

7

Vibration Design and Control

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Summary

There are desirable and undesirable types and situations of mechanical vibration. Undesirable vibrations are those that cause human discomfort and hazards, structural degradation and failure, performance deterioration and malfunction of machinery and processes, and various other problems. This chapter discusses ways of either eliminating or reducing the undesirable effects of vibration. Specifically, some useful topics on design for vibration suppression and the control of vibration are addressed. General approaches to vibration mitigation may be identified from the dynamic systems point of view. Typically, a set of vibration specifications is given as simple threshold values (bounds) or frequency spectra, and the goal is to either design or control the system to meet these

specifications. Frequency-domain techniques based on transfer functions such as transmissibility, and time-domain techniques using the state-space representation, optimal control and modal control are presented. Applications considered here include vibration isolation, balancing of rotating and reciprocating machinery, whirling suppression, and passive and active control of vibration.

7.1 Introduction

Consider the schematic diagram of a vibratory system shown in Figure 7.1. Forcing excitations $f(t)$ to the mechanical system S cause the vibration responses y . Our objective is to suppress y to a level that is acceptable. Clearly, there are three general ways of doing this:

1. *Isolation.* Suppress the excitations of vibration. This method deals with f .
2. *Design modification.* Modify or redesign the mechanical system so that for the same levels of excitation, the resulting vibrations are acceptable. This method deals with S .
3. *Control.* Absorb or dissipate the vibrations using external devices, through implicit or explicit sensing and control. This method deals with y .

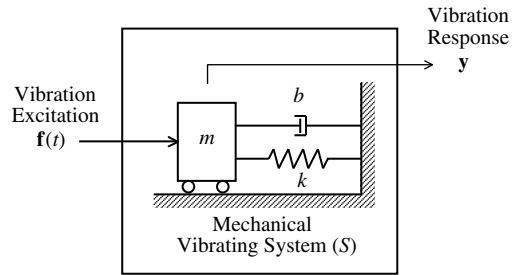


FIGURE 7.1 A vibrating mechanical system.

Within each of these three categories, several approaches can be used to achieve the objective of vibration mitigation. Essentially, each of these approaches involves designing (either complete through redesign or incremental design modification) of the system on the one hand, and controlling the vibration through external means (passive or active devices) on the other. Note that removal of faults (e.g., misalignments and malfunctions by repair or parts replacement) can also remove vibrations. This approach may fall into any of the three categories listed above.

The category of vibration isolation involves “isolating” a mechanical system (S) from vibration excitations (f) so that the excitation signals are “filtered” out or dissipated prior to reaching the system. The use of properly designed suspension systems, mounts, and damping layers falls within this category. The category of design modification will involve making changes to the components and the structure of a mechanical system according to a set of specifications and design guidelines. Balancing of rotating machinery and structural modification through modal analysis and design techniques fall into this category. The category of control will involve either passive devices (which do not use external power), such as dynamic absorbers and dampers, or active control devices (which need external power for operation). In the passive case, the control device implicitly senses the vibration response and dissipates it (as in the case of a damper) or absorbs and stores its energy where it is slowly dissipated (as in the case of a dynamic absorber). In the active case, the vibrations y are explicitly sensed through *sensors* and *transducers*. The forces that should be acted on the system to counteract and suppress vibrations are determined by a controller, and the corresponding forces or torques are applied to the system through one or more *actuators*.

Note that there may be some overlap in the three general categories of vibration mitigation that were mentioned above. For example, the addition of a mount (category 1) may also be interpreted as a design modification (category 2) or as incorporating a passive damper (category 3). It should also be noted that the general approach, commonly known as *source alteration*, may fall into either category 1 or category 2. In this case, the purpose is to alter or remove the source of vibration. The source could either be external (e.g., road irregularities that result in vehicle vibrations) — a category 1 problem, or internal (imbalance or misalignment in rotating devices that results in periodic forces, moments, and vibrations) — a category 2 problem. It can be more difficult to alter external vibration sources (e.g., resurfacing the roadways) than to modify the internal sources (e.g., balancing of rotating machinery).

Furthermore, the external source of vibration may be quite random and may not be accessible for alteration at all (e.g., aerodynamic forces on an aircraft).

7.1.1 Shock and Vibration

Sometimes, response to shock loads is considered separately from response to vibration excitations for the purpose of design and control of mechanical systems. For example, shock isolation and vibration isolation are treated under different headings in some literature. This is actually unnecessary. Even though vibration analysis predominantly involves periodic excitations and responses, transient and random oscillations (vibrations) are also commonly found in practice. The frequency band of the latter two types of signals is much broader than that of a simple periodic signal. A shock signal is transient by definition, and has a very short duration (in comparison to the predominant time constants of the mechanical system to which the shock load is applied). Hence, it will possess a wide band of frequencies. Consequently, frequency-domain techniques are still applicable. Furthermore, time-domain techniques are particularly suited to dealing with transient signals in general and shock signals in particular. In that context, a shock excitation may be treated as an impulse whose effect is to instantaneously change the velocity of an inertia element. Then, in the time domain, a shock load may also be treated as an initial-velocity excitation of an otherwise free (unforced) system.

7.2 Specification of Vibration Limits

Design and control procedures of vibration have the primary objective of ensuring that under normal operating conditions, the system of interest does not encounter vibration levels that exceed the specified values. In this context, then, the ways of specifying vibration limits become important. This section will present some common ways of vibration specification.

7.2.1 Peak Level Specification

Vibration limits for a mechanical system may be specified in either the time domain or the frequency domain. In the time domain, the simplest specification is the peak level of vibration (typically, acceleration in units of g — the acceleration due to gravity). Here, the techniques of isolation, design, or control should ensure that the peak vibration response of the system do not exceed the specified level. In this case, the entire time interval of operation of the system is monitored, and the peak values are checked against the specifications. Note that, in this case, it is the instantaneous peak value at a particular time instant that is of interest, and what is used in representing vibration is an instantaneous amplitude measure rather than an average amplitude or an energy measure.

7.2.2 Root-Mean-Square Value Specification

The root-mean-square (RMS) value of a vibration signal $y(t)$ is given by the square root of the average of the squared signal as

$$y_{\text{RMS}} = \left[\frac{1}{T} \int_0^T y^2 dt \right]^{1/2} \quad (7.1)$$

Note that, by squaring the signal, its sign is eliminated and, essentially, the energy level of the signal is used. The period T , over which the squared signal is averaged, will depend on the problem and the nature of the signal. For a periodic signal, one period is adequate for averaging. For transient signals, several time constants (typically four times the larger time constant) of the vibrating system would be sufficient. For random signals, a value that is as large as feasible should be used.

In the method of RMS value specification, the RMS value of the acceleration response (typically, acceleration in g s) is computed using Equation 7.1 and is compared with the specified value. In this method, instantaneous bursts of vibration do not have a significant effect as they are filtered out because of the integration. It is the average energy or power of the response signal that is considered. The duration of exposure enters into the picture indirectly and in an undesirable manner. For instance, a highly transient vibration signal can initially have a damaging effect. However, the larger the T that is used in Equation 7.1, the smaller the computed RMS value. Hence, in this case, the use of a large value for T would lead to diluting or masking the damage potential. In practice, the longer the exposure to a vibration signal, the greater the harm caused. Hence, when using specifications such as peak and RMS values, they have to be adjusted according to the period of exposure. Specifically, a larger specification should be used for longer periods of exposure.

7.2.3 Frequency-Domain Specification

It is not realistic to specify the limitation to vibration exposure of a complex dynamic system by just a single threshold value. Usually, the effect of vibration on a system depends on at least the following three parameters:

1. Level of vibration (peak, RMS, power, etc.).
2. Frequency content (range) of excitation.
3. Duration of exposure to vibration.

This is particularly true because the excitations that generate the vibration environment may not necessarily be a single-frequency (sinusoidal) signal and may be broadband and random. Furthermore, the response of the system to the vibration excitations will depend on its frequency transfer function, which determines its resonances and damping characteristics. Under these circumstances, it is desirable to provide specifications in a *nomograph* where the horizontal axis gives frequency (Hz) and the vertical axis could represent a motion variable such as displacement (m), velocity (m/s), or acceleration (m/s^2 or g). It is not important which of these motion variables represents the vertical axis of the nomograph. This is true because in the frequency domain

$$\text{Velocity} = j\omega \times \text{displacement}$$

$$\text{Acceleration} = j\omega \times \text{velocity}$$

and one form of motion may be easily converted into one of the remaining two motion representations. In each of the forms, assuming that the two axes of the nomograph are graduated in a logarithmic scale, the constant displacement, constant velocity, and constant acceleration lines are straight lines.

Consider a simple specification of machinery vibration limits as given by the following values:

$$\text{Displacement limit (peak)} = 0.001 \text{ m}$$

$$\text{Velocity limit} = 0.01 \text{ m/s}$$

$$\text{Acceleration limit} = 1.0g$$

This specification may be represented in a velocity vs. frequency nomograph (log-log) as in Figure 7.2.

Usually, such simple specifications in the frequency domain are not adequate. As noted above, the system behavior will vary depending on the excitation frequency range. For example, motion sickness in humans may be predominant in low frequencies in the range of 0.1 to 0.6 Hz and passenger discomfort in

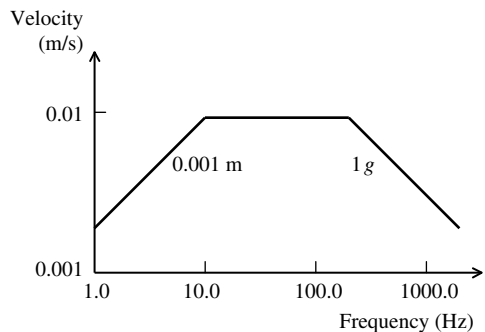


FIGURE 7.2 Operating vibration specification (nomograph) for a machine.

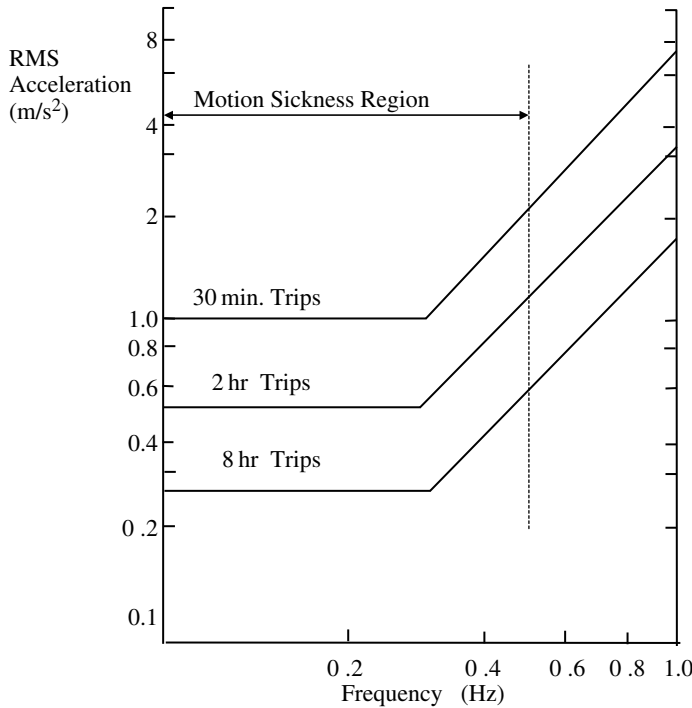


FIGURE 7.3 A severe-discomfort vibration specification for ground transit vehicles.

ground transit vehicles may be most serious in the frequency range of 4 to 8 Hz for vertical motion and 1 to 2 Hz for lateral motion. In addition, for any dynamic system, particularly at low damping levels, the neighborhoods of resonant frequencies should be avoided and, hence, should be specified by low vibration limits in the resonant regions. Furthermore, the duration of vibration exposure should be explicitly accounted for in specifications. For example, Figure 7.3 presents a ride comfort specification for a ground transit vehicle, where lower vibration levels are specified for longer trips.

Finally, it should be noted that the specifications we are concerned with in the present context of design and control are upper bounds of vibration. The system should perform below (within) these specifications under normal operating conditions. Test specifications are lower bounds. The test should be conducted at or above these vibration levels so that the system would meet the test specifications. Some considerations of vibration engineering are summarized in [Box 7.1](#).

7.3 Vibration Isolation

The purpose of vibration isolation is to “isolate” the system of interest from vibration excitations by introducing an *isolator* in between them. Examples of isolators are machine mounts and vehicle suspension systems. Two general types of isolation can be identified:

1. Force isolation (related to force transmissibility)
2. Motion isolation (related to motion transmissibility)

In force isolation, vibration forces that would be ordinarily transmitted directly from a source to a supporting structure (isolated system) are filtered out by an isolator through its flexibility (spring) and dissipation (damping) so that part of the force is routed through an inertial path. Clearly, the concepts of *force transmissibility* are applicable here. In motion isolation, vibration motions that are applied at a moving platform of a mechanical system (isolated system) are absorbed by an isolator through its flexibility and dissipation so that the motion that is transmitted to the system of interest is weakened.

Box 7.1

VIBRATION ENGINEERING

Vibration mitigation approaches:

- Isolation (buffers system from excitation)
- Design modification (modifies the system)
- Control (senses vibration and applies a counteracting force: passive/active)

Vibration specification:

- Peak and RMS values
- Frequency-domain specs on a nomograph
 - *Vibration levels
 - *Frequency content
 - *Exposure duration

Note:

$$|\text{Velocity}| = \omega \times |\text{Displacement}|$$

$$|\text{Acceleration}| = \omega \times |\text{Velocity}|$$

Limiting specifications:

Operation (design) specifications: specify upper bounds

Testing specifications: specify lower bounds

The concepts of motion transmissibility are applicable in this case. The design problem in both cases is to select applicable parameters for the isolator so that the vibrations entering the system are below specified values within a frequency band of interest (the operating frequency range).

Let us revisit the main concepts of force transmissibility and motion transmissibility. Figure 7.4(a) gives a schematic model of force transmissibility through an isolator. Vibration force at the source is $f(t)$. In view of the isolator, the source system (with impedance Z_m) is made to move at the same speed as the isolator (with impedance Z_s). This is a parallel connection of impedances. Hence, the force $f(t)$ is split so that part of it is taken up by the inertial path (broken line) of Z_m . Only the remainder (f_s) is transmitted through Z_s to the supporting structure, which is the isolated system. Force transmissibility is

$$T_f = \frac{f_s}{f} = \frac{Z_s}{Z_m + Z_s} \quad (7.2)$$

Figure 7.4(b) gives a schematic model of motion transmissibility through an isolator. Vibration motion $v(t)$ of the source is applied through an isolator (with impedance Z_s and mobility M_s) to the isolated system (with impedance Z_m and mobility M_m). The resulting force is assumed to transmit directly from the isolator to the isolated system and hence, these two units are connected in series. Consequently, we have the motion transmissibility:

$$T_m = \frac{v_m}{v} = \frac{M_m}{M_m + M_s} = \frac{Z_s}{Z_s + Z_m} \quad (7.3)$$

It can be seen that, according to these two models, we have

$$T_f = T_m \quad (7.4)$$

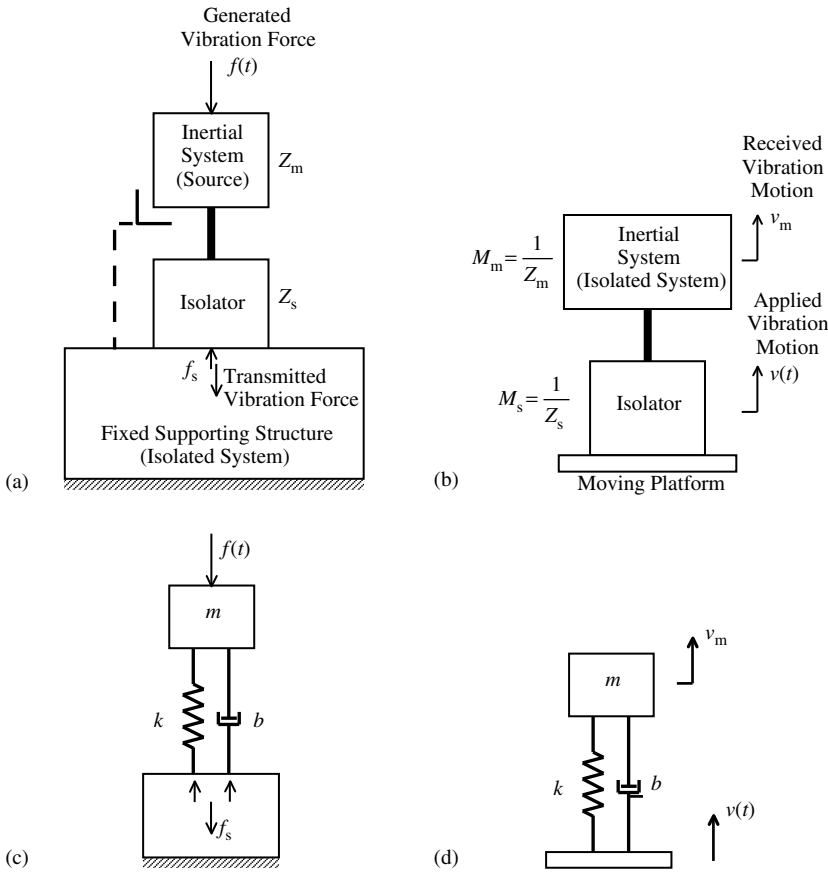


FIGURE 7.4 (a) Force isolation; (b) motion isolation; (c) force isolation example; (d) motion isolation example.

As a result, the concepts of force transmissibility and motion transmissibility may be studied using just one common transmissibility function T .

Simple examples of force isolation and motion isolation are shown in Figure 7.4(c) and (d). For both cases, the transmissibility function is given by

$$T = \frac{k + bj\omega}{(k - m\omega^2 + bj\omega)} \quad (7.5)$$

where ω is the frequency of vibration excitation. Note that the model (Equation 7.5) is not restricted to sinusoidal vibrations. Any general vibration excitation may be represented by a Fourier spectrum, which is a function of frequency ω . Then, the response vibration spectrum is obtained by multiplying the excitation spectrum by the transmissibility function T . The associated design problem is to select the isolator parameters k and b to meet the specifications of isolation.

Equation 7.5 may be expressed as

$$T = \frac{\omega_n^2 + 2\zeta\omega_n\omega j}{(\omega_n^2 - \omega^2 + 2\zeta\omega_n\omega j)} \quad (7.6)$$

where

$$\omega_n = \sqrt{k/m} = \text{undamped natural frequency of the system}$$

$$\zeta = \frac{b}{2\sqrt{km}} = \text{damping ratio of the system}$$

Equation 7.6 may be written in the nondimensional form:

$$T = \frac{1 + 2\zeta rj}{1 - r^2 + 2\zeta rj} \quad (7.7)$$

where the nondimensional excitation frequency is defined as

$$r = \omega/\omega_n$$

The transmissibility function has a phase angle as well as magnitude. In practical applications, the level of attenuation of the vibration excitation (rather than the phase difference between the vibration excitation and the response) is of primary importance. Accordingly, the transmissibility magnitude

$$|T| = \sqrt{\frac{1 + 4\zeta^2 r^2}{(1 - r^2)^2 + 4\zeta^2 r^2}} \quad (7.8)$$

is of interest. It can be shown that $|T| < 1$ for $r > \sqrt{2}$, which corresponds to the isolation region. Hence, the isolator should be designed such that the operative frequencies ω are greater than $\sqrt{2}\omega_n$. Furthermore, a threshold value for $|T|$ would be specified, and the parameters k and b of the isolator should be chosen so that $|T|$ is less than the specified threshold in the operating frequency range (which should be given). This procedure may be illustrated using an example.

Example 7.1

A machine tool and its supporting structure are modeled as the simple mass–spring–damper system shown in Figure 7.5.

1. Draw a mechanical-impedance circuit for this system in terms of the impedances of the three elements: mass (m), spring (k), and viscous damper (b).
2. Determine the exact value of the frequency ratio r in terms of the damping ratio ζ , at which the force transmissibility magnitude will peak. Show that for small ζ , this value is $r = 1$.
3. Plot $|T_f|$ vs. r for the interval $r = [0, 5]$, with one curve for each of the five ζ values 0.0, 0.3, 0.7, 1.0, and 2.0, on the same plane. Discuss the behavior of these transmissibility curves.
4. From part (3), determine for each of the five ζ values and the excitation frequency range with respect to ω_n , for which the transmissibility magnitude is:
 - Less than 1.05
 - Less than 0.5
5. Suppose that the device in Figure 7.5 has a primary, undamped natural frequency of 6 Hz and a damping ratio of 0.2. It is necessary that the system has a force transmissibility magnitude of less than 0.5 for operating frequency values greater than 12 Hz. Does the existing system meet this requirement? If not, explain how you should modify the system to meet the requirement.

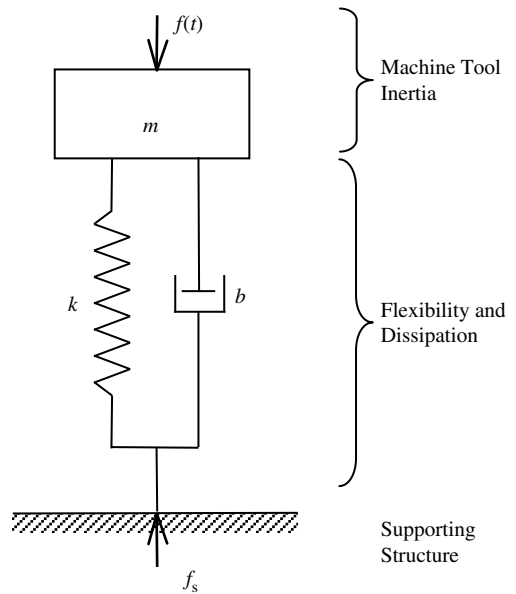


FIGURE 7.5 A simplified model of a machine tool and its supporting structure.

Solution

1. Here, the elements m , b , and f are in parallel with a common velocity v across them, as shown in Figure 7.6. In the circuit, $Z_m = mj\omega$, $Z_b = b$, and $Z_k = k/j\omega$.

Force transmissibility

$$T_f = \frac{F_s}{F} = \frac{F_s/V}{F/V} = \frac{Z_s}{Z_s + Z_0} = \frac{Z_b + Z_k}{Z_m + Z_b + Z_k} \quad (i)$$

Substitute the element impedances. We obtain

$$T_f = \frac{b + \frac{k}{j\omega}}{mj\omega + b + \frac{k}{j\omega}} = \frac{bj\omega + k}{-\omega^2 m + bj\omega + k} = \frac{j\omega b/m + k/m}{-\omega^2 + j\omega b/m + k/m} \quad (ii)$$

The last expression is obtained by dividing the numerator and the denominator by m . Now, use the fact that

$$\frac{k}{m} = \omega_n^2 \quad \text{and} \quad \frac{b}{m} = 2\zeta\omega_n$$

and divide Equation ii throughout by ω_n^2 . We obtain

$$T_f = \frac{\omega_n^2 + 2\zeta\omega_n j\omega}{\omega_n^2 - \omega^2 + 2\zeta\omega_n j\omega} = \frac{1 + 2\zeta r j}{1 - r^2 + 2\zeta r j} \quad (iii)$$

The transmissibility magnitude is

$$|T_f| = \sqrt{\frac{1 + 4\zeta^2 r^2}{(1 - r^2)^2 + 4\zeta^2 r^2}} \quad (7.9)$$

where $r = \omega/\omega_n$ is the normalized frequency.

2. To determine the peak point of $|T_f|$, differentiate the expression within the square-root sign in Equation iv and equate to zero:

$$\frac{[(1 - r^2)^2 + 4\zeta^2 r^2]8\zeta^2 r - [1 + 4\zeta^2 r^2][2(1 - r^2)(-2r) + 8\zeta^2 r]}{[(1 - r^2)^2 + 4\zeta^2 r^2]^2} = 0$$

Hence,

$$4r\{[(1 - r^2)^2 + 4\zeta^2 r^2]2\zeta^2 + [1 + 4\zeta^2 r^2][(1 - r^2) - 2\zeta^2]\} = 0$$

which simplifies to

$$r(2\zeta^2 r^4 + r^2 - 1) = 0$$

The roots are

$$r = 0 \quad \text{and} \quad r^2 = \frac{-1 \pm \sqrt{1 + 8\zeta^2}}{4\zeta^2}$$

The root $r = 0$ corresponds to the initial stationary point at zero frequency. That does not represent a peak. Taking only the positive root for r^2 and then its positive square root, the peak point of the transmissibility magnitude is given by

$$r = \frac{[\sqrt{1 + 8\zeta^2} - 1]^{1/2}}{2\zeta} \quad (iv)$$

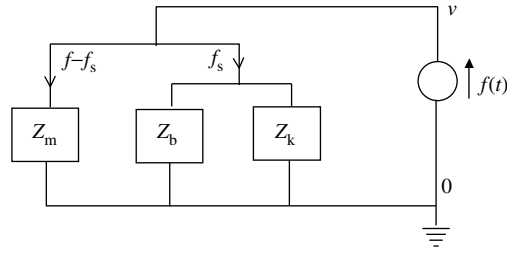


FIGURE 7.6 The mechanical impedance circuit of the force isolation problem.

For small ζ , Taylor series expansion gives

$$\sqrt{1 + 8\zeta^2} \approx 1 + \frac{1}{2} \times 8\zeta^2 = 1 + 4\zeta^2$$

With this approximation, Equation iv equates to 1. Hence, for small damping, the transmissibility magnitude will have a peak at $r = 1$, and from Equation 7.9, its value is

$$|T_f| \approx \frac{\sqrt{1 + 4\zeta^2}}{2\zeta} \approx \frac{1 + \frac{1}{2} \times 4\zeta^2}{2\zeta}$$

or

$$|T_f| \approx \frac{1}{2\zeta} + \zeta \approx \frac{1}{2\zeta} \quad (v)$$

3. The five curves of $|T_f|$ vs. r for $\zeta = 0, 0.3, 0.7, 1.0$, and 2.0 are shown in Figure 7.7. Note that these curves use the exact expression (see Equation 7.9).

From the curves, we observe the following:

1. There is always a nonzero frequency value at which the transmissibility magnitude will peak. This is the resonance.
 2. For small ζ , the peak transmissibility magnitude is obtained at approximately $r = 1$. As ζ increases, this peak point shifts to the left (i.e., a lower value for peak frequency).
 3. The peak magnitude decreases with increasing ζ .
 4. All the transmissibility curves pass through the magnitude value 1.0 at the same frequency $r = \sqrt{2}$.
 5. The isolation (i.e., $|T_f| < 1$) is given by $r > \sqrt{2}$. In this region, $|T_f|$ increases with ζ .
 6. The transmissibility magnitude decreases for large r .
4. From the curves in Figure 7.7, we obtain:
- For $|T_f| < 1.05$; $r > \sqrt{2}$ for all ζ .
 - For $|T_f| < 0.5$; $r > 1.73, 1.964, 2.871, 3.77, 7.075$ for $\zeta = 0.0, 0.3, 0.7, 1.0$, and 2.0 , respectively.

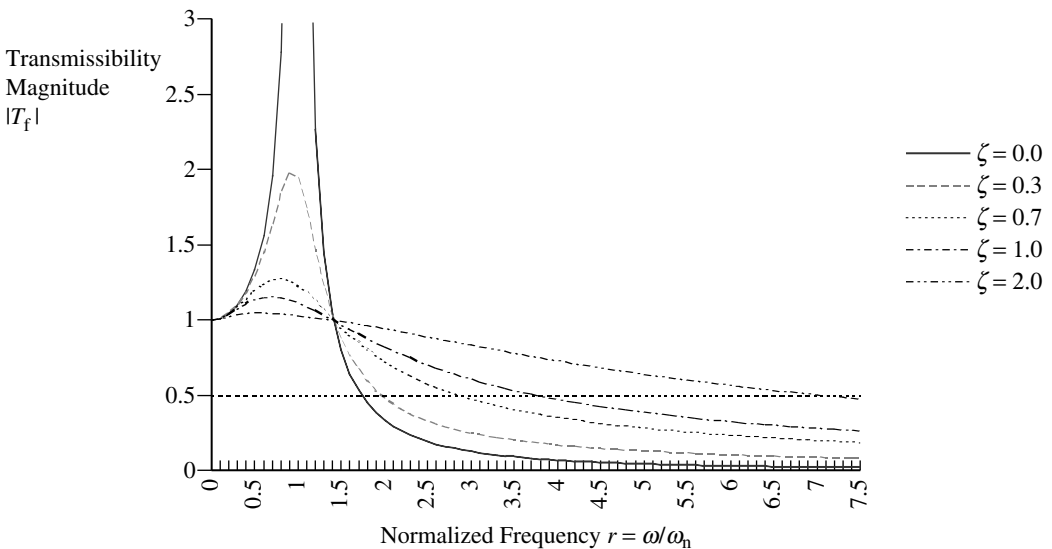


FIGURE 7.7 Transmissibility curves for a simple oscillator model.

5. We need

$$\sqrt{\frac{1 + 4\zeta^2 r^2}{(1 - r^2)^2 + 4\zeta^2 r^2}} < \frac{1}{2}$$

or

$$\frac{1 + 4\zeta^2 r^2}{(1 - r^2)^2 + 4\zeta^2 r^2} < \frac{1}{4}$$

or

$$4 + 16\zeta^2 r^2 < (1 - r^2)^2 + 4\zeta^2 r^2$$

or

$$r^4 - 2r^2 - 12\zeta^2 r^2 - 3 > 0$$

For $\zeta = 0.2$ and $r = 12/6 = 2$, the left-hand-side expression computes to

$$2^4 - 2 \times 2^2 - 12 \times (0.2)^2 \times 2^2 - 3 = 3.08 > 0$$

Hence, the requirement is met. In fact, since, for $r = 2$

$$\text{LHS} = 2^4 - 2 \times 2^2 - 12 \times 2^2 \zeta^2 - 3 = 5 - 48\zeta^2$$

it follows that the requirement would be met for

$$5 - 48\zeta^2 > 0$$

or

$$\zeta < \sqrt{\frac{5}{48}} = 0.32$$

If the requirement was not met (e.g., if $\zeta = 0.4$), the option would be to reduce damping.

