping

Internal damping of materials originates from the energy dissipation associated with microstructure defects, such as grain boundaries and impurities; thermoelastic effects caused by local temperature gradients resulting from nonuniform stresses, as in vibrating beams; eddy current effects in ferromagnetic

materials; dislocation motion in metals; and chain motion in polymers. Several models have been employed to represent energy dissipation caused by internal damping. This variety of models is primarily a result of the vast range of engineering materials; no single model can satisfactorily represent the internal damping characteristics of all materials. Nevertheless, two general types of internal damping can be identified: viscoelastic damping and hysteretic damping. The latter term is actually a misnomer, because all types of internal damping are associated with hysteresis loop effects. The stress (σ) and strain (ϵ) relations at a point in a vibrating continuum possess a hysteresis loop, such as the one shown in Figure 1.1. The area of the hysteresis loop gives the energy dissipation per unit volume of the material, per stress cycle. This is termed the per-unit-volume damping capacity, and is denoted by d . It is clear that d is given by the cyclic integral

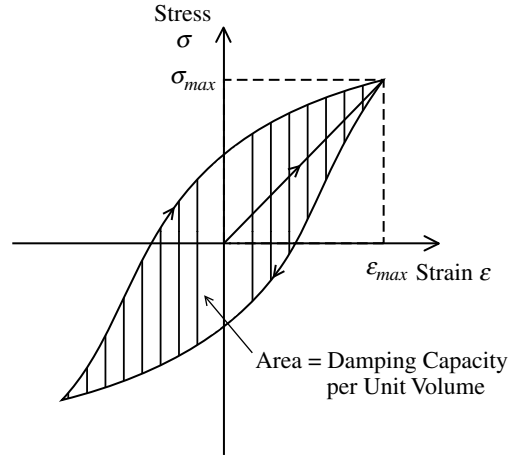


FIGURE 1.1 A typical hysteresis loop for mechanical damping.

$$d = \oint \sigma \, d\epsilon \quad (1.1)$$

In fact, for any damped device, there is a corresponding hysteresis loop in the displacement–force plane as well. In this case, the cyclic integral of force with respect to the displacement, which is the area of the hysteresis loop, is equal to the work done against the damping force. It follows that this integral (loop area) is the energy dissipated per cycle of motion. This is the *damping capacity* which, when divided by the material volume, gives the per-unit-volume damping capacity as before.

It should be clear that, unlike a pure elastic force (e.g., a spring force), a damping force cannot be a function of displacement (q) alone. The reason is straightforward. Consider a force $f(q)$ which depends on q alone. Then, for a particular displacement point, q , of the component the force will be the same regardless of the direction of motion (i.e., the sign of $\dot{q}$). It follows that, in a loading and unloading cycle, the same path will be followed in both directions of motion. Hence, a hysteresis loop will not be formed. In other words, the net work done in a complete cycle of motion will be zero. Next consider a force $f(q, \dot{q})$ which depends on both q and $\dot{q}$. Then, at a given displacement point, q , the force will depend on $\dot{q}$ as well. Hence, force in one direction of motion will be different from that in the opposite direction. As a result, a hysteresis loop will be formed, which corresponds to work done against the damping force (i.e., energy dissipation). We can conclude then that the damping force has to depend on a relative velocity, $\dot{q}$, in some manner. In particular, Coulomb friction, which does not depend on the magnitude of $\dot{q}$, does depend on the sign (direction) of $\dot{q}$.

1.2.1.1 Viscoelastic Damping

For a linear viscoelastic material, the stress–strain relationship is given by a linear differential equation with respect to time, having constant coefficients. A commonly employed relationship is

$$\sigma = E\epsilon + E^* \frac{d\epsilon}{dt} \quad (1.2)$$

which is known as the Kelvin–Voigt model. In Equation 1.2, E is Young's modulus and E^* is a viscoelastic parameter that is assumed to be time independent. The elastic term $E\epsilon$ does not contribute to damping, and, as noted before, mathematically, its cyclic integral vanishes. Consequently, for the Kelvin–Voigt model, damping capacity per unit volume is

$$d_v = E^* \oint \frac{d\epsilon}{dt} d\epsilon \pi \quad (1.3)$$

For a material that is subjected to a harmonic (sinusoidal) excitation, at steady state, we have

$$\varepsilon = \varepsilon_{\max} \cos \omega t \quad (1.4)$$

When Equation 1.4 is substituted in Equation 1.3, we obtain

$$d_v = \pi \omega E^* \varepsilon_{\max}^2 \quad (1.5)$$

Now, $\varepsilon = \varepsilon_{\max}$ when $t = 0$ in Equation 1.4, or when $d\varepsilon/dt = 0$. The corresponding stress, according to Equation 1.2, is $\sigma_{\max} = E\varepsilon_{\max}$. It follows that

$$d_v = \frac{\pi \omega E^* \sigma_{\max}^2}{E^2} \quad (1.6)$$

These expressions for d_v depend on the frequency of excitation, ω .

Apart from the Kelvin–Voigt model, two other models of viscoelastic damping are also commonly used. They are, the Maxwell model given by

$$\sigma + c_s \frac{d\sigma}{dt} = E^* \frac{d\varepsilon}{dt} \quad (1.7)$$

and the standard linear solid model given by

$$\sigma + c_s \frac{d\sigma}{dt} = E\varepsilon + E^* \frac{d\varepsilon}{dt} \quad (1.8)$$

It is clear that the standard linear solid model represents a combination of the Kelvin–Voigt model and the Maxwell model, and is the most accurate of the three. But, for most practical purposes, the Kelvin–Voigt model is adequate.

1.2.1.2 Hysteretic Damping

It was noted above that the stress, and hence the internal damping force, of a viscoelastic damping material depends on the frequency of variation of the strain (and consequently the frequency of motion). For some types of material, it has been observed that the damping force does not significantly depend on the frequency of oscillation of strain (or frequency of harmonic motion). This type of internal damping is known as hysteretic damping.

Damping capacity per unit volume (d_h) for hysteretic damping is also independent of the frequency of motion and can be represented by

$$d_h = J \sigma_{\max}^n \quad (1.9)$$

A simple model that satisfies Equation 1.9, for the case of $n = 2$, is given by

$$\sigma = E\varepsilon + \frac{\tilde{E}}{\omega} \frac{d\varepsilon}{dt} \quad (1.10)$$

which is equivalent to using a viscoelastic parameter, E^* , that depends on the frequency of motion in Equation 1.2 according to $E^* = \tilde{E}/\omega$.

Consider the case of harmonic motion at frequency ω , with the material strain given by

$$\varepsilon = \varepsilon_0 \cos \omega t \quad (1.11)$$

Then, Equation 1.10 becomes

$$\sigma = E\varepsilon_0 \cos \omega t - \tilde{E}\varepsilon_0 \sin \omega t = E\varepsilon \cos \omega t + \tilde{E}\varepsilon_0 \cos\left(\omega t + \frac{\pi}{2}\right) \quad (1.12)$$

Note that the material stress has two components, as given by the right-hand side of Equation 1.12. The first component corresponds to the linear elastic behavior of a material and is in phase with the strain. The second component of stress, which corresponds to hysteretic damping, is 90° out of phase. (This stress component leads the strain by 90° .) A convenient mathematical representation is possible, by using the usual complex form of the response according to

$$\varepsilon = \varepsilon_0 e^{j\omega t} \quad (1.13)$$

Then, Equation 1.10 becomes

$$\sigma = (E + j\tilde{E})\varepsilon \quad (1.14)$$

It follows that this form of simplified hysteretic damping may be represented by using a complex modulus of elasticity, consisting of a real part which corresponds to the usual linear elastic (energy storage) modulus (or Young's modulus) and an imaginary part which corresponds to the hysteretic loss (energy dissipation) modulus.

By combining Equation 1.2 and Equation 1.10, a simple model for combined viscoelastic and hysteretic damping may be given by

$$\sigma = E\varepsilon + \left(E^* + \frac{\tilde{E}}{\omega}\right)\frac{d\varepsilon}{dt} \quad (1.15)$$

The equation of motion for a system whose damping is represented by Equation 1.15 can be deduced from the pure elastic equation of motion by simply substituting E by the operator

$$E + \left(E^* + \frac{\tilde{E}}{\omega}\right)\frac{\partial}{\partial t}$$

in the time domain.

Example 1.1

Determine the equation of flexural motion of a nonuniform slender beam whose material has both viscoelastic and hysteretic damping.

Solution

The Bernoulli–Euler equation of bending motion on an undamped beam subjected to a dynamic load of $f(x, t)$ per unit length, is given by

$$\frac{\partial^2}{\partial x^2}EI\frac{\partial^2 q}{\partial x^2} + \rho A\frac{\partial^2 q}{\partial t^2} = f(x, t) \quad (1.16)$$

Here, q is the transverse motion at a distance, x , along the beam. Then, for a beam with material damping (both viscoelastic and hysteretic) we can write

$$\frac{\partial^2}{\partial x^2}EI\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2}{\partial x^2}\left(E^* + \frac{\tilde{E}}{\omega}\right)I\frac{\partial^3 q}{\partial t\partial x^2} + \rho A\frac{\partial^2 q}{\partial t^2} = f(x, t) \quad (1.17)$$

in which ω is the frequency of the external excitation $f(x, t)$ in the case of steady forced vibrations. In the case of free vibration, however, ω represents the frequency of free vibration decay. Consequently, when analyzing the modal decay of free vibrations, ω in Equation 1.17 should be replaced by the appropriate frequency (ω_i) of modal vibration in each modal equation. Hence, the resulting damped vibratory system possesses the same normal mode shapes as the undamped system. The analysis of the damped case is very similar to that for the undamped system.

1.2.2 Structural Damping

Structural damping is a result of mechanical energy dissipation caused by friction due to the relative motion between components and by impacting or intermittent contact at the joints in a mechanical system or structure. Energy dissipation behavior depends on the details of the particular mechanical system. Consequently, it is extremely difficult to develop a generalized analytical model that would satisfactorily describe structural damping. Energy dissipation caused by rubbing is usually represented by a Coulomb friction model. Energy dissipation caused by impacting, however, should be determined from the coefficient of restitution of the two members that are in contact.

The most common method of estimating structural damping is by measurement. The measured values, however, represent the overall damping in the mechanical system. The structural damping component is obtained by subtracting the values corresponding to other types of damping, such as material damping, present in the system (estimated by environment-controlled experiments, previous data, and so forth) from the overall damping value.

