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International Centre
for Mechanical Sciences

Cellular and Porous Materials in Structures and Processes

CISM Courses and Lectures, vol. 521



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INTERNATIONAL CENTRE FOR MECHANICAL SCIENCES

COURSES AND LECTURES - No. 521



CELLULAR AND POROUS MATERIALS IN STRUCTURES AND PROCESSES

EDITED BY

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Printed in Italy

SPIN 80016755

All contributions have been typeset by the authors.

ISBN 978-3-7091-0296-1 SpringerWienNewYork

PREFACE

Metallic and polymer foams are widely used in modern industries, e.g., the aircraft and the automotive industries, but also with other application fields like biomechanics. The reason for this is some specific properties of these advanced materials. They are very light, but the specific strength is comparable with the classical structural materials. If they are applied as sandwiches, the specific properties can be even much better. In addition, they are able to absorb energy which allows the use of these materials as crash elements.

In general, technological parameters in foam production are adjusted such that a uniform effective foam density is achieved throughout the products. Some technologies, e.g., injecting foam in a cavity or filling a mold with foam by an expansion process, naturally result in non-uniform distributions of the effective density. These inhomogeneities of the effective foam density may be exploited in structural design, essentially treating the foam as a functionally graded material.

FGMs are composite materials where the composition or the microstructure is locally varied so that a certain variation of the local material properties is achieved. Modern FGMs are constructed for complex requirements, such as the heat shield of a rocket or implants for humans. In these cases the analysis of the material and the structures made of FGMs cannot be only limited to the mechanical behavior. FGMs can be modeled as a porous material with non-homogeneous distribution of porosity.

Engineering structures made of porous materials, especially metal foams, have been used in different applications in the last decades. A metal foam is a cellular structure consisting of a solid metal, for example aluminium, steel, copper, etc., containing a large volume fraction of gas-filled pores. There are two types of metal foams. One is the closed-cell foam, while the second one is the open-cell foam. The defining characteristics of metal foams are a very high porosity: typically well over 80%, 90% and even 98% of the volume consists of void spaces.

The course "Cellular and Porous Materials: Modeling - Testing - Application" was devoted to cellular and porous materials, and the modeling and simulation of the material behavior, but also the structural analysis of structures made of foams, the testing and the appli-

cations. The main question was on which scale one has to model the material. It is well-known that there are as a minimum three scales:

- *micro-mechanical scale*

The starting point in this case is a unit cell which can be open or closed (open-cell foams, closed-cell foams). The description is very fine (the unit cell is modeled by beams, plates or shells), but the computational effort increases significantly if high accuracy (e.g. in the case of plasticity) is required, and the identification of the properties is still under question.

- *meso-mechanical scale*

Now the starting point is an agglomerate of cells (representative volume), computational effort is quite expensive. In addition, the periodicity assumption is under question.

- *macro-mechanical scale*

Now one has the size of structural elements or specimens. Using standard numerical techniques the computational effort is as in the case of classical structural materials. The disadvantage is that "smeared" properties should be introduced (that means the constitutive behavior will be represented by phenomenological equations) and local effects cannot be investigated.

During the 6 lectures

- *Fracture Mechanics of Foams (Livia Marsavina)*
- *Finite Element Modeling of Foams (Thomas Darner)*
- *Plasticity of Three-dimensional Foams (Andreas Öchsner)*
- *Thin-walled Structures Made of Foams (Holm Altenbach)*
- *Plasticity of Porous and Powder Metals (Sergei Alexandrov)*
- *Impact of Cellular Materials (Henry Tan)*

the advantages and disadvantages of each approach were discussed.

The basics of the analysis of structures made of foams is the continuum mechanics. Briefly were presented the foundations. In addition, the introduction in the theory of elasticity and plasticity was given. Special attention was paid to the yield criteria, to anisotropy and to the different behavior in tension and compression. Many structural elements made of foams can be presented by beam, plate or shell models. As an example, a plate theory based on the direct approach was presented. One of the basic elements of this theory is the effective property concept. Such a theory is suitable for the global analysis of plates (deflections, frequencies, etc.).

Holm Altenbach and Andreas Öchsner

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Fracture Mechanics of Cellular Solids

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Abstract Most of foam materials crush in compression, while in tension fail by propagating of single cracks. Rigid polymer foams have a linear-elastic behavior in tension up to the brittle fracture, so they can be treated using Linear Elastic Fracture Mechanics. This chapter presents the Linear Elastic Fracture Mechanics parameters and criteria with application for foam materials. Methods for experimental determination for tear strength and fracture toughness are presented. The obtained experimental results are discussed and compared with other published data. Main micromechanical models for predicting fracture toughness of cellular materials are also discussed.

1 Fundamentals of Fracture Mechanics

1.1 Introduction

Most of the structures contain small flaws whose size and distribution are dependent on the material and its processing. These flaws can extend rapidly at applied load levels which are nominally well within the linear elastic part of load-displacement response of the structure. Several structural failures can be associated with the fracture of one or more of its components. When such events occur, they are mostly unexpected, sudden, and unfortunate, and it is natural for us to focus attention on minimizing the undesired consequences when designing and analyzing modern-day structures. The study of crack behavior, prevention and analysis of fracture of materials is known as fracture mechanics. The goal of fracture mechanics is to predict the critical loads that will cause catastrophic failure in a structure or component. The influence of pre-existing cracks on the strength of components needs to be understood and quantified. The initiation and growth of crack like defects during service (fatigue, creep) needs to be understood and quantified. A defect tolerant design and maintenance philosophy is necessary to be developed.

The most famous failure produced by foam fracture is the Space Shuttle Columbia disaster occurred on February 1, 2003, when disintegrated over

Palestine, Texas during re-entry into the Earth's atmosphere, with the loss of all seven crew members. The Columbia accident was a result of damage sustained during launch when a piece of foam insulation with the size of a small briefcase broke off the Space Shuttle external tank under the aerodynamic forces of launch. The debris struck the leading edge of the left wing, damaging the Shuttle's thermal protection system (TPS), Fig. 1, which protects it from heat generated with the atmosphere during re-entry. While Columbia was still in orbit, some engineers suspected damage, but

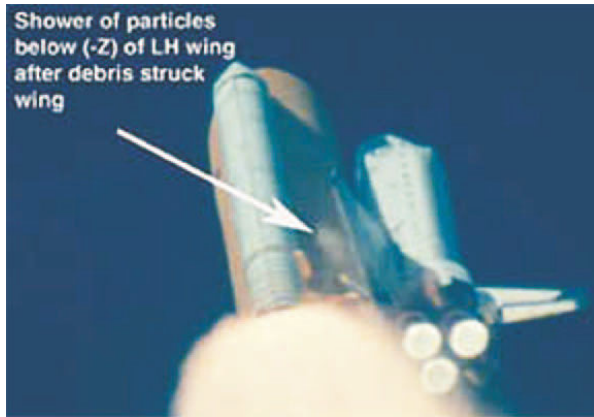


Figure 1. The foam from the fuel tank hit the left wing of the shuttle KSC-03PD-0250: <http://mediaarchive.ksc.nasa.gov/>

NASA managers limited the investigation, on the grounds that little could be done even if problems were found. NASA engineers reviewed the launch tapes to reveal a 500 mm piece of hardened insulation foam breaking off the main fuel tank and hitting the shuttle's left wing. When the shuttle broke up during reentry, no one could ascertain if it was because of the damaged wing hit by the foam.

The reason to apply fracture mechanics for cellular materials can be summarized as follows:

- The Columbia disaster is an example of catastrophic failure caused by fracture of foam,
- The rigid foams have a brittle fracture in tension, and the failure is produced by single crack propagation,
- LEFM approaches describe the fracture of rigid polyurethane foams and expanded polystyrene foams, while tearing occurs for semi-rigid low-densities polyethylene, or flexible polyurethane foams.

1.2 Linear Elastic Fracture Mechanics

Many textbooks present the principle of linear elastic fracture mechanics (LEFM) (Broek, 1986; Ewalds and Wanhill, 1989; Anderson, 1995; Shukla, 2005). Here only a review of the main fracture criteria for LEFM will be presented.

Griffith Energy. Griffith (1921) proposed an energy balance approach to study the fracture phenomenon in cracked bodies. He proposed that the reduction in strain energy of a body when the crack propagates could be equated to the increase in surface energy due to the increase in the surface area. The Griffith theory assumed that the fracture strength was limited by the existence of initial cracks and that brittle materials contain elliptical microcracks, which introduce high stress concentrations near their tips. He developed a relationship between crack length (a), surface energy connected with traction-free crack surfaces ($2\gamma_e$), Young's modulus E and applied stress σ , which is given by

$$\sigma\sqrt{a} = \sqrt{\frac{2E\gamma_e}{\pi}} \quad (1)$$

According with Eq. (1), which is valid for plane stress, the fracture occurs when the product $\sigma\sqrt{a}$ reaches a particular value depending on material constants E and γ_e .

Later Irwin (1957) introduced the energy release rate $\mathcal{G}$ defined as the rate of change in potential energy with crack area for a linear elastic material. For an infinite plate (Fig. 2), with a crack of length $2a$, loaded with a remote tensile stress σ the energy release rate is:

$$\mathcal{G} = \frac{\pi\sigma^2 a}{E} \quad \text{- for plane stress} \quad (2)$$

$$\mathcal{G} = \frac{\pi\sigma^2 a}{E}(1 - \nu^2) \quad \text{- for plane strain} \quad (3)$$

Irwin (1957) proposed a fracture criteria based on the energy release rate which states that the fracture occurs when the energy release rate reaches a critical value $\mathcal{G}_c$, known also as fracture toughness

$$\mathcal{G} = \mathcal{G}_c \quad (4)$$

The critical combination of stress and crack length for fracture occurrence can be expressed in the form

$$\mathcal{G}_c = \frac{\pi\sigma_f^2 a_c}{E} \quad (5)$$

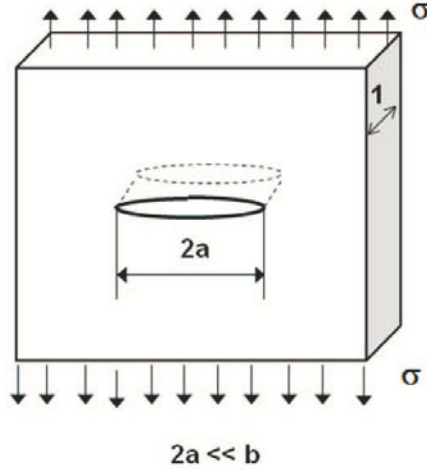


Figure 2. Crack of length $2a$ in an infinite plate remote loaded with tensile stress σ

where σ_f represents the fracture stress and a_c is the critical crack length. Eq. (1) can also be expressed in the form

$$\frac{\pi \sigma^2 a}{E} = 2\gamma_e \quad \text{- for plane stress} \quad (6)$$

$$\frac{\pi \sigma^2 a}{E}(1 - \nu^2) = 2\gamma_e \quad \text{- for plane strain} \quad (7)$$

where the left hand side represents the energy release rate $\mathcal{G}$, while the right hand side represents the energy needed for incremental crack growth. In addition, it is a measure of resistance to crack advance. So, the condition for unstable crack growth occurs when the energy available for crack extension (crack driving force) equals the resistance of the material to crack advance

$$\mathcal{G} = \mathcal{R} \quad (8)$$

and is graphically represented in Fig. 3.

Orowan (1955) extended the Griffith criterion valid for brittle materials to materials with limited plastic deformations at the crack tip. He postulated that the resistance to crack growth in engineering materials is equal to the sum of elastic surface energy γ_e and the plastic dissipation or plastic

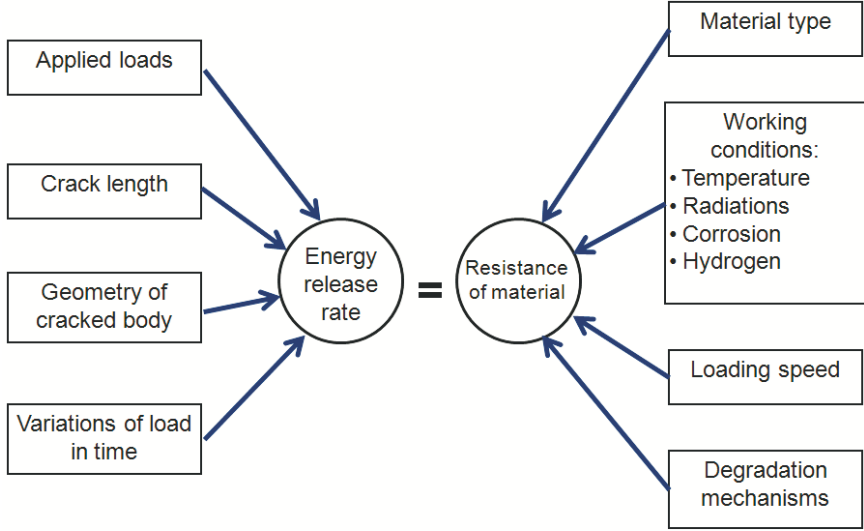


Figure 3. Unstable crack growth condition

work γ_p accompanying crack extension

$$\frac{\pi\sigma^2 a}{E} = 2(\gamma_e + \gamma_p) \quad (9)$$

Asymptotic stress field at the crack tip. In general, consideration of three different loading symmetries is sufficient to decompose any arbitrary loading with respect to a crack, Fig. 4. Mode I corresponds to opening mode, Mode II to in-plane shear and Mode III to out - of - plane shear. In most engineering applications the brittle fracture of structures occurs in Mode I.

According with linear elastic fracture mechanics approach the stress field near the crack tip depends on the crack length a , applied stress σ and a coefficient β which accounts for the geometry of cracked body (finite size, geometric features) and type of loading (tensile, bending, torsion). These are expressed by the Stress Intensity Factor (SIF)

$$K_I = \beta\sigma\sqrt{\pi a} \quad (10)$$

and represents the strength of the stress singularity near the crack tip. The nature of the stress field near the crack tip in a linearly elastic solid

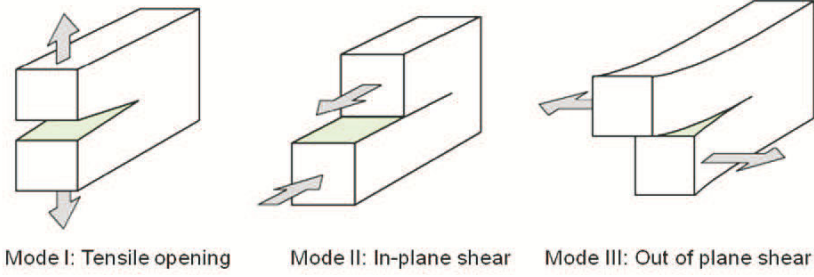


Figure 4. Types of loadings

was established through the efforts of Westergaard (1939); Irwin (1957); Williams (1957) and many other investigators.

For a homogeneous, isotropic, and linearly elastic solid, the structure of the solutions to the equations of equilibrium subjected to traction-free crack surfaces can easily be determined for each of the three modes of loading. This stress field in the vicinity of the crack, cf. Fig. 5 for general mixed mode loading can be written in the following form

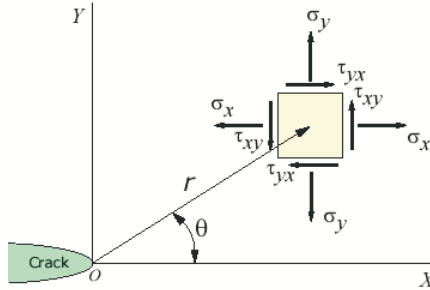


Figure 5. Stress components near crack tip

$$\sigma_{ij}(r, \theta) = \frac{1}{\sqrt{2\pi r}} [K_I f_{ij}^I(\theta) + K_{II} f_{ij}^{II}(\theta) + K_{III} f_{ij}^{III}(\theta)] + \text{higher order terms}, \quad (11)$$

where r, θ are the polar coordinates defining the position of an element in front of a crack tip, cf. Fig. 5, K_I, K_{II} and K_{III} are the Stress Intensity Factors corresponding to the three modes of loading, and $f_{ij}^I, f_{ij}^{II}, f_{ij}^{III}$ are non-dimensional functions indicated in Table 1. The stress field described by Eq. (11) shows that at the crack tip the stresses are theoretically infinite.

Table 1. Non-dimensional functions

f_{ij}	Mode I	Mode II	Mode III
f_{xx}	$\cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right)$	$-\sin \frac{\theta}{2} \left(2 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \right)$	0
f_{yy}	$\cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right)$	$\sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2}$	0
f_{zz}	$\left\{ \begin{array}{cc} 0 & - \text{pl. stress} \\ \nu(\sigma_{xx} + \sigma_{yy}) & - \text{pl. strain} \end{array} \right.$	$\left\{ \begin{array}{cc} 0 & - \text{pl. stress} \\ \nu(\sigma_{xx} + \sigma_{yy}) & - \text{pl. strain} \end{array} \right.$	0
f_{xy}	$\sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2}$	$\cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right)$	0
f_{xz}	0	0	$-\sin \frac{\theta}{2}$
f_{yz}	0	0	$\cos \frac{\theta}{2}$

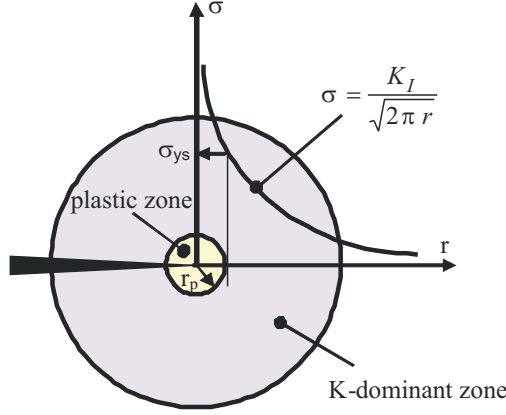


Figure 6. Validity of linear elastic stress field

Most engineering materials when the actual stress reaches the yield stress (σ_{ys}) and have a finite strength. The damaged zone is known as plastic zone for metals and fracture process zone for ceramics and some composite materials. The size of the plastic zone is denoted by r_p and represents the radius corresponding to the yield stress, cf. Fig. 6. The linear elastic stress field is valid only in K -dominant zone, for example when r_p is less than $1/25^{\text{th}}$ of the crack length a .

The crack tip displacement fields for linear elastic isotropic materials are

- for Mode I

$$\begin{aligned} u &= \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} \left(\kappa - 1 + 2 \sin^2 \frac{\theta}{2} \right) \\ v &= \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} \left(\kappa + 1 - 2 \cos^2 \frac{\theta}{2} \right) \end{aligned} \quad (12)$$

- for Mode II

$$\begin{aligned} u &= \frac{K_{II}}{2\mu} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} \left(\kappa + 1 + 2 \cos^2 \frac{\theta}{2} \right) \\ v &= -\frac{K_{II}}{2\mu} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} \left(\kappa - 1 - 2 \sin^2 \frac{\theta}{2} \right) \end{aligned} \quad (13)$$

- for Mode III

$$\begin{aligned} w &= \frac{K_{III}}{\mu} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} \\ v &= v = 0 \end{aligned} \quad (14)$$

where μ is the shear modulus and $\kappa = 3 - 4\nu$ for plane strain and $\kappa = (3 - \nu)/(1 + \nu)$ for plane stress.

The Stress Intensity Factors are very important because under conditions of small-scale yielding, all crack front fields are dominated by the stress intensity factor. Therefore, all crack behavior like stability (will the crack tip propagate?), trajectory (in which direction will grow), and rate (how fast will grow) are controlled by the stress intensity factor and, maybe by higher order terms.

The Stress Intensity Factors could be determined analytically using complex stress functions, weight functions, Green's functions, numerically with Finite Element Method or Boundary Element Method or experimentally by strain gauges, photoelastic stress analysis, thermoelastic stress analysis.

The fracture criterion based on the SIF states that unstable fracture occurs when the stress intensity factor reaches a critical value K_{Ic} , also called fracture toughness. K_{Ic} represents the inherent ability of a material to withstand a given stress field intensity at the tip of a crack and to resist to progressive tensile crack extension. Thus a crack will propagate (under pure mode I), whenever the stress intensity factor K_I (which characterizes the strength of the singularity for a given problem) reaches a material constant K_{Ic} . Hence, under the assumptions of linear elastic fracture mechanics (LEFM), at the point of incipient crack growth

$$K_I = \beta\sigma\sqrt{\pi a} = K_{Ic} \quad (15)$$

This fracture criteria is the most commonly used in LEFM because a lot SIF's solutions are available for different cracked bodies and the fracture toughness of materials is a common mechanical property. The measured value of K_c is high for small specimen thicknesses and reaches a nearly constant lower plateau at large specimen thicknesses; this value of the critical stress intensity factor, labeled K_{Ic} , represents the plane strain fracture toughness, cf. Fig. 7.

Generally, determination of fracture toughness is carried on experimentally using standard test procedures, and will be presented in Section 2. Micromechanical models were developed based on analytical solutions (Section 3) or on finite element methods in order to predict the fracture toughness of cellular materials.

The chart of the fracture toughness versus Young modulus was plotted by Ashby, Fig. 8. The range of fracture toughness is large: from less than 0.01 to over 100 MPa m^{0.5}. At the lower end of this range are brittle materials, which, when loaded, remain elastic until they fracture. For these materials the linear-elastic fracture mechanics works well, and the fracture toughness itself is a well-defined property. At the upper end lie the hi-tough

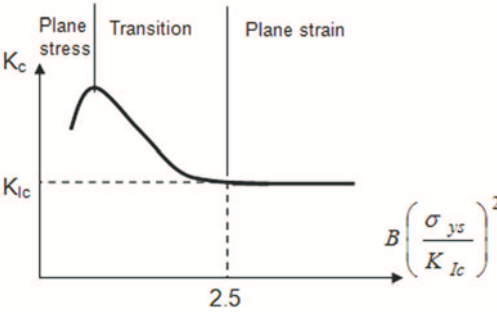


Figure 7. Variation of fracture toughness with normalized specimen thickness

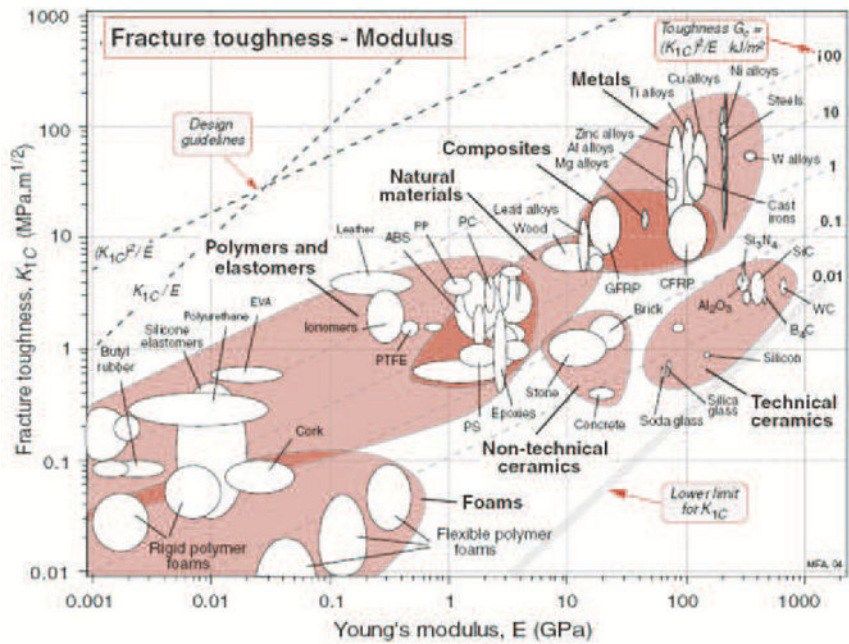


Figure 8. Fracture toughness K_{IC} , plotted against Young's modulus, E , after Ashby (2005)

materials, all of which show substantial plasticity before they break. For these materials the Elasto-Plastic Fracture Mechanics should be applied. Cellular materials like rigid polymer foams are placed on the bottom left part in this chart, showing low fracture toughness and low Young's modulus. The lines corresponding to different values of $(K_{Ic})^2/E$ and constant K_{Ic}/E , could be used in design against fracture. The shaded band corresponding to the lower limit of K_{Ic} shows the "necessary condition" for fracture. Fracture can, in fact, occur below this limit under conditions of corrosion, or cyclic loading.

Comparing Eqs (2) and (8) it can be seen the correlation between energy release rate and stress intensity factor:

$$\mathcal{G} = \frac{K_I^2}{E} \quad - \text{ for plane stress} \quad (16)$$

$$\mathcal{G} = (1 - \nu^2) \frac{K_I^2}{E} \quad - \text{ for plane strain} \quad (17)$$

Design based on LEFM. Assessment of integrity and reliability of structures requires a detailed analysis of the stress and deformation that they experience. Early designs were based on strength of materials criteria which states that $\sigma = \sigma_L$ (Fig. 9), where σ_L is the limit stress, representing the ratio between yield or tensile strength and a safety factor. The Fracture

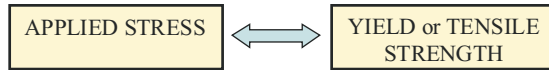


Figure 9. Classical Strength of Materials design criteria

Mechanics approach takes into account the flaw size, cf. Eq. (15).

Thus, for the design of a cracked, or potentially cracked, structure, the engineer would have to decide what design variables can be selected, as only, two of these variables can be fixed, and the third must be determined. The design variables shown in Fig. 10 are:

Material properties: fracture toughness K_{Ic} ,

Design stress level: applied stress σ ,

Flaw size: crack length a .

In assessing the safety of a cracked body, it is essential that the crack length a is properly known. In most cases it is not. Thus assumptions must be made for its value, and those assumptions are dependent upon the

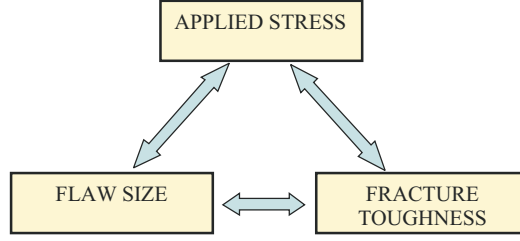


Figure 10. Fracture Mechanics design criteria

crack detection methodology adopted. The presence of a crack, equal to the smallest one that can be detected, must be assumed.

1.3 Crack tip stress and displacement fields in anisotropic materials

In many cases the behavior of the cellular materials is anisotropic. Usually, the in-plane properties differ from those in out-of-plane direction for manufactured foams. Natural cellular solids are more anisotropic (Gibson and Ashby, 1997). The generalized Hooke's Law can be expressed for an anisotropic material in the form

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} \quad (18)$$

For this general case we have 36 independent constants a_{ij} . However, taking into account the symmetry $a_{ij} = a_{ji}$ this reduces to 21 independent constants. If the material has one plane of symmetry, it reduces to 13 independent constants. For the case of an orthogonal anisotropic material, known also orthotropic, there are three orthogonal planes of elastic symmetry and $a_{16} = a_{26} = a_{36} = a_{45} = 0$, resulting in 9 independent constants

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} \quad (19)$$

Considering the engineering constants Young's moduluae E_1, E_2, E_3 , Poisson's ratios $\nu_{12}, \nu_{21}, \nu_{13}, \nu_{31}, \nu_{23}, \nu_{32}$ and shear moduluae G_{12}, G_{13}, G_{23} for an orthotropic material, Eq. (19) can be expressed

$$\begin{aligned}
 \varepsilon_1 &= \frac{1}{E_1}(\sigma_1 - \nu_{12}\sigma_2 - \nu_{13}\sigma_3) \\
 \varepsilon_2 &= \frac{1}{E_2}(\sigma_2 - \nu_{21}\sigma_1 - \nu_{23}\sigma_3) \\
 \varepsilon_3 &= \frac{1}{E_3}(\sigma_3 - \nu_{31}\sigma_1 - \nu_{32}\sigma_2) \\
 \gamma_{12} &= \frac{\sigma_{12}}{G_{12}} \\
 \gamma_{13} &= \frac{\sigma_{13}}{G_{13}} \\
 \gamma_{23} &= \frac{\sigma_{23}}{G_{23}}
 \end{aligned} \tag{20}$$

allowing to identify the coefficients a_{ij}

$$\begin{aligned}
 a_{11} &= \frac{1}{E_1}, a_{22} = \frac{1}{E_2}, a_{33} = \frac{1}{E_3}, \\
 a_{12} = a_{21} &= -\frac{\nu_{12}}{E_1} = -\frac{\nu_{21}}{E_2}, a_{23} = a_{32} = -\frac{\nu_{23}}{E_2} = -\frac{\nu_{32}}{E_3}, \\
 a_{13} = a_{31} &= -\frac{\nu_{31}}{E_3} = -\frac{\nu_{13}}{E_1}, \\
 a_{44} &= \frac{1}{G_{12}}, a_{55} = \frac{1}{G_{31}}, a_{66} = \frac{1}{G_{12}}
 \end{aligned} \tag{21}$$

For a transversally isotropic material, Fig. 11 like some man-made foams or corks, we have only 5 independent elastic constants

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(a_{11} - a_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{44} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}, \tag{22}$$

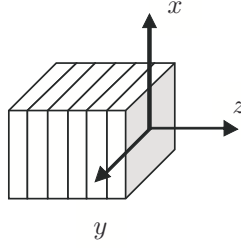


Figure 11. Transversally isotropic material

Table 2. Number of independent and nonzero coefficients

Material Type	Independent coefficients	Nonzero coefficients
General Anisotropic	21	36
Orthotropic	9	12
Transversally isotropic	5	12
Isotropic	2	12

which in terms of engineering constants become

$$\begin{aligned}
 \varepsilon_1 &= \frac{1}{E}\sigma_1 - \frac{\nu}{E}\sigma_2 - \frac{\nu'}{E'}\sigma_3 \\
 \varepsilon_2 &= \frac{1}{E}\sigma_2 - \frac{\nu}{E}\sigma_1 - \frac{\nu'}{E'}\sigma_3 \\
 \varepsilon_3 &= \frac{1}{E'}\sigma_3 - \frac{\nu'}{E'}(\sigma_1 + \sigma_2) \\
 \gamma_{12} &= \frac{2(1 + \nu)\sigma_{12}}{E} \\
 \gamma_{13} &= \frac{\sigma_{13}}{G'} \\
 \gamma_{23} &= \frac{\sigma_{23}}{G'}
 \end{aligned} \tag{23}$$

where E and ν represents the elastic properties in the plane of isotropy and E' , G' and ν' elastic properties for the plane normal to the plane of isotropy.

Table 2 presents the number of coefficients for different types of materials.

For a cracked homogeneous anisotropic material loaded in mixed mode loading (I and II) the stress field can be expressed using the stress intensity

factors in the form (Sih et al., 1965)

$$\begin{aligned}
 \sigma_{xx} &= \frac{K_{Ia}}{\sqrt{2\pi r}} \Re \left[\frac{s_1 s_2}{s_1 - s_2} \left(\frac{s_2}{\sqrt{\cos \theta + s_2 \sin \theta}} - \frac{s_1}{\sqrt{\cos \theta + s_1 \sin \theta}} \right) \right] \\
 &= \frac{K_{IIa}}{\sqrt{2\pi r}} \Re \left[\frac{1}{s_1 - s_2} \left(\frac{s_2^2}{\sqrt{\cos \theta + s_2 \sin \theta}} - \frac{s_1^2}{\sqrt{\cos \theta + s_1 \sin \theta}} \right) \right] \\
 \sigma_{yy} &= \frac{K_{Ia}}{\sqrt{2\pi r}} \Re \left[\frac{1}{s_1 - s_2} \left(\frac{s_1}{\sqrt{\cos \theta + s_2 \sin \theta}} - \frac{s_2}{\sqrt{\cos \theta + s_1 \sin \theta}} \right) \right] \\
 &= \frac{K_{IIa}}{\sqrt{2\pi r}} \Re \left[\frac{1}{s_1 - s_2} \left(\frac{1}{\sqrt{\cos \theta + s_2 \sin \theta}} - \frac{1}{\sqrt{\cos \theta + s_1 \sin \theta}} \right) \right] \\
 \sigma_{xy} &= \frac{K_{Ia}}{\sqrt{2\pi r}} \Re \left[\frac{s_1 s_2}{s_1 - s_2} \left(\frac{1}{\sqrt{\cos \theta + s_2 \sin \theta}} - \frac{1}{\sqrt{\cos \theta + s_1 \sin \theta}} \right) \right] \\
 &= \frac{K_{IIa}}{\sqrt{2\pi r}} \Re \left[\frac{1}{s_1 - s_2} \left(\frac{s_1}{\sqrt{\cos \theta + s_2 \sin \theta}} - \frac{s_2}{\sqrt{\cos \theta + s_1 \sin \theta}} \right) \right],
 \end{aligned} \tag{24}$$

where $\Re$ represents the real part, s_1 and s_2 are roots of the general equation (Saouma, 1997)

$$a_{11}s^4 - 2a_{16}s^3 + (2a_{12} + a_{66})s^2 - 2a_{26}s + a_{22} = 0 \tag{25}$$

in the form $s_j = \alpha_j + i\beta_j$, for $j = 1, 2$, a_{ij} are the components of the compliant matrix.

The most important observation is that for the anisotropic case like in the isotropic case the stress field has a singularity of order $r^{-0.5}$. Regarding the stress intensity factors values for anisotropic materials differ from those for isotropic materials (Murakami, 1987).

The displacement field for mode I loading conditions can be expressed in the form

$$\begin{aligned}
 u_1 &= K_{Ia} \sqrt{\frac{2r}{\pi}} \Re \left\{ \frac{1}{s_1 - s_2} \left[s_1 p_2 \sqrt{\cos \theta + s_2 \sin \theta} - s_2 p_1 \sqrt{\cos \theta + s_1 \sin \theta} \right] \right\}, \\
 v_1 &= K_{Ia} \sqrt{\frac{2r}{\pi}} \Re \left\{ \frac{1}{s_1 - s_2} \left[s_1 q_2 \sqrt{\cos \theta + s_2 \sin \theta} - s_2 q_1 \sqrt{\cos \theta + s_1 \sin \theta} \right] \right\}, \\
 w_1 &= 0
 \end{aligned} \tag{26}$$

with

$$p_j = a_{11}s_j^2 + a_{12} - a_{16}s_j, \quad q_j = a_{12}s_j + \frac{a_{22}}{s_j} - a_{26}, \quad j = 1, 2 \tag{27}$$

For orthotropic materials the characteristic equation becomes

$$s^4 + \left(\frac{E_1}{G_{12}} - 2\nu_{12} \right) s^2 + \frac{E_1}{E_2} = 0 \tag{28}$$

A study of the solutions for the characteristic equation is presented by Saouma (1997).

2 Experimental Determination of Fracture Toughness of Foam Materials

It is well known that the foam crushes in compression, Fig. 12 (*left*), while in tension fails by propagating of single crack, Fig. 12 (*right*). Most of the

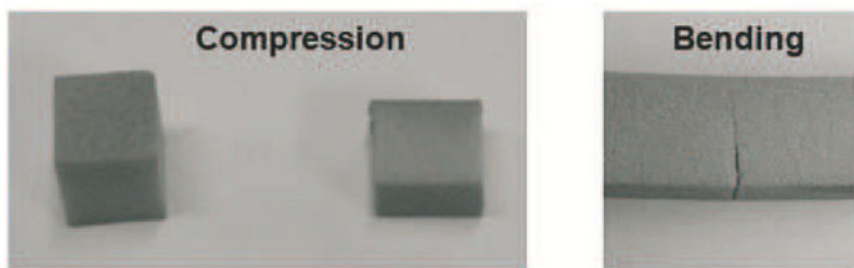


Figure 12. Mechanisms of foam damage in compression and bending

rigid polymer foams have a linear - elastic behavior in tension up to fracture, and a brittle fracture, so they can be treated using fracture criteria's of Linear Elastic Fracture Mechanics (LEFM). The case of metallic foams with elasto-plastic fracture behavior will not be treated here, but was extensively investigated during the last 10 years by McCullough et al. (1999); Ashby et al. (2000); Olurin et al. (2000); Andrews and Gibson (2001); Fleck et al. (2001); Motz and Pippan (2001); Motz (2002); Degischer and Kriszt (2002); Onck et al. (2005).

Methods for testing of polymeric foams are reviewed by Landrock (1995); Brown (1999, 2002); Ward and Sweeney (2004). Different procedures for determination of fracture toughness for cellular materials were developed for flexible and rigid foams.

2.1 Tear Test for Flexible Cellular Materials

This test method is standardized by ASTM D 3574-03: Standard Test Methods for Flexible Cellular Materials-Slab, Bonded, and Molded Urethane Foams (Test F), ASTM D 3575-00: Standard Test Methods for Flexible Cellular Materials Made from Olefin Polymers (Test G), and BS EN ISO 8067-89: Flexible Cellular Polymeric Materials - Determination of Tear Strength.

Scope. The test method covers determination of the tear propagation resistance of foam, by measuring the tear resistance under the conditions of this particular test. The tear strength represents the force needed to rip a foam specimen.

Apparatus. The tear resistance should be measured on a power-driven apparatus which will indicate the force at which fracture of the specimen takes place. An automatic machine can be used which draws the actual curve, or, a style or scale can be used having an indicator that remains at the point of maximum force after fracture.

Test Specimens. The test specimens shall be a block shape free of skin, voids, and densification lines, as shown in Fig. 13. They may be cut on

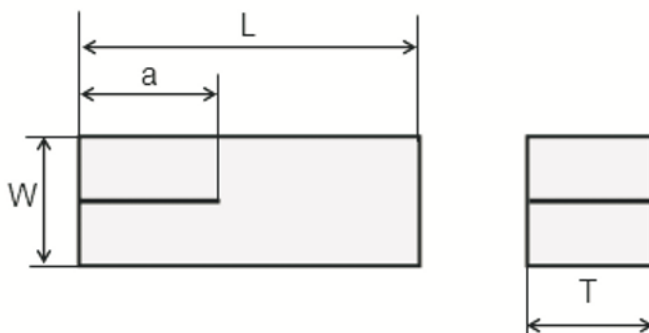


Figure 13. Tear strength specimens

a saw or die cut from sheet material so that the sides are parallel and perpendicular to each other. The block dimensions should be $T = W = 25$ mm, $L = 125$ mm (after EN ISO 8067-89). A nominal $a = 50$ mm cut shall be produced, with a razor blade or knife, in one side as shown in Fig. 13. The tests should be performed at least 72 hours after manufacture at room temperature. At least, three specimens per sample must be tested. The values reported are to be the mean of those tested.

Procedure. The specimen legs are clamped and pulled apart with a speed of 50 to 500 mm/min. After the fracture of the specimen, or after at least a 50-mm length is torn, record the maximum force in Newtons and note also the thickness of the specimen

Calculation. Calculate the tear strength from the maximum force registered on the testing machine and the average thickness of the specimen T , as follows

$$R = \frac{F_{\max}}{T} 10^3, \quad (29)$$

where R is the tear strength in [N/m], $F_{\max}$ - maximum force, [N], T - thickness, [mm]. Some Tear Strength results together with other mechanical characteristics of flexible ethylene-vinyl acetate (EVA) foam are presented in Table 3 from www.rueylung.com.tw/products/EVA_Foam.

2.2 Standard Test Methods for Plane-Strain Fracture Toughness and Strain Energy Release Rate of Plastic Materials

The test procedure for determining the plane strain fracture toughness of plastic materials is standardized by the American Society for Testing and Materials, ASTM D5045-99 which is based on ASTM E399-97 Test Method for Plane-Strain Fracture Toughness of Metallic Materials, but includes specific consideration for plastic materials.

Scope. The test method is designed to determine the toughness of plastics in terms of the critical stress intensity factor, K_{Ic} , and the energy per unit area of crack surface or critical strain energy release rate, $\mathcal{G}_{Ic}$, at fracture initiation. Two types of specimens are proposed: Single Edge Notch Bending (SENB) specimen and Compact Tension (CT) specimen. Linear elastic behavior under plane strain conditions is assumed, certain restrictions on linearity and specimen dimensions are considered in order to validate the experimental results

Apparatus. A constant displacement rate device shall be used such as an electromechanical, screw-driven machine, or a closed loop, feedback-controlled servo-hydraulic load frame. For SENB, a rig with either stationary or moving rollers of sufficiently large diameter to avoid excessive plastic indentation is required. A suitable arrangement for loading the SENB specimen is presented in Fig. 14 (*left*). A loading clevis suitable for loading compact tension specimens is shown in Fig. 14 (*right*). Loading is produced through pins on the specimen holes (Fig. 14 (*right*)). An accurate displacement measurement must be obtained to assure accuracy of the $\mathcal{G}_{Ic}$ value.

For either SENB or CT specimen configurations, the displacement measurement can be performed using the machine's position transducer. The load-displacement data must be corrected for system compliance, loading-

Table 3. Mechanical characteristics of EVA foam materials

Foam Type	L-2500	C-1600	C-3000	H-25	H-35	A-25
Density [kg/m ³]	21-27	35-41	23-29	65-75	70-80	80-90
Original Sheet Thickness [mm]	80	80	80	36	32	38
Tensile Strength [MPa]	0.196	0.314	0.274	0.932	1.157	0.932
Elongation at Break [%]	180	180	340	220	160	120
Tear Strength [N/m]	1863	2353	2059	3334	3726	3236
Compressive Strength at 25% Deflection [MPa]	0.039	0.059	0.029	0.069	0.078	0.069
Compressive Strength at 50% Deflection [MPa]	0.098	0.127	0.078	0.147	0.176	0.157

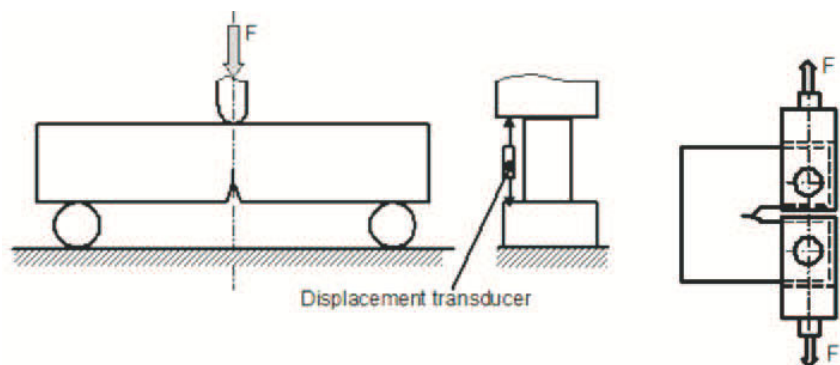


Figure 14. Arrangement of loadings: SENB specimen (left), CT specimen (right)

pin penetration and sample compression by performing a calibration of the testing system.

If an internal displacement transducer is not available, or has insufficient precision, then an externally applied displacement-measuring device may be used as illustrated in Fig. 14 (*right*) for the SENB configuration. For CT specimens, a clip gage can be mounted across the loading pins. For both the SENB and CT specimens, the displacement should be taken at the load point.

Test Specimens. SENB and CT geometries are recommended over other configurations because these have predominantly bending stress states which allow smaller specimen sizes to achieve plane strain conditions. If the material is supplied in the form of a sheet, the specimen thickness, B , should be identical with the sheet thickness. The criteria require that B must be sufficient to ensure plane strain and that $(W - a)$ be sufficient to avoid excessive plasticity in the ligament. If $(W - a)$ is too small and non-linearity in loading occurs, then increasing the W/B ratio to a maximum of 4 can be attempted for SENB specimens. Initially, prepare a sharp notch by machining. Subsequently, initiate a natural crack by inserting a fresh razor blade and tapping. If a natural crack cannot be successfully initiated by tapping, a sufficiently sharp crack can alternatively be generated by sliding or sawing a new razor blade across the notch root. The dimensions of standard specimens are shown in Fig. 15. At least three specimens for each material condition should be tested.

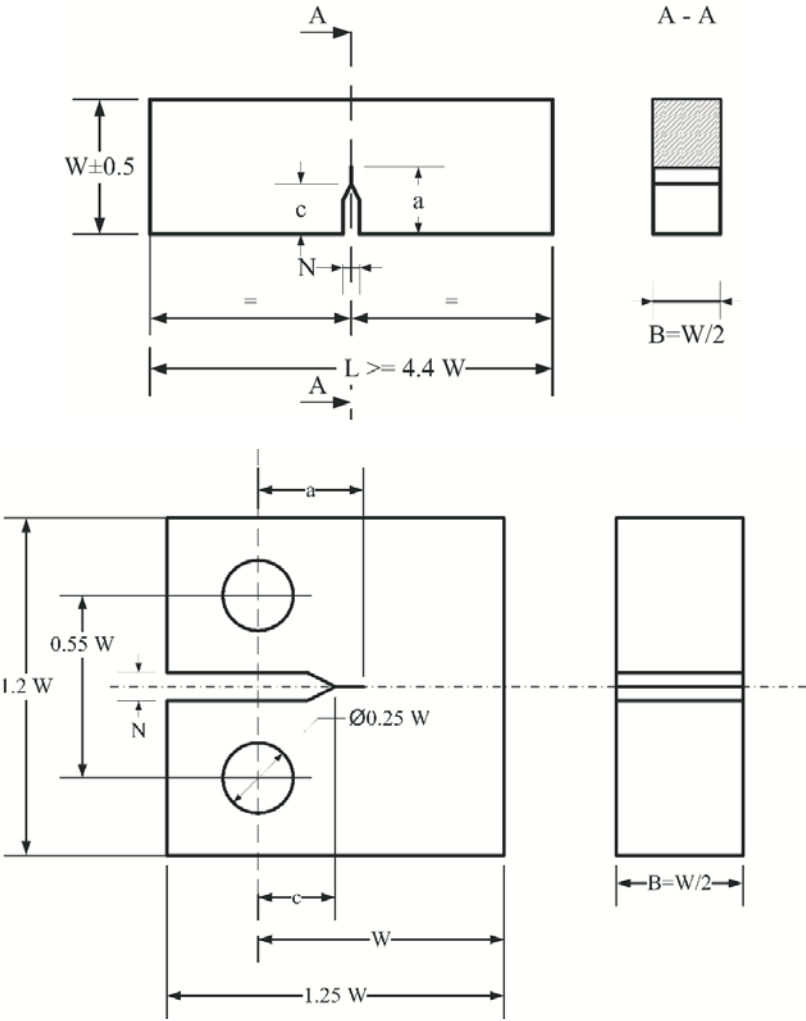


Figure 15. Specimens configuration for D 5045-99 test: SENB specimen (up), CT specimen (down)

Procedure. Since plastics are viscoelastic materials, it is necessary to specify both the temperature and time scale under which the result was obtained. It is recommended that tests to be performed at a temperature of 23°C, and a crosshead rate of 10 mm/min.

The load versus loading-point displacement curve is recorded during test. In the ideal case this is a linear diagram with an abrupt drop of load to zero at the instant of crack growth initiation. In some cases this occurs and K_Q can be found from the maximum load.

The determination of $\mathcal{G}_{Ic}$ requires an accurate integration of the load versus loading point displacement curve, which necessitates an accurate displacement determination using a displacement transducer.

Calculation and results interpretation. Calculation value of stress intensity factors K_Q is obtained based on force-displacement curve, cf. Fig. 16. First, draw a straight line (AB) to determine the initial compliance

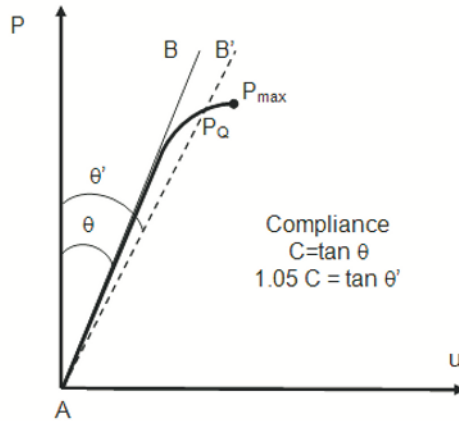


Figure 16. Typical load - displacement curve

$C = \tan \theta$, which represents the slope of line (AB). Draw a second line (AB') with the compliance 5% greater than that of line (AB). If the maximum load that the specimen was able to sustain, P_{max} , falls within lines (AB) and (AB'), use P_{max} to calculate K_Q . If P_{max} falls outside line (AB) and line (AB'), then use the intersection of line (AB') and the load curve as P_Q . Furthermore, if $P_{max}/P_Q < 1.1$ use P_Q in the calculation of K_Q . However, if $P_{max}/P_Q > 1.1$ the test is invalid.

Calculation of K_Q :

- for CT specimen

$$K_Q = P_Q \frac{f\left(\frac{a}{W}\right)}{B\sqrt{W}}, \quad (30)$$

where for $0.2 < a/W < 0.8$

$$f\left(\frac{a}{W}\right) = \left(2 + \frac{a}{W}\right) \times \frac{0.886 + 4.64\frac{a}{W} - 13.32\left(\frac{a}{W}\right)^2 + 14.72\left(\frac{a}{W}\right)^3 - 5.6\left(\frac{a}{W}\right)^4}{\sqrt[3]{\left(1 - \frac{a}{W}\right)^2}}$$

and P_Q - load [N], B - specimen thickness [mm], W - specimen width [mm], a - crack length [mm].

- for SENB specimen with $L/W = 4$

$$K_Q = P_Q \frac{f\left(\frac{a}{W}\right)}{B\sqrt{W}}, \quad (31)$$

where ($0 < a/W < 1$)

$$f\left(\frac{a}{W}\right) = 6\sqrt{\frac{a}{W}} \frac{1.99 - \left(\frac{a}{W}\right) \left(1 - \frac{a}{W}\right) \left(2.15 - 3.93\left(\frac{a}{W}\right) + 2.7\left(\frac{a}{W}\right)\right)^2}{\left(1 + 2\frac{a}{W}\right) \sqrt[3]{\left(1 - \frac{a}{W}\right)^2}}$$

and P_Q - load [N], B - specimen thickness [mm], W - specimen width [mm], a - crack length [mm].

After K_Q calculation the validity is checked

$$a, B, (W - a) \geq 2.5 \left(\frac{K_Q}{\sigma_y}\right)^2 \quad (32)$$

If condition (32) is fulfilled then the fracture toughness is equal with K_Q

$$K_{Ic} = K_Q \quad (33)$$

Otherwise the test is not valid, and should be repeated on larger specimens (increasing the dimensions in increments with 1.5 times).

In principle, $\mathcal{G}_{Ic}$ can be obtained using (16), with mention that for plastics E must be obtained at the same time and temperature conditions as the fracture test because of viscoelastic effects. Many uncertainties are introduced by this procedure and it is considered preferable to determine $\mathcal{G}_{Ic}$ directly from the energy derived from integration of the load versus displacement curve up to the same load point as used for K_{Ic} according with ASTM 5045-99.

Table 4. Fracture toughness results for PVC foam, after Viana and Carlson (2002)

Foam grade	Density ρ^* [kg/m ³]	Specimen dimensions [mm]				Fracture toughness K_{Ic} [MPa m ^{0.5}]
		Width B [mm]	Height W [mm]	Span L [mm]	Crack length a [mm]	
H30	36	19.2	39.0	153	19.5	0.064
H80	80	20.0	38.1	152	18.5	0.117
H100	100	28.0	58.3	232	27.6	0.168
H200	200	24.5	50.2	197	24.0	0.370
H400	400	12.7	25.5	108	13.5	0.450

2.3 Fracture Toughness Experimental Results

McIntyre and Anderson (1979), using single edge notch specimen, made of rigid closed-cell PU foams measured the K_{Ic} for different densities. They found that the fracture toughness is independent of crack length and a linear correlation of K_{Ic} with density for foams with densities less than 200 kg/m³. At higher densities the correlation became non-linear. Linear relationship between K_{Ic} and densities (in the range 90 - 235 kg/m³) was also obtained by Danielsson (1996) for PVC Divinycell HD using three point bend specimens with two different span-to-depth ratios and centre cracked sheet specimens.

Burman (1998) presented fracture toughness results for two commercial foams Rohacell WF51 (density 52 kg/m³) and Divinycell H100 (density 100 kg/m³). The mode I fracture toughness K_{Ic} was obtained on SENB specimens and has values 0.08 MPa m^{0.5} for WF51, respectively 0.21 MPa m^{0.5} for H100. He also determined the Mode II fracture toughness using End-Notch Flexure (ENF) specimen with values of 0.13 MPa m^{0.5} for WF51, respectively 0.21 MPa m^{0.5} for H100.

Viana and Carlson (2002) presented results of fracture toughness for PVC foams of different densities ($\rho^* = 36, 80, 100, 200$ and 400 kg/m³). They used the standard procedure described by ASTM D5045. The specimen dimensions and fracture toughness results are shown in Table 4.

They also show that the fracture toughness increases with increasing specimen thickness, which is opposite behavior to metals, where a decrease of thickness leads to plane stress conditions and increases the size of plastic zone and consequently the toughness. For foams the cellular structure induces additional deformation mechanisms. Plane stress conditions indicate that cells are allowed to bend, which apparently decreases the cohesion

Table 5. Fracture toughness results for PVC foams, after Kabir and Saha (2006)

	Type of foam		
	R75	HD130	R260
Density ρ^* [kg/m ³]	75	130	260
Loading rate [mm/s]	Fracture toughness K_{Ic} [MPa ^{0.5}]		
0.0245	0.09	0.30	0.66
0.254	0.10	0.34	0.72
2.54	0.11	0.36	0.78

strength of the foam, explaining the low toughness of small thickness specimens.

Kabir and Saha (2006) used the procedure described by ASTM D5045 for determining the fracture toughness of polyvinyl chloride (PVC) and polyurethane (PU) foams. Specimens with dimensions width B , height $W = 2B$, span $L = 4W$ and crack length $a = 0.5W$ were considered, where $B = (8.5 - 25.4)$ mm. They investigated the effect of density, specimen size, loading rate, and cell orientation. Density has a significant effect on fracture toughness (Table 5), which increases more than 7 times when the foam density increases 3.5 times. A correlation between fracture toughness and relative density was proposed by authors

$$K_{Ic} = 10.47 \left(\frac{\rho^*}{\rho_s} \right)^{1.59}, \quad (34)$$

where the density of PVC polymer was $\rho_s = 1400$ kg/m³.

It was also presented the influence of the specimen size on the fracture toughness. It was observed that fracture toughness decreases with increase of specimen height until the specimen height reaches 25 mm. In contrast, specimen thickness appears to have no effect on fracture toughness. Experimental results in Table 5 show that the fracture toughness increases by 18% when the loading rate is increased by two orders of magnitude.

Investigating the effect of crack orientation it can be observed that higher values were obtained when the crack is parallel to the rise direction, cf. Table 6.

Comparing two different closed cell foams PVC foam R260 and PU foam PUR240 with almost the same density, shows a considerable difference in fracture toughness.

$$K_{Ic}^{PUR240} = 0.30 \text{ MPa}\sqrt{\text{m}} < K_{Ic}^{R260} = 0.72 \text{ MPa}\sqrt{\text{m}}, \quad (35)$$

Table 6. Fracture toughness results for H130 foam (Kabir and Saha, 2006)

Orientation	Flow direction	Rise direction
Foam type	H130	H130
Density ρ^* [kg/m ³]	130	130
Loading rate [mm/s]	Fracture toughness K_{Ic} [MPa m ^{0.5}]	
0.0245	0.22	0.28
0.245	0.23	0.27
2.45	0.24	0.28

which could be explained by higher fracture toughness of solid PVC polymer (2.45 MPa m^{0.5}) compared to the fracture toughness of the PU solid (0.3 - 0.4 MPa m^{0.5}).

Jin et al. (2007) tested rigid closed cell PU foams (320 kg/m³ density), using SENB specimens. The dimensions of the SENB specimens were: width $B = 12.7$ mm, height $W = 25$ mm, specimen length $L = 5 W = 127$ mm, crack length $a = 0.4 W = 10.2$ mm (initial notch of 7.8 mm was obtained by saw cut with cut width of 0.63 mm, and the sharp crack of 2.4 mm was produced by a sharp thin razor blade). Tests were conducted using displacement control with 0.025 mm/s). Load versus displacement were recorded during tests. For fracture toughness determination the maximum load was considered. The obtained result for fracture toughness was 0.215 MPa m^{0.5}. They recorded the displacement and strain field during tests using the Digital Image Correlation technique.

Prior to the determination of the fracture toughness an extensive experimental program was carried out for determination of mechanical properties of close cell polyurethane (PU) foams. Tensile, compression and three point bending tests were performed on three different laboratories, i.e. Lublin University of Technology - Poland, POLITEHNICA University of Technology Timisoara and University POLITEHNICA Bucharest from Romania. The obtained results were reported by Marsavina et al. (2008, 2009). The experimental determination of fracture toughness was performed on PU foams of three different densities (40, 140 and 200 kg/m³) following the procedure proposed by ASTM D5045-99. The mechanical properties on tensile tests for un-cracked specimens are given in Table 7. Single Edge Notch Bending specimens were used and the crack was inserted using a razor blade.

The tests were performed at room temperature with a loading rate of 2 mm/min. The load versus loading-point displacement curve is recorded during tests performed with Zwick/Roell 5 kN testing machine, cf. Fig. 17.

Typical load displacement curve is shown in Fig. 18. For all specimens, linear diagram with an abrupt drop of load to zero after the crack initiation was obtained. The K_Q was evaluated using the maximum load value. Brittle fracture was observed for all tested specimens. A fracture surface and the optical microscope detail are shown in Fig. 19. The linear elastic behavior was confirmed during the tests when no cushioning occurs and no plastic deformations remain after the test.

The fracture toughness values and the fulfillment of the plane strain condition are presented in Table 8. The experimental results of fracture toughness are between $0.0276 \text{ MPa m}^{0.5}$ for 40 kg/m^3 density, to $0.448 \text{ MPa m}^{0.5}$ for 200 kg/m^3 density. For the PU foam of 40 kg/m^3 density a study of the orientation of specimens was performed. Similar results were obtained for both orientations in-plane and out-of plane.

Based on the experimental data a correlation between fracture toughness and relative density was proposed, which is indicated in Fig. 20, with coefficient of determination close to 1 indicating a good correlation. Fig. 21 shows the fracture toughness results versus density obtained by different researchers. It could be observed that the order of magnitude for fracture toughness of plastic foams is $10^{-1} - 10^{-2} \text{ MPa m}^{0.5}$. The fracture toughness is strongly dependent on foam density. Experimental results show that fracture toughness depends on the loading rate and in some cases on the crack orientation. Different values of fracture toughness were obtained for different plastic foams having the same density.