7.3.1 Design Considerations

The *level of isolation* is defined as $1 - T$. It was noted that in the isolation region ($r > \sqrt{2}$) the transmissibility decreases (hence, the level of isolation increases) as the damping ratio ζ decreases. Thus, the best conditions of isolation are given by $\zeta = 0$. This is not feasible in practice, but we should maintain ζ as small as possible. For small ζ in the isolation region, Equation 7.8 may be approximated by

$$T = \frac{1}{(r^2 - 1)} \quad (7.10)$$

Note that T is real, in this case, of $\zeta \cong 0$, and also is positive since $r > \sqrt{2}$. However, in general, T may denote the magnitude of the transmissibility function. Substitute

$$r^2 = \omega^2/\omega_n^2 = \omega^2 m/k$$

We get

$$k = \frac{\omega^2 m T}{(1 + T)} \quad (7.11)$$

This equation may be used to determine the design stiffness of the isolator for a specified level of isolation ($1 - T$) in the operating frequency range $\omega > \omega_0$ for a system of known mass (including the isolator mass). Often, the static deflection δ_s of spring is used in design procedures and is

given by

$$\delta_s = \frac{mg}{k} \quad (7.12)$$

Substituting Equation 7.12 into Equation 7.11, we obtain

$$\delta_s = (1 + T) \frac{g}{\omega^2 T} \quad (7.13)$$

Since the isolation region is $\omega > \sqrt{2}\omega_n$ it is desirable to make ω_n as small as possible in order to obtain the widest frequency range of operation. This is achieved by making the isolator as soft as possible (k as low as possible). However, there are limits to this in terms of structural strength, stability, and availability of springs. Then, m may be increased by adding an inertia block as the base of the system, which is then mounted on the isolator spring (with a damping layer) or an air-filled pneumatic mount. The inertia block will also lower the centroid of the system, thereby providing added desirable effects of stability and reducing rocking motions and noise transmission. For improved load distribution, instead of just one spring of design stiffness k , a set of n springs, each with stiffness k/n and uniformly distributed under the inertia block, should be used.

Another requirement for good vibration isolation is low damping. Usually, metal springs have very low damping (typically ζ less than 0.01). On the other hand, higher damping is needed to reduce resonant vibrations that will be encountered during start-up and shutdown conditions when the excitation frequency will vary and pass through the resonances. In addition, vibration energy has to be effectively dissipated, even under steady operating conditions. Isolation pads made of damping material such as cork, natural rubber, and neoprene may be used for this purpose. They can provide damping ratios of the order of 0.01.

The basic design steps for a vibration isolator in force isolation are as follows:

1. The required level of isolation (1 to T) and the lowest frequency of operation (ω_0) are specified. The mass of the vibration source (m) is known.
2. Use Equation 7.11 with $\omega = \omega_0$ to compute the required stiffness k of the isolator.
3. If the component k is not satisfactory, then increase m by introducing an inertia block and recompute k .
4. Distribute k over several springs.
5. Introduce a mounting pad of known stiffness and damping. Modify k and b accordingly and compute T using Equation 7.8. If the specified T is exceeded, then modify the isolator parameters as appropriate and repeat the design cycle.

Box 7.2 gives some relations that are useful in a design for vibration isolation.

Example 7.2

Consider a motor and fan unit of a building ventilation system weighing 50 kg and operating in the speed range of 600 to 3600 rpm. Since offices are located directly underneath the motor room, a 90% vibration isolation is desired. A set of mounting springs, each having a stiffness of 100 N/cm, is available. Design an isolation system to mount the motor-fan unit on the room floor.

Solution

For an isolation level of 90%, the required force transmissibility is $T = 0.1$. The lowest frequency of operation is $\omega = (600/60)2\pi$ rad/s. First, we try four mounting points. The overall spring stiffness is $k = 4 \times 100 \times 10^2$ N/m. Substitute in Equation 7.11.

$$4 \times 100 \times 100 = \frac{(10 \times 2\pi)^2 \times m \times 0.1}{1.1}$$

Box 7.2

VIBRATION ISOLATION

Transmissibility (force/force or motion/motion):

$$|T| = \sqrt{\frac{1 + 4\zeta^2 r^2}{(1 - r^2)^2 + 4\zeta^2 r^2}} \cong \frac{1}{(r^2 - 1)} \text{ for } r > 1 \text{ and small } \zeta$$

Properties:

1. $T_{\text{peak}} \cong \frac{\sqrt{1 + 4\zeta^2}}{2\zeta} \cong \frac{1}{2\zeta}$ for small ζ
2. T_{peak} occurs at $r_{\text{peak}} = \frac{[\sqrt{1 + 8\zeta^2} - 1]^{1/2}}{2\zeta} \cong 1$ for small ζ
3. All $|T|$ curves coincide at $r = \sqrt{2}$ for all ζ
4. Isolation region: $r > \sqrt{2}$
5. In isolation region:
 $|T|$ decreases with r (i.e., better isolation at higher frequencies)
 $|T|$ increases with ζ (i.e., better isolation at lower damping)

Design formulas:

Level of isolation = $1 - T$

Isolator stiffness:

$$k = \frac{\omega^2 m T}{(1 + T)}$$

where

m = system mass

ω = operating frequency

Static deflection:

$$\delta_s = \frac{mg}{k} = (1 + T) \frac{g}{\omega^2 T}$$

which gives $m = 111.5$ kg. Since the mass of the unit is 50 kg, we should use an inertia block of mass 61.5 kg or more.

7.3.2 Vibration Isolation of Flexible Systems

The simple model shown in Figure 7.4(c) and (d) may not be adequate in the design of vibration isolators for sufficiently flexible systems. A more appropriate model for this situation is shown in Figure 7.8. Note that the vibration isolator has an inertia block of mass m in addition to damped flexible mounts of stiffness k and damping constant b . The vibrating system itself has a stiffness K and damping constant B in addition to its mass M .

In the absence of K , B , and the inertia block (m) as in Figure 7.4(c), the vibrating system becomes a simple inertia (M). Then, y_a and y are the same and the equation of motion is

$$M\ddot{y} + b\dot{y} + ky = f(t) \quad (7.14)$$

with the force transmitted to the support structure, f_s , given by

$$f_s = b\dot{y} + ky \quad (7.15)$$

The force transmissibility in this case is

$$T_{\text{inertial}} = \frac{f_s}{f} = \frac{bs + k}{Ms^2 + bs + k} \quad (7.16)$$

with $s = j\omega$

For the flexible system and isolator shown in Figure 7.8, the equation of motion:

$$M\ddot{y}_a + B(\dot{y}_a - \dot{y}) + K(y_a - y) = f(t) \quad (7.17)$$

$$m\ddot{y} + B(\dot{y} - \dot{y}_a) + K(y - y_a) + b\dot{y} + ky = 0 \quad (7.18)$$

Hence, in the frequency domain, we have

$$(Ms^2 + Bs + K)y_a - (Bs + K)y = f \quad (7.19)$$

$$[ms^2 + (B + b)s + K + k]y = (Bs + K)y_a$$

with $s = j\omega$ (7.20)

Substitute Equation 7.20 into Equation 7.19 for eliminating y_a . We obtain

$$\left\{ (Ms^2 + Bs + K) \frac{[ms^2 + (B + b)s + K + k]}{(Bs + K)} - (Bs + K) \right\} y = f$$

which simplifies

$$\left\{ \frac{Ms^2[ms^2 + (B + b)s + K + k] + (Bs + K)(ms^2 + bs + k)}{(Bs + K)} \right\} y = f \quad (7.21)$$

The force transmitted to the supporting structure is still given by Equation 7.15. Hence, the transmissibility with the flexible system is

$$T_{\text{flexible}} = \frac{(Bs + K)(bs + k)}{\left\{ Ms^2[ms^2 + (B + b)s + K + k] + (Bs + K)(ms^2 + bs + k) \right\}} \quad \text{with } s = j\omega \quad (7.22)$$

From Equation 7.16 and Equation 7.22, the transmissibility magnitude ratio is

$$\frac{T_{\text{flexible}}}{T_{\text{inertial}}} = \left| \frac{(Bs + K)(Ms^2 + bs + k)}{Ms^2[ms^2 + (B + b)s + K + k] + (Bs + K)(ms^2 + bs + k)} \right| \quad \text{with } s = j\omega \quad (7.23)$$

or

$$\frac{T_{\text{flexible}}}{T_{\text{inertial}}} = \left| \frac{(Ms^2 + bs + k)}{Ms^2(ms^2 + bs + k)/(Bs + K) + Ms^2 + ms^2 + bs + k} \right| \quad s = j\omega \quad (7.24)$$

In the nondimensional form, we have

$$\frac{T_{\text{flexible}}}{T_{\text{inertial}}} = \left| \frac{1 - r^2 + 2j\zeta_b r}{-r^2(1 - r_m r^2 + 2j\zeta_b r)/(r_\omega^2 + 2j\zeta_a r_\omega r) + 1 - (1 + r_m)r^2 + 2j\zeta_b r} \right| \quad (7.25)$$

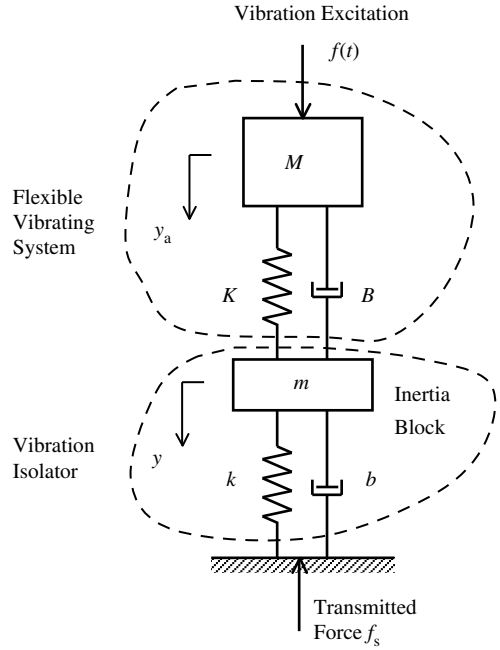


FIGURE 7.8 A model for vibration isolation of a flexible system.

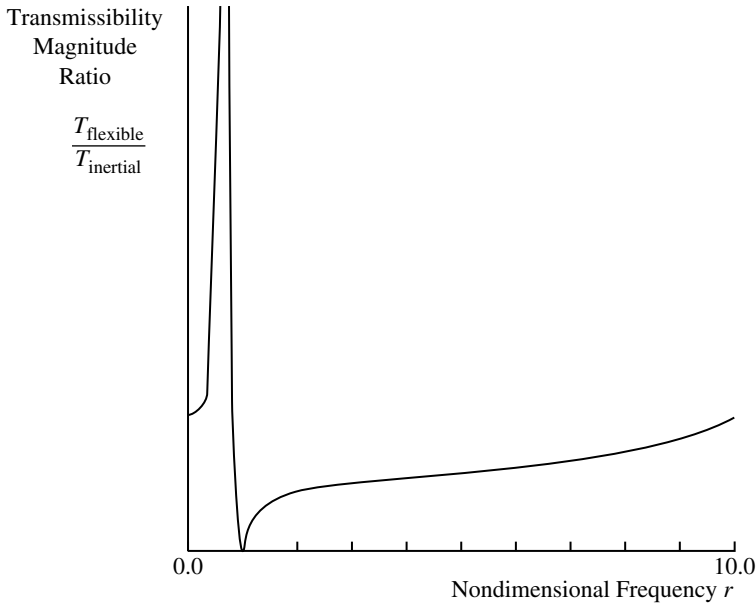


FIGURE 7.9 The effect of system flexibility on the transmissibility magnitude in the undamped case (mass ratio = 1.0; natural frequency ratio = 10.0).

where

$$r = \frac{\omega}{\sqrt{k/M}}$$

$$r_m = \frac{m}{M}$$

$$r_\omega = \frac{\sqrt{K/M}}{\sqrt{k/M}} = \sqrt{\frac{K}{k}}$$

$$\zeta_a = \frac{B}{2\sqrt{KM}}$$

$$\zeta_b = \frac{b}{2\sqrt{kM}}$$

Again, the design problem of vibration isolation is to select the parameters r_m , r_ω , ζ_a , and ζ_b so that the required level of vibration isolation is realized for an operating frequency range of r .

A plot of Equation 7.25 for the undamped case with $r_m = 1.0$ and $r_\omega = 10.0$ is given in Figure 7.9. Generally, the transmissibility ratio will be zero at $r = 1$ (the resonance of the inertial system) and there will be two values of r (the resonances of the flexible system), for which the ratio will become infinity in the undamped case. The latter two neighborhoods should be avoided under steady operating conditions.

7.4 Balancing of Rotating Machinery

Many practical devices that move contain rotating components. Examples are wheels of vehicles, shafts, gear transmissions of machinery, belt drives, motors, turbines, compressors, fans, and rollers. An unbalance (imbalance) is created in a rotating part when its center of mass does not coincide with the axis

of rotation. The reasons for this *eccentricity* include the following:

- 1. Inaccurate production procedures (machining, casting, forging, assembly, etc.)
- 2. Wear and tear
- 3. Loading conditions (mechanical)
- 4. Environmental conditions (thermal loads and deformation)
- 5. Use of inhomogeneous and anisotropic material (which does not have a uniform density distribution)
- 6. Component failure
- 7. Addition of new components to a rotating device

For a component of mass m , eccentricity e , and rotating at angular speed ω , the centrifugal force that is generated is $m\omega^2e$. Note the quadratic variation with ω . This rotating force may be resolved into two orthogonal components, which will be sinusoidal with frequency ω . It follows that harmonic forcing excitations are generated due to the unbalance, which can generate undesirable vibrations and associated problems.

Problems caused by unbalance include wear and tear, malfunction and failure of components, poor quality of products, and undesirable noise. The problem becomes increasingly important given the present trend of developing high-speed machinery. It is estimated that the speed of operation of machinery has doubled during the past 50 years. This means that the level of unbalance forces may have quadrupled during the same period, causing more serious vibration problems.

An unbalanced rotating component may be balanced by adding or removing material to or from the component. We need to know both the magnitude and location of the balancing masses to be added to, or removed. The present section will address the problem of component balancing for vibration suppression.

Note that the goal to remove the source of vibration (namely, the mass eccentricity) typically by adding one or more balancing mass elements. Two methods are available:

- 1. Static (single-plane) balancing
- 2. Dynamic (two-plane) balancing

The first method concerns balancing of planar objects (e.g., pancake motors, disks) whose longitudinal dimension about the axis of rotation is not significant. The second method concerns balancing of objects that have a significant longitudinal dimension. We will discuss both methods.

7.4.1 Static Balancing

Consider a disk rotating at angular velocity ω about a fixed axis. Suppose that the mass center of the disk has an eccentricity e from the axis of rotation, as shown in Figure 7.10(a). Place a fixed coordinate frame

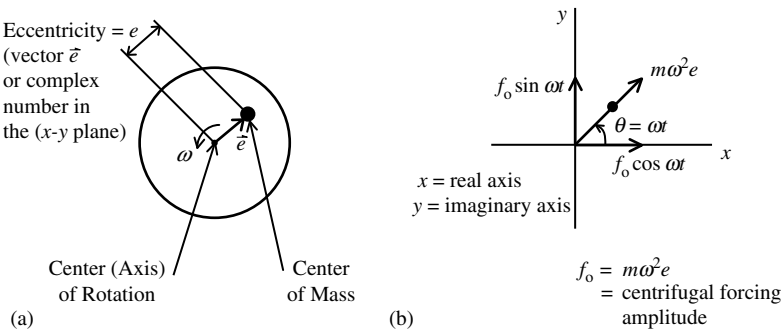


FIGURE 7.10 (a) Unbalance in a rotating disk due to mass eccentricity; (b) rotating vector (phasor) of centrifugal force due to unbalance.

x - y at the center of rotation. The position $\vec{e}$ of the mass center in this coordinate frame may be represented as:

1. A position vector rotating at angular speed ω .
2. A complex number, with x -coordinate denoting the real part and y -coordinate denoting the imaginary part.

The centrifugal force due to the mass eccentricity is also a vector in the direction of $\vec{e}$, but with a magnitude $f_o = m\omega^2 e$, as shown in Figure 7.10(b). It is seen that harmonic excitations result in both x and y directions, given by $f_o \cos \omega t$ and $f_o \sin \omega t$, respectively, where $\theta = \omega t$ = orientation of the rotating vector with respect to the x -axis. To balance the disk, we should add a mass m at $-\vec{e}$. However, we do not know the value of m and the location of $\vec{e}$.

7.4.1.1 Balancing Approach

1. Measure the amplitude V_u and the phase angle ϕ_1 (e.g., by the signal from an accelerometer mounted on the bearing of the disk) of the unbalance centrifugal force with respect to some reference.
2. Mount a known mass (trial mass) M_t at a known location on the disk. Suppose that its own centrifugal force is given by the rotating vector $\vec{V}_w$, and the resultant centrifugal force due to both the original unbalance and the final mass is $\vec{V}_r$.
3. Measure the amplitude V_r and the phase angle ϕ_2 of the resultant centrifugal force as in step 1, with respect to the same phase reference.

A vector diagram showing the centrifugal forces $\vec{V}_u$ and $\vec{V}_w$ due to the original unbalance and the trial-mass unbalance, respectively, is shown in Figure 7.11. The resultant unbalance is $\vec{V}_r = \vec{V}_u + \vec{V}_w$. Note that $-\vec{V}_u$ represents the centrifugal force due to the balancing mass. Therefore, if we determine the angle ϕ_b in Figure 7.11, it will give the orientation of the balancing mass. Suppose also that the balancing mass is M_b and it is mounted at an eccentricity equal to that of the trial mass M_t . Then,

$$\frac{M_b}{M_t} = \frac{V_u}{V_w}$$

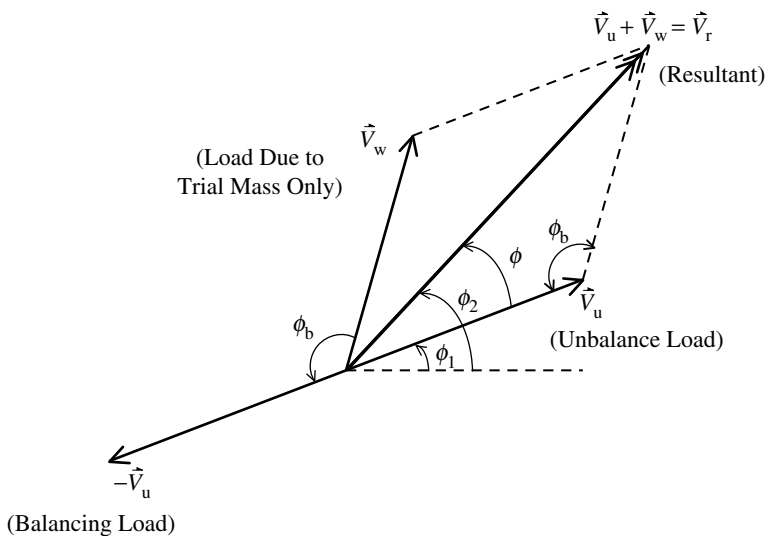


FIGURE 7.11 A vector diagram of the single-plane (static) balancing problem.

We need to determine the ratio V_u/V_w and the angle ϕ_b . These values can be derived as follows:

$$\phi = \phi_2 - \phi_1 \quad (7.26)$$

The cosine rule gives

$$V_w^2 = V_u^2 + V_r^2 - 2V_u V_r \cos \phi \quad (7.27)$$

This will provide V_w since V_u , V_r , and ϕ are known. Apply the cosine rule again:

$$V_r^2 = V_u^2 + V_w^2 - 2V_u V_w \cos \phi_b$$

Hence,

$$\phi_b = \cos^{-1} \left[\frac{V_u^2 + V_w^2 - V_r^2}{2V_u V_w} \right] \quad (7.28)$$

Note: One may think that since we measure ϕ_1 , we know exactly where $\bar{V}_u$ is. This is not the case because we do not know the reference line from which ϕ_1 is measured. We only know that this reference is kept fixed (through strobe synchronization of the body rotation) during measurements. Hence, we need to know ϕ_b , which gives the location of $-\bar{V}_u$ with respect to the known location of $\bar{V}_w$ on the disk.

7.4.2 Complex Number/Vector Approach

Again, suppose that the imbalance is equivalent to a mass of M_b that is located at the same eccentricity (radius) r as the trial mass M_t . Define complex numbers (mass location vectors in a body frame)

$$\bar{M}_b = M_b \angle \theta_b \quad (7.29)$$

$$\bar{M}_t = M_t \angle \theta_t \quad (7.30)$$

as shown in Figure 7.12.

Associated force vectors are

$$\bar{V}_u = \omega^2 r e^{j\omega t} M_b \angle \theta_b \quad (7.31)$$

$$\bar{V}_w = \omega^2 r e^{j\omega t} M_t \angle \theta_t \quad (7.32)$$

or

$$\bar{V}_u = \bar{A} \bar{M}_b \quad (7.33)$$

$$\bar{V}_w = \bar{A} \bar{M}_t \quad (7.34)$$

where $\bar{A} = \omega^2 r e^{j\omega t}$ is the conversion factor (complex) from the mass to the resulting dynamic force (rotating). This factor is the same for both cases since r is the same. We need to determine $\bar{M}_b$.

From Equation 7.33

$$\bar{M}_b = \frac{\bar{V}_u}{\bar{A}} \quad (7.35)$$

Substitute Equation 7.34:

$$\bar{M}_b = \frac{\bar{V}_u}{\bar{V}_w} \cdot \bar{M}_t \quad (7.36)$$

However, since

$$\bar{V}_r = \bar{V}_u + \bar{V}_w \quad (7.37)$$

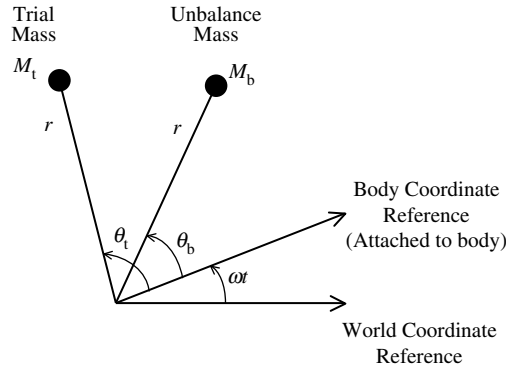


FIGURE 7.12 Rotating vectors of mass location.

we have

$$\vec{M}_b = \frac{\vec{V}_u}{(\vec{V}_r - \vec{V}_u)} \cdot \vec{M}_t \quad (7.38)$$

Since we know $\vec{M}_t$ and we measure $\vec{V}_u$ and $\vec{V}_r$ to the same scaling factor, we can compute $\vec{M}_b$. Locate the balancing mass at $-\vec{M}_b$ (with respect to the body frame).

Example 7.3

Consider the following experimental steps:

Measured: Accelerometer amplitude (oscilloscope reading) of 6.0 with a phase lead (with respect to strobe signal reference, which is synchronized with the rotating body frame) of 50° .

Added: Trial mass $M_t = 20$ g at angle 180° with respect to a body reference radius.

Measured: Accelerometer amplitude of 8.0 with a phase lead of 60° (with respect synchronized strobe signal).

Determine the magnitude and location of the balancing mass.

Solution

Method 1:

We have the data

$$\phi = 60 - 50 = 10^\circ$$

$$V_u = 6.0; \quad V_r = 8.0$$

Hence, from Equation 7.27:

$$V_w = \sqrt{6^2 + 8^2 - 2 \times 6 \times 8 \cos 10^\circ} = 2.37$$

Balancing mass:

$$M_b = \frac{6.0}{2.37} \times 20 = 50.63 \text{ g}$$

Equation 7.28 gives

$$\phi_b = \cos^{-1} \left[\frac{6^2 + 2.37^2 - 8^2}{2 \times 6 \times 2.37} \right] = \cos^{-1}(-0.787) = 142^\circ \text{ or } 218^\circ$$

Pick the result $0^\circ \leq \phi_b \leq 180^\circ$, as clear from the vector diagram shown in [Figure 7.11](#).

Hence,

$$\phi_b = 142^\circ$$

However,

$$\vec{M}_t = 20 \angle 180^\circ \text{ g}$$

It follows that

$$-\vec{M}_b = 50.63 \angle (180^\circ + 142^\circ) \text{ g} = 50.63 \angle 322^\circ \text{ g}$$

Method 2:

We have

$$\vec{M}_t = 20 \angle 180^\circ \text{ g}$$

$$\vec{V}_u = 6.0 \angle 50^\circ$$

$$\vec{V}_r = 8.0 \angle 60^\circ$$

Then, from Equation 7.38 we obtain

$$\vec{M}_b = \frac{6.0 \angle 50^\circ}{(8.0 \angle 60^\circ - 6.0 \angle 50^\circ)} 20 \angle 180^\circ \text{ g}$$

First, we compute

$$\begin{aligned} 8.0 \angle 60^\circ - 6.0 \angle 50^\circ &= (8.0 \cos 60^\circ + j8.0 \sin 6.0^\circ) - (6.0 \cos 50^\circ + j6.0 \sin 50^\circ) \\ &= (8.0 \cos 60^\circ - 6.0 \cos 50^\circ) + j(8.0 \sin 60^\circ - 6.0 \sin 50^\circ) = 0.1433 + j2.332 \\ &= 2.336 \angle 86.48^\circ \end{aligned}$$

Hence,

$$\vec{M}_b = \frac{6.0 \angle 50^\circ}{2.336 \angle 86.48^\circ} 20 \angle 180^\circ = \frac{6.0 \times 20}{2.336} \angle (50^\circ + 180^\circ - 86.48^\circ) = 51 \angle 143.5^\circ \text{ g}$$

The balancing mass should be located at

$$-\vec{M}_b = 51 \angle 323.5^\circ \text{ g}$$

Note: This angle is measured from the same body reference as for the trial mass.

7.4.3 Dynamic (Two-Plane) Balancing

Instead of an unbalanced disk, consider an elongated rotating object supported at two bearings, as shown in Figure 7.13. In this case, in general, there may not be an equivalent single unbalance force at a single plane normal to the shaft axis. To show this, recall that a system of forces may be represented by a single force at a specified location and a couple (two parallel forces that are equal and opposite). If this single force (resultant force) is zero, we are left with only a couple. The couple cannot be balanced by a single force.

All the unbalance forces at all the planes along the shaft axis can be represented by an equivalent single unbalance force at a specified plane and a couple. If this equivalent force is zero, then to balance the couple we will need two equal and opposite forces at two different planes.

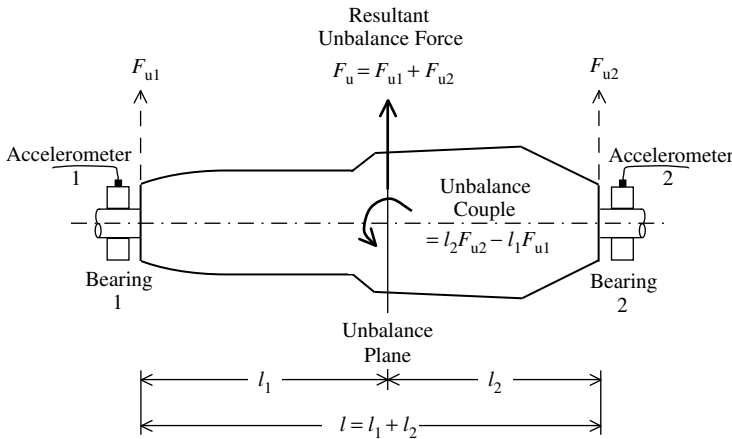


FIGURE 7.13 A dynamic (two-plane) balancing problem.

On the other hand, if the couple is zero, then a single force in the opposite direction at the same plane of the resultant unbalance force will result in complete balancing. However, this unbalance plane may not be reachable, even if it is known, for the purpose of adding the balancing mass.

In the present (two-plane) balancing problem, the balancing masses are added at the two bearing planes so that both the resultant unbalance force and couple are balanced, in general. It is clear from Figure 7.13 that even a sole unbalance mass $\bar{M}_b$ at a single unbalance plane may be represented by two unbalance masses $\bar{M}_{b1}$ and $\bar{M}_{b2}$ at the bearing planes 1 and 2. Likewise, in the presence of an unbalance couple, we can simply add two equal and opposite forces at the planes 1 and 2 so that its couple is equal to the unbalance couple. Hence, a general unbalance can be represented by the two unbalance masses $\bar{M}_{b1}$ and $\bar{M}_{b2}$ at planes 1 and 2, as shown in Figure 7.13. As for the single-plane balancing problem, the resultant unbalance forces at the two bearings (which would be measured by the accelerometers at 1 and 2) are

$$\bar{V}_{u1} = \bar{A}_{11}\bar{M}_{b1} + \bar{A}_{12}\bar{M}_{b2} \quad (7.39)$$

$$\bar{V}_{u2} = \bar{A}_{21}\bar{M}_{b1} + \bar{A}_{22}\bar{M}_{b2} \quad (7.40)$$

Suppose that a trial mass of $\bar{M}_{t1}$ (at a known location with respect to the body reference line) was added at plane 1. The resulting unbalance forces at the two bearings are

$$\bar{V}_{r11} = \bar{A}_{11}(\bar{M}_{b1} + \bar{M}_{t1}) + \bar{A}_{12}\bar{M}_{b2} \quad (7.41)$$

$$\bar{V}_{r21} = \bar{A}_{21}(\bar{M}_{b1} + \bar{M}_{t1}) + \bar{A}_{22}\bar{M}_{b2} \quad (7.42)$$

Next, suppose that a trial mass of $\bar{M}_{t2}$ (at a known location with respect to the body reference line) was added at plane 2, after removing $\bar{M}_{t1}$. The resulting unbalance forces at the two bearings are

$$\bar{V}_{r12} = \bar{A}_{11}\bar{M}_{b1} + \bar{A}_{12}(\bar{M}_{b2} + \bar{M}_{t2}) \quad (7.43)$$

$$\bar{V}_{r22} = \bar{A}_{21}\bar{M}_{b1} + \bar{A}_{22}(\bar{M}_{b2} + \bar{M}_{t2}) \quad (7.44)$$

The following subtractions of equations are now made.