Usually, internal damping is negligible compared to structural damping. A large proportion of mechanical energy dissipation in tall buildings, bridges, vehicle guideways, and many other civil engineering structures and in machinery, such as robots and vehicles, takes place through the structural damping mechanism. A major form of structural damping is the slip damping that results from energy dissipation by interface shear at a structural joint. The degree of slip damping that is directly caused by Coulomb (dry) friction depends on such factors as joint forces (for example, bolt tensions), surface properties and the nature of the materials of the mating surfaces. This is associated with wear, corrosion, and general deterioration of the structural joint. In this sense, slip damping is time-dependent. It is a common practice to place damping layers at joints to reduce undesirable deterioration of the joints. Sliding causes shear distortions in the damping layers, causing energy dissipation by material damping and also through Coulomb friction. In this way, a high level of equivalent structural damping can be maintained without causing excessive joint deterioration. These damping layers should have a high stiffness (as well as a high specific-damping capacity) in order to take the structural loads at the joint.

For structural damping at a joint, the damping force varies as slip occurs at the joint. This is primarily caused by local deformations at the joint, which occur with slipping. A typical hysteresis loop for this case is shown in Figure 1.2(a). The arrows on the hysteresis loop indicate the direction of relative velocity. For idealized Coulomb friction, the frictional force (F) remains constant in each direction of relative motion. An idealized hysteresis loop for structural Coulomb damping is shown in Figure 1.2(b). The corresponding constitutive relation is

$$f = c \operatorname{sgn}(\dot{q}) \quad (1.18)$$

in which f is the damping force, q is the relative displacement at the joint and c is a friction parameter. A simplified model for structural damping caused by local deformation may be given by

$$f = c|q|\operatorname{sgn}(\dot{q}) \quad (1.19)$$

The corresponding hysteresis loop is shown in Figure 1.2(c). Note that the *signum function* is defined by

$$\operatorname{sgn}(v) = \begin{cases} 1 & \text{for } v \geq 0 \\ -1 & \text{for } v < 0 \end{cases} \quad (1.20)$$

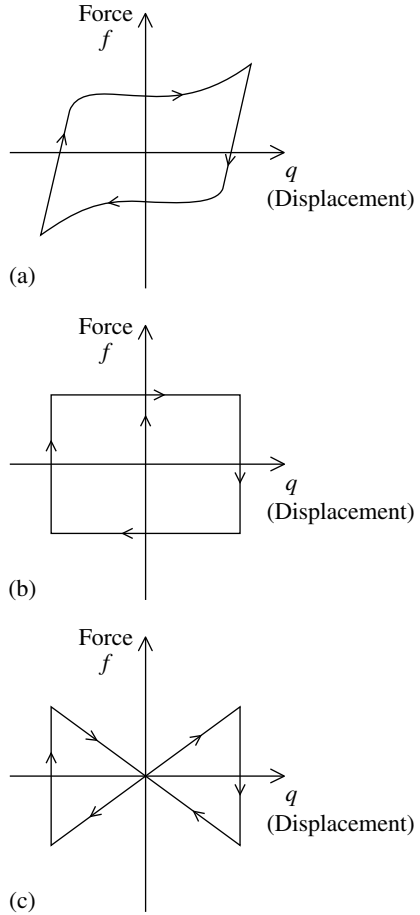


FIGURE 1.2 Some representative hysteresis loops: (a) typical structural damping; (b) Coulomb friction model; and (c) simplified structural damping model.

1.2.3 Fluid Damping

Consider a mechanical component moving in a fluid medium. The direction of relative motion is shown parallel to the y -axis in Figure 1.3. Local displacement of the element relative to the surrounding fluid is denoted by $q(x, y, t)$.

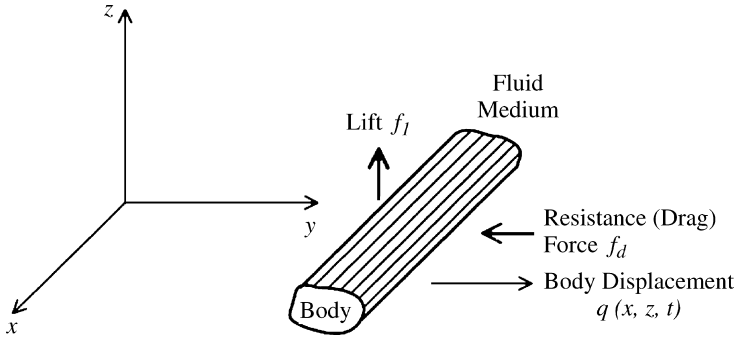


FIGURE 1.3 A body moving in a fluid medium.

The resulting drag force per unit area of projection on the x - z plane is denoted by f_d . This resistance is the cause of mechanical energy dissipation in fluid damping. It is usually expressed as

$$f_d = \frac{1}{2} c_d \rho \dot{q}^2 \text{sgn}(\dot{q}) \quad (1.21)$$

in which $\dot{q} = \partial q(x, z, t) / \partial t$ is the relative velocity. The drag coefficient, c_d , is a function of the Reynold's number and the geometry of the structural cross section. A net damping effect is

generated by viscous drag produced by the boundary layer effects at the fluid–structure interface, and by pressure drag produced by the turbulent effects resulting from flow separation at the wake. The two effects are illustrated in Figure 1.4. Fluid density is ρ . For fluid damping, the damping capacity per unit volume associated with the configuration shown in Figure 1.3 is given by

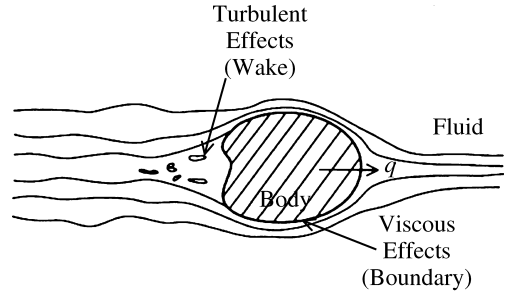


FIGURE 1.4 Mechanics of fluid damping.

$$d_f = \frac{\oint \int_0^{L_x} \int_0^{L_z} f_d \, dz \, dx \, dq(x, z, t)}{L_x L_z q_0} \quad (1.22)$$

in which, L_x and L_z are cross-sectional dimensions of the element in the x and y -directions, respectively, and q_0 is a normalizing amplitude parameter for relative displacement.

Example 1.2

Consider a beam of length L and uniform rectangular cross section that is undergoing transverse vibration in a stationary fluid. Determine an expression for the damping capacity per unit volume for this fluid–structure interaction.

Solution

Suppose that the beam axis is along the x -direction and the transverse motion is in the z -direction. There is no variation in the y -direction, and hence, the length parameters in this direction cancel out.

$$d_f = \frac{\oint \int_0^L f_d \, dx \, dq(x, t)}{L q_0}$$

or

$$d_f = \frac{\int_0^T \int_0^L f_d \dot{q}(x, t) \, dx \, dt}{L q_0} \quad (1.23)$$

in which T is the period of the oscillations. Assuming constant c_d , we substitute Equation 1.21 into Equation 1.23:

$$d_f = \frac{1}{2} \frac{c_d \rho}{L q_0} \int_0^L \int_0^T |\dot{q}|^3 dt \, dx \tag{1.24}$$

For steady-excited harmonic vibration at frequency ω and shape function $Q(x)$ (or for free-modal vibration at natural frequency ω and mode shape $Q(x)$) we have

$$q(x, t) = q_{\max} Q(x) \sin \omega t \tag{1.25}$$

In this case, with the change of variable $\theta = \omega t$, Equation 1.24 becomes

$$d_f = 2 c_d \rho \frac{q_{\max}^3}{L q_0} \int_0^L |Q(x)|^3 dx \, \omega^2 \int_0^{\pi/2} \cos^3 \theta \, d\theta$$

or

$$d_f = \frac{4}{3} c_d \rho q_{\max}^3 \omega^2 \frac{\int_0^L |Q(x)|^3 dx}{L q_0}$$

Note. The integration interval of $t = 0$ to T becomes $\theta = 0$ to 2π or four times that from $\theta = 0$ to $\pi/2$. If the normalizing parameter is defined as

$$q_0 = \frac{1}{L} q_{\max} \int_0^L |Q(x)|^3 \, dx$$

then, we get

$$d_f = \frac{4}{3} c_d \rho q_{\max}^2 \omega^2 \tag{1.26}$$

A useful classification of damping is given in Box 1.1.

Box 1.1		
DAMPING CLASSIFICATION		
Type of Damping	Origin	Typical Constitutive Relation
Internal damping	Material properties	Viscoelastic $\sigma = E \varepsilon + E^* \frac{d\varepsilon}{dt}$
		Hysteretic $\sigma = E \varepsilon + \frac{\check{E}}{\omega} \frac{d\varepsilon}{dt}$
Structural damping	Structural joints and interfaces	Structural deformation $f = c q \operatorname{sgn}(\dot{q})$
		Coulomb $f = c \operatorname{sgn}(\dot{q})$
		General interface $f = \begin{cases} f_s & \text{for } v = 0 \\ f_{sb}(v) \operatorname{sgn}(v) & \text{for } v \neq 0 \end{cases}$
Fluid damping	Fluid–structure interactions	$f_d = \frac{1}{2} c_d \rho \dot{q}^2 \operatorname{sgn}(\dot{q})$

1.3 Representation of Damping in Vibration Analysis

It is not practical to incorporate detailed microscopic representations of damping in the dynamic analysis of systems. Instead, simplified models of damping that are representative of various types of energy dissipation are typically used. Consider a general n -degree-of-freedom mechanical system. Its motion can be represented by the vector $\mathbf{x}$ of n generalized coordinates, x_i , representing the independent motions of the inertia elements. For small displacements, linear spring elements can be assumed. The corresponding equations of motion may be expressed in the vector matrix form

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{d} + \mathbf{K}\mathbf{x} = \mathbf{f}(t) \tag{1.27}$$

in which $\mathbf{M}$ is the mass (inertia) matrix and $\mathbf{K}$ is the stiffness matrix. The forcing-function vector is $\mathbf{f}(t)$. The damping force vector $\mathbf{d}(\mathbf{x}, \dot{\mathbf{x}})$ is generally a nonlinear function of $\mathbf{x}$ and $\dot{\mathbf{x}}$. The type of damping used in the system model may be represented by the nature of $\mathbf{d}$ that is employed in the system equations. The various damping models that may be used, as discussed in the previous section, are listed in Table 1.1. Only the linear viscous damping term given in Table 1.1 is amenable to simplified mathematical analysis. In simplified dynamic models, other types of damping terms are usually replaced by an equivalent viscous damping term. Equivalent viscous damping is chosen so that its energy dissipation per cycle of oscillation is equal to that for the original damping. The resulting equations of motion are expressed by

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}(t) \tag{1.28}$$

In modal analysis of vibratory systems, the most commonly used model is proportional damping, where the damping matrix satisfies

$$\mathbf{C} = c_m\mathbf{M} + c_k\mathbf{K} \tag{1.29}$$

The first term on the right-hand side of Equation 1.29 is known as the inertial damping matrix. The corresponding damping force on each concentrated mass is proportional to its momentum. It represents the energy loss associated with a change in momentum (for example, during an impact). The second term is known as the stiffness damping matrix. The corresponding damping force is proportional to the rate of change of the local deformation forces at joints near the concentrated mass elements. Consequently, it represents a simplified form of linear structural damping. If damping is of the proportional type, it follows that the damped motion can be uncoupled into individual modes. This means that, if the damping model is of the proportional type, the damped system (as well as the undamped system) will possess real modes.