Legan et al. (2005) performed tests to determine the quasi-brittle fracture of foam-polystyrene plates of trademark PSB-25 involving some holes, cuts and also without them. The plates of dimensions $1 \times 1 \text{ m}$ and thickness 5 cm were used. The ultimate strength of material, Young's modulus, Poisson's ratio, pre-fracturing strain and fracture toughness were determined. A value of $0.017 \text{ MPa m}^{0.5}$ was obtained for the fracture toughness.

Table 7. Mechanical properties from three point bending test

Foam density [kg/m^3]	Tensile strength [MPa]	Yield strength [MPa]	Tensile modulus [MPa]
40	1.60	0.34	53
140	4.90	3.00	140
200	8.40	6.50	212



Figure 17. Testing machine and detail of a fractured specimen during test: Zwick/Roell testing machine (left), detail of fractured specimen (right)

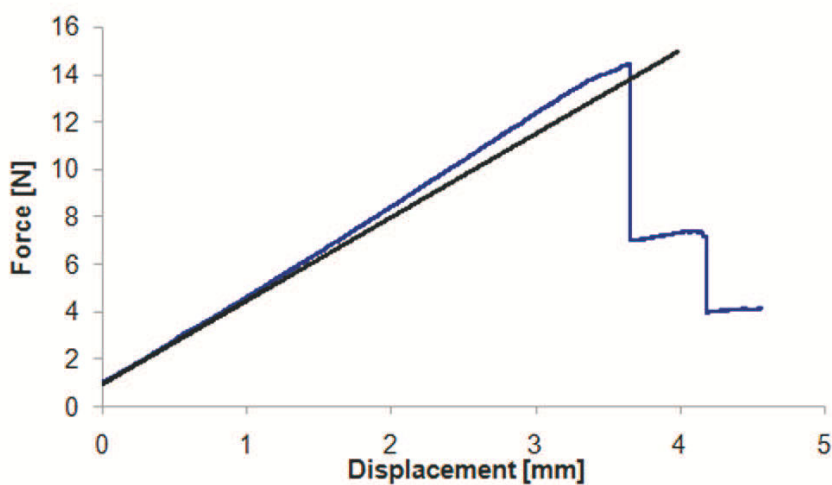


Figure 18. Typically force - displacement curve for PU foam of 40 kg/m^3 density - out of plane direction

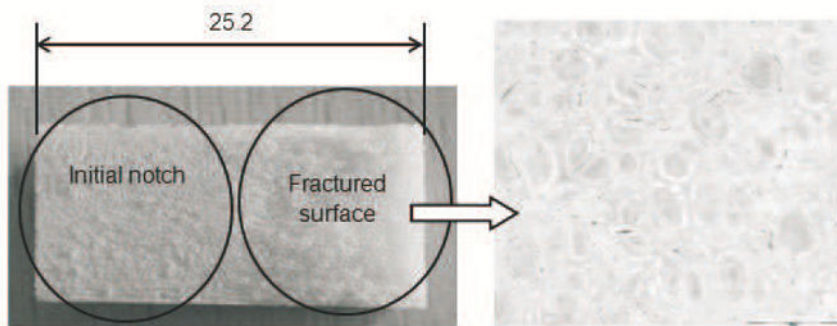


Figure 19. Fracture surface and optical microscope image

2.4 Impact Fracture Toughness

One of the most important characteristic of foams is the high capacity for energy absorption; consequently the foams are used for packing, in constructions and composite structures. Of particular interest is the impact fracture toughness of such foams because foam cracking weakens the structure's capacity for carrying loads and absorbing energy.

Kabir and Saha (2006) investigated the dynamic fracture toughness and found a maximum value of $2.74 \text{ MPa m}^{0.5}$ for PVC foam with 260 kg/m^3 density which is 3.75 times higher than the static fracture toughness of the same foam. A brittle fracture, produced in Mode I, without yielding was observed. Mills and Kang (1994) used a falling mass of 2.04 kg (velocity of 1.85 m/s) with a compact tension specimen (thickness of 20 mm) in order to determine the dynamic fracture toughness, and Mills (2007) also proposed a correlation between the dynamic fracture toughness and foam density

$$K_{Ic} = 0.955 \left(\frac{\rho}{\rho_s} \right)^{1.119} \quad (36)$$

Impact fracture toughness results for closed cell (the characteristic dimensions were cell - wall size $200 - 500 \mu\text{m}$ and wall thickness $3 - 4 \mu\text{m}$.) PU foam of 200 kg/m^3 density, cf. Fig. 22, was also reported by Marsavina and Sadowski (2008). Three point bend specimens ($12 \times 12 \times 60 \text{ mm}$) were adopted with a crack of 1.5 mm (cut with a razor blade). A span of 40 mm was used for the test and the impact load was applied in the transverse direction.

The principle of impact and instrumented impact tests of plastic mate-

Table 8. Fracture toughness experimental results

Foam density [kg/m ³]	Orien- tation ρ	Specimen dimensions [mm]				Fracture toughness K_{Ic} [MPa m ^{0.5}]	Yield stress σ_{ys} [MPa]	$2.5 \left(\frac{K_{Ic}}{\sigma_y} \right)^2$ [mm]
		Width B [mm]	Height W [mm]	Span L [mm]	Crack length a [mm]			
40	in- plane	25.3	50.50	200	25	0.0276	0.34	16.10
40	out-of- plane	25.2	49.75	200	25	0.0279	0.34	16.83
140	in- plane	13.7	25.17	100	12.5	0.168	3.00	7.85
200	in- plane	12.2	27.25	100	12.2	0.448	6.50	11.79

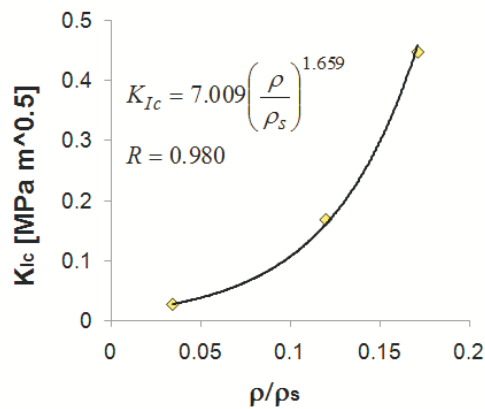


Figure 20. Fracture toughness versus relative density

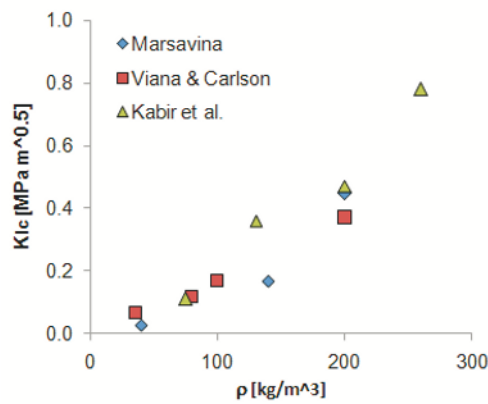


Figure 21. Comparison of the fracture toughness results

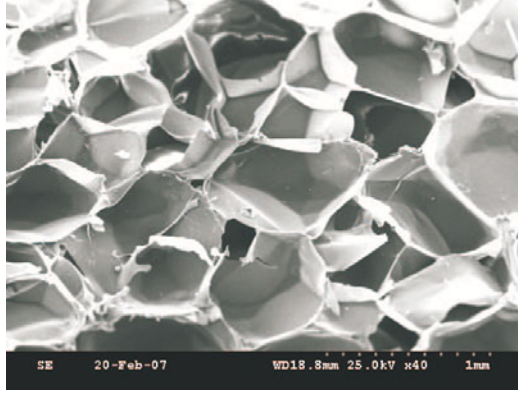


Figure 22. PU foam microstructure

rials are given in the EN ISO 179. A KB Prüftechnik pendulum (Germany) was used for the instrumented impact tests, with the following main characteristics: pendulum mass 2.04 kg, pendulum length 0.386 m, drop height 0.742 m, drop angle 157.32° , pendulum energy 7.5 J, impact velocity 3.815 m/s. The tup has a built-in electronic sensor which allows recording the load with 1 MHz frequency. A four-channel data acquisition A/D card (AdLink NuDAQ PCI-9812) was used for recording the load in time. Tests were performed at room temperature. The load history $F(t)$ was recorded by using strain gages bonded near the striker edge. A typical load-time curve is shown in Fig. 23.

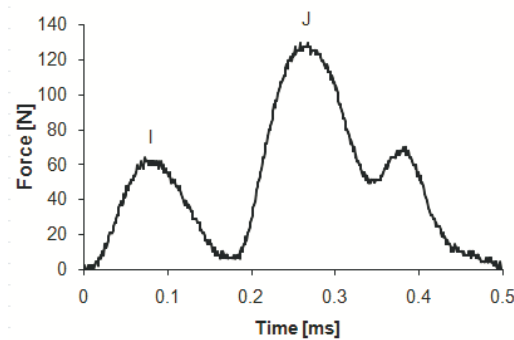


Figure 23. Load history during instrumented impact test

Following points are well represented: an inertial peak (point I) and the maximum force peak, representing the failure of the foam specimen (point J). The obtained maximum force values were around 120 - 130 N for all specimens, cf. Fig. 23. The impact fracture toughness was estimated using Eq. (31) in which the maximum load was considered. The obtained value of the impact fracture toughness was $0.368 \pm 0.010 \text{ MPa m}^{0.5}$. Comparing the dynamic fracture toughness with the static fracture toughness of rigid PUR of density 200 kg/m^3 ($K_{Ic} = 0.448 \text{ MPa m}^{0.5}$) it can be observed that for the investigated foam the dynamic fracture toughness is lower approximately 18 %.

At the end of this part we can conclude that:

- The experimentally values for fracture toughness of plastic foams are between $10^{-2} - 10^{-1} \text{ MPa m}^{0.5}$,
- Fracture toughness is strongly dependent on foam density,
- Fracture toughness depends on loading rate, and possibly on crack orientation,
- Different values of fracture toughness were obtained for different plastic foams having the same density,
- The fracture of polyurethane foams is quasi brittle,
- For the same foam the impact fracture toughness differs to the fracture toughness determined under static loading.

3 Micromechanical Models for Foams Fracture

Micromechanics of cellular materials is the analysis on the level of the individual phases that constitute these materials. Given the properties of the constituent phases, an important goal of micromechanics consists in predicting the properties of the cellular materials on the basis of the geometries and properties of the individual phases, known as homogenization. The benefit of homogenization is that the behavior of cellular materials can be determined without testing. Lattice models are often used as micromechanical tools (Ostoja-Starzewski, 2002). Further, micromechanics can predict the full multi-axial properties and response of cellular materials, which are usually anisotropic (Grenestedt, 1999; Onck et al., 2001; Saint-Michel et al., 2006b,a; Christensen, 2007). Such properties are often difficult to measure experimentally, but they are required for design of such structures. The particular micromechanics theory must be validated through comparison to experimental data (Lin, 1997; Zhu et al., 1997a,b; Andrews and Gibson, 2001).

Micromechanical analysis allows evaluating the local stress and strain fields in the phases for given macroscopic load states, phase properties and

phase geometries. Such knowledge is especially important in understanding and describing material damage and failure (Mills and Zhu, 1999; Gupta and Ricci, 2006; Ren and Silberschmidt, 2008).

Extensive studies of micromechanical models for cellular materials are presented by Gibson and Ashby (1997); Mills (2007). Here only the main formulations relating to prediction of fracture mechanics for plastic foams will be related.

Micromechanical models relate the fracture toughness of the foam K_{Ic} to the fracture strength of the cell walls σ_{fs} , and the relative density ρ^*/ρ_s . The geometric characteristics of an open cell are the cell length l , and the thickness of strut h , cf. Fig. 24.

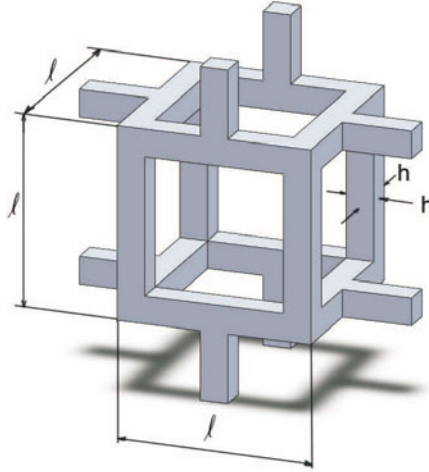


Figure 24. Cubic cell geometry, cf. Gibson and Ashby (1997)

Gibson and Ashby (1997) considered that the crack advances discrete with one cell width, Fig. 25. Loading the foam in tension the cell wall deforms elastically. The load is transmitted through the foam as a set of forces and moments acting on the cell edges.

Gibson and Ashby (1997) assumed that the crack tip is located at half-edge length and considered an elastic mode I stress field at the crack tip. They start from the stress singularity at the tip of a crack of length $2a$ and normal to remote loading σ in an elastic continuum solid, at distance r (on

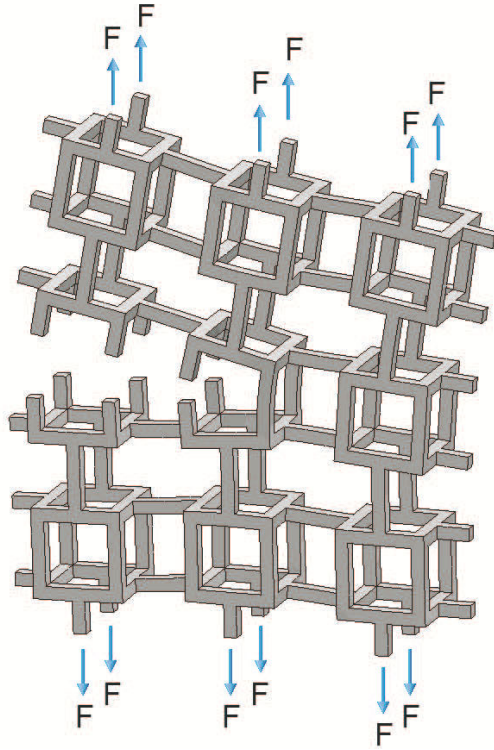


Figure 25. Cracked cellular structure, cf. Gibson and Ashby (1997)

a direction $\theta = 0$) from the crack tip, cf. Eq. (11)

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} = \frac{\beta\sigma\sqrt{\pi a}}{\sqrt{2\pi r}} \quad (37)$$

with β a constant depending on the geometry of the cracked body.

The first unbroken edge is placed at distance $l/2$ in front of the crack and is subjected to a force F

$$F = l \int_0^l \sigma_{yy} dr = \sqrt{2}\beta\sigma l^2 \sqrt{\frac{a}{l}} \quad (38)$$

The edges fail when the stress produced by the bending moment $M = Fl/2$

reaches the ultimate tensile strength σ_{fs}

$$\sigma_{fs} = \frac{\frac{Fl}{2}}{\frac{h^3}{6}} = \frac{3Fl}{h^3} = \frac{3\sqrt{2}\beta K_I l^{5/2}}{\sqrt{\pi}h^3} \Rightarrow K_{Ic} = \frac{\sqrt{\pi l}}{3\sqrt{2}\beta} \sigma_{fs} \left(\frac{h}{l}\right)^3 \quad (39)$$

Taking into account the proportionality between relative density and cell dimensions ratio for open cell foam (Gibson and Ashby, 1997)

$$\left(\frac{\rho^*}{\rho_s}\right) \propto \left(\frac{h}{l}\right)^2, \quad (40)$$

results in

$$K_{Ic} = C \sigma_{fs} \sqrt{\pi l} \left(\frac{\rho^*}{\rho_s}\right)^{3/2}, \quad (41)$$

where C contains all the proportionality constants.

Comparing with experimental data for open cells and different materials Fowlkes (1974) used polyurethane, McIntyre and Anderson (1979) for polyurethane, Maiti et al. (1984) for PMMA, Brezny and Green (1989) for ceramic foams results in $C = 0.65$, and correlation (41) can be given as

$$\frac{K_{Ic}}{\sigma_{fs} \sqrt{\pi l}} = 0.65 \left(\frac{\rho^*}{\rho_s}\right)^{3/2} \quad (42)$$

Experimental results on reticulated carbon foams to validate relation (42) show that this is valid for $a/l > 10$ (Huang and Gibson, 1991).

A similar correlation was proposed by Green (1985) considering elastic deformation in the shell theory of a hollow sphere model for foam cells

$$\frac{K_{Ic}}{\sigma_{fs} \sqrt{\pi l}} = 0.28 \left(\frac{\rho^*}{\rho_s}\right)^{1.3} \quad (43)$$

Choi and Sankar (2005) using LEFM relates the stress intensity factors to the stress filed in the crack tip ligament of the foam. They consider the crack length

$$a = \alpha l, \quad (44)$$

when α is a non-dimensional factor and l is cell size.

Starting again from the stress field across of the crack ($\theta = 0$) loaded in Mode I is

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \quad (45)$$

with r distance from the crack tip.

The stress produces axial forces and a bending moment in the crack tip strut, cf. Fig. 26, which can be assimilated with the stress distribution in front of a crack, cf. Fig. 16. The axial force can be obtained by integrating

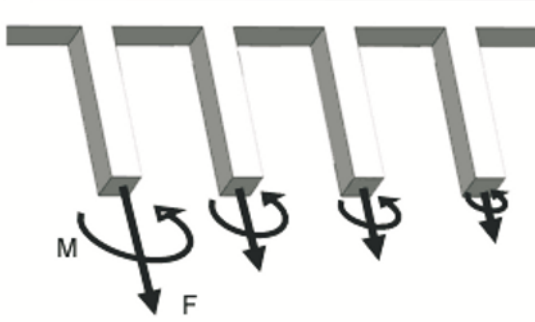


Figure 26. Forces and moments at the crack in foam, after Choi and Sankar (2005)

over the effective crack length

$$F = l \int_0^a \sigma_{yy} dr = K_I l \sqrt{\frac{2a}{\pi}} \quad (46)$$

Similarly, the bending moment in the crack tip strut is

$$M = l \int_0^a r \sigma_{yy} dr = K_{II} l \sqrt{\frac{2}{\pi}} \frac{a^{3/2}}{3} \quad (47)$$

Assuming that fracture of strut occurs when the maximum stress (for combined load bending and tensile) equals the ultimate tensile strength of the ligament material σ_{fs} Choi and Sankar (2005) relates the fracture toughness K_{Ic} to the tensile strength

$$\sigma_{fs} = \sigma_{bend} + \sigma_{tens} = \frac{M \frac{h}{2}}{\frac{h^4}{12}} + \frac{F}{h^2} = \frac{K_{II} l}{h^2} \sqrt{\frac{2a}{\pi}} \left(1 + 2 \frac{a}{h}\right) \quad (48)$$

resulting in

$$K_{Ic} = \sigma_{fs} \frac{h^2}{l} \sqrt{\frac{\pi}{2a}} \frac{1}{1 + 2 \frac{a}{h}}$$

and replacing $a = \alpha l$

$$K_{\text{Ic}} = \sigma_{\text{fs}} \frac{h^2}{l^2} \sqrt{l} \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{\alpha} \left(1 + 2\alpha \frac{l}{h}\right)} \quad (49)$$

Similarly for mode II the shear stress ahead of the crack tip can be expressed as

$$\tau_{xy} = \frac{K_{\text{II}}}{\sqrt{2\pi r}} \quad (50)$$

The shear force over the effective distance a can be found as

$$F_{\text{sh}} = l \int_0^a \tau_{xy} dr = K_{\text{II}} l \sqrt{\frac{2a}{\pi}} \quad (51)$$

The maximum bending moment in the crack tip element is produced by a shear force F_{sh} at a moment arm equal to $l/2$

$$M = F_{\text{sh}} \frac{l}{2} = K_{\text{II}} \frac{l^2}{2} \sqrt{\frac{2a}{\pi}} \quad (52)$$

The maximum bending stress is

$$\sigma_{\text{bend}} = \frac{M}{\frac{h^3}{6}} = 3K_{\text{II}} \frac{l^2}{h^3} \sqrt{\frac{2a}{\pi}} = \sigma_{\text{fs}} \quad (53)$$

The fracture toughness for mode II can be found

$$K_{\text{IIc}} = \sigma_{\text{fs}} \frac{h^3}{3l^2} \sqrt{\frac{\pi}{2a}},$$

and considering $a = \alpha l$, results

$$K_{\text{IIc}} = \sigma_{\text{fs}} \frac{h^3}{3l^2} \sqrt{\frac{\pi}{2\alpha l}} = \sigma_{\text{fs}} \frac{h^3}{3l^{3/2}} \sqrt{\frac{\pi}{2\alpha}} \quad (54)$$

Another model was proposed by Choi and Lakes (1996). They considered that due to crack blunting the stress field around crack in foam is non-singular. The stress field around a crack, with $2a$ length and a crack tip radius r_{tip} is, at distance $r > r_{\text{tip}}/2$

$$\sigma = \frac{K_{\text{I}}}{\sqrt{2\pi r}} + \frac{K_{\text{I}}}{\sqrt{2\pi r}} \frac{r_{\text{tip}}}{2r} \quad (55)$$

The force produced in the first unbroken strut is

$$F = \int_{r_{\text{tip}}/2}^{r_{\text{tip}}/2+h} \left(\frac{K_I}{\sqrt{2\pi r}} + \frac{K_I}{\sqrt{2\pi r}} \frac{r_{\text{tip}}}{2r} \right) r_{\text{tip}} dr \quad (56)$$

For a regular tetrakaidecahedron (14-sided polyhedron with 6 square and 8 hexagonal faces) with cell size l the crack tip radius is $r_{\text{tip}} = \sqrt{2}l$.

Integration Eq. (56) results in

$$F = 2.38K_I l \sqrt{\frac{l}{\pi}} \frac{h}{l} \quad (57)$$

Considering that the maximum stress in bending reaches the tensile strength of the solid results in

$$K_{Ic} = 0.20\sigma_{fs}\sqrt{\pi l} \left(\frac{h}{l} \right)^2 \quad (58)$$

For regular tetrakaidehedron cell packing with the relation between relative density and cell dimensions given as

$$\frac{\rho^*}{\rho_s} = 1.06 \left(\frac{h}{l} \right)^2 \quad (59)$$

the fracture toughness will be

$$\frac{K_I}{\sigma_{fs}\sqrt{\pi l}} = 0.19 \frac{\rho^*}{\rho_s} \quad (60)$$

All the above micromodels relate the foam fracture toughness to the tensile strength of the solid material and microstructure parameters: cell length and relative density. The methodology assumes that the load is transmitted through the foam as a set of discrete forces and moments acting on cell struts. Different integration limits were used in order to determine the forces and moments. Singular (Gibson and Ashby, 1997; Choi and Sankar, 2005) and non-singular stress fields (Choi and Lakes, 1996) in front of the crack were considered. The fracture toughness was obtained by considering that the crack extends when the stress in the first strut in front of the crack reaches the tensile strength of the solid. When micromechanical models are used the size effect regarding the variation of the tensile strength of the solid material σ_{fs} with strut size should be considered (Green, 1985). Huang and Gibson (1991) proposed a statistical analysis based on a Weibull distribution in order to show the effect of cell size on fracture toughness.

Lipperman et al. (2007) considered a lattice model consisting of rigidly connected Euler beams which can fail when the skin stress reaches a critical value. The conventional Mode I and Mode II fracture toughness were calculated and the influence of the relative density is shown. Long finite length cracks, modeled by several broken beams, were considered in the infinite lattice model. Four different layouts were considered kagome, triangular, square and hexagonal honeycombs, cf. Fig. 27.

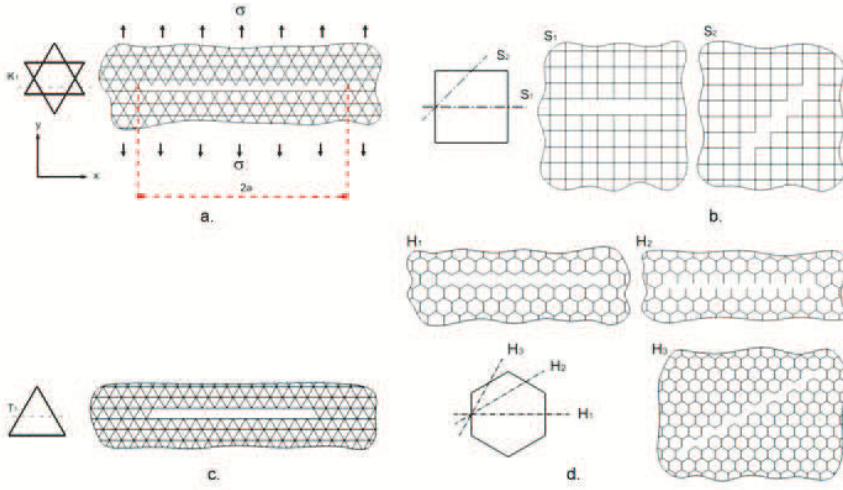


Figure 27. Lattice cracked models (a. kagome, b. square, c. triangular, d. hexagonal), after Lipperman et al. (2007)

The solution was obtained analytically using discrete Fourier transform reducing the initial problem for unbounded domain to the analysis of a finite repetitive module in the transform space. The variation of fracture toughness with relative density is investigated for the different types of cell lattices. This is close to linear and is in agreement with other published data. The Mode II fracture toughness is smaller than the Mode I for almost all investigated cases.

Another foam cracked model is to consider a central void of rectangular shape with size h_i , cf. Fig. 28 in the struts of thickness h . For this configuration, the fracture toughness can be expressed (Gibson and Ashby, 1997) as

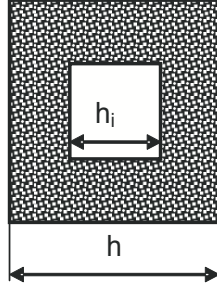


Figure 28. Strut with a box-like central crack

$$K_{Ic} = C \sigma_{fs} \sqrt{\pi l} \left(\frac{\rho^*}{\rho_s} \right)^{3/2} \frac{1 + \left(\frac{h_i}{h} \right)^2}{\sqrt{1 - \left(\frac{h_i}{h} \right)^2}} \quad (61)$$

Choi and Lakes (1996) highlight the correlation between fracture toughness of the foam, expressed in a generalized form as

$$\frac{K_I}{\sigma_{fs} \sqrt{\pi l}} = C \left(\frac{\rho^*}{\rho_s} \right)^{1+n} \quad (62)$$

with $n \geq 0$ and the fracture toughness of the solid

$$K_{Ics} = \sigma_{fs} \sqrt{\pi a} \quad (63)$$

with a as a small defect in the bulk material.

Substituting σ_{fs} between Eqs (62) and (63) we obtain

$$\frac{K_{Ic}}{\frac{K_{Ics}}{\sqrt{\pi a}} \sqrt{\pi l}} = C \left(\frac{\rho^*}{\rho_s} \right)^{1+n}$$

resulting in

$$\frac{K_{Ic}}{\rho^*} = \frac{K_{Ics}}{\rho_s} \sqrt{\frac{l}{a}} C \left(\frac{\rho^*}{\rho_s} \right)^n \quad (64)$$

Some important conclusions can be drawn from Eq. (64):

- The relative foam toughness K_{Ic}/ρ^* can be increased by increasing the foam cell size l . In order to obtain a certain value of foam fracture toughness the cell size l of foams can be changed accordingly.

- The size of defects in the bulk material a can also vary among materials. The defect size a can be of magnitude order of strut thickness h , which is less than the cell size l . Consequently $l > a$, and with large cell foams we could even have $l \gg a$.
- The relative density term may be equal to or less than one, depending on the cell packing. If a model with $n > 0$ applies, the foam cell size would have to be increased further to achieve a given level of foam toughness.
- Fracture toughness of foam materials per unit weight is similar to or even larger than that of bulk materials.
- The voids from the foam structure can act as built-in crack stoppers.

In recent years Finite Element Analysis is used extensively in order to solve the micromechanical models (Ableidinger, 2000; Choi and Sankar, 2005; Lee et al., 2007; Arakere et al., 2008). Finite Element Analysis is specially used for closed cell foams where the analytical formulations are more complex.

This topic will be treated in the Chapter "Finite Element Modeling of Foams" of the present book.

4 Concluding Remarks

At the end of this chapter we can conclude that Linear Elastic Fracture Mechanics can be applied for rigid plastic foams. In order to use the LEFM in designing of such structures the fracture toughness of materials should be known. Experimental tests for fracture toughness determination are presented, together with some experimental results. An alternative approach to tests is the micromechanical modeling. A review of the main micromechanical models to predict fracture toughness for open cell foams is shown. However, the fracture toughness values resulting after micromechanical analysis should be validated using experimental data.

Acknowledgements

The experiments carried out for determination the fracture toughness of polyurethane foams were supported by the Marie Curie Transfer of Knowledge project MTKD-CT-2004-014058: *Modern Composite Materials Applied in Aerospace, Civil and Mechanical Engineering: Theoretical Modelling and Experimental Verification* - coordinated by Prof. Tomasz Sadowski from Lublin University of Technology, Poland and by Research for Excellence grant 202/2006 awarded from Romanian National Authority for Scientific Research - coordinated by Prof. Dan M. Constantinescu from Uni-

versity POLITEHNICA of Bucharest, Romania. The author is also grateful to colleagues Marcin Kneć from Lublin University of Technology, Radu Negru and Linul Emanoil from POLITEHNICA University of Technology for their help in performing the experiments.

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Finite Element Modeling of Cellular Materials

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Abstract Cellular materials are characterized by a low apparent density and a discrete micro-structure which is on a distinctly lower length-scale than the one of components made from them. Consequently, the effective behavior of cellular materials is rooted in the mechanical behavior of the struts and/or cell walls on the length scale of individual cells. These lecture notes describe methods of modeling cellular materials by the finite element method. Topics include the setup of micro-mechanical models for open and closed-cell foams as well as sintered hollow sphere foam, the transition between the different mechanical length scales, and the optimization of the density distribution in components made from functionally graded foam.

1 Introduction

Finite Element (FE) methods are employed widely for the prediction of the mechanical response of fluids and structures. They are based on the approximation of the behavior of a continuum of potentially complex shape by subdividing it into geometrically simple regions of finite size, the so-called *Finite Elements*, which are simple enough to be described analytically. A great number of solution procedures can then be applied to predicting the elastic or inelastic, as well as the static or dynamic response of the problem described by the FE model, compare, for example, Bathe (1996); Hughes (1987); Zienkiewicz and Taylor (2000); Belytschko et al. (2000).

Because of their great versatility with regard to the modeling of complex material and structural behavior, FE methods have been applied heavily to simulations of the mechanical behavior of cellular materials. The modeling and the simulation of cellular materials by means of the FE method are the focus of the present section.

Cellular materials are materials that exhibit a discrete structure at the ‘microscopic’ level, with a skeleton of solid material being surrounded by a gas or a liquid. The size of the individual cells ranges from tenths of

millimeters to several millimeters and is typically much smaller than the size of the components that are made of or which include the cellular material. Another important characteristic of a cellular material is the low relative density ρ_{rel} , which can be calculated by dividing the averaged, effective density ρ^* of the cellular material by the density ρ_S of the solid material that the cell walls or the cell struts consist of:

$$\rho_{\text{rel}} = \frac{\rho^*}{\rho_S} \quad (1)$$

Cellular materials are defined by their low relative density, which is typically in the range of a few percent of the solid volume fraction. For higher relative densities, the classification *porous material* is more appropriate.

Cellular materials offer distinct advantages with respect to their high weight-specific stiffness and strength. In addition to their good weight-specific mechanical properties they can often be used in a multi-functional manner, for example in heat-exchangers or catalytic converters if they are permeable to fluids, as it is the case for open-cell foams, or as materials providing insulation and buoyancy, if the closed cells are impermeable to fluids. Some related aspects like lightweight design, energy absorption and heat dissipation are discussed in Evans et al. (1999, 2001).

A rich variety of cellular materials of natural origin emerged as a result of evolutionary competition. Cellular tissue can be found in most living beings, where it appears in the form of wood, sponge, trabecular bone, or cork. Understanding the mechanics of natural cellular material is not only interesting from the theoretical point of view, but also important for designing bio-compatible materials for bone-replacement or for use as porous scaffolds. A review of the bio-mechanics of cellular solids can be found in Gibson (2005). The behavior of wood cells is discussed, for example, in Müller et al. (2003). General reviews of the mechanics of cellular materials are given in Gibson (1989) and Gibson and Ashby (1997).

Cellular materials are also an important class of engineering materials, as the wide variety of polymeric foams, metallic foams and ceramic foams shows. Most cellular engineering materials can be classified as having either an open-cell or a closed-cell topology. Besides the availability of materials with stochastic micro-structures, production technologies such as rapid prototyping offer the possibility of producing or assembling geometrically regular and periodic micro-structures.

The main focus of this contribution lies on Finite Element (FE) models of the micro-structure of man-made cellular materials such as polymer foams and metallic foams. Methods for the setup of such models are presented, and the transition between the length-scales of the struts and cell-

walls, in the following called the ‘micro-mechanical length scale’, and the length scale of whole components made of or containing foam, denoted as ‘macro-mechanical length scale’, are discussed. While most of the presented material deals with topics related to micro-mechanics, the last Section 5 is devoted to a selected macro-mechanical issue: the optimization of the apparent density distribution in a component made of functionally graded foam.

2 Homogenization and the Unit Cell Method

The effective physical behavior of cellular structures on the macroscopic level, where the cellular structure can be treated as a homogenized pseudo-material rather than a discrete structure, is certainly governed by the physical processes on the micro-scopic level, where the behavior of individual struts or cell walls has to be observed or predicted.

Deriving the macro-mechanical properties of a micro-structured material from the micro-mechanical behavior is a process called *homogenization*. The aim of homogenization is the reduction of the number of the degrees-of-freedom in the mechanical model, from the displacements and rotations of countless individual micro-structural members to a much smaller number of macro-scopic displacements and rotations. Homogenization relations usually take the form of volume averages, which give the homogenized value of a variable $f(\mathbf{x})$, $\langle f \rangle$, in the form of the weighted volume integral

$$\langle f \rangle = \frac{1}{\Omega} \int_{\Omega} f(\mathbf{x}) \, d\Omega \quad , \quad (2)$$

where $\mathbf{x}$ stands for a position vector and Ω represents the integration volume, that is, the inhomogeneous micro-scopic volume element under consideration.

The opposite process, namely the transition from macro-mechanical stresses or strains to local deformations and micro-mechanical stress and strain fields is called *localization*. Examples of localization can be found in fracture analysis of cellular materials, where macro-mechanical displacement fields are imposed on discrete, micro-structures, compare Section 4.5.

Approaches which are suitable for performing homogenization and localization with respect to the mechanical behavior of micro-structured materials can be divided into two main groups, compare Böhm et al. (2009):

Mean Field Approaches, which describe the physical field variables on the micro-scopic level only by means of their volume average in the different constituents. These methods require statistical methods for

the introduction of micro-geometrical parameters. Mean Field Approaches typically give estimates or bounds for the effective properties of non-homogeneous materials with a minimum of numerical effort, since they mostly rely on analytical relationships.

Discrete Microfield Approaches resolve the physical fields on the micro-scopic level fully, using analytical methods (for example beam theory for open-cell foams) or numerical methods to predict the actual behavior of the micro-structure in as much detail as necessary.

Mean field and Hashin–Shtrikman-type methods, which play an important role in continuum micro-mechanics of composites and materials with a small volume fraction of pores, compare Ponte Castañeda and Suquet (1998), have seen only limited use for cellular materials. Hashin–Shtrikman upper bounds for the elastic moduli of macro-scopically isotropic cellular materials can be evaluated by prescribing a vanishing stiffness for the void phase. Typically, the predicted elastic moduli of foams lie considerably below these upper bounds. The lower bounds do not give useful information, since they vanish trivially.

Since mean field methods are of limited use for predicting the behavior of cellular materials, most micro-mechanical studies of these materials are based on discrete micro-geometrical models.

From the point of view of micro-mechanical modeling, cellular metals are usually divided into two main groups:

Open-cell materials, which have a solid scaffold that consists mainly of beam-like members, such that the material is more or less permeable to fluids.

Closed-cell materials, which consist of cells that are formed by interconnected cell walls. Depending on the porosity of the cell walls, these material are more or less impermeable to fluids.

Cellular materials typically show a limited elastic range, and their effective mechanical behavior tends to be dominated by micro-scopic deformation mechanisms such as bending of struts, buckling of cell walls, formation of yield joints or folds, and fracture of cell struts and cell walls. These deformation mechanisms are influenced by the micro-geometry, which, in practice, tends to be complex and highly non-uniform in the case of solid foams.

As long as the real micro-geometry is radically simplified and idealized, it is possible to use analytical methods for describing the micro-mechanical behavior. For open-cell foams, for example, beam theory was employed successfully in some studies. These methods, however, soon reach their limits, and most generic micro-geometries have been, consequently, investigated

by numerical methods. Because of its versatility and robustness, the finite element method has proved to be the most popular numerical tool.

With respect to the representation of a micro-structured material by means of micro-mechanical finite element models, the following principal modeling approaches exist:

Multi-cell models of finite samples, which represent a real foam sample in the most accurate manner possible with the available hard- and software. The aim of this approach is to predict the behavior of an actual foam sample under realistic loading conditions, which, in most cases, correspond to uni-axial compression. Thus, direct comparison to experimental results is possible. One aspect of this kind of models, which is not as strongly present in the other approaches, is the modeling of the boundary conditions on the surfaces that are not free. The interaction between the sample and loading platens, for example, is an issue here, as it may constrain the boundary layer of cells in different ways. The fact that the boundary layer potentially behaves differently to the rest of the sample may induce a size-effect that causes the effective behavior to be dependent on the number of cells in the sample.

Unit-cell models are more concerned with an efficient treatment of the material as an infinite medium. The unit-cells typically consist of a regular, periodic arrangement of a limited number of base cells, which are modeled in high geometrical detail. Geometrical periodicity and periodicity boundary conditions ensure that the unit cell is space-filling both in the undeformed and the deformed states. This kind of models is particularly suitable for a controlled study of the influence of a variation of geometrical parameters on the effective behavior. Furthermore, unit-cells with random cell geometry also benefit from the application of periodicity constraints, because these boundary conditions give the best prediction of the mechanical behavior of an random, but periodic, infinite material.

Embedded-cell models are hybrids which feature a micro-geometrically fully resolved “core” (similar to a periodic unit cell model) which is embedded in a matrix of surrounding material, the effective behavior of which corresponds closely to the effective behavior of the embedded cell. This way, it is possible to look at micro-mechanical processes in a specific section of a component without having to model the whole component in full detail. One useful field of application of this method is the investigation of the phenomena around crack tips in cellular materials, as will be discussed in Section 4.5.

These different modeling strategies, which have well-established equivalents in continuum micro-mechanics of composite materials, give information about the local deformation mechanisms on the one hand and the corresponding overall behavior, which can be obtained by homogenization, on the other hand.

In the following, the treatment of periodic unit-cell models is given a closer look. The unit-cell method assumes that the behavior of an infinite periodic structure can be represented at least in the sense of a good approximation by a model of finite size. This model is a combination of a geometrically representative building block, which allows for a seamless, periodic tiling of space, and appropriate boundary conditions, which ensure the periodicity of the displacement field of the structure. The space-filling structure that is thought to be represented by the unit-cell model is obtained by shifting the original unit-cell along linear combinations of the *vectors of periodicity*. In general, two and three vectors of periodicity are required for the description of periodic 2/D and 3/D structures, respectively. For mechanical problems the displacement components of those points on the boundary of the unit cell, which can be mapped onto each other by a translation along a linear combination of integer multiples the vectors of periodicity, have to be coupled by kinematic constraints.

Compared to the high geometrical complexity and irregularity of actual, random cellular materials, unit-cell models tend to have highly idealized geometries.

The formulation of periodicity boundary conditions for the displacement field on the boundary of a rectangular unit-cell will be described in the following. Figure 1 shows the undeformed and a deformed configuration of an originally rectangular 2/D unit cell. The boundary of the rectangular unit cell in Fig. 1 consists of the four edges, which are identified by the letters N, S, W, and E, as well as the four corners, which are marked by the letter-codes NW, SW, NE, SE. The letters in this naming scheme represent the first letters of the four cardinal directions North, South, West, and East.

The unit-cell has to be constrained against rigid body movement modes. In Fig. 1 the translational degrees-of-freedom of the corner node SW is constrained to suppress rigid body translations. Additionally, the vertical displacement component of the finite element node SE is constrained to prevent a rigid body rotation. Even though rigid body movement modes are suppressed by these constraints, the unit cell can still deform without violating the condition of geometrical periodicity by shortening or elongation in horizontal and vertical directions, respectively, as well as in a shear-like mode. It is convenient to relate these global deformation modes

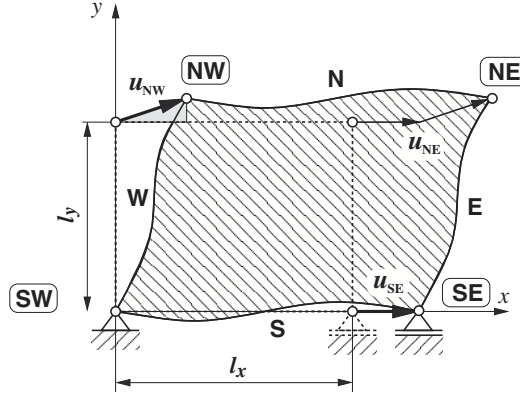


Figure 1. Idealized 2/D unit cell in the undeformed and a deformed configuration. Displacement vectors and letter codes are given for the corner and edge nodes. Single letter codes denote edges, two-letter codes denote corners

to selected displacement components of specific finite element nodes at the corners of the model. These nodes are commonly called “master nodes”. The nodes NW and SE in Fig. 1 are master nodes, and their contribution to the deformation field of the unit cell is indicated by bold arrows.

In order to maintain geometrical periodicity in the deformed state of a rectangular unit cell, the displacements of opposite boundary edges have to be coupled. The degrees of freedom (DOF) of one of these coupled edges remain unconstrained. These DOFs are so-called “master” DOFs. The nodes on the opposite, “slave” edge of the unit cell, which have a partner node on the “master” edge, are coupled in to the master nodes in terms of their displacement vector. In addition, an additional displacement vector is superimposed on the displacement field of the slave nodes. This displacement vector is related to the macro-scopic DOFs of the unit cell. In the idealized unit cell in Fig. 1 the offset vectors for the edges N and E are given by $\mathbf{u}_{NW} = \{u_{NW}, v_{NW}\}$ and $\mathbf{u}_{SE} = \{u_{SE}, 0\}$, respectively. The components of these displacement vectors are related to the macro-scopic strain tensor, which is described in vector notation by the vector of its components $\epsilon = \{\epsilon_{xx}, \epsilon_{yy}, \gamma_{xy}\}$:

$$\epsilon_{xx} = \frac{u_{SE}}{l_x}, \quad \epsilon_{yy} = \frac{v_{NW}}{l_y}, \quad \gamma_{xy} = \frac{u_{NW}}{l_y}, \quad (3)$$

Note, that this choice of master and slave entities is not the only possible

one. In fact, any suitable DOF in the finite element model can be used as a master DOF; it does not even have to belong to any node of the actual unit cell. Otherwise, an automated generation scheme for periodicity boundary conditions would fail as soon as there happen to be no finite element nodes in the corners of the unit cell.

Coupling conditions have to be defined separately for slave edge nodes and slave corner nodes. In the presented 2/D setup, the displacements of the corner node SW are fully constrained to avoid rigid body translations, and the vertical displacement of the corner node SE is constrained for preventing a rigid body rotation of the whole unit cell. The horizontal DOF of the node SE is related to the macro-scopic normal strain in the x direction and the horizontal and vertical displacements of the corner node SW define the macro-scopic shear strain and the macro-scopic normal strain in the y direction. The edges S and W remain unconstrained. The displacements of the slave edges N and E are then obtained by the coupling equations:

$$\mathbf{u}_E(y) = \mathbf{u}_W(y) + \mathbf{u}_{SE} \quad (4)$$

$$\mathbf{u}_N(x) = \mathbf{u}_S(x) + \mathbf{u}_{NW} \quad (5)$$

The displacement vector of the corner node NE, on the other hand, is a linear combination of the displacement vectors $\mathbf{u}_{NW}$ and $\mathbf{u}_{SE}$ of the two master nodes NW and SE:

$$\mathbf{u}_{NE} = \mathbf{u}_{NW} + \mathbf{u}_{SE} \quad (6)$$

For finite element models involving structural finite elements such as beam or shell elements, which have rotational DOFs, additional coupling equations have to be provided. In the case of the rectangular 2/D unit cell the rotational DOFs φ of all corner nodes are constrained to be identical:

$$\varphi_{NW} = \varphi_{NE} = \varphi_{SE} = \varphi_{SW} \quad (7)$$

Here, the DOF φ_{SW} acts as the master DOF for the rotational DOF in the four corners of the unit cell. The rotational DOFs of opposite edges have to be coupled by:

$$\varphi_E(y) = \varphi_W(y), \quad \varphi_N(x) = \varphi_S(x) \quad (8)$$

Since the displacements of all nodes along the slave edges on the boundary of the unit cell are coupled to individual master DOFs, the forces acting on these master DOFs are distributed over the whole edge that follows the master DOF's movement. This means, that the surface tractions along the

slave edge are automatically summed up to give a resultant force acting on the master DOF. Analogously, the reaction forces at the constrained DOFs have to set the unit cell into static equilibrium and correspond to the sum of the surface traction vectors at the master edges.

Thus, the unit cell model reacts to concentrated loads that are applied to master DOFs just as the infinite periodic structure would react to homogenized applied stresses, see Smit et al. (1998). From that, a relationship similar to Equation (3) follows between the homogenized stress state $\sigma = \{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}\}$ and the concentrated horizontal and vertical nodal forces H and V ; this relationship involves the side lengths l_x and l_y of the rectangular unit cell:

$$\sigma_{xx} = \frac{H_{SE}}{l_y}, \quad \sigma_{yy} = \frac{V_{NW}}{l_x}, \quad \sigma_{xy} = \frac{H_{NW}}{l_x}, \quad (9)$$

So far, l_x and l_y are defined as the initial side lengths of the unit cell, so that these stress components are nominal, or engineering, stress measures. The extension to nonlinear stress measures in the large strain domain is straightforward and involves the consideration of the actual side lengths.

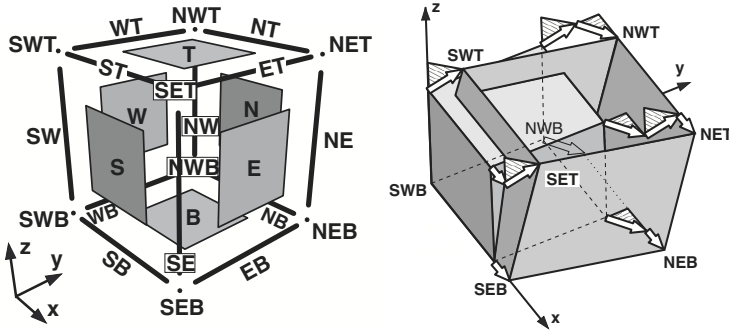


Figure 2. Default identifiers for the node sets on the unit cell cube (*left*) and general deformation state of the 3D unit cube (*right*)

The scheme outlined above can be extended to three-dimensional unit cells. Figure 2 (*left*) presents a naming convention for identifying corners, edges, and faces on a cube-shaped unit cell. Two additional identifiers Top and Bottom are introduced to account for the third spatial dimension. Three-letter identifiers designate vertices, two-letter identifiers describe edges, and single letter identifiers denote faces. Figure 2 (*right*) shows the cube-shaped 3/D unit cell in a state of general macroscopic deformation, with three normal and three shear deformation DOFs being active.

Unlike Fig. 1, this figure does not show the periodic fluctuations of the local deformation field.

The most complex constraint equation in terms of the number of independent DOFs involved (three) can be found for the x -displacement u_{NET} of the corner NET:

$$u_{\text{NET}} = u_{\text{SEB}} + u_{\text{NWB}} + u_{\text{SWT}} \quad (10)$$

Figure 3 shows the successful application of the proposed kinematic boundary conditions to a 3D unit cell model containing multiple cells with a closed-cell tetrakaidecahedral topology. Since the cell walls are modeled us-

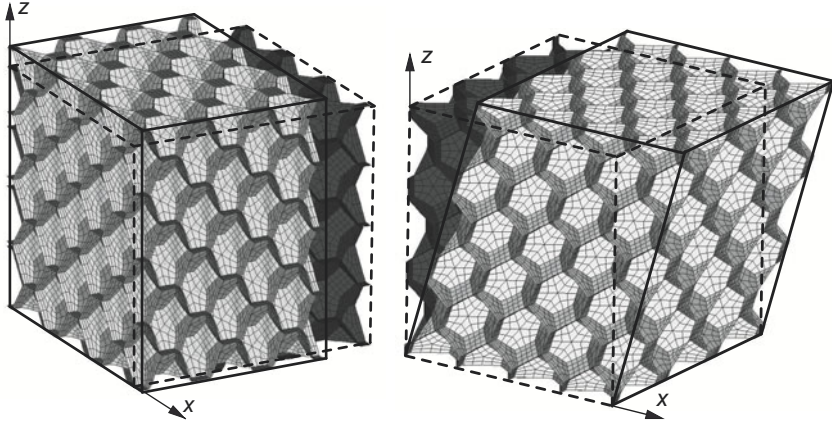


Figure 3. Predicted undeformed (dark) and deformed configurations of a 3D unit cell model composed of multiple tetrakaidecahedral cells. The deformed configurations for uniaxial compression (*left*) and combined shear (*right*) are shown

ing shell elements, periodicity coupling conditions analogous to Eqs (7) and (8) have to be provided for rotational DOFs, too. Figure 3 (*left*) shows a deformed configuration predicted for uniaxial compression along the y -axis. The undeformed unit cell is indicated by the elements which are shaded in dark gray and surrounded by a dashed bounding box. The deformed cube, which has a light gray color and is surrounded by a solid-line bounding box, clearly shows the Poisson's effect. Figure 3 (*right*) shows the deformed configuration for combined macroscopic shear loading, $\sigma_{zx} > 0$ and $\sigma_{yz} > 0$. Both deformation modes, uniaxial deformation and shear deformation, are shown to be handled correctly by the periodicity boundary conditions. Since

the predicted displacements in Fig. 3 are the result of a linear finite element analysis they had to be scaled up for plotting.

As soon as a unit cell model is not intended to represent only a regular and geometrically periodic micro-structure, but rather a random material with all kinds of irregularities and imperfections, the question arises, how many cells a unit cell model is required to have in order to be representative of the actual micro-structure. This question — which can cause rather heated discussions — can only be answered by the application of statistical measures, for example the maximum allowable standard deviation of a particular effective property evaluated over a series of unit cells. Clearly, it is desirable to include as many cells as possible in the model while maintaining enough detail in the individual cells. A discussion of this issue can be found in Hashin (1983).

It is important for the designer of unit cell models to keep in mind that the results are periodic in all respects. However, some phenomena such as

- strain localization by the collapse of a weak layer of cells,
- crack propagation, or
- wave propagation

are not strictly periodic in all spatial directions. For example, the densification of cellular materials will always be predicted as a periodic pattern by unit cell models; the collapse of a cell layer in the periodic unit cell models would correspond to the simultaneous collapse of multiple layers in the real material which is unphysical in most cases. The wavelength of buckling modes also depends strongly on the size of a periodic unit cell, and cannot exceed it, unless special mathematical methods are applied, see Section 4.3. Furthermore, single crack tips (as opposed to periodic patterns of cracks) cannot be handled by periodic micro-geometries, and free surfaces are restricted to layer-like geometries in which at least one direction is non-periodic. For these types of problems, embedded cell approaches are a possible simulation approach (Ableidinger, 2000; Rammerstorfer and Böhm, 2002).

3 Micro-Mechanical Finite Element Models of Cellular Materials

3.1 Introduction

In this section we describe micro-mechanical finite element models which have been proposed as suitable representations of the micro-structure of cellular materials. We first concentrate on cellular materials that are the result of solidifying a liquid foam, for example, polymer foam or metal foam.

Solidified Liquid Foams. Liquid foams are governed by the laws of Laplace and Plateau. The Law of Laplace states that the mean local curvature $H = \frac{1}{2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$ of a cell wall, R_1 and R_2 being the minimum and the maximum local radii of curvature, respectively, is proportional to the pressure difference Δp between the cells that are separated by the cell wall: $H = \frac{\Delta p}{2\gamma}$, where γ stands for the surface tension. Therefore, the mean local curvature H is constant within a given cell wall.

The cells of the solid foam correspond to bubbles of the liquid foam. These bubbles have the shape of polyhedra, with cell walls meeting in edges and edges meeting in vertices. In the liquid state material is drained from the cell walls by surface tension and redistributed into the edges, which assume a cross-section which can be described as a triangle with concave sides. This specific kind of edge geometry is known as Plateau border. The typical shape of the Plateau borders can be seen from a micro-section of a metal foam in Fig. 4 (*left*) With regard to the distribution of material be-

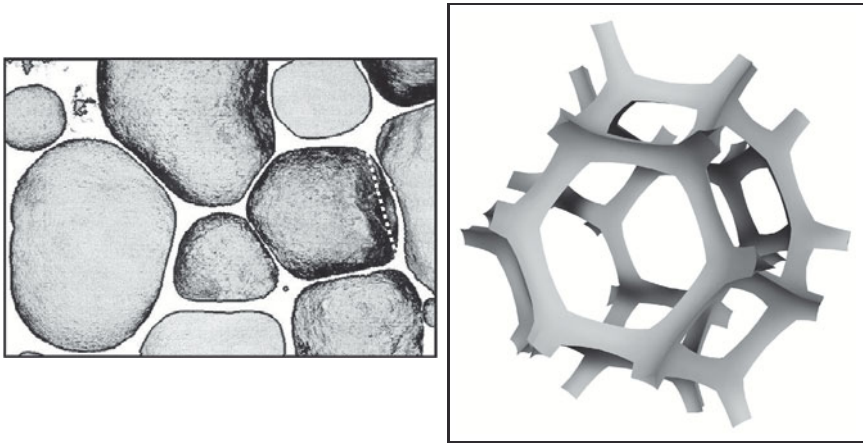


Figure 4. Micro-section through metal foam (*left*) and geometrical model of a Plateau border network (*right*), from Bitsche (2005)

tween cell walls and cell edges in closed-cell foams, the following distinction can be made:

- *wet* foam: a foam where most material is concentrated in the edges, which have a plateau border cross-section.
- *dry* foam: a low-density foam, where most material is situated in the thin cell walls, and the amount of material in the edges is determined by the intersection of the cell walls.

With these definitions, the Laws of Plateau can be summarized:

- The cell wall films in a dry wall can intersect only three at a time, and must do so at an angle of 120° .
- No more than four intersection lines may meet at the vertices of a dry foam. These tetrahedral vertices are perfectly symmetric, i.e., the angle between two edges is $\arccos(-1/3) \approx 109.47^\circ$.
- The transition from a Plateau border to an adjacent film is smooth, that is, the surface normal is the same.

In a dry, liquid foam, the stored energy is proportional to the total surface area of the liquid films. Trying to minimize the potential energy of the foam is, therefore, the same problem as trying to minimize the total surface area for a given arrangement of bubbles.

Minimizing the surface area of a given arrangement of bubbles is a task that can be tackled with the program *Surface Evolver*, which was developed by Brakke (1992). This program is a powerful tool for deriving physically sound geometrical templates for finite element meshes of cellular materials originating from liquid foams.

The *Surface Evolver* program can be used to create models of the two most commonly used arrangements of polyhedra of identical volume, which are spatially periodic and have minimum surface area, thus fulfilling the need for both simplicity and physical relevance:

Kelvin Cell: Lord Kelvin (Sir W. Thomson) proposed this periodic structure as an answer to the question which space-filling arrangement of cells of equal volume has minimal surface area. In Thomson (1887) he described the tetrakaidecahedron, see Fig. 5 (*left*), which appears to be the solution as long as only arrangements of identical polyhedra are considered.

Weaire-Phelan Cells: Weaire and Phelan (1994), by using the *Surface Evolver* program, found a periodic structure consisting of two different kinds of polyhedra of equal volume (see Fig. 5) that has a slightly smaller surface area than a Kelvin foam. It can, thus, be seen as a more realistic model for liquid foam.

In addition to these generic periodic structures, random arrangements of cells of equal volume (mono-disperse foam), of two families of cells of different volume (bi-disperse foam) and of cells of random volume have been investigated with the *Surface Evolver* program, see, for example, Kraynik et al. (2003, 2004, 2005), respectively.

Voronoi Diagrams. Another popular method for creating models of cellular structures is the Voronoi diagram. Here, seed points are randomly placed in space, and the symmetry planes between individual seed points

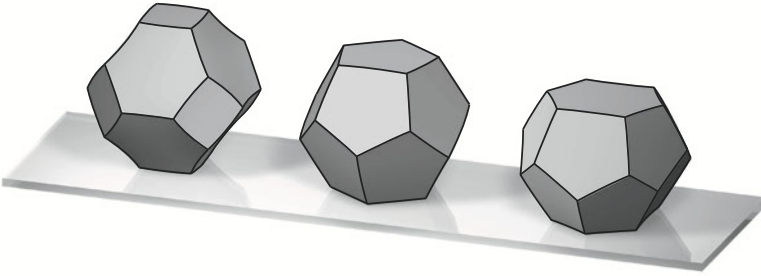


Figure 5. Image of the three polyhedra that are commonly used in periodic unit cell models for foam: tetrakaidecahedron (*left*) as the base cell of Kelvin foam, and the two different polyhedra that constitute the Weaire-Phelan foam (*middle/right*); from Bitsche (2005)

and their next neighbors are generated. Intersecting these planes produces an arrangement of convex polyhedra that can be used as the basis of open or closed-cell foam models, compare Fig. 6. This is the structure that will be obtained if a uniform growth rate of bubbles nucleating at the seed points is assumed. The connectivity of Voronoi models does not necessarily correspond to the one based on the physical laws governing liquid foam. This disadvantage can be removed by treating the Voronoi structure with *Surface*

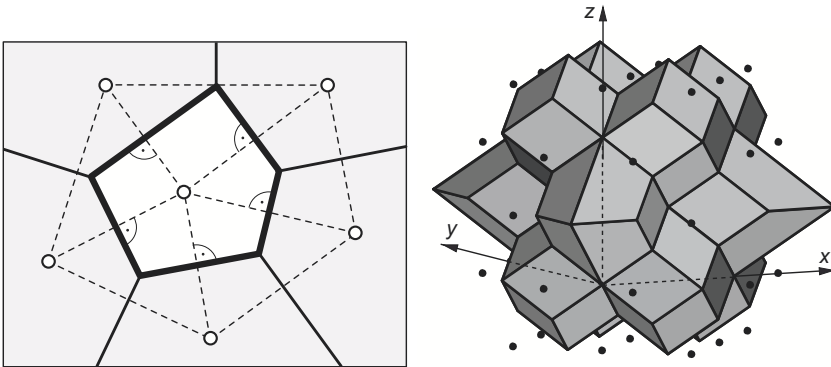


Figure 6. Construction of a five-sided polyhedral cell from a Voronoi diagram of six seed points (*left*) and a 3D Voronoi diagram for the construction of the unit cell for a ‘3D honeycomb’ (*right*)

Evolver, in order to obtain a liquid foam that obeys mechanical equilibrium conditions, see Kraynik (2003).