Equation 7.41 minus Equation 7.39:

$$\bar{V}_{r11} - \bar{V}_{u1} = \bar{A}_{11}\bar{M}_{t1} \quad \text{or} \quad \bar{A}_{11} = \frac{\bar{V}_{r11} - \bar{V}_{u1}}{\bar{M}_{t1}} \quad (7.45)$$

Equation 7.42 minus Equation 7.40:

$$\bar{V}_{r21} - \bar{V}_{u2} = \bar{A}_{21}\bar{M}_{t1} \quad \text{or} \quad \bar{A}_{21} = \frac{\bar{V}_{r21} - \bar{V}_{u2}}{\bar{M}_{t1}} \quad (7.46)$$

Equation 7.43 minus Equation 7.39:

$$\bar{V}_{r12} - \bar{V}_{u1} = \bar{A}_{12}\bar{M}_{t2} \quad \text{or} \quad \bar{A}_{12} = \frac{\bar{V}_{r12} - \bar{V}_{u1}}{\bar{M}_{t2}} \quad (7.47)$$

Equation 7.44 minus Equation 7.40:

$$\bar{V}_{r22} - \bar{V}_{u2} = \bar{A}_{22}\bar{M}_{t2} \quad \text{or} \quad \bar{A}_{22} = \frac{\bar{V}_{r22} - \bar{V}_{u2}}{\bar{M}_{t2}} \quad (7.48)$$

Hence, generally

$$\bar{A}_{ij} = \frac{\bar{V}_{rij} - \bar{V}_{ui}}{\bar{M}_{tj}} \quad (7.49)$$

These parameters A_{ij} are called *influence coefficients*.

Next, in Equation 7.39 and Equation 7.40 eliminate $\bar{M}_{b2}$ and $\bar{M}_{b1}$ separately to determine the other. Thus,

$$\begin{aligned}\bar{A}_{22}\bar{V}_{u1} - \bar{A}_{12}\bar{V}_{u2} &= (\bar{A}_{22}\bar{A}_{11} - \bar{A}_{12}\bar{A}_{21})\bar{M}_{b1} \\ \bar{A}_{21}\bar{V}_{u1} - \bar{A}_{11}\bar{V}_{u2} &= (\bar{A}_{21}\bar{A}_{12} - \bar{A}_{11}\bar{A}_{22})\bar{M}_{b2}\end{aligned}$$

or

$$\bar{M}_{b1} = \frac{\bar{A}_{22}\bar{V}_{u1} - \bar{A}_{12}\bar{V}_{u2}}{(\bar{A}_{22}\bar{A}_{11} - \bar{A}_{12}\bar{A}_{21})} \quad (7.50)$$

$$\bar{M}_{b2} = \frac{\bar{A}_{21}\bar{V}_{u1} - \bar{A}_{11}\bar{V}_{u2}}{(\bar{A}_{21}\bar{A}_{12} - \bar{A}_{11}\bar{A}_{22})} \quad (7.51)$$

Substitute Equation 7.45 to Equation 7.48 into Equation 7.50 and Equation 7.51 to determine $\bar{M}_{b1}$ and $\bar{M}_{b2}$. Balancing masses that should be added are $-\bar{M}_{b1}$ and $-\bar{M}_{b2}$ in planes 1 and 2, respectively.

The single-plane and two-plane balancing approaches are summarized in [Box 7.3](#).

Example 7.4

Suppose that the following measurements are obtained.

Without trial mass:

Accelerometer at 1: amplitude = 10.0; phase lead = 55°.

Accelerometer at 2: amplitude = 7.0; phase lead = 120°

With trial mass 20 g at location 270° of plane 1:

Accelerometer at 1: amplitude = 7.0; phase lead = 120°

Accelerometer at 2: amplitude = 5.0; phase lead = 225°

With trial mass 25 g at location 180° of plane 2:

Accelerometer at 1: amplitude = 6.0; phase lead = 120°

Accelerometer at 2: amplitude = 12.0; phase lead = 170°

Determine the magnitude and orientation of the necessary balancing masses in planes 1 and 2 in order to completely balance (dynamic) the system.

Solution

In the phasor notation, we can represent the given data as follows:

$$\bar{V}_{u1} = 10.0 \angle 55^\circ; \quad \bar{V}_{u2} = 7.0 \angle 120^\circ$$

$$\bar{V}_{r11} = 7.0 \angle 120^\circ; \quad \bar{V}_{r21} = 5.0 \angle 225^\circ$$

$$\bar{V}_{r12} = 6.0 \angle 120^\circ; \quad \bar{V}_{r22} = 12.0 \angle 170^\circ$$

$$\bar{M}_{t1} = 20 \angle 270^\circ \text{ g}; \quad \bar{M}_{t2} = 25 \angle 180^\circ \text{ g}$$

From Equation 7.45 to Equation 7.48, we have

$$\bar{A}_{11} = \frac{7.0 \angle 120^\circ - 10.0 \angle 55^\circ}{20 \angle 270^\circ}; \quad \bar{A}_{21} = \frac{5.0 \angle 225^\circ - 7.0 \angle 120^\circ}{20 \angle 270^\circ}$$

$$\bar{A}_{12} = \frac{6.0 \angle 120^\circ - 10.0 \angle 55^\circ}{25 \angle 180^\circ}; \quad \bar{A}_{22} = \frac{12.0 \angle 170^\circ - 7.0 \angle 120^\circ}{25 \angle 180^\circ}$$

These phasors are computed as below:

$$\begin{aligned}\bar{A}_{11} &= \frac{(7.0 \cos 120^\circ - 10 \cos 55^\circ) + j(7 \sin 120^\circ - 10 \sin 55^\circ)}{20 \angle 270^\circ} = \frac{-9.235 - j2.129}{20 \angle 270^\circ} = \frac{9.477 \angle 193^\circ}{20 \angle 270^\circ} \\ &= 0.474 \angle -77^\circ\end{aligned}$$

Box 7.3

BALANCING OF ROTATING COMPONENTS

Static or single-plane balancing (balances a single equivalent dynamic force)

Experimental approach:

1. With respect to a body reference line of accelerometer signal at bearing, measure magnitude (V) and phase (ϕ):
 - (a) Without trial mass: (V_u, ϕ_1) or $\vec{V}_u = V_u \angle \phi_1$
 - (b) With trial mass M_t : (V_r, ϕ_2) or $\vec{V}_r = V_r \angle \phi_2$.
2. Compute balancing mass M_b and its location with respect to M_t .
3. Remove M_t and add M_b at determined location.

Computation approach 1:

$$V_w = [V_u^2 + V_r^2 - 2V_u V_w \cos(\phi_2 - \phi_1)]^{1/2}$$

$$\phi_b = \cos^{-1} \left[\frac{V_u^2 + V_w^2 - V_r^2}{2V_u V_w} \right] \text{ and } M_b = \frac{V_u}{V_w} M_t$$

Locate M_b at ϕ_b from M_t .

Computation approach 2:

Unbalance mass phasor

$$\vec{M}_b = \frac{\vec{V}_u}{(\vec{V}_r - \vec{V}_u)} \vec{M}_t$$

where $\vec{M}_t = M_t \angle \theta_t$ (trial mass phasor).

Locate balancing mass at $-\vec{M}_b$.

Dynamic or two-plane balancing (balances an equivalent dynamic force and a couple)

Experimental approach:

1. Measure $\vec{V}_{ui}$ at bearings $i = 1, 2$, with a trial mass.
2. Measure $\vec{V}_{rij}$ at bearings $i = 1, 2$, with only one trial mass $\vec{M}_{tj}$ at $j = 1, 2$.
3. Compute unbalance mass phasor $\vec{M}_{bi}$ in planes $i = 1, 2$.
4. Remove trial mass and place balancing masses $-\vec{M}_{bi}$ in planes $i = 1, 2$.

Computations:

Influence coefficients: $\vec{A}_{ij} = (\vec{V}_{rij} - \vec{V}_{ui})/\vec{M}_{tj}$

Unbalance mass phasors:

$$\vec{M}_{b1} = \frac{\vec{A}_{22} \vec{V}_{u1} - \vec{A}_{12} \vec{V}_{u2}}{(\vec{A}_{22} \vec{A}_{11} - \vec{A}_{12} \vec{A}_{21})} \text{ and } \vec{M}_{b2} = \frac{\vec{A}_{21} \vec{V}_{u1} - \vec{A}_{11} \vec{V}_{u2}}{(\vec{A}_{21} \vec{A}_{12} - \vec{A}_{11} \vec{A}_{22})}$$

$$\begin{aligned} \vec{A}_{21} &= \frac{(5 \cos 225^\circ - 7 \cos 120^\circ) + j(5 \sin 225^\circ - 7 \sin 120^\circ)}{20 \angle 270^\circ} = \frac{-7.036 - j9.6}{20 \angle 270^\circ} = \frac{11.9 \angle 234^\circ}{20 \angle 270^\circ} \\ &= 0.595 \angle -36^\circ \end{aligned}$$

$$\begin{aligned} \vec{A}_{12} &= \frac{(6 \cos 120^\circ - 10 \cos 55^\circ) + j(6 \sin 120^\circ - 10 \sin 55^\circ)}{25 \angle 180^\circ} = \frac{-8.736 - j3.0}{25 \angle 180^\circ} = \frac{9.237 \angle 199^\circ}{25 \angle 180^\circ} \\ &= 0.369 \angle 19^\circ \end{aligned}$$

$$\begin{aligned}\bar{A}_{22} &= \frac{(12 \cos 170^\circ - 7 \cos 120^\circ) + j(12 \sin 170^\circ - 7 \sin 120^\circ)}{25 \angle 180^\circ} = \frac{-8.318 - j4.0}{25 \angle 180^\circ} = \frac{9.23 \angle 205.7^\circ}{25 \angle 180^\circ} \\ &= 0.369 \angle 25.7^\circ\end{aligned}$$

Next, the denominators of the balancing mass phasors (in Equation 7.50 and Equation 7.51) are computed as

$$\begin{aligned}\bar{A}_{22}\bar{A}_{11} - \bar{A}_{12}\bar{A}_{21} &= (0.369 \angle 25.7^\circ \times 0.474 \angle -77^\circ) - (0.369 \angle 19^\circ \times 0.595 \angle -36^\circ) \\ &= 0.1749 \angle -51.3^\circ - 0.2196 \angle -17^\circ \\ &= (0.1749 \cos 51.3^\circ - 0.2196 \cos 17^\circ) - j(0.1749 \sin 51.3^\circ - 0.2196 \sin 17^\circ) \\ &= -0.1 - j0.0723 = 0.1234 \angle 216^\circ\end{aligned}$$

and, hence

$$-(\bar{A}_{22}\bar{A}_{11} - \bar{A}_{12}\bar{A}_{21}) = 0.1234 \angle 36^\circ$$

Finally, the balancing mass phasors are computed using Equation 7.50 and Equation 7.51 as

$$\begin{aligned}\bar{M}_{b1} &= \frac{0.369 \angle 25.7^\circ \times 10 \angle 55^\circ - 0.369 \angle 19^\circ \times 7.0 \angle 120^\circ}{0.1234 \angle 216^\circ} = \frac{3.69 \angle 80.7^\circ - 2.583 \angle 139^\circ}{0.1234 \angle 216^\circ} \\ &= \frac{(3.69 \cos 80.7^\circ - 2.583 \cos 139^\circ) + j(3.69 \sin 80.7^\circ - 2.583 \sin 139^\circ)}{0.1234 \angle 216^\circ} = \frac{2.546 + j1.947}{0.1234 \angle 216^\circ} \\ &= \frac{3.205 \angle 37.4^\circ}{0.1234 \angle 216^\circ} = 26 \angle -178.6^\circ \\ \bar{M}_{b2} &= \frac{0.595 \angle -36^\circ \times 10 \angle 55^\circ - 0.474 \angle -77^\circ \times 7.0 \angle 120^\circ}{0.1234 \angle 36^\circ} = \frac{5.95 \angle 19^\circ - 3.318 \angle 43^\circ}{0.1234 \angle 36^\circ} \\ &= \frac{(5.95 \cos 19^\circ - 3.318 \cos 43^\circ) - j(5.95 \sin 19^\circ - 3.318 \sin 43^\circ)}{0.1234 \angle 36^\circ} = \frac{3.2 + j0.326}{0.1234 \angle 36^\circ} \\ &= \frac{1.043 \angle 5.8^\circ}{0.1234 \angle 36^\circ} = 8.45 \angle -30.0^\circ\end{aligned}$$

Finally, we have

$$-\bar{M}_{b1} = 26 \angle 1.4^\circ \text{ g}; \quad -\bar{M}_{b2} = 8.45 \angle 150^\circ \text{ g}$$

7.4.4 Experimental Procedure of Balancing

The experimental procedure for determining the balancing masses and locations for a rotating system should be clear from the analytical developments and examples given above. The basic steps are:

1. Determine the magnitude and the phase angle of accelerometer signals at the bearings with and without trial masses at the bearing planes.
2. Using this data, compute the necessary balancing masses (magnitude and location) at the bearing planes.
3. Place the balancing masses.
4. Check whether the system is balanced. If not, repeat the balancing cycle.

A laboratory experimental setup for two-plane balancing is shown schematically in Figure 7.14. A view of the system is shown in Figure 7.15. The two disks rigidly mounted on the shaft are driven by a DC motor. The drive speed of the motor is adjusted by the manual speed controller. The (two) shaft

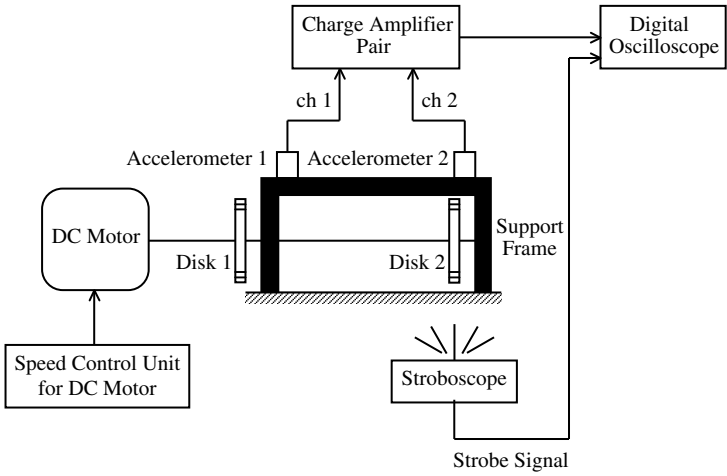


FIGURE 7.14 Schematic arrangement of a rotor balancing experiment.

bearings are located very close to the disks, as shown in Figure 7.14. Two accelerometers are mounted on the top of the bearing housing so that the resulting vertical accelerations can be measured. The accelerometer signals are conditioned using the two-channel charge amplifier and read and displayed through two channels of the digital oscilloscope. The output of the stroboscope (tachometer) is used as the reference signal with respect to the phase angles of the accelerometer signals that are measured.

In Figure 7.15, the items of equipment are seen from left to right. The first item is the two-channel digital oscilloscope. The manual speed controller with control knob for the DC motor follows. Next is the pair of charge amplifiers for the accelerometers. The strobe-light unit (strobe-tacho) is placed on top of the common housing of the charge amplifier pair. The two-disk rotor system with the drive motor is shown as the last item to the right. Also, note the two accelerometers (seen as small vertical projections) mounted on the bearing frame of the shaft directly above the two bearings.

Because this reference always has to be fixed prior to reading the oscilloscope data, the strobe-tacho is synchronized with the disk rotation. This is achieved as follows (note that all the readings are taken with

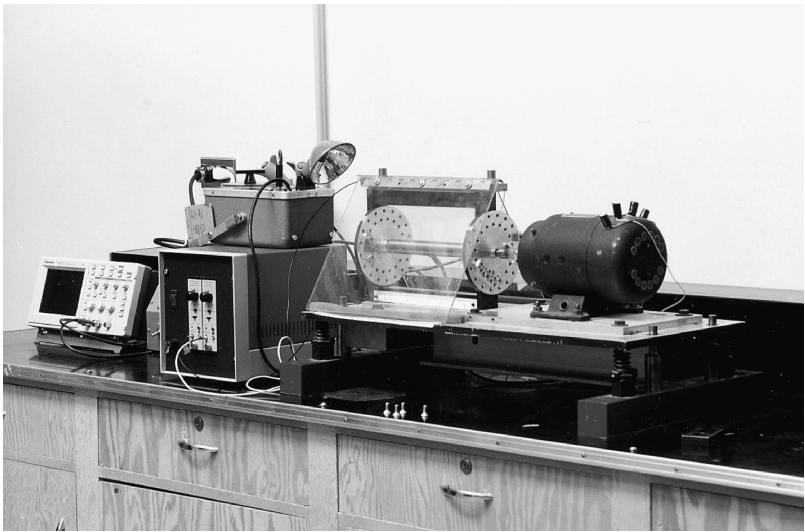


FIGURE 7.15 A view of the experimental setup for two-plane balancing at the University of British Columbia.

the same rotating speed, which is adjusted by the manual speed controller): First, make a physical mark (e.g., a black spot in a white background) on one of the disks. Aim the strobe flash at this disk. As the motor speed is adjusted to the required fixed value, the strobe flash is synchronized such that the mark on the disk “appears” stationary at the same location (e.g., at the uppermost location of the circle of rotation). This ensures not only that the strobe frequency is equal to the rotating speed of the disk, but also that the same phase angle reference is used for all readings of accelerometer signals.

The two disks have slots at locations whose radius is known, and whose angular positions in relation to a body reference line (a radius representing the 0° reference line) are clearly marked. Known masses (typically, bolts and nuts of known mass) can be securely mounted in these slots. Readings obtained through the oscilloscope are:

1. Amplitude of each accelerometer signal
2. Phase lead of the accelerometer signal with respect to the synchronized and reference-fixed strobe signal (note: a phase lag should be represented by a negative sign in the data)

The measurements taken and the computations made in the experimental procedure should be clear from Example 7.4.

7.5 Balancing of Reciprocating Machines

A reciprocating mechanism has a slider that moves rectilinearly back and forth along some guideway. A piston-cylinder device is a good example. Often, reciprocating machines contain rotatory components in addition to the reciprocating mechanisms. The purpose is to either convert a reciprocating motion to a rotary motions (as in the case of an automobile engine), or to convert a rotary motion to a reciprocating motions (as in the opto-slider mechanism of a photocopier). Irrespective of the reciprocating machine employed, it is important to remove the vibratory excitations that arise in order to realize the standard design goals of smooth operation, accuracy, low noise, reliability, mechanical integrity, and extended service life. Naturally, in view of their rotational asymmetry, reciprocating mechanisms with rotary components are more prone to unbalance than purely rotary components. Removing the “source of vibration” by proper balancing of the machine would be especially applicable in this situation.

7.5.1 Single-Cylinder Engine

A practical example of a reciprocating machine with integral rotary motion is the internal combustion (IC) engine of an automobile. A single-cylinder engine is sketched in Figure 7.16. Observe the nomenclature of the components. The reciprocating motion of the piston is transmitted through the connecting rod and crank into a rotatory motion of the crankshaft. The crank, as sketched in Figure 7.16, has a counterbalance mass, the purpose of which is to balance the rotary force (centrifugal).

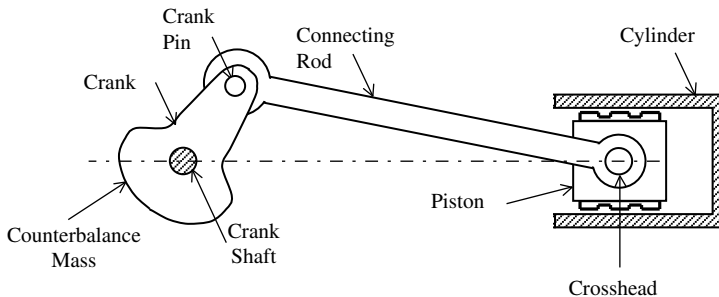


FIGURE 7.16 A single-cylinder reciprocating engine.

We will ignore this in our analysis because the goal is to determine the unbalance forces and ways to balance them.

Clearly, both the connecting rod and the crank have distributed mass and moment of inertia. To simplify the analysis, we approximate as follows:

1. Represent the crank mass by an equivalent lumped mass at the crank pin (equivalence may be based on either centrifugal force or kinetic energy).
2. Represent the mass of the connecting rod by two lumped masses, one at the crank pin and the other at the cross head (piston pin).

The piston itself has a significant mass, which is also lumped at the crosshead. Hence, the equivalent system has a crank and a connecting rod, both of which are considered massless, with a lumped mass m_c at the crank pin and another lumped mass m_p at the piston pin (crosshead).

Furthermore, under normal operation, the crankshaft rotates at a constant angular speed (ω). Note that this steady speed is realized not by natural dynamics of the system, but rather by proper speed control (a topic which is beyond the scope of the present discussion).

It is a simple matter to balance the lumped mass m_c at the crank pin. Simply place a counterweight m_c at the same radius in the radially opposite location (or a mass in inverse proportion to the radial distance from the crankshaft, but remaining in the radially opposite direction). This explains the presence of the counterweight in the crank shown in Figure 7.16. Once complete balancing of the rotating inertia (m_c) is thus achieved, we still need to completely eliminate the effect of the vibration source on the crankshaft. To achieve this, we must compensate for the forces and moments on the crankshaft that result from:

1. The reciprocating motion of the lumped mass m_p
2. Time-varying combustion (gas) pressure in the cylinder

Both types of forces act on the piston in the direction of its reciprocating (rectilinear) motion. Hence, their influence on the crankshaft can be analyzed in the same way, except that the combustion pressure is much more difficult to determine.

The above discussion justifies the use of the simplified model shown in Figure 7.17 for analyzing the balancing of a reciprocating machine. The characteristics of this model are as follows:

1. A light crank OC of radius r rotates at constant angular speed ω about O, which is the origin of the x - y coordinate frame.
2. A light connecting rod CP of length l is connected to the connecting rod at C and to the piston at P with frictionless pins. Since the rod is light and the joints are frictionless, the force f_c supported by it will act along its length. (Assume that the force f_c in the connecting rod is compressive, for the purpose of the sign convention). Connecting rod makes an angle ϕ with OP (the negative x axis).
3. A lumped mass m_p is present at the piston. A force f acts at P in the negative x direction. This may be interpreted as either the force due to the gas pressure in the cylinder or the inertia force $m_p a$ where a is the acceleration m_p in the positive x direction. These two cases of forcing are considered separately.
4. A lateral force f_l acts on the piston by the cylinder wall, in the positive y direction.

Again, note that the lumped mass m_c at C is not included in the model of Figure 7.17 because it is assumed to be completely balanced by a counterweight in the crank. Furthermore, the lumped mass m_p includes both the mass of the piston and also part of the inertia of the connecting rod.

There are no external forces at C. Furthermore, the only external forces at P are f and f_l , where f is interpreted as either the inertia force in m_p or the gas force on the piston. Hence, there should be

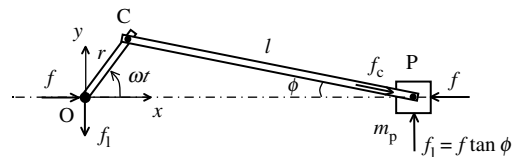


FIGURE 7.17 The model used to analyze balancing of a reciprocating engine.

equal and opposite forces at the crankshaft O, as shown in Figure 7.17, to support the forces acting at P. Now, let us determine f_1 .

Equilibrium at P gives

$$f = f_c \cos \phi$$

$$f_1 = f_c \sin \phi$$

Hence,

$$f_1 = f \tan \phi \quad (7.52)$$

This lateral force f_1 acting at both O and P, albeit in the opposite directions, forms a couple $\tau = xf_1$ or, in view of Equation 7.52:

$$\tau = xf \tan \phi \quad (7.53)$$

This couple acts as a torque on the crankshaft. It follows that, once the rotating inertia m_c at the crank is completely balanced by a counter mass, the load at the crankshaft is due only to the piston load f and it consists of:

1. A force f in the direction of the piston motion (x)
2. A torque $\tau = xf \tan \phi$ in the direction of rotation of the crankshaft (z)

As discussed below, the means of removing f at the crankshaft will also remove τ to some extent. Hence, we will discuss only the approach of balancing f .

7.5.2 Balancing the Inertia Load of the Piston

First, consider the inertia force f due to m_p . Here,

$$f = m_p a \quad (7.54)$$

where a is the acceleration $\ddot{x}$, with the coordinate x locating the position P of the piston (in other words $OP = x$). We notice from Figure 7.17 that

$$x = r \cos \omega t + l \cos \phi \quad (7.55)$$

However,

$$r \sin \omega t = l \sin \phi \quad (7.56)$$

Hence,

$$\cos \phi = \left[1 - \left(\frac{r}{l} \right)^2 \sin^2 \omega t \right]^{1/2} \quad (7.57)$$

which can be expanded up to the first term of Taylor series as

$$\cos \phi \cong 1 - \frac{1}{2} \left(\frac{r}{l} \right)^2 \sin^2 \omega t \quad (7.58)$$

This approximation is valid because l is usually several times larger than r and, hence, $(r/l)^2$ is much small than unity. Next, in view of

$$\sin^2 \omega t = \frac{1}{2} [1 - \cos 2\omega t] \quad (7.59)$$

we have

$$\cos \phi \cong 1 - \frac{1}{4} \left(\frac{r}{l} \right)^2 [1 - \cos 2\omega t] \quad (7.60)$$

Substitute Equation 7.60 into Equation 7.55. We get, approximately

$$x = r \cos \omega t + \frac{l}{4} \left(\frac{r}{l} \right)^2 \cos 2\omega t + l - \frac{l}{4} \left(\frac{r}{l} \right)^2 \quad (7.61)$$

Differentiate Equation 7.61 twice with respect to t to get the acceleration

$$a = \ddot{x} = -r\omega^2 \cos \omega t - l \left(\frac{r}{l} \right)^2 \omega^2 \cos 2\omega t \quad (7.62)$$

Hence, from Equation 7.54, the inertia force at the piston (and its reaction at the crankshaft) is

$$f = -m_p r \omega^2 \cos \omega t - m_p l \left(\frac{r}{l} \right)^2 \omega^2 \cos 2\omega t \quad (7.63)$$

It follows that the inertia load of the reciprocating piston exerts a vibratory force on the crankshaft which has a *primary component* of frequency ω and a smaller *secondary component* of frequency 2ω , where ω is the angular speed of the crank. The primary component has the same form as that created by a rotating lumped mass at the crank pin. However, unlike the case of a rotating mass, this vibrating force acts only in the x direction (there is no $\sin \omega t$ component in the y direction) and, hence, cannot be balanced by a rotating counter mass. Similarly, the secondary component cannot be balanced by a counter mass rotating at double the speed. To eliminate f , we use multiple cylinders whose connecting rods and cranks are connected to the crankshaft with their rotations properly phased (delayed), thus canceling out the effects of f .

7.5.3 Multicylinder Engines

A single-cylinder engine generates a primary component and a secondary component of vibration load at the crankshaft, and they act in the direction of piston motion (x). Because there is no complementary orthogonal component (y), it is inherently unbalanced and cannot be balanced using a rotating mass. It can be balanced, however, by using several piston-cylinder units with their cranks properly phased along the crankshaft. This method of balancing multicylinder reciprocating engines is addressed now.

Consider a single cylinder whose piston inertia generates a force f at the crankshaft in the x direction given by

$$f = f_p \cos \omega t + f_s \cos 2\omega t \quad (7.64)$$

Note that the primary and secondary forcing amplitudes f_p and f_s , respectively, are given by Equation 7.63. Suppose that there is a series of cylinders in parallel, arranged along the crankshaft, and the crank of cylinder i makes an angle α_i with the crank of cylinder 1 in the direction of rotation, as schematically shown in Figure 7.18(a). Hence, force f_i on the crankshaft (in the x direction, shown as vertical in Figure 7.18) due to cylinder i is

$$f_i = f_p \cos(\omega t + \alpha_i) + f_s \cos(2\omega t + 2\alpha_i) \quad \text{for } i = 1, 2, \dots, \text{ with } \alpha_1 = 0 \quad (7.65)$$

Not only do the cranks need to be properly phased, but the cylinders should also be properly spaced along the crankshaft to obtain the necessary balance. Consider two examples.

7.5.3.1 Two-Cylinder Engine

Consider the two-cylinder case, as shown schematically in Figure 7.18(b) where the two cranks are in radially opposite orientations (i.e., 180° out of phase). In this case, $\alpha_2 = \pi$. Hence,

$$f_1 = f_p \cos \omega t + f_s \cos 2\omega t \quad (7.66)$$

$$f_2 = f_p \cos(\omega t + \pi) + f_s \cos(2\omega t + 2\pi) = -f_p \cos \omega t + f_s \cos 2\omega t \quad (7.67)$$

It follows that the primary force components cancel out. However, they form a couple $z_0 f_p \cos \omega t$ where z_0 is the spacing of the cylinders. This causes a bending moment on the crankshaft, and it will not vanish

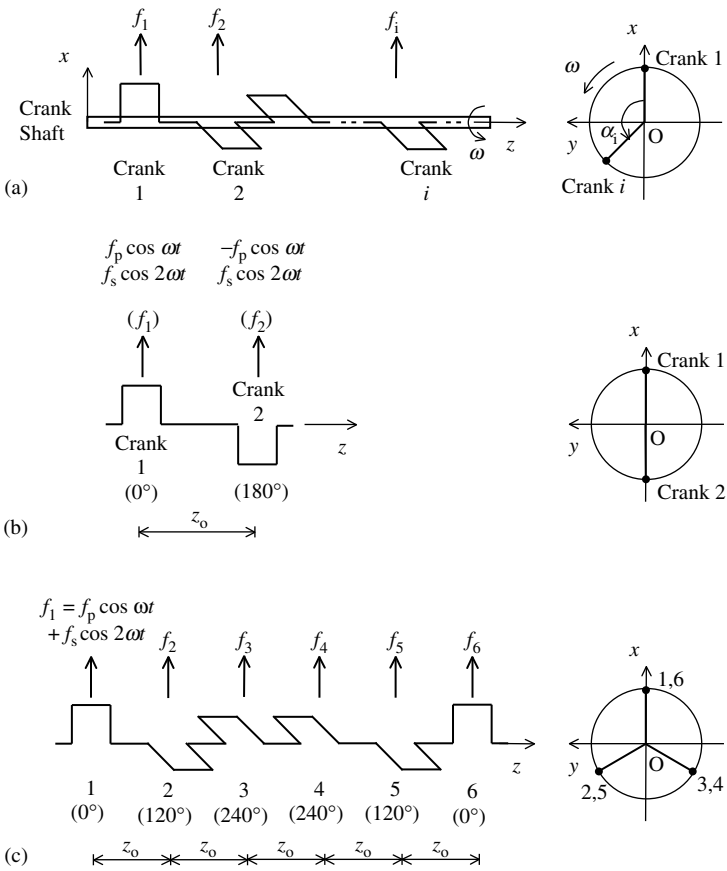


FIGURE 7.18 (a) Crank arrangement of a multicylinder engine; (b) two-cylinder engine; (c) six-cylinder engine (balanced).

unless the two cylinders are located at the same point along the crankshaft. Furthermore, the secondary components are equal and additive to $2f_s \cos 2\omega t$. This resultant component acts at the midpoint of the crankshaft segment between the two cylinders. There is no couple due to the secondary components.