TABLE 1.1 Some Common Damping Models Used in Dynamic System Equations

Damping Type	Simplified Model d_i
Viscous	$\sum_j c_{ij}\dot{x}_j$
Hysteretic	$\sum_j \frac{1}{\omega} c_{ij}\dot{x}_j$
Structural	$\sum_j c_{ij} x_j \operatorname{sgn}(\dot{x}_j)$
Structural Coulomb	$\sum_j c_{ij} \operatorname{sgn}(\dot{x}_j)$
Fluid	$\sum_j c_{ij} \dot{x}_j \dot{x}_j$

1.3.1 Equivalent Viscous Damping

Consider a linear, single-DoF system with viscous damping, subjected to an external excitation. The equation of motion, for a unit mass, is given by

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = \omega_n^2u(t) \quad (1.30)$$

If the excitation force is harmonic, with frequency ω , we have

$$u(t) = u_0 \cos \omega t \quad (1.31)$$

Then, the response of the system at steady state is given by

$$x = x_0 \cos(\omega t + \phi) \quad (1.32)$$

in which the response amplitude is

$$x_0 = u_0 \frac{\omega_n^2}{[(\omega_n^2 - \omega^2) + 4\zeta^2\omega_n^2\omega^2]^{1/2}} \quad (1.33)$$

and the response phase lead is

$$\phi = -\tan^{-1} \frac{2\zeta\omega_n\omega}{(\omega_n^2 - \omega^2)} \quad (1.34)$$

The energy dissipation (i.e., damping capacity), ΔU , per unit mass in one cycle is given by the net work done by the damping force, f_d ; thus,

$$\Delta U = \oint f_d dx = \int_{-\phi/\omega}^{(2\pi-\phi)/\omega} f_d \dot{x} dt \quad (1.35)$$

Since the viscous damping force, normalized with respect to mass (see Equation 1.30), is given by

$$f_d = 2\zeta\omega_n\dot{x} \quad (1.36)$$

the damping capacity, ΔU_v , for viscous damping, can be obtained as

$$\Delta U_v = 2\zeta\omega_n \int_0^{2\pi/\omega} \dot{x}^2 dt \quad (1.37)$$

Finally, using Equation 1.32 in Equation 1.37 we get

$$\Delta U_v = 2\pi x_0^2 \omega_n \omega \zeta \quad (1.38)$$

For any general type of damping (see [Table 1.1](#)), the equation of motion becomes

$$\ddot{x} + \mathbf{d}(x, \dot{x}) + \omega_n^2x = \omega_n^2u(t) \quad (1.39)$$

The energy dissipation in one cycle (Equation 1.35) is given by

$$\Delta U = \int_{-\phi/\omega}^{(2\pi-\phi)/\omega} d(x, \dot{x}) \dot{x} dt \quad (1.40)$$

Various damping force expressions, $d(x, \dot{x})$, normalized with respect to mass, are given in [Table 1.2](#). For fluid damping, for example, the damping capacity is

$$\Delta U_f = \int_{-\phi/\omega}^{(2\pi-\phi)/\omega} c|\dot{x}| \dot{x}^2 dt \quad (1.41)$$

By substituting Equation 1.32 in Equation 1.41 for steady, harmonic motion we obtain

$$\Delta U_f = \frac{8}{3} c x_0^3 \omega^2 \quad (1.42)$$

TABLE 1.2 Equivalent Damping Ratio Expressions for Some Common Types of Damping

Damping Type	Damping Force, $d(x, \dot{x})$, per Unit Mass	Equivalent Damping Ratio, ζ_{eq}
Viscous	$2\zeta\omega_n\dot{x}$	ζ
Hysteretic	$\frac{c}{\omega}\dot{x}$	$\frac{c}{2\omega_n\omega}$
Structural	$c x \text{sgn}(\dot{x})$	$\frac{c}{\pi\omega_n\omega}$
Structural Coulomb	$c\text{sgn}(\dot{x})$	$\frac{2c}{\pi x_0\omega_n\omega}$
Fluid	$c \dot{x} \dot{x}$	$\frac{4}{3\pi}\left(\frac{\omega}{\omega_n}\right)x_0c$

By comparing Equation 1.42 with Equation 1.38, the equivalent damping ratio for fluid damping is obtained as

$$\zeta_f = \frac{4}{3\pi}\left(\frac{\omega}{\omega_n}\right)x_0c \tag{1.43}$$

in which x_0 is the amplitude of steady-state vibrations, as given by Equation 1.33. For the other types of damping listed in Table 1.1, expressions for the equivalent damping ratio can be obtained in a similar manner. The corresponding equivalent damping ratio expressions are given in Table 1.2. It should be noted that, for nonviscous damping types, ζ is generally a function of the frequency of oscillation, ω , and the amplitude of excitation, u_0 . It should be noted that the expressions given in Table 1.2 are derived assuming harmonic excitation. Engineering judgment should be exercised when employing these expressions for nonharmonic excitations.

For multi-DoF systems that incorporate proportional damping, the equations of motion can be transformed into a set of one-DoF equations (modal equations) of the type given in Equation 1.30. In this case, the damping ratio and natural frequency correspond to the respective modal values and, in particular, $\omega = \omega_n$.

1.3.2 Complex Stiffness

Consider a linear spring of stiffness k connected in parallel with a linear viscous damper of damping constant c , as shown in Figure 1.5(a). Suppose that a force, f , is applied to the system, moving it through distance x from the relaxed position of the spring. Then we have

$$f = kx + c\dot{x} \tag{1.44}$$

Suppose that the motion is harmonic, as given by

$$x = x_0 \cos \omega t \tag{1.45}$$

It is clear that the spring force, kx , is in phase with the displacement, but the damping force, $c\dot{x}$, has a 90° phase lead with respect to the displacement. This is because the velocity, $\dot{x} = -x_0\omega \sin \omega t = x_0\omega \cos(\omega t + \pi/2)$, has a 90° phase lead with respect to x . Specifically, we have

$$f = kx_0 \cos \omega t + cx_0\omega \cos\left(\omega t + \frac{\pi}{2}\right) \tag{1.46}$$

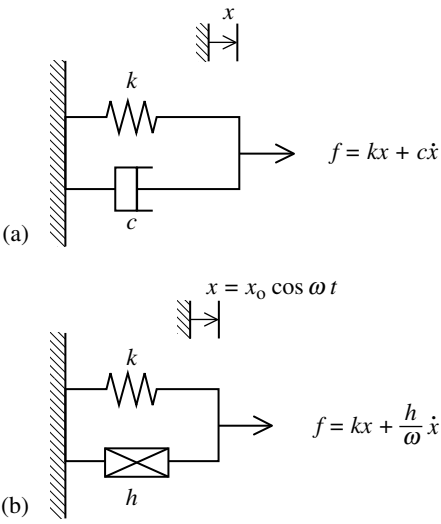


FIGURE 1.5 Spring element in parallel with (a) a viscous damper and (b) a hysteretic damper.

This same fact may be represented by using complex numbers, where the in-phase component is considered as the real part and the 90° phase lead component is considered as the imaginary part with each component oscillating at the same frequency ω . Then, we can write Equation 1.46 in the equivalent form

$$f = kx + j\omega cx \quad (1.47)$$

This is exactly what we get by starting with the complex representation of the displacement

$$x = x_0 e^{j\omega t} \quad (1.48)$$

and substituting it in Equation 1.44. We note that Equation 1.47 may be written as

$$f = k^* x \quad (1.49)$$

where k^* is a “complex” stiffness, given by

$$k^* = k + j\omega c \quad (1.50)$$

Clearly, the system itself and its two components (spring and damper) are real. Their individual forces are also real. The complex stiffness is simply a mathematical representation of the two force components (spring force and damping force), which are 90° out of phase, when subjected to harmonic motion. It follows that the linear damper may be “mathematically” represented by an “imaginary” stiffness. In the case of viscous damping this imaginary stiffness (and hence, the damping force magnitude) increases linearly with the frequency, ω , of the harmonic motion. The concept of complex stiffness when dealing with discrete dampers is analogous to the use of complex elastic modulus in material damping, as discussed earlier in this chapter.

We have noted that, for hysteretic damping, the damping force (or damping stress) is independent of the frequency in harmonic motion. It follows that a hysteretic damper may be represented by an equivalent damping constant of

$$c = \frac{h}{\omega} \quad (1.51)$$

which is valid for a harmonic motion (e.g., modal motion or forced motion) of frequency ω . This situation is shown in Figure 1.5(b). It can be seen that the corresponding complex stiffness is

$$k^* = k + jh \quad (1.52)$$

Example 1.3

A flexible system consists of a mass, m , attached to the hysteretic damper and spring combination shown in Figure 1.5(b). What is the frequency response function of the system relating an excitation force, f , applied to the mass and the resulting displacement response, x ? Obtain the resonant frequency of the system. Compare the results with the case for viscous damping.