Simulations of the Foaming Process. It would appear to be desirable to simulate the foaming process directly in order to obtain suitable micro-structural models of the solidified foam. Such simulations have been presented, but appear to be still in their infancy. Applications of the lattice Boltzmann method for simulating the nucleation, growth and coalescence of bubbles have been proposed, see, for example, Körner (2008). Unfortunately, these simulations are still restricted to 2D settings. Bikard et al. (2005) present a 3D simulation of the evolution of polymer foam during the foaming process. This simulation is based on the finite element techniques implemented in the program *REM3D*.

Micro-Computed Tomography. The most direct approach for obtaining micro-mechanical models of cellular materials is a 3D scan by means of micro-computed tomography (μ -CT), followed by a suitable surface extraction and meshing procedure.

Early works in this field use the geometrical information contained in the voxel cloud that represents the solid phase of foam materials in μ -CT scans directly by translating the voxels into cube-shaped continuum finite elements. Corresponding results are presented in Kinney et al. (2001); Maire et al. (2003). A very thorough investigation of the yield surface of a small cubic sample of open-cell aluminum foam under multi-axial loading is presented in Wicklein and Thoma (2005). Their model was also created by the voxel-to-element method.

Youssef et al. (2005) observed the deformation of a polyurethane foam sample by μ -CT and discussed issues related to the quality of finite element models of this sample, such as size of a representative volume element, mesh refinement, and boundary conditions. They used a unstructured mesh of second-order, tetrahedral solid elements. Their simulation results captured well the overall mechanical behavior of the sample as well as some very local deformation mechanisms.

Jeon et al. (2009) presented an interesting approach for the determination of the mechanical properties of the cell wall material of Aluminum foam. They produced small foam samples consisting of only a few cells each, scanned them in a μ -CT and constructed appropriate FE models such that the parameters of a power law for the hardening behavior of the cell wall material could be obtained by fitting the predicted overall force-displacement behavior to the experimentally measured one.

Brydon et al. (2005) used a particle model based on a μ -CT scan for

simulating the densification of open-cell carbon foam.

Hollow Sphere Structures (HSS). Bonding small hollow spheres together is a possible way of obtaining a cellular structure with a very well defined micro-geometry. Generic models for such HSS are often based on cubic close packings of spheres, with face-centered cubic or body-centered cubic packings being the alternatives.

Overlapping Issues when using Structural Elements. Slender struts in open cell foams and thin cell walls in closed-cell foams are not well suited for being discretized with continuum finite elements, because a great number of elements with unfavorable aspect ratios would be necessary to mesh them. For those structural entities, structural finite elements such as beam elements or shell elements are better suited.

Unfortunately, the use of structural finite elements causes complications in models of high relative density, because in the FE model the beams and shells meet in point-like vertices and edges that are collapsed into lines, respectively. This causes the fictitious material in the joined structural finite elements to overlap, leading to a under-estimation of the wall thickness in the model and a under-estimation of the stiffness of the vertices and edges. Figure 7 illustrates the overlap resulting from joining four finite beam elements of circular cross-section in a tetrahedral vertex.

Methods such as representing the vertex volume in open-cell foams by beam elements with artificially high stiffness have been proposed. In any case, the (virtual) overlapping of material in structural elements has to be accounted for when the thickness or the cross-sections of the elements are

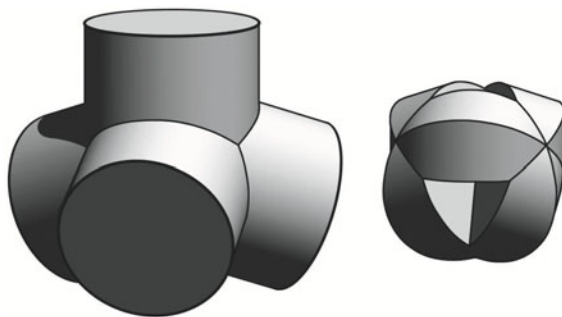


Figure 7. Virtual overlap of material at a tetrahedral vertex joining four finite beam elements with circular cross-section

calculated.

The described problems could, perhaps, be circumvented by using a hybrid model that models the vertices/edges with continuum elements while the struts and cell walls are modeled by structural elements. No such model has been proposed so far.

A thorough discussion of different approaches to modeling open-cell foam can be found in Jang et al. (2008), where the issue of material overlap in the vertices is discussed in detail.

Models of Samples of Finite Size. Unit cell models suffer from some deficiencies with regard to the consideration of boundary effects or the proper simulation of localization mechanisms. Therefore, full-size finite element models of finite-sized samples have been proposed. Because of the large computational effort that is necessary to simulate a structure with hundreds of individual cells, these works have so far been restricted to 2D problems (most notably crushing of honeycombs), open-cell 3D models, for which 3D beam models can be employed, compare Gong and Kyriakides (2006), and continuum element models in the elastic regime, Kirca et al. (2007).

3.2 Open-Cell Foams

Introduction. Open-cell foams are foams which are formed by a micro-structural networks or lattices of slender, beam-like members. Perhaps it would be more appropriate to call open-cell foams “sponges”, but this designation did not stick so far. Just as real sponges, open-cell foams are permeable to fluids.

Examples of natural open-cell materials comprise spongy/trabecular bone and natural sponges, see Gibson and Ashby (1997). Many engineering materials such as polymeric foams, cellular ceramics or some metallic cellular materials also show an open-cell micro-structure. Open-cell micro-structures with a regular, periodic geometry have been produced by rapid prototyping, compare Stampfl et al. (2004); Wöß et al. (2004).

The primary mode of deformation of open-cell materials is bending of their struts. These networks of struts are often the cell edges or Plateau borders that are left when a previously closed-cell material loses its cell wall faces during a production process.

Figure 8 shows the deformed shape of a strut in an open-cell metallic foam. The bending mode of deformation is clearly visible. The figure also shows the characteristic triangular Plateau border cross-section of the struts which was already explained by the laws of physics of wet foams in

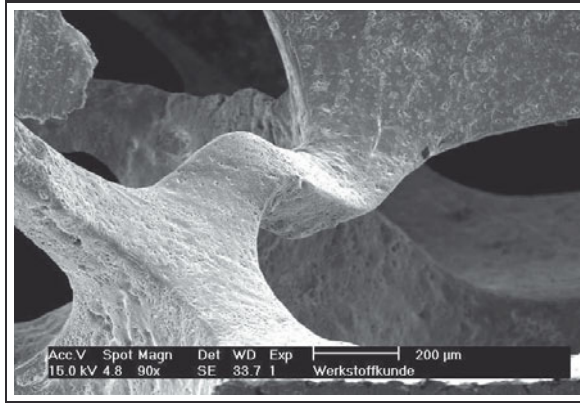


Figure 8. REM image of a bent strut in an open-cell metallic foam; (image courtesy of Institute of Materials Science and Testing, Vienna University of Technology)

Section 3.1.

Because the effective stiffness of the Plateau border network is so important for the overall stiffness of an open-cell foam, Warren et al. (1997) calculated the torsional stiffness of a Plateau border cross-section.

Beam Finite Element Models for Open-Cell Foams. Beam elements are the natural choice for the discretization of open-cell foams. They are computationally inexpensive and can be used to compose models with many cells. Care must be taken to avoid issues arising from overlapping of beam elements at the vertices. Also, beam-to-beam contact can cause problems in high-strain compression situations or, worse, is not supported at all by some FE codes.

Most studies involving 3D beam models of open-cell foam deal with the latter's (hyper-)elastic properties. The reason for this restriction to the elastic regime of the strut material can be found in the difficulty of correctly representing an expanding plastic zone in a Plateau border cross-section with beam elements from standard element libraries. Indeed, relevant studies use circular cross-sections in the elastic-plastic regime, compare Jang and Kyriakides (2009).

Shulmeister et al. (1998) investigated the nonlinear tension/compression tests for open-cell Kelvin foam, a rhombic dodecahedron structure and a random Voronoi foam, using different elastic material laws. The work is

an extension of van der Burg et al. (1997), where only the linear elastic response was considered.

Zhu et al. (2000) investigated the influence of cell shape regularity in periodic Voronoi unit cell models on the elastic properties. Zhu and Windle (2002) crushed periodic Voronoi foam models with linear elastic strut material to up to 60% compressive strain. Plateau border cross-sections with constant area were assumed for the struts in both studies.

Gan et al. (2005) predicted elastic properties of random, periodic Voronoi models in the presence of imperfections. Furthermore, they determined failure surfaces for compressive loading which were defined by the limit load that was the result of geometrical nonlinearities and (elastic) buckling of struts. Constant circular cross-sections were assumed for the struts.

Li et al. (2006) investigated regular and irregular periodic models of carbon foams with respect to the influence of cell disorder and the shape of the strut cross-section.

Gong and Kyriakides (2006) studied the collapse of large-scale Kelvin foam models using nonlinear springs as substitutes for rigorous contact conditions. While they used linear elastic strut material in this study, the same methodology was applied to aluminum foam with elasto-plastic strut material behavior in Jang and Kyriakides (2009); circular cross-sections of varying area along the strut axis were modeled. Related papers dealing with long-wavelength buckling in anisotropic Kelvin foams will be discussed in Section 4.3.

Jang et al. (2008) investigated the elastic properties of polyester urethane and Duocel aluminum foam by means of different FE models, offering the opportunity to compare results obtained by different modeling approaches.

Continuum Finite Element Models for Open-Cell Foam. Continuum solid element models for open-cell foam are rare because of the high computational cost that results from a fine discretization of the slender struts or Plateau border network.

Roberts and Garboczi (2002) presented a voxel-to-element technique for obtaining various random 3D open-cell micro-structures and predicting their elastic properties.

Open-cell models with fully modeled Plateau border geometries can be found in Boomsma et al. (2003) and Mills (2005) in the context of permeability analyzes using computational fluid dynamics. A simulation of crushing of open-cell polyurethane foam based on a wet Kelvin foam unit cell models is presented in Mills (2007).

Kirca et al. (2007) discussed a large-scale continuum FE model for a whole sample of open-cell carbon foam considering the linear elastic response

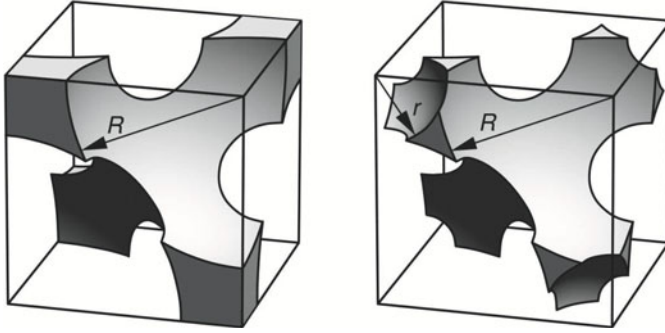


Figure 9. Unit cell models for open-cell aluminum foam produced by infiltrating a close packing of spheres of uniform radius R (*left*) and a packing of spheres of radii R and r (*right*) proposed by Kou et al. (2008)

of the model to uniaxial compression.

Kou et al. (2008) propose a unit cell model for open-cell aluminum foam which is produced by infiltrating closely packed spheres, see Fig. 9. Prescribing elastic/perfectly plastic bulk material behavior, they calculate the stiffness and the yield under uniaxial compression. A very similar model was used by Sihn and Roy (2004) for predicting the elastic constants of open-cell carbon foam. A different generic model for open-cell foam is presented by Oliveira et al. (2008), see Fig. 10. This model is generated by intersecting a cube with cylinders to obtain a simple cubic lattice. The paper deals with

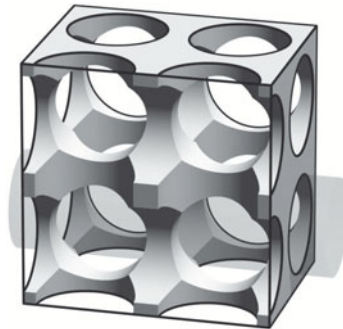


Figure 10. Generic model for open-cell foam based on the intersection of a cube with cylinders, Oliveira et al. (2008)

uniaxial compression of such structures made from aluminum in simulation and experiment.

Particle Models for Open-Cell Foam. Brydon et al. (2005) proposed a particle model for open-cell foam which allows to study the crushing of a finite foam sample into the densification regime. The model was based on CT scan data of an actual open-cell carbon foam.

3.3 Closed-Cell Foams

Introduction. Closed-cell micro-structures are arrangements of closed cells, which are clearly identifiable by the cell walls that are present between the skeleton of cell edges or the Plateau border network. While the individual cell walls may have holes or pores, the closed-cell material as a whole is effectively impermeable to fluids.

Since solid foams are often produced by the solidification of liquid foams, which can be compared to soap foam or beer foam, closed-cell topologies are found frequently as the results of many production processes for cellular materials. A review of different production methods for metallic foams can be found in Banhart (2001). The deformation mechanisms of closed-cell foam differ from those of open-cell foam or sponges: While sponges deform primarily by bending of the struts, the dominant deformation mechanisms in closed-cell foams are cell wall stretching, shearing, and buckling, as discussed in Gibson and Ashby (1997).

Models for closed-cell foam are often based on the periodic, space-filling Kelvin and Weaire-Phelan structures, the former being the most popular. If these structures are generated by means of the *Surface Evolver* program then the laws of equilibrium in liquid foam cause the faces of the polyhedra to be slightly curved, see Fig. 11. These distortions can act as natural imperfections in nonlinear analyzes. Figure 12 shows a multi-cell foam model based on dry Kelvin base cells. This model was created by the Voronoi diagram of a body-centered cubic seed point arrangement. Shifting the seed points a little bit allows for the introduction of irregularity in the model, see Fig. 12 (*right*). Figure 13 gives an impression of what a dry closed-cell Weaire-Phelan foam with several cells looks like.

Another generic geometrical representation of a special closed-cell material, which is sometimes referred to as a “three-dimensional honeycomb”, will be presented in the following. This closed-cell structure is the result of a production process which involves the expansion of polymer spheres that are coated with a metal powder slurry (Reinfried et al., 2004; Friedl et al., 2006). The expansion process merges the coating layers of adjacent

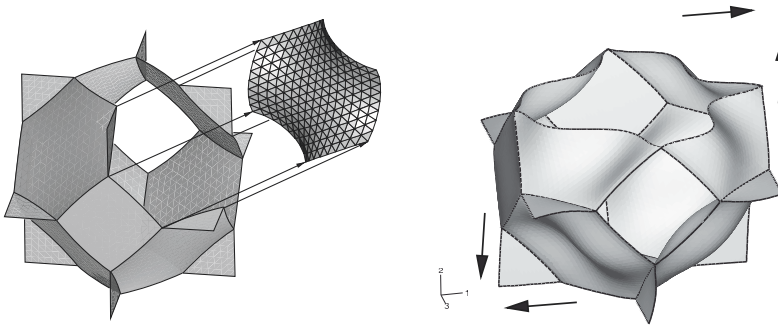


Figure 11. *Left:* Kelvin Cell with the distortion of one hexagonal face being magnified by a factor of 10, from Bitsche (2005). *Right:* predicted linear elastic deformation mode under a macro-scopic shear load

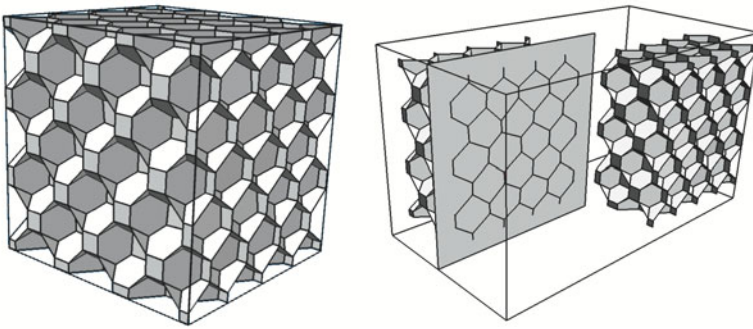


Figure 12. Perfect and disturbed closed-cell foam model based on Kelvin base cells, see Denzer and Maier (1999); Daxner et al. (2000a)

spheres to form the cell walls of a closed-cell structure. After removing the precursor spheres by chemical and heat treatment, the remaining cell wall structure is stabilized by sintering.

By putting the necessary effort into a regular arrangement of the precursor spheres, remarkably regular cellular structures can be obtained. A simple, generic model of a 3/D honeycomb can be generated by arranging the precursor spheres in a regular face-centered cubic arrangement, corresponding to a close packing of spheres in an infinite cellular medium, see Fig. 14.

An expansion of the spheres in the face-centered cubic sphere assembly

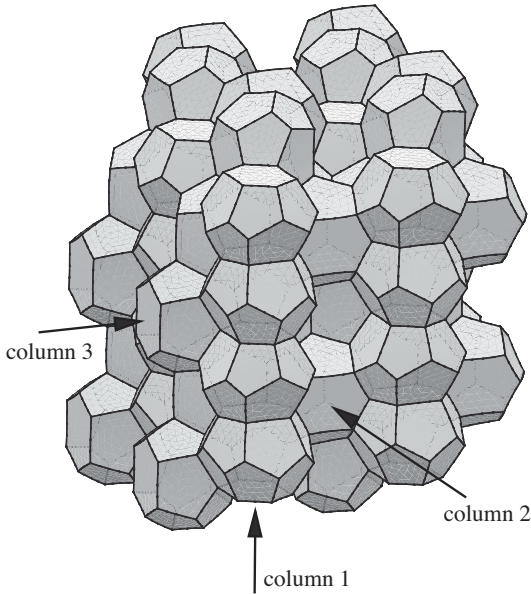


Figure 13. Stacking of several Weaire-Phelan base cells, from Bitsche (2005)

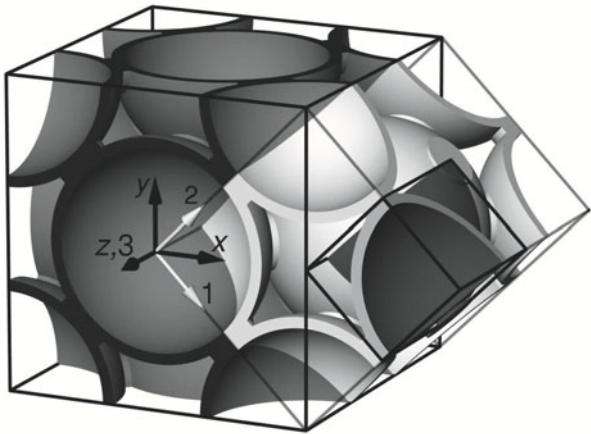


Figure 14. Face-centered cubic arrangement of hollow spheres

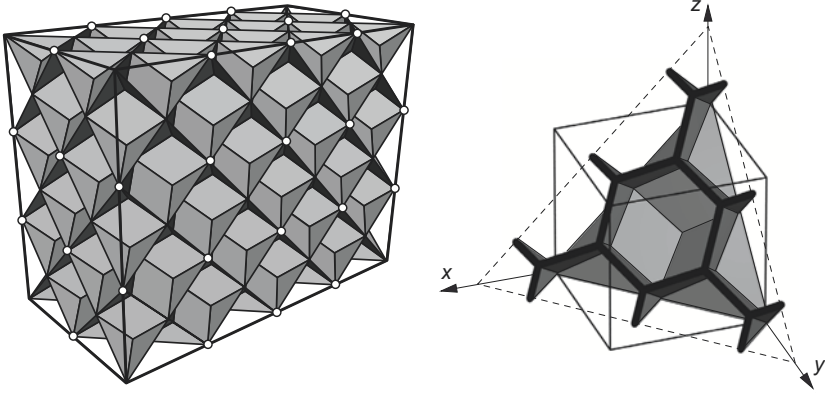


Figure 15. A periodic array of rhombic dodecahedra, building up a ‘three-dimensional honeycomb’ (left), and a hexagonal cross-section thereof (right)

results in a thin-walled structure that corresponds to a Voronoi tessellation of the face-centered cubic lattice, see Fig. 15 (*left*). The building block of this tessellation is the rhombic dodecahedron, which is a polyhedron with 12 rhombic faces. The nickname “three-dimensional honeycomb” for the cellular structure originates from the fact that sections along specific planes give regular hexagonal cross-sections, compare Fig. 15 (*right*).

Shell Finite Element Models for Closed-Cell Foam. Finite shell elements are an efficient means for modeling closed-cell foams. Virtual overlapping of material where the faces meet has to be accounted for when calculating the shell thickness. Attempts to introduce the stiffness of Plateau borders into the model include the superposition of a finite beam element model along the edges of the shell model.

The earliest finite shell element model reported here (Santosa and Wierzbicki, 1998) was actually quite advanced, because it allowed for the crushing of several cells of a closed-cell foam model, the basic geometry of which can be described as cubes that are truncated at their corners by double-pyramids.

A sophisticated shell model of closed-cell Kelvin foam was proposed by Simone and Gibson (1998b) for predicting the influence of curved and wavy cell walls on the elastic stiffness and the limit load. The combined use of symmetry and antisymmetry boundary conditions made the model very

small and efficient.

Grenestedt and Tanaka (1999) investigated the influence of irregularity on the elastic stiffness of a Voronoi foam created by disturbing a regular Kelvin foam structure. Grenestedt and Bassinet (2000) studied the influence of cell wall thickness variations on the elastic stiffness of a regular, closed-cell Kelvin foam with periodic boundary conditions applied to $2 \times 2 \times 2$ base cells.

A family of finite shell element models that are based on intersecting planar faces as well as spheres or ellipsoids were suggested by Meguid et al. (2002), Czekanski et al. (2005a), Czekanski et al. (2005b), and Kim et al. (2006). All these models were used to predict the uniaxial force-displacement behavior of closed-cell foam during crushing. A dry Weaire-Phelan foam model was investigated in terms of its elastic properties and its yield surface in Daxner et al. (2006b). Finally, Mills et al. (2009) presented a shell model for simulating the crushing of two layers of dry Kelvin foam cells, taking into account the pressure of the enclosed gas.

Continuum Finite Element Models for Closed-Cell Foam. Simone and Gibson (1998a) were the first to propose a continuum finite element model for predicting the influence of the material distribution between the Plateau borders and the cell walls on the effective elastic modulus and the effective peak stress. Using a combination of symmetric and anti-symmetric boundary conditions they came up with a very small unit cell model which is still a model of numerical efficiency.

Roberts and Garboczi (2001) used a voxel-to-element method for setting up random 3D models of closed-cell foam. Quite unusually, they did not use μ -CT scans as a source for the voxel data, but rather Voronoi diagrams and Gaussian random fields. Only elastic properties were predicted.

Wang et al. (2009) used an interesting Kelvin foam model for predicting the stiffening effect of reinforcing a closed-cell polypropylene foam with randomly distributed short fibers. The model consisted of 1729 tetrakaidecahedral cells which were discretized by continuum solid elements. The short fibers were introduced into the model in the form of beam elements.

Maire et al. (2003) demonstrated the use of the voxel-to-element method for creating a model of closed-cell polyurethane foam from CT scans. Youssef et al. (2005) also created a model of closed-cell PU foam, but used an unstructured mesh of second-order tetrahedra. Jeon et al. (2009) discretized μ -CT scans of two small samples of closed-cell aluminum foam in great detail. Second-order tetrahedral finite continuum elements were used.

3.4 Open-Cell Foams with Hollow Struts.

Most materials with a cellular micro-structure can be classified as either open-cell sponges or closed-cell foams. There are, however, topologies that do not fit into either of those two categories. An example of such a mixed topology will be presented in the following, namely an open-cell foam (sponge) the struts of which are hollow and form a closed network of channels.

The production of open-cell polymer foams is relatively straightforward, as the extensive use of such foams in packaging, insulation and crash protection applications demonstrates. Open-cell polymer sponges can be used as templates for coating processes such as the application of a slurry or metallic vapor deposition. Such processes create hollow metallic shells with a geometry which is affine to that of the precursor sponge. One of these technologies involves the electro-deposition of metal ions from an electrolyte onto polymeric foam, while another technology uses vapor deposition to form a metallic layer from gaseous metals or gaseous metallic compounds. The coating or deposition processes are generally followed by removing the polymer precursor structure by thermal or chemical treatment, see Banhart (2001).

The metallic coating covers the precursor foam as a layer of near constant thickness. Removing the polymer precursors' struts leaves behind a structure that can be described as a network of hollow metallic struts with roughly triangular, shell-like cross-sections owing to the shape of the Plateau borders of the polymer template sponge (see Fig. 16). A detailed description of open-cell nickel foams with hollow struts produced by the company NiTECH, including information about the deformation of the foam under loading can be found in Dillard et al. (2005). An equivalent material with the trade name Incofoam is presented in Paserin et al. (2004), compare Fig. 16 (*left*).

Owing to the fact that the struts of these metal sponges themselves represent hollow beams, instabilities can occur on different micro-structural levels: one hierarchical level is the level of the strut walls, or the ribs that are formed by the junction of the strut walls, while a higher hierarchical level is the level of the strut skeleton, which behaves, effectively, like a conventional open-cell sponge, that is, by bending of the struts.

A periodic finite element unit cell model for sponges with a hollow strut topology was proposed in Daxner et al. (2006a). This FE model, is composed of continuum shell and solid finite elements. The solid elements are used for joining the continuum shell elements along the ribs that enclose the cusps of the Plateau border cross-sections. A comparison of the resulting finite element model to the real micro-structure is given in Fig. 16. The

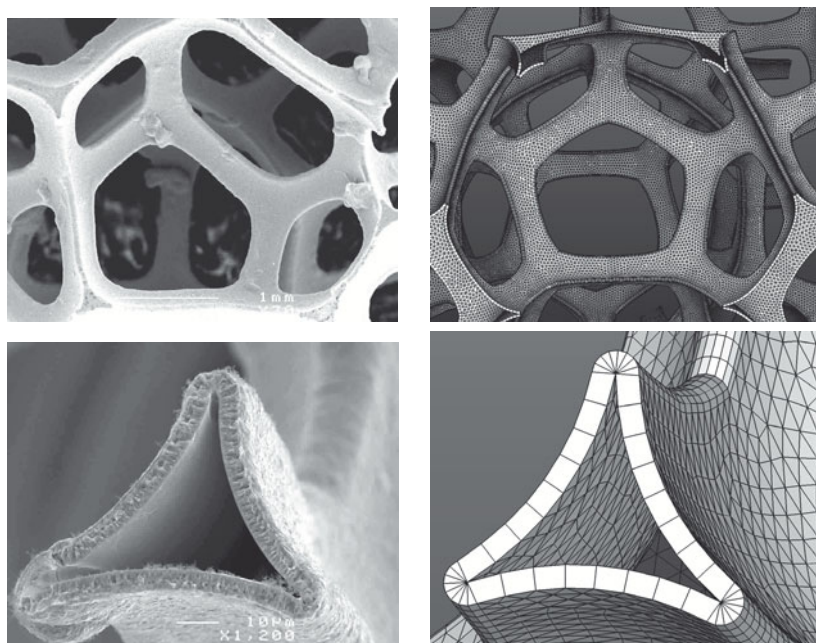


Figure 16. Comparison of a commercial nickel sponge with hollow struts (left; from Paserin et al. (2004), courtesy of V. Paserin, INCO Ltd.) with subsections of the proposed finite element unit cell model of a generic sponge with hollow struts

connectivity and geometry of the struts' mid-axes correspond to that of a Weaire-Phelan foam (Weaire and Phelan, 1994; Weaire, 1996). The inner surface of the hollow struts is equivalent to the outer surface of the Plateau border network of a wet liquid foam, which was predicted with the program '*Surface Evolver*' (Brakke, 1992).

4 Micro-Mechanical Models — Methods and Results

In the following sections, selected results that can be obtained by micro-mechanical finite element analyses of cellular materials are described.

4.1 Elastic Properties

The elastic properties of cellular materials are difficult to measure experimentally, because a uniform load introduction is difficult to achieve and the boundary conditions influence the result.

For micro-mechanical finite element analyses the prediction of elastic properties is more straightforward. Depending on the symmetry of the material, see Nye (1985), a sufficient number of independent load cases has to be imposed on the models for determining the unknown elastic constants. The determination of the effective Young's modulus E and the effective Poisson's ratio ν is usually comparatively simple and can be achieved with unit-cell models that have symmetry boundary conditions, compare Fig. 17 (*center*). Predicting the effective shear modulus G can prove somewhat harder, because it may involve the generation of periodic boundary conditions, see Fig. 17 (*right*), or the simulation of a bi-axial tension-compression load case on a rotated unit-cell, compare the rotated coordinate system 1-2-3 in Fig. 14.

While random, multi-cell micro-mechanical foam models tend towards isotropic elastic behavior, generic unit-cell models can show a considerable degree of elastic anisotropy as is demonstrated by the plot of the direction-dependence of the effective Young's modulus E of a Weaire-Phelan foam model in Fig. 18.

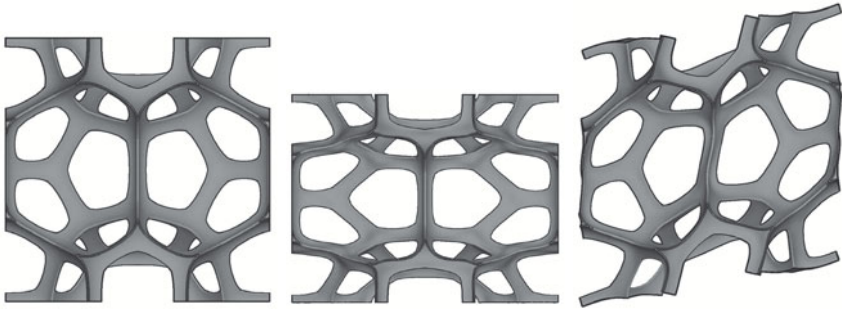


Figure 17. Undeformed state (*left*) and deformed configurations under compressive uniaxial (*center*) and shear loading (*right*) of the unit cell of the Weaire-Phelan model for an open cell foam with hollow struts (Daxner et al., 2007)

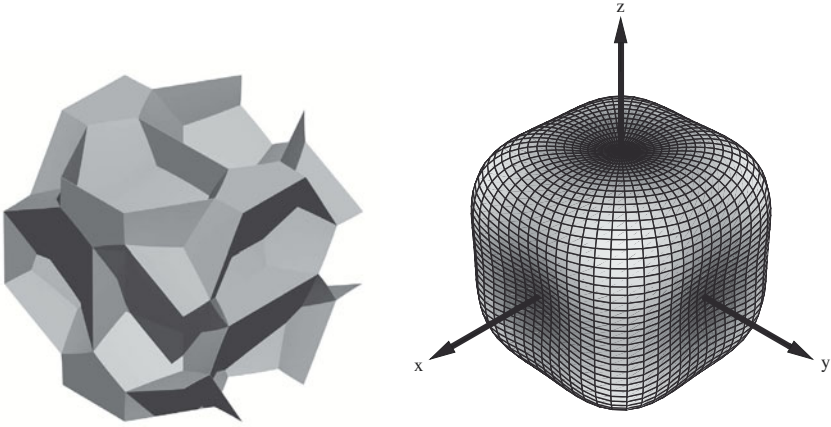


Figure 18. Orientation dependence of the effective Young's modulus (*right*) of a Weaire-Phelan foam (*left*) with $\rho_{\text{rel}} = 0.1\%$ relative density. Bright shades of gray refer to large values of E_n ; from Bitsche (2005)

4.2 Yielding

Introduction. The evaluation of yield and collapse surfaces in the macro-mechanical stress space will be discussed in the following. Yield surfaces allow for the prediction of the onset of inelastic behavior under multi-axial macro-scopic stress states. Collapse surfaces are defined by the set of macro-scopic stress states that correspond to limit load states, which indicate the onset of collapse of cells.

Macro-scopic yielding in micro-mechanical finite element models can be defined by different criteria:

- Yielding at the integration point level, or
- Yielding defined by a threshold value of the residual macro-scopic plastic deformation or the dissipated energy.

Yielding in Two-Dimensional Models. First, some aspects of yield and collapse surfaces will be discussed for an irregular 2D micro-geometry. A unit cell was generated on the basis of a section through a sample of closed cell aluminum foam given by Grenestedt (1998), suitable adaptations being introduced to support periodic boundary conditions, see Fig. 19 (*left*).

The criterion of first yielding at the integration point level was used as the macro-scopic yield criterion. This typically gives rise to initial yield

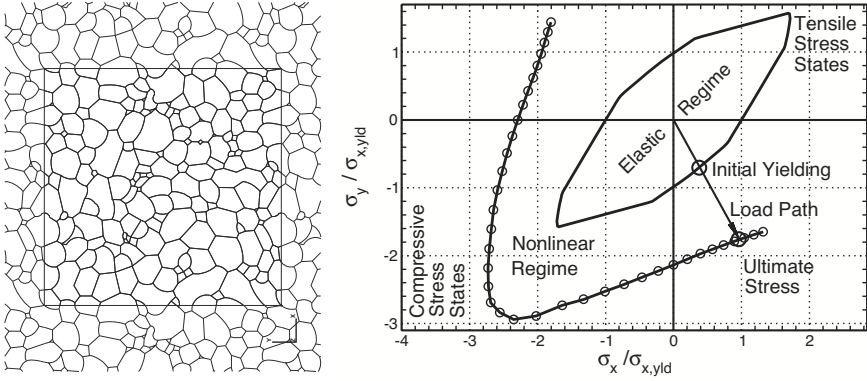


Figure 19. Overall yield and collapse surfaces (*right*) predicted for a real structure 2D honeycomb model (*left*) (Daxner, 2003)

surfaces with distinct corners, as can be seen from Fig. 19 (*right*).

If the macro-scopic load is increased radially in the macro-scopic stress space, a limit load will be reached either due to cell collapse under compressive or fracture under tensile load cases. Macro-scopic stress states that correspond to limit loads under compression are connected to form a collapse envelope in the macro-scopic stress space, compare Fig. 19 (*right*). For the irregular 2/D micro-structure, the predicted collapse surface surrounds the initial yield surface, and the shapes of the two surfaces are similar, compare Fig. 19 (*right*).

The collapse surface presented in Fig. 19 (*right*) is not defined for all possible loading paths, but only for those which exhibit a clear limit load due to buckling and collapse of cells. This implies, that this collapse surface comprises only the purely compressive third quadrant of the stress space and those parts of the second and fourth shear quadrants, for which the larger principal stress is compressive. For other loading directions the predicted stress versus strain responses do not allow the extraction of limit stresses, since the geometrical non-linearity causes the micro-structure to deform such that the structural members align with the loading direction and the response becomes stiffer rather than unstable. In order to obtain peak stresses from such tension dominated loading paths additional failure criteria, for instance cell wall fracture, would have to be introduced.

Predicting the shape of an initial yield surface on the basis of yielding at integration points allows the detection of plastic micro-yielding by

super-posing the results of linear analysis for unit load cases. The obtained macroscopic yield surface is an important ingredient for the development of a macroscopic elastoplastic constitutive model.

More information, however, is necessary for the calculation of the macro-mechanical material response at large strains, which may occur in crushing or crash analyses, for which detailed information on the evolution of the yield surface under loading has to be provided by a hardening law. Additionally, strain rate effects may have to be accounted for.

One of the macroscopic hardening laws used in connection with metal foams is based on isotropic hardening, compare Deshpande and Fleck (2000). For isotropic hardening, the origin and the shape of the yield surface stay the same, but its size increases driven by an accumulated equivalent plastic strain. Other foam models assume mixed hardening, meaning that the yield surface keeps the same shape, but is allowed to grow proportionally (isotropic hardening) and to shift its origin (kinematic hardening).

Micro-mechanical analysis of the real structure model shown in Fig. 19 (*left*) give a more complex picture of the evolution of the yield surface of a cellular material under uniaxial compression in x -direction. Figure 20 presents predicted yield surfaces based on the integration point yield criterion for several states of progressive loading. At several points during the loading sequence the analysis was interrupted, the model was unloaded and analyzed by the linear superposition of unit load cases. This way, the residual stress state in the micro-structure was accounted for in the prediction of the initial micro-yield surfaces at different loading stages.

Initially, the macroscopic hardening behavior was predicted to be essentially kinematic, and the yield surface was shifted in the direction of the applied stress without major changes in shape or size. As the compressive stress increases, however, the evolving yield surface changes its shape and begins to contract. The results in Fig. 20 are based on the superposition of solutions from linear analysis and, thus, cannot account for geometrical nonlinearities, that have to be expected as the applied stress approaches the limit stress. In this regime, a more complex geometrically nonlinear analysis becomes necessary, which, unfortunately, means losing the benefits of being able to superpose linear elastic calculation results. Nevertheless, the results show that for small stresses kinematic hardening has to be taken into consideration. The hardening behavior appears to become more complex as the stress level approaches the limit stress.

Yielding in Three-Dimensional Models. Results obtained by planar models are applicable to 2/D problems such as the in-plane loading of 2/D honeycombs, but cannot necessarily be transferred directly to 3/D micro-

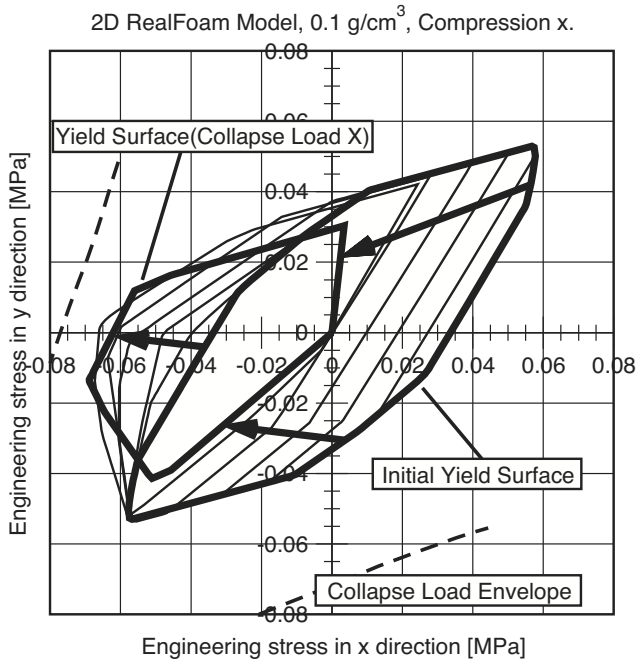


Figure 20. Evolution of the overall yield surface under uniaxial compressive loading in x -direction predicted for the real structure honeycomb model presented in Fig. 19, from Daxner (2003)

structures. The latter require true three-dimensional simulations for making quantitative predictions.

Denzer and Maier (1999) developed three-dimensional closed cell microgeometries based on a body-centered cubic arrangement of tetrakaidecahedral cells. In Daxner et al. (2000a) initial overall micro-yield surfaces were evaluated for configurations with perfect cell geometry as well as for models with statistically perturbed vertex positions, see Fig. 12. The resulting yield surfaces are presented in Fig. 21. Their overall shape in the principal stress space is ellipsoid-like. Their major axes are aligned with the hydrostatic axis. This can be explained by the stress states on the micro-scopic level: The stress state in the cell walls is mainly membrane-like for hydrostatic loads (even though regular tetrakaidecahedra are not elastically isotropic) whereas uniaxial loads tend to give rise to bending stress distributions close

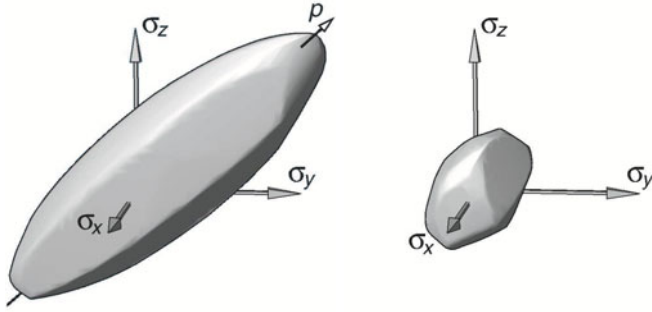


Figure 21. Initial macroscopic yield surface predicted for the regular (*left*) and the irregular (*right*) Kelvin models shown in Fig. 12; from Daxner et al. (2000a)

to the vertices. As long as buckling is not an issue, membrane loading on the micro-level allows for more macro-scopic load to be carried elastically by the material. For the imperfect models, Fig. 21 (*right*), the “aspect ratio” of the yield surface is lower, because the irregular geometry induces bending even under purely hydrostatic macro-scopic loading.

The prediction of macro-mechanical yield surfaces based on the first occurrence of yielding on the integration point level can lead to a strong mesh dependence when the model exhibits artificial stress concentrations, for example, in the vertex regions of shell models. In these cases, it is better to define a yield criterion based on residual effective strain or the energy dissipated by yielding in the model, both approaches being discussed in Bitsche (2005). Figure 22 shows a yield surface that was obtained for a dry Weaire-Phelan model. At the pole of the yield surface that corresponds to the dominance of hydrostatic pressure, a dark cap indicates stress states, for which the stiffness matrices exhibited negative eigenvalues. These stress states may not be physical, because they are obviously preceded by buckling of the cell faces.

Recent works by other authors include Wicklein and Thoma (2005), who evaluated the yield surface of a real foam sample based on a μ -CT scan and using a macroscopic yield criterion characterized by the detection of a macroscopic effective plastic strain of 0.1%. In addition, they determined the increments of the vectors of plastic strain, which, in turn indicate the directions of plastic flow. They found the vectors of plastic flow to be approximately perpendicular to the yield surface. This would support the assumption of associated plastic flow for the considered foam model.

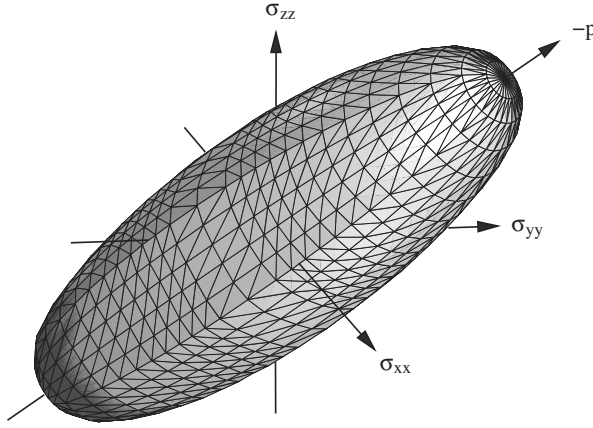


Figure 22. Initial yield surface of a Weaire-Phelan foam of $\rho_{\text{rel}} = 10\%$ relative density, from Bitsche (2005). The dark cap at the pole of positive hydrostatic pressure indicates deformation state where yielding may be preceded by buckling

Demiray et al. (2007) went a step further and predicted the evolution of the yield surface for regular and irregular Kelvin foam models. Their results are based on micro-yielding and are qualitatively similar to those discussed above for the 2D real-foam model.

4.3 Buckling

Introduction. Buckling instabilities affect structures which contain slender structural members or thin, shell-like components. Cellular materials of low effective density certainly exhibit such buckling-prone structural members. Therefore, attention is devoted to the prediction of the onset of buckling and the post-buckling behavior of cellular materials.

A micro-mechanical buckling analysis of a cellular structure can be based on unit-cell models. As soon as periodicity boundary conditions are imposed on these unit cell models, the prediction of buckling loads and buckling modes becomes fairly straightforward, at least as long as the buckling process is an elastic one.

There is, however, a possible complication: The unit cell may contain fewer cells than are necessary for predicting buckling modes of long wavelength. In this case, the periodic unit cell model acts as a filter, which

filters out buckling modes the wave-lengths of which is larger than the unit cell dimensions. As a consequence, the critical buckling mode may not be detected.

Laroussi et al. (2002); Gong et al. (2005); Gong and Kyriakides (2005) have demonstrated that the actual buckling mode of a periodic model of open-cell polyurethane foam can span over several base cells. In Fig. 23, a similar phenomenon is predicted in Todt (2008) for a square honeycomb lattice under uniaxial compression along a principal direction. The (critical)

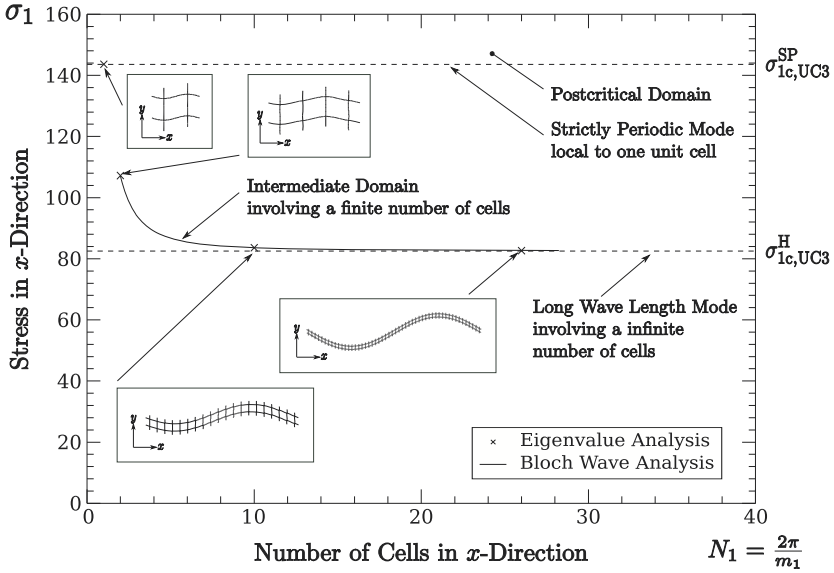


Figure 23. Critical stress of a square honeycomb with regard to buckling under uniaxial compression plotted over the wavelength of the buckling eigenform, expressed in terms of the number of cells in loading direction, from Todt (2008)

stress in loading direction x is plotted over the number of cells contained in loading direction by the unit cell model. The diagram shows that the predicted buckling load is dependent on the number of cells in the model, and decreases as the cell count is increased. Finally, the buckling stress asymptotically approaches a lower bound which corresponds to an infinite buckling wave-length.

Determining the number of base cells (along all vectors of periodicity) corresponding to the critical buckling eigenvalue can be a very expensive

numerical process, since unit cell models with a large number of cells are involved. As has been shown above, even an unlimited number of cells may be necessary for finding the correct buckling stress.

Fortunately, alternatives to brute-force unit-cell analyses exist, as was demonstrated in Gong et al. (2005), where the critical buckling stresses was be predicted very efficiently based on the (tangent) stiffness matrix of a *single* (deformed) periodic base cell.

The corresponding numerical procedure is based on Bloch wave theory. The most important ingredient of this theory is the assumption of a bifurcation mode that takes the following form, see Gong et al. (2005):

$$\tilde{\mathbf{u}}(\mathbf{x}) = \mathbf{u}_{\text{per}}(\mathbf{x}) \exp \left(i \left[\frac{m_1 x_1}{l_1} + \frac{m_2 x_2}{l_2} + \frac{m_3 x_3}{l_3} \right] \right), \quad (11)$$

where $\tilde{\mathbf{u}}$ is the displacement vector as a function of material coordinates $\mathbf{x}$, $\mathbf{u}_{\text{per}}(\mathbf{x})$ is a displacement field that has the same periodicity as the base cell, that is:

$$\mathbf{u}_{\text{per}}(x_1 + n_1 l_1, x_2 + n_2 l_2, x_3 + n_3 l_3) = \mathbf{u}_{\text{per}}(x_1, x_2, x_3), \quad (12)$$

where l_i ($i = 1, 2, 3$) are the dimensions of the base cell, $i = \sqrt{-1}$ is the imaginary number, and $m_i/l_i = \omega_i$ ($i = 1, 2, 3$) are the wave numbers.

With the ansatz for the buckling mode shape, Eq. (11), and the extraction of the instantaneous stiffness matrix of the base cell, it is possible to formulate an eigenvalue problem that yields long wavelength buckling modes and corresponding eigenvalues. For details the reader is directed to Gong et al. (2005).

Not taking into account the occurrence of long wavelength buckling modes can lead to an overestimation of the macroscopic buckling loads. In Gong et al. (2005) the buckling load of a single, slightly elongated tetrakaidecahedral beam element unit cell was found to be approximately 16% higher than the predicted buckling eigenvalue for a stack of 12 base cells.

Post-Buckling. Reaching a bifurcation point on the load-displacement path may cause a stable or unstable post-buckling process. For the case of bifurcation buckling with a stable post-buckling response an increase of the applied load is possible until a structural limit load is reached. In such situations a post-buckling analysis of the structure for determining the limit load can be performed.

This is usually achieved by imposing small geometric imperfections on the initial geometry, which are typically chosen as being affine to the bifurcation displacement vector corresponding to the lowest buckling eigenvalue.

Then, the non-linear response of the imperfect structure can be calculated, since the bifurcation problem is converted into a stress problem by imposing the imperfection. This procedure is the method of choice for dealing with post-buckling problems in standard finite element codes.

In the case of an unstable buckling response the same procedure of imposing an initial imperfection can be used for assessing the sensitivity of the structure to geometrical imperfections. Here, the bifurcation problem is turned into a snap-through problem, which can be treated by arc-length methods or dynamic simulation.

The method of crossing (known) bifurcation points by imposing initial geometrical imperfections usually works well if the initial bifurcation is not followed by secondary bifurcation points at a later stage of loading or deformation. The prediction of the post-buckling behavior of structures becomes more difficult if the equilibrium path has to cross more than one bifurcation point. In this case, the buckling mode may switch shape in the post-buckling regime, and it is difficult to determine the correct corresponding geometrical imperfection *a priori*. Mathematically rigorous methods for switching branches on the equilibrium path have been documented, but are usually not implemented in standard finite element codes. Potential candidates for jumps of the buckling mode in the post-buckling domain are thin cell walls that are subjected to shear loads.

A further complication is the occurrence of elastoplastic buckling, because elastic unloading during buckling can change the tangent modulus of the material in the unloaded region from the tangent modulus of the hardening regime to the elastic (unloading) modulus. This change of the local material stiffness is usually not considered by the buckling eigenvalue prediction. Again, this problem can be circumvented by imposing an initial geometrical imperfection that is affine to the relevant buckling mode.

Crossing a bifurcation point with an unstable static post-buckling equilibrium path triggers a dynamical process that can only be predicted when a dynamic finite element procedure is used. Because of their robustness, explicit finite element codes are well suited for this task. Often, round-off errors are large enough to act as ‘numerical imperfections’ and to trigger buckling during a quasi-static, explicit analysis. More sophisticated methods perform a static analysis until a bifurcation point is detected, determine the buckling mode shape, impose it as a small geometrical imperfection on the model and then switch to a dynamic procedure to capture the post-buckling behavior, see examples in Falzon and Aliabadi (2008).

Buckling in Open-Cell Foams. Instabilities in open-cell foams typically take place in the form of progressive snap-through and/or bifurcation

buckling of struts or groups of struts. The former is perhaps the more frequently observed (and predicted) mode of deformation, because bifurcation buckling in the form of elastic Euler buckling or inelastic buckling requires an orientation of the concerned strut which is nearly parallel to the loading direction in the uniaxial case. Figure 8 shows a deformed strut in an open-cell metallic foam. The deformation may be the result of snap-through buckling of the cell that is bordered by this strut.

The finite element method has been applied for studying buckling of open-cell foams, for example, by Shulmeister et al. (1998), where finite element analysis of a random Voronoi tessellation based open-cell structure are carried out for compression loading. This study did not include dedicated buckling eigenvalue analysis, making it unclear if any bifurcation points were crossed in the course of the simulated deformation. Such buckling eigenvalue analysis were performed very thoroughly on periodic tetrakaidecahedral unit cell models of open-cell foams in Gong et al. (2005) and augmented with experimental results in Gong and Kyriakides (2005). Their ability to detect long wavelength buckling modes by means of Bloch wave methods was already discussed above. A further discussion of the buckling mode of periodic Kelvin structures under uniaxial and hydrostatic loading, including post-buckling analyses, can be found in Demiray et al. (2006).

The influence of imperfections on the behavior of open-cell micro-structures can be assessed by producing and testing such structures with well-defined imperfections by means of rapid prototyping (Stampfl et al., 2004; Wöß et al., 2004). Such methods are demonstrated in Luxner et al. (2005, 2006); Luxner (2006); Luxner et al. (2007) along with numerical experiments on finite sized cubical and cylindrical samples of perfect and imperfect simple cubic lattices.

Buckling in Closed-Cell Foams. The initial mode of instability in closed-cell cellular materials is buckling of the cell walls along the direction(s) of compressive principal stress(es), see Fig. 24. The cell face can typically sustain tensile principal stresses beyond the onset of buckling, but this post-buckling load carrying capacity is limited by cell wall fracture, see McCullough et al. (1999). Cell walls can also fracture wherever plastic hinges form during the collapse of individual cells (Thornton and Magee, 1975). As soon as the load-carrying capacity of the cell walls is exhausted at higher compressive strains, the edge regions start to deform in modes that are similar to those that were described for open-cell structures.

In the following, some issues pertaining to buckling in closed-cell foams will be discussed on the basis of the 3D-honeycomb structure that was introduced in Section 3.

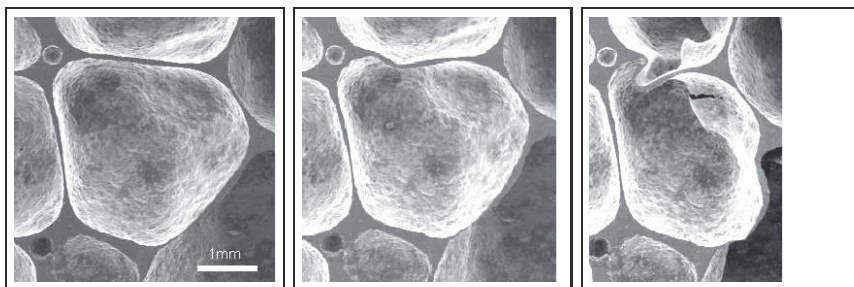


Figure 24. Deformation sequence of a wrought alloy aluminum foam showing post-buckling collapse of a cell wall oriented in deformation direction; images courtesy of Institute of Materials Science and Testing, Vienna University of Technology Kriszt et al. (1999)

If this polyhedral structure is subjected to macroscopic loading, the deformation of the rhombic faces can be characterized as being dominated by in-plane shear. Cell wall bending plays a subordinate role, and becomes important only at high relative densities, or for thick cell walls, respectively.

Naturally, the shear-dominated deformation of the cell walls also influences the buckling pattern. Since most macroscopic stress states, including tensile ones, translate to shear deformations on the cell wall level, the respective elastic buckling modes are very similar. As an example, the buckling eigenform for a 3D-HC unit cell under uniaxial compression in the '1'-direction is given in Fig. 25(a) as the result of a linear initial stability analysis. A periodic unit cell model composed of linear shell elements was used for obtaining this result.

The buckling eigenform shows that the rhombic cell walls buckle like plates which are clamped on all sides. No significant macroscopic deformation is visible nor do the cell walls show any sign of interacting with each other. It may be concluded that the buckling of this specific structure can be attributed to the buckling of the individual faces of the polyhedra.

The shear deformation of the cell walls can lead to cell wall buckling even when the macroscopic stress state is a tensile one. Fig. 26 shows a short wavelength buckling mode on faces of the 3D-HC which are aligned with the macroscopic loading direction (indicated by arrows in the positive and negative '1' direction). This buckling mode is triggered by the compressive principal stress component that is acting perpendicularly to the macroscopic loading direction in the concerned faces. The depicted short wavelength mode appears shortly before the uniaxial, tensile macroscopic limit load

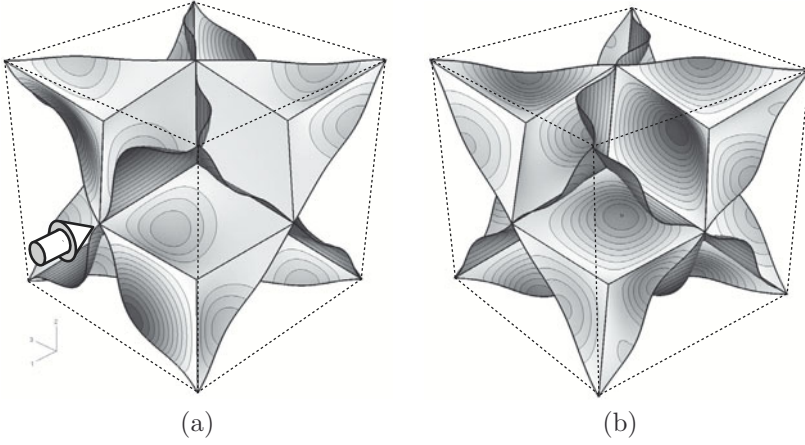


Figure 25. Predicted buckling eigenmode for a 3D-honeycomb under (a) macroscopic uniaxial compression, and (b) macroscopic hydrostatic compression

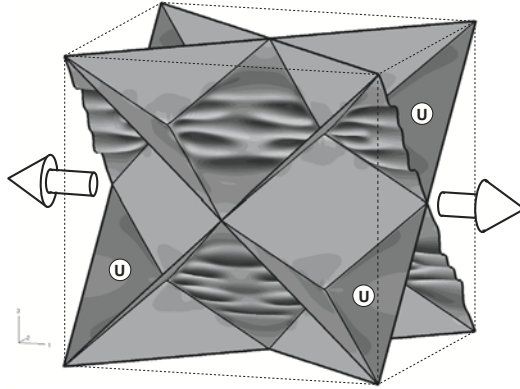


Figure 26. Contour plot of the predicted maximum principal stress in a 3D honeycomb plotted on a deformed configuration (displacements magnified by a factor of 10) close to the tensile uniaxial limit load. Faces marked with (U) will be the next to buckle

of the model is reached. This can be seen from the fact that some of the faces parallel to the macroscopic loading direction, marked with the letter 'U' in Fig. 26, have not yet buckled. This buckling mode affects a configuration that is already severely affected by plastic yielding. The initial, elastic buckling mode exhibits individual dimples in the concerned cell walls, similar to those displayed in Fig. 25(a). These dimples, however, vanish as soon as the tensile principal stress along the face diagonal in the loading direction increases. The macroscopic stress has to be increased further for the short wavelength elasto-plastic buckling pattern to appear progressively on faces aligned with the macroscopic loading direction.