7.5.3.2 Six-Cylinder Engine

Consider the six-cylinder arrangement shown schematically in Figure 7.18(c). Here, the cranks are arranged such that $\alpha_2 = \alpha_5 = 2\pi/3$, $\alpha_3 = \alpha_4 = 4\pi/3$, and $\alpha_1 = \alpha_6 = 0$. Furthermore, the cylinders are equally spaced, with spacing z_0 . In this case, we have

$$f_1 = f_6 = f_p \cos \omega t + f_s \cos 2\omega t \tag{i}$$

$$f_2 = f_5 = f_p \cos(\omega t + 2\pi/3) + f_s \cos(2\omega t + 4\pi/3) \tag{ii}$$

$$f_3 = f_4 = f_p \cos(\omega t + 4\pi/3) + f_s \cos(2\omega t + 8\pi/3) \tag{iii}$$

Now, we use the fact that

$$\cos \theta + \cos\left(\theta + \frac{2\pi}{3}\right) + \cos\left(\theta + \frac{4\pi}{3}\right) = 0 \tag{iv}$$

which may be proved either by straightforward trigonometric expansion or by using geometric interpretation (i.e., three sides of an equilateral triangle, the sum of whose components in any

direction vanishes). The relation iv holds for any θ , including $\theta = \omega t$ and $\theta = 2\omega t$. Furthermore,

$$\cos(2\omega t + 8\pi/3) = \cos\left(2\omega t + \frac{2\pi}{3}\right)$$

Thus, from Equation i to Equation iii, we can conclude that

$$f_1 + f_2 + f_3 + f_4 + f_5 + f_6 = 0 \quad (7.68)$$

This means that the lateral forces on the crankshaft that are exerted by the six cylinders will completely balance. Furthermore, by taking moments about the location of crank 1 of the crankshaft, we have

$$\begin{aligned} & (z_0 + 4z_0) \left[f_p \cos\left(\omega t + \frac{2\pi}{3}\right) + f_s \cos\left(2\omega t + \frac{4\pi}{3}\right) \right] \\ & + (2z_0 + 3z_0) \left[f_p \cos\left(\omega t + \frac{4\pi}{3}\right) + f_s \cos\left(2\omega t + \frac{8\pi}{3}\right) \right] \\ & + 5z_0 [f_p \cos \omega t + f_s \cos 2\omega t] \end{aligned} \quad (v)$$

which also vanishes in view of relation iv. Hence, the set of six forces is in complete equilibrium and, as a result, there will be neither a reaction force nor a bending moment on the bearings of the crankshaft from these forces.

In addition, it can be shown that the torques $x_i f_i \tan \phi_i$ on the crankshaft due to this set of inertial forces f_i will add to zero, where x_i is the distance from the crankshaft to the piston of the i th cylinder and ϕ_i is the angle ϕ of the connecting rod of the i th cylinder. Hence, this six-cylinder configuration is in complete balance with respect to the inertial load.

Example 7.5

An eight-cylinder in-line engine (with identical cylinders that are placed in parallel along a line) has its cranks arranged according to the phasing angles 0, 180, 90, 270, 270, 90, 180, and 0° on the crankshaft. The cranks (cylinders) are equally spaced, with spacing z_0 . Show that this engine is balanced with respect to primary and secondary components of reaction forces and bending moments of inertial loading on the bearings of the crankshaft.

Solution

The sum of the reaction forces on the crankshaft are

$$\begin{aligned} & 2 \left[f_p \cos \omega t + f_s \cos 2\omega t + f_p \cos(\omega t + \pi) + f_s \cos(2\omega t + 2\pi) + f_p \cos\left(\omega t + \frac{\pi}{2}\right) + f_s \cos(2\omega t + \pi) \right. \\ & \left. + f_p \cos\left(\omega t + \frac{3\pi}{2}\right) + f_s \cos(2\omega t + 3\pi) \right] = 2 \left[f_p \cos \omega t - f_p \cos \omega t - f_p \sin \omega t + f_p \sin \omega t \right. \\ & \left. + f_s \cos 2\omega t + f_s \cos 2\omega t - f_s \cos 2\omega t - f_s \cos 2\omega t \right] = 0 \end{aligned}$$

Hence, both primary forces and secondary forces are balanced. The moment of the reaction forces about the crank 1 location of the crankshaft is

$$\begin{aligned} & (z_0 + 6z_0) [f_p \cos(\omega t + \pi) + f_s \cos(2\omega t + 2\pi)] + (2z_0 + 5z_0) \left[f_p \cos\left(\omega t + \frac{\pi}{2}\right) + f_s \cos(2\omega t + \pi) \right] \\ & + (3z_0 + 4z_0) \left[f_p \cos\left(\omega t + \frac{3\pi}{2}\right) + f_s \cos(2\omega t + 3\pi) \right] + 7z_0 [f_p \cos \omega t + f_s \cos 2\omega t] \\ & = 7z_0 [-f_p \cos \omega t + f_s \cos 2\omega t - f_p \sin \omega t - f_s \cos 2\omega t + f_p \sin \omega t - f_s \cos 2\omega t + f_p \cos \omega t + f_s \cos 2\omega t] = 0 \end{aligned}$$

Hence, both primary bending moments and secondary bending moments are balanced. Therefore, the engine is completely balanced.

The formulas applicable for balancing reciprocating machines are summarized in [Box 7.4](#).

Box 7.4

BALANCING OF RECIPROCATING MACHINES

Single cylinder engine:

Inertia force at piston (and its reaction on crankshaft)

$$f = -m_p r \omega^2 \cos \omega t - m_p l \left(\frac{r}{l} \right)^2 \omega^2 \cos 2\omega t = f_p \cos \omega t + f_s \cos 2\omega t$$

where

ω = rotating speed of crank

m_p = equivalent lumped mass at piston

r = crank radius

l = length of connecting rod

f_p = amplitude of the primary unbalance force (frequency ω)

f_s = amplitude of the secondary unbalance force (frequency 2ω)

Multicylinder engine:

Net unbalance reaction force on crankshaft = $\sum_{i=1}^n f_i$

Net unbalance moment on crankshaft = $\sum_{i=1}^n z_i f_i$

where

$$f_i = f_p \cos(\omega t + \alpha_i) + f_s \cos(2\omega t + 2\alpha_i)$$

α_i = angular position of the crank of i th cylinder, with respect to a body (rotating) reference (i.e., crank phasing angle)

z_i = position of the i th crank along the crankshaft, measured from a reference point on the shaft

n = number of cylinders (assumed identical)

Note: For a completely balanced engine, both the net unbalance force and the net unbalance moment should vanish.

Finally, it should be noted that, in the configuration considered above, the cylinders are placed in parallel along the crankshaft. These are termed in-line engines. Their resulting forces f_i act in parallel along the shaft. In other configurations such as V6 and V8, the cylinders are placed symmetrically around the shaft. In this case, the cylinders (and their inertial forces, which act on the crankshaft) are not parallel. Here, a complete force balance may be achieved without having to phase the cranks. Furthermore, the bending moments of the forces can be reduced by placing the cylinders at nearly the same location along the crankshaft. Complete balancing of the combustion/pressure forces is also possible by such an arrangement.

7.5.4 Combustion/Pressure Load

In the balancing approach presented above, the force f on the piston represents the inertia force due to the equivalent reciprocating mass. Its effect on the crankshaft is an equal reaction force f in the lateral direction (x) and a torque $\pi = x f \tan \phi$ about the shaft axis (z). The balancing approach is to use a series of cylinders so that their reaction forces f_i on the crankshaft from an equilibrium set so that no net reaction or bending moment is transmitted to the bearings of the shaft. The torques τ_i also can be balanced by the same approach, which is the case, for example, in the six-cylinder engine.

Another important force that acts along the direction of piston reciprocation is the drive force due to gas pressure in the cylinder (e.g., created by combustion of the fuel–air mixture of an internal combustion engine). As above, this force may be analyzed by denoting it as f . However, several important observations should be made first:

1. The combustion force f is not sinusoidal of frequency ω . It is reasonably periodic but the shape is complex and depends on the firing/fuel-injection cycle and the associated combustion process.
2. The reaction forces f_i on the crankshaft, which are generated from cylinders i , should be balanced to avoid the transmission of reaction forces and bending moments to the shaft bearings (and hence, to the supporting frame — the vehicle). However, the torques τ_i in this case are in fact the drive torques. Obviously, they are the desired output of the engine and should not be balanced, unlike the inertia torques.

Therefore, although the analysis completed for balancing the inertia forces cannot be directly used here, we can employ similar approaches to the use of multiple cylinders for reducing the gas-force reactions. This is a rather difficult problem given the complexity of the combustion process itself. In practice, much of the leftover effects of the ignition cycle are suppressed by properly designed engine mounts. Experimental investigations have indicated that in a properly balanced engine unit, much of the vibration transmitted through the engine mounts is caused by the engine firing cycle (internal combustion) rather than the reciprocating inertia (sinusoidal components of frequency ω and 2ω). Hence, active mounts, where stiffness can be varied according to the frequency of excitation, are being considered to reduce engine vibrations in the entire range of operating speeds (e.g. 500 to 2500 rpm).

7.6 Whirling of Shafts

In the previous two sections, we studied the vibration excitations caused on rotating shafts and their bearings due to some form of mass eccentricity. Methods of balancing these systems to eliminate the undesirable effects were also presented. One limitation of the given analysis is the assumptions that the rotating shaft is rigid and, thus, does not deflect from its axis of rotation due to the unbalanced excitations. In practice, however, rotating shafts are made lighter than the components they carry (rotors, disks, gears, etc.) and will undergo some deflection due to the unbalanced loading. As a result, the shaft will bow out and this will further increase the mass eccentricity and associated unbalanced excitations and gyroscopic forces of the rotating elements (disks, rotors, etc.). The nature of damping of rotating machinery (which is rather complex and incorporates effects of rotation at bearings, structural deflections, and lateral speeds) will further affect the dynamic behavior of the shaft under these conditions. In this context, the topic of whirling of rotating shafts becomes relevant.

Consider a shaft that is driven at a constant angular speed ω (e.g., by using a motor or some other actuator). The central axis of the shaft (passing through its bearings) will bow out. The deflected axis itself will rotate, and this rotation is termed *whirling* or *whipping*. The whirling speed is not necessarily equal to the drive speed ω (at which the shaft rotates about its axis with respect to a fixed frame). However, when the whirling speed is equal to ω , the condition is called *synchronous whirl*, and the associated deflection of the shaft can be quite excessive and damaging.

To develop an analytical basis for whirling, consider a light shaft supported on two bearings carrying a disk of mass m in between the bearings, as shown in Figure 7.19(a). Note that C is the point on the disk at which it is mounted on the shaft. Originally, in the neutral configuration when the shaft is not driven ($\omega = 0$), the point C coincides with point O on the axis joining the two bearings. If the shaft were rigid, then the points C and O would continue to coincide during motion. The mass center (centroid or center of gravity for constant g) of the disk is denoted as G in Figure 7.19. During motion, C will move away from O due to the shaft deflection. The whirling speed (speed of rotation of the shaft axis) is the speed of rotation of the radial line OC with respect to a fixed reference. Denoting the angle of OC with respect to a fixed reference as θ , the whirling speed is $\dot{\theta}$. This is explained in Figure 7.19(b) where an end view of the

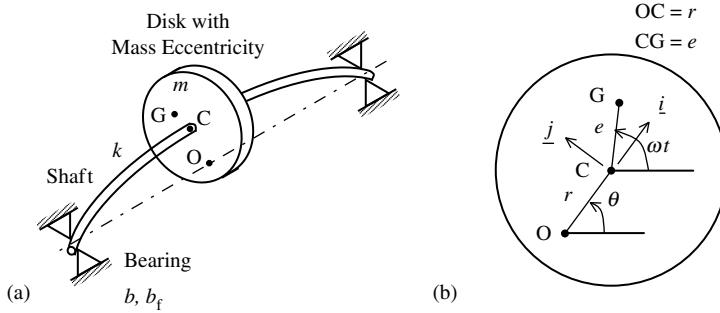


FIGURE 7.19 (a) A whirling shaft carrying a disk with mass eccentricity; (b) end view of the disk and whirling shaft.

disk is given under deflected conditions. The constant drive speed ω of the shaft is the speed of the shaft spin with respect to a fixed reference, and is the speed of rotation of the radial line CG with respect to the fixed horizontal line shown in Figure 7.19(b). Hence, the angle of shaft spin is ωt , as measured with respect to this line. The angle of whirl, θ , is also measured from the direction of this fixed line, as shown.

7.6.1 Equations of Motion

Under practical conditions, the disk moves entirely in a single plane. Hence, its complete set of equations of motion consists of two equations for translatory (planar) motion of the centroid (with lumped mass m), and one equation for rotational motion about the fixed bearing axis. The latter equation depends on the motor torque that drives the shaft at constant speed ω , and is not of interest in the present context. Thus, we will limit our development to the two translatory equations of motion. The equations may be written either in a Cartesian coordinate system (x, y) or a polar coordinate system (r, θ) . Here, we will use the polar coordinate system.

Consider a coordinate frame $(\mathbf{i}, \mathbf{j})$ that is fixed to the disk with its $\mathbf{i}$ axis lying along OC as shown in Figure 7.19(b). Note that the angular speed of this frame is $\dot{\theta}$ (about the $\mathbf{k}$ axis that is orthogonal to $\mathbf{i}$ and $\mathbf{j}$). Hence, as is well known, we have

$$\frac{d\mathbf{i}}{dt} = \dot{\theta}\mathbf{j} \quad \text{and} \quad \frac{d\mathbf{j}}{dt} = -\dot{\theta}\mathbf{i} \quad (7.69)$$

The position vector of the mass point G from O is

$$\vec{OG} = \vec{r}_G = \vec{OC} + \vec{CG} = r\mathbf{i} + e \cos(\omega t - \theta)\mathbf{i} + e \sin(\omega t - \theta)\mathbf{j} \quad (7.70)$$

The velocity vector $\mathbf{v}_G$ of the mass point G can be obtained simply by differentiating Equation 7.70 with the use of Equation 7.69. However, this can be simplified because ω is constant. Here, line CG has a velocity $e\omega$ that is perpendicular to it about C . This can be resolved along the axes $\mathbf{i}$ and $\mathbf{j}$. Hence, the velocity of G relative to C is

$$\mathbf{v}_{G/C} = -e\omega \sin(\omega t - \theta)\mathbf{i} + e\omega \cos(\omega t - \theta)\mathbf{j}$$

However, the velocity of point C is

$$\mathbf{v}_C = \frac{d}{dt}r\mathbf{i} = \dot{r}\mathbf{i} + r\frac{d\mathbf{i}}{dt} = \dot{r}\mathbf{i} + r\dot{\theta}\mathbf{j}$$

Hence, the velocity of G , which is given by $\mathbf{v}_G = \mathbf{v}_C + \mathbf{v}_{G/C}$, can be expressed as

$$\mathbf{v}_G = \dot{r}\mathbf{i} + r\dot{\theta}\mathbf{j} - e\omega \sin(\omega t - \theta)\mathbf{i} + e\omega \cos(\omega t - \theta)\mathbf{j} \quad (7.71)$$

Similarly, the acceleration of C is

$$\mathbf{a}_C = \frac{d}{dt} \mathbf{v}_C = \frac{d}{dt} [\dot{r}\mathbf{i} + r\dot{\theta}\mathbf{j}] = \ddot{r}\mathbf{i} + \dot{r}\dot{\theta}\mathbf{j} + \dot{r}\dot{\theta}\mathbf{j} + r\ddot{\theta} - r\dot{\theta}^2\mathbf{i} = (\ddot{r} - r\dot{\theta}^2)\mathbf{i} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{j}$$

Also, since line CG rotates at constant angular speed ω about C, the point G has only a radial (centrifugal) acceleration $e\omega^2$ along GC. This can be resolved along $\mathbf{i}$ and $\mathbf{j}$ as before. Hence, the acceleration of G relative to C is

$$\mathbf{a}_{G/C} = -e\omega^2 \cos(\omega t - \theta)\mathbf{i} - e\omega^2 \sin(\omega t - \theta)\mathbf{j}$$

It follows that the acceleration of point G, given by $\mathbf{a}_G = \mathbf{a}_C + \mathbf{a}_{G/C}$, may be expressed as

$$\mathbf{a}_G = (\ddot{r} - r\dot{\theta}^2)\mathbf{i} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{j} - e\omega^2 \cos(\omega t - \theta)\mathbf{i} - e\omega^2 \sin(\omega t - \theta)\mathbf{j} \quad (7.72)$$

The forces acting on the disk are as follows:

Restraining elastic force due to lateral deflection of the shaft = $-k\mathbf{r}\mathbf{i}$

Viscous damping force (proportional to the velocity of C) = $-b\dot{r}\mathbf{i} - br\dot{\theta}\mathbf{j}$

In addition, there is a frictional resistance at the bearing, which is proportional to the reaction and, hence, the shaft deflection is r and also depends on the spin speed ω . The following approximate model may be used:

$$\text{Bearing friction force} = -b_f r \omega \mathbf{j}$$

Here,

k = lateral deflection stiffness of the shaft at the location of the disk

b = viscous damping constant for lateral motion of the shaft

b_f = bearing frictional coefficient

The overall force acting on the disk is

$$\mathbf{f} = -b\dot{r}\mathbf{i} - (br\dot{\theta} + b_f r \omega)\mathbf{j} \quad (7.73)$$

The equation of rectilinear motion

$$\mathbf{f} = m\mathbf{a}_G \quad (7.74)$$

on using Equation 7.72 and Equation 7.73, reduces to the following pair in the $\mathbf{i}$ and $\mathbf{j}$ directions:

$$-kr - b\dot{r} = m[\ddot{r} - r\dot{\theta}^2 - e\omega^2 \cos(\omega t - \theta)] \quad (7.75)$$

$$-br\dot{\theta} - b_f r \omega = m[r\ddot{\theta} + 2\dot{r}\dot{\theta} - e\omega^2 \sin(\omega t - \theta)] \quad (7.76)$$

These equations may be expressed as

$$\ddot{r} + 2\zeta_v \omega_n \dot{r} + (\omega_n^2 - \dot{\theta}^2)r = e\omega^2 \cos(\omega t - \theta) \quad (7.77)$$

$$r\ddot{\theta} + 2(\zeta_v \omega_n r + \dot{r})\dot{\theta} + 2\zeta_f \omega_n \omega r = e\omega^2 \sin(\omega t - \theta) \quad (7.78)$$

where the undamped natural frequency of lateral vibration is

$$\omega_n = \sqrt{\frac{k}{m}} \quad (7.79)$$

and

ζ_v = viscous damping ratio of lateral motion

ζ_f = frictional damping ratio of the bearings

Equation 7.77 and Equation 7.78, which govern the whirling motion of the shaft-disk system, are a pair of coupled nonlinear equations, with excitations (depending on ω) that are coupled with a motion variable (θ). Hence, a general solution would be rather complex. A relatively simple solution is possible, however, under steady-state whirling.

7.6.2 Steady-State Whirling

Under steady-state conditions, the whirling speed $\dot{\theta}$ is constant at $\dot{\theta} = \omega_w$, hence, $\ddot{\theta} = 0$. Also, the lateral deflection of the shaft is constant, hence, $\dot{r} = \ddot{r} = 0$. Therefore, Equation 7.77 and Equation 7.78 become

$$(\omega_n^2 - \omega_w^2)r = e\omega^2 \cos(\omega t - \theta) \quad (7.80)$$

$$2\zeta_v \omega_n \omega_w r + 2\zeta_f \omega_n \omega r = e\omega^2 \sin(\omega t - \theta) \quad (7.81)$$

In Equation 7.80 and Equation 7.81, the left-hand side is independent of t . Hence, the right-hand side should also be independent of t . For this, we must have

$$\theta = \omega t - \phi \quad (7.82)$$

where ϕ is interpreted as the phase lag of whirl with respect to the shaft spin (ω), and should be clear from Figure 7.19(b). It follows from Equation 7.82 that, for steady-state whirl, the whirling speed $\dot{\theta} = \omega_w$ is

$$\omega_w = \omega \quad (7.83)$$

This condition is called synchronous whirl because the whirl speed (ω_w) is equal to the shaft spin speed (ω). It follows that under steady-state conditions, we should have the state of synchronous whirl. The equations governing steady-state whirl are

$$(\omega_n^2 - \omega^2)r = e\omega^2 \cos \phi \quad (7.84)$$

$$2\zeta \omega_n \omega r = e\omega^2 \sin \phi \quad (7.85)$$

along with Equation 7.82 and, hence, Equation 7.83. Here, $\zeta = \zeta_v + \zeta_f$ is the overall damping ratio of the system. Note that the phase angle ϕ and the shaft deflection r are determined from Equation 7.84 and Equation 7.85. In particular, squaring these two equations and adding to eliminate ϕ , we obtain

$$r = \frac{e\omega^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta \omega_n \omega)^2}} \quad (7.86)$$

which is of the form of magnitude of the frequency transfer function of a simple oscillator with an acceleration excitation. Divide Equation 7.85 by Equation 7.84 to get the phase angle:

$$\phi = \tan^{-1} \frac{2\zeta \omega_n \omega}{(\omega_n^2 - \omega^2)} \quad (7.87)$$

Using simple calculus (differentiate the square and equate to zero), we can show that the maximum deflection occurs at the critical spin speed ω_c given by

$$\omega_c = \frac{\omega_n}{\sqrt{1 - 2\zeta^2}} \quad (7.88)$$

This *critical speed* corresponds to a resonance. For light damping, we have approximately $\omega_c = \omega_n$. Hence, critical speed for low damping is equal to the undamped natural frequency of bending vibration of the shaft-rotor unit. The corresponding shaft deflection is (see Equation 7.86)

$$r_c = \frac{e}{2\zeta} \quad (7.89)$$

which is also a good approximation of r at critical speed, with light damping. From Equation 7.84 and Equation 7.85, we can see that, at critical speed (with low damping), $\sin \phi = 1$ and $\cos \phi = 0$, which gives $\phi = \pi/2$. Also, note from Equation 7.86 that the steady-state shaft deflection is almost zero at low speeds and approaches e at very high speeds. However, Equation 7.87 shows that, for small ω , $\tan \phi$ is positive and small. We can see from Equation 7.85 that $\sin \phi$ is positive. This means ϕ itself is small for small ω . For large ω , we can see from Equation 7.86 that r approaches e . Thus, we can see from

Equation 7.87 that $\tan \phi$ is small and negative, whereas Equation 7.85 shows that $\sin \phi$ is positive. Hence, ϕ approaches π for large ω .

It is seen from Equation 7.89 that, at critical speed, the shaft deflection increases with mass eccentricity and decreases with damping. This indicates that the approaches for reducing the damaging effects of whirling are:

1. Eliminate or reduce the mass eccentricity through proper construction practices and balancing.
2. Increase damping.
3. Increase shaft stiffness.
4. Avoid operation near critical speed.

There will be limitations to the use of these approaches, particularly making the shaft stiffer. Note also that our analysis did not include the mass distribution of the shaft. A Bernoulli–Euler type beam analysis has to be incorporated for a more accurate analysis of whirling for shafts whose mass cannot be accurately represented by a single parameter that is lumped at the location of the rotor. Formulas related to whirling of shafts are summarized in [Box 7.5](#).

Example 7.6

The fan of a ventilation system has a normal operating speed of 3600 rpm. The blade set of the fan weighs 20 kg and is mounted in the mid-span of a relatively light shaft that is supported on lubricated bearings at its two ends. The bending stiffness of the shaft at the location of the fan is 4.0×10^6 N/m. Equivalent damping ratio that acts on the possible whirling motion of the shaft is 0.05. Owing to fabrication error, the centroid of the fan has an eccentricity of 1.0 cm from the neutral axis of rotation of the shaft:

1. Determine the critical speed of the fan system and the corresponding shaft deflection at the location of the fan at steady state.
2. What is the steady-state shaft deflection at the fan during normal operation?

The fan was subsequently balanced using a mass of 5 kg. The centroid eccentricity was reduced to 2 mm by this means. What is the shaft deflection at the fan during normal operation now? Comment on the improvement that has been realized.

Solution

1. The system is lightly damped. Hence, the critical speed is given by the undamped natural frequency; thus

$$\omega_c \cong \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4 \times 10^6}{20}} \text{ rad/s} = 447.2 \text{ rad/s}$$

The corresponding shaft deflection is

$$r_c = \frac{e}{2\zeta} = \frac{1.0}{2 \times 0.05} \text{ cm} = 10.0 \text{ cm}$$

2. Operating speed $\omega = (3600/60) \times 2\pi \text{ rad/s} = 377 \text{ rad/s}$. Using Equation 7.86, the corresponding shaft deflection, at steady state, is

$$r = \frac{1.0 \times (377)^2}{[(447.2^2 - 377^2)^2 + (2 \times 0.05 \times 447.2 \times 377)^2]^{1/2}} \text{ cm} = 2.36 \text{ cm}$$

After balancing, the new eccentricity $e = 0.2 \text{ cm}$.

The new natural frequency (undamped) is

$$\omega_n = \sqrt{\frac{4 \times 10^6}{25}} \text{ rad/s} = 400 \text{ rad/s}$$

Box 7.5

WHIRLING OF SHAFTS

Whirling: A shaft spinning at speed ω about its axis, may bend due to flexure. The bent (bowed out) axis will rotate at speed ω_w . This is called whirling.

Equations of motion:

$$\ddot{r} + 2\zeta_v \omega_n \dot{r} + (\omega_n^2 - \dot{\theta}^2)r = e\omega^2 \cos(\omega t - \theta)$$

$$r\ddot{\theta} + 2(\zeta_v \omega_n r + \dot{r})\dot{\theta} + 2\zeta_f \omega_n \omega r = e\omega^2 \sin(\omega t - \theta)$$

where (r, θ) are polar coordinates of shaft deflection at the mounting point of lumped mass.

e = eccentricity of the lumped mass from the spin axis of shaft

$\dot{\theta} = \omega_w$ = whirling speed

ω = spin speed of shaft

$\omega_n = \sqrt{k/m}$ = natural frequency of bending vibration of shaft

k = bending stiffness of shaft at lumped mass

m = lumped mass

ζ_v = damping ratio of bending motion of shaft

ζ_f = damping ratio of shaft bearings

Steady-state whirling (synchronous whirl):

Here, whirling speed ($\dot{\theta}$ or ω_w) is constant and equals the shaft spin speed ω (i.e., $\omega_w = \omega$ for steady-state whirling).

Shaft deflection at lumped mass

$$r = \frac{e\omega^2}{[(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2]^{1/2}}$$

Phase angle between shaft deflection (r) and mass eccentricity (e)

$$\phi = \tan^{-1} \frac{2\zeta\omega_n\omega}{(\omega_n^2 - \omega^2)}$$

where $\zeta = \zeta_v + \zeta_f$

Note: For small spin speeds ω , we have small r and ϕ . For large ω , we have $r \cong e$ and $\phi \cong \pi$

Critical speed:

$$\text{Spin speed } \omega = \frac{\omega_n}{\sqrt{1 - 2\zeta^2}} \omega_n \text{ for small } \zeta$$

$$\phi = \pi/2$$

The corresponding shaft deflection during steady-state operation is

$$r = \frac{0.2 \times (377)^2}{[(400^2 - 377^2)^2 + (2 \times 0.05 \times 400 \times 377)^2]^{1/2}} \text{ cm} = 1.216 \text{ cm}$$

Note that, even though the eccentricity has been reduced by a factor of five by balancing, the operating deflection of the shaft has been reduced only by a factor of less than two. The main reason for this is that the operating speed is close to the critical speed. Methods of improving the performance include changing the operating speed, using a smaller mass to balance the fan, using more damping, and making

the shaft stiffer. However, some of these methods may not be feasible. Operating speed is determined by the task requirements. A location may not be available that is sufficiently distant to place a balancing mass that is appropriately small. Increased damping will increase heat generating, cause bearing problems, and will also reduce the operating speed. Replacement or stiffening of the shaft may require too much modification to the system and add cost. A preferable alternative would be to balance the fan by removing some mass. This will move the critical frequency (natural frequency) away from the operating speed rather than closer to it, while reducing the mass eccentricity at the same time. For example, suppose that a mass of 3 kg is removed from the fan, which results in an eccentricity of 2.0 mm. The new natural/critical frequency is

$$\sqrt{\frac{4 \times 10^6}{17}} \text{ rad/s} = 485.1 \text{ rad/s}$$

The corresponding shaft deflection during steady operation is

$$r = \frac{0.2 \times (377)^2}{[(485.1^2 - 377^2)^2 + (2 \times 0.05 \times 485.1 \times 377)^2]^{1/2}} \text{ cm} = 0.3 \text{ cm}$$

In this case, the deflection has been reduced by a factor of eight.

7.6.3 Self-Excited Vibrations

Equation 7.77 and Equation 7.78, which represent the general whirling motion of a shaft, are nonlinear and coupled. In these equations, the motion variables (r and θ) occur as (nonlinear) products of the excitation (ω). Such systems are termed self-excited. Note that, in general (before reaching the steady state) the response variables r and θ will exhibit vibratory characteristics in view of the presence of the excitation functions $\cos(\omega t - \theta)$ and $\sin(\omega t - \theta)$. Hence, a whirling shaft may exhibit self-excited vibrations. Because the excitation forces directly depend on the motion itself, it is possible that a continuous energy flow into the system could occur. This will result in a steady growth of the motion amplitudes and represents an *unstable* behavior.

A simple example of self-excited vibration is provided by a pendulum whose length is time variable. Although the system is stable when the length is fixed, it can become unstable under conditions of variable length. Practical examples of self-excited vibrations with possible exhibition of instability include the flutter of aircraft wings due to coupled aerodynamic forces, wind-induced vibrations of bridges and tall structures, galloping of ice-covered transmission lines due to air flow-induced vibrations, and chattering of machine tools due to friction-related excitation forces. Proper design and control methods, as discussed in this chapter, are important in suppressing self-excited vibrations.