Solution

For a harmonic motion of frequency ω , the equation of motion of the system is

$$m\ddot{x} + \frac{h}{\omega}\dot{x} + kx = f \quad (1.53)$$

With a forcing excitation of $f = f_0 e^{j\omega t}$ and the resulting steady-state response, $x = x_0 e^{j\omega t}$, where x_0 has a phase difference (i.e., it is a complex function) with respect to f_0 . Then, in the frequency domain, substituting the harmonic response $x = x_0 e^{j\omega t}$ into Equation 1.53 we get

$$\left[-\omega^2 m + \frac{h}{\omega} j\omega + k \right] x = f$$

resulting in the frequency transfer function

$$\frac{x}{f} = \frac{1}{[k - \omega^2 m + jh]} \quad (1.54)$$

Note that, as usual, this result is obtained simply by substituting $j\omega$ for d/dt . The magnitude of transfer function is at a maximum at resonance. This corresponds to a minimum value of

$$p(\omega) = (k - \omega^2 m)^2 + h^2$$

If we set $dp/d\omega = 0$, we get,

$$2(k - \omega^2 m)(-2\omega) = 0$$

Hence, the resonant frequency corresponds to the root of

$$k - \omega^2 m = 0$$

This gives the resonant frequency

$$\omega_r = \sqrt{\frac{k}{m}} \quad (1.55)$$

Note that, in the case of hysteretic damping, the resonant frequency is equal to the undamped natural frequency, ω_n , and, unlike in the case of viscous damping, does not depend on the level of damping itself. For convenience consider the system response as the spring force

$$f_s = kx \quad (1.56)$$

Then, a normalized transfer function is obtained, as given by

$$\frac{f_s}{f} = G(j\omega) = \frac{1}{\left[1 - \omega^2 \frac{m}{k} + j \frac{h}{k}\right]} \quad (1.57)$$

or,

$$\frac{f_s}{f} = \frac{1}{[1 - r^2 + j\alpha]} \quad (1.58)$$

where

$$r = \frac{\omega}{\omega_n} \quad \text{and} \quad \alpha = \frac{h}{k} \quad (1.59)$$

which are the normalized frequency and the normalized hysteretic damping, respectively. The magnitude of the transfer function is

$$\left| \frac{f_s}{f} \right| = \frac{1}{\sqrt{(1 - r^2)^2 + \alpha^2}} \quad (1.60)$$

and the phase angle (phase lead) is

$$\angle f_s/f = -\tan^{-1} \frac{\alpha}{(1 - r^2)} \quad (1.61)$$

These results are sketched in [Figure 1.6](#).

1.3.3 Loss Factor

We define the *damping capacity* of a device (damper) as the energy dissipated in a complete cycle of motion; specifically

$$\Delta U = \oint f_d dx \quad (1.62)$$

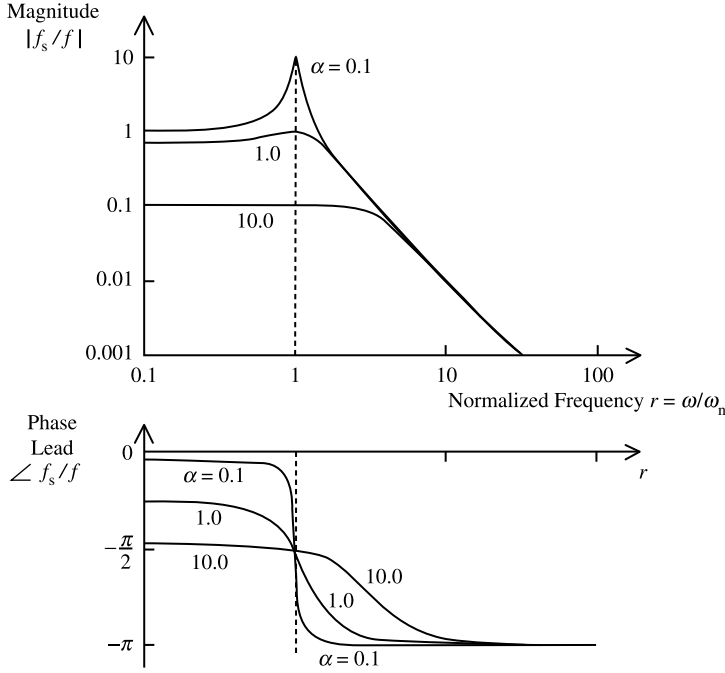


FIGURE 1.6 Frequency transfer function of a simple oscillator with hysteretic damping.

This is given by the area of the hysteresis loop in the displacement force plane. If the initial (total) energy of the system is denoted by $U_{\max}$, then the *specific damping capacity*, D , is given by the ratio

$$D = \frac{\Delta U}{U_{\max}} \quad (1.63)$$

The *loss factor*, η , is the specific damping capacity per radian of the damping cycle. Hence,

$$\eta = \frac{\Delta U}{2\pi U_{\max}} \quad (1.64)$$

Note that $U_{\max}$ is approximately equal to the maximum kinetic energy and also to the maximum potential energy of the device when the damping is low.

Equation 1.38 gives the damping capacity per unit mass of a device with viscous damping as

$$\Delta U = 2\pi x_0^2 \omega_n \omega \zeta \quad (1.65)$$

Here, x_0 is the amplitude and ω is the frequency of harmonic motion of the device, ω_n is the undamped natural frequency and ζ is the damping ratio. The maximum potential energy per unit mass of the system is

$$U_{\max} = \frac{1}{2} \frac{k}{m} x_0^2 = \frac{1}{2} \omega_n^2 x_0^2 \quad (1.66)$$

Hence, from Equation 1.64, the loss factor for a viscous damped simple oscillator is given by

$$\eta = \frac{2\pi x_0^2 \omega_n \omega \zeta}{2\pi \times \frac{1}{2} \omega_n^2 x_0^2} = \frac{2\omega \zeta}{\omega_n} \quad (1.67)$$

For free decay of the system, we have $\omega = \omega_d \cong \omega_n$, where the latter approximation holds for low damping. For forced oscillation, the worst response conditions occur when $\omega = \omega_d \cong \omega_n$, which is what one must consider with regard to energy dissipation. In either case, the loss factor is approximately

given by

$$\eta = 2\zeta$$

(1.68)

For other types of damping, Equation 1.68 will still hold when the equivalent damping ratio, ζ_{eq} , (see Table 1.2) is used in place of ζ .

The loss factors of some common materials are given in Table 1.3. Definitions of useful damping parameters, as defined here, are summarized in Table 1.4. Expressions of loss factors for some useful damping models are given in Table 1.5.

TABLE 1.3 Loss Factors of Some Useful Materials

Material	Loss Factor $\eta \cong 2\zeta$
Aluminum	2×10^{-5} to 2×10^{-3}
Concrete	0.02 to 0.06
Glass	0.001 to 0.002
Rubber	0.1 to 1.0
Steel	0.002 to 0.01
Wood	0.005 to 0.01

TABLE 1.4 Definitions of Damping Parameters

Parameter	Definition	Mathematical Formula
Damping capacity (ΔU)	Energy dissipated per cycle of motion (area of displacement–force hysteresis loop)	$\oint f_d \, dx$
Damping capacity per volume (d)	Energy dissipated per cycle per unit material volume (area of strain–stress hysteresis loop)	$\oint \sigma \, d\epsilon$
Specific damping capacity (D)	Ratio of energy dissipated per cycle (ΔU) to the initial maximum energy (U_{max}) <i>Note:</i> for low damping, U_{max} = maximum potential energy = maximum kinetic energy	$\frac{\Delta U}{U_{max}}$
Loss factor (η)	Specific damping capacity per unit angle of cycle. <i>Note:</i> for low damping, $\eta = 2 \times$ damping ratio	$\frac{\Delta U}{2\pi U_{max}}$

TABLE 1.5 Loss Factors for Several Material Damping Models

Material Damping Model	Stress–Strain Constitute Relation	Loss Factor (η)
Viscoelastic Kelvin–Voigt	$\sigma = E\epsilon + E^* \frac{d\epsilon}{dt}$	$\frac{\omega E^*}{E}$
Hysteretic Kelvin–Voigt	$\sigma = E\epsilon + \frac{\tilde{E}}{\omega} \frac{d\epsilon}{dt}$	$\frac{\tilde{E}}{E}$
Viscoelastic standard linear solid	$\sigma + c_s \frac{d\sigma}{dt} = E\epsilon + E^* \frac{d\epsilon}{dt}$	$\frac{\omega E^*}{E} \frac{(1 - c_s E/E^*)}{(1 + \omega^2 c_s)}$
Hysteretic standard linear solid	$\sigma + c_s \frac{d\sigma}{dt} = E\epsilon + \frac{\tilde{E}}{\omega} \frac{d\epsilon}{dt}$	$\frac{\tilde{E}}{E} \frac{(1 - \omega c_s E/\tilde{E})}{(1 + \omega^2 c_s)}$

1.4 Measurement of Damping

Damping may be represented by various parameters (such as specific damping capacity, loss factor, Q -factor, and damping ratio) and models (such as viscous, hysteretic, structural, and fluid). Before attempting to measure damping in a system, we need to decide on a representation (model) that will adequately characterize the nature of mechanical energy dissipation in the system. Next, we should decide on the parameter or parameters of the model that need to be measured.

It is extremely difficult to develop a realistic yet tractable model for damping in a complex piece of equipment operating under various conditions of mechanical interaction. Even if a satisfactory damping model is developed, experimental determination of its parameters could be tedious. A major difficulty arises because it usually is not possible to isolate various types of damping (for example, material, structural, and fluid) from an overall measurement. Furthermore, damping measurements must be conducted under actual operating conditions for them to be realistic.

If one type of damping (say, fluid damping) is eliminated during the actual measurement then it would not represent the true operating conditions. This would also eliminate possible interacting effects of the eliminated damping type with the other types. In particular, overall damping in a system is not generally equal to the sum of the individual damping values when they are acting independently. Another limitation of computing equivalent damping values using experimental data arises because it is assumed for analytical simplicity that the dynamic system behavior is linear. If the system is highly nonlinear, a significant error could be introduced into the damping estimate. Nevertheless, it is customary to assume linear viscous behavior when estimating damping parameters using experimental data.

There are two general ways by which damping measurements can be made: using a time–response record and using a frequency–response function of the system to estimate damping.

1.4.1 Logarithmic Decrement Method

This is perhaps the most popular time–response method that is used to measure damping. When a single-DoF oscillatory system with viscous damping (see Equation 1.30) is excited by an impulse input (or an initial condition excitation), its response takes the form of a time decay (see Figure 1.7), given by

$$y(t) = y_0 \exp(-\zeta \omega_n t) \sin \omega_d t \quad (1.69)$$

in which the damped natural frequency is given by

$$\omega_d = \sqrt{1 - \zeta^2} \omega_n \quad (1.70)$$

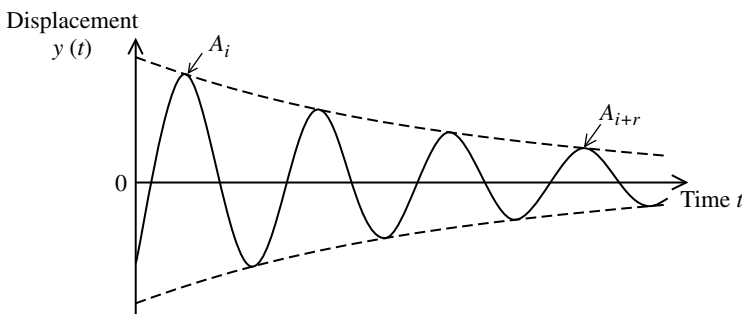


FIGURE 1.7 Impulse response of a simple oscillator.