No shear deformation, but rather bi-axial compression, is involved in the uniform hydrostatic compression of 3D-HC on the macroscopic level. Here, the predicted buckling pattern, which can be found in Fig. 25(b), is much more uniform and all faces seem to buckle at once, owing to the uniformity of the compressive global and local stress states.

Having presented the buckling modes under some important macroscopic load cases we now want to compare the respective macroscopic buckling stresses. Figure 27 shows the macroscopic elastic buckling stresses for uniaxial

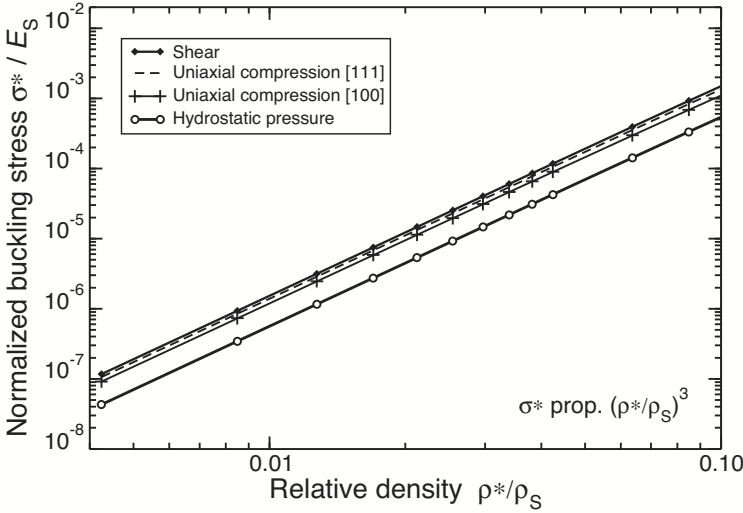


Figure 27. Macroscopic elastic buckling stresses as a function of relative density of 3D-honeycombs under different macroscopic loads

the space diagonal of the unit cell ($[111]$), for pure shear loading, and for loading under hydrostatic pressure. The critical stresses are normalized by the elastic modulus E_S of the bulk material and plotted over the relative density ρ^*/ρ_S . For the simplified study presented here, where no correction for taking into account the volume of the edges and the vertices was performed, the relative density was linearly proportional to the face wall thickness, which enters the bending stiffness of the cell walls with a power of three. Consequently, the normalized elastic macroscopic buckling stresses were also proportional to the third power of the relative density,

$$\frac{\sigma_{el}^*}{E_S} \approx \left[\frac{\rho^*}{\rho_S} \right]^3. \quad (13)$$

Furthermore, the relative differences between the macroscopic buckling stresses for the different loading cases are more or less constant. Taking the macroscopic buckling stress for the uniaxial compression along the unit cell axes as the reference value, the following relations hold: The macroscopic uniaxial elastic buckling stress along the space diagonal is approximately 17% higher than the reference buckling stress. The macroscopic shear buckling stress is approximately 30% higher than the macroscopic uniaxial compressive buckling stress, and, lastly, the macroscopic hydrostatic buckling stress has the lowest of all four values and is, on average, 53% lower than the reference stress.

Certainly a more detailed model could have been used for predicting the buckling behavior of the presented model topology, especially with respect to the more accurate representation of the edge and vertex regions, but the principal features of the buckling and post-buckling of closed-cell foams could be demonstrated nonetheless for this comparatively simple structure. It was most remarkable that, at least for the demonstrated example of the 3D honeycomb, the buckling events are hardly discernible in the overall macroscopic stress-strain curve owing to the stable post-buckling behavior of the buckled faces. Nevertheless, the progressive buckling of the cell faces led to a softening of the structure, the stability of which relies — after the rupture of the cell faces under the further growing tensile principal stresses — on the load carrying capacity of the cell edge regions.

Buckling in Open-Cell Foams with Hollow Struts. Figure 28 shows predicted buckling modes for two different hollow sponges (right) along with the corresponding undeformed configurations (left).

For the model in Fig. 28(a) a template sponge with a comparatively low relative density of 1% was chosen in order to set up a model of a hollow sponge with very slender struts. A hydrostatic pressure load was applied

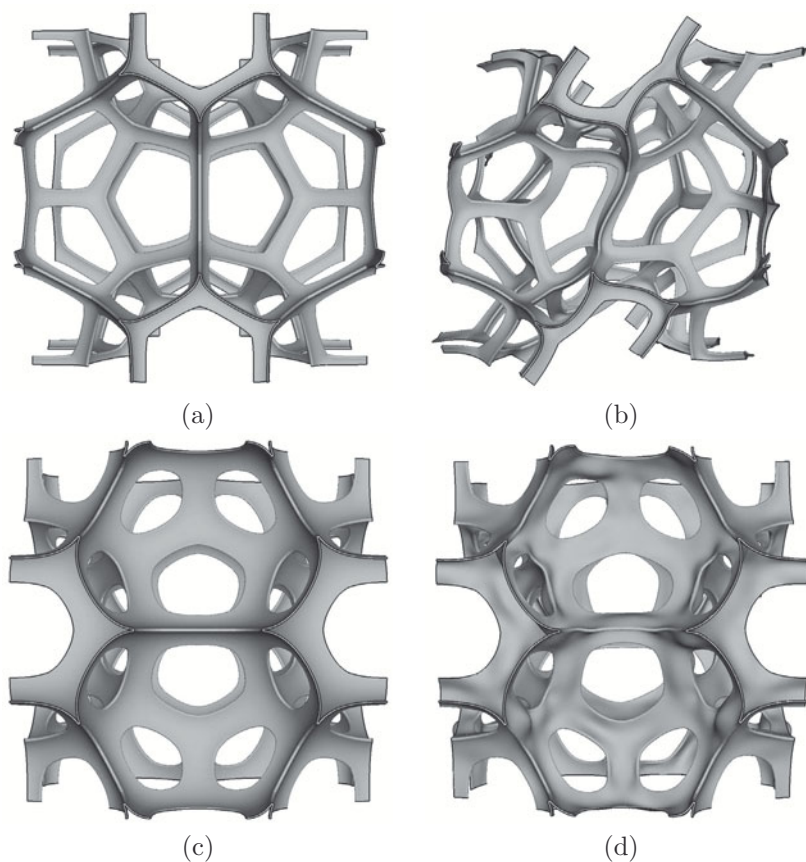


Figure 28. Predicted strut buckling (top row) versus strut wall buckling (bottom row) in a FE model of a sponge with hollow struts under hydrostatic loading

on the macroscopic level and translated to appropriate master node forces. A linear initial stability analysis was performed on this setup, and the corresponding buckling mode is shown in Fig. 28(b). This image shows a buckling pattern that does not show any significant cell wall deformation, but rather a deformation mode of the struts which is in accordance with classical (Bernoulli-Euler) beam theory. The periodic nature of the buckling pattern is clearly visible and reinforces the need for periodic boundary

conditions on the unit cell.

A slightly different result arises when the relative density of the template sponge is set to a higher value of 4%, giving a final finite element configuration as depicted in Fig. 28(c). When a macroscopic pressure load is applied to this model and a linear initial stability analysis is performed, the result looks very different from the strut buckling mode observed in Fig. 28(b), as is documented in Fig. 28(d). Here, the shell of the hollow sponge suddenly forms local bumps and also distinct kinks on the ribs at the junction lines of the cell walls. It can be concluded, that the higher effective bending stiffness of the struts, which is a consequence of their higher moment of inertia owing to the fact that the compactness of the template sponge moves the coating material farther away from the axes of inertia, prevents beam-like buckling. At the same time, the average curvature of the hollow sponge's shell is lower than in the case of the slender, low density template sponge. This leads to a reduced resistance of the shell to local buckling and changes the buckling mode from beam-like buckling to shell-like buckling.

The same effect can be obtained if the coating thickness is increased on a hollow sponge with slender struts. In this case, the bending stiffness of the struts prevents strut buckling while the high curvature of the coating avoids shell buckling. Buckling nevertheless occurs in the form of buckling of the ribs, which experience the highest compressive stresses due to their distance from the neutral axes of the struts. Figure 29 shows the undeformed and the predicted buckled configuration of a vertex in a hollow sponge. Owing to the fact that the cross-sectional bending moments of the struts reach their highest values close to the vertices, buckling has also to be expected

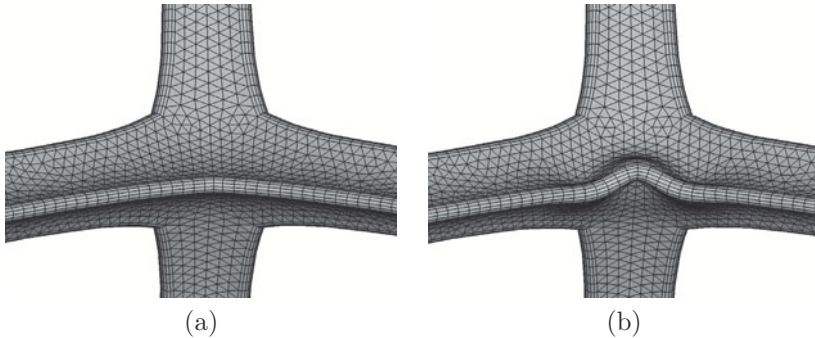


Figure 29. Detailed view of the predicted mode of local buckling of a rib along the struts of a sponge with hollow struts

in these regions. For the presented configuration, the buckling instability is expressed in a lateral buckling of the rib that has formed around the edges of the Plateau borders.

4.4 Densification

The final stage of the mechanical response of cellular solids with regard to compressive loading is marked by the collapse of most or all cells. If a collapsed cell is deformed further and further, self-contact between opposite cell walls will occur on the micro-mechanical level. The formation of additional mechanical bridges between cell walls due to self-contact leads to an increase of the effective stiffness of the material. When the deformation potential of most cells in a structure is exhausted at high compressive strains, the compressive stress will rise steeply, indicating the progressive densification of the material.

Insight into the contact mechanics of collapsing cellular structures can be gained by (periodic) unit-cell models. If periodicity boundary conditions are applied, special care has to be taken to detect contact between cell walls on opposite sides of the unit cell. One method for achieving this is the bounding of the unit cell by the middle surfaces of cell walls, compare Fig. 30, reproduced from Daxner (2003). Since there are no free surfaces anywhere on the unit cell boundary, problems with detecting contact that “wraps around” the unit cell do not occur¹.

Simulations with completely resolved micro-structure are very costly in the large strain compression regime and require finite element codes with robust contact algorithms such as those found in explicit codes for crush and crash analysis.

Early works in this context pertain to the simulation of crushing behavior of 2D structures, most commonly honeycombs, see, e.g., Papka and Kyriakides (1998, 1999), where the layer-wise collapse and the localized deformation of such structures was successfully predicted numerically.

With regard to open cell foams Mills (2007) presented a continuum finite element model based on a wet Kelvin geometry, thereby fully modeling the Plateau borders. For a foam with an effective relative density of $\rho_{\text{rel}} = 0.028$, self-contact was predicted to occur at 66% nominal compressive strain. For a foam of $\rho_{\text{rel}} = 0.0622$, Mills (2007) also predicted local buckling of the sharp edges of the Plateau border network in the middle of the struts. Because the thin edges do not contribute much to the stiffness of the struts, this

¹A periodic unit cell model built around the same principle was used in Ohno et al. (2002) and Okumura et al. (2002) for predicting bifurcation buckling in regular 2D honeycombs.

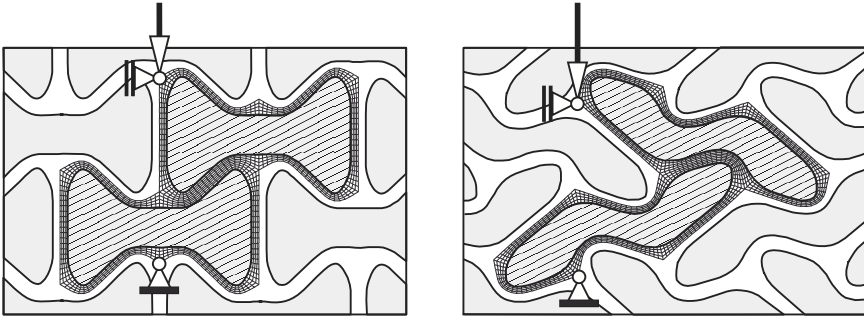


Figure 30. Symmetric (*left*) and periodic (*right*) deformation modes in the post-buckling regime of regular 2D honeycombs with plateau borders, compare Daxner (2003). Bounding the unit cell by the middle surfaces of the cell walls allows for the prediction of self-contact

local buckling did not significantly influence the overall stiffness. Gong and Kyriakides (2006) and Jang and Kyriakides (2009) used finite beam element models of open-cell Kelvin foams with elastic and elastic-plastic strut material, respectively, for simulating the crushing process. They approximated beam-to-beam contact by introducing nonlinear springs between opposite vertices of the vertical ‘faces’ in their model.

Compression of closed-cell Kelvin and Voronoi foams well into the densification regime was presented by Denzer and Maier (1999) who used hybrid beam-shell models to take into account the material distribution between cell walls and Plateau borders. Meguid et al. (2002) demonstrated strain localization in a $5 \times 5 \times 5 = 125$ cell model under uniaxial compression. The individual cells had the shape of cubes that are truncated at their corners by spheres. Localized collapse was triggered by random variations of the local apparent density. A finite sized finite element model consisting of $4 \times 4 \times 5 = 80$ ‘truncated cube’ unit cells, similar to those used by Meguid et al. (2002), was crushed in Kim et al. (2006), where the influence of the ratio between the thicknesses of the cell walls of the cubes and the spheres as well as statistical thickness distributions were investigated. Simulations of the dynamic compression of a thin layer of low-density closed-cell polymer foam were performed by Mills et al. (2009). They used a Kelvin cell model consisting of two cells in thickness direction for representing the layer. The low density, dry foam could be modeled well by shell elements. Nominal compressive strains of over 80% were reached, which corresponded to extensive folding of the cell walls and contact of the cell walls throughout the

thickness of the model. The state of densification was reached in this study.

Alternative numerical methods known as “Particle Methods” have also been applied to simulate the phenomena of foam compaction. Bardenhagen et al. (2005) employed a “particle-in-cell” method to reach a state of full compaction by numerical simulation. They found that the residual porosity in a fully densified open-cell structure is remarkably high, 35% residual porosity being predicted for the compression of a open-cell material with an initial porosity of 90%. Brydon et al. (2005) applied the same method to simulating the densification of a real open-cell carbon foam sample, the micro-structure of which was modeled from μ -CT data.

One issue that arises in the simulation of crushing and, in particular, crashing of cellular materials is the pressure of the gas that is entrapped in cells or flowing through pores and openings. Computational Fluid Dynamics (CFD) analysis can be applied to predict the permeability of discrete micro-structures to fluids. Boomsma et al. (2003) calculated the permeability of a wet Weaire-Phelan foam (relative density $\rho_{\text{rel}} = 0.04$, mean cell diameter 2.3 mm) with regard to water flow. Mills (2005) performed a computational fluid dynamics analysis on a wet Kelvin foam model in order to predict the effective gas-flow permeability for laminar flow as a function of cell size and cell face hole size. They found the permeability to depend almost linearly on the cell diameter and on the area of the holes between adjacent cells.

The influence of the compression of entrapped gas during the crashing of two layers of (dry) Kelvin cells representing a thin layer of low-density closed-cell polyethylene foam was investigated in Mills et al. (2009). They documented that the increase of the predicted effective compressive stress in the engineering strain range of 10% to 60% could be attributed mainly to the resistance of the enclosed gas to compression, whereas at higher compressive strains cell wall contact added strongly to the stress response.

4.5 Fracture

Fracture of cellular materials is the limiting mechanism when critical tensile macroscopic loads are applied. Approaches for predicting the onset and the propagation of cracks in cellular materials are the subject of this section.

Unit cell models with periodicity boundary conditions cannot directly be applied for predicting crack propagation on the micro-scale, because they give rise to periodic deformation modes, periodic stress and strain fields, and, as a consequence, periodic patterns of damage and cracks. This leads to a strong over-prediction of the energy necessary for driving cracks through the material.

One logical solution for this problem consists of modeling whole samples in micro-structural detail, a process which is possible for very small sample sizes. For example, Schaffner et al. (2000) used a Voronoi honeycomb to study fatigue damage of cancellous bone.

A possible compromise between resolving the whole model in full detail and using only a periodic unit cell are embedding methods. In the case of crack propagation, these methods use a detailed finite element representation of the micro-structure only around the crack-tip, whereas the surrounding material is treated only in an homogenized sense.

Such a strategy was followed by Ableidinger (2000) for studying the influence of a number of micro-geometrical and material parameters on the macroscopic fracture mechanical behavior of a compact tension (CT) specimen made of an open cell aluminum foam subjected to monotonic loading, see also Rammerstorfer and Böhm (2000).

The region surrounding the crack tip was modeled with a fully resolved, but highly generic, three-dimensional tetrakaidecahedral open cell micro-geometry discretized with beam elements. This kernel was embedded into the remainder of the CT specimen, compare Fig. 31 (*top, left*), which was treated as a homogeneous structure with an effective material behavior obtained from unit cell analyses of the same tetrakaidecahedral micro-geometry. In the out-of-plane direction the core consisted of a single layer of tetrakaidecahedra, the embedding region having the same thickness, and symmetry boundary conditions were applied at the top and bottom planes, so that the model actually corresponds to an inner layer (plane strain conditions) of the specimen. Tensile loads were applied at the positions of the loading fixtures of the CT specimen, so that the crack proceeded in fracture

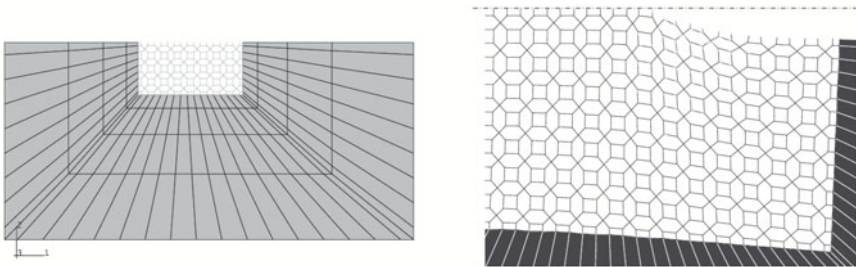


Figure 31. Embedded cell model of an open cell foam using a tetrakaidecahedral core and a homogenized outer region (*left*) and detail of the core around the crack tip (*right*), from Ableidinger (2000). Note that symmetry with respect to the crack plane is assumed

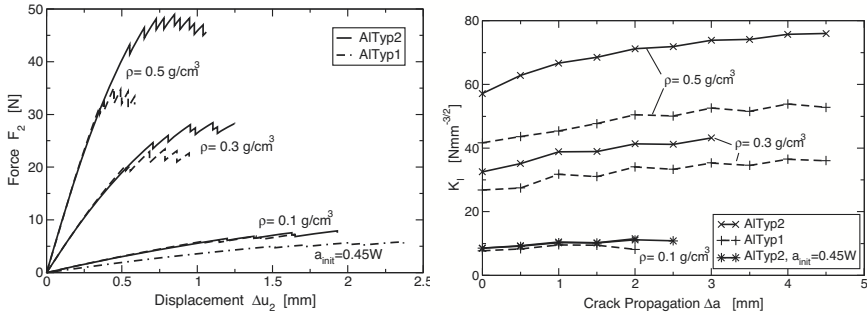


Figure 32. Macroscopic force versus displacement curves (*left*) and macroscopic crack resistance curves (*right*) of CT specimens made of open cell aluminum foams predicted by embedded cell models, see Fig. 6.31, for different effective densities and different aluminum alloys (Ableidinger, 2000)

mode I on the macro-scale. Crack propagation at the micro level as resolved in the kernel region was assumed to be caused by sequential ductile failure of the most highly loaded struts, compare Fig. 31 (*top, right*).

From the predicted force versus displacement diagrams of the CT specimens, see Fig. 32 (*left*), effective K_I versus Δa (crack resistance) curves as shown in Fig. 32 (*right*) were obtained. For this purpose, the macroscopic stress intensity factor K_I was evaluated at each maximum of the force versus displacement curves according to standard expressions for CT specimens, the crack length increments Δa being dictated by the selected micro-geometry. Both Figs 32 (*left/right*) show results for three values of the apparent density of the foam and for two aluminum alloys differing in ductility. Within such a modeling approach parameters such as the mass density of the foam (which determines the void volume fraction), the size of the cells (which influences the fracture behavior in terms of an internal length scale), the yield strength, and the strain to fracture of the bulk material (which govern the ductile damage and failure of the struts) can be varied easily in order to study their respective influences on the fracture behavior of open cell metallic foams.

A somewhat similar approach for predicting the fracture toughness of a rectangular lattice of struts was followed by Choi and Sankar (2005), who derived the effective elastic properties of the lattice and inserted them into analytical relationships for calculating the displacement field along the boundary of a FE model consisting of 100×100 cells. Thus, the homogenized outer region used by Ableidinger (2000) was implicitly represented

by the prescribed displacements. A maximum stress criterion at the strut level was used to predict Mode I and Mode II fracture toughness values. Their analysis shows that the displacements and stresses in the foam near the crack tip are very similar to that in an equivalent homogeneous material, and continuum fracture mechanics concepts can be applied for predicting the fracture of a cellular medium. Lee et al. (2007) extend the method to functionally graded foams. Since no analytical descriptions for the displacement field were available for this more complex case, they used a macro-mechanical continuum model for deriving this field, thereby effectively using the micro-mechanical model as a highly refined sub-model for the macro-mechanical one. They found that the effective foam density at the crack tip has the greatest influence on the fracture toughness which is similar to the one of a homogeneous foam of the same density.

Schmidt and Fleck (2001) investigated the crack growth under Mode I loading and small scale yielding conditions in 2D honeycombs. They prescribed the displacements and rotations at the boundary of a finite beam element model of the region around the crack tip in order to apply the far-field load. Lipperman et al. (2007) studied the propagation of cracks in infinite periodic 2D lattice structures by sophisticated analytical means based on the discrete Fourier transform.

Fatigue in open-cell foam aluminum foam was taken into account in a micro-mechanical model by Demiray et al. (2009), who simulated the effect of cyclic loading on a Kelvin foam model. Damage accumulation and failure in individual struts was evaluated on the basis of the Palmgren-Miner rule.

5 Optimization of Foam Density Distribution

With growing knowledge about the mechanical performance of cellular metals, and in particular of metallic foams, methods for the design of foam components for applications such as packaging, energy absorption and sandwich structures have evolved significantly, see Gibson and Ashby (1988); Gibson (1989); Ashby et al. (1998). The ability of cellular metals to compete with other materials can be underlined by emphasizing their multi-functionality, Evans et al. (1999). Furthermore, it was shown that in certain structures, such as cylindrical sandwich shells, metal foam cores are more weight efficient than other stiffening concepts (Hutchinson and He, 2000).

In studies dealing with the design and optimization of metal foam components usually a foam of uniform density has been assumed. One popular field of research in this respect is the weight-efficient design of thin-walled, foam-filled tubes for energy absorption applications, see, for example, Chen et al. (2002); Kim et al. (2002); Zarei and Kröger (2008a,b)

Optimization of density gradients in thin layers of functionally graded polymeric foams intended for crash energy absorption applications were recently investigated in Forero Rueda et al. (2009); Cui et al. (2009a,b).

Metallic foams and syntactic foams² (Gupta, 2007), show some potential for being produced with controlled spatial variations of their apparent density. In the case of foams produced via powder metallurgical routes the foaming process, and, as a consequence, the final density distribution may be controlled by suitable choices of the distribution of the blowing agent and of process parameters such as temperature and pressure. This suggests employing such foams as graded materials in space filling lightweight structures designed in analogy to cancellous bone, a natural functionally graded cellular material that displays increased density in regions of high loading. Reiter and Rammerstorfer (1993); Reiter (1996) presented algorithms for the optimization of composite materials that are based on procedures for simulating the natural adaptation of bone to applied loads.

Daxner et al. (2000b) implemented the algorithm of Reiter (1996) for a self-adapting material, providing options for converging towards a uniform distribution of the local strain energy density or towards the satisfaction of a local yield criterion by iteratively adapting the local density. The former choice gives rise to configurations of increased stiffness and the latter to density distributions in which the material shows uniform safety against plastic failure. It was shown that strength optimization with material parameters that are typical for metallic foams leads to continuous solutions showing density gradients, compare Fig. 33, which may indeed be realizable by suitable processing technologies. Attempts to optimize the specific

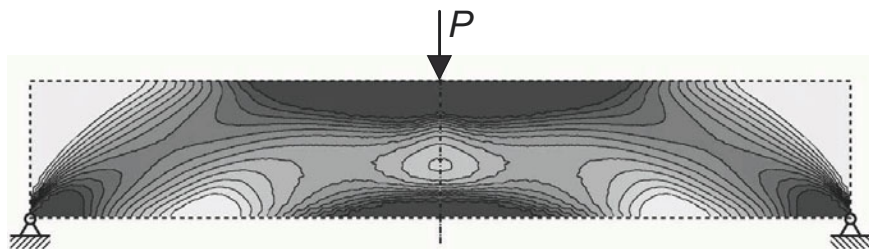


Figure 33. Strength optimized symmetric beam with fixed supports under a central load P . Light and dark areas indicate low and high local foam density, respectively

²Syntactic foams are composites consisting of hollow particles (micro-balloons) embedded in a metal, polymer or ceramic matrix.

stiffness of foam structures, on the other hand, resulted in the formation of discrete structures such as frameworks of struts. This indicates that cellular metals may not be ideal for stiffness-optimized structures, at least in the presence of concentrated loads. For sandwich beams with foam cores as studied by Vonach et al. (2000) smooth mesoscopic density distributions were also obtained for stiffness optimized configurations under distributed loads.

6 Summary

In this section, some aspects of modeling and simulation of cellular materials with the finite element method were discussed. A detailed review of the methods available for the prediction of the mechanical behavior of cellular materials on the micro-mechanical length-scale was given. This review revealed, that the standards in modeling cellular materials on the micro-mechanical level reflect the state of the art, that is, it is fairly easy to predict elastic properties and yield phenomena, but the analyses become increasingly more difficult as soon as buckling instabilities or fracture processes are involved, both of which still pose theoretical and practical challenges to the analyst.

With respect to the effective, macro-mechanical behavior of cellular materials, a special topic was discussed, namely the optimization of the distribution of the effective density in structures containing cellular materials based on the dependence of the effective properties on the local density. The proposed optimization algorithm can provide information regarding the proper design of components in which the effective density of the cellular material can be varied spatially by appropriate production technologies.

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Plasticity of Three-Dimensional Foams

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Abstract This chapter gives in the first part a summary of some important elements in continuum mechanics, i.e. the decomposition of the stress tensor in its spherical and the deviatoric part and the use of stress invariants to describe the physical content of the stress tensor. In the next part, the elastic behaviour of isotropic materials based on generalised Hooke's law is summarised and a notation appropriate for computer implementation is introduced. The constitutive description is then extended to plastic material behaviour and the description based on a yield condition, flow rule and hardening law is introduced. The concept of invariants is consistently applied and explained for the characterisation of yield conditions. A classical simple cubic cell model based on beams (Gibson/Ashby model) is investigated in the next chapter in order to highlight the assumptions and the derivation of the macroscopic material properties (elastic constants and yield stress). In the following, a strategy to determine the influence of the hydrostatic stress on the yield behaviour is proposed and conceptionally realised by a state of plane strain and a state of uniaxial strain. In addition, alternative ways to determine the complete set of elastic constants are shown. The last part covers the implementation of yield conditions into finite element codes. The understanding of the predictor-corrector concept is required to provide new constitutive equations in commercial computational codes.

1 Fundamentals of Continuum Mechanics

1.1 Stress Tensor and Decomposition

It is of great importance in the framework of the plasticity theory of isotropic materials to decompose in a unique manner the stress tensor σ_{ij}

into a pure volume changing (spherical or hydrostatic) tensor σ_{ij}^o and a pure shape changing (deviatoric) stress tensor s_{ij} (cf. Fig. 1)¹:

$$\sigma_{ij} = \sigma_{ij}^o + s_{ij} = \sigma_m \delta_{ij} + s_{ij}. \quad (1)$$

In Eq. (1), $\sigma_m = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$ denotes the mean normal stress² and δ_{ij} the Kronecker tensor (δ_{ij} is equal to 1 if $i = j$, and 0 if $i \neq j$). Furthermore, Einstein's summation convention³ is introduced, Bronstein and Semendjajew (1988).

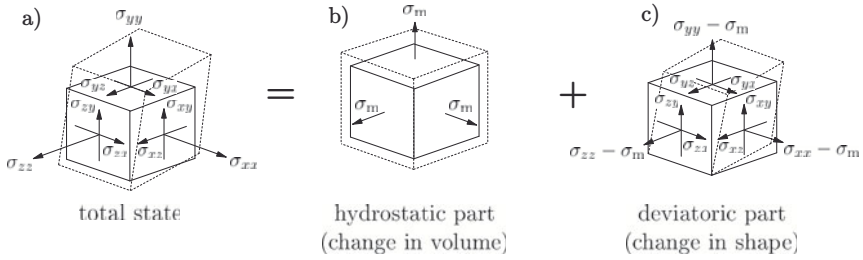


Figure 1. Decomposition of the stress tensor into its spherical and the deviatoric part.

Equation (1) can be written with components in the following way as

$$\underbrace{\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}}_{\text{stress tensor } \sigma_{ij}} = \underbrace{\begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}}_{\text{hydrostatic tensor } \sigma_{ij}^o} + \underbrace{\begin{bmatrix} s_{xx} & s_{xy} & s_{xz} \\ s_{xy} & s_{yy} & s_{yz} \\ s_{xz} & s_{yz} & s_{zz} \end{bmatrix}}_{\text{deviatoric tensor } s_{ij}}. \quad (2)$$

It can be seen that the elements outside the main diagonal, i. e. the shear stresses, are the same for the stress and the deviatoric stress tensor:

$$s_{ij} = \sigma_{ij} \quad \text{for } i \neq j, \quad (3)$$

$$s_{ij} = \sigma_{ij} - \sigma_m \quad \text{for } i = j. \quad (4)$$

¹It should be noted that in the case of anisotropic materials, a hydrostatic stress state may result in a shape change and a deviatoric stress state in a volume change, Betten (2001).

²Also called the hydrostatic stress. In the context of soil mechanics, also the pressure $p = -\sigma_m$ is used. Be aware that some finite element codes use this definition.

³Wherever a subscript occurs twice in the same term, it is understood that the subscript is to be summed from x to z , e.g. $\sigma_{kk} = \sum_{k=x}^z \sigma_{kk} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 3\sigma_m$.

The hydrostatic part of σ_{ij} has in the case of metallic materials (full dense materials) for temperatures approximately under $0.3 \cdot T_m$ (T_m : melting temperature) nearly no influence on the occurrence of inelastic strains since dislocations slip only under the influence of shear stresses. On the other hand, the hydrostatic stress has a considerable influence on the yielding behaviour in the case of cellular materials, in soil or damage mechanics.

1.2 Invariants

To ensure the independence of a stress based description of physical phenomena from the chosen coordinate system (objectiveness), it is meaningful to use instead of the stress tensor components σ_{ij} a set of independent tensor invariants, Backhaus (1983); Jirasek and Bazant (2002), among others. These invariants are independent from the orientation of the coordinate system and represent the physical content of the stress tensor. The so-called characteristic equation⁴

$$\det(\sigma_{ij} - \lambda \delta_{ij}) = 0, \quad (5)$$

or in components

$$\begin{vmatrix} \sigma_{xx} - \lambda & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \lambda & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \lambda \end{vmatrix} = 0, \quad (6)$$

leads to the cubic equation

$$\lambda^3 - I_1(\sigma_{ij})\lambda^2 + I_2(\sigma_{ij})\lambda - I_3(\sigma_{ij}) = 0, \quad (7)$$

for the definition of the three scalar *principal* stress invariants I_1, I_2 and I_3 . The three roots λ_i of Eq. (7) are the principal stresses: $\sigma_I = \max(\lambda_1, \lambda_2, \lambda_3)$, $\sigma_{III} = \min(\lambda_1, \lambda_2, \lambda_3)$, and $\sigma_{II} = (I_1 - \sigma_I - \sigma_{III})$. Let us remind here that the principal stress state is the state which has no shear components, i.e. the only stress components are the principal stresses σ_i ($i = I, II, III$) on the main diagonal of the stress matrix. Another interpretation of the principal invariants is given by:

$$- I_1 = \text{trace}^5 \text{ of } \sigma_{ij}:$$

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}. \quad (8)$$

⁴ $\det(\dots)$ denotes the determinant of a matrix or tensor.

⁵The trace of the stress tensor is defined: $\text{tr}(\sigma_{ij}) = \sigma_{ii}$.

– I_2 = sum of the two-row main subdeterminants of σ_{ij} :

$$I_2 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix}. \quad (9)$$

– I_3 = determinant of σ_{ij} :

$$I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix}. \quad (10)$$

In addition to these principal invariants, there is often another set of invariants used. This set is included in the principal invariants I_i and called *basic* invariants J_i :

$$J_1 = I_1, \quad (11)$$

$$J_2 = \frac{1}{2}I_1^2 - I_2, \quad (12)$$

$$J_3 = \frac{1}{3}I_1^3 - I_1I_2 + I_3. \quad (13)$$

The definition of both of principal and basic invariants is summarised and compared in Tab. 1.

It can be seen from Tab. 1 that the spherical tensor σ_{ij}^o is completely characterised by its first invariant because the second and third invariant are powers of it. The stress deviator tensor s_{ij} is completely characterised by its second and third invariant. Therefore, the physical content of the stress state σ_{ij} can be completely described either by the three invariants or if we use the decomposition in its spherical and deviatoric part by the first invariant of the spherical tensor and the second and third invariant of the stress deviator tensor. It should be noted here that this statement is valid for both, i.e. the principal and basic invariants. In the following, we will only use the basic invariants and thus, the physical content of the stress tensor will be described by the following set of invariants:

$$\sigma_{ij} \rightarrow J_1^o, J_2', J_3'. \quad (14)$$

The following Eq. (15) summarises formulae based on the general stress components σ_{ij} for the practical calculation of the three stress invariants J_1^o, J_2' and J_3' .

Table 1. Definition of principal and basic stress invariants of isotropic materials.

stress tensor		
σ_{ij}	σ_{ij}^o	s_{ij}
principal invariants		
I_1, I_2, I_3	I_1^o, I_2^o, I_3^o	I'_1, I'_2, I'_3
$I_1 = \sigma_{ii}$ $I_2 = \frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji})$ $I_3 = \frac{1}{3}(\frac{1}{2}\sigma_{ii}\sigma_{jj}\sigma_{kk} + \sigma_{ij}\sigma_{jk}\sigma_{ki} - \frac{3}{2}\sigma_{ij}\sigma_{ji}\sigma_{kk})$	$I_1^o = \sigma_{ii}$ $I_2^o = \frac{1}{3}(\sigma_{ii})^2$ $I_3^o = \frac{1}{27}(\sigma_{ii})^3$	$I'_1 = 0$ $I'_2 = -\frac{1}{2}s_{ij}s_{ji}$ $I'_3 = \frac{1}{3}s_{ij}s_{jk}s_{ki}$
$\Rightarrow I_1, I_2, I_3$	$\Rightarrow I_1^o$	$\Rightarrow I'_2, I'_3$
basic invariants		
J_1, J_2, J_3	J_1^o, J_2^o, J_3^o	J'_1, J'_2, J'_3
$J_1 = \sigma_{ii}$ $J_2 = \frac{1}{2}\sigma_{ij}\sigma_{ji}$ $J_3 = \frac{1}{3}\sigma_{ij}\sigma_{jk}\sigma_{ki}$	$J_1^o = \sigma_{ii}$ $J_2^o = \frac{1}{6}(\sigma_{ii})^2$ $J_3^o = \frac{1}{9}(\sigma_{ii})^3$	$J'_1 = 0$ $J'_2 = \frac{1}{2}s_{ij}s_{ji}$ $J'_3 = \frac{1}{3}s_{ij}s_{jk}s_{ki}$
$\Rightarrow J_1, J_2, J_3$	$\Rightarrow J_1^o$	$\Rightarrow J'_2, J'_3$

$$J_1^o = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}, \quad (15)$$

$$J_2^o = \frac{1}{6}[(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2] + \sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2, \quad (16)$$

$$J_3^o = s_{xx}s_{yy}s_{zz} + 2\sigma_{xy}\sigma_{yz}\sigma_{zx} - s_{xx}\sigma_{yz}^2 - s_{yy}\sigma_{zx}^2 - s_{zz}\sigma_{xy}^2, \quad (17)$$

where e.g. the deviatoric stresses can be calculated as

$$s_{xx} = \frac{1}{3}(2\sigma_{xx} - \sigma_{yy} - \sigma_{zz}). \quad (18)$$

Finally, it should be mentioned here that it is also possible to derive the same sets of invariants from the strain tensor ε_{ij} and that further sets of independent invariants can be found e.g. in Altenbach et al. (1995).

1.3 Constitutive Equations

Figure 2 shows the emplacement of the constitutive equation in the framework of the solid mechanical modelling of a material particle. The constitutive equation combines the kinematic and equilibrium equations by indicating the dependence of strains on stresses.

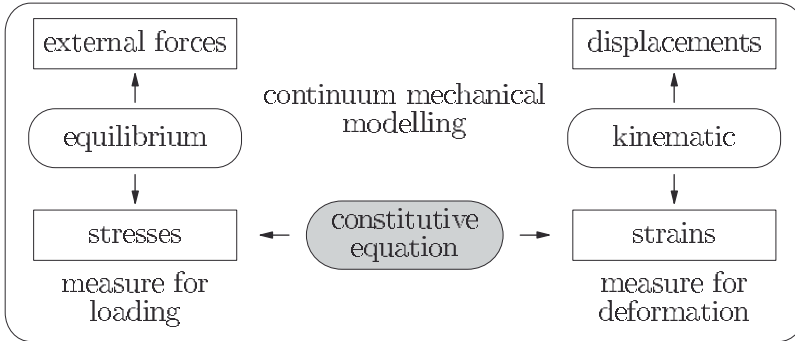


Figure 2. Solid mechanical modelling of engineering materials.

According to the level of description, microscopic, stochastic and macroscopic constitutive equations are distinguished, Lebon (1992). Microscopic constitutive equations describe the material behaviour based on variables (e.g. dislocation density) taken from physics of solids. A transfer of the microscopic material behaviour on the macroscopic level was not successful up to now. Stochastic constitutive equations describe also the material behaviour in the micro-level. For that purpose, probabilistic processes are applied. The transfer on the macroscopic level is due to its mathematical structure possible. Macroscopic constitutive equations are also called phenomenological models. Their field of application is the continuum mechanical modelling of components and structures. The material is idealised as a homogeneous continuum which constitution is described by a few variables such as strain, temperature and occasionally by additional so-called

internal variables. In the case of multiphase or inhomogeneous materials, such as cellular or fibre-reinforced materials, this assumption holds not in the first place. By averaging the different material properties an approximately homogeneity can be assumed, cf. Fig. 3.

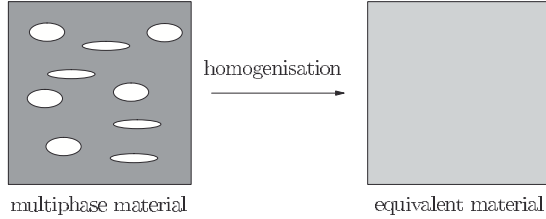


Figure 3. Averaging the materials properties to obtain an equivalent material.

1.4 Linear Elastic Behaviour: Generalised Hooke's Law for Isotropic Materials

In general Cauchy elasticity the stress tensor σ_{ij} is a nonlinear tensor function of the strain tensor ε_{kl}

$$\sigma_{ij} = f_{ij}(\varepsilon_{kl}), \quad (19)$$

whereas in linear elasticity of homogenous materials the Cauchy stress tensor is, in extension to Hooke's law ($\sigma = E\varepsilon$) of the year 1678, proportional to the Cauchy strain tensor through the linear transformation

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}. \quad (20)$$

It may be notated here that Eq. (20) is the simplest generalisation of the linear dependence of stress and strain observed by Hooke in a simple tension test, and consequently Eq. (20) is referred to as the generalised Hooke's law. The fourth-order tensor C_{ijkl} has 81 independent components in total and is called the elasticity tensor. In the case of linear elasticity⁶, the components of the elasticity tensor are material parameters. Due to symmetry of σ_{ij} and ε_{kl} , the pairs of indices ij and kl in C_{ijkl} can be permuted. Hence, the number of independent constants of the elasticity tensor is reduced to 36. A further reduction of constants is obtained if we assume the existence of a strain energy potential $w(\varepsilon_{ij})$ from which the stresses are derived by differentiating as:

⁶Considering a homogeneous material at constant temperature.

$$\sigma_{ij} = \frac{\partial w}{\partial \varepsilon_{ij}}. \quad (21)$$

After partial differentiating of Eq. (20) with respect to ε_{ij} and consideration of Eq. (21), one obtains the components of the elasticity tensor as

$$C_{ijkl} = \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = \frac{\partial^2 w}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}}. \quad (22)$$

There is an additional symmetry due to the commutation of the order of the partial derivative with respect to ε_{ij} and ε_{kl} and the number of independent constants is reduced to 21. A material for which such a constitutive relation is assumed is also called Green elastic material, Altenbach and Altenbach (1994). Under consideration of material symmetries (e. g. crystal systems), a further reduction of the number of constants can be achieved. For an isotropic material, the mechanical properties are the same for all directions and it can be shown, Hahn (1985), that the components of the elasticity tensor are determined by two independent constants, the so-called Lamé's constants λ and μ and the stress-strain relationship is given for this case by

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}. \quad (23)$$

The characterisation of metallic materials is in general based on the engineering constants Young's modulus E and Poisson's ratio ν . On the other hand, the description⁷ of isotropic materials in soil or rock mechanics is rather based on the shear modulus G and the bulk modulus K than E and ν . The conversion of the engineering elastic constants can be done with the aid of the relationships given in Tab. 2.

Mechanical and physical reference values⁸ of some typical engineering materials are summarised in Tab. 3.

In the following description, the tensor notation is abandoned and the engineering notation (so-called Voigt or matrix notation) is now consistently introduced and observed, i.e. the components of the second-order strain tensor ε_{ij} and the stress tensor σ_{ij} are arranged into column matrices

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z \end{bmatrix} \rightarrow \boldsymbol{\varepsilon} = \{\varepsilon_x \ \varepsilon_y \ \varepsilon_z \ 2\varepsilon_{xy} \ 2\varepsilon_{yz} \ 2\varepsilon_{xz}\}^T \quad (24)$$

⁷This is also valid for visco-elastic materials.

⁸Conversion from GPa to MPa: value times 1000; conversion from $\text{g} \cdot \text{cm}^{-3}$ to $\text{kg} \cdot \text{m}^{-3}$ value times 1000; $1 \text{ Pa} = 1 \text{ N/m}^2$.

Table 2. Conversion of elastic constants.

	E, ν	K, ν	G, ν	K, G
K	$\frac{E}{3(1-2\nu)}$	K	$\frac{2G(1+\nu)}{3(1-2\nu)}$	K
E	E	$3K(1-2\nu)$	$2G(1+\nu)$	$\frac{9KG}{3K+G}$
ν	ν	ν	ν	$\frac{3K-2G}{2(3K+G)}$
G	$\frac{E}{2(1+\nu)}$	$\frac{3K(1-2\nu)}{2(1+\nu)}$	G	G

Table 3. Reference values of typical engineering materials (all values are given near room temperature).

Material	E in GPa	ν in –	G in GPa	density in $\text{g} \cdot \text{cm}^{-3}$
Aluminium	70	0.35	26	2.700
Iron	211	0.29	82	7.874
Magnesium	45	0.29	17	1.738
Titanium	116	0.32	44	4.506

and

$$\sigma_{ij} = \begin{bmatrix} \sigma_x & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_y & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_z \end{bmatrix} \quad \rightarrow \quad \boldsymbol{\sigma} = \{\sigma_x \ \sigma_y \ \sigma_z \ \sigma_{xy} \ \sigma_{yz} \ \sigma_{xz}\}^T, \quad (25)$$

and the fourth-order elasticity tensor C_{ijkl} is represented by a square (6×6) -matrix $\mathbf{C}$. This formalism is closer to actual computer implementations than the tensorial notation, Simo and Hughes (1998). Furthermore, we use the unit column matrix $\mathbf{1} = \{1 \ 1 \ 1 \ 0 \ 0 \ 0\}^T$ and the diagonal matrix $\mathbf{L} = [1 \ 1 \ 1 \ 2 \ 2 \ 2]$, which is not a unit matrix because the strain vector is

expressed using the engineering definition of shear strain (e.g. $\gamma_{xy} = 2\varepsilon_{xy}$) rather than the tensor definition, cf. Fig. 4.

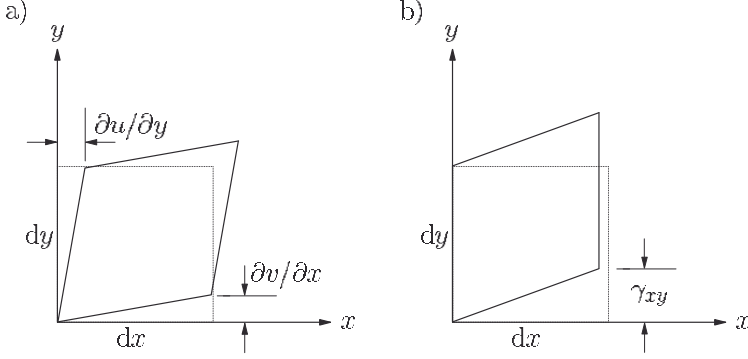


Figure 4. Definition of shear strain: a) tensor definition as the average $\varepsilon_{xy} = \varepsilon_{yx} = (\partial v/\partial x + \partial u/\partial y)/2$; b) engineering definition as the total $\gamma_{xy} = \partial v/\partial x + \partial u/\partial y$.

Now, the generalised Hooke's law for isotropic materials based on Lamé's constants can be written in matrix form as

$$\boldsymbol{\sigma} = (2\mu\mathbf{L}^{-1} + \lambda\mathbf{1} \otimes \mathbf{1})\boldsymbol{\varepsilon}, \quad (26)$$

or in explicit form with all components as

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{pmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \cdot \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \end{pmatrix}. \quad (27)$$

The dyadic product $\otimes$ used in Eq. (26) is represented for two column matrices $\mathbf{a}$ and $\mathbf{b}$ by

$$\mathbf{a} \otimes \mathbf{b} = \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{Bmatrix} \otimes \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Bmatrix} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & \dots & a_1 b_n \\ a_2 b_1 & a_2 b_2 & \dots & a_2 b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n b_1 & \vdots & \dots & a_n b_n \end{bmatrix}, \quad (28)$$

where the same result can be obtained by the multiplication of a column and row matrix $\mathbf{ab}^T$. Alternatively, Eq. (26) can be written in the form of Eq. (23) as

$$\begin{aligned}\boldsymbol{\sigma} &= 2\mu\mathbf{L}^{-1}\boldsymbol{\varepsilon} + \lambda\mathbf{1} \otimes \mathbf{1} (\varepsilon_m\mathbf{1} + \mathbf{e}) = 2\mu\mathbf{L}^{-1}\boldsymbol{\varepsilon} + \lambda\varepsilon_m \underbrace{\mathbf{1}\mathbf{1}^T}_{31} + \lambda \underbrace{\mathbf{1}\mathbf{1}^T}_{0}\mathbf{e} \\ &= 2\mu\mathbf{L}^{-1}\boldsymbol{\varepsilon} + \lambda(3\varepsilon_m)\mathbf{1}.\end{aligned}\quad (29)$$

Substituting for λ and μ in terms of E and ν (cf. Tab. 2) gives the alternative formulation of the stiffness matrix in Eq. (27) based on the engineering constants as

$$\mathbf{C} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}. \quad (30)$$

In a simple tension test, the only nonzero stress component σ_x causes axial strain ε_x and transverse strains $\varepsilon_y = \varepsilon_z$. Thus, Eq. (30) yields

$$\varepsilon_x = \frac{\sigma_x}{E} \quad \text{and} \quad \varepsilon_y = -\nu\varepsilon_x = -\frac{\nu\sigma_x}{E}. \quad (31)$$

By using Eqs. (31), one can calculate the elastic constants, Young's modulus E and Poisson's ratio ν , from a uniaxial tension or compression test. Introducing the shear modulus G and the bulk modulus K according to Tab. 2 yields a further formulation of the stiffness matrix in Hooke's generalised law as

$$\mathbf{C} = \begin{bmatrix} K + \frac{4}{3}G & K - \frac{2}{3}G & K - \frac{2}{3}G & 0 & 0 & 0 \\ K - \frac{2}{3}G & K + \frac{4}{3}G & K - \frac{2}{3}G & 0 & 0 & 0 \\ K - \frac{2}{3}G & K - \frac{2}{3}G & K + \frac{4}{3}G & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix}. \quad (32)$$

The quantity $K + \frac{4}{3}G$ in Eq. (32) is also known as the constraint modulus, Chen and Han (1988).

From the point of view of continuum mechanics, a state under pure shear stress (result: shear modulus G) and a state under pure hydrostatic stress

(result: bulk modulus K) should aim to determine the elastic constants, since the constants are independent from each other in this case. However, the experimental determination of the elastic constants is mostly based on a simple realisable tension or compression test, from which Young's modulus E and Poisson's ratio ν can be obtained.

A summary of the different formulations of Hooke's law in matrix form is given in Tab. 4 and compared with the corresponding tensor form given in common literature, e. g. Hahn (1985); Chen and Saleeb (1982); Flügge (1962); Mang and Hofstetter (2000).

Table 4. Formulations of isotropic Hooke's law (elastic stiffness form).

Hooke's law	
$\boldsymbol{\sigma} = \mathbf{C} \boldsymbol{\varepsilon}$	$\sigma_{ij} = C_{ijkl} \varepsilon_{kl}$
Formulation based on Lamé's constants	
$\boldsymbol{\sigma} = (\lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{L}^{-1}) \boldsymbol{\varepsilon}$	$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$
Formulation based on Young's modulus and Poisson's ratio	
$\boldsymbol{\sigma} = \frac{E}{1+\nu} \cdot \left(\frac{\nu}{1-2\nu} \mathbf{1} \otimes \mathbf{1} + \mathbf{L}^{-1} \right) \boldsymbol{\varepsilon}$	$\sigma_{ij} = \frac{E}{1+\nu} \cdot \left(\varepsilon_{ij} + \frac{\nu}{1-2\nu} \delta_{ij} \varepsilon_{kk} \right)$
Formulation based on bulk modulus and shear modulus	
$\boldsymbol{\sigma} = (K \mathbf{1} \otimes \mathbf{1} + 2G(\mathbf{L}^{-1} - \frac{1}{3} \mathbf{1} \otimes \mathbf{1})) \boldsymbol{\varepsilon}$	$\sigma_{ij} = 2G \cdot \left(\varepsilon_{ij} + \frac{3K-2G}{6G} \delta_{ij} \varepsilon_{kk} \right)$
Decomposition in volumetric and deviatoric part	
$\boldsymbol{\sigma} = \underbrace{3K \varepsilon_m \mathbf{1}}_{\text{volumetric}} + \underbrace{2G \mathbf{L}^{-1} \mathbf{e}}_{\text{deviatoric}}$	$\sigma_{ij} = \underbrace{K \varepsilon_{kk} \delta_{ij}}_{\text{volumetric}} + \underbrace{2G e_{ij}}_{\text{deviatoric}}$

2 Constitutive Relationships for Pressure Sensitive Materials: Systematic Overview

The three essential ingredients of plastic analysis are the yield criterion, the flow rule, and the hardening rule, cf. Armen (1979). The yield criterion relates the state of stress to the onset of yielding. The flow rule relates the state of stress σ_{ij} to the corresponding increments of plastic strain $d\varepsilon_{ij}^p$ when an increment of plastic flow occurs. The hardening rule describes how the yield criterion is modified by straining beyond the initial yield. In the following section, the mathematical and graphical representation of the initial yield criterion will be discussed in detail. The initial yield criterion can generally be expressed for isotropic and anisotropic materials as

$$F = F(\sigma_{ij}), \quad (33)$$

where the state of stress σ_{ij} can be split in its spherical σ_{ij}^o and deviatoric part σ'_{ij} . For an isotropic material, the stress state can then be expressed in terms of combinations of three independent stress invariants. As discussed in one of the previous sections, the set of the so-called *basic* invariants, J_1^o , J_2' and J_3' , is used. Thus, one can replace Eq. (33) for an isotropic material by

$$F = F(J_1^o, J_2', J_3'). \quad (34)$$

The yield condition $F = 0$ represents a hypersurface in the multi-dimensional stress space and is also called the yield surface. A direct *graphical* representation of this yield surface in a Cartesian coordinate system with three coordinates is not possible due to its dimension. However, a reduction of the dimensions is possible if a principal axis transformation is applied to the argument σ_{ij} . The components of the stress tensor are then uniquely reduced for isotropic materials to the principal stresses σ_I , σ_{II} and σ_{III} on the principal diagonal of the stress tensor. In such a principal stress space, it is now possible to graphically represent the yield condition as a three-dimensional surface. A hydrostatic stress state ($\sigma_I = \sigma_{II} = \sigma_{III}$) lies in such a principal stress system on the space diagonal (hydrostatic axis). Any plane perpendicular to the hydrostatic axis is called an octahedral plane. The particular octahedral plane passing through the origin is the deviatoric plane or π -plane, Backhaus (1983). On the basis of the dependency of the yield condition on the invariants, a descriptive classification can be performed. Yield conditions independent of hydrostatic stress can be represented by the invariants J_2' and J_3' . Stress states with $J_2' = \text{const.}$ lie on a circle around the hydrostatic axis in an octahedral plane. A dependency

of the yield condition on J'_3 results in a deviation from the circle shape. A dependency on J'_1 denotes a size change of the cross-section of the yield surface along the hydrostatic axis. However, the shape of the cross-section remains similar in the mathematical sense. Therefore, a dependency on J'_1 can be represented by sectional views through planes along the hydrostatic axis. The geometrical interpretation of stress invariants, Chen and Han (1988), is given in Fig. 5.

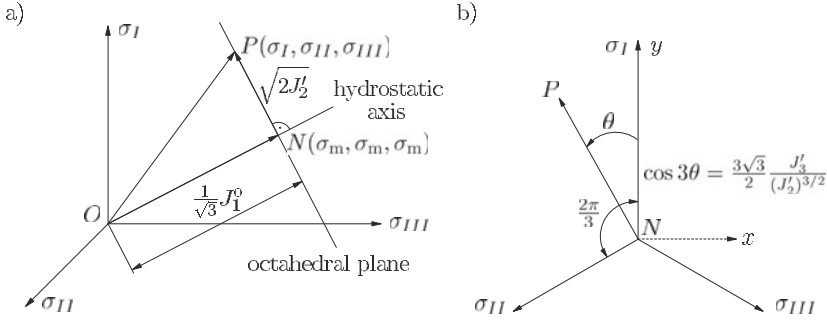


Figure 5. Geometrical interpretation of basic stress invariants in relation to an arbitrary stress state P : a) principal stress space; b) octahedral plane, $J'_1 = \text{const.}$, view along the hydrostatic axis.

It can be seen that an arbitrary stress state P can be expressed by its position along the hydrostatic axis $(\frac{1}{\sqrt{3}} J'_1) = \xi$ and its polar coordinates $(\sqrt{2} J'_2 = \rho, \theta(J'_2, J'_3))$ in the octahedral plane through P . For the set of polar coordinates, the so-called stress Lode angle θ is defined in the range $0 \leq \theta \leq 60^\circ$ as, Nayak and Zienkiewicz (1972),

$$\cos(3\theta) = \frac{3\sqrt{3}}{2} \cdot \frac{J'_3}{(J'_2)^{3/2}}. \quad (35)$$

The set of coordinates $(\xi, \rho, \cos(3\theta))$ is known as the Haigh-Westergaard coordinates. To investigate the shape of the yield surface, multiaxial stress states must be simulated and the shape within the corresponding octahedral plane has to be drawn according to the following transformation which projects a principal stress state first in the octahedral plane (angle of transformation: $\cos \vartheta = 1/\sqrt{3}$) and then to the Cartesian coordinate system (x, y) shown in Fig. 5 b):

$$y = \frac{2}{\sqrt{6}} \cdot (\sigma_I - 0.5 \cdot (\sigma_{II} + \sigma_{III})) , \quad (36)$$

$$x = \frac{1}{\sqrt{2}} \cdot (\sigma_{III} - \sigma_{II}) . \quad (37)$$

In such a case, it is now essential whether the plastic behaviour is pressure sensitive, i.e. depends on J_1^o , or not. If there is no dependency on J_1^o , i.e. a constant shape along the hydrostatic axis, then all evaluated points can be drawn in a single octahedral plane. However, if one expects a dependency, then only stress states with the same hydrostatic stress are allowed to be represented in the same octahedral plane where $J_1^o = \text{const.}$ holds. As a result, e.g. uniaxial tensile ($J_1^o = \sigma_I$) and pure shear tests ($J_1^o = 0$) cannot be represented in the same octahedral plane. In order to draw the shape of the yield surface for a pressure sensitive material, the following differing multiaxial stress states with $J_1^o = 0$, e.g. $\sigma_I = -\sigma_{II}$ ($\sigma_{III} = 0$) or $\sigma_I = -2\sigma_{II} = -2\sigma_{III}$ ($\sigma_I > 0 \vee \sigma_I < 0$), are a possibility to obtain values in the same deviatoric plane. It should be noted here again that the shape changes only its size along the hydrostatic axes but remains similar in the mathematical sense. Typical Lode angles for basic experiments are summarised in Tab. 5 where the graphical representation in the octahedral plane is given.

For an isotropic material, the labels *I*, *II*, *III* attached to the principal coordinate axes are arbitrary. It follows that the yield condition must have threefold symmetry and it is only required to investigate the sector $\theta = 0^\circ$ to 60° , cf. Fig. 6. The other sectors follow directly from the symmetry. Is the uniaxial yield stress in addition the same in tension and compression, a sixfold symmetry is obtained and only the sector $\theta = 0^\circ$ to 30° needs to be investigated, cf. Fig. 6.