7.7 Design through Modal Testing

Experimental modal analysis (EMA) involves extracting modal parameters (natural frequencies, modal damping ratios, mode shapes) of a mechanical system through testing (notably, through excitation-response data) and then developing a dynamic model of the system (mass, stiffness, and damping matrices) on that basis. The techniques of EMA are useful in modeling and model validation (i.e., verification of the accuracy of an existing model that was obtained, for example, through analytical modeling). In addition to these uses, EMA is also a versatile tool for design development. In the context of “design for vibration,” EMA may be employed in the design and design modification of mechanical systems with the goal of achieving desired performance under vibrating conditions. This section will introduce this approach.

In applying EMA for design development of a mechanical system, three general approaches are employed:

1. Component modification
2. Modal response specification
3. Substructuring

The method of component modification allows us to modify (i.e., add, remove, or vary) physical parameters (inertia, stiffness, damping) in a mechanical system, and to determine the resulting effect on the modal response (natural frequencies, damping ratios, and mode shapes) of the system. The method of modal response specification provides the capability to establish the best changes, from the design viewpoint, in system parameters (inertia, stiffness, damping values, and associated directions) in order to realize a specified change in the modal response. In the techniques of substructuring, two or more subsystem models are combined using proper components of interfacing (interconnection), and the overall model of the integrated system is determined. Some of the subsystems used in this approach could be of analytical or computational origin (e.g., finite element models). It should be clear how these methods could be used in the design development of a mechanical system for proper vibration performance. The first method is essentially a trial and error technique of incremental design. Here, some appropriate parameters are changed and the resulting modal behavior is determined. If the resulting performance is not satisfactory, further changes are made in discrete steps until an acceptable performance (with regard to natural frequencies, response magnification factors, etc.) is achieved. The second method is clearly a direct design approach, where the design specifications are first developed in terms of modal characteristics, and then the design procedure will generate the size and type of the physical parameters to meet the specifications. In the third method, a suitable set of subsystems is first designed to meet performance characteristics of each subsystem. Then, these subsystems are linked through suitable mechanical interfacing components, and the performance of the overall system is determined to verify acceptance. In this manner, a complex system may be designed through the systematic design of its subsystems.

7.7.1 Component Modification

The method of component modification involves changing a mass, stiffness, or damping element in the system and determining the corresponding dynamic response, particularly the natural frequencies, modal damping ratios, and mode shapes. This is relatively straightforward because a single modal analysis or modal test (EMA) will give the required information. Because single step of component modification might not be acceptable as an appropriate design (e.g., a natural frequency might be too close to a significant frequency component of a vibration excitation), a number of modifications may be necessary. For such incremental procedures, modal analysis would be more convenient and cost effective than EMA because, in the latter case, physical modification and retesting would be needed, whereas the former involves the same computational steps as before, but with a new set of parameter values.

For example, consider an aluminum I beam that has a number of important modes of vibration, including bending and torsional modes. Figure 7.20(a) shows the fourth mode shape of vibration at natural frequency 678.4 Hz. The dotted line in Figure 7.20(b) shows the transfer function magnitude when the beam is excited at some location in the vertical direction and the response is measured in the vertical direction, at some other location, where neither of the locations are node points. The curve shows the first six natural frequencies.

Next, a lumped mass is added to the top flange at the shown location. The corresponding transfer function magnitude is shown by the solid curve in Figure 7.20(b). Note that all the natural frequencies have decreased due to the added mass, but the effect is larger for higher modes. Similarly, mode shapes also will change. If the new modes are not satisfactory (e.g., a particular natural frequency has not shifted enough) further modification and evaluation will be required.

Consider a mechanical vibrating system whose free response y is described by

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{K}\mathbf{y} = 0 \quad (7.90)$$

Damping has been ignored for simplicity, but the following discussion can also be extended to a damped system (quite directly, for the case of proportional damping). If the mass matrix $\mathbf{M}$ and the stiffness matrix $\mathbf{K}$ are modified by $\delta\mathbf{M}$ and $\delta\mathbf{K}$, respectively, the corresponding response (as well as the natural frequencies and mode shapes) will be different from that of the original system. To illustrate, let the

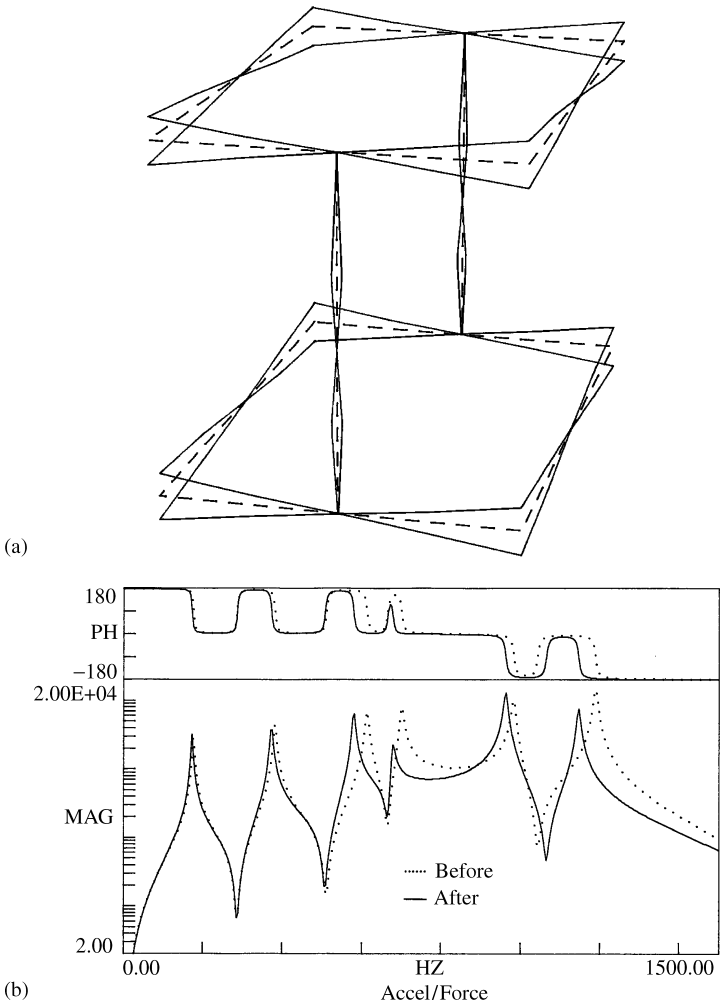


FIGURE 7.20 An example of component modification: (a) the shape of Mode 4 prior to modification; (b) transfer function magnitude before and after modification.

modal matrix (the matrix whose columns are the independent mode shape vectors of the original system) be Ψ . Then, using the modal transformation

$$\mathbf{y} = \Psi \mathbf{q} \tag{7.91}$$

Equation 7.90 can be expressed in the canonical form, with modal generalized coordinates $\mathbf{q}$, as

$$\bar{\mathbf{M}} \ddot{\mathbf{q}} + \bar{\mathbf{K}} \mathbf{q} = 0 \tag{7.92}$$

where

$$\Psi^T \mathbf{M} \Psi = \bar{\mathbf{M}} = \text{diag}[M_1, M_2, \dots, M_n] \tag{7.93}$$

$$\Psi^T \mathbf{K} \Psi = \bar{\mathbf{K}} = \text{diag}[K_1, K_2, \dots, K_n] \tag{7.94}$$

If the same transformation (Equation 7.91) is used for the modified system

$$(\mathbf{M} + \delta \mathbf{M}) \ddot{\mathbf{y}} + (\mathbf{K} + \delta \mathbf{K}) \mathbf{y} = 0 \tag{7.95}$$

we obtain

$$(\bar{\mathbf{M}} + \mathbf{\Psi}^T \delta \mathbf{M} \mathbf{\Psi}) \ddot{\mathbf{q}} + (\bar{\mathbf{K}} + \mathbf{\Psi}^T \delta \mathbf{K} \mathbf{\Psi}) \mathbf{q} = 0 \quad (7.96)$$

Since both $\mathbf{\Psi}^T \delta \mathbf{M} \mathbf{\Psi}$ and $\mathbf{\Psi}^T \delta \mathbf{K} \mathbf{\Psi}$ are not diagonal matrices in general, $\mathbf{\Psi}$ would not remain the modal matrix for the modified system. Furthermore, the original natural frequencies $\omega_i = \sqrt{K_i/M_i}$ will change due to the component modification. For the special case of proportional modifications ($\delta \mathbf{M}$ proportional to $\mathbf{M}$ and $\delta \mathbf{K}$ proportional to $\mathbf{K}$), the mode shapes will not change. However, in general, the natural frequencies will change.

The reverse problem is the modal response specification. Here, a required set of modal parameters (ω_{ir} and ψ_{ir}) is specified and the necessary changes $\delta \mathbf{M}$ and $\delta \mathbf{K}$ to meet the specifications must be determined. Note that the solution is not unique and is more difficult than the direct problem. In this case, a sensitivity analysis may initially be performed to determine the directions and magnitudes of the modal shift for a particular physical parameter shift. Then, the necessary magnitudes of physical shift to achieve the specified modal shift are estimated on that basis. The corresponding modifications are made and the modified system is analyzed/tested to check whether it is within specification. If not, further cycles of modification should be performed.

Example 7.7

As an example of component modification, consider the familiar problem of a two-DoF system, as shown in Figure 7.21. The squared nondimensional natural frequencies $r_i^2 = (\omega_i/\omega_0)^2$ of the systems are given by

$$r_1^2, r_2^2 = \frac{1}{2\alpha} \times \{\alpha + \beta + \alpha\beta\} \left\{ 1 \pm \sqrt{1 - \frac{4\alpha\beta}{(\alpha + \beta + \alpha\beta)^2}} \right\}$$

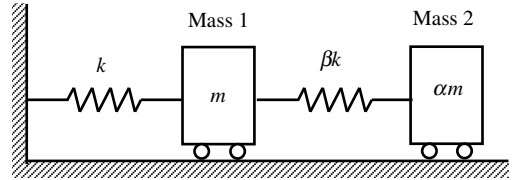


FIGURE 7.21 A two-degree-of-freedom example.

where $\omega_0 = \sqrt{k/m}$. We also showed that the mode shapes, as given by the ratio of the displacement of mass 2 to that of mass 1 at a natural frequency, are

$$\left(\frac{\psi_2}{\psi_1} \right)_i = \frac{1 + \beta - r_i^2}{\beta} \quad \text{for mode } i$$

Consider a system with $\alpha = 0.5$ and $\beta = 0.5$. By direct computation, we can show that $r_1 = 0.71$ and $r_2 = 1.41$. Estimate the modification of β (the relative stiffness of the second spring) that would be necessary to shift the system natural frequencies to approximately $r_1 = 0.8$ and $r_2 = 2.0$. Check the corresponding shift in mode shapes.

Solution

For $\alpha = 0.5$ and $\beta = 0.5$, direct substitution yields $r_1 = 0.71$ and $r_2 = 1.41$ with $(\psi_2/\psi_1)_1 = 2.0$ and $(\psi_2/\psi_1)_2 = -1.0$. Now, consider an incremental change in β by 0.1. Then, $\beta = 0.6$. The corresponding natural frequencies are computed as

$$r_i^2 = \frac{1}{2 \times 0.5} \{0.5 + 0.6 + 0.5 \times 0.6\} \left\{ 1 \mp \left[1 - \frac{4 \times 0.5 \times 0.6}{(0.5 + 0.6 + 0.5 \times 0.6)^2} \right]^{1/2} \right\} = 0.528, 2.272$$

Hence,

$$r_1, r_2 = 0.727, 1.507$$

This step may be interpreted as a way of establishing the sensitivity of the system to the particular component modification. Clearly, the problem of modification is not linear. However, as a first

approximation, assume a linear variation of r_i^2 with β , and make modifications according to

$$\frac{\delta\beta}{\delta\beta_0} = \frac{\delta r_i^2}{\delta r_{i0}^2} \quad (7.97)$$

where the subscript 0 refers to the initial trial variation ($\delta\beta_0 = 0.1$). Equation 7.97 is intuitively satisfying given the nature of the physical problem and the fact that, for a single-DoF problem, squared frequency varies with k_0 . Then, we have

For mode 1:

$$\frac{\delta\beta}{0.1} = \frac{0.8^2 - 0.71^2}{0.727^2 - 0.71^2} = 5.634$$

or

$$\delta\beta = 0.56$$

For mode 2:

$$\frac{\delta\beta}{0.1} = \frac{2^2 - 1.41^2}{1.507^2 - 1.41^2} = 7.09$$

or

$$\delta\beta = 0.709$$

Therefore, we use $\delta\beta = 0.71$, which is the larger of the two. This corresponds to

$$\beta = 0.5 + 0.71 = 1.21$$

The natural frequencies are computed as usual:

$$r_1^2, r_2^2 = \frac{1}{2 \times 0.5} \{0.5 + 1.21 + 0.5 \times 1.21\} \left\{ 1 \mp \left[1 - \frac{4 \times 0.5 \times 1.21}{(0.5 + 1.21 + 0.5 \times 1.21)^2} \right]^{1/2} \right\} = 0.60, 4.03$$

or

$$r_1, r_2 = 0.78, 2.01$$

In view of the nonlinearity of the problem, this shift in frequencies is satisfactory. The corresponding mode shapes are

$$\left(\frac{\psi_2}{\psi_1} \right)_1 = \frac{(1 + 1.21) - 0.6}{1.21} = 1.33$$

$$\left(\frac{\psi_2}{\psi_1} \right)_2 = \frac{(1 + 1.21) - 4.03}{1.21} = -1.50$$

It follows that, as the stiffness of the second spring is increased, the motions of the two masses become closer in mode 1. Furthermore, in mode 2, the node point becomes closer to mass 1. Note the limitation of this particular component modification. As $\beta \rightarrow \infty$, the two masses become rigidly linked giving a frequency ratio of $r_1 = \sqrt{k/(m + \alpha m)}/\sqrt{k/m} = 1/\sqrt{1 + \alpha} = 1/\sqrt{1.5} = 0.816$, with $r_2 \rightarrow \infty$. Hence, it is unreasonable to expect a frequency ratio that is closer to this value of r_1 by a change in β alone.

7.7.2 Substructuring

For large and complex mechanical systems with many components, the approach of substructuring can make the process of “design for vibration” more convenient and systematic. In this approach, the system is first divided into a convenient set of subsystems that are more amenable to testing and analysis. The subsystems are separately modeled and designed through the approaches of modal analysis and testing, along with any other convenient approaches (e.g., finite element technique). Note that the performance

of the overall system depends on the interface conditions that link the subsystems, as well as the characteristics of the individual subsystems. Hence, it is not possible to translate the design specifications for the overall system into those for the subsystems without taking the interface conditions into account. The overall system is *assembled* from the designed subsystems by using *compatibility* requirements at the assembly locations together with dynamic equations of the interconnecting components such as spring–mass–damper units or rigid linkages. If the assembled system does not meet the design specifications, then modifications should be made to one or more of the subsystems and interfacing (assembly) linkages, and the procedure should be repeated. Thus, the main steps of using the approach of substructuring for vibration design of a complex system are as follows:

1. Divide the mechanical system into convenient subsystems (substructuring) and represent the interconnection points of subsystems by forces/moments.
2. Develop models for the subsystems through analysis, modal testing, and other standard procedures.
3. Design the subsystems so that their performance is well within the performance specifications provided for the overall system.
4. Establish the interconnecting (assembling) linkages for the subsystems, and obtain dynamic equations for them in terms of the linking forces/moments and motions (displacements/rotations).
5. Establish continuity (force balancing) and compatibility (motion consistency) conditions at the assembly locations.
6. Using matrix methods, eliminate the unknown variables and assemble the overall system.
7. Analyze (or test) the overall system to determine its vibration performance. If satisfactory, stop. If not, make modifications to the systems or assembly conditions and repeat step 4 to step 7.

As a simple example, consider two single-DoF systems that are interconnected by a spring linkage, as shown in Figure 7.22. The two subsystems may be represented by

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \ddot{\mathbf{y}} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \mathbf{y} = 0$$

and the corresponding natural frequencies are

$$\omega_{s1} = \sqrt{k_1/m_1} \quad \text{and} \quad \omega_{s2} = \sqrt{k_2/m_2}$$

The overall interconnected system is given by

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \ddot{\mathbf{y}} + \begin{bmatrix} k_1 + k_c & -k_c \\ -k_c & k_2 + k_c \end{bmatrix} \mathbf{y} = 0$$

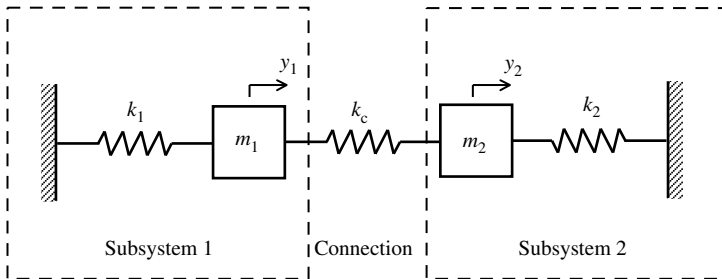


FIGURE 7.22 An example of substructuring.

Its natural frequencies are obtained by solving the equation

$$\det \begin{bmatrix} k_1 + k_c - \omega^2 m_1 & -k_c \\ -k_c & k_2 + k_c - \omega^2 m_2 \end{bmatrix} = 0$$

or

$$(k_1 + k_c - \omega^2 m_1)(k_2 + k_c - \omega^2 m_2) - k_c^2 = 0$$

which simplifies to

$$\omega^4 - \left[\frac{k_1 + k_c}{m_1} + \frac{k_2 + k_c}{m_2} \right] \omega^2 + \frac{k_1 k_2 + k_c(k_1 + k_2)}{m_1 m_2} = 0$$

The sum of the roots is

$$\omega_1^2 + \omega_2^2 = \frac{k_1 + k_c}{m_1} + \frac{k_2 + k_c}{m_2} > \omega_{s1}^2 + \omega_{s2}^2$$

The product of the roots is

$$\omega_1^2 \omega_2^2 = \frac{k_1 k_2 + k_c(k_1 + k_2)}{m_1 m_2} > \omega_{s1}^2 \omega_{s2}^2$$

This does not mean that both frequencies will increase due to the interconnection. Note that the limit on the lower frequency, as $k_c \rightarrow \infty$, is given by that of a single-DoF system with mass $m_1 + m_2$ and stiffness $k_1 + k_2$, which is, $\sqrt{(k_1 + k_2)/(m_1 + m_2)}$. This value can be larger or smaller than the natural frequency of a subsystem depending on the relative values of the parameters. Hence, even for this system, exact satisfaction of a set of design natural frequencies would be somewhat challenging because these frequencies depend on the interconnection as well as the subsystems.

Substructuring is a design development technique where complex designs can be accomplished through parallel and separate development of several subsystems and interconnections. Through this procedure, dynamic interactions among subsystems can be estimated and potential problems can be detected, which will allow redesigning of the subsystems or interfacing linkages prior to building the over prototype. Design approaches using EMA that may be used in vibration problems are summarized in [Box 7.6](#).

7.8 Passive Control of Vibration

The techniques discussed in this chapter for reduction of the effects of mechanical vibration fall into the categories of vibration isolation and design for vibration. The third category, vibration control, is addressed now. Characteristic of vibration control is the use of a sensing device to detect the level of vibration in a system, and an actuation (forcing) device to apply a forcing function to the system to counteract the effects of vibration. In some such devices, the sensing and forcing functions are implicit and integrated together.

Vibration control may be subdivided into the following two broad categories:

1. Passive control
2. Active control

Passive control of vibration employs passive controllers. By definition, passive devices do not require external power for their operation. The two passive controllers of vibration discussed in the present section are vibration absorbers (or dynamic absorbers or Frahm absorbers, named after H. Frahm, who first employed the technique for controlling ship oscillations) and dampers. In both types of devices, sensing is implicit and control is achieved through a force generated by the device from its response to the vibration excitation. A dynamic absorber is a mass–spring-type mechanism with little or no damping,

Box 7.6

TEST-BASED DESIGN APPROACHES FOR VIBRATION

1. **Component modification:**

Modify a component (mass, spring, damper) and determine modal parameters (natural frequencies, damping ratios, mode shapes).

- Can determine sensitivity to component changes.
- Can check whether a particular change is satisfactory.

2. **Modal response specification:**

Specify a desired modal response (natural frequencies, damping ratios, mode shapes) and determine the “best” component changes (mass, spring, damper) that will realize the modal specs.

- Can be accomplished by first performing a sensitivity study (as in item 1).

3. **Substructuring:**

- (i) Design subsystems to meet specs (analytically, experimentally, or by a mixed approach).
- (ii) Establish interconnections between subsystems, and obtain continuity (force balance) and compatibility (motion consistency) at assembly locations.
- (iii) Assemble the overall system by eliminating unknown variables at interconnections.
- (iv) Analyze or test the overall system. If satisfactory, stop. Otherwise, make changes to the subsystems or interconnections, and repeat the above steps.

which can “absorb” the vibration excitation through energy transfer into it, thereby reducing the vibrations of the primary system. The energy received by the absorber will be slowly dissipated due to its own damping. A damper is a purely dissipative device which, unlike a dynamic absorber, directly dissipates the energy received from the system rather than storing it. Hence, it is a more wasteful device, which also may exhibit problems related to wear and thermal effects. However, it has advantages over an absorber, having, for example, a wider frequency of operation.

7.8.1 Undamped Vibration Absorber

A dynamic vibration absorber (or a dynamic absorber, vibration absorber, or Frahm absorber) is a simple mass–spring oscillator with very low damping. An absorber that is tuned to a frequency of vibration of a mechanical system and is able to receive a significant portion of the vibration energy from the primary system at that frequency. In effect, the resulting vibration of the absorber applies an oscillatory force opposing the vibration excitation of the primary system and thereby virtually cancels the effect. In theory, the vibration of the system can be completely removed while the absorber itself undergoes vibratory motion. Since damping is quite low in practical vibration absorbers, we will first consider the case of an undamped absorber.

A vibration absorber may be used for vibration control in two common types of situations, as shown in [Figure 7.23](#). Here, the primary system whose vibration needs to be controlled is modeled as an

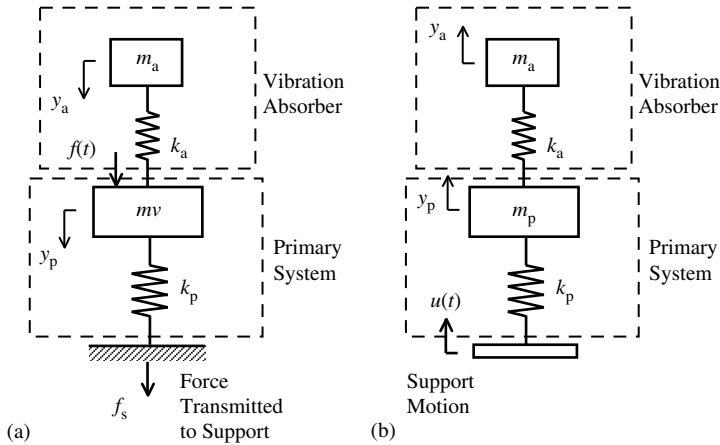


FIGURE 7.23 Two types of applications of a vibration absorber: (a) reduction of the response to forcing excitation (or reducing the force transmitted to support structure); (b) reduction of the response to support motion.

undamped, single-DoF mass–spring system (denoted by the subscript p). An undamped vibration absorber is also a single-DoF mass–spring system (denoted by the subscript a). In the application shown in Figure 7.23(a), the objective of the absorber is to reduce the vibratory response y_p of the primary system as a result of a vibration excitation $f(t)$. The force f_s that is transmitted to the support structure due to the vibratory response of the system is given by

$$f_s = k_p y_p \quad (7.98)$$

Therefore, the objective of reducing y_p may also be interpreted as one of reducing this transmitted vibratory force (a goal of vibration isolation). In the second type of application, represented in Figure 7.23(b), the primary system is excited by a vibratory support motion and the objective of the absorber is to reduce the resulting vibratory motions y_p of the primary system. Note that, in both classes of application, the purpose is to reduce the vibratory responses. Hence, static loads (e.g., gravity) are not considered in the analysis.

Table 7.1 shows the development of the equations of motion for the two systems shown in Figure 7.23. Because we are interested mainly in the control of oscillatory responses to oscillatory excitations, the frequency-domain model is particularly useful. Note from Table 7.1 that the transfer function f_s/f of System a is simply k times the transfer function y_p/f , and is in fact identical to the transfer function y_p/u of System b. The two problems are essentially identical and, thus, we need only address only one of them.

Before investigating the common transfer function for the two types of problems, let us look closely at the frequency-domain equations for the system shown in Figure 7.23(a). We have

$$(k_p + k_a - \omega^2 m_p) y_p - k_a y_a = f \quad (7.99)$$

$$(k_a - \omega^2 m_a) y_a = k_a y_p \quad (7.100)$$

along with Equation 7.98. Here, m_p and k_p are the mass and the stiffness of the primary system, m_a and k_a are the mass and the stiffness of the absorber, f is the excitation amplitude, ω is the excitation frequency, y_p is the primary mass response, and y_a is the absorber response. Now note from Equation 7.100 that if $\omega = \sqrt{k_a/m_a}$ then $y_p = 0$. This means that if the absorber is tuned so that its natural frequency is equal to the excitation frequency (drive frequency), the primary system will not (ideally) undergo any vibratory motion, and is perfectly controlled. The reason for this should be clear from Equation 7.99 which, when $y_p = 0$ is substituted, gives $k_a y_a = -f$. In other words, a tuned absorber applies to the primary system a spring force that is exactly equal and opposite to the excitation force, thereby neutralizing the effect. The absorber mass moves, albeit 180° out of phase with the excitation.

TABLE 7.1 Equations for the Two Types of Absorber Applications

	Absorber Application for the Reduction of Response to a:	
	Forcing Excitation	Support Motion
Time-domain equations	$m_p \ddot{y}_p = -k_p y_p - k_a (y_p - y_a) + f(t)$ $m_a \ddot{y}_a = k_a (y_p - y_a)$	$m_p \ddot{y}_p = k_p (u(t) - y_p) - k_a (y_p - y_a)$ $m_a \ddot{y}_a = k_a (y_p - y_a)$
Frequency-domain equations	$(-\omega^2 m_p + k_p + k_a) y_p = k_a y_a + f$ $(-\omega^2 m_a + k_a) y_a = k_a y_p$	$(-\omega^2 m_p + k_p + k_a) y_p = k_a y_a + k_p u$ $(-\omega^2 m_a + k_a) y_a = k_a y_p$
Matrix form	$\begin{bmatrix} k_p + k_a - \omega^2 m_p & -k_a \\ -k_a & k_a - \omega^2 m_a \end{bmatrix} \begin{bmatrix} y_p \\ y_a \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$	$\begin{bmatrix} k_p + k_a - \omega^2 m_p & -k_a \\ -k_a & k_a - \omega^2 m_a \end{bmatrix} \begin{bmatrix} y_p \\ y_a \end{bmatrix} = k_p \begin{bmatrix} u \\ 0 \end{bmatrix}$
Transfer-function matrix form	$\begin{bmatrix} y_p \\ y_a \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} k_a - \omega^2 m_a & k_a \\ k_a & k_p k_a - \omega^2 m_p \end{bmatrix} \begin{bmatrix} f \\ 0 \end{bmatrix}$	$\begin{bmatrix} y_p \\ y_a \end{bmatrix} = \frac{k}{\Delta} \begin{bmatrix} k_a - \omega^2 m_a & k_a \\ k_a & k_p k_a - \omega^2 m_p \end{bmatrix} \begin{bmatrix} u \\ 0 \end{bmatrix}$
Vibration-control transfer function	$\frac{f_s}{f} = \frac{k_p y_p}{f} = \frac{k_p}{\Delta} (k_a - \omega^2 m_a)$	$\frac{y_p}{u} = \frac{k_p}{\Delta} (k_a - \omega^2 m_a)$
Characteristic polynomial	$\Delta = (k_p + k_a - \omega^2 m_p)(k_a - \omega^2 m_a) - k_a^2$ $= m_p m_a \omega^4 - [k_a(m_p + m_a) + k_p m_a] \omega^2 + k_p k_a$	

The frequency of these motions will be ω (the same as that of the excitation) and the amplitude is proportional to that of the excitation (f) and inversely proportional to the stiffness of the absorber spring. It follows that a vibration absorber “absorbs” vibration energy from the primary system. Furthermore, note from Equation 7.98 that with a tuned absorber the vibration force transmitted to the support structure is (ideally) zero as well. All this information is observed without any mathematical manipulation of the equations of motion.

Note that we are dealing with vibratory excitations and responses. Therefore, static loading (such as gravity and spring preloads) is not considered (we investigate responses with respect to the static equilibrium configuration of the system). In summary, we are now able to state the characteristics of a vibration absorber (undamped) as follows:

1. It is effective only for a single excitation frequency (i.e., a sinusoidal excitation).
2. For the best effect, it should be “tuned” such that its natural frequency $\sqrt{k_a/m_a}$ is equal to the excitation frequency.
3. In the case of forcing vibration excitation, a tuned absorber can (ideally) make the vibratory response of the primary system and the vibratory force transmitted to the support structure zero.
4. In the case of a vibratory support motion, a tuned absorber can make the resulting response of the primary system zero.
5. It functions by acquiring vibration energy from the primary system and storing it (as kinetic energy of the mass or potential energy of the spring) rather than by directly dissipating the energy.
6. It functions by applying a vibration force to the primary system that is equal and opposite to the excitation force, thereby neutralizing the excitation.
7. The amplitude of motion of the vibration absorber is proportional to the excitation amplitude and is inversely proportional to the absorber stiffness. The frequency of the absorber motion is the same as the excitation frequency.