If the response at $t = t_i$ is denoted by y_i , and the response at $t = t_i + 2\pi r/\omega_d$ is denoted by y_{i+r} , then, from Equation 1.69, we have

$$\frac{y_{i+r}}{y_i} = \exp\left(-\zeta \frac{\omega_n}{\omega_d} 2\pi r\right), \quad i = 1, 2, \dots, n$$

In particular, suppose that y_i corresponds to a peak point in the time decay function, having magnitude A_i , and that y_{i+r} corresponds to the peak point r cycles later in the time history, and its magnitude is denoted by A_{i+r} (see Figure 1.7). Even though the above equation holds for any pair of points that are r periods apart in the time history, the peak points seem to be the appropriate choice for measurement in the present procedure, as these values would be more prominent than any arbitrary points in a response–time history. Then,

$$\frac{A_{i+r}}{A_i} = \exp\left(-\zeta \frac{\omega_n}{\omega_d} 2\pi r\right) = \exp\left[-\frac{\zeta}{\sqrt{1-\zeta^2}} 2\pi r\right]$$

where Equation 1.70 has been used. Then, the logarithmic decrement δ is given by (per unit cycle)

$$\delta = \frac{1}{r} \ln\left(\frac{A_i}{A_{i+r}}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \quad (1.71)$$

or the damping ratio may be expressed as

$$\zeta = \frac{1}{\sqrt{1 + (2\pi/\delta)^2}} \quad (1.72)$$

For low damping (typically, $\zeta < 0.1$), $\omega_d \cong \omega_n$ and Equation 1.71 become

$$\frac{A_{i+r}}{A_i} \cong \exp(-\zeta 2\pi r) \quad (1.73)$$

or

$$\zeta = \frac{1}{2\pi r} \ln\left(\frac{A_i}{A_{i+r}}\right) = \frac{\delta}{2\pi} \quad \text{for } \zeta < 0.1 \quad (1.74)$$

This is in fact the “per-radian” logarithmic decrement.

The damping ratio can be estimated from a free-decay record, using Equation 1.74. Specifically, the ratio of the extreme amplitudes in prominent r cycles of decay is determined and substituted into Equation 1.74 to get the equivalent damping ratio.

Alternatively, if n cycles of damped oscillation are needed for the amplitude to decay by a factor of two, for example, then, from Equation 1.74, we get

$$\zeta = \frac{1}{2\pi n} \ln(2) = \frac{0.11}{n} \quad \text{for } \zeta < 0.1 \quad (1.75)$$

For slow decays (low damping), we have

$$\ln\left(\frac{A_i}{A_{i+1}}\right) \cong \frac{2(A_i - A_{i+1})}{(A_i + A_{i+1})} \quad (1.76)$$

Then, from Equation 1.74, we get

$$\zeta = \frac{A_i - A_{i+1}}{\pi(A_i + A_{i+1})} \quad \text{for } \zeta < 0.1 \quad (1.77)$$

Any one of Equation 1.72, Equation 1.74, Equation 1.75, and Equation 1.77 could be employed in computing ζ from test data. It should be noted that the results assume single-DoF system behavior. For multi-DoF systems, the modal damping ratio for each mode can be determined using this method if the initial excitation is such that the decay takes place primarily in one mode of vibration.

In other words, substantial modal separation and the presence of “real” modes (not “complex” modes with nonproportional damping) are assumed.

1.4.2 Step–Response Method

This is also a time–response method. If a unit-step excitation is applied to the single-DoF oscillatory system given by Equation 1.30, its time–response is given by

$$y(t) = 1 - \frac{1}{\sqrt{1 - \zeta^2}} \exp(-\zeta \omega_n t) \sin(\omega_d t + \phi) \quad (1.78)$$

in which $\phi = \cos \zeta$. A typical step–response curve is shown in Figure 1.8. The time at the first peak (peak time), T_p , is given by

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\sqrt{1 - \zeta^2} \omega_n} \quad (1.79)$$

The response at peak time (peak value), M_p , is given by

$$M_p = 1 + \exp(-\zeta \omega_n T_p) = 1 + \exp\left(\frac{-\pi \zeta}{\sqrt{1 - \zeta^2}}\right) \quad (1.80)$$

The percentage overshoot, PO, is given by

$$PO = (M_p - 1) \times 100\% = 100 \exp\left(\frac{-\pi \zeta}{\sqrt{1 - \zeta^2}}\right) \quad (1.81)$$

It follows that, if any one parameter of T_p , M_p or PO is known from a step–response record, the corresponding damping ratio, ζ , can be computed by using the appropriate relationship from the following:

$$\zeta = \sqrt{1 - \left(\frac{\pi}{T_p \omega_n}\right)^2} \quad (1.82)$$

$$\zeta = \frac{1}{\sqrt{1 + \frac{1}{\left[\frac{\ln(M_p - 1)}{\pi}\right]^2}}} \quad (1.83)$$

$$\zeta = \frac{1}{\sqrt{1 + \frac{1}{\left[\frac{\ln(PO/100)}{\pi}\right]^2}}} \quad (1.84)$$

It should be noted that when determining M_p the response curve should be normalized to unit steady-state value. Furthermore, the results are valid only for single-DoF systems and modal excitations in multi-DoF systems.

1.4.3 Hysteresis Loop Method

For a damped system, the force versus displacement cycle produces a hysteresis loop. Depending on the inertial and elastic characteristics and other conservative loading conditions (e.g., gravity) in the system,

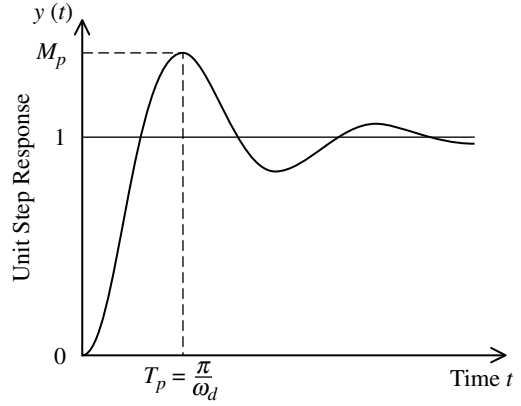


FIGURE 1.8 A typical step–response of a simple oscillator.

the shape of the hysteresis loop will change. But the work done by conservative forces (e.g., inertial, elastic, and gravitational) in a complete cycle of motion will be zero. Consequently, the net work done will be equal to the energy dissipated due to damping only. Accordingly, the area of the displacement–force hysteresis loop will give the damping capacity, ΔU (see Equation 1.62). The maximum energy in the system can also be determined from the displacement–force curve. Then, the loss factor, η , can be computed using Equation 1.64, and the damping ratio from Equation 1.68. This method of damping measurement may also be considered basically as a time domain method.

Note that Equation 1.65 is the work done against (i.e., energy dissipation in) a single loading–unloading cycle per unit mass. It should be recalled that $2\zeta\omega_n = c/m$, where c = viscous damping constant and m = mass. Accordingly, from Equation 1.65, the energy dissipation per unit mass and per hysteresis loop is $\Delta U = \pi x_0^2 \omega c/m$. Hence, without normalizing with respect to mass, the energy dissipation per hysteresis loop of viscous damping is

$$\Delta U_v = \pi x_0^2 \omega c \quad (1.85)$$

Equation 1.85 can be derived by performing the cyclic integration indicated in Equation 1.62 with the damping force $f_d = c\dot{x}$, harmonic motion $x = x_0 e^{j\omega t}$ and the integration interval $t = 0$ to $2\pi/\omega$.

Similarly, in view of Equation 1.51, the energy dissipation per hysteresis loop of hysteretic damping is

$$\Delta U_h = \pi x_0^2 h \quad (1.86)$$

Now, since the initial maximum energy may be represented by the initial maximum potential energy, we have

$$U_{\max} = \frac{1}{2} k x_0^2 \quad (1.87)$$

Note that the stiffness, k , may be measured as the average slope of the displacement–force hysteresis loop. Hence, in view of Equation 1.64, the loss factor for hysteretic damping is given by

$$\eta = \frac{h}{k} \quad (1.88)$$

Then, from Equation 1.68, the equivalent damping ratio for hysteretic damping is

$$\zeta = \frac{h}{2k} \quad (1.89)$$

Example 1.4

A damping material was tested by applying a loading cycle of -900 to 900 N and back to -900 N to a thin bar made of the material and measuring the corresponding deflection. The smoothed load vs. deflection curve obtained in this experiment is shown in [Figure 1.9](#). Assuming that the damping is predominantly of the hysteretic type, estimate

1. The hysteretic damping constant
2. The equivalent damping ratio

Solution

Approximating the top and the bottom segments of the hysteresis loop by triangles, we estimate the area of the loop as

$$\Delta U_h = 2 \times \frac{1}{2} \times 2.5 \times 900 \text{ N}\cdot\text{mm}$$

Alternatively, we may obtain this result by counting the squares within the hysteresis loop. The deflection amplitude is

$$x_0 = 8.5 \text{ mm}$$

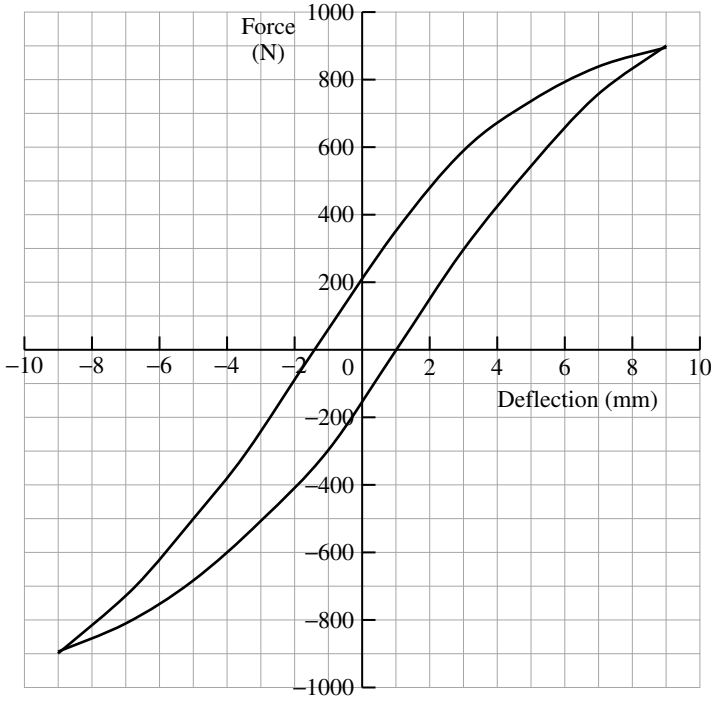


FIGURE 1.9 An experimental hysteresis loop of a damping material.