Further restrictions of the shape of the yield condition in the octahedral plane are based on the requirement of a convex shape⁹, cf. Fig. 7. The inner bound corresponds to the Tresca yield criterion. An influence of the third invariant would result in a deviation from the circular shape (e.g. von Mises).

The maximum distance between the circle and the outer bound is obtained for $\theta = 30^\circ$ as (cf. Fig. 7 b))

$$\frac{AC}{k} = \frac{AO - k}{k} = \frac{1}{\cos 30^\circ} - 1 = \frac{2}{\sqrt{3}} - 1 = 15.47\% , \quad (38)$$

⁹The convexity of the yield surface can be derived from Drucker's stability postulate, Betten (2001); Lubliner (1990). It is assumed here that the postulate is valid!

Table 5. Values of the stress Lode angle for basic tests.

case	component	θ in Fig. 5
uniax. tension	$\sigma_I = \sigma$ or	0°
	$\sigma_{II} = \sigma$ or	120° (-60°)
	$\sigma_{III} = \sigma$	240° (60°)
uniax. compression	$\sigma_I = -\sigma$ or	0° (180°)
	$\sigma_{II} = -\sigma$ or	-60° (120°)
	$\sigma_{III} = -\sigma$	60°
biax. tension	$\sigma_I = \sigma_{II} = \sigma$ or	60°
	$\sigma_I = \sigma_{III} = \sigma$ or	-60°
	$\sigma_{II} = \sigma_{III} = \sigma$	180°
biax. compression	$\sigma_I = \sigma_{II} = -\sigma$ or	240°
	$\sigma_I = \sigma_{III} = -\sigma$ or	120°
	$\sigma_{II} = \sigma_{III} = -\sigma$	0°
triax. tension	$\sigma_I = \sigma_{II} = \sigma_{III} = \sigma$	—
triax. compression	$\sigma_I = \sigma_{II} = \sigma_{III} = -\sigma$	—
pure shear	$\sigma_I = \tau; \sigma_{II} = -\tau$ ($\sigma_{xy} = \tau$)	-30°
	$\sigma_I = \tau; \sigma_{III} = -\tau$ ($\sigma_{xz} = \tau$)	30°
	$\sigma_{II} = \tau; \sigma_{III} = -\tau$ ($\sigma_{yz} = \tau$)	90°

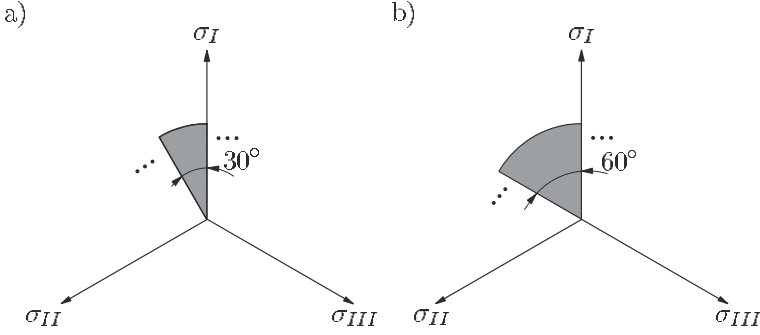


Figure 6. Schematic representation of periodic segments in the octahedral plane: a) 30° symmetry segment; 60° symmetry segment.

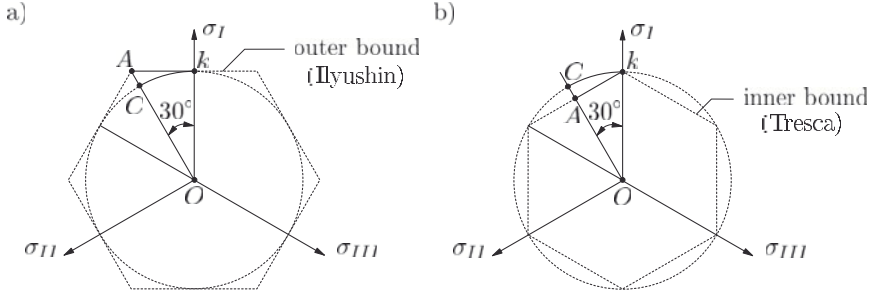


Figure 7. Schematic representation of outer (a) and inner bound (b) of a yield condition in the octahedral space.

of the circle diameter. The maximum distance between the circle and the inner bound is obtained under the same angle as (cf. Fig. 7 b))

$$\frac{AC}{k} = \frac{k - AO}{k} = 1 - \cos 30^\circ = 1 - \frac{\sqrt{3}}{2} = 13.40\%. \quad (39)$$

In the case that a material has the same uniaxial yield stress in tension and compression, the influence of the third invariant is often disregarded since the maximum error is in the indicated range of the outer and inner bound, cf. Eqs. (38) and (39). In such a case, the mathematical description can be performed in a J_1° - $\sqrt{3J_2^\circ}$ invariant space as indicated in Fig. 8.

In the J_1° - $\sqrt{3J_2^\circ}$ invariant space, basic material tests can be identified as lines through the origin as indicated in Fig. 8. Table 6 summarises for these basic experiments the linear equations. As one can see, a uniaxial tensile

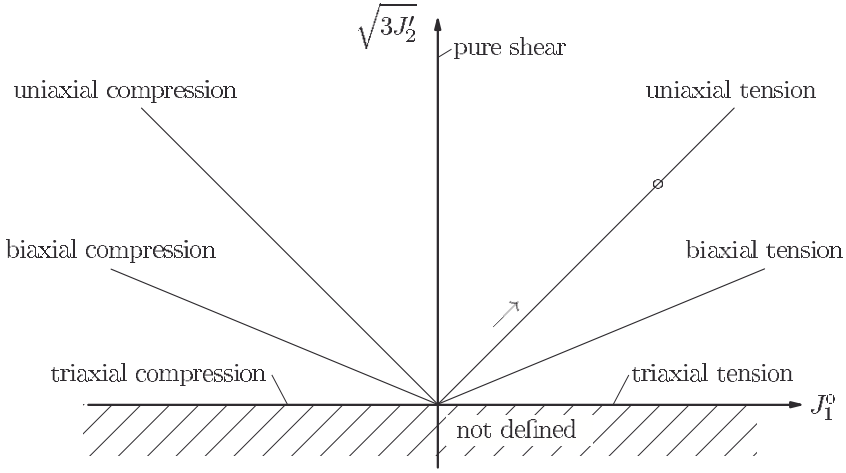


Figure 8. Schematic representation of basic tests in the $J_1^o - \sqrt{3J_2'}$ invariant space.

test is represented for example by the bisecting line in the first quadrant of the Cartesian $J_1^o - \sqrt{3J_2'}$ coordinate system. Performing a uniaxial tensile test would mean to 'walk' from the origin along this straight line (by monotonically increasing the load) until a material limit, e.g. initial yield or failure, is reached. This point (indicated in Fig. 8 by $\circ$) makes part of the yield or limit surface and the connection of all such points obtained from different experiments (i.e. from walks along different lines) gives the final shape which must be described in terms of the variables to obtain the expression $F(J_1^o, J_2') = 0$.

Some typical representations of yield or limit surfaces are schematically shown in Fig. 9. A representation which depends only on J_2' is shown in part a): A constant diameter is obtained along the hydrostatic axis and a circular shape in the octahedral plane. Incorporating a dependency on the hydrostatic stress (part b)) changes the diameter along the hydrostatic axis but the octahedral plain still displays a circle. Only the third invariant (part c)) can deform the circle in the octahedral plane. For such a dependency, sectional views for $\theta = 0^\circ$ ('tensile meridian'), $\theta = 60^\circ$ ('compression meridian') or $\theta = 30^\circ$ ('shear meridian') are common representations in the $J_1^o - \sqrt{3J_2'}$ space. Along these meridians, a hydrostatic stress state is superimposed to the respective stress states.

Table 6. Definition of basic tests in the J_1° - $\sqrt{3J_2'}$ invariant space.

case	J_1°	$\sqrt{3J_2'}$	comment
uniax. tension (σ)	σ	σ	slope: 1
uniax. compression ($-\sigma$)	$-\sigma$	σ	slope: -1
biax. tension (σ)	2σ	σ	slope: 0.5
biax. compression ($-\sigma$)	-2σ	σ	slope: -0.5
triax. tension (σ)	3σ	0	horiz. axis
triax. compression ($-\sigma$)	-3σ	0	horiz. axis
pure shear (τ)	0	$\sqrt{3}\tau$	vertical axis

The flow rule is in general stated in terms of a function Q , which is described in units of stress, and is called a *plastic potential*. With $d\lambda$ a scalar called *plastic multiplier*, the plastic strain increments are given by

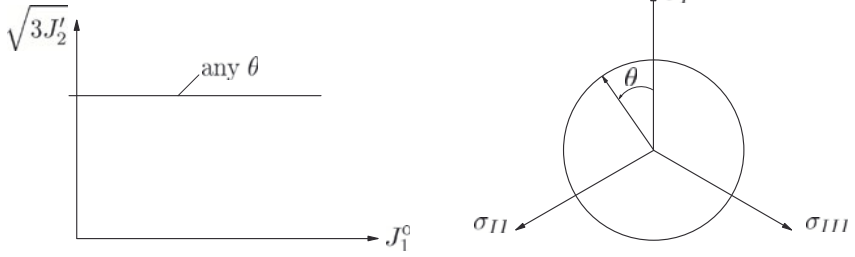
$$d\boldsymbol{\epsilon} = d\lambda \cdot \frac{\partial Q}{\partial \boldsymbol{\sigma}}. \quad (40)$$

The flow rule given in Eq. (40) is said to be *associated* if $Q = F$, otherwise it is *non-associated*. Hardening can be modelled as isotropic (i.e. initial yield surface expands uniformly without distortion and translation as plastic flow occurs) or as kinematic (i.e. initial yield surface translates as a rigid body in stress space, maintaining its size, shape and orientation), either separately or in combination. With the isotropic hardening parameter κ and the kinematic hardening parameters $\boldsymbol{\alpha} = \{\alpha_x, \dots, \alpha_{xz}\}^T$, the hardening vector $\mathbf{q} = \{\kappa, \boldsymbol{\alpha}\}^T$ can be composed and the yield condition under consideration of hardening effects results as:

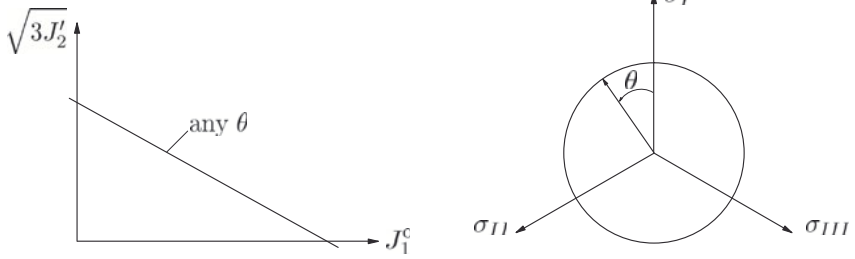
$$F = F(\boldsymbol{\sigma}, \mathbf{q}) = 0. \quad (41)$$

Equation (41) can be further extended e.g. by incorporating of damage effects or anisotropy. More details on modelling of plastic material behaviour can be found in the standard books Altenbach et al. (1995); Zyczkowski

a)



b)



c)

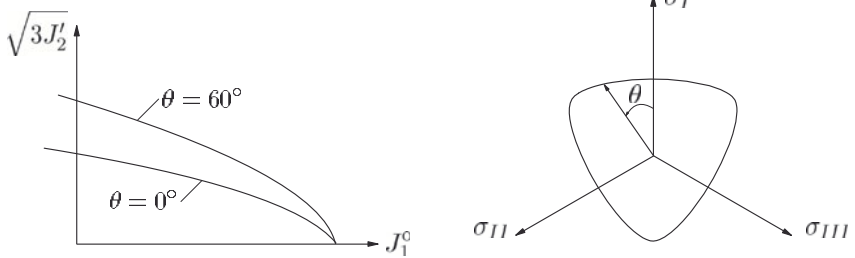


Figure 9. Schematic representation of different yield/failure conditions: a) $F = F(J'_2)$; b) $F = F(J_1^c, J'_2)$; c) $F = F(J_1^c, J'_2, J'_3)$.

(1981); Betten (2005); Kolupaev (2006).

Let us have in the following a look on a typical yield criterion proposed for cellular materials. For metallic and polymer foams, the following yield criterion which incorporates the hydrostatic stress

$$F(J_1^o, J_2') = \frac{1}{27}\alpha^2 (J_1^o)^2 + J_2' - k_s^2 = 0 \quad (42)$$

has been proposed by Deshpande and Fleck (2000, 2001). A graphical representation in the $\sqrt{J_2'} - J_1^o$ -plane is shown in Fig. 10 a). Since the hydrostatic stress (J_1^o) is given as a quadratic function, i.e. to power two, the shape is symmetric to the $\sqrt{J_2'}$ -axis. The general shape is an ellipse with major axes coincident with the coordinate axes.

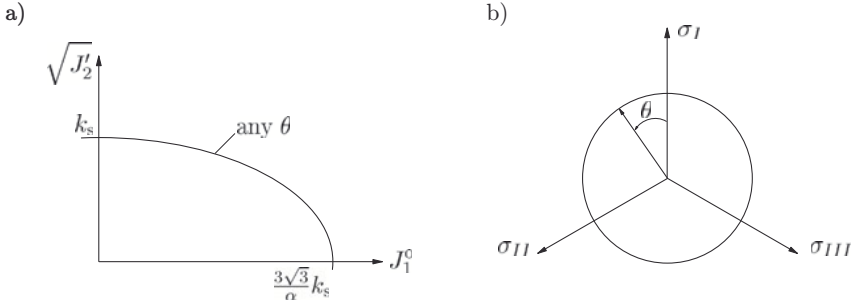


Figure 10. Graphical representation of the yield condition according to Deshpande-Fleck: a) $\sqrt{J_2'} - J_1^o$ -coordinate system; b) octahedral plane.

In the octahedral plane, a circular shape is obtained since Eq. (42) does not incorporate J_3' , cf. Fig. 10 b). To further investigate the yield condition, let us consider a biaxial stress state where only the two normal stresses σ_I and σ_{II} are acting. Substituting the relationships from Eqs. (15) for the basic invariants into Eq. (42) and using the identity

$$k_s^2 = \frac{k_s^2}{1 - 12\alpha^2} - \frac{12\alpha^2 k_s^2}{1 - 12\alpha^2}, \quad (43)$$

yields after some transformations the equation of an ellipse in the σ_I - σ_{II} -space as:

$$F_{\sigma_I - \sigma_{II}} = \left(\frac{x}{\sqrt{2}k_s} \right)^2 + \left(\frac{\sqrt{\frac{2}{27}\alpha^2 + \frac{1}{6}} \cdot y}{k_s} \right)^2 - 1 = 0, \quad (44)$$

where

$$x = \frac{1}{\sqrt{2}}(\sigma_I + \sigma_{II}) \quad \text{and} \quad y = \frac{1}{\sqrt{2}}(\sigma_I - \sigma_{II}). \quad (45)$$

The same procedure yields also to an ellipse in the σ - τ -space:

$$\left(\frac{\sqrt{\frac{1}{27} \alpha^2 + \frac{1}{3}} \cdot \sigma}{k_s} \right)^2 + \left(\frac{\tau}{k_s} \right)^2 - 1 = 0, \quad (46)$$

from which by substituting $\tau \rightarrow 0$ and $\sigma \rightarrow k_t$ the relationship between the tensile yield stress k_t and the shear yield stress k_s can be obtained as:

$$k_t = \frac{k_s}{\sqrt{\frac{1}{27} \alpha^2 + \frac{1}{3}}}. \quad (47)$$

The elliptical shapes in the two-dimensional stress spaces (σ_I - σ_{II} and σ - τ) are shown in Fig. 11.

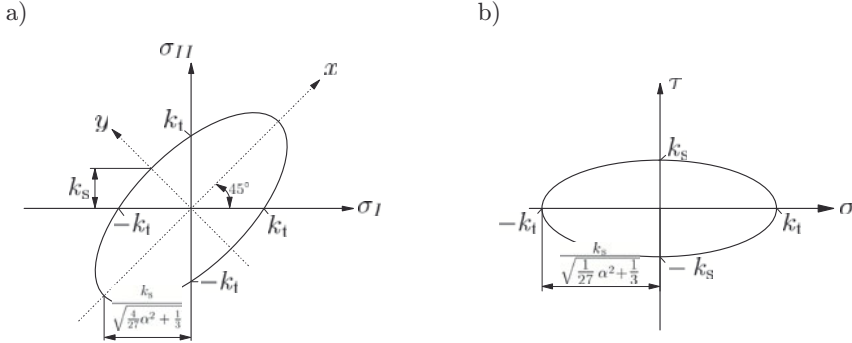


Figure 11. Graphical representation of the yield condition according to Deshpande-Fleck: a) σ_I - σ_{II} -space; b) σ - τ -space.

It should be mentioned here that there are similar yield conditions known in literature, e.g. the Mahrenholtz-Ismar yield criterion, Mahrenholtz and Ismar (1979, 1981),

$$F(J_1^\circ, J_2') = \alpha(J_1^\circ)^2 + J_2' - (k_s)^2, \quad (48)$$

which can be easily described by the transformation $\frac{1}{27} \alpha^2 \rightarrow \alpha$ in the same way. Let us note at the end of this section that Eq. (42) was originally published with the tensile yield stress k_t as:

$$F = \frac{1}{\left[1 + \left(\frac{\alpha}{3}\right)^2\right]} \cdot [\sigma_e^2 + \alpha^2 \sigma_m^2] - k_t^2 = 0. \quad (49)$$

3 Simple Cubic Cell Models based on Beams and Shells for Open and Closed Cell Materials

The approaches found in literature to model the physical behaviour of cellular materials and to derive respective constitutive equations to predict their behaviour can be broadly divided in four directions:

- the use of phenomenological models,
- the application of equations developed for composite materials, Hashin (1962); Cohen and Ishai (1967),
- the analysis of model structures representative of the physical structure,
- the analysis of the real structure e.g. based on micro-computed tomography (μ CT) images, Feldkamp et al. (1989); Müller and Rügsegger (1995).

The last approach, i.e. consideration of the real geometry, would require the lowest degree of simplifications since the geometry would be exactly represented for a certain volume. However, there are still problems connected with this most advanced approach:

- It is required to consider a representative volume element (RVE) which comprises a certain number of cells in order to be representative for the heterogeneous or random cell structure. In addition, a minimum number of cells is required to avoid any influence of the edge effects occurring at the 'free' surface of the volume (which can be the boundary of the scanned volume) in order to represent macroscopic values for a larger arrangements of cells, Lemaitre (1996). The precise number depends, however, on the corresponding cell structure and should be examined separately;
- The standard engineering tool to simulate the geometry is the finite element method. The conversion of the geometrical information based on μ CT scans, Müller et al. (1985), is normally automatically done and so-called tetrahedral elements are generated. However, these elements reveal inferior performance in the case of non-linear material behaviour. Strategies to apply brick (hexaeder) elements by converting voxels into finite elements are given e.g. in Keyak et al. (1990); van Rietbergen et al. (1995). However, this approach results in unsmoothed surfaces with sharp geometric discontinuities which may give erroneous results, Marks and Gardner (1993), especially in the case on non-linear material behaviour;
- The numerical approach based on the finite element method with real geometries implies high demands with respect to the computational

hardware and e.g. the detailed representation of the size of a mechanical test specimen still reaches the limits of actual computer memory (RAM). In addition, access to expensive scanning hardware (CT) is required.

In the following, we will concentrate on the third approach, i.e. analysis of model structures. In this case, the real geometry is strongly simplified. On the other hand, analytical (e.g. Bernoulli-Euler beam theory) or numerical methods (e.g. finite element method) at a much lower demand may be applied to investigate the physical properties. The major idea is to create a model structure which represents some characteristic of the real geometry (e.g. porosity) and behaves (e.g. deforms) in a similar way as the real structure (e.g. major deformation behaviour represented by beams under bending). However, such a modelling approach cannot aim to determine the precise values of the macroscopic material parameters since a dependency on the pore geometry and arrangement exists. Nevertheless, this approach may aim to determine the principle characteristic of the constitutive equation while the exact values of the respective free material parameters need to be determined experimentally for a given real material. In other words, the approach may be able to derive an equation to describe the material behaviour e.g. as a function of the porosity but there are parameters in the equation which must be evaluated by experiments. This approach is e.g. common for classical engineering material where ductile metals can be described by the von Mises yield condition (known mathematical equation) whereas the yield stress (free material parameter) must be experimentally determined for each specific ductile metal.

Typical expressions for the average mechanical properties such as stiffness of cellular material are obtained as power laws of density or relative density with the following structure

$$\text{mechanical property} = C_1 \cdot (\text{density})^{C_2}, \quad (50)$$

where C_1 and C_2 are real numbers to be fitted by experimental values. In the following, some classical modelling approaches for the geometry will be described.

The so-called ball-and-strut model by Gent and Thomas (1959) was applied to represent the cellular morphology of open-cell rubber foams, cf. Fig. 12 a). It consists of circular struts joined at their intersections by polymer spheres. The connecting elements (spheres) were assumed to be undeformable and the orientation distribution of the struts was taken to be random. In addition, it was assumed that the entire surface of the spherical dead volume is covered by strut elements. This model was later applied by

Lederman (1971) who assumed that in general the dead volume does not need to be entirely covered by strut elements. The important geometrical parameter in this model is the ratio of the sphere diameter to the free strut length.

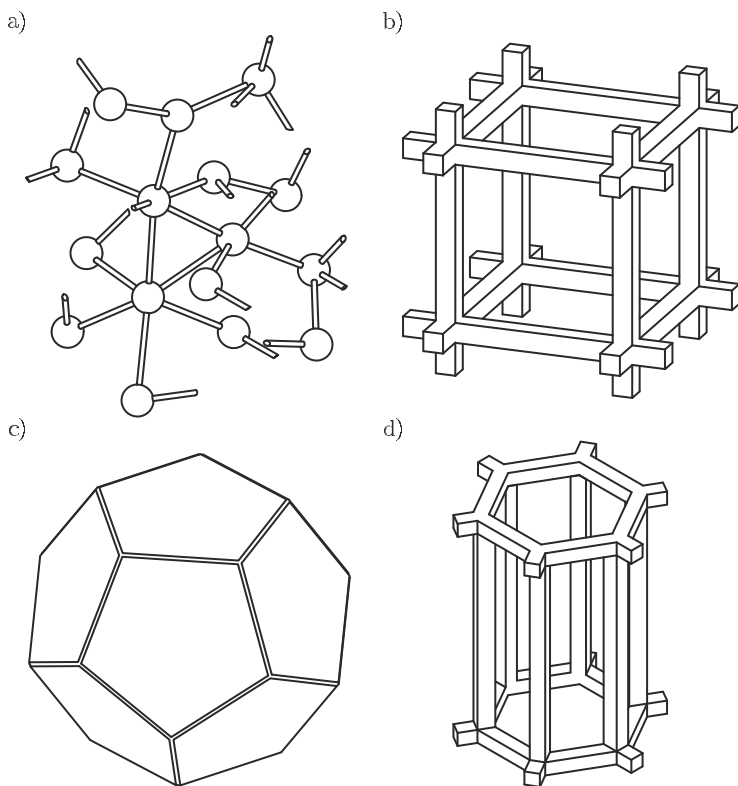


Figure 12. Simple idealizations of open cell structures: a) the ball and strut model, Gent and Thomas (1959); b) the cubic strut model, Gent and Thomas (1963); c) pentagonal dodecahedron, Gibson and Ashby (1997); d) hexagonal model of rod-like columnar structure, Gibson (1985).

A cubical arrangement of square-section struts (cf. Fig. 12 b)) was proposed by Gent and Thomas (1963) who investigated the elastic behaviour and the tensile rupture of open-cell foamed elastic materials. In their analysis they assumed that the junctions of the struts are essentially undeformable compared with the struts themselves, and they referred to this as

dead volume. An equivalent model for closed-cell systems was proposed by Matonis (1964).

A different modelling approach is based on so-called polyhedral cells, Gibson and Ashby (1997). Typical representatives are the tetrakaidecahedron which is composed of 6 squares and 8 hexagons and the pentagonal dodecahedron which is build up of 12 regular pentagons (cf. Fig. 12 c)). It should be noted here that only the tetrakaidecahedron is a true space-filling body and the pentagonal dodecahedron does not pack properly unless distorted or combined with other structures.

Columnar models with a hexagonal cross-section are shown in Fig. 12 d) for a rod-like structure cancellous bone. This model is motivated from bones where the loading is mainly uniaxial (e.g. as the vertebrae) and the trabeculae often develop a columnar structure with cylindrical symmetry, Weaver and Chalmers (1966).

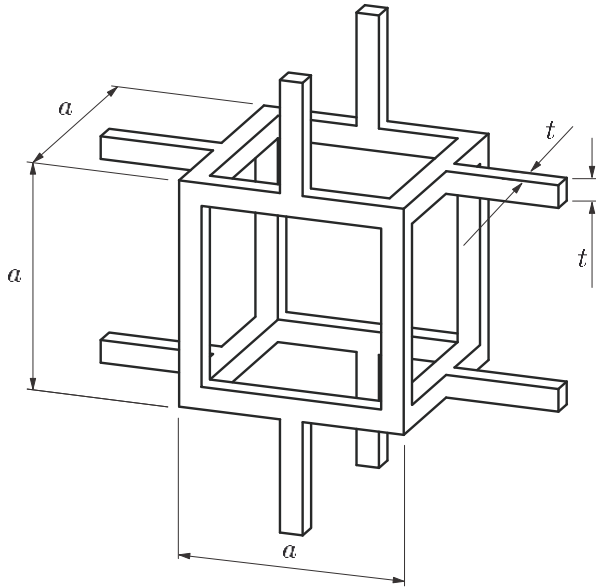


Figure 13. A cubic model for an open-cell foam with edge length a and edge thickness t ; cf. Gibson and Ashby (1997).

Cubic strut models have been extensively applied by Gibson and Ashby to different types of materials (bone, polymer and metal foams, e.g. Gibson and Ashby (1997); Gibson (1985); Gibson and Ashby (1982)) to investigate different macroscopic properties (e.g. elastic and plastic behaviour)

by consideration of beam bending of the struts. Anisotropy can be simply incorporated in this cubical model by uniformly stretching it in one of the principal direction, e.g. Kanakkanatt (1973). Some important properties will be derived in the following in order to highlight the modelling approach and the involved assumptions.

3.1 Relative Density

In the case of cellular materials, physical properties are commonly described as a function of their relative density, Gibson and Ashby (1997),

$$\varrho_{\text{rel}} = \frac{\varrho}{\varrho_{\text{So}}}, \quad (51)$$

where ϱ is the density of the cellular material and ϱ_{So} is the density of the solid material from which the cells are made. Such scaling relations or power-laws which account for the density dependency can be found e.g. in Ashby et al. (2000). Since the mass of the solid material and the mass of the cellular material itself are identical, the right-hand side (RHS) of Eq. (51) can be divided by the mass to express the relative density in terms of volumes as:

$$\varrho_{\text{rel}} = \frac{V_{\text{So}}}{V_{\text{UC}}}. \quad (52)$$

The value of Eq. (52) is also named the relative solid volume

$$V_{\text{rel,So}} = \frac{V_{\text{So}}}{V_{\text{UC}}}, \quad (53)$$

where V_{So} is the volume of the solid material (e.g. material of the cell walls, spheres, bond material etc.) and V_{UC} is the volume of the unit cell or of the entire specimen. In the case of open-cell cellular structures where a gas or liquid can flow through the open structure, the relative free volume

$$V_{\text{rel,free}} = \frac{V_{\text{free}}}{V_{\text{UC}}}, \quad (54)$$

is a useful quantity. In the context of damage mechanics, $V_{\text{rel,free}}$ is called the porosity, Lemaitre (1996). It should be noted that the volume of the unit cell (UC) in Eqs. (53) and (54) can be expressed as the sum of solid and free volume as:

$$V_{\text{UC}} = V_{\text{So}} + V_{\text{free}}. \quad (55)$$

Looking at Fig. 14 a), one can calculate the volume of the unit cell as $V_{\text{UC}} = a^3$. The free volume is composed of the inner cube with side length $(a - 2t)$ and the space between the struts. Thus, the entire free volume of the unit cell can be expressed as:

$$\begin{aligned}
 V_{\text{free}} &= (a - 2t)^3 + 6 \cdot (a - 2t)^2 t \\
 &= a^3 - 3a^2 2t + 3a 4t^2 - 8t^3 + 6t(a^2 - 4at + 4t^2) \\
 &= a^3 - 12at^2 + 16t^3.
 \end{aligned} \tag{56}$$

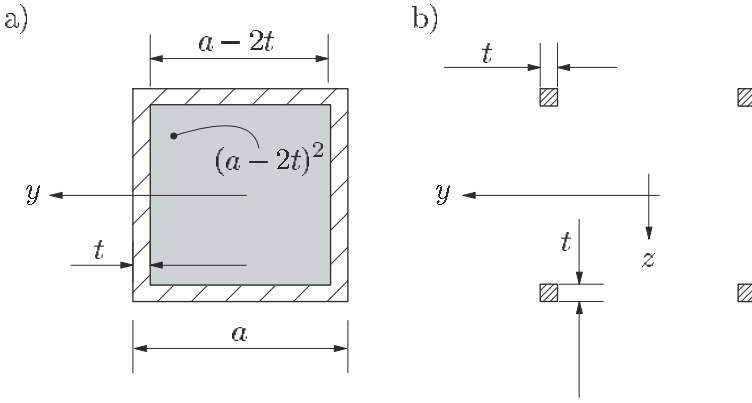


Figure 14. Different cross sections of the cubic unit cell shown in Fig. 13.

Combining Eqs. (53) and (55) gives the relative density as:

$$\rho_{\text{rel}} = 1 - \frac{a^3 - 12at^2 + 16t^3}{a^3} = 12 \cdot \frac{t^2}{a^2} - 16 \cdot \frac{t^3}{a^3}. \tag{57}$$

The last term (to power 3) in Eq. (57) can be neglected for small strut dimensions¹⁰ and the relative density of the unit cell shown in Fig. 13 can be approximated as:

$$\frac{\rho}{\rho_{\text{So}}} \sim \left(\frac{t}{a} \right)^2. \tag{58}$$

¹⁰Taking $t = 1$ mm and $a = 50$ mm, the ratio $(t/a)^2/(t/a)^3 = 50$ is obtained while $t = 1$ mm and $a = 100$ mm gives a ratio of 100.

3.2 Geometrical Moment of Inertia

The second moment of area I depends on the considered cross section area and its distribution. Assuming a quadratic cross section of side length t , the area moment is equal to $I_{y,\text{square}} = \frac{1}{12}t^4$ while the assumption of a circular cross section of diameter t gives $I_{y,\text{circle}} = \frac{1}{64}\pi t^4$. In addition, it must be considered that the cross sections of the struts are distributed from the centre of the unit cell (cf. Fig. 14 b)). Applying the parallel axis theorem and considering the arrangement as shown in Fig. 14 b), the area moment is given as:

$$I_y = 4 \cdot \left(\frac{1}{12}t^4 + \left\{ \frac{a}{2} - \frac{t}{2} \right\}^2 t^2 \right), \quad (59)$$

which is in Gibson and Ashby (1997) approximated as

$$I_y \sim t^4. \quad (60)$$

3.3 Young's Modulus

Determination of the macroscopic Young's modulus requires the realisation of e.g. a uniaxial tensile or compression test as schematically shown in Fig. 15 a). To model the deformation under the acting force F , a simple beam¹¹ under a centric point load is considered as shown in Fig. 15 b). Depending on the specific boundary condition at the left and right side of the beam, different maximum deflections can be obtained in the middle of the beam. In general, it can be concluded that the deflection is proportional to the factor $\frac{Fa^3}{E_{\text{So}}I_y}$. Considering that the stress can be calculated by the ratio of applied force and cross cross sectional area and strain by the ratio of deformation and initial length, Young's modulus can be approximated as:

$$E \sim \frac{\Delta\sigma}{\Delta\varepsilon} = \frac{\frac{F}{a^2}}{\frac{\delta}{a}} = \frac{F}{\delta a} = \frac{F E_{\text{So}} I_y}{a F a^3} = \frac{E_{\text{So}} I_y}{a^4} \stackrel{\text{Eq. (60)}}{=} E_{\text{So}} \frac{t^4}{a^4}. \quad (61)$$

Introducing a factor c_1 which accounts for the unknown boundary conditions and the other simplifications and considering Eq. (58), Young's modulus is expressed as:

$$E = c_1 \left(\frac{\rho}{\rho_{\text{So}}} \right)^2. \quad (62)$$

¹¹The shear contribution is neglected in this simplified approach.

It must be noted here that in spite of the single beam consideration as shown in Fig. 15 b), the force is related to the entire unit cell cross sectional area (a^2) and the deflection is related to the corresponding unit cell length of a , cf. Fig. 15 a). This is the key idea of this modelling approach: considering only a single beam (which makes only one twelfth of the entire cubic unit cell structure) for the deformation but referring the results to the entire unit cell. Parameter c_1 is obtained in Gibson and Ashby (1997) by *fitting* experimental values in the range¹² $c_1 \approx 0.3 \dots 1.0$.

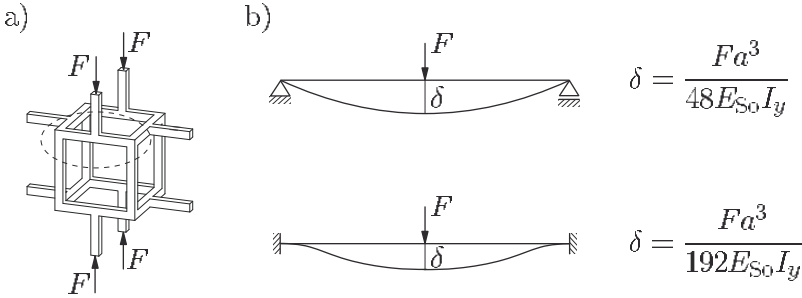


Figure 15. a) Unit cell loaded to determine Young's modulus (figure adopted from Gibson and Ashby (1997)); b) simplified beam models and maximum deflections.

3.4 Shear Modulus and Poisson's Ratio

The unit cell shown in Fig. 13 has orthotropic material behaviour where three mutually orthogonal axes are equal. Such a material is known as an orthotropic materials with cubic structure which can be described in the elastic range by generalised Hooke's law with three independent material constants. Doing the strong simplification and assuming that the cell reveals isotropic material behaviour, only two elastic constants are required to complete the set of material parameters. It should be noted here that the consideration of an orthotropic material with cubic structure *as isotropic* is as long valid as the loads (stresses) are applied in the same principal coordinate system¹³. Looking at Table 2, e.g. the shear modulus or Poisson's

¹²It is important to note here that theses values of c_1 are obtained by fitting experimental results and that the range does not have a theoretical basis!

¹³From Fig. 13 it follows one principal coordinate system may be aligned with the cell edges (i.e. three orthogonal "beams") or a second principal coordinate system can be aligned to the space diagonals which are connecting opposite corners of the cube.

ratio can be chosen as the second elastic parameter. The simplest experiment would be to apply a torque to a cylindrical specimen (cf. Fig. 16 a)) and to measure the shear strain. Such an experiment would result in a pure state of shear stress (cf. Fig. 16 b)) but requires in general non-standard experimental equipment, i.e. a torsion testing machine. In addition, the application of a torque or shear stresses to the cubic cell in a computational approach e.g. based on the finite element method is not so easy.

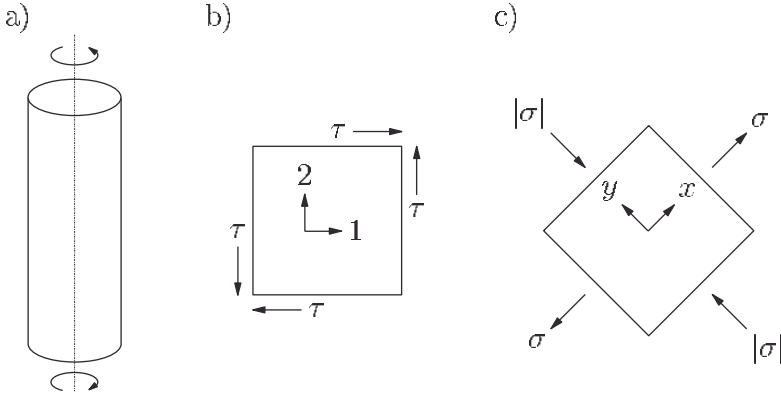


Figure 16. Pure shear stress states: a) torsion test; b) shear state based on shear stresses; c) shear state based on normal stresses (biaxial tension/compression).

A more elegant way especially for computational approaches can be based on the following strategy: Applying a principal axis transformation (PAT) to the pure state of shear stress shown in Fig. 16 b) gives due to rotation of the coordinate system by 45° a shear stress free state under pure normal stresses ($\pm\sigma$) as shown in Fig. 16 c). In the case of a regular structured material as shown in Fig. 17, it is important to consider the proper principal coordinate system for the application of the biaxial stress state $\pm\sigma$. If the uniaxial compression test is realised as shown in Fig. 15, i.e. normal force (stress) aligned to the cell struts, the pure shear state must be performed as shown in Fig. 17 d). Only then are normal stresses applied in the same principal coordinate system and the pure shear stress must be sketched as shown in Fig. 17 b). Would the normal stresses being applied under 45° as shown in Fig. 17 c), then the normal stresses of the uniaxial compression test and of the pure shear test would be applied in different principal coordinate systems and the classical relationship for isotropic materials, i.e. that the shear modulus results from Young's modulus and Pois-

son's ratio by $G = \frac{E}{2(1+\nu)}$, is no more valid. Thus, the orthotropic structure could no more serve as a model for an isotropic material. Considering the definition of shear modulus, i.e. ratio between shear stress increment and shear strain increment, a principal axis transformation must be applied to the shear stresses (and resulting shear strains) which are shown in Fig. 17 b) to get the relationship between shear and normal quantities. Assuming that the shear strains (cf. τ in Fig. 17 b)) are given, the required normal strains are obtained ($\sigma_x = \sigma_y = 0$) as:

$$\sigma_I = \frac{1}{2}(\sigma_x + \sigma_y) + \sqrt{\frac{1}{2} \cdot (\sigma_x + \sigma_y)^2 + \tau_{xy}} = \tau_{xy}, \quad (63)$$

$$\sigma_{II} = \frac{1}{2}(\sigma_x + \sigma_y) - \sqrt{\frac{1}{2} \cdot (\sigma_x + \sigma_y)^2 + \tau_{xy}} = -\tau_{xy}. \quad (64)$$

Based on the angle of rotation (45°), the relationship between the shear and normal strains are obtained as:

$$|\gamma_{xy}| = |2\varepsilon_I| = |2\varepsilon_{II}| = |\varepsilon_I| + |\varepsilon_{II}|. \quad (65)$$

Thus, the shear modulus can be approximated as

$$G = \frac{\Delta\tau}{\Delta\gamma} = \frac{|\tau_{xy}|}{|\gamma_{xy}|} = \frac{|\sigma_I|}{|2\varepsilon_I|} = \frac{|\sigma_{II}|}{|2\varepsilon_{II}|} \sim \frac{Fa}{a^2 2\delta} = \frac{Fa E_{So} I_y}{2a^2 F a^3} = \frac{E_{So} t^4}{2a^4}. \quad (66)$$

Introducing a factor c_2 which accounts for all the assumptions and simplifications and considering the definition of the relative density (cf. Eq. (58)), the macroscopic shear modulus of the simple cubic unit cell is given by:

$$\frac{G}{E_{So}} = c_2 \left(\frac{\rho}{\rho_{So}} \right)^2. \quad (67)$$

A value of $c_2 = \frac{3}{8}$ is given in Gibson and Ashby (1997) based on experimental values. Under the assumption that the cubic cell is considered as an isotropic material, the relationships between elastic constants (cf. Tab. 2) can be used to calculate Poisson's ratio based on Young's and shear modulus as:

$$\nu = \frac{E}{2G} - 1 = \frac{c_1 E_{So} \left(\frac{\rho}{\rho_{So}} \right)^2}{2c_2 E_{So} \left(\frac{\rho}{\rho_{So}} \right)^2} - 1 = \frac{c_1}{2c_2} - 1 = c_3. \quad (68)$$

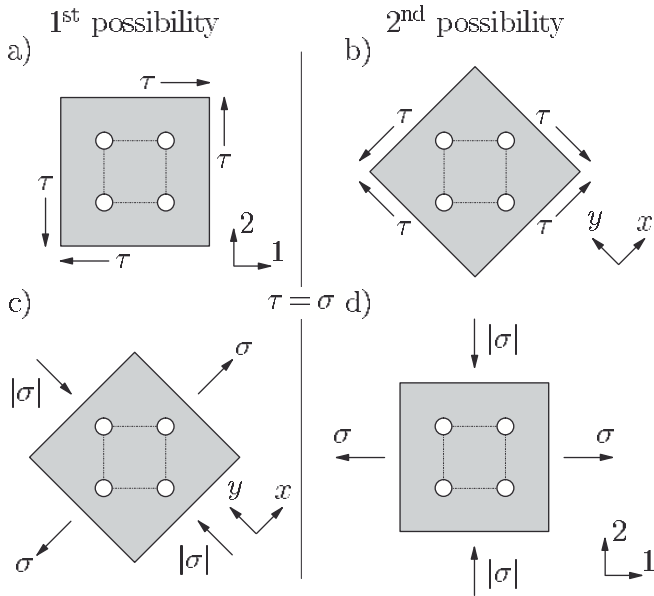


Figure 17. Orthotropic materials and principal axis respectively states of pure shear stress.

A value of $c_3 \approx 0.33$ is given in Gibson and Ashby (1997). However, the experimental values show a strong scattering and it is difficult to extrapolate a constant value for c_3 . The schematic realisation of the shear stress state due to biaxial tension/compression is shown in Fig. 18.

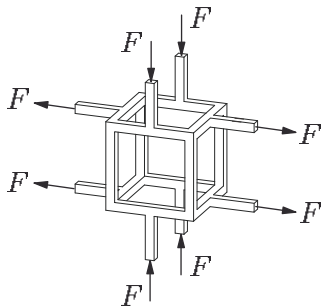


Figure 18. Cubic unit cell under biaxial tension/compression to realise a pure shear stress state; figure adopted from Gibson and Ashby (1997).

3.5 Yield Stress

The bending stress, i.e. the normal stress distribution, for the beams shown in Fig. 15 b) is generally given in the elastic range as:

$$\sigma(x, y) = \frac{M_y(x)}{I_y} \cdot z, \quad (69)$$

where z is the distance from the neutral fibre. In the elastic range, a linear distribution of the normal stress is obtained where the stress maximum is located at the outer fibre of the cross section, cf. Fig. 19 c). The bending moment distribution $M_y(x)$ is dependent on the boundary conditions and two cases are given in Fig. 19 a) and b) where for both cases the maximum is obtained in the middle of the beam. Assuming a squared cross section with side length t , the second moment of area is given by $I_y = \frac{1}{12}t^4$. However, the stress distribution over the height of the cross section looks in the plastic range different. Assuming ideal plastic material behaviour, i.e. no hardening, with a yield stress of k_{So} , the stress distribution shown in Fig. 19 d) is obtained in the fully plastic range. Now, the moment can be calculated based on the couple of resulting forces F_{res} . Multiplying the yield stress with the respective area, i.e. $\frac{t^2}{2}$, gives the resulting forces as $F_{\text{res}} = k_{\text{So}} \frac{t^2}{2}$ which are acting at a distance of $\frac{t}{2}$. Thus, the acting moment in the plastic range is equal to $M_y^{\text{P}} = \frac{t}{2} \cdot k_{\text{So}} \frac{t^2}{2} = \frac{k_{\text{So}} t^3}{4}$. Considering that the plastic moment is equal to the maximum moment, the macroscopic yield stress of the cubic unit cell is proportional to $\frac{k_{\text{So}} t^3}{4a^3}$. Introducing a factor c_4 which accounts for all the assumptions and simplifications and considering the definition of the relative density (cf. Eq. (58)), the macroscopic yield stress of the simple cubic unit cell is given by:

$$\frac{k}{k_{\text{So}}} = c_4 \left(\frac{\rho}{\rho_{\text{So}}} \right)^{\frac{3}{2}}, \quad (70)$$

where constant c_4 is given in Gibson and Ashby (1997) as 0.3.

The macroscopic yield stress k must be related to the external load by $k \sim \frac{F}{a^2}$ while the maximum moment is $M_{\text{max}} \sim Fa$, cf. Fig. 19 a). Combining these two relationships gives that the macroscopic yield stress k is proportional to $\frac{M_{\text{max}}}{a^3}$.

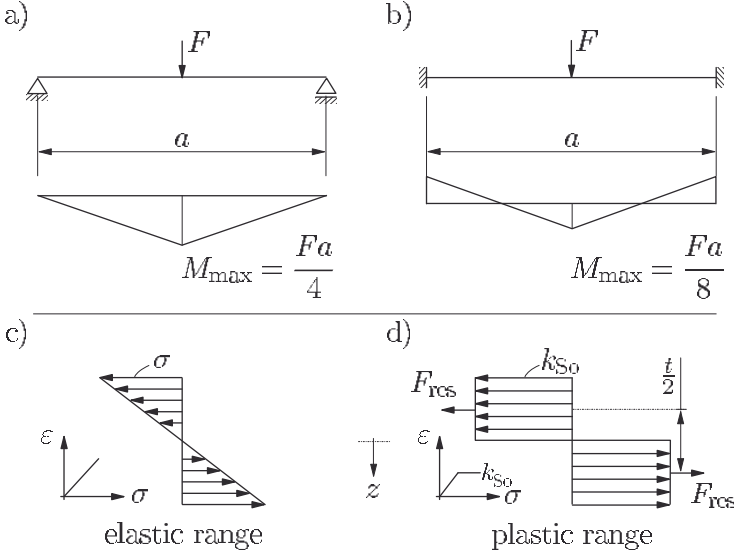


Figure 19. a)-b) Bending moment distributions; c)-d) stress distributions for elastic and ideal plastic material behaviour.

4 Procedures to Determine the Influence of the Hydrostatic Stress on the Yield Behaviour

In the following, two procedures will be introduced to determine the influence of the hydrostatic stress on the yield behaviour of cellular materials, Fiedler et al. (2006); Öchsner et al. (2006).

Under the classical assumption of small strains, the total strain increment $d\varepsilon_{ij}$ is assumed to be the sum of the elastic strain increment $d\varepsilon_{ij}^e$ and the plastic strain increment $d\varepsilon_{ij}^p$

$$d\varepsilon_{ij} = d\varepsilon_{ij}^e + d\varepsilon_{ij}^p, \quad (71)$$

where the elastic strain increment can be obtained from Hooke's law

$$\varepsilon_{ij} = \frac{1+\nu}{E} \left(\sigma_{ij} - \frac{\nu}{1+\nu} \delta_{ij} \sigma_{kk} \right) = C_{ijkl}^{-1} \sigma_{kl}, \quad (72)$$

and the plastic strain increment increment from the associated flow rule, cf. Eq. (40).

Application of the chain rule with respect to the stress tensor σ_{ij} gives the derivative of the general scalar yield criterion of the form $F = g_1(\alpha) \cdot f_1(J_1^\circ) + f_2(J_2') + f_3(k_s) = 0$ as:

$$\frac{\partial F}{\partial \sigma_{ij}} = \frac{\partial F}{\partial J_1^\circ} \frac{\partial J_1^\circ}{\partial \sigma_{ij}} + \frac{\partial F}{\partial J_2'} \frac{\partial J_2'}{\partial \sigma_{ij}} = g_1(\alpha) \frac{\partial f_1}{\partial J_1^\circ} \cdot \delta_{ij} + \frac{\partial f_2}{\partial J_2'} \cdot s_{ij}. \quad (73)$$

In the following, only the yield criterion according to Eq. (42) is considered. The derivative of F with respect to the stress tensor σ_{ij} is obtained as:

$$\frac{\partial F}{\partial \sigma_{ij}} = \frac{2}{27} \alpha^2 \cdot J_1^\circ \cdot \delta_{ij} + 1 \cdot s_{ij}. \quad (74)$$

If Hooke's law (72) is applied for the elastic component and the associated flow rule (40) for the plastic component, the complete stress-strain relationship for a material obeying a yield criterium of form (42) is expressed as

$$d\varepsilon_{ij} = d\lambda \left(\frac{2}{27} \alpha^2 \cdot J_1^\circ \cdot \delta_{ij} + 1 \cdot s_{ij} \right) + \frac{1+\nu}{E} \left(d\sigma_{ij} - \frac{\nu}{1+\nu} \delta_{ij} d\sigma_{kk} \right). \quad (75)$$

For a state of plane strain (no wall friction, principal directions are I, II, III), we have (cf. Fig. 20 a))

$$\sigma_I < 0 \quad \wedge \quad \varepsilon_I \neq 0, \quad (76)$$

$$\sigma_{II} \neq 0 \quad \wedge \quad \varepsilon_{II} = 0, \quad (77)$$

$$\sigma_{III} = 0 \quad \wedge \quad \varepsilon_{III} \neq 0, \quad (78)$$

while a state of uniaxial strain (cf. Fig. 20 b)) is characterised by:

$$\sigma_I < 0 \quad \wedge \quad \varepsilon_I \neq 0, \quad (79)$$

$$\sigma_{II} \neq 0 \quad \wedge \quad \varepsilon_{II} = 0, \quad (80)$$

$$\sigma_{III} \neq 0 \quad \wedge \quad \varepsilon_{III} = 0. \quad (81)$$

Using Eqs. (76)-(81) with the associated flow rule (40) and (75), the increments of the plastic strain are given by:

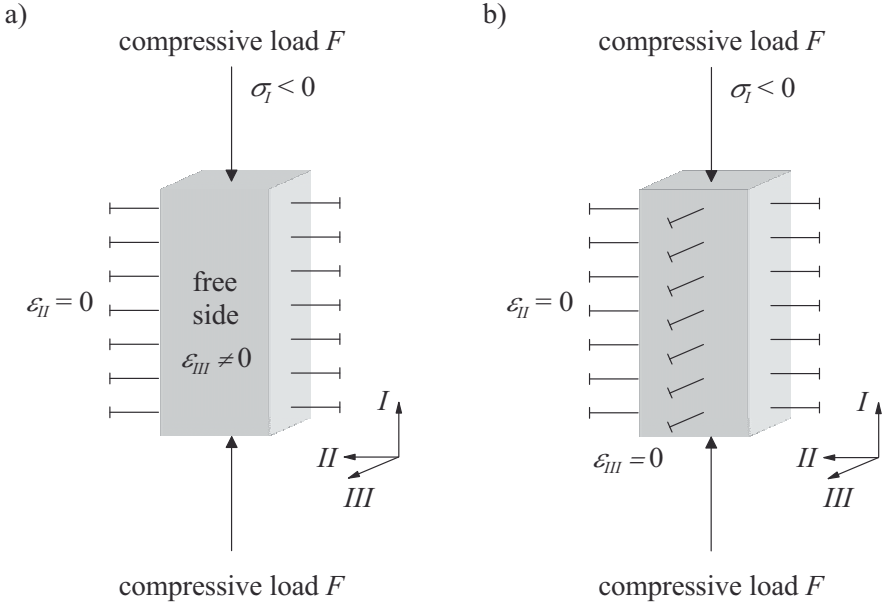


Figure 20. Schematic representation of a state of plane strain a) and a state of uniaxial strain b).

$$d\varepsilon_I^p = d\lambda \left[\frac{2}{27} \alpha^2 \cdot J_1^o + \cdot s_I \right], \quad (82)$$

$$d\varepsilon_{II}^p = d\lambda \left[\frac{2}{27} \alpha^2 \cdot J_1^o + \cdot s_{II} \right]. \quad (83)$$

Introducing the principal stresses in Eq. (82) and (83), one can obtain:

$$d\varepsilon_I^p = d\lambda \left[\frac{2}{27} \alpha^2 (\sigma_I + \sigma_{II} + \kappa \sigma_{III}) + \frac{1}{3} (2\sigma_I - \sigma_{II} - \kappa \sigma_{III}) \right], \quad (84)$$

$$d\varepsilon_{II}^p = d\lambda \left[\frac{2}{27} \alpha^2 (\sigma_I + \sigma_{II} + \kappa \sigma_{III}) + \frac{1}{3} (-\sigma_I + 2\sigma_{II} - \kappa \sigma_{III}) \right]. \quad (85)$$

where the parameter κ distinguishes between the plane strain and uniaxial strain case:

$$\kappa = \begin{cases} 0 : & \text{plane strain,} \\ 1 : & \text{uniaxial strain.} \end{cases} \quad (86)$$

Dividing Eq. (85) by (84) and rearranging for α yields to an equation of the form

$$\alpha^2 = \frac{9}{2} \cdot \frac{\sigma_I \left(1 + 2 \frac{d\varepsilon_{II}^p}{d\varepsilon_I^p}\right) - \sigma_{II} \left(2 + \frac{d\varepsilon_{II}^p}{d\varepsilon_I^p}\right) + \kappa \sigma_{III} \left(1 - \frac{d\varepsilon_{II}^p}{d\varepsilon_I^p}\right)}{(\sigma_I + \sigma_{II} + \kappa \sigma_{III}) \left(1 - \frac{d\varepsilon_{II}^p}{d\varepsilon_I^p}\right)}. \quad (87)$$

For small plastic strains (respectively at the beginning of the yielding), the quotient is $d\varepsilon_{II}^p/d\varepsilon_I^p \ll 1$ (respectively = 0) and the quantity α can be approximated by

$$\alpha \approx \frac{9}{2} \cdot \frac{\sigma_I - 2\sigma_{II} + \kappa \sigma_{III}}{\sigma_I + \sigma_{II} + \kappa \sigma_{III}}. \quad (88)$$

Evaluation of Eq. (87) requires the determination of the plastic strain increments $d\varepsilon_I^p$ and $d\varepsilon_{II}^p$. These increments can be determined from Eq. (71)

$$d\varepsilon_I^p = d\varepsilon_I - d\varepsilon_I^e, \quad (89)$$

$$d\varepsilon_{II}^p = \underbrace{d\varepsilon_{II}}_{=0} - d\varepsilon_{II}^e = -d\varepsilon_{II}^e, \quad (90)$$

$$d\varepsilon_{III}^p \begin{cases} \kappa = 1 : & \underbrace{d\varepsilon_{III}}_{=0} - d\varepsilon_{III}^e = -d\varepsilon_{III}^e, \\ \kappa = 0 : & d\varepsilon_{III} - d\varepsilon_{III}^e, \end{cases} \quad (91)$$

while the elastic strain increments can be obtained from Hooke's law (72).

In the hardening theory of plasticity, the hardening parameter in the yield criterion must be related to the experimental uniaxial stress-strain curve. To this end, one needs to define a stress variable, called effective stress, which is a function of the stresses and some strain variables, called effective strain, which is a function of the plastic strains, so that they can be plotted and used to correlate the test results obtained by different loading conditions, Chen and Han (1988). Since the effective stress should reduce to the stress σ_I in a uniaxial test, it follows that the function $f = g_1(\alpha) \cdot f_1(J_1^p) + f_2(J_2^p)$ must be some constant c multiplied by the effective stress σ_{eff} to some power n

$$f(\sigma_{ij}, \varepsilon_{\text{eff}}^{\text{p}}) = c \cdot \sigma_{\text{eff}}^n. \quad (92)$$

For the uniaxial test with $\sigma_{\text{eff}} = \sigma_I$, $\sigma_{II} = \sigma_{III} = 0$ and $J_1^o = \sigma_I$; $J_2' = \frac{1}{3}\sigma_I^2$, coefficient comparison, i.e.

$$g_1(\alpha)f_1(\sigma_I) + f_2(\sigma_I) = c \cdot \sigma_I^n, \quad (93)$$

gives the parameters c and n . The effective plastic strain increment $d\varepsilon_{\text{eff}}^{\text{p}}$ can be defined in terms of the plastic work per unit volume in the form

$$dw^{\text{p}} = \sigma_{\text{eff}} d\varepsilon_{\text{eff}}^{\text{p}} = \sigma_{ij} d\varepsilon_{ij}^{\text{p}} = \sigma_I d\varepsilon_I^{\text{p}} + \sigma_{II} d\varepsilon_{II}^{\text{p}} + \kappa \sigma_{III} d\varepsilon_{III}^{\text{p}}. \quad (94)$$

It follows from Eq. (94) using Eq. (72) and the results from Eq. (93) that in the case of the axial compaction the effective plastic strain increment is given by the following equation

$$d\varepsilon_{\text{eff}}^{\text{p}} = \frac{1}{\sigma_{\text{eff}}} \left[\sigma_I d\varepsilon_I^{\text{p}} - \sigma_{II} \frac{1}{E} (-\nu d\sigma_I + d\sigma_{II} - 2\kappa \nu d\sigma_{III}) - \kappa \sigma_{III} \frac{1}{E} (-\nu d\sigma_I - \nu d\sigma_{II} + d\sigma_{III}) \right]. \quad (95)$$

It should be mentioned here that the elastic range is independent of the yield criterion and the following incremental relation can be derived for the uniaxial strain case:

$$\frac{d\sigma_I}{d\varepsilon_I} = K + \frac{4}{3}G, \quad (96)$$

where $K + \frac{4}{3}G$ is known as the constrained modulus. The last equation may play an important role for the experimental determination of the elastic material parameters of cellular materials. If Young's modulus E is obtained from a uniaxial tensile or compression test, then Poisson's ratio ν can be calculated from the slope $\frac{d\sigma_I}{d\varepsilon_I}$ in the elastic range of an axial compression test. Using the well-known relationships between elastic constants (e.g. Chen and Han (1988)), Eq. (96) can be rewritten as:

$$\frac{d\sigma_I}{d\varepsilon_I} = \frac{E}{3(1-2\nu)} + \frac{4}{3} \cdot \frac{E}{2(1+\nu)}. \quad (97)$$

The plain strain test reveals also an interesting relationship for the elastic range. Introducing $\varepsilon_{II}^e = 0$ and $\sigma_{III} = 0$ into Hooke's law, a new equation for the determination of Poisson's ratio is obtained:

$$\nu = \frac{\sigma_{II}}{\sigma_I}. \quad (98)$$

It should be highlighted here that the application of Eq. (98) requires only force measurement to determine the stresses and is independent of any strain measurement.