Now, consider the transfer function (f_s/f or y_p/u) of an undamped vibration absorber, as given in Table 7.1. We have

$$G(\omega) = \frac{k_p(k_a - \omega^2 m_a)}{m_p m_a \omega^4 - [k_a(m_p + m_a) + k_p m_a] \omega^2 + k_p k_a}$$

(7.101)

It is convenient to use a nondimensional form in analyzing this frequency-transfer function. To that end, we define the following nondimensional parameters and frequency variable:

Fractional mass of the absorber $\mu = m_a/m_p$

Nondimensional natural frequency of the absorber $\alpha = \omega_a/\omega_p$

Nondimensional excitation (drive) frequency $r = \omega/\omega_p$

where

$\omega_a = \sqrt{k_a/m_a}$ = natural frequency of the absorber

$\omega_p = \sqrt{k_p/m_p}$ = natural frequency of the primary system

It is straightforward to divide the numerator and the denominator by $k_p k_a$ and then carry out simple algebraic manipulations to express the transfer function of Equation 7.101 in the nondimensional form as

$$G(r) = \frac{\alpha^2 - r^2}{r^4 - [\alpha^2(1 + \mu) + 1]r^2 + \alpha^2} \quad (7.102)$$

For this undamped system, there is no difference between the resonant frequencies (where the magnitude of the transfer function peaks) and the natural frequencies (roots of the characteristic equation that correspond to the “natural” or free time response oscillations). These are obtained by solving the equation

$$r^4 - [\alpha^2(1 + \mu) + 1]r^2 + \alpha^2 = 0 \quad (7.103)$$

which gives

$$r_1^2, r_2^2 = \frac{1}{2}[\alpha^2(1 + \mu) + 1] \mp \frac{1}{2}\sqrt{[\alpha^2(1 + \mu) + 1]^2 - 4\alpha^2} \quad (7.104)$$

These are squared frequencies, both of which are positive as clear from Equation 7.104. The actual, nondimensional natural frequencies are their square roots. The magnitude of the transfer function becomes infinite at either of these two natural/resonant frequencies. Furthermore, it is clear from Equation 7.102 that the transfer function magnitude becomes zero at $r = \alpha$, where the excitation frequency (ω) is equal to the natural frequency of the absorber (ω_a) as noted above. In the present undamped case, the transfer function $G(r)$ is real but it can be either positive or negative. The magnitude is thus the absolute value of $G(r)$, which is positive. The magnitude plot given in Figure 7.24 shows the resonant and control characteristics of a system with an undamped vibration absorber. Originally, the primary system had a resonance at $r = 1$ (i.e., $\omega = \omega_p$). When the absorber, which also has a resonance at $r = 1$, is added, the original resonance becomes an *antiresonance* with a zero response. Here, however, two new resonances are created, one at $r = 0.854$ and the other at $r = 1.171$, which are on either side of the tuned frequency ($r = 1$) of the absorber.

Owing to these two resonances, the effective region of the absorber is limited to a narrow frequency band centering its tuned frequency. Specifically, the absorber is not effective unless $|G| < 1$. The effective frequency band of a vibration absorber may be determined using this condition.

Example 7.8

A high-precision, yet high-power positioning system uses a hydraulic actuator and a valve. The pressurized oil to this hydraulic servo system is provided by a gear-type rotary pump. The pump and the positioning system are mounted on the same workbench. The mass of the pump is 25 kg. The normal operating speed of the pump is 3600 rpm. During operation, it was observed that the pump exhibits a vertical resonance at this speed and it affects the accuracy of the position servo system. To control the vibrations of the pump at its operating speed, a vibration absorber of mass 1.25 kg tuned to the normal operating speed of the pump is attached, as shown schematically in Figure 7.25. Because the speed of the pump normally fluctuates during operation, we must determine the speed range within which the vibration absorber is effective. What are the new resonant frequencies of the system? (Neglect damping.)

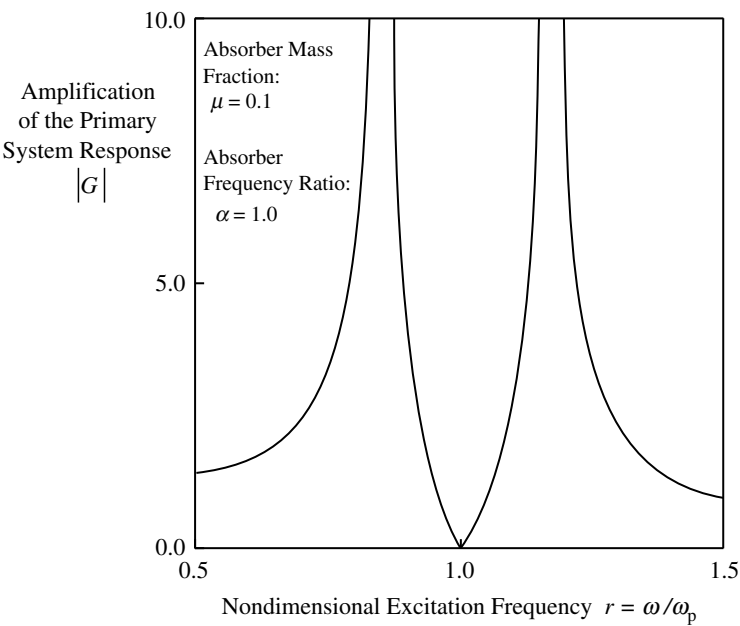


FIGURE 7.24 The effect of an undamped vibration absorber on the vibration response of a primary system.

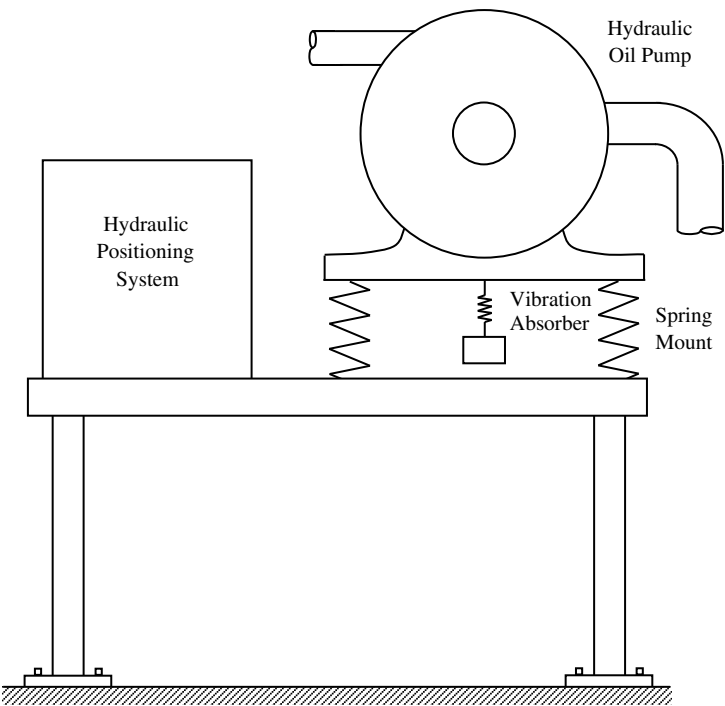


FIGURE 7.25 A hydraulic positioning system with a gear pump.

Solution

For this problem, the fractional mass $\mu = 1.25/25.0 = 0.05$. Since the absorber is tuned to the resonant frequency of the pump, $\alpha = 1.0$. From Equation 7.103, the characteristic equation of the modified system becomes

$$r^4 - 2.05r^2 + 1 = 0$$

which has roots $r_1 = 0.854$ and $r_2 = 1.171$. It follows that the new resonances are at 0.854×3600 and 1.171×3600 rpm. These are 3074.4 and 4215.6 rpm, which should be avoided. From Equation 7.102, the system transfer function is

$$G(r) = \frac{1 - r^2}{(r^4 - 2.05r^2 + 1)}$$

The effective frequency band of the absorber corresponds to $|G(r)| < 1.0$. Since a sign reversal of $G(r)$ occurs at $r = 1$, we need to solve both

$$\frac{1 - r^2}{(r^4 - 2.05r^2 + 1)} = 1 \quad \text{and} \quad -1$$

The first equation gives the roots $r = 0$ and 1.025. The second equation gives the roots $r = 0.977$ and 1.45.

Hence, the effective frequency band corresponds to $\Delta r = [0.977, 1.025]$. In terms of the operating speed of the pump, we have an effective band of 3517.2 to 3690 rpm. Thus, a speed fluctuation of about ± 80 rpm is acceptable.

Finally, recall that the presence of the absorber generates two new resonances on either side of the resonance of the original system (to which the absorber is normally tuned). It is also clear from Equation 7.104 that these two resonances become farther and farther apart as the fractional mass μ of the vibration absorber is increased.

7.8.2 Damped Vibration Absorber

Damping is not the primary means by which vibration control is achieved in a vibration absorber. As noted above, the absorber acquires vibration energy from the primary system (and in turn, exerts a force on the system that is equal and opposite to the vibration excitation), thereby suppressing the vibratory motion. The energy received by the absorber has to be dissipated gradually and, hence, some damping should be present in the absorber. Furthermore, the two resonances that are created by adding the absorber have an infinite magnitude in the absence of damping. Hence, damping has the added benefit of lowering these resonant peaks.

The analysis of a vibratory system with a damped absorber is similar to but somewhat more complex than, that involving an undamped absorber. Furthermore, an extra design parameter — the damping ratio of the absorber — enters into the scene. Consider the model shown in Figure 7.26. Another version of application of a damped absorber, which corresponds to Figure 7.23(b), may also be presented. However, because the two types of application have the same transfer function, it is sufficient to consider Figure 7.26 alone.

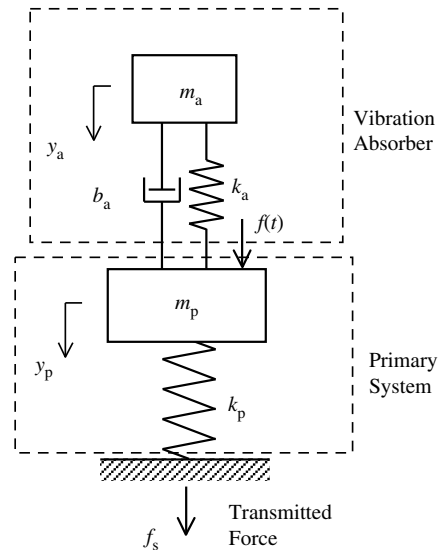


FIGURE 7.26 Primary system with a damped vibration absorber.

Again, the transfer function of vibration control may be taken as either y_a/f or f_s/f , the latter being simply k_p times the former. Although we will consider the dimensionless case of f_s/f , the results are equally valid for y_p/f , except that the responses must be converted from force to displacement by dividing by k_p .

There is no need to derive the transfer function anew for the damped system. Simply replace k_a in Equation 7.101 by the complex stiffness $k_a + j\omega b_a$, which incorporates the viscous damping constant b_a and the excitation frequency ω . Hence, the transfer function of the damped system is

$$G(\omega) = \frac{k_p(k_a + j\omega b_a - \omega^2 m_a)}{m_p m_a \omega^4 - [(k_a + j\omega b_a)(m_p + m_a) + k_p m_a] \omega^2 + k_p(k_a + j\omega b_a)} \quad (7.105)$$

With the parameters defined as before, the nondimensional form of this transfer function is obtained by dividing throughout by $k_p k_a$ and then substituting the appropriate parameters. In particular, we use the fact that

$$\frac{b_a}{k_a} = \frac{2b_a}{2\sqrt{k_a m_a}} \sqrt{\frac{m_a}{k_a}} = \frac{2\zeta_a}{\omega_a} = \frac{2\zeta_a}{\omega_p} \frac{\omega_p}{\omega_a} = \frac{2\zeta_a}{\alpha \omega_p} \quad (7.106)$$

where the damping ratio ζ_a of the absorber is given by

$$\zeta_a = \frac{b_a}{2\sqrt{k_a m_a}} \quad (7.107)$$

as usual. Then, we follow the same procedure that used to derive Equation 7.102 from Equation 7.101 to get

$$G(r) = \frac{\alpha^2 - r^2 + 2j\zeta_a \alpha r}{r^4 - [(\alpha^2 + 2j\zeta_a \alpha r)(1 + \mu) + 1]r^2 + (\alpha^2 + 2j\zeta_a \alpha r)} \quad (7.108)$$

Note that this result is equivalent to simply replacing α^2 by $\alpha^2 + 2j\zeta_a \alpha r$ in Equation 7.102.

It is important to note that the undamped natural frequencies are obtained by solving the characteristics equation with $\zeta_a = 0$. These are the same as before and given by the square roots of Equation 7.104. The damped natural frequencies are obtained by first setting $j\tau = \lambda$ (hence, $r^2 = -\lambda^2$ and $r^4 = \lambda^4$) and then solving the resulting characteristics equation (see the denominator of Equation 7.108).

$$\lambda^4 + 2\zeta_a \alpha (1 + \mu) \lambda^3 + (\alpha^2 + \alpha^2 \mu + 1) \lambda^2 + 2\zeta_a \alpha \lambda + \alpha^2 = 0 \quad (7.109)$$

and then taking the imaginary parts of the roots of λ . These depend on ζ_a and are different from those obtained from Equation 7.104. The resonant frequencies correspond to the r values where the magnitude of $G(r)$ will peak. Generally, these are not the same as the undamped or damped natural frequencies. However, for low damping (small ζ_a compared with 1), these three types of system characteristics frequencies are almost identical.

The magnitude of the transfer function (Equation 7.108) is plotted in Figure 7.27 for the case $\mu = 1.0$ and $\alpha = 1.0$, as in Figure 7.24, but for damping ratios $\zeta_a = 0.01, 0.1$, and 0.5 . Note that the curve for $\zeta_a = 0.01$ is very close to that in Figure 7.24 for the undamped case. When ζ_a is large, as shown in the case of $\zeta_a = 0.5$, the two masses m_p and m_a tend to become locked together and appear to behave like a single mass. Then, the system tends to act like a single-DoF one, and the primary system is modified only in its mass (which increases). Consequently, only one resonant frequency is produced, which is smaller than that of the original primary system. Furthermore, as expected in this high-damping case, the effect of a vibration absorber is no longer present.

All three curves in Figure 7.27 pass through the two common points A and B, as shown. This is true for all curves corresponding to all values of ζ_a , and particularly for the extreme cases of $\zeta_a = 0$ and $\zeta_a \rightarrow \infty$. Hence, these points can be determined as the points of intersection of the transfer function magnitude curves for the limiting cases $\zeta_a = 0$ and $\zeta_a \rightarrow \infty$.

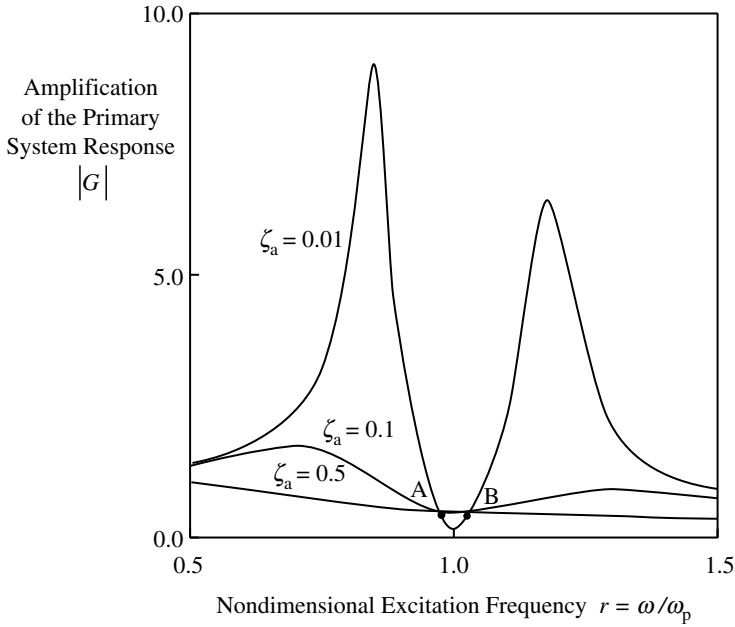


FIGURE 7.27 Vibration amplification (transfer function magnitude) curves for damped vibration absorbers (absorber mass $\mu = 0.1$, absorber resonant frequency $\alpha = 1.0$).

Equation 7.102 gives $G(r)$ for $\zeta_a = 0$. Next, from Equation 7.108, we note that, as $\zeta_a \rightarrow \infty$, all the terms not containing ζ_a can be neglected. Hence,

$$G(r) = \frac{2j\zeta_a\alpha r}{-2j\zeta_a\alpha r(1 + \mu)r^2 + 2j\zeta_a\alpha r}$$

Cancel the common term and we get (for $r \neq 0$)

$$G(r) = \frac{1}{1 - (1 + \mu)r^2} \quad \text{for } \zeta_a \rightarrow \infty \quad (7.110)$$

Note that this is the normalized transfer function of a single-DoF system of natural frequency $1/\sqrt{1 + \mu}$. This result confirms the fact that as $\zeta_a \rightarrow \infty$, the two masses m_p and m_a become locked together and act as a single mass $(m_p + m_a)$ supported on a spring of stiffness k_p . Its natural frequency is $\sqrt{k_p/(m_p + m_a)}$ which, when normalized with respect to $\sqrt{k_p/m_p}$, becomes

$$\sqrt{\frac{k_p}{(m_p + m_a)} \frac{m_p}{k_p}} = \sqrt{\frac{m_p}{(m_p + m_a)}} = \frac{1}{\sqrt{1 + \mu}}$$

In determining the points of intersection between the functions (see Equation 7.102 and Equation 7.110), we should first note that at the first point of intersection (A), the function in Equation 7.102 is negative and positive in Equation 7.110, while the reverse is true for the second point of intersection (B). For either point, this means that the sign of one of the functions should be reversed before equating them. Thus,

$$\frac{\alpha^2 - r^2}{r^4 - [\alpha^2(1 + \mu) + 1]r^2 + \alpha^2} = -\frac{1}{1 - (1 + \mu)r^2}$$

which gives

$$(2 + \mu)r^4 - 2[\alpha^2(1 + \mu) + 1]r^2 + 2\alpha^2 = 0 \quad (7.111)$$

This is the equation whose roots (e.g., r_1 and r_2) give the points A and B. Then, we have the sum of the squared roots equal to the negative coefficient of r^2 in the quadratic (in r^2) Equation 7.111. Thus,

$$r_1^2 + r_2^2 = \frac{2[\alpha^2(1 + \mu) + 1]}{(2 + \mu)} \quad (7.112)$$

In addition, the product of the squared roots is equal to the constant term in the quadratic (r^2) in Equation 7.111. Hence,

$$r_1^2 r_2^2 = \frac{2\alpha^2}{(2 + \mu)} \quad (7.113)$$

7.8.2.1 Optimal Absorber Design

It has been pointed out (primarily by J.P. Den Hartog) that an optimal absorber design should not only have equal response magnitudes at the common points of intersection (i.e., equal ordinates of points A and B in Figure 7.27), but also that the resonances should occur at these points to achieve some balance and uniformity in the response amplification in the region surrounding the tuned frequency of the absorber. It is expected that these (intuitive) design conditions would give relations between the parameters α , μ , and ζ_a , corresponding to an optimal absorber.

Consider the first requirement of equal transfer function magnitudes at A and B. As noted earlier, because these two points do not depend on ζ_a , we use Equation 7.110 to satisfy the requirement. Thus, keeping in mind the sign reversal of the transfer function between A and B (i.e., as the transfer function passes through the resonance), we have

$$\frac{1}{1 - (1 + \mu)r_1^2} = -\frac{1}{1 - (1 + \mu)r_2^2}$$

which gives

$$r_1^2 + r_2^2 = \frac{2}{1 + \mu} \quad (7.114)$$

Substituting this result (for equal ordinates) in the intersection-point condition (see Equation 7.112), we have

$$\frac{2}{1 + \mu} = \frac{2[\alpha^2(1 + \mu) + 1]}{(2 + \mu)}$$

On simplification, we get the simple result

$$\alpha = \frac{1}{1 + \mu} \quad (7.115)$$

Next, we turn to the task of achieving peak magnitudes of the transfer function at the points of intersection (A and B). Generally, when one point peaks the other does not. As reported by Den Hartog, with straightforward but lengthy analysis, we obtain

$$\zeta_a^2 = \frac{\mu[3 - \sqrt{\mu(\mu + 2)}]}{8(1 + \mu)^3} \quad (7.116)$$

for peak at the first intersection point, and

$$\zeta_a^2 = \frac{\mu[3 + \sqrt{\mu(\mu + 2)}]}{8(1 + \mu)^3} \quad (7.117)$$

for peak at the second intersection point.

So, for design purposes, a balance is obtained by taking the average value of the results of Equation 7.116 and Equation 7.117 as

$$\zeta_a^2 = \frac{3\mu}{8(1+\mu)^3} \quad (7.118)$$

Thus, Equation 7.115 and Equation 7.118 correspond to an optimal vibration absorber. In addition, practical requirements and limitations need to be addressed in any design procedure. In particular, since μ is considerably less than unity (i.e., absorber mass is a small fraction of the primary mass), the absorber mass should undergo relatively large amplitudes at the operating frequency in order to receive the energy of the primary system. The absorber spring must be designed accordingly, while meeting the tuning frequency conditions that determine the ratio m_a/k_a .

Example 7.9

The air compressor of a wind tunnel weighs 48 kg and normally operates at 2400 rpm. The first major resonance of the compressor unit occurs at 2640 rpm, with severe vibration amplitudes that are quite dangerous. Design a vibration absorber (damped) for installation on the mounting base of the compressor. What are the vibration amplifications of the compressor unit at the new resonances of the modified system? Compare these with the vibration amplitude of the original system in normal operation.

Solution

As usual, we will tune the absorber to the normal operating speed (2400 rpm). Then, we have the nondimensional resonant frequency of the absorber:

$$\alpha = \frac{\omega_a}{\omega_p} = \frac{2400}{2640} = \frac{12}{13}$$

Now, for an optimal absorber, from Equation 7.115

$$\mu = \frac{1}{\alpha} - 1 = \frac{13}{12} - 1 = \frac{1}{12}$$

Hence, the absorber mass

$$m_a = 48 \times \frac{1}{12} \text{ kg} = 4.0 \text{ kg}$$

Then, from Equation 7.118, the damping ratio of the absorber is

$$\zeta_a = \left[\frac{3/12}{8(1+1/12)^3} \right]^{1/2} = 0.157$$

Now,

$$\omega_a = \sqrt{\frac{k_a}{m_a}} = \sqrt{\frac{k_a}{4.0}} = \frac{2400}{60} \times 2\pi \text{ rad/s} = 88\pi \text{ rad/s}$$

Hence,

$$k_a = (88\pi)^2 \times 4.0 \text{ N/m} = 2.527 \times 10^5 \text{ N/m}$$

Also,

$$\zeta_a = \frac{1}{2} \frac{b_a}{\sqrt{m_a k_a}}$$

Then, we have

$$b_a = 2 \times 0.157 \sqrt{4.0 \times 2.527 \times 10^5} \text{ N s/m} = 315.7 \text{ N s/m}$$

This gives us the damped absorber. Now, let us check its performance. We know that, in theory, the vibration amplitude at the operating speed should be almost zero now. However, two resonances are created around the operating point. Since damping is small, we use the undamped characteristic Equation 7.103 to compute these resonances:

$$r^4 - \left[\frac{12^2}{13^2} \left(1 + \frac{1}{12} \right) + 1 \right] r^2 + \frac{12^2}{13^2} = 0$$

which gives

$$r^4 - \frac{25}{13} r^2 + \frac{12^2}{13^2} = 0$$

The roots of r^2 are 0.692 and 1.231. The (positive) roots of r are 0.832 and 1.109.

These correspond to compressor speeds of (multiply r by 2640 rpm) 2196 and 2929 rpm. Although they are approximately at -10% and $+20\%$ of the operating speed, the first resonance will be encountered during startup and shutdown conditions. To determine the corresponding vibration amplifications (force/force), use Equation 7.108 which, when the undamped characteristic equation is substituted into the denominator, becomes

$$G(r) = \frac{\alpha^2 - r^2 + 2j\zeta_a \alpha r}{[-2j\zeta_a \alpha r(1 + \mu)r^2 + 2j\zeta_a \alpha r]} = \frac{1 - j(\alpha^2 - r^2)/(2\zeta_a \alpha r)}{1 - (1 + \mu)r^2} \quad (7.119)$$

Substitute the resonant frequencies $r_1 = 0.832$ and $r_2 = 1.109$. We get $|G(r_1)| = 4.223$ and $|G(r_2)| = 4.634$.

Without the absorber, we approximate the system by a simple undamped oscillator with transfer function

$$G_p(r) = \frac{1}{1 - r^2}$$

The corresponding vibration amplification at the operating speed is

$$|G_p(r_0)| = \frac{1}{|1 - 12^2/13^2|} = 6.76$$

It is observed that after adding the absorber, the resonant vibrations are smaller than even the operating vibrations of the original system. Hence, the design is satisfactory. Note that we used the force/force transfer functions. To get the displacement/force transfer functions we divide by k_p . However, we have

$$\sqrt{\frac{k_p}{m_p}} = \frac{2640}{60} \times 2\pi \text{ rad/s} = 88\pi \text{ rad/s}$$

Hence,

$$k_p = (88\pi)^2 \times 48 \text{ N/m} = 3.67 \times 10^6 \text{ N/m} = 3.67 \times 10^3 \text{ N/mm}$$

Thus, the amplitude of operating vibrations of the original system is

$$\frac{6.76}{3.67 \times 10^3} \text{ mm/N} = 1.84 \times 10^{-3} \text{ mm/N}$$

The amplitudes of the resonant vibrations of the modified system are

$$\frac{4.223}{3.67 \times 10^3} \quad \text{and} \quad \frac{4.634}{3.67 \times 10^3} \text{ mm/N or } 1.15 \times 10^{-3} \text{ and } 1.26 \times 10^{-3} \text{ mm/N}$$

Vibration absorbers are simple and passive devices, which are commonly used in the control of narrowband vibrations (limited to a very small interval of frequencies). Applications are found in vibration suppression of transmission wires (e.g., a stockbridge damper, which simply consists of a piece of cable carrying two masses at its ends), consumer appliances, automobile engines, and industrial machinery. It should be noted that the concepts presented for a rectilinear vibration absorber may be directly extended to a rotary vibration absorber. Figure 7.28 provides a schematic representation of a rotary vibration absorber. This model corresponds to vibration force excitations (compare with Figure 7.23(a)). The case of rotational support-motion excitations (see Figure 7.23(b)), which has essentially the same transfer function, may also be addressed. Approaches of vibration control are summarized in Box 7.7.

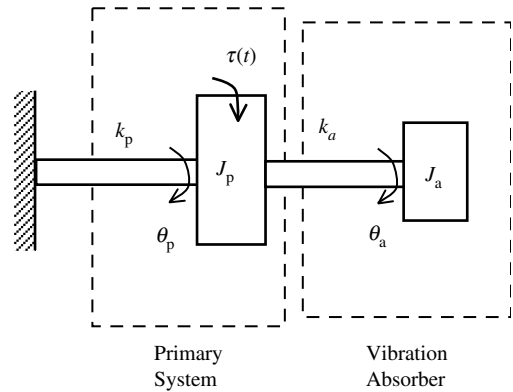


FIGURE 7.28 The application of a rotary vibration absorber.

7.8.3 Vibration Dampers

As discussed above, vibration absorbers are simple and effective passive devices, which are used in vibration control. They have the added advantage of being primarily nondissipative. The main disadvantage of a vibration absorber is that it is only effective over a very narrow band of frequencies enclosing its resonant frequency (tuned frequency). When passive vibration control over a wide band of frequencies is required, a damper would be a preferable choice.

Vibration dampers are dissipative devices. They control vibration through direct dissipation of the vibration energy of the primary (vibrating) system. As a result, however, there will be substantial heat generation, and associated thermal problems and component wear. Consequently, methods of cooling (e.g., use of a fan, coolant circulation, and thermal conduction blocks) may be required in some special situations.

Consider a vibrating system modeled as an undamped single-DoF mass–spring system (simple oscillator). The magnitude of the excitation-response transfer function will have a resonance with a theoretically infinite magnitude in this case. Operation in the immediate neighborhood of such a resonance would be destructive. Adding a simple viscous damper, as shown in Figure 7.29(a), will correct the situation. The equation of motion (about the static equilibrium position) is

$$m\ddot{y} + b\dot{y} + ky = f(t) \quad (7.120)$$

with the dynamic force that is transmitted through the support base (f_s) given by

$$f_s = ky + b\dot{y} \quad (7.121)$$

Hence, the transfer function between the forcing excitation f and the vibration response y is

$$\frac{y}{f} = \frac{1}{k - \omega^2 m + j\omega b} \quad (7.122)$$

and that between the forcing excitation and the force transmitted to the support structure is

$$\frac{f_s}{f} = \frac{k + j\omega b}{k - \omega^2 m + j\omega b} \quad (7.123)$$

Using the nondimensional frequency variable $r = \omega/\omega_n$ where $\omega_n = \sqrt{k/m}$ is the undamped natural frequency of the system and the damping ratio $\zeta = b/(2\sqrt{km})$, we can express Equation 7.122 and

Box 7.7

VIBRATION CONTROL

Passive control (no external power):

1. Dampers
 - A dissipative approach (thermal problems, degradation)
 - Useful over a wide frequency band
2. Vibration absorbers (dynamic absorbers, Frahm absorbers)
 - Absorbs energy from vibrating system and applies counteracting force
 - Useful over a very narrow frequency band (near the tuned frequency)
 - Absorber executes large motions

Undamped absorber design:

$$\text{Transfer function of system with absorber} = \frac{\alpha^2 - r^2}{r^4 - [\alpha^2(1 + \mu) + 1]r^2 + \alpha^2}$$

where

μ = absorber mass/primary system mass

α = absorber natural frequency/primary system natural frequency

r = excitation frequency/primary system natural frequency

The most effective operating frequency $r_{\text{op}} = \alpha$.

Avoid the two resonances.

Optimal damped absorber design:

Mass ratio

$$\mu = \frac{1}{\alpha} - 1$$

Damping ratio

$$\zeta_a = \frac{3\mu}{8(1 + \mu)^3}$$

Active control (needs external power):

1. Measure vibration response using sensors/transducers.
2. Apply control forces to vibrating system through actuators, according to a suitable control algorithm.

Equation 7.123 in the form

$$\frac{y}{f} = \frac{1}{k(1 - r^2 + 2j\zeta r)} \quad (7.124)$$

$$\frac{f_s}{f} = \frac{1 + 2j\zeta r}{(1 - r^2 + 2j\zeta r)} \quad (7.125)$$

When vibration control of the primary system is desired, we use the transfer function in Equation 7.124. However, when force transmissibility is the primary consideration, we use Equation 7.125.

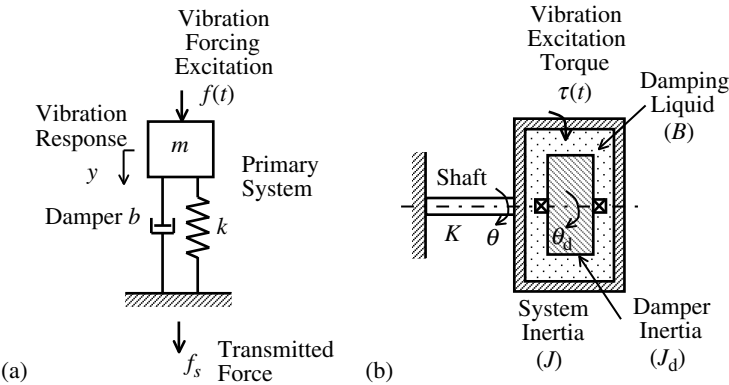


FIGURE 7.29 (a) A system with a linear viscous damper; (b) a rotary system with a Houdaille damper.