Hence, from Equation 1.86 we have

$$h = \frac{2 \times \frac{1}{2} \times 2.5 \times 900}{\pi \times 8.5^2} \text{ N/mm} = 9.9 \text{ N/mm}$$

The stiffness of the damping element is estimated as the average slope of the hysteresis loop; thus

$$k = \frac{600}{4.5} \text{ N/mm} = 133.3 \text{ N/mm}$$

Hence, from Equation 1.89, the equivalent damping ratio is

$$\zeta = \frac{9.9}{2 \times 133.3} \approx 0.04$$

1.4.4 Magnification Factor Method

This is a frequency–response method. Consider a single-DoF oscillatory system with viscous damping. The magnitude of its frequency–response function is

$$|H(\omega)| = \frac{\omega_n^2}{[(\omega_n^2 - \omega^2)^2 + 4\zeta^2 \omega_n^2 \omega^2]^{1/2}} \quad (1.90)$$

A plot of this expression with respect to ω , the frequency of excitation, is given in [Figure 1.10](#). The peak value of magnitude occurs when the denominator of the expression is at its minimum. This corresponds to

$$\frac{d}{d\omega} [(\omega_n^2 - \omega^2)^2 + 4\zeta^2 \omega_n^2 \omega^2] = 0 \quad (1.91)$$

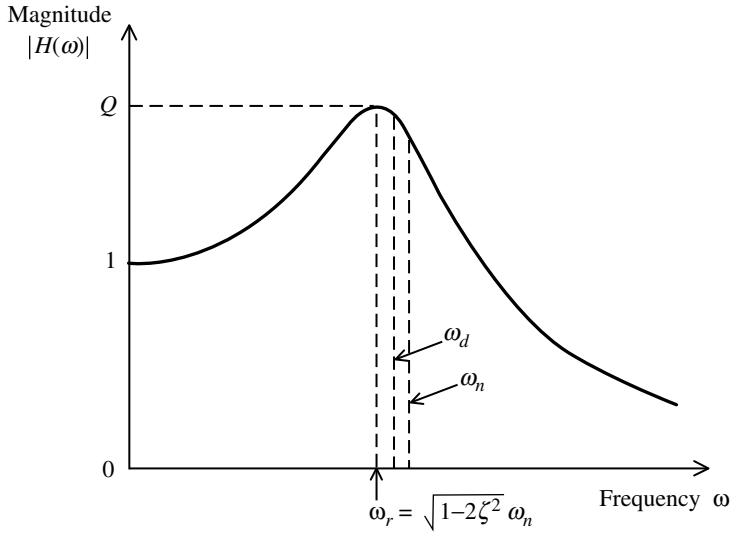


FIGURE 1.10 The magnification factor method of damping measurement applied to a single-DoF system.

The resulting solution for ω is termed the resonant frequency, ω_r

$$\omega_r = \sqrt{1 - 2\zeta^2} \omega_n \quad (1.92)$$

It is noted that $\omega_r < \omega_d$ (see Equation 1.70), but for low damping ($\zeta < 0.1$), the values of ω_n , ω_d , and ω_r are nearly equal. The amplification factor, Q , which is the magnitude of the frequency–response function at resonant frequency, is obtained by substituting Equation 1.92 in Equation 1.90:

$$Q = \frac{1}{2\zeta\sqrt{1 - \zeta^2}} \quad (1.93)$$

For low damping ($\zeta < 0.1$), we have

$$Q = \frac{1}{2\zeta} \quad (1.94)$$

In fact, Equation 1.94 corresponds to the magnitude of the frequency–response function at $\omega = \omega_n$.

It follows that, if the magnitude curve of the frequency–response function (or a Bode plot) is available, then the system damping ratio, ζ , can be estimated using Equation 1.94. When using this method, the frequency–response curve must be normalized so that its magnitude at zero frequency (termed *static gain*) is unity.

For a multi-DoF system modal damping values may be estimated from the magnitude of the Bode plot of its frequency–response function, provided that the modal frequencies are not too closely spaced and the system is lightly damped. Consider the logarithmic (to the base ten) magnitude plot shown in Figure 1.11. The magnitude is expressed in decibels (dB), which is calculated by multiplying the $\log_{10}(\text{magnitude})$ by a factor of 20. At the i th resonant frequency, ω_i , the amplification factor, q_i (in dB), is obtained by drawing an asymptote to the preceding segment of the curve and measuring the peak value from the asymptote. Then,

$$Q_i = (10)^{q_i/20} \quad (1.95)$$

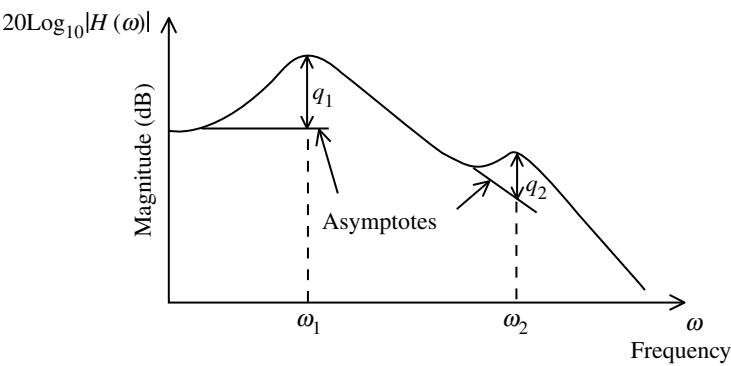


FIGURE 1.11 Magnification factor method applied to a multi-DoF system.

and the modal damping ratio

$$\zeta = \frac{1}{2Q_i}, \quad i = 1, 2, \dots, n \tag{1.96}$$

If the significant resonances are closely spaced, curve-fitting to a suitable function may be necessary in order to determine the corresponding modal damping values. The Nyquist plot may also be used in computing damping using frequency domain data.

1.4.5 Bandwidth Method

The bandwidth method of damping measurement is also based on frequency–response. Consider the frequency–response function magnitude given by Equation 1.90 for a single-DoF, oscillatory system with viscous damping. The peak magnitude is given by Equation 1.94 for low damping. Bandwidth (half-power) is defined as the width of the frequency–response magnitude curve when the magnitude is $(1/\sqrt{2})$ times the peak value. This is denoted by $\Delta\omega$ (see Figure 1.12). An expression

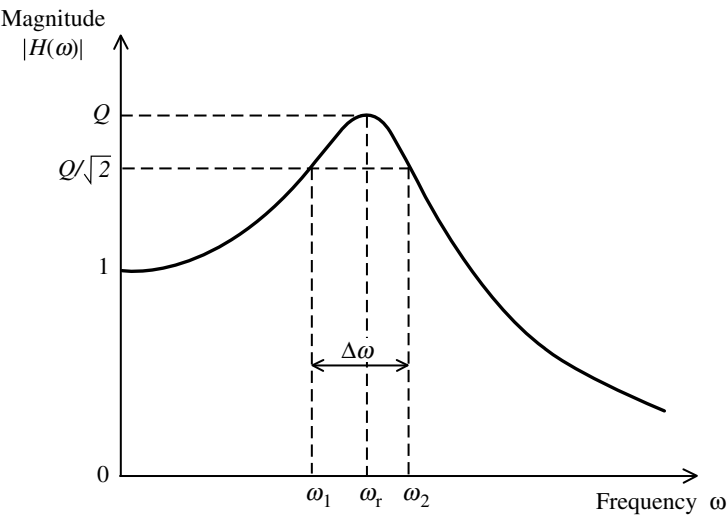


FIGURE 1.12 Bandwidth method of damping measurement in a single-DoF system.

for $\Delta\omega = \omega_2 - \omega_1$ is obtained below using Equation 1.90. By definition, ω_1 and ω_2 are the roots of the equation

$$\frac{\omega_n^2}{\left[(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2\right]^{1/2}} = \frac{1}{\sqrt{2} \times 2\zeta} \quad (1.97)$$

for ω . Equation 1.97 can be expressed in the form

$$\omega^4 - 2(1 - 2\zeta^2)\omega_n^2\omega^2 + (1 - 8\zeta^2)\omega_n^4 = 0 \quad (1.98)$$

This is a quadratic equation in ω^2 , having roots ω_1^2 and ω_2^2 , which satisfy

$$(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2) = \omega^4 - (\omega_1^2 + \omega_2^2)\omega^2 + \omega_1^2\omega_2^2 = 0$$

Consequently,

$$\omega_1^2 + \omega_2^2 = 2(1 - 2\zeta^2)\omega_n^2 \quad (1.99)$$

and

$$\omega_1^2\omega_2^2 = (1 - 8\zeta^2)\omega_n^4 \quad (1.100)$$

It follows that

$$(\omega_2 - \omega_1)^2 = \omega_1^2 + \omega_2^2 - 2\omega_1\omega_2 = 2(1 - 2\zeta^2)\omega_n^2 - 2\sqrt{1 - 8\zeta^2}\omega_n^2$$

For small ζ (in comparison to 1), we have

$$\sqrt{1 - 8\zeta^2} \cong 1 - 4\zeta^2$$

Hence,

$$(\omega_2 - \omega_1)^2 \cong 4\zeta^2\omega_n^2$$

or, for low damping

$$\Delta\omega = 2\zeta\omega_n = 2\zeta\omega_r \quad (1.101)$$

From Equation 1.101 it follows that the damping ratio can be estimated from the bandwidth using the relation

$$\zeta = \frac{1}{2} \frac{\Delta\omega}{\omega_r} \quad (1.102)$$

For a multi-DoF system with widely spaced resonances, the foregoing method can be extended to estimate modal damping. Consider the frequency–response magnitude plot (in dB) shown in [Figure 1.13](#).