5 Implementation of New Constitutive Equations into Commercial Finite Element Codes

5.1 One-Dimensional Drucker-Prager Yield Condition

The general three-dimensional version of the Drucker-Pragers yield condition, cf. Chen and Han (1988), can be simplified under the conditions of a uniaxial stress (and strain) state with $J_1^o \rightarrow \sigma$ and $J_2' \rightarrow \frac{1}{3} \cdot \sigma^2$ to:

$$F(\sigma, \kappa) = \alpha(\kappa)\sqrt{3}\sigma_x + |\sigma_x| - (1 + \sqrt{3}\alpha(\kappa))k_t(\kappa) \leq 0. \quad (99)$$

Let us assume in the following the case of strain hardening, i.e. $\kappa = |\varepsilon_x^{\text{pl}}|$, and that the parameter α is constant. Thus, Eq. (99) can be written¹⁴ in the following form:

$$F(\sigma, \kappa) = \alpha\sqrt{3}\sigma + |\sigma| - (1 + \sqrt{3}\alpha)k_t(\kappa) \leq 0. \quad (100)$$

A graphical representation of the yield condition in the $\sqrt{3J_2'}-J_1^o$ -space is shown in Fig. 21 where the two possible loading conditions, i.e. uniaxial tension ($+\sigma$) and uniaxial compression ($-\sigma$), are shown. To perform a uniaxial tensile test with increasing load would mean to move from the origin of the coordinate system along the space diagonal. At the end of the elastic range, this space diagonal crosses the initial yield surface. It can be clearly seen that yielding under compression initiates much later than under tensile conditions. The initial yield surface is represented in the $\sqrt{3J_2'}-J_1^o$ -space as a straight line with the slope $-\sqrt{3}\alpha$. Under the assumption that the parameter α is constant, the subsequent yield locus is obtained by parallel movement of the initial yield surface. Should α be a function e.g. of the equivalent plastic strain, then the inclination of the subsequent yield surface changes compared to the initial yield surface.

Let us assume in addition that the flow rule is based on the yield condition F , i.e. an associated flow rule, which gives based on Eq. (100):

$$d\varepsilon^{\text{pl}} = d\lambda \frac{\partial F}{\partial \sigma} = d\lambda r = d\lambda(\alpha\sqrt{3} + \text{sgn}(\sigma)), \quad (101)$$

¹⁴Let us abandon in the following the index x , i.e. $\sigma_x \rightarrow \sigma$.

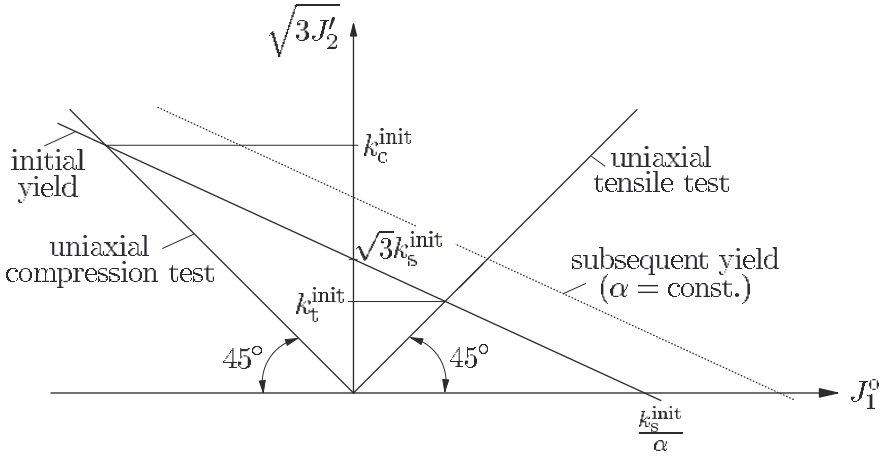


Figure 21. Graphical representation of the Drucker-Prager yield condition.

where sgn is the signum function which takes the following values:

$$\text{sgn}(\sigma) = \begin{cases} -1 & \text{if } \sigma < 0 \\ 0 & \text{if } \sigma = 0 \\ +1 & \text{if } \sigma > 0 \end{cases} . \quad (102)$$

Furthermore, we consider only the case of linear isotropic hardening:

$$k(\kappa) = k_t^{\text{init}} + E^{\text{pl}}\kappa = k_t^{\text{init}} + E^{\text{pl}}|\varepsilon^{\text{pl}}| . \quad (103)$$

The evolution of the hardening parameter κ is obtained by combining Eq. (103) with the flow rule according to (101) to give:

$$d\kappa = d|\varepsilon^{\text{pl}}| = d\lambda h = d\lambda|\alpha\sqrt{3} + \text{sgn}(\sigma)| \quad (104)$$

The total differential of the yield condition (100) is obtained as:

$$dF = \left(\frac{\partial F}{\partial \sigma} \right) d\sigma + \left(\frac{\partial F}{\partial \kappa} \right) d\kappa . \quad (105)$$

Following e.g. the procedure given in Öchsner (2003), the consistency parameter takes the form

$$d\lambda = \frac{E \cdot \frac{\partial F}{\partial \sigma}}{\left(\frac{\partial F}{\partial \sigma} \right)^2 E - \left(\frac{\partial F}{\partial \sigma} \right) \left(\frac{\partial F}{\partial \kappa} \right)} \quad (106)$$

and the elasto-plastic modulus can be calculated according to:

$$E^{\text{elpl}} = \frac{d\sigma}{d\varepsilon} = \frac{E \cdot \left(\frac{\partial F}{\partial \kappa} \right) \cdot \left| \frac{\partial F}{\partial \sigma} \right|}{\left(\frac{\partial F}{\partial \kappa} \right) \cdot \left| \frac{\partial F}{\partial \sigma} \right| - E \cdot \left(\frac{\partial F}{\partial \sigma} \right)^2}, \quad (107)$$

where

- $\frac{\partial F}{\partial \sigma} = \alpha\sqrt{3} + \text{sgn}(\alpha),$
- $\left| \frac{\partial F}{\partial \sigma} \right| = |\alpha\sqrt{3} + \text{sgn}(\alpha)|,$
- $\frac{\partial F}{\partial \kappa} = -(\alpha\sqrt{3} + 1) \cdot E^{\text{pl}}.$

5.2 Integration of the Constitutive Equations

Compared to a pure linear-elastic finite element simulation, the computation of plastic material behaviour can no more be performed in a single step since there is no more a unique relationship between stress and strain¹⁵. The load must be incrementally applied and a nonlinear system of equations has to be solved in each increment (Newton-Raphson algorithm). Based on the stress state at the end of the previous increment (n) and the given strain increment $\Delta\varepsilon_n$, the field variables (e.g. stress σ_{n+1}) must be calculated for each increment ($n+1$) at each integration point (Gauss point), cf. Fig. 22.

To this end, it is required to numerically integrate the differential form of the explicit constitutive behaviour. Explicit integration schemes, such as the Forward-Euler procedure, are however inaccurate and under circumstances instable since a global error can be accumulated, Crisfield (2001). Instead of explicit integration schemes, so-called predictor-corrector algorithm (cf. Fig. 23) which consists of an initial explicit elastic-predictor step, involving a deviation away from the yield surface, and a plastic-corrector step (implicit correction) which returns the stress to the updated yield surface.

Under the assumption of pure linear-elastic material behaviour, an elastic predictor calculates a trial state (so-called trial stress):

$$\sigma_{n+1}^{\text{trial}} = \sigma_n + \underbrace{E\Delta\varepsilon_n}_{\text{predictor } \Delta\sigma_n^{\text{el}}} \quad (108)$$

¹⁵Only in the special cases of proportional or radial loading, the unique relationship is maintained. In the general case with six stress and six strain components, there is only a unique relationship between the effective stress and effective plastic strain given.

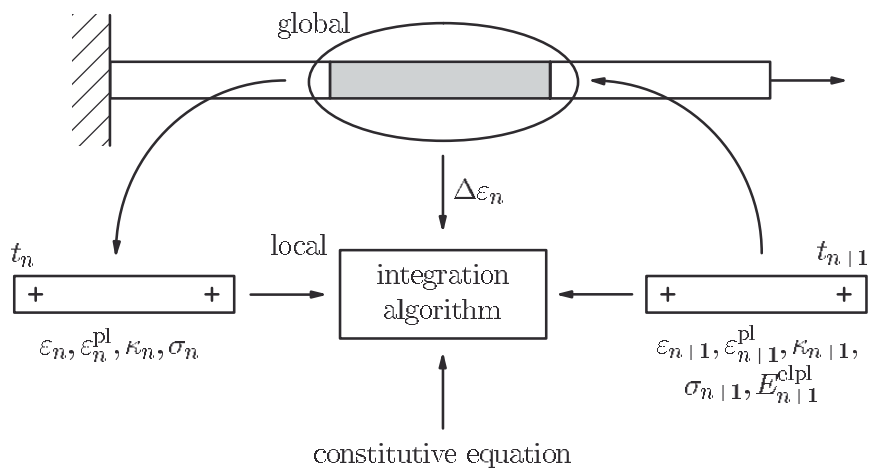


Figure 22. Schematic representation of the integration algorithm for plastic material behaviour in the FEM.

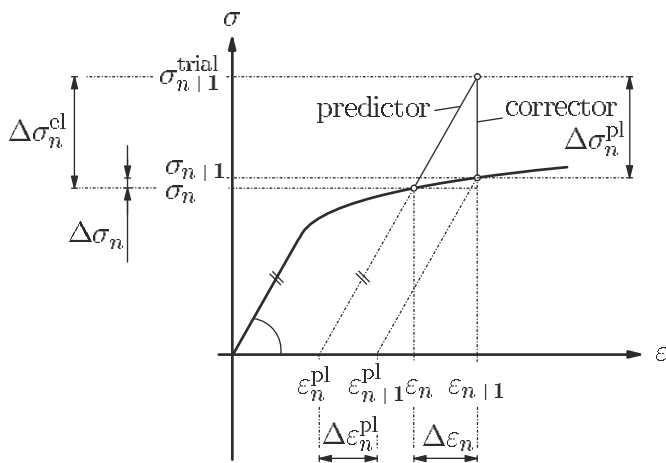


Figure 23. Schematic representation of the integration algorithm in the stress-strain diagram.

The corresponding hardening state corresponds to the state of the previous increment. Thus, it is assumed that the load step is pure elastic, i.e. without plastic deformation and thus, without hardening:

$$\kappa_{n+1}^{\text{trial}} = \kappa_n. \quad (109)$$

Dependent on the location of the trial state in the stress space, the yield condition allows to distinguish two elemental states:

- a) The stress state is in the elastic range or on the yield surface (valid stress state):

$$F(\sigma_{n+1}^{\text{trial}}, \kappa_{n+1}^{\text{trial}}) \leq 0. \quad (110)$$

In this case, the trial state can be taken as the new stress and hardening state since it corresponds to the real state:

$$\sigma_{n+1} = \sigma_{n+1}^{\text{trial}}, \quad (111)$$

$$\kappa_{n+1} = \kappa_{n+1}^{\text{trial}}. \quad (112)$$

Finally, one can move to the next increment.

- b) The stress state is outside the yield surface (invalid stress state, cf. Fig. 22):

$$F(\sigma_{n+1}^{\text{trial}}, \kappa_{n+1}^{\text{trial}}) > 0. \quad (113)$$

In this case, the second part of the algorithm calculates from the invalid state a valid on the yield surface ($F(\sigma_{n+1}, \kappa_{n+1}) = 0$). The required stress difference

$$\Delta\sigma^{\text{pl}} = \sigma_{n+1}^{\text{trial}} - \sigma_{n+1} \quad (114)$$

is called plastic corrector.

Alternative expressions for the predictor-corrector method are return-mapping or catching-up. In the following, we are going to have a closer look on the return mapping procedure. Detailed descriptions can be found in Simo and Hughes (1998); Belytschko et al. (2000); Crisfield (2001, 2000); de Souza Neto et al. (2008).

The difference between the initial and target state (stress increment)

$$\Delta\sigma_n = \sigma_{n+1} - \sigma_n \quad (115)$$

results according to Hooke's law from the elastic part of the strain increment which can be obtained as the difference between the total strain increment and the plastic part as:

$$\Delta\sigma_n = E\Delta\varepsilon_n^{\text{el}} = E(\Delta\varepsilon_n - \Delta\varepsilon_n^{\text{pl}}). \quad (116)$$

The total strain increment can be expressed with the trial stress state according to Fig. 22 as:

$$\Delta\varepsilon_n = \varepsilon_{n+1} - \varepsilon_n = E^{-1}(\sigma_{n+1}^{\text{trial}} - \sigma_n). \quad (117)$$

Introducing the last equation and the flow rule¹⁶ in Eq. (116), the final state σ_{n+1} can be obtained in dependence of the trial stress state $\sigma_{n+1}^{\text{trial}}$ as:

$$\sigma_{n+1} = \sigma_{n+1}^{\text{trial}} - \Delta\lambda_{n+1}E \cdot \left(\alpha\sqrt{3} + \text{sgn}(\sigma) \right). \quad (118)$$

Depending on the point for the evaluation of function $\text{sgn}(\sigma)$, different methods can be obtained (cf. Tab. 7) to calculate the initial value for the plastic corrector and to iteratively determine the final state. To obtain the initial value for the plastic corrector, $\text{sgn}(\sigma)$ can be either evaluated in the trial stress state (Backward-Euler) or on the yield surface (Forward-Euler; for the transition from the elastic to the plastic range, this is the initial yield stress). Evaluating the function $\text{sgn}(\sigma)$ during the iteration in the final state ensures that the normal rule is fulfilled in the final state. For this implicit Backward-Euler algorithm (also known as Closest-Point-Projection (CPP)), Simo and Hughes (1998), it is required to calculate higher order derivatives in the general three-dimensional case. For the so-called cutting-plane algorithm, Simo and Ortiz (1985), the function $\text{sgn}(\sigma)$ is evaluated for the stress state of the i th iteration step. The normal rule will be not exactly fulfilled in the final state, however there is no need to calculate higher derivatives. The function $\text{sgn}(\sigma)$ is for the midpoint rule in both states, i.e. the final state and on the yield surface, evaluated and equally weighted. The evaluation of the $\text{sgn}(\sigma)$ only on the yield surface yields the semi-implicit Backward-Euler algorithm, Moran et al. (1990), which requires only the determination of first order derivatives. Considering the dependency of the yield condition on the hardening variable, requires the application of a further equation to describe the hardening. The following incremental relationship for the determination of the hardening variable can be obtained from the evolution equation (104):

$$\kappa_{n+1} = \kappa_n + \Delta\lambda_{n+1}|\alpha\sqrt{3} + \text{sgn}(\sigma_{n+1})|. \quad (119)$$

Finally, it should be noted that three of the integration procedures in Tab. 7 can be summarised based on the following equation:

$$\sigma_{n+1} = \sigma_{n+1}^{\text{trial}} - \Delta\lambda_{n+1}E \left([1 - \eta][\alpha\sqrt{3} + \text{sgn}(\sigma_n)] + \eta[\alpha\sqrt{3} + \text{sgn}(\sigma_{n+1})] \right), \quad (120)$$

¹⁶At this point, we change the notation from $d\lambda$ to $\Delta\lambda$.

Table 7. Summary of predictor-corrector algorithms.

State for evaluation of $\text{sgn}(n)$	σ_{n+1} acc. Eq. (118)
Initial value for corrector	
trial state	$\sigma_{n+1}^{\text{trial}} - \Delta\lambda_{n+1}E(\alpha\sqrt{3} + \text{sgn}(\sigma_{n+1}^{\text{trial}}))$
on the flow curve	$\sigma_{n+1}^{\text{trial}} - \Delta\lambda_{n+1}E(\alpha\sqrt{3} + \text{sgn}(\sigma_n))$
during iteration	
final state (full.-impl. Backward-Euler alg.) (Closest-Point-Projection)	$\sigma_{n+1}^{\text{trial}} - \Delta\lambda_{n+1}E(\alpha\sqrt{3} + \text{sgn}(\sigma_{n+1}))$
on flow curve (semi-impl. Backward-Euler alg.)	$\sigma_{n+1}^{\text{trial}} - \Delta\lambda_{n+1}E(\alpha\sqrt{3} + \text{sgn}(\sigma_n))$
final state and on flow curve (midpoint-rule)	$\sigma_{n+1}^{\text{trial}} - \Delta\lambda_{n+1}E\frac{1}{2} \times$ $(2\alpha\sqrt{3} + \text{sgn}(\sigma_{n+1}) + \text{sgn}(\sigma_n))$
stress state of i th iteration step (Cutting-Plane algorithm)	$\sigma_{n+1}^{\text{trial}} - \Delta\lambda_{n+1}E(\alpha\sqrt{3} + \text{sgn}(\sigma^{(i)}))$

where parameter η takes then values of 1, 0 or $\frac{1}{2}$.

5.3 Mathematical Derivation of the Fully Implicit Backward Euler Algorithm

This return mapping algorithm calculates the closest point projection of the trial state onto the yield surface in the complementary energy. It is not as one could assume the calculation of the geometrical closest point. The basis of this procedure is the assumption that the plastic work takes its maximum for a given strain state. Together with the elemental request that the final stress state must be located on the yield surface, the return mapping algorithm can be understood as the solution of a optimisation problem (maximum of plastic work) under a convex constraint (final stress state must be on the yield surface). The procedure is implicit for the calculation of the

function $\text{sgn}(\sigma)$ since the evaluation is performed for the final state $n + 1$. Closest point projection (CPP) or fully implicit Backward Euler algorithm are common expressions for this stress update procedure. In the final state ($n + 1$), the equations

$$\sigma_{n+1} = \sigma_{n+1}^{\text{trial}} - \Delta\lambda_{n+1}E \cdot \left(\alpha\sqrt{3} + \text{sgn}(\sigma_{n+1}) \right), \quad (121)$$

$$\kappa_{n+1} = \kappa_n + \Delta\lambda_{n+1} \cdot \left| \alpha\sqrt{3} + \text{sgn}(\sigma_{n+1}) \right|, \quad (122)$$

$$F = F(\sigma_{n+1}, \kappa_{n+1}) = 0, \quad (123)$$

are fulfilled. However, a residuum r remains for each of these equation outside the final state:

$$\begin{aligned} r_\sigma(\sigma, \kappa, \Delta\lambda) &= \sigma - \sigma_{n+1}^{\text{trial}} + \Delta\lambda E \cdot \left(\alpha\sqrt{3} + \text{sgn}(\sigma) \right) \neq 0 \quad \text{or} \\ &= E^{-1}\sigma - E^{-1}\sigma_{n+1}^{\text{trial}} + \Delta\lambda \cdot \left(\alpha\sqrt{3} + \text{sgn}(\sigma) \right) \neq 0, \end{aligned} \quad (124)$$

$$\begin{aligned} r_\kappa(\kappa, \Delta\lambda) &= \kappa - \kappa_n - \Delta\lambda \cdot \left| \alpha\sqrt{3} + \text{sgn}(\sigma) \right| \neq 0 \quad \text{or} \\ &= -\kappa + \kappa_n + \Delta\lambda \cdot \left| \alpha\sqrt{3} + \text{sgn}(\sigma) \right| \neq 0, \end{aligned} \quad (125)$$

$$r_F(\sigma, \kappa) = F(\sigma, \kappa) = \alpha\sqrt{3}\sigma + |\sigma| - (1 + \sqrt{3}\alpha)k(\kappa) \neq 0. \quad (126)$$

Thus, the final stress and hardening state is the root of a vector function $\mathbf{m}$ which is composed of the residual functions. Furthermore, it is useful to collect all variables in a single variable vector $\mathbf{v}$:

$$\mathbf{m}(\mathbf{v}) \in (\mathbb{R}^3 \rightarrow \mathbb{R}^3) = \begin{Bmatrix} r_\sigma(\mathbf{v}) \\ r_\kappa(\mathbf{v}) \\ r_F(\mathbf{v}) \end{Bmatrix}, \quad \vec{v} = \begin{Bmatrix} \sigma \\ \kappa \\ \Delta\lambda \end{Bmatrix}. \quad (127)$$

The root is found by a Newton iteration (iteration index: i):¹⁷

$$\mathbf{v}^{(i+1)} = \mathbf{v}^{(i)} - \left[\frac{d\mathbf{m}}{d\mathbf{v}}(\mathbf{v}^{(i)}) \right]^{-1} \mathbf{m}(\mathbf{v}^{(i)}), \quad (128)$$

where

$$\mathbf{v}^{(0)} = \begin{Bmatrix} \sigma^{(0)} \\ \kappa^{(0)} \\ \Delta\lambda^{(0)} \end{Bmatrix} = \begin{Bmatrix} \sigma_{n+1}^{\text{trial}} \\ \kappa_n \\ 0 \end{Bmatrix} \quad (129)$$

¹⁷For a one-dimensional function $f(x)$, the Newton procedure takes normally the following form: $x^{(i+1)} = x^{(i)} - \left[\frac{df}{dx}(x^{(i)}) \right]^{-1} \times f(x^{(i)})$.

can be used as an initial value. The Jacobian matrix $\frac{\partial \mathbf{m}}{\partial \mathbf{v}}$ of the residual functions is obtained from the partial derivatives of Eqs. (124) to (126) as:

$$\begin{aligned} \frac{\partial \mathbf{m}}{\partial \mathbf{v}}(\sigma, \kappa, \Delta\lambda) &= \begin{bmatrix} \frac{\partial r_\sigma}{\partial \sigma} & \frac{\partial r_\sigma}{\partial \kappa} & \frac{\partial r_\sigma}{\partial \Delta\lambda} \\ \frac{\partial r_\kappa}{\partial \sigma} & \frac{\partial r_\kappa}{\partial \kappa} & \frac{\partial r_\kappa}{\partial \Delta\lambda} \\ \frac{\partial r_F}{\partial \sigma} & \frac{\partial r_F}{\partial \kappa} & \frac{\partial r_F}{\partial \Delta\lambda} \end{bmatrix} = \\ &= \begin{bmatrix} E^{-1} & 0 & (\alpha\sqrt{3} + \text{sgn}(\sigma)) \\ 0 & -1 & |\alpha\sqrt{3} + \text{sgn}(\sigma)| \\ (\alpha\sqrt{3} + \text{sgn}(\sigma)) & -(\alpha\sqrt{3} + 1)E^{\text{pl}} & 0 \end{bmatrix}. \end{aligned} \quad (130)$$

In addition to the fulfilment of the plasticity equations (121) to (123) in each integration point, the global equilibrium must be fulfilled. To be able to apply the Newton iteration for this task, it is required even for small strains in the general three-dimensional case to derive the appropriate elasto-plastic tangent modulus¹⁸, Wriggers (2001). The consistent elasto-plastic modulus results for the one-dimensional case as:

$$E_{n+1}^{\text{elpl}} = \frac{\partial \sigma_{n+1}}{\partial \varepsilon_{n+1}} = \frac{\partial \Delta \sigma_n}{\partial \varepsilon_{n+1}}. \quad (131)$$

The inversion of the Jacobian matrix $\frac{\partial \mathbf{m}}{\partial \mathbf{v}}$ which has to be evaluated in the converged state of the above mentioned Newton iteration gives

$$\left[\left(\frac{\partial \mathbf{m}}{\partial \mathbf{v}} \right)_{n+1} \right]^{-1} = \begin{bmatrix} \tilde{m}_{11} & \tilde{m}_{12} & \tilde{m}_{13} \\ \tilde{m}_{21} & \tilde{m}_{22} & \tilde{m}_{23} \\ \tilde{m}_{31} & \tilde{m}_{32} & \tilde{m}_{33} \end{bmatrix}_{n+1}, \quad (132)$$

with the following matrix elements:

¹⁸ Alternative notation is consistent tangent modulus or algorithmic modulus.

$$\begin{aligned}
\tilde{m}_{11} &= |\alpha\sqrt{3} + \text{sgn}(\sigma)|(\alpha\sqrt{3} + 1)E^{\text{pl}} \cdot D, \\
\tilde{m}_{12} &= -(\alpha\sqrt{3} + \text{sgn}(\sigma))(\alpha\sqrt{3} + 1)E^{\text{pl}} \cdot D, \\
\tilde{m}_{13} &= (\alpha\sqrt{3} + \text{sgn}(\sigma)) \cdot D, \\
\tilde{m}_{21} &= |\alpha\sqrt{3} + \text{sgn}(\sigma)|(\alpha\sqrt{3} + 1) \cdot D, \\
\tilde{m}_{22} &= -(\alpha\sqrt{3} + \text{sgn}(\sigma))^2 \cdot D, \\
\tilde{m}_{23} &= -|\alpha\sqrt{3} + \text{sgn}(\sigma)|E^{-1} \cdot D, \\
\tilde{m}_{31} &= (\alpha\sqrt{3} + \text{sgn}(\sigma)) \cdot D, \\
\tilde{m}_{32} &= (\alpha\sqrt{3} + 1)E^{-1}E^{\text{pl}} \cdot D, \\
\tilde{m}_{33} &= -E^{-1} \cdot D,
\end{aligned} \tag{133}$$

where

$$D = \frac{E}{|\alpha\sqrt{3} + \text{sgn}(\sigma)| \cdot (\alpha\sqrt{3} + 1) \cdot E^{\text{pl}} + (\alpha\sqrt{3} + \text{sgn}(\sigma))^2 \cdot E}. \tag{134}$$

The consistent elasto-plastic modulus can be obtained from (132) as:

$$E_{n+1}^{\text{elpl}} = \tilde{m}_{11}. \tag{135}$$

For the special case of linear hardening, i.e. $\frac{\partial k}{\partial \kappa} = E^{\text{pl}} = \text{const.}$, Eq. (128) does not require an iterative solution and the required solution vector $\mathbf{v}_{n+1}$ is directly obtained with the initial value (129) as:

$$\mathbf{v}_{n+1} = \mathbf{v}^{(0)} - \left[\frac{d\mathbf{m}}{d\mathbf{v}}(\mathbf{v}^{(0)}) \right]^{-1} \vec{m}(\mathbf{v}^{(0)}), \tag{136}$$

or with components:

$$\begin{Bmatrix} \sigma_{n+1} \\ \kappa_{n+1} \\ \Delta\lambda_{n+1} \end{Bmatrix} = \begin{Bmatrix} \sigma_{n+1}^{\text{trial}} \\ \kappa_n \\ 0 \end{Bmatrix} - \begin{bmatrix} \tilde{m}_{11} & \tilde{m}_{12} & \tilde{m}_{13} \\ \tilde{m}_{21} & \tilde{m}_{22} & \tilde{m}_{23} \\ \tilde{m}_{31} & \tilde{m}_{32} & \tilde{m}_{33} \end{bmatrix}_0 \begin{Bmatrix} 0 \\ 0 \\ F_{n+1}^{\text{trial}} \end{Bmatrix}. \tag{137}$$

The third equation of (137) gives the consistency parameter in the case of linear hardening as:

$$\Delta\lambda_{n+1} = \frac{F_{n+1}^{\text{trial}}}{|\alpha\sqrt{3} + \text{sgn}(\sigma)| \cdot (\alpha\sqrt{3} + 1) \cdot E^{\text{pl}} + (\alpha\sqrt{3} + \text{sgn}(\sigma))^2 \cdot E}. \tag{138}$$

The first equation of (137) gives the stress in the final state as:

$$\sigma_{n+1} = \sigma_{n+1}^{\text{trial}} - D \cdot \left(\alpha\sqrt{3} + \text{sgn}(\sigma_{n+1}^{\text{trial}}) \right) \cdot F_{n+1}^{\text{trial}}. \quad (139)$$

And the second equation of (137) yields the isotropic hardening variable in the final state as:

$$\kappa_{n+1} = \kappa_n + D \cdot \left| \alpha\sqrt{3} + \text{sgn}(\sigma_{n+1}^{\text{trial}}) \right| \cdot F_{n+1}^{\text{trial}}. \quad (140)$$

Based on the flow rule, the plastic strain can be calculated as:

$$\varepsilon_{n+1}^{\text{pl}} = \varepsilon_n^{\text{pl}} + \Delta\lambda_{n+1} \left| \alpha\sqrt{3} + \text{sgn}(\sigma_{n+1}) \right|, \quad (141)$$

and the consistent elasto-plastic modulus is obtained from Eq.(135). Finally, it should be recalled that

$$\sigma_{n+1}^{\text{trial}} = \sigma_n + E \cdot \Delta\varepsilon_n, \quad (142)$$

$$F_{n+1}^{\text{trial}} = \alpha\sqrt{3}\sigma_{n+1}^{\text{trial}} + |\sigma_{n+1}^{\text{trial}}| - (\alpha\sqrt{3} + 1)(k_t^{\text{init}} + E^{\text{pl}}\varepsilon_n^{\text{pl}}). \quad (143)$$

At the end of this section, the computation steps of the closest point algorithm are presented in a compact form:

I. Compute Elastic Trial State

$$\begin{aligned} \sigma_{n+1}^{\text{trial}} &= \sigma_n + E\Delta\varepsilon_n \\ \kappa_{n+1}^{\text{trial}} &= \kappa_n \end{aligned}$$

II. Check Yield Condition

IF $F(\sigma_{n+1}^{\text{trial}}, \kappa_{n+1}^{\text{trial}}) \leq 0$ THEN

$$\begin{aligned} \sigma_{n+1} &= \sigma_{n+1}^{\text{trial}} \\ \kappa_{n+1} &= \kappa_{n+1}^{\text{trial}} \end{aligned}$$

ELSE

return mapping ($\Rightarrow$ step III)

ENDIF

III. Return Mapping

Initial values:

$$\mathbf{v}^{(0)} = \begin{Bmatrix} \sigma^{(0)} \\ \kappa^{(0)} \\ \Delta\lambda^{(0)} \end{Bmatrix} = \begin{Bmatrix} \sigma_{n+1}^{\text{trial}} \\ \kappa_n \\ 0 \end{Bmatrix}$$

Determination of the root by a Newton iterative procedure:

REPEAT

$$\mathbf{v}^{(i+1)} = \mathbf{v}^{(i)} - \left[\frac{d\mathbf{m}}{d\mathbf{v}}(\mathbf{v}^{(i)}) \right]^{-1} \mathbf{m}(\mathbf{v}^{(i)})$$

UNTIL

$$\|\mathbf{v}^{(i+1)} - \mathbf{v}^{(i)}\| < t_{\text{end}}^{\mathbf{v}}$$

The termination criteria uses a vector norm (or length of a vector) which is obtained for an arbitrary vector $\mathbf{x}$ as $\|\mathbf{x}\| = \sqrt{\sum_{i=1}^n x_i^2}$.

IV. Update Variables

$$\begin{aligned} \sigma_{n+1} &= \sigma^{(i+1)} \\ \kappa_{n+1} &= \kappa^{(n+1)} \\ E_{n+1}^{\text{elpl}} &= \tilde{m}_{11} \end{aligned}$$

As the termination criteria $t_{\text{end}}^{\mathbf{v}}$ for the Newton iteration, one may assign the internal accuracy of the used finite element system.

As one can conclude from Fig. 23, the total strain increment can be expressed in dependence of the trial stress state as:

$$\Delta\varepsilon_n = \varepsilon_{n+1} - \varepsilon_n = E^{-1}(\sigma_{n+1}^{\text{trial}} - \sigma_n). \quad (144)$$

Introducing the last equation and the flow rule (101) in Eq (116), gives the final stress state in dependance of the trial stress state σ_{n+1} as:

$$\sigma_{n+1} = \sigma_{n+1}^{\text{trial}} - \Delta\lambda_{n+1} E \cdot \left(\alpha\sqrt{3} + \text{sgn}(\sigma_{n+1}^{\text{trial}}) \right). \quad (145)$$

5.4 Example Problem: Return Mapping for Ideal Plasticity and Linear Hardening

Figure 24 a) shows an idealised engineering stress-strain diagram which can be experimentally obtained e.g. from a uniaxial *tensile* test. The deformation of a tensile rod (cf. Fig. 24 b)) is to be investigated based on this material behaviour. The right end of the rod is displaced by $u = 8 \cdot 10^{-3}$ m in 10 equal increments. The rod should be treated as a continuum and is not to be discretised with the finite element method. Calculate with the return mapping algorithm the stress state and other variables in each increment for the Drucker-Prager yield condition ($\alpha = 0.2$)

- (a) for an ideal plastic material ($E^{\text{elpl}} = 0$),
- (b) for linear hardening with E^{elpl} .

Give a graphical representation of the updated stress path.

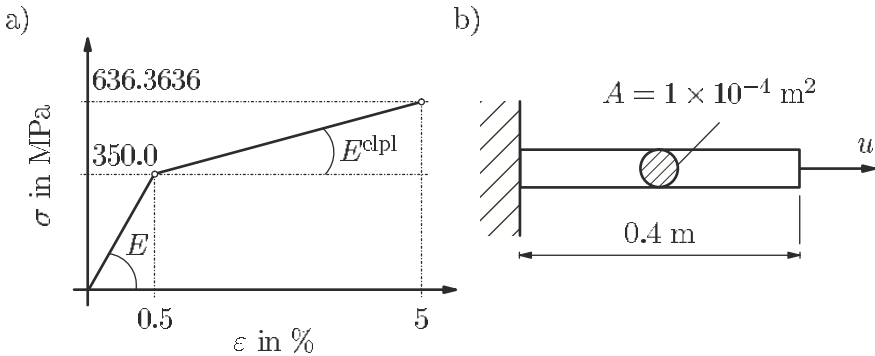


Figure 24. Example Problem: return mapping stress update, a) stress-strain diagram; b) geometry and boundary conditions.

Solution:

The return mapping algorithm requires several material parameters of the elastic and plastic range. Young's modulus is obtained as the fraction between the stress and strain increment in the elastic range as

$$E = \frac{\Delta\sigma}{\Delta\epsilon} = \frac{350 \text{ MPa}}{0.005} = 70000 \text{ MPa}. \quad (146)$$

The plastic modulus E^{pl} is the ratio between the flow stress and the plastic

strain increment:

$$E^{\text{pl}} = \frac{\Delta k}{\Delta \varepsilon^{\text{pl}}} = \frac{636.3636 \text{ MPa} - 350 \text{ MPa}}{(0.05 - 636.3636 \text{ MPa}/E) - (0.005 - 350 \text{ MPa}/E)} = 7000 \text{ MPa}. \quad (147)$$

Thus, the elasto-plastic modulus can be calculated based on Eq. (107) as:

$$E^{\text{elpl}} = \begin{cases} 0.0 \text{ MPa} & \text{for ideal plasticity,} \\ 6363.636 \text{ MPa} & \text{for linear hardening.} \end{cases} \quad (148)$$

Finally, the equation of the flow curve results with the initial yield stress as:

$$k_t(\kappa) = \begin{cases} 350 \text{ MPa} = \text{const.} & \text{for ideal plasticity,} \\ 350 \text{ MPa} + 7000 \text{ MPa} \cdot \kappa & \text{for linear hardening.} \end{cases} \quad (149)$$

A graphical representation of the flow curves is given in Fig. 25.

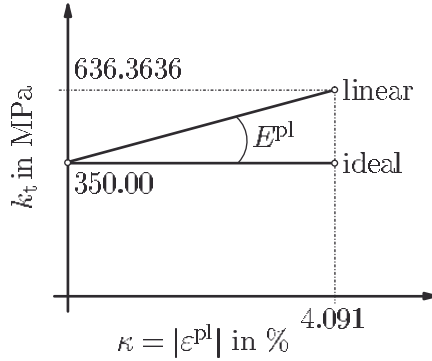


Figure 25. Schematic representation of a flow curve of an ideal plastic and a linear hardening material.

The return mapping algorithm still requires the strain increment. For a total displacement of $8 \cdot 10^{-3} \text{ m}$ which is to be applied in 10 equal steps, the strain increment is obtained as:

$$\Delta \varepsilon = \frac{1}{10} \cdot \frac{0.008 \text{ m}}{0.4 \text{ m}} = 0.002. \quad (150)$$

Tables 8 and 9 summarise the results for a simulated tensile and compression test in the case of the ideal plastic material behaviour. For tensile

Table 8. Numerical values of the incremental integration for a uniaxial tensile test (ideal plasticity, 10 increments; $\Delta\varepsilon = 0.002$).

inc	ε	σ^{trial}	σ	κ	$d\kappa$	$d\lambda$	E^{elpl}
–	10^{-2}	MPa	MPa	10^{-3}	10^{-3}	10^{-3}	MPa
1	0.2	140.0	140.0	0.0	0.0	0.0	0.0
2	0.4	280.0	280.0	0.0	0.0	0.0	0.0
3	0.6	420.0	350.0	1.0	1.0	0.742716	0.0
4	0.8	490.0	350.0	3.0	2.0	1.485431	0.0
5	1.0	490.0	350.0	5.0	2.0	1.485431	0.0
6	1.2	490.0	350.0	7.0	2.0	1.485431	0.0
7	1.4	490.0	350.0	9.0	2.0	1.485431	0.0
8	1.6	490.0	350.0	11.0	2.0	1.485431	0.0
9	1.8	490.0	350.0	13.0	2.0	1.485431	0.0
10	2.0	490.0	350.0	15.0	2.0	1.485431	0.0

Table 9. Numerical values of the incremental integration for a uniaxial compression test (ideal plasticity, 10 increments; $\Delta\varepsilon = -0.002$).

inc	ε	σ^{trial}	σ	κ	$d\kappa$	$d\lambda$	E^{elpl}
–	10^{-2}	MPa	MPa	10^{-3}	10^{-3}	10^{-3}	MPa
1	-0.2	-140.0	-140.0	0.0	0.0	0.0	0.0
2	-0.4	-280.0	-280.0	0.0	0.0	0.0	0.0
3	-0.6	-420.0	-420.0	0.0	0.0	0.0	0.0
4	-0.8	-560.0	-560.0	0.0	0.0	0.0	0.0
5	-1.0	-700.0	-700.0	0.0	0.0	0.0	0.0
6	-1.2	-840.0	-721.008	1.69989	1.69989	2.60084	0.0
7	-1.4	-861.008	-721.008	3.69989	2.00000	3.06002	0.0
8	-1.6	-861.008	-721.008	5.69989	2.00000	3.06002	0.0
9	-1.8	-861.008	-721.008	7.69989	2.00000	3.06002	0.0
10	-2.0	-861.008	-721.008	9.69989	2.00000	3.06002	0.0

testing, the two first trial states are elastic ($F < 0$) and the resulting stress can be calculated based on Hooke's law. Looking at the results of the compression test, one can see that yielding occurs much later in the negative stress regime. Only for increment 6, the trial state violates the yield condition ($F > 0$) and the stress state must be updated according to Eq. (145).

The results for linear hardening are summarised in Tabs. 10 and 11. It can be observed that yielding initiates at the same stress as in the case of ideal plasticity. However, the yield stress increases now with ongoing plastic deformation.

Table 10. Numerical values of the incremental integration for a uniaxial tensile test (linear hardening, 10 increments; $\Delta\varepsilon = 0.002$).

inc –	ε 10^{-2}	σ^{trial} MPa	σ MPa	κ 10^{-3}	$d\kappa$ 10^{-3}	$d\lambda$ 10^{-3}	E^{elpl} MPa
1	0.2	140.0	140.0	0.0	0.0	0.0	0.0
2	0.4	280.0	280.0	0.0	0.0	0.0	0.0
3	0.6	420.0	356.364	0.90909	0.90909	0.67520	6363.636
4	0.8	496.364	369.091	2.72727	1.81818	1.35039	6363.636
5	1.0	509.091	381.818	4.54546	1.81818	1.35039	6363.636
6	1.2	521.818	394.545	6.36363	1.81818	1.35039	6363.636
7	1.4	534.545	407.273	8.18181	1.81818	1.35039	6363.636
8	1.6	547.273	420.000	10.00000	1.81818	1.35039	6363.636
9	1.8	560.000	432.727	11.81818	1.81818	1.35039	6363.636
10	2.0	572.727	445.455	13.63636	1.81818	1.35039	6363.636

Table 11. Numerical values of the incremental integration for a uniaxial compression test (linear hardening, 10 increments; $\Delta\varepsilon = -0.002$).

inc –	ε 10^{-2}	σ^{trial} MPa	σ MPa	κ 10^{-3}	$d\kappa$ 10^{-3}	$d\lambda$ 10^{-3}	E^{elpl} MPa
1	-0.2	-140.0	-140.0	0.0	0.0	0.0	0.0
2	-0.4	-280.0	-280.0	0.0	0.0	0.0	0.0
3	-0.6	-420.0	-420.0	0.0	0.0	0.0	0.0
4	-0.8	-560.0	-560.0	0.0	0.0	0.0	0.0
5	-1.0	-700.0	-700.0	0.0	0.0	0.0	0.0
6	-1.2	-840.0	-741.334	1.40952	1.40952	2.15658	11956.993
7	-1.4	-881.334	-765.248	3.06789	1.65837	2.53733	11956.993
8	-1.6	-905.248	-789.162	4.72626	1.65837	2.53733	11956.993
9	-1.8	-925.162	-813.076	6.38464	1.65837	2.53733	11956.993
10	-2.0	-953.076	-836.990	8.04300	1.65837	2.53733	11956.993

A graphical representation of the stress update path is given in Fig. 26 for the ideal plastic material behaviour (a) and the linear hardening (b) case.

Finally, it must be noted that a comparison with the von Mises yield condition ($\alpha = 0$) gives absolutely the same results as presented in Tabs. 8 to 11 for the *uniaxial* cases. To explain this observation, let us have a look on the uniaxial Drucker-Prager yield condition for a tensile test:

$$\alpha\sqrt{3}\sigma + |\sigma| - (\alpha\sqrt{3} + 1)k_t \leq 0. \quad (151)$$

Since we assume here a tensile test (i.e. $\sigma > 0$), the simplification $|\sigma| \rightarrow \sigma$ holds and Eq. (151) can be written as

$$(\alpha\sqrt{3} + 1)\sigma - (\alpha\sqrt{3} + 1)k_t \leq 0, \quad (152)$$

which reduces immediately to the von Mises yield condition for a uniaxial tensile test:

$$F_{\text{Mises},t} = \sigma - k_t \leq 0. \quad (153)$$

The same way, we can have a look on the Drucker-Prager yield condition for a compression ($\sigma < 0$) test which gives in the formulation with the compression yield stress k_c the following formulation:

$$\alpha\sqrt{3}\sigma + |\sigma| - (1 - \alpha\sqrt{3})k_c \leq 0. \quad (154)$$

Since we have now $\sigma < 0$ while $k_c > 0$ holds by definition, the last equation reduces to

$$(\alpha\sqrt{3} - 1)\sigma - (1 - \alpha\sqrt{3})k_c \leq 0, \quad (155)$$

which gives again the von Mises yield condition for a uniaxial compression test:

$$F_{\text{Mises},c} = \sigma + k_c \leq 0. \quad (156)$$

Be aware that these conclusions are only valid for *uniaxial* stress states and that a multiaxial stress states gives in general a deviation of the Drucker-Prager yield criterion from the von Mises yield condition.

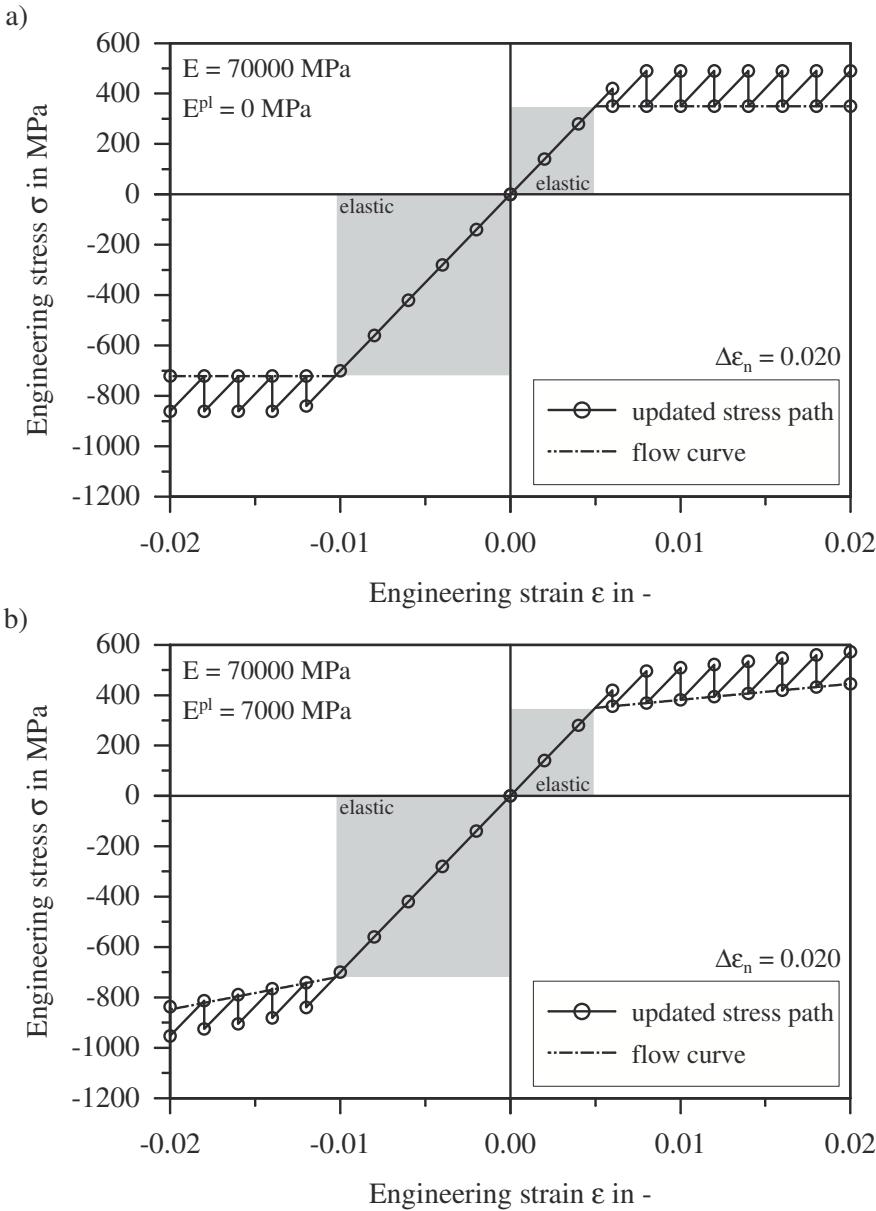


Figure 26. Representation of the return mapping scheme for the one-dimensional plasticity problem of the Drucker-Prager yield condition: a) ideal plasticity; b) linear hardening.

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Thin-walled Structures Made of Foams

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Abstract Thin-walled structures made of foams, composites, sandwiches or functionally graded materials have many applications in aircraft, automotive, and other industries since they combine functionality, high specific stiffness with light weight. The analysis of such structures can be performed on the base of the three-dimensional theory, but often theories with a lower dimension are applied.

The classical plate theory elaborated by Kirchhoff can be applied to thin plates made of classical materials like steel. But this approach is connected with various limitations (e.g., constant material properties in the thickness direction). In addition, mathematical inconsistencies (for example, the order of the governing equation does not correspond to the number of boundary conditions) exist. Many suggestions for improvements of the classical plate theory are made. The engineering direction of improvements was ruled by applications (e.g., the use of laminates or sandwiches as the plate material), and new hypotheses are introduced. In addition, some mathematical approaches like power series expansions or asymptotic integration techniques are applied.

A conceptional different direction is the direct approach in the plate theory which is applied to plates made of the above mentioned advanced materials. Within this theory static and the vibration problems are solved assuming linear-elastic and linear-viscoelastic behavior. The material properties are changing in the thickness direction only. It is shown that the results based on the direct approach differ significantly from the classical estimates which can be explained by the non-classical estimation of the transverse shear stiffness. In the last part a general linear six-parametric theory is presented. The kinematics of the plate is described by using the vector of translation and the vector of rotation as the independent variables. The relations between the equilibrium conditions of a three-dimensional micropolar plate-like body and the two-dimensional equilibrium equations of the deformable surface are established.

1 Introduction

1.1 Plates as Structural Elements

The main goal of engineering mechanics is the prediction of the strength, the dynamical behavior, the stability, etc. of structural elements. Structural elements are the basic models of constructions, parts of machines or in civil engineering applications. The following classification of structural models can be given

- one-dimensional models like rods and beams,
- two-dimensional models like plates and shells, and
- three-dimensional models like spheres, cubes, etc.

In-between these classes there are other models. For example, membranes are two-dimensional models, but they have no resistance against bending loads. Another example is a model which contains elements of beams and shells - the thin-walled beam with box-girt cross section.

For all model classes special theories are developed simplifying the three-dimensional continuum mechanics equations. Below we focus our attention on plates as one class of structural models which can be presented by two-dimensional equations and simple geometrical assumption (no curvature in the reference state). In comparison with the shell theory the theory of plates is much simpler. But in the methodological sense many details are similar. For example, the formulation of the basic equations applying the direct approach and the identification procedure of the effective stiffness properties can be generalized without any difficulties - only the effort is much greater.

The classical plate theory can be applied to thin plates made of materials like steel. The first theory allowing a satisfying analysis of such plates was elaborated by Kirchhoff (1850). But this approach is connected with various limitations (e.g., constant material properties in the thickness direction). In addition, some mathematical inconsistencies like the order of the governing equation and the number of boundary conditions are accepted. During the last century many suggestions for improvements of the classical plate theory were made by Reissner (1944, 1945, 1947); Mindlin (1951); Levinson (1980); Reddy (1984) among others. The engineering direction of improvements is ruled by applications (e.g., the use of laminates or sandwiches as the plate material) and so new hypotheses for the derivation of the governing equations are introduced. In addition, some mathematical approaches like power series expansions (Kienzler, 2002) or asymptotic integration techniques are applied. A conceptional other direction in the formulation of a plate theory is connected with the direct approach (Naghdi, 1972; Reissner, 1985).

1.2 Foam as a Material for Structural Elements

Engineering structures made of porous materials, especially metal or polymer foams, have had different applications for the last decades, see the books of Lee and Ramesh (2004); Landrock (1995); Banhart (1997); Gibson and Ashby (1997); Banhart et al. (1999); Mills (2007); Ashby et al. (2000); Degischer and Kriszt (2002) and the review papers of Banhart (2000, 2006); Ashby and Tianjian (2003); Salimon et al. (2005); Ashby (2006); Lefebvre et al. (2008) among others. A foam is a cellular material consisting of a solid material (metal, polymer or ceramics) and containing a large volume fraction of gas-filled pores. There are two types of metal or polymer foams. One is the closed-cell foam, while the second one is the open-cell foam. The defining characteristic of the foams is the very high porosity: typically well over 80%, 90% and even 98% of the volume consist of void spaces. Examples of closed-cell foams with different porosity are presented in Fig. 1, while the open-cell polymer foams are presented in Fig. 2. In addition, perspective

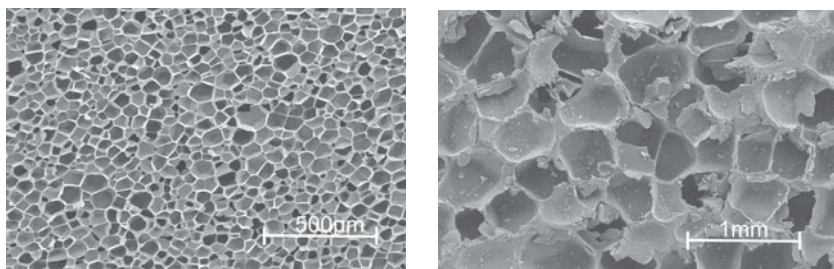


Figure 1. Closed-cell polymer foams with different density (photos by courtesy of Arnim Kraatz, German Institute for Polymers, Darmstadt)

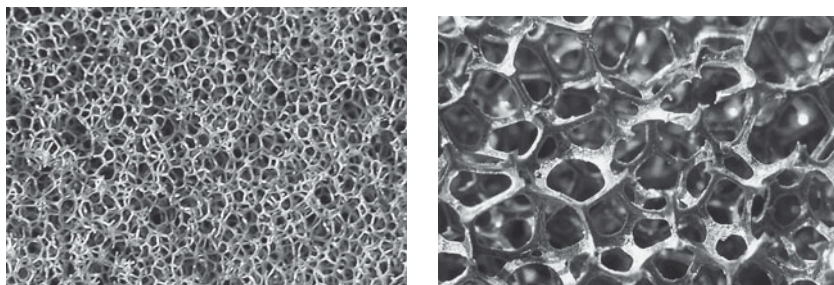


Figure 2. Open-cell polymer foams with different density

materials are composite materials synthesized by filling a metal, polymer or ceramic matrix with hollow particles called microballoons (see, e.g. Sanders and Gibson (2003a,b); El Hadek and Tippur (2003); Gupta and Ricci (2006); Fiedler (2008)). This type of foams is named syntactic foam.

Foams are applied in the automotive or aerospace industries since they combine low weight, high specific strength, and excellent possibilities to absorb energy. The technical realization is mostly performed as sandwich panels (plates or shells with hard and stiff face sheets and a core layer made of a foam), see the cited above books and, for example, the papers of Deshpande and Fleck (2001); Deshpande et al. (2001); McCormack et al. (2001); Bart-Smith et al. (2002); Kesler and Gibson (2002); Kesler et al. (2003); Kardomateas et al. (2002); Hohe (2003); Benderly and Putter (2004); Chang et al. (2004); Kulkarni et al. (2004); Sha and Yip (2004); Sha et al. (2006); Gibson (2005); Mamalis et al. (2005); Queheillalt and Wadley (2005); Rizov et al. (2005); Rammerstorfer et al. (2006); Rice et al. (2006); Ionita and Weitsman (2007); Mohan et al. (2007); Nemat-Nasser et al. (2007); Qin and Wang (2007); Raj et al. (2007); Tagarielli et al. (2007); Tian et al. (2007); Torsakul (2007); Raj and Ghosn (2008); Ambrogio et al. (2008); Banhart and Seeliger (2008); Caner and Bažant (2009); Caprino et al. (2009); Reany and Grenestedt (2009); Rubino et al. (2009); Zhang and He (2009); Zhu et al. (2009).

From the mechanical point of view, a foam is a very complex material which exhibits many non-classical properties. We mentioned already their high porosity. The other “exotic properties” are:

- *Negative Poisson’s ratio.* Some polymer and metal foams have a negative Poisson’s ratio even in the case of isotropic material behavior, see Lakes (1987); Friis et al. (1988), and Gibson and Ashby (1997). Such foams are named re-entrant foams.
- *Size effect.* Foams demonstrate size-effect, see Fleck et al. (2001); Chen and Fleck (2002); Kesler and Gibson (2002); Bažant et al. (2003); Tekoglu and Onck (2005). For example, the ratio of the specimen length to the radius of cells has an influence on the stressed state or fracture (Fleck et al., 2001; Bažant et al., 2003).
- *Micropolar properties.* Besides that some foams are examples of the Cosserat continuum, see the experimental data presented by Lakes (1986, 1991, 1995), and the results published by Diebels (1999); Diebels and Steeb (2003); Dillard et al. (2006). In the Cosserat continuum or micropolar continuum each material particle has six degrees of freedom, i.e. the translations and rotations fields are independently, and there exist force and couple stresses.

Special interest is focussed on foams with a nonhomogeneous distribu-

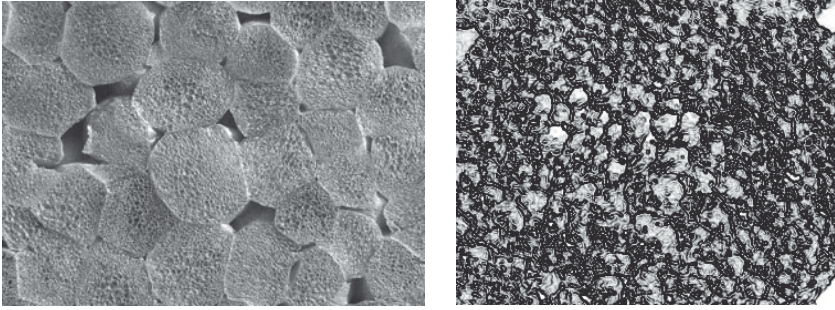


Figure 3. Nonhomogeneous closed-cell foams

tion of porosity. In this case the foam may be considered as a functionally graded material (FGM), see Fig. 3. The foam itself can be modeled also as a functionally graded material with mechanical properties changing over the thickness direction. Let us mention here the papers where the nonhomogeneous distribution of porosity is taken into account, see Beals and Thompson (1997); El Hadek and Tippur (2003); Zhu et al. (2004); Gupta and Ricci (2006); Gonthier and Cox (2007); Hsu et al. (2007). Beals and Thompson (1997), discussed a functionally graded aluminium foam, Hsu et al. (2007) investigated bioceramics, El Hadek and Tippur (2003); Gupta and Ricci (2006) researched syntactic foams, and Zhu et al. (2004) investigated the temperature distribution in functionally graded metal foam.

2 Direct Two-dimensional Plate Theory

2.1 Classical Approaches in the Plate Theory

The Kirchhoff's plate theory suggested more than 150 years ago (Kirchhoff, 1850) was worked out for thin plates made of homogeneous isotropic materials and the case of bending. Introducing the deflection of the midplane the theory is limited only by the assumption that each material point of the midplane has one independent degree of freedom. The governing equation for the bending problem can be deduced as

$$D\Delta\Delta w = q \quad (1)$$

with the deflection w , the given external surface load q , the Laplacian Δ and the bending stiffness D . Equation (1) is a two-dimensional partial differential equation of fourth order reducing the behavior of a three-dimensional plate-like body to the behavior of a two-dimensional surface (here the mid-

plane). In the background of this theory one can see very helpful assumptions for engineering applications: instead of the stresses the stress resultants (averaged stresses in the thickness direction) are introduced. Forces and the moments, which have a clear mechanical interpretation, and the individual mechanical properties can be presented by another averaged property – the bending stiffness which is an effective property combining the Young's modulus E , the Poisson's ratio ν and the plate thickness h

$$D = \frac{Eh^3}{12(1-\nu^2)} \quad (2)$$

On the other hand one observes very rough limitations which are not helpful in modern applications. Examples of such limitations are the thinness hypothesis, the isotropy assumption, the homogeneity assumption, the small deflection assumption and the assumption that the plate behavior (out-of-plane) and the plane stress behavior (in-plane) are decoupled (the last one is ignored in the Kirchhoff's theory). In addition, the order of the governing equation (1) does not correspond to the number of boundary conditions in some cases.