Furthermore, it is convenient to use the transfer function in Equation 7.124 in the nondimensional form:

$$\frac{ky}{f} = G(r) = \frac{1}{(1 - r^2 + 2j\zeta r)} \tag{7.126}$$

The magnitude of this transfer function is plotted in Figure 7.30 for several values of damping ratio. Note that the addition of significant levels of damping considerably lowers the resonant peak and flattens the overall response. This example illustrates the broadband nature of the effect of a damper. However, unlike a vibration absorber, it is not possible with a simple damper to bring the vibration levels to a theoretical zero. However, a damper is able to bring the response uniformly close to the static value (unity in Figure 7.30).

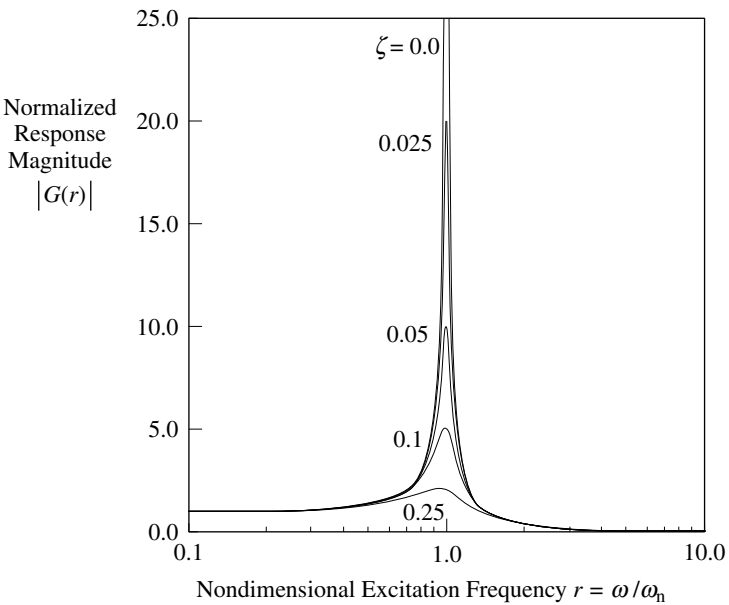


FIGURE 7.30 Frequency response of a system containing a linear damper.

Another common application of damper is connecting it through a free inertia element. For a rotational system, such an arrangement is known as the Houdaille damper, and is modeled as in Figure 7.29(b). The equations of motion are

$$J\ddot{\theta} + B(\dot{\theta} - \dot{\theta}_d) + K\theta = \tau(t) \quad (7.127)$$

$$J_d\ddot{\theta}_d + B(\dot{\theta}_d - \dot{\theta}) = 0 \quad (7.128)$$

In this case, the transfer function between the vibratory excitation torque τ and the response angle θ is given by

$$\frac{\theta}{\tau} = \frac{B + J_d j\omega}{KB - B(J + J_d)\omega^2 - J_d J \omega^3 + KJ_d j\omega} \quad (7.129)$$

Again, we use the normalized form of $K\theta/\tau$. Then, we obtain

$$\frac{K\theta}{\tau} = G(r) = \frac{2\zeta + jr\mu}{2\zeta[1 - (1 + \mu)r^2] + jr\mu(1 - r^2)} \quad (7.130)$$

where $r = \omega/\omega_n$, $\zeta = B/(2\sqrt{KJ})$, $\mu = J_d/J$, and $\omega_n = \sqrt{K/J}$.

Note the two extreme cases. When $\zeta = 0$, the system becomes the original undamped system, as expected. When $\zeta \rightarrow \infty$, the system becomes an undamped simple oscillator, but with a lower natural frequency of $r = 1/\sqrt{1 + \mu}$, instead of $r = 1$ that was present in the original system. This is to be expected because as $\zeta \rightarrow \infty$, the two inertia elements become locked together and act as a single combined inertia $J + J_d$. Clearly, in these two extreme systems, the effect of damping is not present. Optimal damping occurs somewhere in between, as is clear from the curves of response magnitude shown in Figure 7.31 for the case of $\mu = 0.2$.

Proper selection of the nature and values of damping is crucial for the effective use of a damper in vibration control. Damping in physical systems is known to be nonlinear and frequency dependent, as well as time-variant and dependent on the environment (e.g., temperature). Various models are available for different types of damping, but these are only approximate representations. In practice, such considerations as the type of damper used, the nature of the system, the specific application, and the

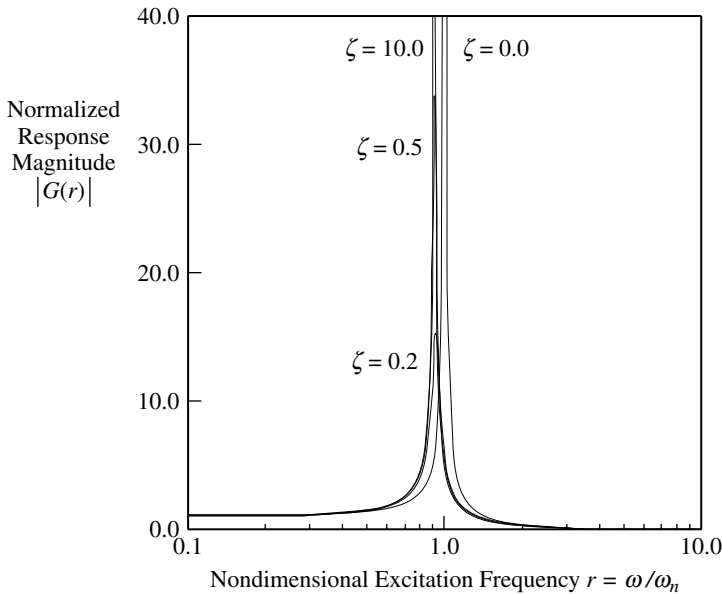


FIGURE 7.31 Response curves for a rotary system with Houdaille damper of inertia ratio $\mu = 0.2$.

speed of operation, determine which particular model (linear viscous, hysteric, Coulomb, Stribeck, quadratic aerodynamic, etc.) is suitable. In addition to the simple linear theory of viscous damper, specific properties of physical damping should be taken into consideration in practical designs.

7.9 Active Control of Vibration

Passive control of vibration is relatively simple and straightforward. Although it is robust, reliable, and economical, it has its limitations. Note that the control force that is generated in a passive device depends entirely on the natural dynamics. Once the device is designed (i.e., after the parameter values for mass, damping constant, stiffness, location, etc. are chosen), it is not possible to adjust the control forces that are naturally generated in real time. Furthermore, in a passive device there is no supply of power from an external source. Hence, even the magnitude of the control forces cannot be changed from their natural values. Since a passive device senses the response of the system as an integral process of the overall dynamics of the system, it is not always possible to directly target the control action at particular responses (e.g., particular modes). This can result in incomplete control, particularly in complex and high-order (e.g., distributed-parameter) systems. These shortcomings of passive control can be overcome using active control. Here, the system responses are directly sensed using sensor-transducer devices, and control actions of specific desired values are applied to desired locations/modes of the system.

7.9.1 Active Control System

Figure 7.32 presents a schematic diagram of an active control system. The mechanical dynamic system whose vibrations need to be controlled is the *plant* or *process*. The controller is the device that generates the signal (or command) according to some scheme (or control law) and controls vibrations of the plant. The plant and the controller are the two essential components of a *control system*. Usually, the plant must

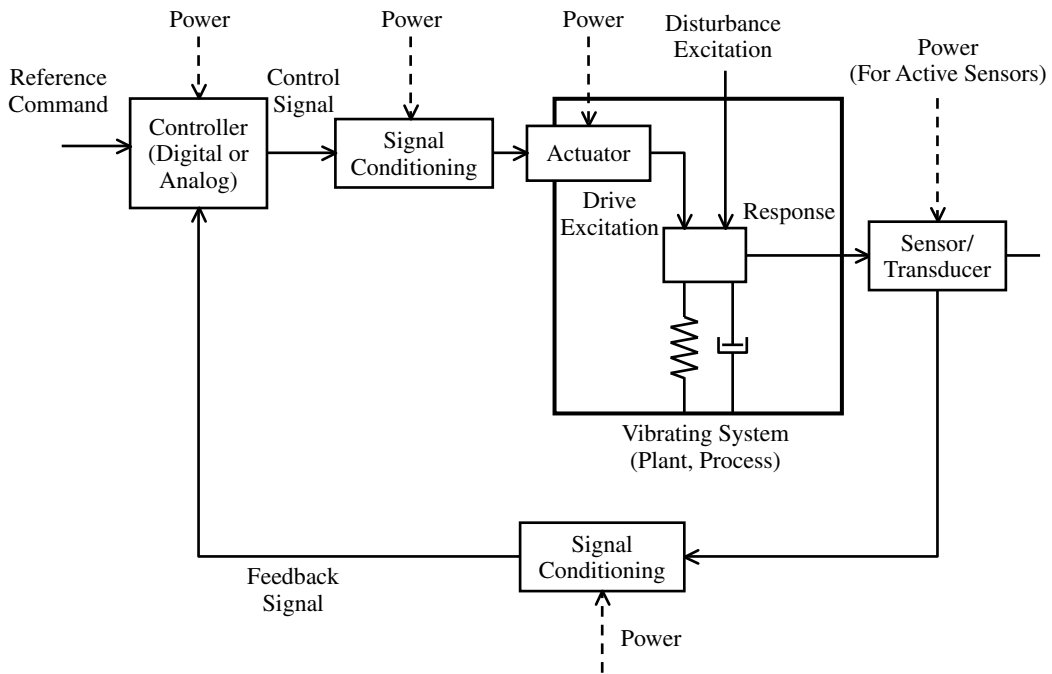


FIGURE 7.32 A system for active control of vibration.

be monitored and its response must be measured using sensors providing feedback into the controller. Then, the controller compares the sensed signal with a desired response specified externally, and uses the error to generate a proper control signal. In this manner, we have a feedback control system. In the absence of a sensor and feedback, we have an open-loop control system. In *feed-forward control*, the excitation (i.e., input signal), not the response (i.e., output signal), is measured and used (i.e., fed forward into the controller) for generating the control signal. Both feedback and feedforward schemes may be used in the same control system.

The actuator that receives a control signal and drives the plant may be an integral part of the plant (e.g., the motor that drives the blade of a saw). Alternatively, it may have to be added specifically as an external component for the control actuation (e.g., a piezoelectric or electromagnetic actuator for controlling blade vibrations of a saw). In the former case, in particular, proper signal conditioning is needed to convert the control signal to a form that is compatible with the existing actuator. In the latter case, both the controller and the actuator must be developed in parallel for integration into the plant. In digital control, the controller is a digital processor. The control signal is in digital form and, typically, it has to be converted into the analog form prior to using in the actuator. Hence, digital-to-analog conversion (DAC) is a form of signal conditioning that is useful here. Furthermore, the analog signal that is generated may have to be filtered and amplified to an appropriate level for use in the actuator. It follows that filters and amplifiers are signal conditioning devices, which are useful in vibration control. In *software control*, the control signal is generated by a computer, which functions as the digital controller. In *hardware control*, the control signal is rapidly generated by digital hardware without using software programs. Alternatively, *analog control* may be used where the control signal is generated directly using analog circuitry. In this case, the controller is quite fast and it does not require DAC. Note that the actuator may need high levels of power. Furthermore, the controller and associated signal conditioning will require some power. The need for an external power source for control distinguishes active control from passive control.

In a feedback control system, sensors are used to measure the plant response, which enables the controller to determine whether the plant operates properly. A sensor unit that “senses” the response may automatically convert (transduce) this “measurement” into a suitable form. A piezoelectric accelerometer senses acceleration and converts it into an electric charge, an electromagnetic tachometer senses velocity and converts it into a voltage, and a shaft encoder senses a rotation and converts it into a sequence of voltage pulses. Hence, the terms *sensor* and *transducer* are used interchangeably to denote a sensor-transducer unit. The signal that is generated in this manner may need conditioning before feeding into the controller. For example, the charge signal from a piezoelectric accelerometer has to be converted to a voltage signal of appropriate level using a charge amplifier, and then it has to be digitized using an analog-to-digital converter (ADC) for use in a digital controller. Furthermore, filtering may be needed to remove measurement noise. Hence, signal conditioning is usually needed between the sensor and the controller as well as between the controller and the actuator. External power is required to operate active sensors (e.g., potentiometer) whereas passive sensors (e.g., electromagnetic tachometer) employ self-generation and do not need an external power source. External power may be needed for conditioning the sensor signals. Finally, as indicated in [Figure 7.32](#), a vibrating system may have unknown disturbance excitations, which can make the control problem particularly difficult. Removing such excitations at the source level through proper design or vibration isolation is desirable, as discussed above. However, in the context of control, if these disturbances can be measured or some information about them is available, then they can be compensated for within the controller itself. This is, in fact, the approach of feedforward control.

7.9.2 Control Techniques

The purpose of a vibration controller is to excite (activate) a vibrating system in order to control its vibration response in a desired manner. In the present context of active feedback control, the controller uses measured response signals and compares them with their desired values in its task of determining an

appropriate action. The relationship that generates the control action from a measured response (and a desired value for the response) is called a *control law*. Sometimes, a *compensator* (analog or digital, hardware or software) is employed to improve the system performance or to enhance the controller so that the task of control is easier. However, for our purpose, we may consider a compensator as an integral part of the controller and thus a distinction between the two is not made.

Various control laws, both linear and nonlinear, have been developed for practical applications. Many of them are suitable in vibration control. A comprehensive presentation of all such control laws is outside the scope of this book. We will give several linear control laws that are common and representative of what is available. These techniques are based on a linear representation (linear model) of the vibrating system (plant). Even when the overall operating range of a plant (e.g., robotic manipulator) is nonlinear, it is often possible to linearize the vibration response (e.g., link vibrations and joint vibrations of a robot) about a reference configuration (e.g., robot trajectory). These linear control techniques would be still suitable even though the overall dynamics of the system is nonlinear.

7.9.2.1 State-Space Models

In applying many types of control techniques, it is convenient to represent the vibrating system (plant) by a state-space model. This is simply a set of ordinary first-order differential equations, which could be coupled or nonlinear, and could have time-varying parameters (time-variant models). Here, we limit our discussion to linear and time-invariant state-space models. Such a model is expressed as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (7.131)$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \quad (7.132)$$

where

$\mathbf{x} = [x_1, x_2, \dots, x_n]^T$ = state vector (n th order column)

$\mathbf{u} = [u_1, u_2, \dots, u_r]^T$ = input vector (r th order column)

$\mathbf{y} = [y_1, y_2, \dots, y_m]^T$ = output vector (m th order column)

$\mathbf{A}$ = system matrix ($n \times n$ square)

$\mathbf{B}$ = input gain matrix ($n \times r$)

$\mathbf{C}$ = measurement gain matrix ($m \times n$)

$\mathbf{D}$ = feedforward gain matrix ($m \times r$)

Usually, for vibrating systems, it is possible to make $\mathbf{D} = 0$, and hence we will drop this matrix in the sequel. Furthermore, although a state variable x_i need not have a direct physical meaning, an output variable y_j should have some physical meaning and, in typical situations, should be measurable as well. The input variables are the “control variables” and are used for controlling the system (plant). The output variables are the “controlled variables,” which correspond to the system response and are measured for feedback control.

It can be verified that the eigenvalues of the system matrix $\mathbf{A}$ occur in complex conjugates of the form $-\zeta_i\omega_i \pm j\sqrt{1-\zeta_i^2}\omega_i$ in the damped oscillatory case, or as $\pm j\omega_i$ in the undamped case, where ω_i is the i th natural frequency of the system and ζ_i is the corresponding damping ratio (of the i th mode). The mathematical verification requires some linear algebra. An intuitive verification can be made since Equation 7.131 is an equivalent model for a system having the traditional mass–spring–damper model

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{f}(t) \quad (7.133)$$

where $\mathbf{M}$ = mass matrix, $\mathbf{C}$ = damping matrix, $\mathbf{K}$ = stiffness matrix, $\mathbf{f}(t)$ = forcing input vector, and $\mathbf{y}$ = displacement response vector. Where both models (Equation 7.131 and Equation 7.133), are equivalent they should have the same characteristic equation, which by its roots determines the natural frequencies and modal damping ratios. This is the case because we are simply looking at two different mathematical representations of the same system. Hence, the parameters of its dynamics, such as ω_i and ζ_i , should remain unchanged. In fact, the state-space mode (Equation 7.131) is not unique, and different versions of state vectors and corresponding models are possible. Of course, all of them should have the

same characteristics polynomial (and hence, the same ω_i and ζ_i). One such state-space model may be derived from Equation 7.133 as follows:

Define the state vector as

$$\mathbf{x} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \text{ and } \mathbf{u} = \mathbf{f}(t) \quad (7.134)$$

Since (for nonsingular $\mathbf{M}$, as required), Equation 7.133 may be written as

$$\ddot{\mathbf{y}} = -\mathbf{M}^{-1}\mathbf{K}\mathbf{y} - \mathbf{M}^{-1}\mathbf{C}\dot{\mathbf{y}} + \mathbf{M}^{-1}\mathbf{f}(t) \quad (7.135)$$

we have

$$\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1} \end{bmatrix} \mathbf{u} \quad (7.136)$$

This is a state-space model that is equivalent to the conventional model (Equation 7.133), and can be shown to have the same characteristic equation. The development of a state-space model for a vibrating system can be illustrated using an example.

Example 7.10

Consider a machine mounted on a support structure, modeled as in Figure 7.33. Using the excitation forces $f_1(t)$ and $f_2(t)$ as the inputs and the displacements y_1 and y_2 of the masses m_1 and m_2 as the outputs, develop a state-space model for this system.

Solution

Assume that the displacements are measured from the static equilibrium positions of the masses. Hence, the gravity forces do not enter into the formulation. Newton's Second law is applied to the two masses; thus

$$m_1\ddot{y}_1 = f_1 - k_1(y_1 - y_2) - b_1(\dot{y}_1 - \dot{y}_2)$$

$$m_2\ddot{y}_2 = f_2 - k_1(y_2 - y_1) - b_1(\dot{y}_2 - \dot{y}_1) - k_2y_2 - b_2\dot{y}_2$$

The following state variables are defined:

$$x_1 = y_1; \quad x_2 = \dot{y}_1; \quad x_3 = y_2; \quad x_4 = \dot{y}_2$$

Also, the input vector is $\mathbf{u} = [u_1 \ u_2]^T$ and the output vector is $\mathbf{y} = [y_1 \ y_2]^T$. Then, we have

$$\dot{x}_1 = x_2$$

$$m_1\dot{x}_2 = u_1 - k_1(x_1 - x_3) - b_1(x_2 - x_4)$$

$$\dot{x}_3 = x_4$$

$$m_2\dot{x}_4 = u_2 - k_1(x_3 - x_1) - b_1(x_4 - x_2) - k_2x_3 - b_2x_4$$

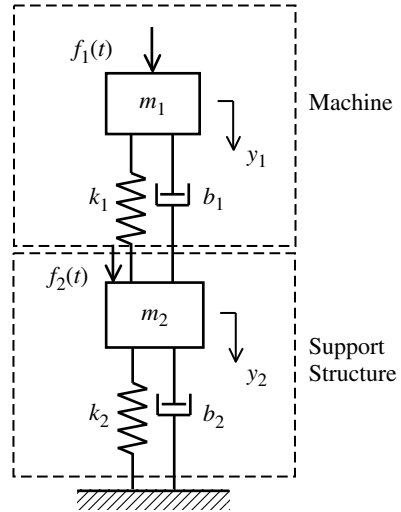


FIGURE 7.33 A model of a machine mounted on support structure.

Accordingly, the state-space model is given by Equation 7.131 and Equation 7.132 with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_1/m_1 & -b_1/m_1 & k_1/m_1 & b_1/m_1 \\ 0 & 0 & 0 & 1 \\ k_1/m_2 & b_1/m_2 & (k_1 + k_2)/m_2 & (b_1 + b_2)/m_2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1/m_1 & 0 \\ 0 & 0 \\ 0 & 1/m_2 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \text{and } \mathbf{D} = 0$$

Also, note that the system can be expressed as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \ddot{\mathbf{y}} + \begin{bmatrix} b_1 & -b_1 \\ -b_1 & (b_1 + b_2) \end{bmatrix} \dot{\mathbf{y}} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & (k_1 + k_2) \end{bmatrix} \mathbf{y} = \mathbf{f}(t)$$

Its characteristic equation may be expressed as the determinant equation:

$$\det \begin{bmatrix} m_1 s^2 + b_1 s + k_1 & -b_1 s - k_1 \\ -b_1 s - k_1 & m_2 s^2 + (b_1 + b_2)s + (k_1 + k_2) \end{bmatrix} = 0$$

It can be verified through direct expansion of the determinants that this equation is equivalent to the characteristic equation of the matrix $\mathbf{A}$, as given by $\det(\lambda \mathbf{I} - \mathbf{A}) = 0$, or

$$\det \begin{bmatrix} \lambda & -1 & 0 & 0 \\ k_1/m_1 & \lambda + b_1/m_1 & -k_1/m_1 & -b_1/m_1 \\ 0 & 0 & \lambda & -1 \\ k_1/m_2 & -b_1/m_2 & -(k_1 + k_2)/m_2 & \lambda - (b_1 + b_2)/m_2 \end{bmatrix} = 0$$

Note that, in the present context, x and y represent the vibration response of the plant and the control objective is to reduce these to zero. We will give some common control techniques that can achieve this goal.

7.9.2.2 Position and Velocity Feedback

In this technique, the position and velocity of each DoF is measured and fed into the system with sign reversal (negative feedback) and amplification by a constant gain. Because velocity is the derivative of position and since the gains are constant (i.e., proportional), this method falls into the general category of proportional-plus-derivative (PD or PPD) control. In this approach, it is tacitly assumed that the degrees of freedom are uncoupled. Then, control gains are chosen so that the DoF in the controlled system are nearly uncoupled, thereby justifying the original assumption. To explain this control method, suppose that a DoF of a vibrating system is represented by

$$m\ddot{y} + b\dot{y} + ky = u(t) \quad (7.137)$$

where y is the displacement (position) of the DoF and u is the excitation input that is applied. Now suppose that u is generated according to the (active) control law

$$u = -k_c y - b_c \dot{y} + u_r \quad (7.138)$$

where k_c is the position feedback gain and b_c is the velocity feedback gain. The implication here is that the position y and the velocity $\dot{y}$ are measured and fed into the controller which in turn generates u according to Equation 7.138. Also, u_r is some reference input that is provided externally to the controller.

Then, substituting Equation 7.138 into Equation 7.137, we obtain

$$m\ddot{y} + (b + b_c)\dot{y} + (k + k_c)y = u_r \quad (7.139)$$

The closed-loop system (the controlled system) now behaves according to Equation 7.139. The control gains b_c and k_c can be chosen arbitrarily (subject to the limitations of the physical controller, signal conditioning circuitry, the actuator, etc.) and may even be negative. In particular, by increasing b_c , the damping of the system can be increased. Similarly, by increasing k_c the stiffness (and the natural frequency) of the system can be increased. Even though a passive spring and damper with stiffness k_c and damping constant b_c can accomplish the same task, once the devices are chosen it is not possible to conveniently change their parameters. Furthermore, it will not be possible to make k_c or b_c negative in this case of passive physical devices. The method of PPD control is simple and straightforward, but the assumptions of linear uncoupled DoF place a limitation on its general use.

7.9.2.3 Linear Quadratic Regulator Control

This is an optimal control technique. Consider a vibrating system that is represented by the linear state-space model:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (7.131)$$

Assume that all the states $\mathbf{x}$ are measurable and all the system modes are controllable. Then, we use the constant-gain feedback control law:

$$\mathbf{u} = \mathbf{K}\mathbf{x} \quad (7.140)$$

The choice of parameter values for the feedback gain matrix $\mathbf{K}$ is infinite. Therefore, we can use this freedom to minimize the cost function:

$$J = \frac{1}{2} \int_t^\infty [\mathbf{x}^T \mathbf{Q} \mathbf{x} + \mathbf{u}^T \mathbf{R} \mathbf{u}] d\tau \quad (7.141)$$

This is the time integral of a quadratic function in both state and input variables, and the optimization goal may be interpreted as bringing $\mathbf{x}$ down to zero (regulating $\mathbf{x}$ to 0), but without spending a rather high control effort. Hence, the name linear quadratic regulation (LQR). In addition, $\mathbf{Q}$ and $\mathbf{R}$ are weighting matrices, with the former being at least positive semidefinite and the latter positive definite. Typically, $\mathbf{Q}$ and $\mathbf{R}$ chosen as diagonal matrices with positive diagonal elements whose magnitudes are determined by the degree of relative emphasis that should be given to various elements of $\mathbf{x}$ and $\mathbf{u}$. It is well known that $\mathbf{K}$ that minimizes the cost function (Equation 7.141) is given by

$$\mathbf{K} = -\mathbf{R}^{-1} \mathbf{B}^T \mathbf{K}_r \quad (7.142)$$

where $\mathbf{K}_r$ is the positive-definite solution of the matrix Riccati algebraic equation

$$\mathbf{K}_r \mathbf{A} + \mathbf{A}^T \mathbf{K}_r - \mathbf{K}_r \mathbf{B} \mathbf{R}^{-1} \mathbf{B}^T \mathbf{K}_r + \mathbf{Q} = 0 \quad (7.143)$$

It is also known that the resulting closed-loop control system is stable. Furthermore, the minimum (optimal) value of the cost function (Equation 7.141) is given by

$$J_m = \frac{1}{2} \mathbf{x}^T \mathbf{K}_r \mathbf{x} \quad (7.144)$$

where $\mathbf{x}$ is the present value of the state vector. Major computational burden of the LQR method is in the solution (Equation 7.143). Other limitations of the technique arise due to the need to measure all the state variables (which may be relaxed to some extent). Although stability of the controlled system is guaranteed, the level of stability that is achieved (i.e., stability margin or the level of modal damping) cannot be directly specified. Further, robustness of the control system in the presence of model errors, unknown disturbances and so on, may be questionable. Besides, the cost function incorporates an integral over an infinite time duration, which does not typically reflect the practical requirement of rapid vibration control.

7.9.2.4 Modal Control

The LQR control technique has the serious limitation of not being able to directly achieve specified levels of modal damping, which may be an important goal in vibration control. The method of modal control that accomplishes this objective is *pole placement*, where poles (eigenvalues) of the controlled system are placed at specified values. Specifically, consider the plant (Equation 7.131) and the feedback control law (Equation 7.140). Then, the closed-loop system is given by

$$\dot{\mathbf{x}} = (\mathbf{A} + \mathbf{BK})\mathbf{x} \quad (7.145)$$

It is well known that if the plant ($\mathbf{A}$, $\mathbf{B}$) is controllable, then a control gain matrix $\mathbf{K}$ can be chosen that will arbitrarily place the eigenvalues of the closed-loop system matrix $\mathbf{A} + \mathbf{BK}$. Based on the given assumptions, the modal control technique assigns not only the modal damping but also the damped natural frequencies at specified values. The assumptions given above are quite stringent but they can be relaxed to some degree. However, a shortcoming of this method is the fact that it does not place a restriction on the control effort, for example, as the LQR technique does, in achieving a specified level of modal control.

7.10 Control of Beam Vibrations

Beam is a distributed-parameter system, which in theory has an infinite number of modes of vibration with associated mode shapes and natural frequencies. In this sense, it is an “infinite order” system with infinite DoF. Hence, the computation of modal quantities and associated control inputs can be quite complex. Fortunately, however, just a few modes may be retained in a dynamic model without sacrificing a great deal of accuracy, thereby facilitating simpler control. Some concepts of controlling vibrations in a beam are considered in this section. The present treatment is intended as an illustration of the relevant techniques and is not meant to be exhaustive. These techniques may be extended to other types of continuous system such as beams with different boundary conditions and plates. Because the control techniques that were outlined previously depend on a model, we will first illustrate the procedure of obtaining a state-space model for a beam.

7.10.1 State-Space Model of Beam Dynamics

Consider a Bernoulli–Euler-type beam with Kelvin–Voigt-type internal (material) damping. The beam equation may be expressed as

$$ELv(x, t) + E^*L \frac{\partial v(x, t)}{\partial t} + \rho A(x) \frac{\partial^2 v(x, t)}{\partial t^2} = f(x, t) \quad (7.146)$$

in which L is the partial differential operator given by

$$L = \frac{\partial^2 I(x)}{\partial x^2} \frac{\partial^2}{\partial x^2} \quad (7.147)$$

and

$f(x, t)$ = distributed force excitation per unit length of the beam

$v(x, t)$ = displacement response at location x along the beam at time t

$I(x)$ = second moment of area of the beam cross section about the neutral axis

E = Young's modulus of the beam material

E^* = Kelvin–Voigt material damping parameter

Note that a general beam with nonuniform characteristics is assumed and, hence, the variations of $I(x)$ and $\rho A(x)$ with x are retained in the formulation.