Since a factor of $\sqrt{2}$ corresponds to 3 dB, the bandwidth corresponding to a resonance is given by the width of the magnitude plot at 3 dB below that resonant peak. For the i th mode, the damping ratio is given by

$$\zeta_i = \frac{1}{2} \frac{\Delta\omega_i}{\omega_i} \quad (1.103)$$

The bandwidth method of damping measurement indicates that the bandwidth at a resonance is a measure of the energy dissipation in the system in the neighborhood of that resonance. The simplified

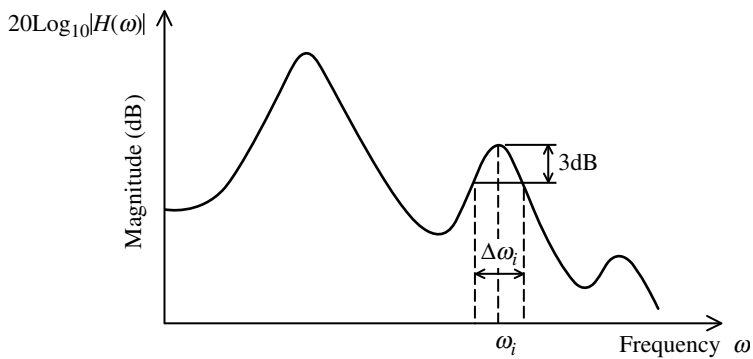


FIGURE 1.13 Bandwidth method of damping measurement in a multi-DoF system.

relationship given by Equation 1.103 is valid for low damping, however, and is based on linear system analysis. Several methods of damping measurement are summarized in [Box 1.2](#).

1.4.6 General Remarks

There are limitations to the use of damping values that are experimentally determined. For example, consider time–response methods for determining the modal damping of a device for higher modes. The customary procedure is to first excite the system at the desired resonant frequency, using a harmonic exciter, and then to release the excitation mechanism. In the resulting transient vibration, however, there invariably will be modal interactions, except in the case of proportional damping. In this type of test, it is tacitly assumed that the device can be excited in the particular mode. In essence, proportional damping is assumed in modal damping measurements. This introduces a certain amount of error into the measured damping values.

Expressions used in computing damping parameters from test measurements are usually based on linear system theory. However, all practical devices exhibit some nonlinear behavior. If the degree of nonlinearity is high, the measured damping values will not be representative of the actual behavior of the system. Furthermore, testing to determine damping is usually performed at low amplitudes of vibration. The corresponding responses could be an order of magnitude lower than, for instance, the amplitudes exhibited under extreme operating conditions. Damping in practical devices increases with the amplitude of motion, except for relatively low amplitudes (see [Figure 1.14](#) illustrating nonlinear behavior). Consequently, the damping values determined from experiments should be extrapolated when they are used to study the behavior of the system under various operating conditions. Alternatively, damping could be associated with a stress level in the device. Different components in a device are subjected to varying levels of stress, however, and it might be difficult to obtain a representative stress value for the entire device. One of the methods recommended for estimating damping in structures under seismic disturbances, for example, is by analyzing earthquake response records for structures that are similar to the one being considered. Some typical damping ratios that are applicable under operating basis earthquake (OBE) and safe-shutdown earthquake (SSE) conditions for a range of items are given in [Table 1.6](#).

When damping values are estimated using frequency–response magnitude curves, accuracy becomes poor at very low damping ratios (< 1%). The main reason for this is the difficulty in obtaining a sufficient number of points in the magnitude curve near a poorly damped resonance when the frequency–response function is determined experimentally. As a result, the magnitude curve is poorly defined in the neighborhood of a weakly damped resonance. For low damping (< 2%), time–response methods are particularly useful. At high damping values, the rate of decay can be so fast that the measurements contain large errors. Modal interference in closely spaced modes can also affect measured damping results.

Box 1.2

DAMPING MEASUREMENT METHODS

Method	Measurements	Formulas
Logarithmic decrement method	A_i = first significant amplitude; A_{i+r} = amplitude after r cycles	Logarithmic decrement $\delta = \frac{1}{r} \ln \frac{A_i}{A_{i+r}} \text{ (per cycle)}$ $\frac{\delta}{2\pi} = \frac{\zeta}{\sqrt{1 - \zeta^2}} \text{ (per radian)}$ or, $\zeta = \frac{1}{\sqrt{1 + (2\pi/\delta)^2}}$ For low damping $\zeta = \frac{\delta}{2\pi}$ $\zeta = \frac{A_i - A_{i+1}}{\pi(A_i + A_{i+1})}$
Step response method	M_p = first peak value normalized r.t. steady-state value; PO = percentage overshoot (over steady-state value)	$M_p = 1 + \exp\left[\frac{-\pi\zeta}{\sqrt{1 - \zeta^2}}\right]$ $\text{PO} = 100 \exp\left[\frac{-\pi\zeta}{\sqrt{1 - \zeta^2}}\right]$
Hysteresis loop method	ΔU = area of displacement–force hysteresis loop; x_0 = maximum displacement of the hysteresis loop; k = average slope of the hysteresis loop	Hysteretic damping constant $h = \frac{\Delta U}{\pi x_0^2}$ Loss factor $\eta = \frac{h}{k}$ Equivalent damping ratio $\zeta = \frac{h}{2k}$
Magnification factor method	Q = amplification at resonance, w.r.t. zero-frequency value	$Q = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}$ For low damping $\zeta = \frac{1}{2Q}$
Bandwidth method	$\Delta\omega$ = bandwidth at $1/\sqrt{2}$ of resonant peak (i.e., half-power bandwidth); ω_i = resonant frequency	$\zeta = \frac{\Delta\omega}{2\omega_i}$

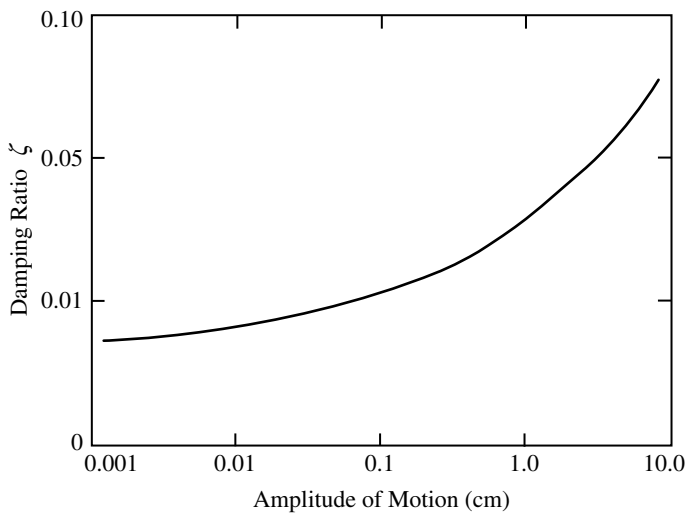


FIGURE 1.14 Effect of vibration amplitude on damping in structures.

TABLE 1.6 Typical Damping Values Suggested by ASME for Seismic Applications

System	Damping Ratio ($\zeta\%$)	
	OBE	SSE
Equipment and large diameter piping systems (> 12 in. diameter)	2	3
Small diameter piping systems (≤ 12 in. diameter)	1	2
Welded steel structures	2	4
Bolted steel structures	4	7
Prestressed concrete structures	2	5
Reinforced concrete structures	4	7

1.5 Interface Damping

In many practical applications damping is generated at the interface of two sliding surfaces. This is the case, for example, in bearings, gears, screws, and guideways. Even though this type of damping is commonly treated under structural damping, due to its significance we will consider it again here in more detail, as a category of its own.

Interface damping was formally considered by DaVinci in the early 1500s and again by Coulomb in the 1700s. The simplified model used by them is the well-known Coulomb friction model as given by

$$f = \mu R \operatorname{sgn}(v) \tag{1.104}$$

where

- f = the frictional force that opposes the motion
- R = the normal reaction force between the sliding surfaces
- v = the relative velocity between the sliding surfaces
- μ = the coefficient of friction

Note that the signum function “sgn” is used to emphasize that f is in the opposite direction of v . This simple model is not expected to provide accurate results in all cases of interface damping. It is known that, apart from the loading conditions, interface damping depends on a variety of factors such as material properties, surface characteristics, nature of lubrication, geometry of the moving parts, and the magnitude of the relative velocity.

A somewhat more complete model for interface damping, incorporating the following characteristics, is shown in Figure 1.15:

1. Static and dynamic friction, with stiction and stick–slip behavior.
2. Conventional Coulomb friction (Region 1).
3. A drop in dynamic friction, with a negative slope, before increasing again. This is known as the “Stribeck effect” (Region 2).
4. Conventional viscous damping (Region 3).

These characteristics cover the behavior of interface damping that is commonly observed in practice. In particular, suppose that a force is exerted to generate a relative motion between two surfaces. For small values of the force, there will not be a relative motion, in view of friction. The minimum force, f_s , that is needed for the motion to start is the static frictional force. The force that is needed to maintain the motion will drop instantaneously to f_d , as the motion begins. It is as though initially the two surfaces were “stuck” and f_s is the necessary breakaway force. Hence, this characteristic is known as stiction. The minimum force f_d that is needed to maintain the relative motion between the two surfaces is called dynamic friction. In fact, under dynamic conditions, it is possible for “stick–slip” to occur where repeated sticking and breaking away cycles of intermittent motion take place. Clearly, such “chattering” motion corresponds to instability (for example, in machine tools). It is an undesirable effect and should be avoided.

After the relative motion begins, conventional Coulomb type damping behavior may dominate for small relative velocities, as represented in Region 1. For lubricated surfaces, at low relative velocities, there will be some solid-to-solid contact that generates a Coulomb-type damping force. As the relative speed increases, the degree of this solid-to-solid contact will decrease and the damping force will drop, as in Region 2 of Figure 1.15. This characteristic is known as the Stribeck effect. Since the slope of the friction curve is negative in Regions 1 and 2, this corresponds to the unstable region. As the relative velocity is further increased, in fully lubricated surfaces, viscous-type damping will dominate, as shown in Region 3 of Figure 1.15. This is the stable region. It follows that a combined model of interface damping may be expressed as

$$f = \begin{cases} f_s & \text{for } v = 0 \\ f_{sb}(v)\text{sgn}(v) + bv & \text{for } v \neq 0 \end{cases} \quad (1.105)$$

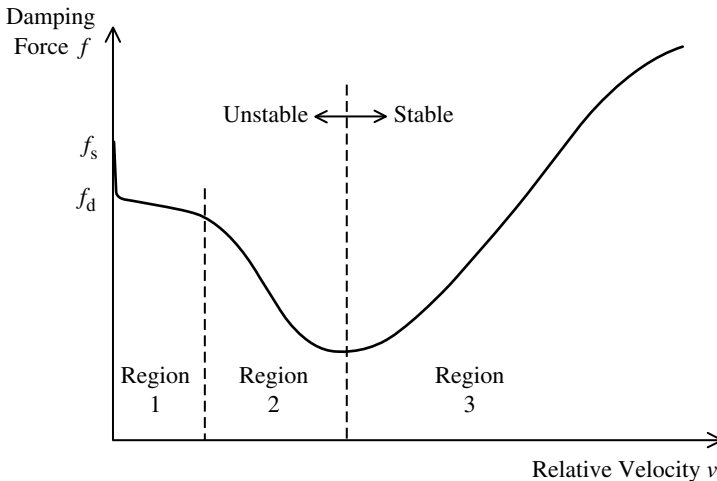


FIGURE 1.15 Main characteristics of interface damping.