Focussing the attention on the applications to modern materials (laminates, sandwiches, foams among others) only various improvements of the classical theory are suggested. For example, considering sandwich structures with soft cores Eric Reissner worked out a theory which takes into account the neglected by Kirchhoff transverse shear (Reissner, 1944, 1945, 1947). Similar governing equations (only some details are not the same) were derived by Mindlin introducing two additional degrees of freedom (rotations) for the material points of the midplane (Mindlin, 1951). On the classical approaches and their improvements one can get more information, for example, from Naghdi (1972); Grigolyuk and Selezov (1973); Reissner (1985); Altenbach et al. (1998); Altenbach (2000a,b); Kienzler (2002); Kienzler et al. (2004).

Summarizing some monographs and reviews of the last 50 years (Naghdi, 1972; Grigolyuk and Selezov, 1973; Rothert, 1973; Reissner, 1985; Altenbach and Zhilin, 1988; Libai and Simmonds, 1998; Chróścielewski et al., 2004; Zhilin, 2006; Eremeyev and Zubov, 2008) among others) one can conclude that there are two different possibilities to deduce the plate equations:

- Starting with the three-dimensional equations of solid mechanics the two-dimensional equations are established with the help of some hypotheses or mathematical treatments.
- Introducing a two-dimensional deformable surface the two-dimensional plate equations are derived by the direct approach.

The advantages/disadvantages are evident. In the first case we get the constitutive equations by the same way like other governing equations applying the hypotheses or the mathematical treatments to the, for example in the elastic case, Hooke's law. The correctness and the approximation degree are often unknown. The direct approach is in this sense better - the two-dimensional equations are physically-based and as exactly as the three-dimensional continuum mechanics equations. But the constitutive equations must be identified separately which is a non-trivial problem. It was shown in Altenbach (1988); Altenbach and Zhilin (1988); Zhilin (1976, 2006) among others that the concept of effective properties can support the handling of the identification.

The discussion below will be limited by the consideration of linear elastic and viscoelastic material behavior and the geometrically linear relations. In contrast to the classical plate theory the deformations will be taken into account as in the theory of laminated plates with unsymmetrical cross-section properties, that means the in-plane behavior and the out-of-plane behavior are coupled in the general case (Altenbach et al., 2004). The introduced theory is not restricted by the assumption that the transverse shear strains and rotator inertia effects are neglected. This chapter is based on the results published earlier in, for example, Altenbach and Eremeyev (2008a,b, 2009a,b,c,d).

2.2 Governing Equations

The direct approach in continuum mechanics is based on the Cosserat theory (Cosserat and Cosserat, 1909; Ericksen and Truesdell, 1958; Rubin, 2000; Zhilin, 2006; Eremeyev and Zubov, 2008). In continuum mechanics any theory is based on the method of sections, the Newton' axiom of reciprocal actions and some governing equations. Two different models can be introduced. In the first case only force actions resulting in translations are assumed. From this follows the symmetry of the stress tensor. But in the basic course of Mechanics the following statements are given: the static equilibrium of forces and moments or the equation of motion as the balance of momentum and the balance of moment of momentum should be considered in the general case. It must be noted that the last relations are independent, which was shown, for example, by Truesdell (1964); Truesdell and Noll (1965). Since for this model are assumed force and moment actions, it follows that there are a symmetric and a nonsymmetric stress tensor and translations and rotations can be introduced independently as force and moment responses. Now any continuum (three-, two- or one-dimensional) can be introduced by a natural way: geometrical relations

(kinematics), material independent balances of mass, momentum, moment of momentum, energy, entropy and the material dependent equations (constitutive equations and evolution equations). Finally, one needs boundary and may be initial conditions.

2.3 Material-independent Equations

Since the direct approach is a natural way to describe the behavior of plates (the stress resultants which are used in most of plate theories can be interpreted as forces and moments) a two-dimensional plate theory which allows to model homogeneous and inhomogeneous plates can be presented as follows. Let us assume that the plate can be represented by a deformable surface. In the undeformed (reference) state this surface is plane.

Two basic assumptions can be considered as the starting point for the formulation of the linear theory. The plate (homogeneous or inhomogeneous in transverse direction) can be represented by a deformable surface. The interaction between different parts of this surface is described by forces and moments only. For the sake of simplicity we consider that each material point is an infinitesimal rigid body with 5 degrees of freedom (3 translations and 2 rotations). In addition, the theory presented here is limited to small displacements and rotations and the quadratic strain energy. The basics are presented in Zhilin (1976, 2006); Altenbach and Zhilin (1988, 2004). The following governing equations can be introduced:

- Balances of momentum and moment of momentum

$$\nabla \cdot \mathbf{T} + \mathbf{q} = \rho \ddot{\mathbf{u}} + \rho \Theta_1 \cdot \ddot{\boldsymbol{\varphi}}, \quad \nabla \cdot \mathbf{M} + \mathbf{T}_\times + \mathbf{m} = \rho \Theta_1^T \cdot \ddot{\mathbf{u}} + \rho \Theta_2 \cdot \ddot{\boldsymbol{\varphi}} \quad (3)$$

Here $\mathbf{T}$, $\mathbf{M}$ are the tensors of forces and moments, $\mathbf{q}$, $\mathbf{m}$ are the surface vectors of loads (forces and moments), $\mathbf{T}_\times$ is the vector invariant of the force tensor (Lurie, 2005), ∇ is the nabla (Hamilton) operator (pseudovector), $\mathbf{u}$, $\boldsymbol{\varphi}$ are the vectors of displacements and rotations, Θ_1 , Θ_2 are the first and the second tensor of inertia, ρ is the density (effective property of the deformable surface), $(\dots)^T$ denotes the transposed tensor and $(\dots)$ the time derivative, respectively.

- Linear geometrical relations

$$\boldsymbol{\mu} = (\nabla \mathbf{u} \cdot \mathbf{a})^{\text{sym}}, \quad \boldsymbol{\gamma} = \nabla \mathbf{u} \cdot \mathbf{n} + \mathbf{c} \cdot \boldsymbol{\varphi}, \quad \boldsymbol{\kappa} = \nabla \boldsymbol{\varphi}, \quad (4)$$

where $\mathbf{a}$ is the first metric tensor (plane tensor), $\mathbf{n}$ the unit normal vector, $\mathbf{c} = -\mathbf{a} \times \mathbf{n}$ the discriminant tensor, $\boldsymbol{\mu}$, $\boldsymbol{\gamma}$, and $\boldsymbol{\kappa}$ the strain tensors (tensor of in-plane strains, vector of transverse shear strains and the tensor of the out-of-plane strains), $(\dots)^{\text{sym}}$ denotes the symmetric part of the tensor.

- Boundary conditions

$$\boldsymbol{\nu} \cdot \mathbf{T} = \mathbf{f}, \quad \boldsymbol{\nu} \cdot \mathbf{M} = \mathbf{l} \quad (\mathbf{l} \cdot \mathbf{n} = 0) \quad (5)$$

or

$$\mathbf{u} = \mathbf{u}^0, \quad \boldsymbol{\varphi} = \boldsymbol{\varphi}^0 \quad \text{along } S. \quad (6)$$

Here $\mathbf{f}$ and $\mathbf{l}$ are external force and couple vectors acting along the boundary of the plate S , while $\mathbf{u}^0$ and $\boldsymbol{\varphi}^0$ are prescribed displacements and rotations of the plate boundary, respectively. $\boldsymbol{\nu}$ is the unit normal vector to S ($\boldsymbol{\nu} \cdot \mathbf{n} = 0$). Equations (5) and (6) are the natural and essential boundary conditions, respectively. Other types of boundary conditions are possible. For example, the simple support boundary conditions are given by

$$\boldsymbol{\nu} \cdot \mathbf{M} \cdot \boldsymbol{\tau} = \mathbf{0}, \quad \mathbf{u} = \mathbf{0}, \quad \boldsymbol{\varphi} \cdot \boldsymbol{\tau} = 0 \quad (7)$$

Here $\boldsymbol{\tau}$ is the unit tangent vector to S ($\boldsymbol{\tau} \cdot \mathbf{n} = \boldsymbol{\tau} \cdot \boldsymbol{\nu} = 0$).

2.4 Two-dimensional Constitutive Equations

Limiting our discussion to the elastic behavior and small strains the following statements for the constitutive modeling can be made. At first, the strain energy can be expanded in a Taylor series limited by quadratic terms. In addition, we assume that the eigenstresses can be neglected. In this case the linear terms in the series can be disregarded. At second, the positive definiteness is guaranteed.

- Strain energy of the deformable surface W

$$\begin{aligned} W(\boldsymbol{\mu}, \boldsymbol{\gamma}, \boldsymbol{\kappa}) &= \frac{1}{2} \boldsymbol{\mu} \cdot \mathbf{A} \cdot \boldsymbol{\mu} + \boldsymbol{\mu} \cdot \mathbf{B} \cdot \boldsymbol{\kappa} + \frac{1}{2} \boldsymbol{\kappa} \cdot \mathbf{C} \cdot \boldsymbol{\kappa} \\ &+ \frac{1}{2} \boldsymbol{\gamma} \cdot \boldsymbol{\Gamma} \cdot \boldsymbol{\gamma} + \boldsymbol{\gamma} \cdot (\boldsymbol{\Gamma}_1 \cdot \boldsymbol{\mu} + \boldsymbol{\Gamma}_2 \cdot \boldsymbol{\kappa}), \end{aligned} \quad (8)$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are 4th rank tensors, $\boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2$ are 3rd rank tensors, $\boldsymbol{\Gamma}$ is a 2nd rank tensor of the effective stiffness properties. They depend on the material properties and the cross-section geometry. The stiffness tensors include 68 components.

From the general structure of the strain energy (8) the following properties of the stiffness tensors are obvious:

- The tensors $\mathbf{A}, \mathbf{C}$ and $\boldsymbol{\Gamma}$ must be symmetrical that means

$$\mathbf{A} \cdot \mathbf{d} = \mathbf{d} \cdot \mathbf{A}, \quad \mathbf{C} \cdot \mathbf{d} = \mathbf{d} \cdot \mathbf{C}, \quad \boldsymbol{\Gamma} = \boldsymbol{\Gamma}^T,$$

where $\mathbf{d}$ is an arbitrary plane second rank tensor.

- Since $\boldsymbol{\mu}$ is a symmetric tensor the following conditions are valid

$$\mathbf{c} \cdot \mathbf{A} = \mathbf{0}, \quad \mathbf{c} \cdot \mathbf{B} = \mathbf{0}, \quad \boldsymbol{\Gamma}_1 \cdot \mathbf{c} = \mathbf{0},$$

where $\mathbf{c} = -\mathbf{c}^T$.

The number of independent and non-zero components is now 45.

Considering that the three-dimensional material behavior can be presented by a symmetric stress tensor ($\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$) one can conclude that $\mathbf{a} \cdot \mathbf{M} = \mathbf{0}$ which results in

$$\mathbf{a} \cdot \mathbf{C} = \mathbf{0}, \quad \mathbf{B} \cdot \mathbf{a} = \mathbf{0}, \quad \boldsymbol{\Gamma}_2 \cdot \mathbf{a} = \mathbf{0}$$

From this it follows that in the general case the tensors contain 36 different values - a reduction is possible assuming some symmetries. In Table 1 the number of non-zero components with different values for different cases of anisotropy are indicated.

Table 1. Number of non-zero components

Type of anisotropy	$\mathbf{A}$	$\mathbf{B}$	$\mathbf{C}$	$\boldsymbol{\Gamma}_1$	$\boldsymbol{\Gamma}_2$	$\boldsymbol{\Gamma}$	Σ
General case	6	9	6	6	6	3	36
Orthotropy	4	5	4	0	0	2	15
Transversal isotropy	2	2	2	0	0	1	7
Isotropy	2	2	2	0	0	1	7

- Constitutive equations
 - in-plane forces

$$\mathbf{T} \cdot \mathbf{a} = \frac{\partial W}{\partial \boldsymbol{\mu}} = \mathbf{A} \cdot \boldsymbol{\mu} + \mathbf{B} \cdot \boldsymbol{\kappa} + \boldsymbol{\gamma} \cdot \boldsymbol{\Gamma}_1, \quad (9)$$

- transverse forces

$$\mathbf{T} \cdot \mathbf{n} = \frac{\partial W}{\partial \boldsymbol{\gamma}} = \boldsymbol{\Gamma} \cdot \boldsymbol{\gamma} + \boldsymbol{\Gamma}_1 \cdot \boldsymbol{\mu} + \boldsymbol{\Gamma}_2 \cdot \boldsymbol{\kappa}, \quad (10)$$

- moments

$$\mathbf{M}^T = \frac{\partial W}{\partial \boldsymbol{\kappa}} = \boldsymbol{\mu} \cdot \mathbf{B} + \mathbf{C} \cdot \boldsymbol{\kappa} + \boldsymbol{\gamma} \cdot \boldsymbol{\Gamma}_2 \quad (11)$$

2.5 Basic Equations in Cartesian Coordinates

Let us assume the Cartesian coordinate system x_1, x_2 (in-plane coordinates) and z (orthogonal to the midplane). Then the unit normal vectors are $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{n}$. With respect to the introduced coordinate system the following equations are valid:

- Displacement and rotation vectors

$$\mathbf{u} = u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 + w \mathbf{n}, \quad \boldsymbol{\varphi} = -\varphi_2 \mathbf{e}_1 + \varphi_1 \mathbf{e}_2 \quad (12)$$

u_α ($\alpha = 1, 2$) are the in-plane displacements, w is the deflection and φ_α are the rotations around the axes $\mathbf{e}_\alpha$, respectively.

- Force and moment tensors

$$\begin{aligned} \mathbf{T} &= T_1 \mathbf{e}_1 \mathbf{e}_1 + T_{12} (\mathbf{e}_1 \mathbf{e}_2 + \mathbf{e}_2 \mathbf{e}_1) + T_2 \mathbf{e}_2 \mathbf{e}_2 \\ &\quad + T_{1n} \mathbf{e}_1 \mathbf{n} + T_{2n} \mathbf{e}_2 \mathbf{n}, \\ \mathbf{M} &= M_1 \mathbf{e}_1 \mathbf{e}_2 - M_{12} (\mathbf{e}_1 \mathbf{e}_1 - \mathbf{e}_2 \mathbf{e}_2) - M_2 \mathbf{e}_2 \mathbf{e}_1, \end{aligned} \quad (13)$$

where T_α, T_{12} are the in-plane forces, $T_{\alpha n}$ the transverse shear forces, M_α the bending moments, and M_{12} the torsion moment, respectively.

- Strain tensors

$$\begin{aligned} \boldsymbol{\mu} &= \mu_1 \mathbf{e}_1 \mathbf{e}_1 + \mu_{12} (\mathbf{e}_1 \mathbf{e}_2 + \mathbf{e}_2 \mathbf{e}_1) + \mu_2 \mathbf{e}_2 \mathbf{e}_2, \\ \boldsymbol{\gamma} &= \gamma_1 \mathbf{e}_1 + \gamma_2 \mathbf{e}_2, \\ \boldsymbol{\kappa} &= \kappa_1 \mathbf{e}_1 \mathbf{e}_2 - \kappa_{12} \mathbf{e}_1 \mathbf{e}_1 + \kappa_{21} \mathbf{e}_2 \mathbf{e}_2 - \kappa_2 \mathbf{e}_2 \mathbf{e}_1, \end{aligned}$$

where μ_α are the normal in-plane strains, μ_{12} the shear strain, γ_α the transverse shear strains, κ_α the bending deformations, and κ_{12} the torsion deformation, respectively.

- External loads

$$\mathbf{q} = q_1 \mathbf{e}_1 + q_2 \mathbf{e}_2 + q_n \mathbf{n}, \quad \mathbf{m} = -m_2 \mathbf{e}_1 + m_1 \mathbf{e}_2. \quad (14)$$

Here q_α are the in-plane forces, q_n the transverse force, m_α the moments, respectively.

How do the symmetries of the “microstructure” affect the physical properties? The answer comes from the *Curie-Neumann’s principle* in the physics of crystals (Nye, 2000).

The symmetry group of the reason belongs to the symmetry group of the consequence.

The symmetry group of the “reasons” for the plate is the intersection of:

- the symmetry of the material (fibre-reinforced material, rolled sheets, etc.),
- the symmetry of the surface shape (shell or plate)
- the symmetry of the internal structure (laminated plates - symmetry of the stacking sequence with respect to the mid-surface, etc.)

Let us consider orthotropic material behavior and a plane mid-surface. In this case instead of the general form of the effective stiffness tensors one

gets (Zhilin, 2006)

$$\begin{aligned} \mathbf{A} &= A_{11}\mathbf{a}_1\mathbf{a}_1 + A_{12}(\mathbf{a}_1\mathbf{a}_2 + \mathbf{a}_2\mathbf{a}_1) + A_{22}\mathbf{a}_2\mathbf{a}_2 + A_{44}\mathbf{a}_4\mathbf{a}_4, \\ \mathbf{B} &= B_{13}\mathbf{a}_1\mathbf{a}_3 + B_{14}\mathbf{a}_1\mathbf{a}_4 + B_{23}\mathbf{a}_2\mathbf{a}_3 + B_{24}\mathbf{a}_2\mathbf{a}_4 + B_{42}\mathbf{a}_4\mathbf{a}_2, \\ \mathbf{C} &= C_{22}\mathbf{a}_2\mathbf{a}_2 + C_{33}\mathbf{a}_3\mathbf{a}_3 + C_{34}(\mathbf{a}_3\mathbf{a}_4 + \mathbf{a}_4\mathbf{a}_3) + C_{44}\mathbf{a}_4\mathbf{a}_4, \\ \mathbf{\Gamma} &= \Gamma_1\mathbf{a}_1 + \Gamma_2\mathbf{a}_2, \quad \mathbf{\Gamma}_1 = \mathbf{0}, \quad \mathbf{\Gamma}_2 = \mathbf{0} \end{aligned}$$

with

$$\begin{aligned} \mathbf{a}_1 &= \mathbf{a} = \mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2, & \mathbf{a}_2 &= \mathbf{e}_1\mathbf{e}_1 - \mathbf{e}_2\mathbf{e}_2, \\ \mathbf{a}_3 &= \mathbf{c} = \mathbf{e}_1\mathbf{e}_2 - \mathbf{e}_2\mathbf{e}_1, & \mathbf{a}_4 &= \mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1 \end{aligned}$$

In addition, one obtains the orthogonality condition for $\mathbf{a}_i$ ($i = 1, 2, 3, 4$)

$$\frac{1}{2}\mathbf{a}_i \cdot \mathbf{a}_j = \delta_{ij}, \quad \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Now the balances of momentum and of moment of momentum, the geometrical relations and the constitutive equations take the form (static case):

- Equilibrium equations

$$\begin{aligned} T_{1,1} + T_{12,2} + q_1 &= 0, \\ T_{12,1} + T_{2,2} + q_2 &= 0, \\ T_{1n,1} + T_{2n,2} + q_n &= 0, \\ M_{1,1} + M_{12,2} - T_{1n} + m_1 &= 0, \\ M_{12,1} + M_{2,2} - T_{2n} + m_2 &= 0 \end{aligned} \tag{15}$$

- Boundary conditions (for brevity, they are presented for the case that S is a part of line $x_1 = \text{const}$ with $\boldsymbol{\nu} = \mathbf{e}_1$, $\boldsymbol{\tau} = \mathbf{e}_2$):

Static boundary conditions

$$T_1 = f_1, \quad T_{12} = f_2, \quad T_{1n} = f_n, \quad M_1 = l_1, \quad M_{12} = l_2 \tag{16}$$

Kinematic boundary conditions

$$u_1 = u_1^0, \quad u_2 = u_2^0, \quad w = w^0, \quad \varphi_1 = \varphi_1^0, \quad \varphi_2 = \varphi_2^0 \tag{17}$$

Boundary conditions for a hinge

$$M_1 = 0, \quad u_1 = u_2 = w = 0, \quad \varphi_2 = 0 \tag{18}$$

- Geometrical relations

$$\begin{aligned} \mu_1 &= u_{1,1}, & \mu_2 &= u_{2,2}, & \mu_{12} &= \frac{1}{2}(u_{1,2} + u_{2,1}), \\ \gamma_1 &= w_{,1} + \varphi_1, & \gamma_2 &= w_{,2} + \varphi_2, \\ \kappa_1 &= \varphi_{1,1}, & \kappa_2 &= \varphi_{2,2}, & \kappa_{12} &= \varphi_{2,1}, \quad \kappa_{21} = \varphi_{1,2} \end{aligned} \tag{19}$$

- Constitutive equations

$$\begin{aligned}
T_1 &= (A_{11} + 2A_{12} + A_{22})\mu_1 + (A_{11} - A_{22})\mu_2 \\
&\quad - (B_{13} + B_{23} - B_{14} - B_{24})\kappa_1 \\
&\quad - (B_{13} + B_{23} + B_{14} + B_{24})\kappa_2, \\
T_2 &= (A_{11} - A_{22})\mu_1 + (A_{11} + 2A_{12} + A_{22})\mu_2 \\
&\quad - (B_{13} - B_{23} - B_{14} + B_{24})\kappa_1 \\
&\quad - (B_{13} - B_{23} + B_{14} - B_{24})\kappa_2, \\
T_{12} &= 2\mu_{12}A_{44} + (\kappa_{12} + \kappa_{21})B_{42}, \\
T_{1n} &= (\Gamma_1 + \Gamma_2)\gamma_1, \\
T_{2n} &= (\Gamma_1 - \Gamma_2)\gamma_2, \\
M_1 &= (-B_{13} - B_{23} + B_{14} + B_{24})\mu_1 \\
&\quad - (B_{13} - B_{23} - B_{14} + B_{24})\mu_2 \\
&\quad + (C_{33} - 2C_{34} + C_{44})\kappa_1 + (C_{33} - C_{44})\kappa_2, \\
M_2 &= -(B_{13} + B_{23} + B_{14} + B_{24})\mu_1 \\
&\quad - (B_{13} - B_{23} + B_{14} - B_{24})\mu_2 \\
&\quad + (C_{33} - C_{44})\kappa_1 + (C_{33} + 2C_{34} + C_{44})\kappa_2, \\
M_{12} &= 2\mu_{12}B_{42} + C_{22}(\kappa_{12} + \kappa_{21})
\end{aligned} \tag{20}$$

From the last equations it can be seen that the A_{ij} are the effective in-plane stiffness tensor components, the B_{ij} are the components of the coupling stiffness tensor, the C_{ij} are the effective plate stiffness tensor components, and the Γ_α are the components of the transverse shear stiffness tensor.

3 Stiffness Identification

The advantage of the presented theory is that it is so strong as any continuum mechanics theory. The problem is the identification of the constitutive parameters - the effective properties - in the relevant equations. This problem can be solved by different approaches:

- the comparison of the stress resultants of a three-dimensional static boundary-value problem with the forces and moments in the corresponding two-dimensional boundary value problem,
- the comparison of the eigenfrequencies of a three-dimensional and the corresponding two-dimensional eigenvibration problem

Note that the values of the constitutive parameters should be independent from the kind of identification procedure.

3.1 Orthotropic Material Behavior

The individuality of each class of plates in the framework of the direct approach is expressed by the effective properties (stiffness, density, inertia terms, etc.). Let us focus our attention on the stiffness expressions. The identification of the effective stresses should be performed on the base of the properties of the real material. Let us assume the Hooke's law as

$$\begin{aligned}
 \varepsilon_1 &= u_{1,1}^* = \frac{1}{E_1} \sigma_1 - \frac{\nu_{21}}{E_2} \sigma_2 - \frac{\nu_{n1}}{E_n} \sigma_n, \\
 \varepsilon_2 &= u_{2,2}^* = \frac{1}{E_2} \sigma_2 - \frac{\nu_{12}}{E_1} \sigma_1 - \frac{\nu_{n2}}{E_n} \sigma_n, \\
 \varepsilon_n &= u_{n,n}^* = \frac{1}{E_n} \sigma_n - \frac{\nu_{1n}}{E_1} \sigma_1 - \frac{\nu_{2n}}{E_2} \sigma_2, \\
 \gamma_{12} &= u_{1,2}^* + u_{2,1}^* = \frac{\tau_{12}}{G_{12}}, \\
 \gamma_{n1} &= u_{1,n}^* + u_{n,1}^* = \frac{\tau_{n1}}{G_{n1}}, \\
 \gamma_{2n} &= u_{2,n}^* + u_{n,2}^* = \frac{\tau_{2n}}{G_{2n}}
 \end{aligned} \tag{21}$$

with $\nu_{ij}E_j = \nu_{ji}E_i$. $\sigma_i, \epsilon_i, \tau_{ij}, \gamma_{ij}$ are the normal stresses, the normal strains, the shear stresses and the shear strains, respectively; E_i, ν_{ij}, G_{ij} are the Young's moduli, the Poisson's ratio and the shear moduli. The axes of orthotropy coincide with the co-ordinate axes.

Here Eqs (21) are valid in the following cases:

Case 1: Homogeneous plates – all properties are constant (no dependency on z).

Case 2: Inhomogeneous plates (sandwich, multilayered, made of foams, functionally graded) – all properties are functions of the co-ordinate in the thickness direction z , e.g., $E_i = E_i(z)$.

The identification of the effective properties can be performed with the help of static boundary value problems (two-dimensional, three-dimensional) and the comparison of the forces and moments (in the sense of averaged stresses or stress resultants)

$$\mathbf{T} = \langle \mathbf{a} \cdot \boldsymbol{\sigma} \rangle, \quad \mathbf{M} = \langle \mathbf{a} \cdot \boldsymbol{\sigma} z \cdot \mathbf{c} \rangle$$

$\boldsymbol{\sigma}$ is the stress tensor, and $\langle \dots \rangle$ denotes the integration over the plate thickness

$$\langle \dots \rangle = \int_{-h/2}^{h/2} (\dots) dz$$

3.2 Classical Stiffness Values

Problem 1 - Tension and bending: The following two-dimensional kinematical field is given

$$\mathbf{u} = D_1 x_1 \mathbf{e}_1 + D_2 x_2 \mathbf{e}_2 - \frac{1}{2} \left(\frac{x_1^2}{R_1} + \frac{x_2^2}{R_2} \right) \mathbf{n}, \quad \boldsymbol{\varphi} = -\frac{x_2}{R_2} \mathbf{e}_1 + \frac{x_1}{R_1} \mathbf{e}_2,$$

where D_α and R_α are unknown constants ($\alpha = 1, 2$). The two-dimensional strain tensors are

$$\boldsymbol{\mu} = D_1 \mathbf{e}_1 \mathbf{e}_1 + D_2 \mathbf{e}_2 \mathbf{e}_2, \quad \boldsymbol{\gamma} = 0, \quad \boldsymbol{\kappa} = \frac{1}{R_1} \mathbf{e}_1 \mathbf{e}_2 - \frac{1}{R_2} \mathbf{e}_2 \mathbf{e}_1,$$

and the two-dimensional constitutive equations can be calculated by

$$\begin{aligned} T_1 &= D_1(A_{11} + 2A_{12} + A_{22}) + D_2(A_{11} - A_{22}) \\ &\quad - \frac{1}{R_1}(B_{13} + B_{23} - B_{14} - B_{24}) \\ &\quad - \frac{1}{R_2}(B_{13} + B_{23} + B_{14} + B_{24}), \\ T_2 &= D_1(A_{11} - A_{22}) + D_2(A_{11} - 2A_{12} + A_{22}) \\ &\quad - \frac{1}{R_1}(B_{13} - B_{23} - B_{14} + B_{24}) \\ &\quad - \frac{1}{R_2}(B_{13} - B_{23} + B_{14} - B_{24}), \\ M_1 &= D_1(-B_{13} - B_{23} + B_{14} + B_{24}) \\ &\quad - D_2(B_{13} - B_{23} - B_{14} + B_{24}) \\ &\quad + \frac{1}{R_1}(C_{33} - 2C_{34} + C_{44}) + \frac{1}{R_2}(C_{33} - C_{44}), \\ M_2 &= -D_1(B_{13} + B_{23} + B_{14} + B_{24}) \\ &\quad - D_2(B_{13} - B_{23} + B_{14} - B_{24}) \\ &\quad + \frac{1}{R_1}(C_{33} - C_{44}) + \frac{1}{R_2}(C_{33} + 2C_{34} + C_{44}) \end{aligned} \quad (22)$$

Assuming the three-dimensional strain tensor components corresponding to the two-dimensional problem

$$\varepsilon_1 = D_1 + \frac{z}{R_1}, \quad \varepsilon_2 = D_2 + \frac{z}{R_2},$$

the stress tensor components considering the plane stress state and the Hooke's law (21) are

$$\sigma_1 = \frac{E_1}{1 - \nu_{12}\nu_{21}}(\varepsilon_1 + \nu_{21}\varepsilon_2), \quad \sigma_2 = \frac{E_2}{1 - \nu_{12}\nu_{21}}(\varepsilon_2 + \nu_{12}\varepsilon_1)$$

Finally, the stress resultants can be estimated as follows

$$\begin{aligned}
 T_1 &= D_1 \left\langle \frac{E_1}{1 - \nu_{12}\nu_{21}} \right\rangle + \frac{1}{R_1} \left\langle \frac{E_1 z}{1 - \nu_{12}\nu_{21}} \right\rangle \\
 &\quad + D_2 \left\langle \frac{\nu_{21} E_1}{1 - \nu_{12}\nu_{21}} \right\rangle + \frac{1}{R_2} \left\langle \frac{\nu_{21} E_1 z}{1 - \nu_{12}\nu_{21}} \right\rangle, \\
 T_2 &= D_2 \left\langle \frac{E_2}{1 - \nu_{12}\nu_{21}} \right\rangle + \frac{1}{R_2} \left\langle \frac{E_2 z}{1 - \nu_{12}\nu_{21}} \right\rangle \\
 &\quad + D_1 \left\langle \frac{\nu_{12} E_2}{1 - \nu_{12}\nu_{21}} \right\rangle + \frac{1}{R_1} \left\langle \frac{\nu_{12} E_2 z}{1 - \nu_{12}\nu_{21}} \right\rangle, \\
 M_1 &= D_1 \left\langle \frac{E_1 z}{1 - \nu_{12}\nu_{21}} \right\rangle + \frac{1}{R_1} \left\langle \frac{E_1 z^2}{1 - \nu_{12}\nu_{21}} \right\rangle \\
 &\quad + D_2 \left\langle \frac{\nu_{21} E_1 z}{1 - \nu_{12}\nu_{21}} \right\rangle + \frac{1}{R_2} \left\langle \frac{\nu_{21} E_1 z^2}{1 - \nu_{12}\nu_{21}} \right\rangle, \\
 M_2 &= D_2 \left\langle \frac{E_2 z}{1 - \nu_{12}\nu_{21}} \right\rangle + \frac{1}{R_2} \left\langle \frac{E_2 z^2}{1 - \nu_{12}\nu_{21}} \right\rangle \\
 &\quad + D_1 \left\langle \frac{\nu_{12} E_2 z}{1 - \nu_{12}\nu_{21}} \right\rangle + \frac{1}{R_1} \left\langle \frac{\nu_{12} E_2 z^2}{1 - \nu_{12}\nu_{21}} \right\rangle
 \end{aligned} \tag{23}$$

The comparison of the stress resultants (22) and (23) allows the estimation of the following stiffness tensor components

$$\begin{aligned}
 \{A_{11}, -B_{13}; C_{33}\} &= \frac{1}{4} \left\langle \frac{E_1 + E_2 + 2E_1\nu_{21}}{1 - \nu_{12}\nu_{21}} \{1; z; z^2\} \right\rangle, \\
 \{A_{12}; B_{23} = -B_{14}; C_{34}\} &= \frac{1}{4} \left\langle \frac{E_1 - E_2}{1 - \nu_{12}\nu_{21}} \{1; -z; z^2\} \right\rangle, \\
 \{A_{22}, B_{24}; C_{44}\} &= \frac{1}{4} \left\langle \frac{E_1 + E_2 - 2E_1\nu_{21}}{1 - \nu_{12}\nu_{21}} \{1; z; z^2\} \right\rangle
 \end{aligned} \tag{24}$$

Problem 2 - Plane shear: Let us assume the two-dimensional kinematical field

$$\mathbf{u} = S_1 x_2 \mathbf{e}_1 + S_1 x_1 \mathbf{e}_2 - S_2 x_1 x_2 \mathbf{n}, \quad \varphi = -S_2 (x_1 \mathbf{e}_1 - x_2 \mathbf{e}_2)$$

with the arbitrary constants S_α . The corresponding three-dimensional problem results in the following strain tensor component

$$\gamma_{12} = u_{1,2}^* + u_{2,1}^* = S_1 + S_2 z$$

The comparison of the computed stress resultants yields

$$\{A_{44}; B_{42}; C_{22}\} = \langle G_{12} \{1; -z; z^2\} \rangle \tag{25}$$

The classical stiffness values estimated within the **problems 1 & 2** correspond to the classical estimates.

3.3 Non-classical Stiffness Values

Problem 3 - Torsion: Let us introduce a deformable strip ($|x_1| \leq l_1$, $|x_2| < \infty$) under constant torsion moments at the boundaries $x_1 = \pm l_1$. Then one gets the two-dimensional kinematical field

$$\mathbf{u} = u_2(x_1)\mathbf{e}_2, \quad \boldsymbol{\varphi} = -\varphi_1(x_1)\mathbf{e}_1$$

and the force and moment tensors

$$\mathbf{T} = (A_{44}u_{2,1} - B_{42}\varphi_{2,1})\mathbf{a}_4 + (\Gamma_1 - \Gamma_2)\varphi_2\mathbf{e}_2\mathbf{n}, \quad \mathbf{M} = (B_{42}u_{2,1} - C_{22}\varphi_{2,1})\mathbf{a}_2.$$

Applying the equilibrium equations to the force and the moment tensor one gets after some manipulations

$$\varphi_{2,11} + \lambda^2\varphi_1 = 0$$

with

$$\lambda^2 = \frac{(\Gamma_1 - \Gamma_2)A_{44}}{A_{44}C_{22} - B_{42}^2}$$

The dual three-dimensional problem (strip $|x_1| \leq l_1$, $|x_2| < \infty$, $|z| \leq h/2$) results the displacements

$$u_1^* = w^* = 0, \quad u_2^* = u_2^*(x_1, z),$$

the stress tensor

$$\boldsymbol{\sigma} = G_{12}\frac{\partial u_2^*}{\partial x_1}\mathbf{a}_4 + G_{2n}\frac{\partial u_2^*}{\partial z}(\mathbf{e}_2\mathbf{n} + \mathbf{n}\mathbf{e}_2)$$

and the equilibrium equation

$$G_{12}\frac{\partial^2 u_2^*}{\partial x_1^2} + \frac{\partial}{\partial z}\left(G_{2n}\frac{\partial u_2^*}{\partial z}\right) = 0$$

The solution with respect to the boundary conditions $\sigma_n = \tau_{1n} = \tau_{2n} = 0$ at $|z| = h/2$ can be obtained by the following Fourier's ansatz:

$$u_2^*(x_1, z) = X(x_1)Z(z),$$

which yields a Sturm-Liouville problem

$$\frac{d}{dz}\left(G_{2n}\frac{dZ}{dz}\right) + \lambda_*^2 G_{12}Z = 0, \quad \frac{dZ}{dz}\bigg|_{|z|=\frac{h}{2}} = 0$$

and

$$\frac{d^2 X}{dx_1^2} - \lambda_*^2 X = 0$$

With the lowest non-trivial positive solution λ_* one obtains

$$X(x_1) = B \frac{\sinh \lambda_* x_1}{\lambda_* \cosh \lambda_* l_1} \quad \text{and} \quad u_2^* = BZ(z) \frac{\sinh \lambda_* x_1}{\lambda_* \cosh \lambda_* l_1}$$

Finally, after comparison of the two-dimensional and the three-dimensional solutions one gets

$$\lambda = \lambda_* = \sqrt{\frac{(\Gamma_1 - \Gamma_2)A_{44}}{A_{44}C_{22} - B_{42}^2}} \quad (26)$$

T_{12} and M_{12} obtained by the two-dimensional and the three-dimensional approaches are in a full agreement. For the kinematical fields one gets

$$\langle G_{12}(u_2^* - u_2 - z\varphi_2)^2 \rangle = \min(u_2, \varphi_2) \quad (27)$$

$$\begin{aligned} u_2 &= \frac{M_{12}^* \langle G_{12} z \rangle}{\langle G_{12} \rangle \langle G_{12} z^2 \rangle - \langle G_{12} z^2 \rangle \lambda_* \cosh \lambda_* l_1} \frac{\sinh \lambda_* x_1}{\lambda_* \cosh \lambda_* l_1}, \\ \varphi_2 &= - \frac{M_{12}^* \langle G_{12} \rangle}{\langle G_{12} \rangle \langle G_{12} z^2 \rangle - \langle G_{12} z^2 \rangle \lambda_* \cosh \lambda_* l_1} \frac{\sinh \lambda_* x_1}{\lambda_* \cosh \lambda_* l_1} \end{aligned}$$

It can be seen that the two-dimensional and the three-dimensional results are in agreement only in the averaged sense. The dimension reduction results in some constraints. In our case these constraints are following from the least square condition (27).

In addition, one has to analyze the similar two-dimensional problem for the second direction ($|x_1| < \infty, |x_2| \leq l_2$) and the three-dimensional case ($|x_1| < \infty, |x_2| \leq l_2, |z| \leq h/2$) with the constant torsion moment at the boundaries $|x_2| \leq l_2$. Now one obtains

$$\frac{d}{dz} \left(G_{12} \frac{d\tilde{Z}}{dz} \right) + \eta^2 G_{12} \tilde{Z} = 0, \quad \left. \frac{d\tilde{Z}}{dz} \right|_{|z|=\frac{h}{2}} = 0, \quad (28)$$

$$\eta = \sqrt{\frac{(\Gamma_1 + \Gamma_2)A_{44}}{A_{44}C_{22} - B_{42}^2}} \quad (29)$$

Considering Eqs (26) and (29) we get the following expressions for the transverse shear stiffness tensor components

$$\Gamma_1 = \frac{1}{2}(\lambda^2 + \eta^2) \frac{A_{44}C_{22} - B_{42}^2}{A_{44}}, \quad \Gamma_2 = \frac{1}{2}(\eta^2 - \lambda^2) \frac{A_{44}C_{22} - B_{42}^2}{A_{44}} \quad (30)$$

The values of the transverse shear stiffness are following from the comparison of a two-dimensional and the corresponding three-dimensional boundary value problem. It is easy to see that the structure of these stiffness expressions are different from the classical values since the λ, η are following from the non-traditional identification procedure including a Sturm-Liouville problem. In most of the classical approaches the factors

$$\frac{1}{2}(\lambda^2 + \eta^2) \quad \text{or} \quad \frac{1}{2}(\eta^2 - \lambda^2)$$

are the analogues of the shear corrections (Timoshenko, 1921).

3.4 Special Case – Isotropic Behavior

In the case of isotropic material behavior to set in Eq. (21) one has

$$E_1 = E_2 = E_n = E(z), \quad \nu_{ij} = \nu(z),$$

$$G_{12} = G_{n1} = G_{2n} = G(z) = \frac{E(z)}{2[1 + \nu(z)]}$$

Instead of Eqs (24)–(30) the following non-zero components of the stiffness tensors are valid:

- the in-plane stiffness tensor components

$$A_{11} = \frac{1}{2} \left\langle \frac{E}{1 - \nu} \right\rangle, \quad A_{22} = \frac{1}{2} \left\langle \frac{E}{1 + \nu} \right\rangle = A_{44} = \langle G \rangle \quad (31)$$

- the coupling stiffness tensor components

$$B_{13} = -\frac{1}{2} \left\langle \frac{E}{1 - \nu} z \right\rangle, \quad B_{24} = \frac{1}{2} \left\langle \frac{E}{1 + \nu} z \right\rangle = -B_{42} = \langle Gz \rangle \quad (32)$$

- the plate stiffness tensor components

$$C_{33} = \frac{1}{2} \left\langle \frac{E}{1 - \nu} z^2 \right\rangle, \quad C_{44} = \frac{1}{2} \left\langle \frac{E}{1 + \nu} z^2 \right\rangle = C_{22} = \langle Gz^2 \rangle \quad (33)$$

- the transverse shear stiffness tensor components

$$\Gamma_1 = \lambda^2 \frac{A_{44}C_{22} - B_{42}^2}{A_{44}} \quad (34)$$

with λ following from

$$\frac{d}{dz} \left(G \frac{dZ}{dz} \right) + \lambda^2 GZ = 0, \quad \left. \frac{dZ}{dz} \right|_{|z|=\frac{h}{2}} = 0 \quad (35)$$

4 Examples of Effective Stiffness Properties Estimates

4.1 Homogeneous Plate

The simplest test for the correctness of the estimated stiffness properties is the homogeneous isotropic plate. The basic geometrical property is the plate thickness h , the plate is symmetrical with respect to the mid-plane which results in $\mathbf{B} \equiv \mathbf{0}$. Let us assume the following material data: the Young's modulus E , the Poisson's ratio ν and the shear modulus $G = E/2(1 + \nu)$. All material properties are constant, that means they do not depend on the thickness coordinate z . The non-zero components of the classical stiffness tensors are

$$\begin{aligned} A_{11} &= \frac{Eh}{2(1 - \nu)}, & A_{22} &= \frac{Eh}{2(1 + \nu)} = A_{44} = Gh, \\ C_{33} &= \frac{Eh^3}{24(1 - \nu)}, & C_{44} &= \frac{Eh^3}{24(1 + \nu)} = C_{22} = \frac{Gh^3}{12} \end{aligned}$$

The classical plate (bending) stiffness follows as

$$C_{33} + C_{44} = \frac{Eh^3}{12(1 - \nu^2)} \quad (36)$$

and can be found in the textbooks, e.g. Timoshenko and Woinowsky-Krieger (1985). The transverse shear stiffness follows from

$$\Gamma = \lambda^2 C_{22}$$

with

$$\frac{d^2 Z}{dz^2} + \lambda^2 Z = 0, \quad \left. \frac{dZ}{dz} \right|_{|z|=\frac{h}{2}} = 0$$

The solution $\cos \lambda z = 0$ yields the smallest positive eigenvalue $\lambda = \frac{\pi}{h}$, and finally one obtains

$$\Gamma_1 = \frac{\pi^2}{h^2} \frac{Gh^3}{12} = \frac{\pi^2}{12} Gh,$$

where $\pi^2/12$ is a factor which is similar to the shear correction factor for beams (Timoshenko, 1921). Here this factor is a result of the non-classical establishment of the transverse shear stiffness. Comparing the proposed value with $\pi^2/12$ (Mindlin, 1951) and $5/6$ (Reissner, 1944) one concludes that the direct approach yields the same value like in Mindlin's theory (note that Mindlin's shear correction is based on the solution of a dynamic

problem, here the solution of a static problem is used). The Reissner's value slightly differs.

In addition, the density and the rotatory inertia coefficient are

$$\rho = \rho_0 h, \quad \Theta = \frac{\rho_0 h^3}{12} \quad (37)$$

4.2 Classical Sandwich Plate in Reissner's Sense

Assuming classical sandwich we have the following geometry: h_c is the core thickness and h_f the thickness of the face sheets ($h_c \gg h_f$). The material properties of the core and the face sheets are E_c, E_f, G_c, G_f with $E_c \ll E_f, G_c \ll G_f$. In this case one has a symmetry with respect to the mid-plane that means $\mathbf{B} \equiv \mathbf{0}$. With the thickness $h = h_c + h_f$ one gets

$$\begin{aligned} A_{11} &= \frac{1}{2} \left(\frac{E_f h_f}{1 - \nu_f} + \frac{E_c h_c}{1 - \nu_c} \right), \\ A_{22} &= \frac{1}{2} \left(\frac{E_f h_f}{1 + \nu_f} + \frac{E_c h_c}{1 + \nu_c} \right) = A_{44}, \\ C_{33} &= \frac{1}{24} \left[\frac{E_f (h^3 - h_c^3)}{1 - \nu_f} + \frac{E_c h_c^3}{1 - \nu_c} \right], \\ C_{44} &= \frac{1}{24} \left[\frac{E_f (h^3 - h_c^3)}{1 + \nu_f} + \frac{E_c h_c^3}{1 + \nu_c} \right] = C_{22} \end{aligned}$$

The bending stiffness results in

$$C_{33} + C_{44} = \frac{1}{12} \left[\frac{E_f (h^3 - h_c^3)}{1 - \nu_f^2} + \frac{E_c h_c^3}{1 - \nu_c^2} \right]$$

The transverse shear stiffness can be computed by

$$\Gamma_1 = \lambda^2 C_{22}$$

with λ as the smallest positive solution of

$$Z(z) = \begin{cases} A \cos \lambda \left(z - \frac{h}{2} \right) & \frac{h_c}{2} \leq z \leq \frac{h}{2}, \\ B \sin \lambda z & |z| \leq \frac{h_c}{2}, \\ -A \cos \lambda \left(z + \frac{h}{2} \right) & -\frac{h}{2} \leq z \leq -\frac{h_c}{2} \end{cases}$$

This yields a transcendent equation

$$\mu \cos \lambda \frac{h_f}{2} \cos \lambda \frac{h_c}{2} - \sin \lambda \frac{h_f}{2} \sin \lambda \frac{h_c}{2} = 0$$

or

$$\mu \cos \gamma (1 - \alpha) \cos \gamma \alpha - \sin \gamma (1 - \alpha) \sin \gamma \alpha = 0$$

with

$$\gamma = \lambda \frac{h}{2}, \quad \alpha = \frac{h_c}{h}$$

μ and α take the values $0 \leq \mu = G_c/G_f < \infty$ and $0 \leq \alpha \leq 1$. A typical sandwich structure has a very weak core. Then the bending stiffness can be approximated by

$$C_{33} + C_{44} = \frac{1}{4} \frac{E_f h^2 h_f}{1 - \nu_f^2} \quad (38)$$

and the transverse shear stiffness by $\Gamma = G_c h$. This solution was firstly obtained by Reissner (1947).

In addition, the density and the rotatory inertia coefficient are now

$$\rho = \rho_{c0} h_c + \rho_{f0} h_f, \quad \Theta = \frac{\rho_{c0} h_c^3}{12} + \frac{\rho_{f0} (h^3 - h_c^3)}{12} \quad (39)$$

4.3 Functionally Graded Materials and Foams

Functionally graded materials are composite materials with locally varied composition or microstructure so that a certain variation of the local material properties is achieved. In this paragraph we consider small deformations of a functionally graded plate made of a foam. For the panel made of a porous foam the distribution of the pores over the thickness can be inhomogeneous (see, for example, Fig. 2). Let us introduce h as the thickness of the panel, ρ_s as the density of the bulk material and ρ_p as the minimum value of the density of the foam. Examples of distribution functions of the porosity are shown in Fig. 4.

Linear Symmetric Distribution. Let us introduce a plate material with changing mechanical properties in the thickness direction. The material is assumed to be isotropic. The mechanical properties are changing linear and the changes are symmetrical with respect to the mid-plane. In addition, it will be suggested that the changing properties are related to the Young's and the shear modulus. To make the calculation as simple as possible the Poisson's ratio is set to be approximately constant ($\nu \approx \text{const}$). Then the distribution of the Young's and the shear modulus can be given as

$$E(z) = E_o \left[\frac{2}{h} |z| + \frac{E_m}{E_o} \left(1 - \frac{2}{h} |z| \right) \right], \quad (40)$$

$$G(z) = G_o \left[\frac{2}{h} |z| + \frac{G_m}{G_o} \left(1 - \frac{2}{h} |z| \right) \right] \quad (41)$$

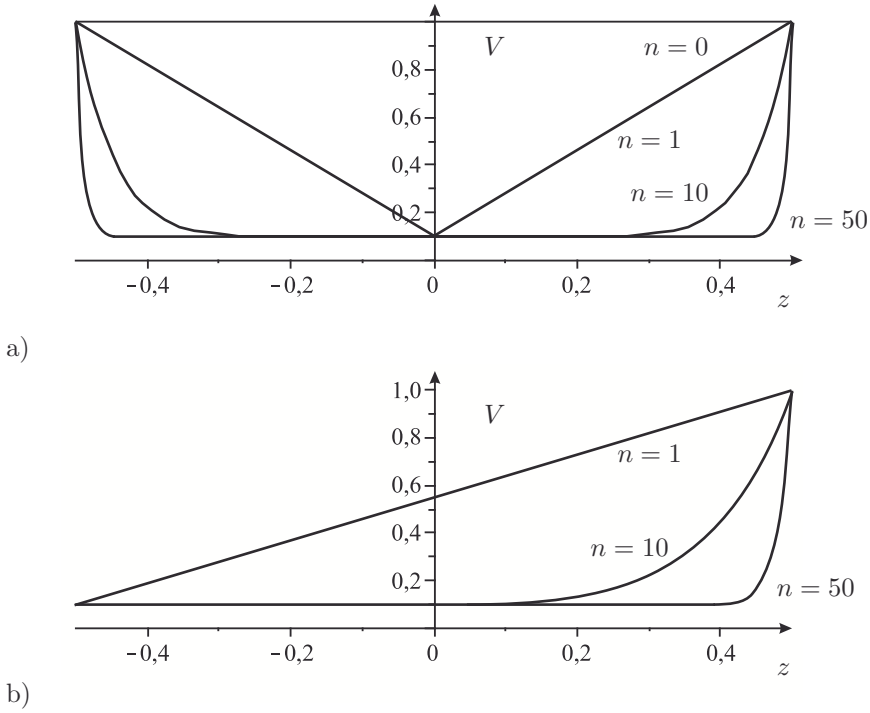


Figure 4. Distribution of the porosity V : a) symmetric, b) non-symmetric

Here the index o denotes the value on the outer surfaces ($z = \pm h/2$) and m at the mid-plane ($z = 0$). Due to the symmetry to the mid-plane $\mathbf{B} \equiv \mathbf{0}$. The classical stiffness tensor components can be computed analytically

$$\begin{aligned} A_{11} &= \frac{E_o h}{4(1-\nu)} \left(1 + \frac{E_m}{E_o} \right), & A_{22} &= \frac{E_o h}{4(1+\nu)} \left(1 + \frac{E_m}{E_o} \right) = A_{44}, \\ C_{33} &= \frac{E_o h^3}{96(1-\nu)} \left(3 + \frac{E_m}{E_o} \right), & C_{44} &= \frac{E_o h^3}{96(1+\nu)} \left(3 + \frac{E_m}{E_o} \right) = C_{22} \end{aligned} \quad (42)$$

The transverse shear stiffness tensor components results from

$$I_1 = \lambda^2 C_{22} = \lambda^2 \frac{G_o h^3}{48} \left(3 + \frac{G_m}{G_o} \right)$$

with λ following from

$$\frac{d}{dz} \left[G(z) \frac{dZ}{dz} \right] + \lambda^2 G(z) Z = 0, \quad \left. \frac{dZ}{dz} \right|_{|z|=\frac{h}{2}} = 0$$

Instead of $G(z)$ one has to take Eq. (41). This spectral problem can be solved numerically by using the shooting method (Stoer and Bulirsch, 1980). The dependence of normalized Γ_1 versus $g_m \equiv G_m/G_o$ is shown in Fig. 5 a) (solid line), the Reissner's solution is presented by the dashed line. The Reissner's solution gives understated values of the transverse stiffness when the difference between elastic moduli is significantly high. On the other

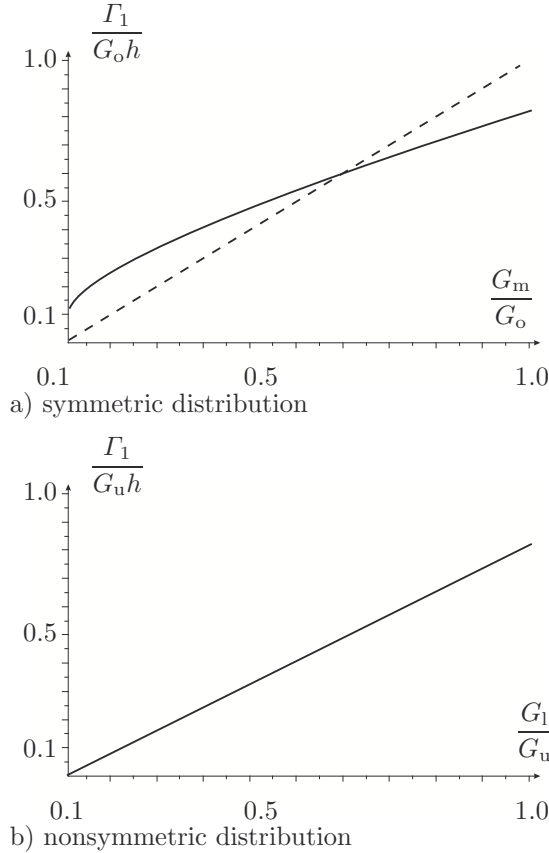


Figure 5. Normalized transverse shear stiffness

hand, the Reissner's solution gives overstated values when the elastic moduli are approximately the same. The Reissner's formula gives us good coincidence with our results only for $g_m \approx 0.6$.

Linear nonsymmetric distribution. If the indices u and l denote the values of the Young's modulus and the shear modulus on the upper surface ($z = h/2$) and the lower surface ($z = -h/2$), respectively, one can represent the linear distribution by

$$E(z) = \frac{1}{2}(E_u - E_l)z\frac{2}{h} + \frac{1}{2}(E_u + E_l), \quad (43)$$

$$G(z) = \frac{1}{2}(G_u - G_l)z\frac{2}{h} + \frac{1}{2}(G_u + G_l) \quad (44)$$

and with the assumption of the constant Poisson's ratio one gets

$$\begin{aligned} A_{11} &= \frac{(E_u + E_l)h}{4(1 - \nu)}, & A_{22} &= \frac{(E_u + E_l)h}{4(1 + \nu)} = A_{44}, \\ B_{13} &= -\frac{(E_u - E_l)h^2}{24(1 - \nu)}, & B_{24} &= \frac{(E_u - E_l)h^2}{24(1 + \nu)} = B_{42}, \\ C_{33} &= \frac{(E_u + E_l)h^3}{24(1 - \nu)}, & C_{44} &= \frac{(E_u + E_l)h^3}{48(1 + \nu)} = C_{22} \end{aligned}$$

The transverse shear stiffness can be estimated by

$$\Gamma_1 = \lambda^2 \frac{A_{44}C_{22} - B_{42}^2}{A_{44}} = \frac{\lambda^2}{72} \frac{[3(G_u + G_l)^2 - (G_u - G_l)^2]h^3}{G_u + G_l}$$

with λ following from

$$\frac{d}{dz} \left[G(z) \frac{dZ}{dz} \right] + \lambda^2 G(z) Z = 0, \quad \left. \frac{dZ}{dz} \right|_{|z|=\frac{h}{2}} = 0$$

Instead of $G(z)$ one has to take Eq. (44). The solution of this spectral problem may be written by using the Kummer functions (Abramowitz and Stegun, 1972). The dependence of the normalized transverse shear stiffness Γ_1 vs. $g_l \equiv G_l/G_u$ is shown in Fig. 5 b). It is easy to prove that a good approximation of this curve is given by

$$\frac{\Gamma_1}{G_u h^3} = \frac{\pi^2}{12} + 0.416(1 - g_l)$$

Nonlinear symmetric distribution. For the description of the symmetric distribution of the porosity we assume the power law (Altenbach and Eremeyev, 2008a)

$$V(z) = \alpha + (1 - \alpha) \left| \frac{2z}{h} \right|^n, \quad (45)$$

where $\alpha = \rho_p/\rho_s$. The power $n = 0$ corresponds to the homogeneous plate, for $n = 1$ we have the linear distribution of the porosity through the thickness. The distribution can be established, for example, if the plate is made of two symmetric layers. If $n > 1$ one has a more complex distribution. If $n \gg 1$ the plate core has an approximately constant porosity, but the distribution of the density in the face layers is significant inhomogeneous. Examples of the distribution functions are shown in Fig. 4 a) for $n = 0, 1, 10, 50$ with $\alpha = 0.1$.

The elastic properties of metallic foams highly depend on the porosity and the cell structure. The dependence of the Young's and the shear modulus on the porosity is given by the power law (Ashby et al., 2000; Gibson and Ashby, 1997)

$$\frac{E_p}{E_s} \sim \alpha^m, \quad \frac{G_p}{G_s} \sim \alpha^m, \quad (46)$$

where E_p and G_p are the Young's and the shear modulus of the foam, respectively, E_s and G_s are the Young's and the shear modulus of the bulk material which is used to synthesize the foam. The value m depends on the structure of the foam. For the closed-cell foam $m = 1$, for the open-cell foam $m \approx 2$. For the isotropic case, the graphs of the normalized stiffness parameters A_{11} , A_{44} , C_{33} , C_{22} are given in Fig. 6. Here $m = 2$, $\alpha = 0.1$, $\nu = 0.3$ are assumed.

To obtain the dependence of transverse shear stiffness we have to solve Eq. (35). The solution of the spectral problem (35) can be performed again numerically by using the shooting method (Stoer and Bulirsch, 1980). The values of normalized transverse shear stiffness Γ_1 vs. n are given in Fig. 7.

The density and the rotatory inertia coefficient are

$$\frac{1}{\rho_s} \rho = \alpha h + \frac{(1 - \alpha) h}{n + 1}, \quad \frac{1}{\rho_s} \Theta = \frac{1}{12} \alpha h^3 + \frac{1}{4} \frac{(1 - \alpha) h^3}{3 + n} \quad (47)$$

Setting $n \gg 1$ in Eq. (47) one can see that for $\rho \approx \rho_p h$ and $\Theta \approx \frac{1}{12} \rho_p h^3$ which coincide with Eq. (39) if $h_f \rightarrow 0$.