Using the approach of modal expansion, the response of the beam may be expressed by

$$v(x, t) = \sum_{i=1}^{\infty} Y_i(x) q_i(t) \quad (7.148)$$

where $Y_i(x)$ is the i th mode shape of the beam, which satisfies

$$LY_i(x) = \frac{\rho A(x)}{E} \omega_i^2 Y_i(x) \quad (7.149)$$

and ω_i is the i th undamped natural frequency. The orthogonality condition for this general example of a nonuniform beam is

$$\int_{x=0}^l \rho A Y_i Y_j dx = \begin{cases} 0 & \text{for } i \neq j \\ \alpha_j & \text{for } i = j \end{cases} \quad (7.150)$$

Suppose that the forcing excitation on the beam is a set of r point forces $u_k(t)$ located at $x = l_k$, $k = 1, 2, \dots, r$. Then, we have

$$f(x, t) = \sum_{k=1}^r u_k \delta(x - l_k) \quad (7.151)$$

where $\delta(x - l_i)$ is the *Dirac delta function*. Now, substitute Equation 7.148 and Equation 7.151 into Equation 7.146, use Equation 7.149, multiply throughout by $Y_j(x)$, and integrate over $x[0, l]$, using Equation 7.150. This gives

$$\ddot{q}_j + \gamma_j \dot{q}_j(t) + \omega_j^2 q_j = \frac{1}{\alpha_j} \sum_{k=1}^r u_k Y_j(l_k) \quad \text{for } j = 1, 2, \dots \quad (7.152)$$

where

$$\gamma_j = \frac{E^*}{E} \omega_j^2 \quad (7.153)$$

Now, define the state variables x_j according to

$$x_{2j-1} = \omega_j q_j, \quad x_{2j} = \dot{q}_j \quad \text{for } j = 1, 2, \dots \quad (7.154)$$

Assuming that only the first m modes are retained in the expansion, we then have the state equations

$$\dot{x}_{2j-1} = \omega_j x_{2j}, \quad \dot{x}_{2j} = -\omega_j x_{2j-1} - \gamma_j x_{2j} + \frac{1}{\alpha_j} \sum_{k=1}^r u_k Y_j(l_k) \quad \text{for } j = 1, 2, \dots, m \quad (7.155)$$

This can be put in the matrix–vector form of a state-space model

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (7.131)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & \omega_1 & & & 0 \\ -\omega_1 & -\gamma_1 & & & \\ & & \ddots & & \\ & & & 0 & \omega_m \\ 0 & & & -\omega_m & -\gamma_m \end{bmatrix}_{n \times n} \quad (7.156)$$

and

$$\mathbf{B} = \begin{bmatrix} 0 & & 0 \\ Y_1(l_1)/\alpha_1 & \cdots & Y_1(l_r)/\alpha_1 \\ \vdots & & \vdots \\ 0 & & 0 \\ Y_m(l_1)/\alpha_m & \cdots & Y_m(l_r)/\alpha_m \end{bmatrix}_{n \times r} \quad (7.157)$$

with $n = 2m$, where m is the number of modes retained in the modal expansion. Note that, as the number of modes used in this model increases, both the accuracy and the computational effort that is needed for the control problem increase because of the proportional increase of the system order. At some point, the potential improvement in accuracy by further increasing the model size would be insignificant in comparison with added computational burden. Hence, a balance must be struck in this tradeoff.

7.10.2 Control Problem

The state-space model (Equation 7.131) for the beam dynamics, with matrices (Equation 7.156 and Equation 7.157), is known to be *controllable*. Hence, it is possible to determine a constant-gain feedback controller $\mathbf{u} = \mathbf{K}\mathbf{x}$ that minimizes a quadratic-integral cost function of the form in Equation 7.141. Also, a similar controller can be determined that places the eigenvalues of the system at specified locations thereby achieving not only specified levels of modal damping but also a specified set of natural frequencies. However, there is a practical obstacle to achieving such an active controller. Note that, in the model given in Equation 7.156 and Equation 7.157, the state variables are proportional to the modal variables q_i and their time derivatives $\dot{q}_i$. They are not directly measurable. However, the displacements and velocities at a set of discrete locations along the beam can usually be measured. Let these locations (s) be denoted by $p_1, p_2, \dots, p_s$. Thus, in view of the modal expansion (Equation 7.148), the measurements can be expressed as

$$v(p_j, t) = \sum_{i=1}^m Y_i(p_j)q_i(t), \quad \dot{v}(p_j, t) = \sum_{i=1}^m Y_i(p_j)\dot{q}_i(t) \quad \text{for } j = 1, 2, \dots, s \quad (7.158)$$

Now, define the output (measurement) vector $\mathbf{y}$ according to

$$\mathbf{y} = [v(p_1, t), \dot{v}(p_1, t), \dots, v(p_s, t), \dot{v}(p_s, t)]^T \quad (7.159)$$

In view of Equation 7.158 and the definitions of the state variable in Equation 7.154, we can write

$$\mathbf{y} = \mathbf{C}\mathbf{x} \quad (7.160)$$

with

$$\mathbf{C} = \begin{bmatrix} Y_1(p_1)/\omega_1 & 0 & \cdots & Y_m(p_1)/\omega_m & 0 \\ 0 & Y_1(p_1) & \cdots & 0 & Y_m(p_1) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ Y_1(p_s)/\omega_1 & 0 & \cdots & Y_m(p_s)/\omega_m & 0 \\ 0 & Y_1(p_s) & \cdots & 0 & Y_m(p_s) \end{bmatrix}_{2s \times n} \quad (7.161)$$

Hence, an active controller is possible of the form:

$$\mathbf{u} = \mathbf{H}\mathbf{y} \quad (7.162)$$

which is an output feedback controller. Therefore, in view of Equation 7.160, we have

$$\mathbf{u} = \mathbf{H}\mathbf{C}\mathbf{x} \quad (7.163)$$

This is not the same as complete state feedback $\mathbf{u} = \mathbf{K}\mathbf{x}$ where $\mathbf{K}$ can take any real value (and, hence, the LQR solution in Equation 7.142 and the complete pole placement solution cannot be applied directly). In Equation 7.163, only $\mathbf{H}$ can be arbitrarily chosen, and $\mathbf{C}$ is completely determined according to Equation 7.161. The resulting product $\mathbf{H}\mathbf{C}$ will not usually correspond to either the LQR solution or the complete pole assignment solution. Still, the output feedback controller in Equation 7.162 can provide a satisfactory performance. However, a sufficient number of displacement and velocity sensors (s) have to be used in conjunction with a sufficient number of actuators (r) for active control. This will increase the system complexity and cost. Furthermore, due to added components and their active nature, the reliability of fault-free operation may degrade somewhat. A satisfactory alternative would be to use passive control devices such as dampers and dynamic absorbers, which is illustrated below. Note that in the matrices $\mathbf{B}$ and $\mathbf{C}$ given by Equation 7.157 and Equation 7.161, both the actuator locations l_j and the sensor locations p_j are variable. Hence, there exists an additional design freedom (or optimization parameters) in selecting the sensor and actuator locations in achieving satisfactory control.

7.10.3 Use of Linear Dampers

Now, consider the use of a discrete set of linear dampers for controlling beam vibration. Suppose that r linear dampers with damping constants b_j are placed at locations l_j , $j = 1, 2, \dots, r$ along the beam, as schematically shown in Figure 7.34. The damping forces are given by

$$u_j = -b_j \dot{v}(l_j, t) \quad \text{for } j = 1, 2, \dots, r \quad (7.164)$$

Substituting the truncated modal expansion (m modes)

$$\dot{v}(l_j, t) = \sum_{i=1}^m Y_i(l_j) \dot{q}_i(t) \quad (7.165)$$

we get, in view of Equation 7.154, the passive feedback control action

$$\mathbf{u} = -\mathbf{K}\mathbf{x} \quad (7.166)$$

with

$$\mathbf{K} = \begin{bmatrix} 0 & b_1 Y_1(l_1) & \cdots & 0 & b_1 Y_m(l_1) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & b_r Y_1(l_r) & \cdots & 0 & b_r Y_m(l_r) \end{bmatrix}_{r \times n} \quad (7.167)$$

By substituting Equation 7.166 into Equation 7.131, we have the closed-loop system equation

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{F})\mathbf{x} = \mathbf{A}_c \mathbf{x} \quad (7.168)$$

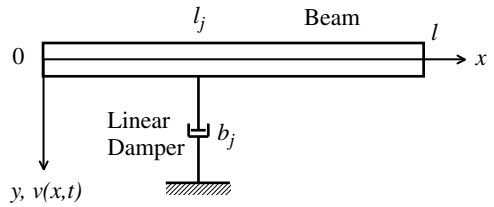


FIGURE 7.34 Use of linear dampers in beam vibration control.

where $\mathbf{F} = \mathbf{BK}$ and is given by

$$\mathbf{F} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & \sum b_i Y_{11}(l_i)/\alpha_1 & \cdots & 0 & \sum b_i Y_{1m}(l_i)/\alpha_1 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & \sum b_i Y_{m1}(l_i)/\alpha_m & \cdots & 0 & \sum b_i Y_{mm}(l_i)/\alpha_m \end{bmatrix}_{n \times n} \quad (7.169)$$

with

$$Y_{ij}(x) = Y_i(x)Y_j(x) \quad (7.170)$$

In this case, the controller design involves the selection of the damping constants b_i and the damper locations l_j to achieve the required performance. This may be achieved, for example, by seeking to make the eigenvalues of the closed-loop system matrix $\mathbf{A}_c$ reach a set of desired values. This achieves the desired modal damping and natural frequency characteristics. However, given that the structure of the $\mathbf{F}$ matrix is fixed, as seen in Equation 7.169, this is not equivalent to complete state feedback (or complete output feedback). Hence, generally, it is not possible to place the poles of the system at the exact desired locations.

7.10.3.1 Design Example

In realizing a desirable modal response of a beam using a set of linear dampers, one may seek to minimize a cost function of the form

$$J = \text{Re}(\boldsymbol{\lambda} - \boldsymbol{\lambda}_d)^T \mathbf{Q} \text{Re}(\boldsymbol{\lambda} - \boldsymbol{\lambda}_d) + \text{Im}(\boldsymbol{\lambda} - \boldsymbol{\lambda}_d)^T \mathbf{R}(\boldsymbol{\lambda} - \boldsymbol{\lambda}_d) \quad (7.171)$$

where $\boldsymbol{\lambda}$ are the actual eigenvalues of the closed-loop system matrix ($\mathbf{A}_c$), and $\boldsymbol{\lambda}_d$ are the desired eigenvalues that will give the required modal performance (damping ratios and natural frequencies). “Re” denotes the real part and “Im” denotes the imaginary part. Weighting matrices $\mathbf{Q}$ and $\mathbf{R}$, which are real and diagonal with positive diagonal elements, should be chosen to relatively weight various eigenvalues. This allows the emphasis of some eigenvalues over others, with real parts and the imaginary parts weighting separately.

Various computational algorithms are available for minimizing the cost function (Equation 7.171). Although the precise details are beyond the scope of this book, we will present an example result. Consider a uniform simply supported 12×5 American Standard beam, with the following pertinent specifications: $E = 2 \times 10^8$ kPa (29×10^6 psi), $\rho A = 47$ kg/m (2.6 lb/in.), length $l = 15.2$ m (600 in.), $I = 9 \times 10^{-5}$ m⁴ (215.8 in.⁴). The internal damping parameter for the j th mode of vibration is given by

$$E^*(\omega_j) = (g_1/\omega_j) + g_2 \quad (7.172)$$

in which ω_j is the j th undamped natural frequency given by

$$\omega_j = (j\pi/l)^2 \sqrt{EI/\rho A} \quad (7.173)$$

The numerical values used for the damping parameters are $g_1 = 88 \times 10^4$ kPa (12.5×10^4 psi) and $g_2 = 3.4 \times 10^4$ kPa s (5×10^3 psi s). For the present problem, $Y_i(x) = \sqrt{2}\sin(j\pi x/l)$ and $\alpha_j = \rho A l$ for all j .

First, ω_j and γ_j are computed using Equation 7.173 and Equation 7.153, respectively, along with Equation 7.172. Next, the open-loop system matrix $\mathbf{A}$ is formed according to Equation 7.156 and its eigenvalues are computed. These are listed in Table 7.2, scaled to the first undamped natural frequency (ω_1). Note that in view of the very low levels of internal material damping of the beam, the actual natural frequencies, as given by the imaginary parts of the eigenvalues, are almost identical to the undamped natural frequencies.

Next, we attempt to place the real parts of the (scaled) eigenvalues all at -0.20 while exercising no constraint on the imaginary parts (i.e., damped natural frequencies) by using: (a) single damper, and

TABLE 7.2 Eigenvalues of the Open-Loop (Uncontrolled) Beam

Mode	Eigenvalue (rad/sec) (Multiply by 26.27)
1	$-0.000126 \pm j1.0$
2	$-0.000776 \pm j4.0$
3	$-0.002765 \pm j9.0$
4	$-0.007453 \pm j16.0$
5	$-0.016741 \pm j25.0$
6	$-0.033.75 \pm j36.0$

(b) two dampers. In the cost function (Equation 7.171), the first three modes are more heavily weighted than the remaining three. Initial values of the damper parameters are $b_1 = b_2 = 0.1$ lbf s/in. (17.6 N s/m) and the initial locations $l_1/l = 0.0$ and $l_2/l = 0.5$. At the end of the numerical optimization, using a modified gradient algorithm, the following optimized values were obtained:

1. Single-damper control

$$b_1 = 36.4 \text{ lbf s/in. } (6.4 \times 10^3 \text{ N s/m})$$
$$l_1/l = 0.3$$

The corresponding normalized eigenvalues (of the closed-loop system) are given in Table 7.3.

2. Two-damper control

$$b_1 = 22.8 \text{ lbf s/in. } (4.0 \times 10^3 \text{ N s/m})$$
$$b_2 = 12.1 \text{ lbf s/in. } (2.1 \times 10^3 \text{ N s/m})$$
$$l_1/l = 0.25, \quad l_2/l = 0.43$$

The corresponding normalized eigenvalues are given in Table 7.4.

It would be overly optimistic to expect perfect assignment all real parts at -0.2 . However, note that good levels of damping have been achieved for all modes except for Mode 3 in the single-damper control and Mode 4 in the two-damper control. In any event, because the contribution of the higher modes towards the overall response, is relatively smaller, it is found that the total response (e.g., at point $x = l/12$) is well damped in both cases of control.

TABLE 7.3 Eigenvalues of the Beam with an Optimal Single Damper

Mode	Eigenvalue (rad/s) (Multiply by 26.27)
1	$-0.225 \pm j0.985$
2	$-0.307 \pm j3.955$
3	$-0.037 \pm j8.996$
4	$-0.119 \pm j15.995$
5	$-0.355 \pm j24.980$
6	$-0.158 \pm j35.990$

TABLE 7.4 Eigenvalues of the Beam with Optimized Two Dampers

Mode	Eigenvalue (rad/s) (Multiply by 26.27)
1	$-0.216 \pm j0.982$
2	$-0.233 \pm j3.974$
3	$-0.174 \pm j8.997$
4	$-0.079 \pm j15.998$
5	$-0.145 \pm j24.999$
6	$-0.354 \pm j35.989$

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Appendix 7A

MATLAB[®] Control Systems Toolbox

7A.1 Introduction

Modeling, analysis, design, data acquisition, and control are important activities within the field of vibration. Computer software tools and environments are available for effectively carrying out, both at the learning level and at the professional application level. Several such environments and tools are commercially available.

MATLAB¹ is an interactive computer environment with a high-level language and tools for scientific and technical computation, modeling and simulation, design, and control of dynamic systems. SIMULINK¹ is a graphical environment for modeling, simulation, and analysis of dynamic systems, and is available as an extension to MATLAB. The Control Systems Toolbox of MATLAB is suitable in the analysis, design, and control of mechanical vibrating systems.

¹MATLAB and SIMULINK are registered trademarks and products of The MathWorks, Inc. LabVIEW is a product of National Instruments, Inc.

7A.2 MATLAB

MATLAB interactive computer environment is very useful in computational activities in Mechatronics. Computations involving scalars, vectors, and matrices can be carried out and the results can be graphically displayed and printed. MATLAB toolboxes are available for performing specific tasks in a particular area of study such as control systems, fuzzy logic, neural network, data acquisition, image processing, signal processing, system identification, optimization, model predictive control, robust control, and statistics. User guides, Web-based help, and on-line help from the parent company, Math-Works, Inc., and various other sources. What is given here is a brief introduction to get started in MATLAB for tasks that are particularly related to Control Systems and Mechatronics.

7A.2.1 Computations

Mathematical computations can be done by using the MATLAB command window. Simply type in the computations against the MATLAB prompt “>>” as illustrated next.

7A.2.2 Arithmetic

An example of a simple computation using MATLAB is given below:

```
>> x = 2; y = - 3;
>> z = x^2 - x * y + 4
z =
14
```

In the first line, we have assigned values 2 and 3 to two variables x and y . In the next line, the value of an algebraic function of these two variables is indicated. Then, MATLAB provides the answer as 14. Note that if you place a “;” at the end of the line, the answer will not be printed/displayed.

Table 7A.1 gives the symbols for common arithmetic operations used in MATLAB. Following example shows the solution of the quadratic equation $ax^2 + bx + c = 0$:

```
>> a = 2; b = 3; c = 4;
>> x = (-b + sqrt(b^2 - 4*a*c))/(2*a)
x =
- 0.7500 + 1.1990i
```

The answer is complex, where i denotes $\sqrt{-1}$. Note that the function `sqrt()` is used, which provides the positive root only. Some useful mathematical functions are given in [Table 7A.2](#).

7A.2.3 Arrays

An array may be specified by giving the start value, increment, and the end value limit. An example is given below.

```
>> x = (0.9:-0.1:0.42)
x =

0.9000 0.8000 0.7000 0.6000 0.5000
```

TABLE 7A.1 MATLAB Arithmetic Operations

Symbol	Operation
+	Addition
-	Subtraction
*	Multiplication
/	Division
^	Power

TABLE 7A.2 Useful Mathematical Functions in MATLAB

Function	Description
abs()	Absolute value/magnitude
acos()	Arc-cosine (inverse cosine)
acosh()	Arc-hyperbolic-cosine
asin()	Arc-sine
atan()	Arc-tan
cos()	Cosine
cosh()	Hyperbolic cosine
exp()	Exponential function
imag()	Imaginary part of a complex number
log()	Natural logarithm
log10()	Log to base 10 (common log)
real()	Real part of a complex number
sign()	Signum function
sin()	Sine
sqrt()	Positive square root
tan()	Tan function

Note that MATLAB is case sensitive.

The entire array may be manipulated. For example, all the elements are multiplied by π as below:

```
>> x = x*pi
x =
2.8274 2.5133 2.1991 1.8850 1.5708
```

The second and the fifth elements are obtained by

```
>> x([2 5])
ans =
2.5133 1.5708
```

Next, we form a new array y using x , and then plot the two arrays, as shown in Figure 7A.1:

```
>> y = sin(x);
>> plot(x,y)
```

A polynomial may be represented as an array of its coefficients. For example, the quadratic equation $ax^2 + bx + c = 0$ as given before, with $a = 2$, $b = 3$, and $c = 4$, may be solved using the function “roots” as below:

```
>> p = [2 3 4];
>> roots(p)
ans =
-0.7500 + 1.1990i
-0.7500 - 1.1990i
```

The answer is the same as we obtained before.

7A.2.4 Relational and Logical Operations

Useful relational operations in MATLAB are given in Table 7A.3. Basic logical operations are given in Table 7A.4.

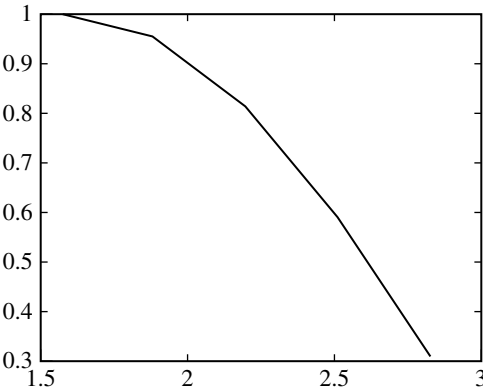


FIGURE 7A.1 A plot using MATLAB.

TABLE 7A.3 Some Relational Operations

Operator	Description
<	Less than
<=	Less than or equal to
>	Greater than
>=	Greater than or equal to
==	Equal to
~=	Not equal to

TABLE 7A.4 Basic Logical Operations

Operator	Description
&	AND
	OR
~	NOT

Consider the following example:

```
>> x = (0:0.25:1)*pi
x =
    0    0.7854    1.5708    2.3562    3.1416
>> cos(x) > 0
ans =
    1    1    1    0    0
>> (cos(x) > 0)&(sin(x) > 0)
ans =
    0    1    1    0    0
```

In this example, first an array is computed. Then the cosine of each element is computed. Next it is checked whether the elements are positive. (A truth value of 1 is sent out if true and a truth value of 0 if false.) Finally, the “AND” operation is used to check whether both corresponding elements of two arrays are positive.

7A.2.5 Linear Algebra

MATLAB can perform various computations with vectors and matrices (see Appendix 3A and Appendix 6A). Some basic illustrations are given here.

A vector or a matrix may be specified by assigning values to its elements. Consider the following example:

```
>> b = [ 1.5  -2];
>> A = [2   1;  -1  1];
>> b = b'
b =
    1.5000
   -2.0000
>> x = inv(A) * b
x =
    1.1667
   -0.8333
```

In this example, first a second-order row vector and 2×2 matrix are defined. The row vector is transposed to get a column vector. Finally the matrix–vector equation $\mathbf{Ax} = \mathbf{b}$ is solved according to $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. The determinant and the eigenvalues of $\mathbf{A}$ are determined by

TABLE 7A.5 Some Matrix Operations in MATLAB

Operation	Description
+	Addition
−	Subtraction
*	Multiplication
/	Division
^	Power
'	Transpose

TABLE 7A.6 Useful Matrix Functions in MATLAB

Function	Description
det()	Determinant
inv()	Inverse
eig()	Eigenvalues
[,] = eig()	Eigenvectors and eigenvalues

```
>> det(A)
ans =
    3
>> eig(A)
ans =
    1.5000 + 0.8660i
    1.5000 - 0.8660i
```

Both eigenvectors and eigenvalues of **A** computed as

```
>>[V,P] = eig(A)
V =
    0.7071          0.7071
   -0.3536 + 0.6124i   -0.3536 - 0.6124i
P =
    1.5000 + 0.8660i          0
          0    1.5000 - 0.8660i
```

Here, the symbol **V** is used to denote the matrix of eigenvectors. The symbol **P** is used to denote the diagonal matrix whose diagonal elements are the eigenvalues.

Useful matrix operations in MATLAB are given in Table 7A.5 and several matrix functions are given in Table 7A.6.

7A.2.6 M-Files

The MATLAB commands have to be keyed in on the command window, one by one. When several commands are needed to carry out a task, the required effort can be tedious. Instead, the necessary commands can be placed in a text file, edited as appropriate (using text editor), which MATLAB can use to execute the complete task. Such a file is called an M-file. The file name must have the extension “m” in the form *filename.m*. A toolbox is a collection of such files, for use in a particular application area (e.g., control systems, fuzzy logic). Then, by keying in the M-file name at the MATLAB command prompt, the file will be executed. The necessary data values for executing the file have to be assigned beforehand.

7A.3 Control Systems Toolbox

There are several toolboxes with MATLAB, which can be used to analyze, compute, simulate, and design control problems. Both time-domain representations and frequency-domain representations can be

used. Also, both classical and modern control problems can be handled. The application is illustrated here through several control problems.

7A.3.1 MATLAB Modern Control Examples

Several examples in modern control engineering are given now to illustrate the use of MATLAB in control.

7A.3.1.1 Pole Placement of a Third-Order Plant

A mechanical plant is given by the input–output differential equation $\ddot{x} + \ddot{x} = u$, where u is the input and x is the output. Determine a feedback law that will yield approximately a simple oscillator with a damped natural frequency of 1 unit and a damping ratio of $1/\sqrt{2}$.

To solve this problem, first we define the state variables as $x_1 = x$, $x_2 = \dot{x}_1$, and $x_3 = \dot{x}_2$. The corresponding state-space model is

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{B}} u$$

$$y = \underbrace{[1 \quad 0 \quad 0]}_{\mathbf{C}} \mathbf{x}$$

The open-loop poles and zeros are obtained using the following MATLAB commands:

```
>> A = [0 1 0; 0 0 1; 0 0 -1];
>> B = [0; 0; 1];
>> C = [1 0 0];
>> D = [0];
>> sys_open = ss(A,B,C,D);
>> [nat_freq_open,damping_open,poles_open] = damp(sys_open)
>> pzmap(sys_open)
```

The open-loop poles are: $[0 \ 0 \ -1]^T$.

The step response of the open-loop system is obtained using the command:

```
>> step(sys_open)
```

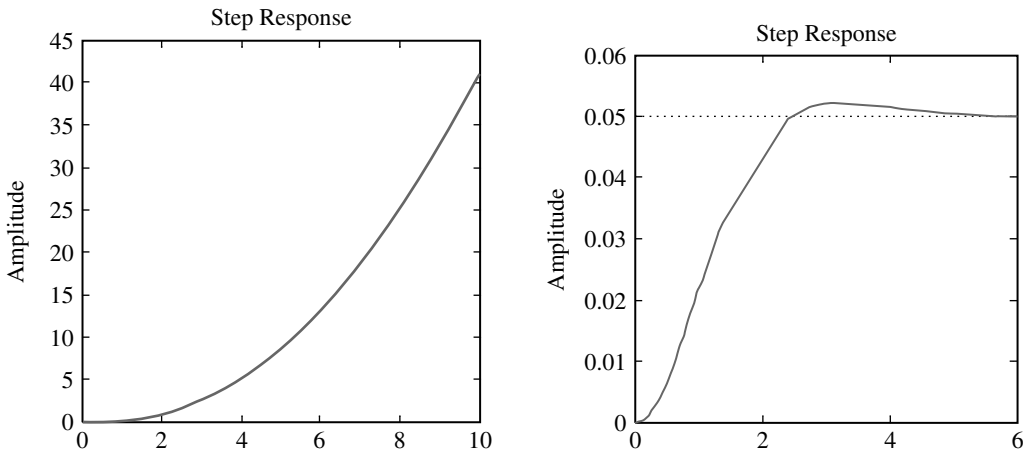


FIGURE 7A.2 (a) Step response of the open-loop system; (b) step response of the third-order system with pole-placement control.

The result is shown in Figure 7A.2(a). Clearly, the system is unstable.

With the desired damped natural frequency $\omega_d = 1$ and damping ratio $\zeta = 1/\sqrt{2}$, we get the undamped natural frequency $\omega_n = \sqrt{2}$ and, hence, $\zeta\omega_n = 1$. It follows that we need to place two poles at $-1 \pm j$. Also the third pole has to be far from these two on the left half plane (LHP); say, at -10 . The corresponding control gain K can be computed using the “place” command in MATLAB:

```
>> p = [-1 + j -1 - j -10];
>> K = place(A,B,p)
place:ndigits = 15
K =
    20.0000    22.0000    11.0000
```

The corresponding step response of the closed-loop system is shown in Figure 7A.2(b).

7A.3.1.2 Linear Quadratic Regulator for a Third-Order Plant

For the third-order plant in the previous example, we design a linear quadratic regulator (LQR), which has a state feedback controller, using MATLAB Control Systems Toolbox. The MATLAB command $K = \text{lqr}(A,B,Q,R)$ computes the optimal gain matrix K such that the state-feedback law $\mathbf{u} = -K\mathbf{x}$ minimizes the quadratic cost function

$$J = \int_0^{\infty} (\mathbf{x}^T \mathbf{Q} \mathbf{x} + \mathbf{u}^T \mathbf{R} \mathbf{u}) dt$$

The weighting matrices $\mathbf{Q}$ and $\mathbf{R}$ are chosen to apply the desired weights to the various states and inputs. The MATLAB commands for designing the controller are

```
>> A = [0 1 0; 0 0 1; 0 0 -1];
>> B = [0; 0; 1];
>> C = [1 0 0];
>> D = [0];
>> Q = [2 0 0; 0 2 0; 0 0 2];
>> R = 2;
>> Klqr = lqr(A,B,Q,R)
>> lqr_closed = ss(A-B*Klqr,B,C,D);
>> step(lqr_closed)
```

The step response of the system with the designed LQR controller is shown in Figure 7A.3.

7A.3.1.3 Modal Analysis Example

Consider the two-DoF mechanical system shown in Figure 7A.4. We now solve the modal analysis problem using MATLAB, for the numerical values

$$\alpha = 0.5, \quad \beta = 0.5, \quad m = 1 \text{ kg}; \quad k = 1 \text{ N/m}$$

For the given mass matrix $\mathbf{M}$ and the stiffness matrix $\mathbf{K}$, the solution steps for the alternative approach of modal analysis are

1. Determine $\mathbf{M}^{1/2}$ and $\mathbf{M}^{-1/2}$.
2. Solve for the eigenvalues λ and the eigenvectors Φ of $\mathbf{M}^{-1/2} \mathbf{K} \mathbf{M}^{-1/2}$. These eigenvalues are the squares of the natural frequencies of the original system.

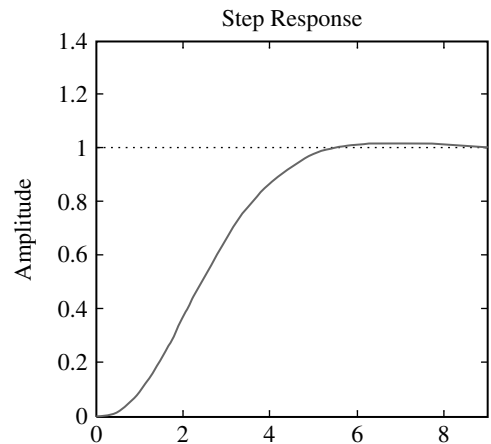


FIGURE 7A.3 Step response of the third-order system with LQR control.

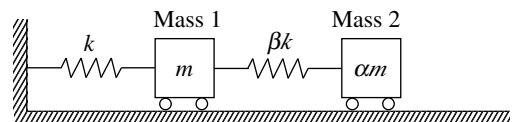


FIGURE 7A.4 A two-DoF system.

3. Determine the modal vectors Ψ of the original system using the transformation $\Psi = M^{-1/2}\Phi$.

The program code is given below:

```
%Modal Analysis Example
clear;
m = 1.0;
k = 1.0;
M = [ m 0; 0 m/2 ];
K = [3/2 * k - k/2;
     - k/2 k/2];
M_sqrt = M^0.5;
M_s_inv = inv(M_sqrt);
lemda = eig(M_s_inv * K * M_s_inv);
[U,D] = eig(M_s_inv * K * M_s_inv);
V = M_s_inv * U;
disp('Natural frequencies')
fprintf('omega1 = %10.3f%14.3f\n',sqrt(lemda(2,1)));
fprintf('\n omega2 = %10.3f%14.3f\n',sqrt(lemda(1,1)));
fprintf('\n')
fprintf('\nMode shapes \n')
fprintf('First mode   Second Mode \n')
for i = 1:2
    fprintf('%10.3f%14.3f\n',V(i,2)/V(1,2),V(i,1)/V(1,1));
end
```

The necessary results are obtained as shown below:

```
>> Natural frequencies
omega1 = 1.414
omega2 = 0.707
```

Mode Shapes

First Mode	Second Mode
1.000	1.000
- 1.000	2.000