Note that $f_{sb}(v)$ is a nonlinear function of velocity that will represent both dynamic friction (for $v > 0$) and the Stribeck effect. Models that have been used to represent this effect include the following

$$f_{sb} = \frac{f_d}{1 + (v/v_c)^2} \quad (1.106)$$

$$f_{sb} = f_d e^{-(v/v_c)^2} \quad (1.107)$$

and

$$f_{sb} = (f_d + \alpha|v|^{1/2})\text{sgn}(v) \quad (1.108)$$

Note that f_d represents dynamic Coulomb friction and v_c and α are modal parameters.

Example 1.5

An object of mass m rests on a horizontal surface and is attached to a spring of stiffness k , as shown in Figure 1.16. The mass is pulled so that the extension of the spring is x_0 , and is moved from rest from that position. Determine the subsequent sliding motion of the object. The coefficient of friction between the object at the horizontal surface is μ .

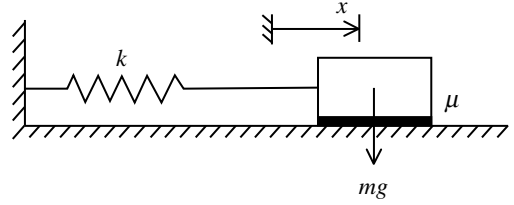


FIGURE 1.16 An object sliding against Coulomb friction.

Solution

Note that, when the object moves to the left, the frictional force, μmg acts to the right, and *vice versa*. Now, consider the first cycle of motion, starting from rest with $x = x_0$ moving to the left, coming to rest with the spring compressed and then moving to the right.

First Half Cycle (Moving to Left)

The equation of motion is

$$m\ddot{x} = -kx + \mu mg \quad (i)$$

or

$$\ddot{x} + \omega_n^2 x = \mu g \quad (ii)$$

where $\omega_n = \sqrt{k/m}$ is the undamped natural frequency. Equation ii has a homogeneous solution of

$$x_h = A_1 \sin(\omega_n t) + A_2 \cos(\omega_n t) \quad (iii)$$

and a particular solution of

$$x_p = \frac{\mu g}{\omega_n^2} \quad (iv)$$

Hence, the total solution is

$$x = A_1 \sin(\omega_n t) + A_2 \cos(\omega_n t) + \frac{\mu g}{\omega_n^2} \quad (v)$$

Using the initial conditions $x = x_0$ and $\dot{x} = 0$ at $t = 0$, we get $A_1 = 0$ and $A_2 = x_0 - (\mu g/\omega_n^2)$. Hence, Equation v becomes

$$x = \left(x_0 - \frac{\mu g}{\omega_n^2}\right) \cos(\omega_n t) + \frac{\mu g}{\omega_n^2} \quad (vi)$$

At the end of this half cycle we have $\dot{x} = 0$ or $\sin \omega_n t = 0$. Hence the corresponding time is $t = \pi/\omega_n$. Substituting this in Equation vi, the corresponding position of the object

is (note: $\cos \pi = -1$)

$$x_{l1} = -\left(x_0 - \frac{2\mu g}{\omega_n^2}\right) \quad (\text{vii})$$

Second Half Cycle (Moving to Right)

The equation of motion is

$$m\ddot{x} = -kx - \mu mg \quad (\text{viii})$$

or

$$\ddot{x} + \omega_n^2 x = -\mu g \quad (\text{ix})$$

The corresponding response is given by

$$x = B_1 \sin(\omega_n t) + B_2 \cos(\omega_n t) - \frac{\mu g}{\omega_n^2} \quad (\text{x})$$

Using the initial conditions $x = -(x_0 - (2\mu g/\omega_n^2))$ and $\dot{x} = 0$ at $t = \pi/\omega_n$, we get $B_1 = 0$ and $B_2 = x_0 - (3\mu g/\omega_n^2)$. Hence, Equation x becomes

$$x = \left(x_0 - \frac{3\mu g}{\omega_n^2}\right) \cos(\omega_n t) - \frac{\mu g}{\omega_n^2} \quad (\text{xi})$$

The object will come to rest ($\dot{x} = 0$) next at $t = 2\pi/\omega_n$, hence the position of the object at the end of the present half cycle would be

$$x_1 = x_0 - \frac{4\mu g}{\omega_n^2} \quad (\text{xii})$$

The response for the next cycle is determined by substituting x_1 as given by Equation xii with Equation vi for the left motion and with Equation xi for the right motion. Then, we can express the general response as

$$\text{left motion in cycle } i: x = [x_0 - (4i - 3)\Delta] \cos \omega_n t + \Delta \quad (\text{xiii})$$

$$\text{right motion in cycle } i: x = [x_0 - (4i - 1)\Delta] \cos \omega_n t - \Delta \quad (\text{xiv})$$

where

$$\Delta = \frac{\mu g}{\omega_n^2} \quad (\text{xv})$$

Note that the amplitude of the harmonic part of the response should be positive for that half cycle of motion to be possible. Hence, we must have

$$x_0 > (4i - 3)\Delta \text{ for left motion in cycle } i$$

$$x_0 > (4i - 1)\Delta \text{ for right motion in cycle } i$$

Also note from Equation xiii and Equation xiv that the equilibrium position for the left motion is $+\Delta$ and for the right motion is $-\Delta$. A typical response curve is sketched in [Figure 1.17](#).

1.5.1 Friction in Rotational Interfaces

Friction in gear transmissions, rotary bearings, and other rotary joints has a somewhat similar behavior. Of course, the friction characteristics will depend on the nature of the devices and also the loading conditions, but, experiments have shown that the frictional behavior of these devices may be represented by the interface damping model given here. Typically, experimental results are presented as

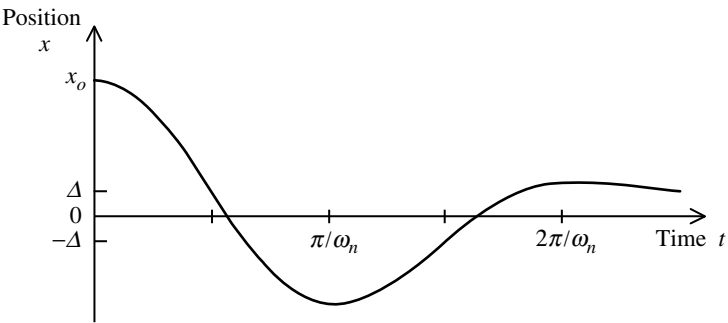


FIGURE 1.17 A typical cyclic response under Coulomb friction.

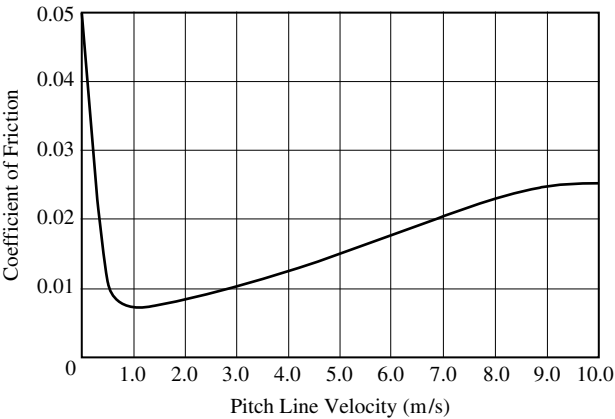


FIGURE 1.18 Frictional characteristics of a pair of spur gears.

curves of coefficient of friction (frictional force/normal force) vs. relative velocity of the two sliding surfaces. In the case of rotary bearings, the rotational speed of the shaft is used as the relative velocity, while for gears; the pitch line velocity is used. Experimental results for a pair of spur gears are shown in Figure 1.18.

What is interesting to notice from the result is the fact that, for this type of rotational device, the damping behavior may be approximated by two straight line segments in the velocity–friction plane; the first segment having a sharp negative slope and the second segment having a moderate positive slope that represents the equivalent viscous damping constant, as shown in Figure 1.19.

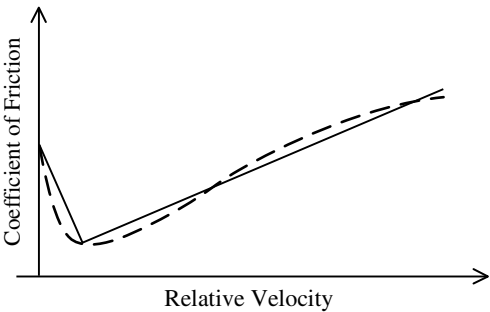


FIGURE 1.19 A friction model for rotatory devices.

1.5.2 Instability

Unstable behavior or self-excited vibrations, such as stick–slip and chatter, that are exhibited by interacting devices such as metal removing tools (e.g., lathes, drills, and milling machines) may be easily

explained using the interface damping model. In particular, it is noted that the model has a region of negative slope (or negative damping constant), which corresponds to low relative velocities, and a region of positive slope, which corresponds to high relative velocities. Consider the single-DoF model:

$$m\ddot{x} + b\dot{x} + kx = 0 \quad (1.109)$$

without an external excitation force. Initially the velocity is $\dot{x} = 0$. But, in this region, the damping constant, b , will be negative and hence the system will be unstable. Hence, a slight disturbance will result in a steadily increasing response. Subsequently, $\dot{x}$ will increase above the critical velocity where b will be positive and the system will be stable. As a result, the response will steadily decrease. This growing and decaying cycle will be repeated at a frequency that primarily depends on the inertia and stiffness parameters (m and k) of the system. Chatter is caused in this manner in interfaced devices.

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