Nonlinear non-symmetric distribution. For the description of the non-symmetric distribution of the porosity we assume the following power

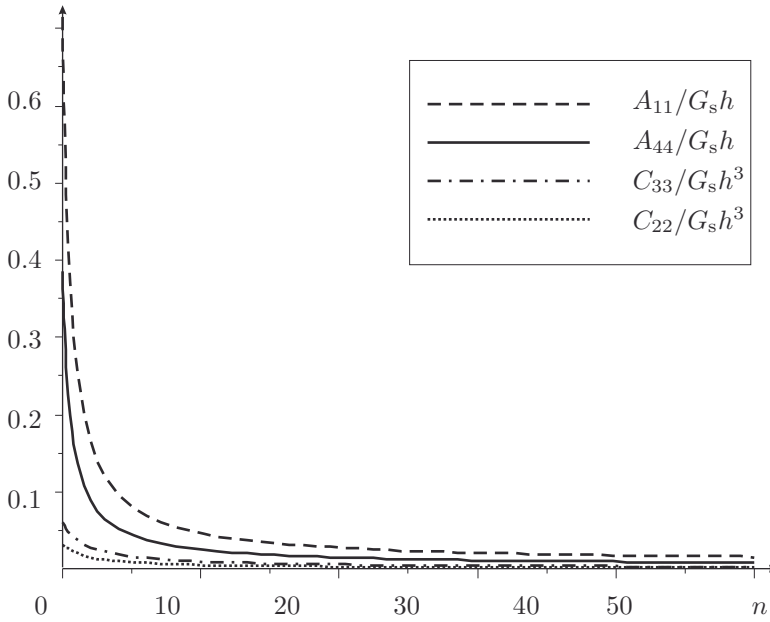


Figure 6. Dimensionless effective elastic moduli

law

$$V(z) = \alpha + (1 - \alpha) \left(\frac{z + h/2}{h} \right)^n \quad (48)$$

Examples of the distribution law are shown in Fig. 4 b) for $n = 0, 10, 50$ with $\alpha = 0.1$. For the isotropic case, the graphs of the normalized stiffness parameters A_{11} , A_{44} , C_{33} , C_{22} , B_{13} , B_{24} are given in Figs 8 and 9. Since the structure is not symmetrically with respect to the mid-plane the in-plane, the coupled and the out-of-plane stiffness parameters must be taken into account. The values of normalized transverse shear stiffness I_1 versus n are given in Fig. 10.

4.4 On the Plates Made of Nanofoams

In this paragraph let us briefly discuss the possible applications of the described above theory to the case of nanoporous materials. Such materials have many applications in the nanotechnology, see Bhushan (2004). The example of the foam-like structure made of ZnO is given in Fig. 11. It is

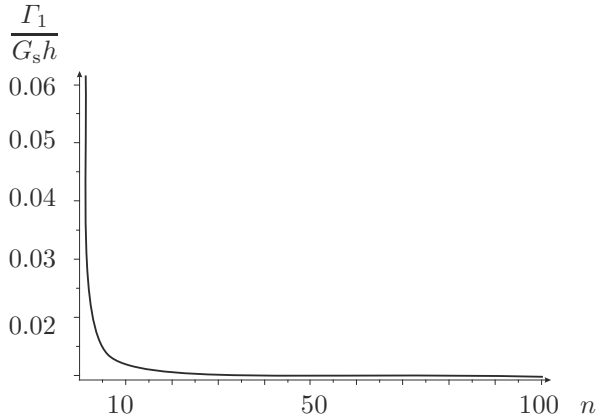


Figure 7. Normalized transverse shear stiffness

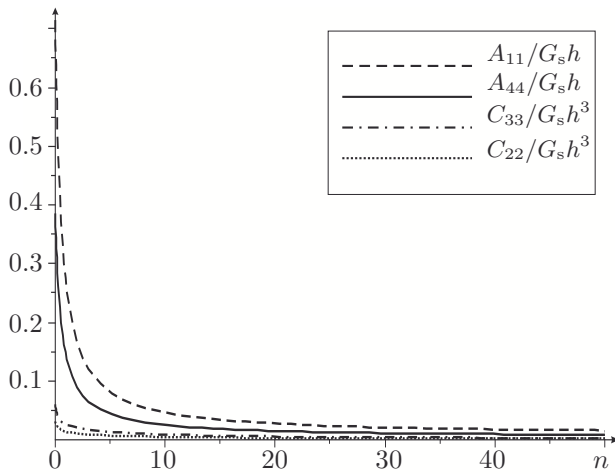


Figure 8. Normalized in-plane and plate stiffness components

seeing that this foam belongs to the type of the open-cell foams. One of the characteristic features of the nanomaterials is the size effect, for example the dependence of the Young's modulus of a nanosize rod on its radius, see Cuenot et al. (2004); Jing et al. (2006); Chen et al. (2006).

One of possible reasons for such phenomenon is the influence of surface stresses, see the review by Duan et al. (2008). Wang et al. (2006) proposed

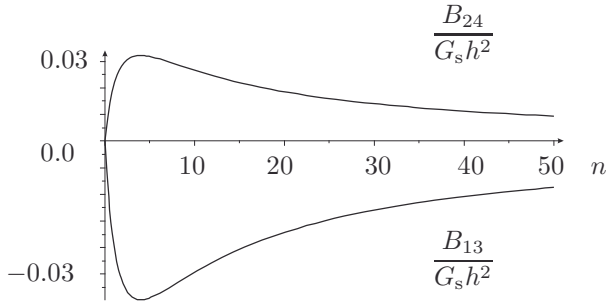


Figure 9. Normalized coupling stiffness components

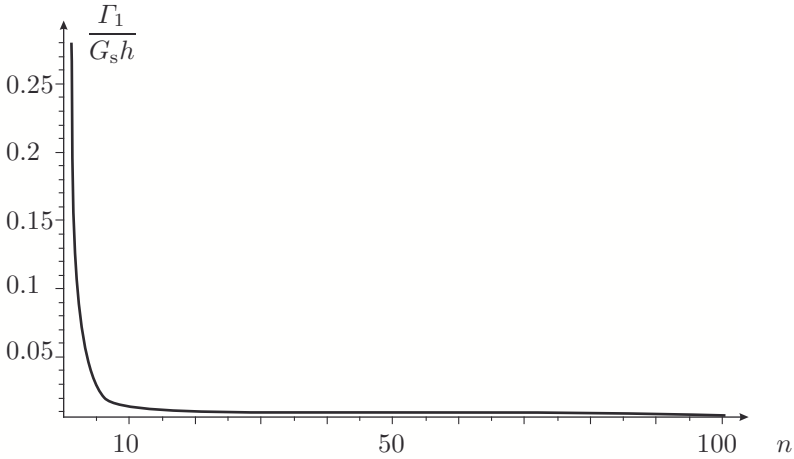


Figure 10. Normalized transverse shear stiffness

the scaling law for the a material property of the nanosize object. Following Wang et al. (2006); Duan et al. (2008) we assume that the Young's modulus E_n of a foam strut with the cross-section of the radius R is given by the following relation

$$E_n = E_s \left(1 + \chi \frac{l_{in}}{R} \right), \quad (49)$$

where E_s is the Young's modulus of a bulk material, l_{in} is an intrinsic length scale related to the surface property, and χ is a nondimensional constant. The formulae (46) were derived by Gibson and Ashby (1997); Ashby et al. (2000) on the consideration of a foam with cubic cells, see also the chapter

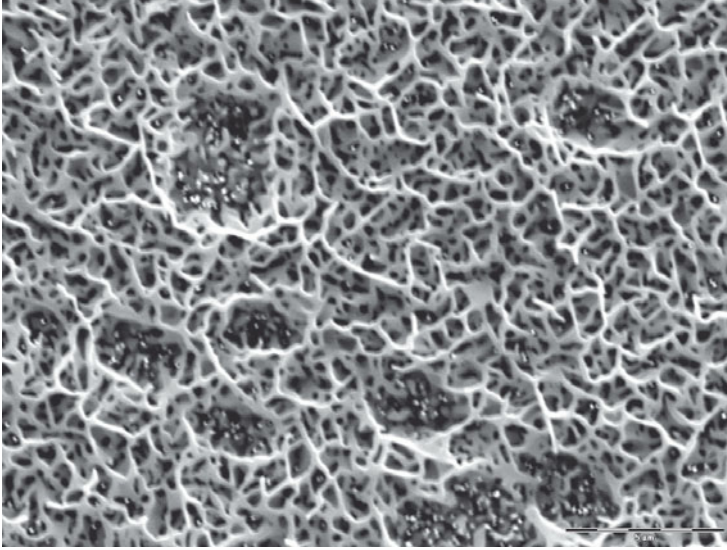


Figure 11. Foam-like structure of ZnO (photos by courtesy of K. Dvadenko, South Scientific Center of RASci, the specimen by E. Kaidashev, South Federal University, Rostov on Don)

of A. Öchsner in this book. Using the scaling law (49) and this technique we modify the dependence of the elastic moduli of a nanofoam on the porosity by the law

$$\frac{E_{np}}{E_s} \sim \left(1 + \chi \frac{l_{in}}{R}\right) \alpha^m, \quad \frac{G_p}{G_s} \sim \left(1 + \chi \frac{l_{in}}{R}\right) \alpha^m, \quad (50)$$

where E_{np} and G_p are the Young's and shear moduli of a nanofoam, respectively. l_{in} is related to the surface effects and is typically in the order of 0.01 – 0.1 nm, see Wang et al. (2006); Duan et al. (2008). Hence, for a foam with nanosize cells one has only one modification for the elastic properties – the elastic moduli are stiffer then in the case of macrosized cells. This conclusion is similar to the presented one in Duan et al. (2008) for nanochannel-array materials.

5 Symmetric Orthotropic Plate - Static Case

Let us consider the equilibrium equations of a symmetric orthotropic plate. In this case Eqs (15) are split into two parts: the in-plane problem for the

tangential displacements u_1 and u_2 , and the bending problem for the w , φ_1 and φ_2 , respectively. The constitutive equations for a symmetric orthotropic plate can be given as follows

$$\begin{aligned}
 T_1 &= (A_{11} + 2A_{12} + A_{22})\mu_1 + (A_{11} - A_{22})\mu_2, \\
 T_2 &= (A_{11} - A_{22})\mu_1 + (A_{11} + 2A_{12} + A_{22})\mu_2, \\
 T_{12} &= 2A_{44}\mu_{12}, \\
 T_{1n} &= (\Gamma_1 + \Gamma_2)\gamma_1, \\
 T_{2n} &= (\Gamma_1 - \Gamma_2)\gamma_2, \\
 M_1 &= (C_{33} - 2C_{34} + C_{44})\kappa_1 + (C_{33} - C_{44})\kappa_2, \\
 M_2 &= (C_{33} - C_{44})\kappa_1 + (C_{33} + 2C_{34} + C_{44})\kappa_2, \\
 M_{12} &= C_{22}(\kappa_{12} + \kappa_{21})
 \end{aligned} \tag{51}$$

With the geometrical relations (19) the equations

$$T_{1,1} + T_{12,2} + q_1 = 0, \quad T_{12,1} + T_{2,2} + q_2 = 0 \tag{52}$$

are reduced to the form

$$\begin{aligned}
 (A_{11} + 2A_{12} + A_{22})u_{1,11} + (A_{11} - A_{22})u_{2,21} \\
 + A_{44}(u_{1,21} + u_{2,11}) + q_1 &= 0, \\
 (A_{11} - A_{22})u_{1,12} + (A_{11} - 2A_{12} + A_{22})u_{2,22} + \\
 + A_{44}(u_{1,22} + u_{2,12}) + q_2 &= 0
 \end{aligned} \tag{53}$$

The equation

$$T_{1n,1} + T_{2n,2} + q_n = 0 \tag{54}$$

takes the following form

$$(\Gamma_1 + \Gamma_2)w_{,11} + (\Gamma_1 - \Gamma_2)w_{,22} + (\Gamma_1 + \Gamma_2)\varphi_{1,1} + (\Gamma_1 - \Gamma_2)\varphi_{2,2} + q_n = 0 \tag{55}$$

The equations

$$M_{1,1} + M_{12,2} - T_{1n} + m_1 = 0, \quad M_{12,1} + M_{2,2} - T_{2n} + m_2 = 0 \tag{56}$$

result in

$$\begin{aligned}
 (C_{33} - 2C_{34} + C_{44})\varphi_{1,11} + (C_{33} - C_{44})\varphi_{2,21} \\
 + C_{22}(\varphi_{1,22} + \varphi_{2,12}) - (\Gamma_1 + \Gamma_2)\varphi_1 - (\Gamma_1 + \Gamma_2)w_{,1} + m_1 &= 0, \\
 (C_{33} + 2C_{34} + C_{44})\varphi_{2,22} + (C_{33} - C_{44})\varphi_{1,12} \\
 + C_{22}(\varphi_{1,21} + \varphi_{2,11}) - (\Gamma_1 - \Gamma_2)\varphi_2 - (\Gamma_1 - \Gamma_2)w_{,2} + m_2 &= 0
 \end{aligned} \tag{57}$$

5.1 Bending Problem - One-dimensional Case

First, let us consider the bending of a rectangular plate when $q_1 = 0$, $q_2 = 0$, $q_n = q(x)$, $m_1 = m_1(x)$, $m_2 = 0$. Then we can assume that $u_1 = u_2 = 0$, $w = w(x)$, $\varphi_1 = \varphi(x)$, $\varphi_2 = 0$. Equation (55) reduces to

$$(\Gamma_1 + \Gamma_2)w'' + (\Gamma_1 + \Gamma_2)\varphi' + q = 0, \quad (58)$$

where $(\dots)'$ denotes $(\dots)_1$. Equations (57) reduce to one equation

$$(C_{33} - 2C_{34} + C_{44})\varphi'' - (\Gamma_1 + \Gamma_2)\varphi - (\Gamma_1 + \Gamma_2)w' + m_1 = 0 \quad (59)$$

Let us transform Eqs (58) and (59) to one equation with respect to w . After some manipulations we obtain

$$(C_{33} - 2C_{34} + C_{44})w'''' = q - \frac{(C_{33} - 2C_{34} + C_{44})}{\Gamma_1 + \Gamma_2}q'' + m_1' \quad (60)$$

If we consider the one-dimensional case for the second direction ($q_1 = 0$, $q_2 = 0$, $q_n = q(y)$, $m_1 = 0$, $m_2 = m_2(y)$), then we can assume that $u_1 = u_2 = 0$, $w = w(y)$, $\varphi_2 = \varphi(y)$, $\varphi_1 = 0$. Finally, Eqs (55) and (57) reduce to

$$(\Gamma_1 - \Gamma_2)w'' + (\Gamma_1 - \Gamma_2)\varphi' + q_n = 0, \quad (61)$$

$$(C_{33} + 2C_{34} + C_{44})\varphi'' - (\Gamma_1 - \Gamma_2)\varphi - (\Gamma_1 - \Gamma_2)w' + m_2 = 0, \quad (62)$$

where now $(\dots)'$ denotes $(\dots)_2$.

By using the same manipulations as in the previous case we obtain the following equation with respect to w

$$(C_{33} + 2C_{34} + C_{44})w'''' = q - \frac{(C_{33} + 2C_{34} + C_{44})}{\Gamma_1 - \Gamma_2}q'' + m_2' \quad (63)$$

Equations (60) or (63) differ from the one-dimensional variant of Eq. (1). If $m_1 = 0$ or $m_2 = 0$ and $\Gamma_1 + \Gamma_2 \rightarrow \infty$ or $\Gamma_1 - \Gamma_2 \rightarrow \infty$ one gets the Kirchhoff's plate equation with $D = C_{33} - 2C_{34} + C_{44}$ or with $D = C_{33} + 2C_{34} + C_{44}$.

5.2 Bending Problem - Two-dimensional Case

Let us eliminate the functions φ_1 and φ_2 from Eqs (55) and (57). For brevity, let us assume that $m_1 = 0$, $m_2 = 0$. By using operator notations, Eqs (57) may be rewritten in the form

$$L_{11}\varphi_1 + L_{12}\varphi_2 = b_1w, \quad L_{21}\varphi_1 + L_{22}\varphi_2 = b_2w, \quad (64)$$

where

$$\begin{aligned}
 L_{11} &= (C_{33} - 2C_{34} + C_{44})\partial_1^2 + C_{22}\partial_2^2 - \Gamma_1 - \Gamma_2, \\
 L_{22} &= (C_{33} + 2C_{34} + C_{44})\partial_2^2 + C_{22}\partial_1^2 - \Gamma_1 + \Gamma_2, \\
 L_{21} &= L_{12} = (C_{33} + C_{22} - C_{44})\partial_1\partial_2, \\
 b_1 &= (\Gamma_1 + \Gamma_2)\partial_1, \\
 b_2 &= (\Gamma_1 - \Gamma_2)\partial_2, \\
 \partial_\alpha(\dots) &\equiv (\dots)_{,\alpha}, \quad \alpha = 1, 2
 \end{aligned}$$

From Eqs (64) we obtain the relations $\mathcal{L}\varphi_1 = \mathcal{L}_1 w$, $\mathcal{L}\varphi_2 = \mathcal{L}_2 w$ with

$$\mathcal{L} = L_{11}L_{22} - L_{12}^2, \quad \mathcal{L}_1 = L_{22}b_1 - L_{12}b_2, \quad \mathcal{L}_2 = L_{11}b_2 - L_{21}b_1$$

Using operator notations Eq. (55) can be rewritten as follows

$$\mathcal{L}_w w + b_1\varphi_1 + b_2\varphi_2 + q_n = 0,$$

where $\mathcal{L}_w = (\Gamma_1 + \Gamma_2)\partial_1^2 + (\Gamma_1 - \Gamma_2)\partial_2^2$. Then we obtain one differential equation of 6th order with respect to w

$$(\mathcal{L}\mathcal{L}_w + b_1\mathcal{L}_1 + b_2\mathcal{L}_2)w + \mathcal{L}q_n = 0 \quad (65)$$

5.3 Bending of an Isotropic Plate

Assuming isotropic behavior we have $C_{22} = C_{44}$, $C_{34} = 0$, $\Gamma_2 = 0$. Then

$$\begin{aligned}
 L_{11} &= (C_{33} + C_{44})\partial_1^2 + C_{44}\partial_2^2 - \Gamma_1, \\
 L_{22} &= (C_{33} + C_{44})\partial_2^2 + C_{44}\partial_1^2 - \Gamma_1, \\
 L_{21} &= C_{33}\partial_1\partial_2 = L_{12}, \\
 b_1 &= \Gamma_1\partial_1, \\
 b_2 &= \Gamma_1\partial_2 \\
 \mathcal{L}_w &= \Gamma_1(\partial_1^2 + \partial_2^2) = \Gamma_1\Delta,
 \end{aligned}$$

and

$$\begin{aligned}
 \mathcal{L} &= [(C_{33} + C_{44})\partial_1^2 + C_{44}\partial_2^2 - \Gamma_1] [(C_{33} + C_{44})\partial_2^2 \\
 &\quad + C_{44}\partial_1^2 - \Gamma_1] - C_{33}^2\partial_1^2\partial_2^2, \\
 \mathcal{L}_1 &= \Gamma_1[(C_{33} + C_{44})\partial_2^2 + C_{44}\partial_1^2 - \Gamma_1]\partial_1 + \Gamma_1 C_{33}\partial_1\partial_2^2, \\
 \mathcal{L}_2 &= -\Gamma_1 C_{33}\partial_1^2\partial_2 + \Gamma_1[(C_{33} + C_{44})\partial_1^2 + C_{44}\partial_2^2 - \Gamma_1]\partial_2
 \end{aligned}$$

Finally, the bending equation (65) has the form

$$(C_{33} + C_{44})\Delta\Delta w = \partial_1 m_1 + \partial_2 m_2 - \frac{C_{33} + C_{44}}{\Gamma_1} \Delta q_n + q_n \quad (66)$$

If $m_1 = m_2 = 0$ and $\Gamma_1 \rightarrow \infty$ one gets again the Kirchhoff's plate equation with the bending stiffness $D = C_{33} + C_{44}$.

5.4 Bending of an Elastic Plate Made of FGM (Symmetric Case)

In this case one gets a decoupling of the in-plane and the plate state. Let us assume the Kirchhoff's plate bending. The governing equation (1) preserves the same form - only instead of the bending stiffness D the effective bending stiffness with respect to Eqs (33) must be introduced as follows

$$D_{\text{eff}} = C_{33} + C_{44} \quad (67)$$

and the external load q should be substituted by

$$\partial_1 m_1 + \partial_2 m_2 - \frac{D_{\text{eff}}}{\Gamma_1} \Delta q_n + q_n$$

To analyze the influence of the transverse shear stiffness on the deflection of the plate let us consider the bending problem of a rectangular plate made of a FGM. For brevity, let us assume that $m_1 = m_2 = 0$, $x_1 \in [0, a]$, $x_2 \in [0, b]$, where a and b are the length and width of the plate. Using the assumption that $\nu = \text{const}$ (Gibson and Ashby, 1997; Ashby et al., 2000) and that the Eqs (33), (34) are valid, we can rewrite Eq. (66) in the following form

$$D_{\text{eff}} \Delta \Delta w = q_n - \frac{2}{1 - \nu} \frac{1}{\lambda^2 h^2} \Delta q_n \quad (68)$$

Introducing dimensionless variables by the formulas

$$\bar{w} = h^{-1} w, \quad \bar{x}_1 = h^{-1} x_1, \quad \bar{x}_2 = h^{-1} x_2, \quad \bar{x}_1 \in \left[0, \frac{a}{h}\right], \quad \bar{x}_2 \in \left[0, \frac{b}{h}\right], \quad \bar{q} = \frac{q_n h^3}{D_{\text{eff}}},$$

Eq. (68) transforms to

$$\Delta \Delta \bar{w} = \bar{q} - \frac{2}{1 - \nu} \frac{1}{\lambda^2 h^2} \Delta \bar{q}, \quad \Delta = \frac{\partial^2}{\partial \bar{x}_1^2} + \frac{\partial^2}{\partial \bar{x}_2^2} \quad (69)$$

Let us consider a sinusoidal load

$$\bar{q} = Q \sin \frac{\pi h \bar{x}_1}{a} \sin \frac{\pi h \bar{x}_2}{b}, \quad Q = \text{const}$$

and the boundary conditions of type (18). Then the solution of Eq. (69) is given by

$$\bar{w} = \frac{K}{\eta^2} Q \sin \frac{\pi h \bar{x}_1}{a} \sin \frac{\pi h \bar{x}_2}{b}, \quad (70)$$

$$K = 1 + \frac{2\eta}{1-\nu} \frac{1}{\lambda h^2}, \quad \eta = \left(\frac{\pi h}{a} \right)^2 + \left(\frac{\pi h}{b} \right)^2$$

Within the Kirchhoff plate theory $K = K_K \equiv 1$, within the Mindlin plate theory

$$K = K_M \equiv 1 + \frac{2\eta}{1-\nu} \frac{1}{\pi^2}$$

Let us consider the maximal deflection as

$$W = \frac{K}{\eta^2} Q$$

Thus, for the Kirchhoff plate theory we obtain that

$$W_K = \frac{1}{\eta^2} Q,$$

for the Mindlin plate theory

$$W_M = \frac{K_M}{\eta^2} Q,$$

and in the general case

$$W_F = \frac{K}{\eta^2} Q$$

Let us use the following values $\nu = 0.3$, $a = b$, $h = 0.05a$. The normalized deflections are presented in the Table 2. It is easy to see that for the functionally graded plates the difference between the theories can be significant. Let us note that for other types of boundary conditions the influence of the plate structure on the deflection may be greater than for the simple support type boundary conditions.

6 Dynamics of Plates Made of an Elastic Foam

6.1 Equations of Motion for a Symmetric Isotropic Plate

In this case Eqs (15) are split into two parts: the in-plane problem for the tangential displacements u_1 and u_2 , and the bending problem for the

Table 2. Maximal deflections of a quadratic plate and corresponding Γ_1

Model	W/W_K	$\Gamma_1/G_s h^3$
Kirchoff's plate	1	∞
Mindlin's plate	1.01	$\frac{\pi^2}{12} \approx 0.82$
FGM plate linear distribution (40)		
$g_m = 0.5$	1.02	0.54
$g_m = 0.1$	1.036	0.25
$g_m = 0.05$	1.045	0.2
FGM plate power law (45)		
$n = 2$	1.12	0.027
$n = 5$	1.15	0.012
$n = 10$	1.05	0.01

w , φ_1 and φ_2 , respectively. In Cartesian coordinates with the geometrical relations (19) Eqs (15)_{1,2} reduce to the form

$$\begin{aligned}
 (A_{11} + A_{22})u_{1,11} + (A_{11} - A_{22})u_{2,21} \\
 + A_{22}(u_{1,21} + u_{2,11}) + q_1 &= \rho \ddot{u}_1, \\
 (A_{11} - A_{22})u_{1,12} + (A_{11} + A_{22})u_{2,22} \\
 + A_{22}(u_{1,22} + u_{2,12}) + q_2 &= \rho \ddot{u}_2
 \end{aligned} \tag{71}$$

The Eq. (15)₃ has the following form

$$\Gamma(w_{,11} + w_{,22}) + \Gamma(\varphi_{1,1} + \varphi_{2,2}) + q_n = \rho \ddot{w} \tag{72}$$

The Eqs (15)_{4,5} result in

$$\begin{aligned}
 (C_{33} + C_{22})\varphi_{1,11} + (C_{33} - C_{22})\varphi_{2,21} \\
 + C_{22}(\varphi_{1,22} + \varphi_{2,12}) - \Gamma\varphi_1 - \Gamma w_{,1} + m_1 &= \Theta \ddot{\varphi}_1, \\
 (C_{33} + C_{22})\varphi_{2,22} + (C_{33} - C_{22})\varphi_{1,12} \\
 + C_{22}(\varphi_{1,21} + \varphi_{2,11}) - \Gamma\varphi_2 - \Gamma w_{,2} + m_2 &= \Theta \ddot{\varphi}_2
 \end{aligned} \tag{73}$$

To eliminate the functions φ_1 and φ_2 from Eqs (72) and (73) let us differentiate the first equation of (73) with respect to x_1 , while the second one with respect to x_2 . Summing up these equations one gets

$$[(C_{33} + C_{22})\Delta - \Gamma](\varphi_{1,1} + \varphi_{2,2}) - \Gamma\Delta w + m_{1,1} + m_{2,2} = \Theta(\ddot{\varphi}_{1,1} + \ddot{\varphi}_{2,2}) \quad (74)$$

Here $\Delta(\dots) = (\dots)_{,11} + (\dots)_{,22}$.

Using Eq. (72) we can eliminate $\varphi_{1,1} + \varphi_{2,2}$ from Eq. (74). Thus, we obtain one equation with respect to w

$$\begin{aligned} & (C_{33} + C_{22})\Delta\Delta w + \rho\ddot{w} + \frac{\rho\Theta}{\Gamma}\ddot{w} - \left(\Theta + \rho\frac{C_{33} + C_{22}}{\Gamma}\right)\Delta\ddot{w} \\ &= q_n + \frac{\Theta}{\Gamma}\ddot{q}_n - \frac{C_{33} + C_{22}}{\Gamma}\Delta q_n + m_{1,1} + m_{2,2} \end{aligned} \quad (75)$$

6.2 Free Oscillations and Dispersion curves of a Rectangular Plate

To analyze the influence of the transverse shear stiffness and the rotatory inertia on the natural frequencies let us consider natural oscillations of a rectangular plate with simple-support type boundary conditions. Let us assume that $m_1 = m_2 = 0$, $q_n = 0$, $x_1 \in [0, a]$, $x_2 \in [0, b]$, where a and b are the length and the width of the plate. Thus, we can rewrite Eq. (75) in the following form

$$D_{\text{eff}}\Delta\Delta w + \rho\ddot{w} + \frac{\rho\Theta}{\Gamma}\ddot{w} - \left(\Theta + \rho\frac{D_{\text{eff}}}{\Gamma}\right)\Delta\ddot{w} = 0 \quad (76)$$

Introducing dimensionless variables by

$$\begin{aligned} \bar{w} &= h^{-1}w, \quad \bar{x}_1 = h^{-1}x_1, \quad \bar{x}_2 = h^{-1}x_2, \\ \bar{x}_1 &\in \left[0, \frac{a}{h}\right], \quad \bar{x}_2 \in \left[0, \frac{b}{h}\right], \quad \bar{t} = T^{-1}t, \quad T^2 = \frac{\rho h^4}{D_{\text{eff}}}, \end{aligned}$$

Eq. (76) transforms to

$$\Delta\Delta\bar{w} + \ddot{\bar{w}} + \beta\ddot{\bar{w}} - \zeta\Delta\ddot{\bar{w}} = 0 \quad (77)$$

Here

$$\Delta = \frac{\partial^2}{\partial \bar{x}_1^2} + \frac{\partial^2}{\partial \bar{x}_2^2},$$

while

$$\beta = \frac{\Theta D_{\text{eff}}}{\rho \Gamma h^4}, \quad \zeta = \frac{1}{h^2} \left(\frac{\Theta}{\rho} + \frac{D_{\text{eff}}}{\Gamma} \right) \quad (78)$$

are dimensionless parameters.

Let us note that, if one considers the limiting case with $\Gamma \rightarrow \infty$ and $\Theta = 0$, then $\beta = \zeta = 0$. Thus, the equation of motion for the Kirchhoff plate follows immediately from Eq. (77) as

$$\Delta \Delta \bar{w} + \ddot{\bar{w}} = 0 \quad (79)$$

Obviously, Eq. (77) contains two small parameters β and ζ and can be considered as a singular perturbation of Eq. (79). While the small parameters are the coefficients of the derivatives with respect to time of higher order, then one can find the difference between the solutions of Eqs (77) and (79) for highly oscillated modes. Equation (77) contains a term with a small parameter on the mixed derivative with respect to time and space.

The solution of Eq. (77) can be given by

$$\bar{w} = W_{mn} \exp(i\omega \bar{t}) \sin \frac{\pi h m \bar{x}_1}{a} \sin \frac{\pi h n \bar{x}_2}{b}, \quad (80)$$

where $m, n = 1, 2, \dots$ are integer numbers, W_{mn} is the dimensionless magnitude of the oscillations.

The solution (80) describes the natural oscillations of the plate where ω and η are connected by the dispersion relation

$$\eta^4 - \omega^2 + \beta \omega^4 - \zeta \eta^2 \omega^2 = 0, \quad \eta^2 = \left(\frac{\pi h m}{a} \right)^2 + \left(\frac{\pi h n}{b} \right)^2 \quad (81)$$

All natural frequencies of the plate have to satisfy the dispersion relation $\omega = \omega(\eta)$ which is the solutions of Eq. (81).

Let us consider the solutions of Eq. (81) for different approaches to the plate theory. For the Kirchhoff plate theory we have $\Gamma \rightarrow \infty$, $\Theta = 0$. Then $\beta = \zeta = 0$ and one gets the standard dispersion relation

$$\omega = \eta^2 \quad (82)$$

For the Mindlin plate theory without rotatory inertia ($\Theta = 0$) we can use Eqs (78). Then we obtain that $\beta = 0$ and $\zeta = 2/(1 - \nu)$. For example, for $\nu = 1/3$ one gets $\zeta = 3$. The dispersion relation is given by

$$\omega = \frac{\eta^2}{\sqrt{1 + \zeta \eta^2}} \quad (83)$$

The dispersion curve given by Eq. (83) coincides with the dispersion curve for the Kirchhoff plate when $\eta \ll 1$, but for $\eta \gg 1$ it coincides with the

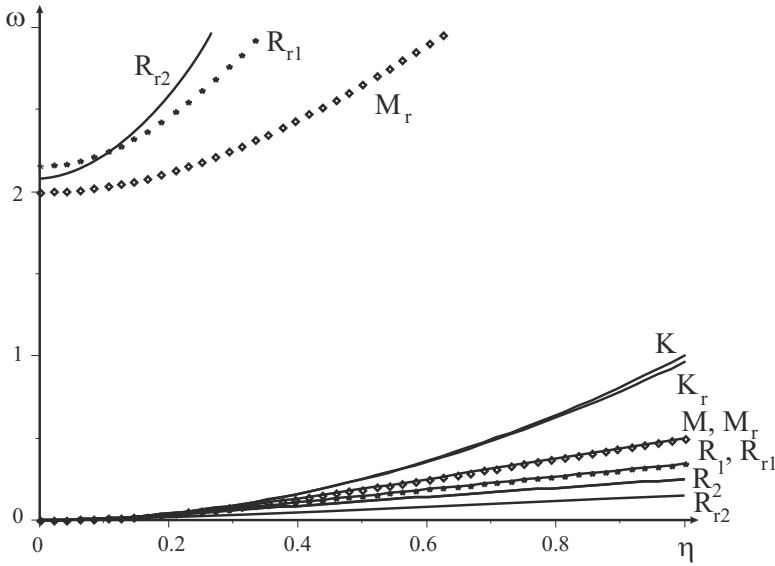


Figure 12. Dispersion curves: K – Kirchhoff plate; M – Mindlin plate; R_1 and R_2 – Reissner sandwich plates with $h_f = 0.1h$, $\alpha = 0.1$, and $h_f = 0.05h$, $\alpha = 0.05$, respectively. K_r , M_r , R_{r1} , and R_{r2} are the corresponding dispersion curves with taken into account rotatory inertia

asymptotic line $\omega = \eta/\sqrt{\zeta}$. The dispersion curves are shown in Fig. 12. The solid curves marked by labels K and M correspond to Eqs (82) and (83), respectively. Here $\nu = 1/3$ is assumed.

For the Reissner's sandwich plate theory we use Eqs (38) and assume that $\Theta = 0$. Then again $\beta = 0$. Let us consider the sandwich plate core made of a metal foam while the faces are made of a material that is used to synthesize the foam. Introducing the notations $E_f = E_s$, $G_c = G_p$, etc., and applying Eqs (46) with the assumption $\nu_c = \nu_f = \nu$ we obtain

$$\zeta = \frac{h_f}{2h(1-\nu)\alpha^m} \quad (84)$$

For instance, if $h_f = 0.1h$, $\alpha = 0.1$, $m = 2$, $\nu = 1/3$ then $\zeta = 7.5$. For $h_f = 0.05h$, $\alpha = 0.05$, $m = 2$, $\nu = 1/3$ one gets $\zeta = 15$. The corresponding dispersion curves are shown in Fig. 12. They are marked by labels R_1 and R_2 , respectively.

Let us investigate the influence of the rotatory inertia ($\Theta \neq 0$). For

the Kirchhoff-type plate theory with $\Gamma \rightarrow \infty$ we obtain from Eq. (78) that $\beta = 0$ while $\zeta = 1/12$. Thus, we have the dispersion relation (83) with a fixed value of ζ . The corresponding dispersion curve is marked by the label K_r in Fig. 12. One can see that, if we neglect the transverse shear stiffness, the influence of the rotatory inertia is not significant.

For the Mindlin-type plate theory we obtain

$$\beta = \frac{1}{6(1-\nu)}, \quad \zeta = \frac{1}{12} + \frac{2}{1-\nu}$$

Thus, in this case we have dispersion equations in the general form (81). For $\nu = 1/3$ the corresponding dispersion curve is marked by the label M_r in Fig. 12 using point style with box markers. One can see that for $\beta \neq 0$ the qualitative behavior of the dispersion curves is quite different. A new branch appears starting from the point $\eta = 0, \omega = 1/\sqrt{\beta}$.

In the case $\beta \neq 0$, Eq. (81) has two solutions given by

$$\omega_{\pm} = \left[\frac{1 + \zeta\eta^2 \pm \sqrt{(1 + \zeta\eta^2)^2 - 4\beta\eta^4}}{2\beta} \right]^{1/2} \quad (85)$$

One solution of Eq. (85) starts from the point $(0, 0)$ in the $\eta - \omega$ diagram, while the second one starts from the point $(0, 1/\sqrt{\beta})$. Both curves have linear asymptotes when $\eta \rightarrow \infty$

$$\omega_{\pm} = \eta \left[\frac{\zeta \pm \sqrt{\zeta^2 - 4\beta}}{2\beta} \right]^{1/2}, \quad \eta \gg 1 \quad (86)$$

The solution ω_- describes the bending low-frequency oscillation modes or bending waves in the case of an infinite plate, while ω_+ describes high-frequency oscillations.

For the sandwich plate with non-zero rotatory inertia we obtain

$$\begin{aligned} \frac{\Theta}{h^2\rho} &= \frac{1}{12} \left[1 - \frac{h_c^3}{h^3}(1-\alpha) \right] \left[1 - \frac{h_c}{h}(1-\alpha) \right]^{-1}, \\ \frac{D_{\text{eff}}}{h^2\Gamma} &= \frac{h_f}{2h(1-\nu)\alpha^m} \end{aligned} \quad (87)$$

The corresponding dispersion curves are shown in Fig. 12 for two sets of values $h_f = 0.1h$, $\alpha = 0.1$, $m = 2$, $\nu = 1/3$ and $h_f = 0.05h$, $\alpha = 0.05$, $m = 2$, $\nu = 1/3$. They are marked by the labels R_{r1} (point style with asterisk markers) and R_{r2} (solid line), respectively.

Note that for the chosen values of the material and geometrical parameters some of the curves are practically coincident with each other. Indeed, the low branch of the M_r -curve coincides with the M -curve, while the low branch of the R_{r1} -curve coincides with the R_1 -curve.

Let us discuss the FGM plates in detail. Using Eqs (47), (33), and (34) we obtain that

$$\frac{\Theta}{h^2 \rho} = \frac{1}{4} \left[\frac{1}{3} \alpha + \frac{1-\alpha}{3+n} \right] \left[\alpha + \frac{1-\alpha}{n+1} \right]^{-1},$$

$$\frac{D_{\text{eff}}}{h^2 \Gamma} = \frac{2}{1-\nu} \frac{1}{h^2 \lambda^2}$$

Note that λ^2 is the minimal eigen-value of the Sturm-Liouville problem (35) (Altenbach, 2000a,b). For the FGM plate it can be solved as in the previous case applying the shooting method. In Fig. 13 we present the dispersion curves for different values of n and α with $m = 2$, $\nu = 0.3$. The

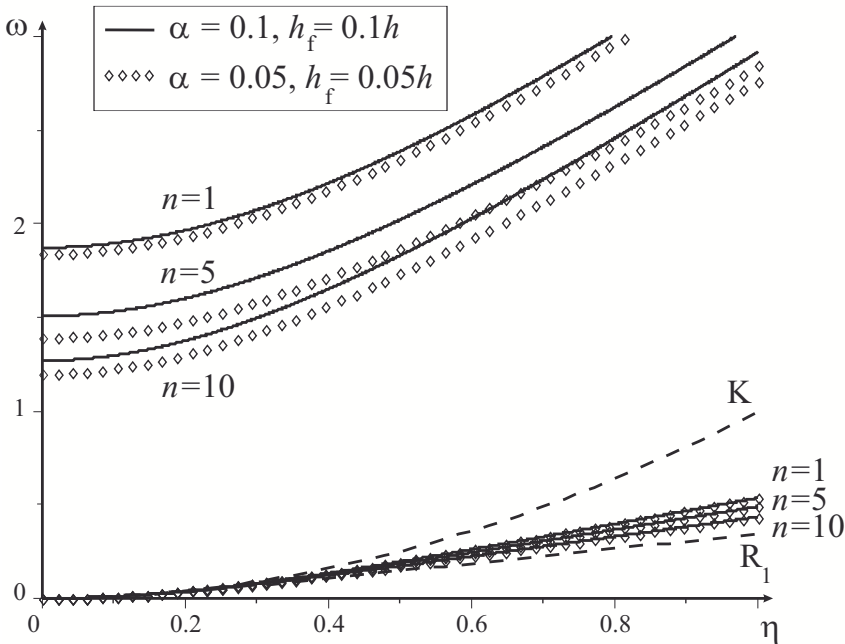


Figure 13. Dispersion curves for a FGM plate for two sets of material and geometrical parameters

dashed curves correspond to Kirchhoff's and Reissner's solutions, marked by labels K and R_1 , respectively. One can see that the influence of α is significant in the case of the high-oscillating branches of the dispersion curves. For the low-oscillating branch, the difference between the dispersion curves corresponding to the values $\alpha = 0.1$ and $\alpha = 0.05$ is very small. For the low dispersion curves, the Kirchhoff's and the Reissner's solutions can be used as the upper and the lower bounds of the eigen-frequencies of a FGM plate. One can see that the Reissner's sandwich-plate approach demonstrates a good qualitative coincidence with our results of an FGM plate as well as the Mindlin's one. On the other hand, the classical plate models can not describe the high-oscillating solutions corresponding to our dispersion curves for ω_+ .

The normalized first eigen-frequencies of a quadratic plate are presented in Table 3. Here $a = b$, $h = 0.1a$, $\nu = 0.3$, $m = 2$, $\Omega = \omega/\omega_K$, and

Table 3. Minimal normalized eigen-frequency of a quadratic plate and corresponding values of the normalized transverse shear stiffness and the rotatory inertia

Model description	$I/G_s h$	$\Theta/\rho h^2$	Ω
Kirchhoff's plate	∞	0	1
Mindlin's plate	$\pi^2/12$	0	0.799
Reissner's sandwich plate			
$\alpha = 0.1$, $h_f = 0.1$	0.01	0	0.644
$\alpha = 0.05$, $h_f = 0.05$	0.025	0	0.512
Kirchhoff-type plate	∞	1/12	0.992
Mindlin-type plate	$\pi^2/12$	1/12	0.797
Reissner-type sandwich plate			
$\alpha = 0.1$, $h_f = 0.1$	0.01	0.028	0.644
$\alpha = 0.05$, $h_f = 0.05$	0.025	0.015	0.511
FGM plate			
$\alpha = 0.1$ $n = 1$	0.061	0.12	0.818
$n = 5$	0.015	0.15	0.734
$n = 10$	0.012	0.14	0.782

ω_K is the first eigen-frequency of the Kirchhoff plate given by (82). One can see that for thick plates the structure of the plate has an influence on the eigen-frequencies. The Kirchhoff theory gives us larger values of the eigen-frequency, while Reissner's theory gives us smaller values of the eigen-

frequency. So they can be used as an estimation for the modal analysis of FGM plates.

The considered example demonstrates that the theory of sandwich plates may be not sufficient to describe the oscillations of FGM plates which are highly non-homogeneous in the thickness direction. The theory of laminate plates (see, e.g. Altenbach et al. (2004)) may be more useful, but here the problem of the determination of the number of laminates and its properties appears.

7 Plate Made of a Linear Viscoelastic Material

7.1 Constitutive Equations

Polymers in the range of their glass transition temperature behave like viscoelastic materials (Lee and Ramesh, 2004; Landrock, 1995; Mills, 2007; Riande et al., 2000; Brinson and Brinson, 2008). That means that the moduli of the polymers depend on the strain-rate or the time of loading. Thus, a foam made of such a polymer behaves viscoelastically, too. Experimental investigations of the viscoelasticity of foams were presented in Gibson and Ashby (1997); Kraatz (2007) among others.

The two-dimensional constitutive equations of a viscoelastic plate were formulated in the general form in Altenbach (1988). A through-the-thickness symmetric structure of the plate and an isotropic material behavior were considered in Altenbach and Eremeyev (2008b). Following Altenbach and Eremeyev (2009c) here we consider the general anisotropic case of two-dimensional viscoelastic constitutive equations. In this case the constitutive equations for the stress resultants can be considered as follows:

- in-plane forces

$$\begin{aligned} \mathbf{T} \cdot \mathbf{a} &= \mathcal{A}\boldsymbol{\mu} + \mathcal{B}\boldsymbol{\kappa} + \tilde{\mathcal{G}}_1\boldsymbol{\gamma} \equiv \int_{-\infty}^t \mathbf{A}(t-\tau) \cdot \dot{\boldsymbol{\mu}}(\tau) d\tau \\ &+ \int_{-\infty}^t \mathbf{B}(t-\tau) \cdot \dot{\boldsymbol{\kappa}}(\tau) d\tau + \int_{-\infty}^t \dot{\boldsymbol{\gamma}}(\tau) \cdot \boldsymbol{\Gamma}_1(t-\tau) d\tau, \end{aligned} \quad (88)$$

- transverse shear forces

$$\begin{aligned} \mathbf{T} \cdot \mathbf{n} &= \mathcal{G}\boldsymbol{\gamma} + \mathcal{G}_1\boldsymbol{\mu} + \mathcal{G}_2\boldsymbol{\kappa} \equiv \int_{-\infty}^t \boldsymbol{\Gamma}(t-\tau) \cdot \dot{\boldsymbol{\gamma}}(\tau) d\tau \\ &+ \int_{-\infty}^t \boldsymbol{\Gamma}_1(t-\tau) \cdot \dot{\boldsymbol{\mu}}(\tau) d\tau + \int_{-\infty}^t \boldsymbol{\Gamma}_2(t-\tau) \cdot \dot{\boldsymbol{\kappa}}(\tau) d\tau, \end{aligned} \quad (89)$$

- moments

$$\begin{aligned}
 \mathbf{M}^T &= \tilde{\mathcal{B}}\boldsymbol{\mu} + \mathcal{C}\boldsymbol{\kappa} + \tilde{\mathcal{G}}_2\boldsymbol{\gamma} \equiv \int_{-\infty}^t \dot{\boldsymbol{\mu}}(\tau) \cdot \cdot \mathbf{B}(t - \tau) d\tau \\
 &+ \int_{-\infty}^t \mathbf{C}(t - \tau) \cdot \cdot \dot{\boldsymbol{\kappa}}(\tau) d\tau + \int_{-\infty}^t \dot{\boldsymbol{\gamma}}(\tau) \cdot \mathbf{F}_2(t - \tau) d\tau
 \end{aligned} \tag{90}$$

Here $\mathcal{A}$, $\mathcal{B}$, $\tilde{\mathcal{B}}$, $\mathcal{C}$, $\mathcal{G}$, $\mathcal{G}_1$, $\mathcal{G}_2$, $\tilde{\mathcal{G}}_1$, $\tilde{\mathcal{G}}_2$, are linear viscoelastic operators, $\mathbf{A}(t)$, $\mathbf{B}(t)$, $\mathbf{C}(t)$ are 4th rank tensors, $\mathbf{F}_1(t)$, $\mathbf{F}_2(t)$ are 3rd rank tensors, $\boldsymbol{\Gamma}(t)$ is a 2nd rank tensor which describes the effective stiffness properties (relaxation functions for the plate). They depend on the material properties and the cross-section geometry.

Using the assumptions that $\mathbf{A}(t)$, $\mathbf{B}(t)$, etc. do not depend on time with the relations $\boldsymbol{\mu}(-\infty) = 0$, $\boldsymbol{\kappa}(-\infty) = 0$, $\boldsymbol{\gamma}(-\infty) = 0$ from Eqs (88) - (90) we obtain the constitutive equations of elastic plates presented in Altenbach (2000a,b); Zhilin (2006); Altenbach and Eremeyev (2008a).

Let us consider orthotropic material behavior and a plane mid-surface. In this case instead of the general form of the effective stiffness tensors one gets similar representations to the elastic case (Altenbach, 2000a,b; Zhilin, 2006; Altenbach and Eremeyev, 2008a).

7.2 Effective Properties

For elastic plates the identification of the components of the effective stiffness tensors is performed in Altenbach (2000a,b); Zhilin (2006); Altenbach and Eremeyev (2008a). By the same technique the analogous viscoelastic stiffness tensor components are computed. Let us consider the three-dimensional viscoelastic constitutive equations (Christensen, 1971; Haupt, 2002)

$$\boldsymbol{\sigma} = \int_{-\infty}^t \mathbf{R}(t - \tau) \cdot \cdot \dot{\boldsymbol{\varepsilon}}(\tau) d\tau \tag{91}$$

or

$$\boldsymbol{\varepsilon} = \int_{-\infty}^t \mathbf{J}(t - \tau) \cdot \cdot \dot{\boldsymbol{\sigma}} d\tau, \tag{92}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are the tensors of stress and strain, $\mathbf{R}(t)$ and $\mathbf{J}(t)$ are the 4th rank tensors of relaxation and creep functions, respectively.

For an isotropic viscoelastic material Eqs (91), (92) reduce to

$$\boldsymbol{\sigma} = \int_{-\infty}^t R_1(t - \tau) \dot{\mathbf{e}}(\tau) d\tau + \int_{-\infty}^t R_2(t - \tau) \mathbf{I} \text{tr} \dot{\mathbf{e}}(\tau) d\tau \quad (93)$$

or

$$\boldsymbol{\varepsilon} = \int_{-\infty}^t J_1(t - \tau) \dot{\mathbf{s}} d\tau + \int_{-\infty}^t J_2(t - \tau) \mathbf{I} \text{tr} \dot{\mathbf{s}} d\tau \quad (94)$$

with two scalar relaxation functions R_1 and R_2 and two scalar creep functions J_1 and J_2 which describe the shear and bulk properties of a viscoelastic isotropic material. $\mathbf{I}$ is the three-dimensional unit tensor. In addition, $\mathbf{s}$ and $\mathbf{e}$ are the deviatoric parts of the stress and the strain tensors, respectively.

Further we consider two cases: homogeneous and inhomogeneous plates. That means that in general $\mathbf{R}$ and $\mathbf{J}$ depend on the thickness coordinate z and on the time t

$$\mathbf{R} = \mathbf{R}(z, t), \quad \mathbf{J} = \mathbf{J}(z, t)$$

In addition, a density function must be considered. Let us assume the simplest case again - the density depends only on the thickness coordinate

$$\rho_0 = \rho_0(z)$$

ρ_0 is the density of the three-dimensional solid.

Using the Laplace transform of a function $f(t)$

$$\bar{f}(s) = \int_0^{\infty} f(t) e^{-st} dt,$$

one can rewrite Eqs (91), (92) in the form (Christensen, 1971; Haupt, 2002)

$$\bar{\boldsymbol{\sigma}} = s \bar{\mathbf{R}}(s) \cdot \cdot \bar{\boldsymbol{\varepsilon}}, \quad \bar{\boldsymbol{\varepsilon}} = s \bar{\mathbf{J}}(s) \cdot \cdot \bar{\boldsymbol{\sigma}} \quad (95)$$

Let us consider an orthotropic viscoelastic material. Using the correspondence principle we may write down the constitutive equations for the Laplace

mappings in the following form

$$\begin{aligned}
 \bar{\varepsilon}_1 &= \frac{1}{s\bar{E}_1} \bar{\sigma}_1 - \frac{\bar{\nu}_{21}}{s\bar{E}_2} \bar{\sigma}_2 - \frac{\bar{\nu}_{n1}}{s\bar{E}_n} \bar{\sigma}_n, \\
 \bar{\varepsilon}_2 &= \frac{1}{s\bar{E}_2} \bar{\sigma}_2 - \frac{\bar{\nu}_{12}}{s\bar{E}_1} \bar{\sigma}_1 - \frac{\bar{\nu}_{n2}}{s\bar{E}_n} \bar{\sigma}_n, \\
 \bar{\varepsilon}_n &= \frac{1}{s\bar{E}_n} \bar{\sigma}_n - \frac{\bar{\nu}_{1n}}{s\bar{E}_1} \bar{\sigma}_1 - \frac{\bar{\nu}_{2n}}{s\bar{E}_2} \bar{\sigma}_2, \\
 \bar{\varepsilon}_{12} &= \frac{\bar{\sigma}_{12}}{s\bar{G}_{12}}, \\
 \bar{\varepsilon}_{n1} &= \frac{\bar{\sigma}_{n1}}{s\bar{G}_{n1}}, \\
 \bar{\varepsilon}_{2n} &= \frac{\bar{\sigma}_{2n}}{s\bar{G}_{2n}}
 \end{aligned} \tag{96}$$

with $\bar{\nu}_{ij}\bar{E}_j = \bar{\nu}_{ji}\bar{E}_i$.

Using the analogy between (95) or (96) and the Hooke's law we can extend the identification procedure (Altenbach, 2000a,b; Zhilin, 2006) to the Laplace mapping of the effective relaxation or creep functions, see Altenbach (1988). The in-plane and the out-of-plane stiffness tensor components are

$$\begin{aligned}
 \{\bar{A}_{11}; -\bar{B}_{13}; \bar{C}_{33}\} &= \frac{1}{4} \left\langle \frac{\bar{E}_1 + \bar{E}_2 + 2\bar{E}_1\bar{\nu}_{21}}{1 - \bar{\nu}_{12}\bar{\nu}_{21}} \{1; z; z^2\} \right\rangle, \\
 \{\bar{A}_{12}; -\bar{B}_{23} = \bar{B}_{14}; -\bar{C}_{34}\} &= \frac{1}{4} \left\langle \frac{\bar{E}_1 - \bar{E}_2}{1 - \bar{\nu}_{12}\bar{\nu}_{21}} \{1; z; z^2\} \right\rangle, \\
 \{\bar{A}_{22}; \bar{B}_{24}; \bar{C}_{44}\} &= \frac{1}{4} \left\langle \frac{\bar{E}_1 + \bar{E}_2 - 2\bar{E}_1\bar{\nu}_{21}}{1 - \bar{\nu}_{12}\bar{\nu}_{21}} \{1; z; z^2\} \right\rangle, \\
 \{\bar{A}_{44}; -\bar{B}_{42}; \bar{C}_{22}\} &= < \bar{G}_{12} \{1; z; z^2\} >,
 \end{aligned} \tag{97}$$

while the transverse shear stiffness tensor components are

$$\bar{T}_1 = \frac{1}{2}(\lambda^2 + \eta^2) \frac{\bar{A}_{44}\bar{C}_{22} - \bar{B}_{42}^2}{\bar{A}_{44}}, \quad \bar{T}_2 = \frac{1}{2}(\eta^2 - \lambda^2) \frac{\bar{A}_{44}\bar{C}_{22} - \bar{B}_{42}^2}{\bar{A}_{44}} \tag{98}$$

where λ and η are the minimal nonzero eigen-values following from the Sturm-Liouville problems

$$\begin{aligned}
 \frac{d}{dz} \left(\bar{G}_{2n} \frac{dZ}{dz} \right) + \lambda^2 \bar{G}_{12} Z &= 0, \quad \frac{dZ}{dz} \Big|_{|z|=\frac{h}{2}} = 0, \\
 \frac{d}{dz} \left(\bar{G}_{1n} \frac{d\tilde{Z}}{dz} \right) + \eta^2 \bar{G}_{12} \tilde{Z} &= 0, \quad \frac{d\tilde{Z}}{dz} \Big|_{|z|=\frac{h}{2}} = 0
 \end{aligned} \tag{99}$$

In the case of isotropic material behavior one has to set in Eqs (96)

$$\begin{aligned}\overline{E}_1 = \overline{E}_2 = \overline{E}_n = \overline{E}(z, s), \quad \overline{\nu}_{ij} = \overline{\nu}(z, s), \\ \overline{G}_{12} = \overline{G}_{n1} = \overline{G}_{2n} = \overline{\mu}(z, s) = \frac{\overline{E}(z, s)}{2[1 + \overline{\nu}(z, s)]}\end{aligned}$$

Instead of (97), (98) the following non-zero components of the stiffness tensors are valid

- the in-plane stiffness tensor components

$$\overline{A}_{11} = \frac{1}{2} \left\langle \frac{\overline{E}}{1 - \overline{\nu}} \right\rangle, \quad \overline{A}_{22} = \frac{1}{2} \left\langle \frac{\overline{E}}{1 + \overline{\nu}} \right\rangle = \overline{A}_{44} = \langle \overline{\mu} \rangle, \quad (100)$$

- the coupling stiffness tensor components

$$\overline{B}_{13} = -\frac{1}{2} \left\langle \frac{\overline{E}}{1 - \overline{\nu}} z \right\rangle, \quad \overline{B}_{24} = \frac{1}{2} \left\langle \frac{\overline{E}}{1 + \overline{\nu}} z \right\rangle = -\overline{B}_{42} = \langle \overline{\mu} z \rangle, \quad (101)$$

- the plate stiffness tensor components

$$\overline{C}_{33} = \frac{1}{2} \left\langle \frac{\overline{E}}{1 - \overline{\nu}} z^2 \right\rangle, \quad \overline{C}_{44} = \frac{1}{2} \left\langle \frac{\overline{E}}{1 + \overline{\nu}} z^2 \right\rangle = \overline{C}_{22} = \langle \overline{\mu} z^2 \rangle, \quad (102)$$

- the transverse shear stiffness tensor components

$$\overline{T}_1 = \overline{T} = \lambda^2 \frac{\overline{A}_{44} \overline{C}_{22} - \overline{B}_{42}^2}{\overline{A}_{44}} \quad (103)$$

with λ following from

$$\frac{d}{dz} \left(\overline{\mu} \frac{dZ}{dz} \right) + \lambda^2 \overline{\mu} Z = 0, \quad \left. \frac{dZ}{dz} \right|_{|z|=\frac{h}{2}} = 0 \quad (104)$$

For the plate which is symmetrical to the midplane the relation $\mathbf{B} = \mathbf{0}$ holds true. The relaxation functions of the isotropic viscoelastic plate with symmetric cross-section is considered in Altenbach and Eremeyev (2008b).

The tensors of inertia and the plate density are given by Altenbach and Zhilin (1988); Zhilin (2006); Altenbach and Eremeyev (2008a)

$$\rho = \langle \rho_0 \rangle, \quad \rho \boldsymbol{\Theta}_1 = -\langle \rho_0 z \rangle \mathbf{c}, \quad \rho \boldsymbol{\Theta}_2 = \Theta \mathbf{a}, \quad \Theta = \langle \rho_0 z^2 \rangle \quad (105)$$

Θ characterizes the rotatory inertia of the cross-section of the plate. Considering the symmetry of the geometry in the thickness direction and of the material properties of the plate from Eqs (105) one gets that $\boldsymbol{\Theta}_1 = \mathbf{0}$.

Note that for an isotropic viscoelastic material we introduced three functions $\overline{E}(s)$, $\overline{\mu}(s)$ and $\overline{\nu}(s)$. They are interlinked by the formula

$$\overline{E} = 2\overline{\mu}(1 + \overline{\nu}) \quad (106)$$

Following Lakes (1992); Lakes and Wineman (2006) one can use Eq. (106) as the definition of the Poisson's ratio for the viscoelastic material.

In the theory of viscoelasticity of solids the assumption

$$\nu(t) = \nu = \text{const}$$

is often used. It is fulfilled in many applications (see arguments in Christensen (1971) where $\nu(t) \approx \text{const}$ is considered). For example, for an incompressible viscoelastic material $\nu = 1/2$. In the general case, ν is a function of the time t . As usual $\nu(t)$ is assumed to be an increasing function of t (Riande et al., 2000; Brinson and Brinson, 2008) or a non-monotonous function of t . The latter case may be realized for cellular materials or foams. Further we consider the influence of $\nu(t)$ on the deflection of a viscoelastic plate and its effective relaxation functions.

7.3 Bounds for the Eigen-values

To obtain the dependence of the transverse shear stiffness relaxation function we have to solve Eq. (104). In the general case, the solution of the spectral problem (104) may be performed numerically. For example, in Altenbach and Eremeyev (2008a) the shooting method (Stoer and Bulirsch, 1980) is used. Let us note that for the viscoelastic plate $\overline{\mu} = \overline{\mu}(z, s)$. Thus, $\lambda = \lambda(s)$. It means that for the determination of $\Gamma(t)$ one has to solve (104) for any arbitrary value of s and with the help of $\lambda = \lambda(s)$ to find numerically the inverse Laplace transform of $\overline{\Gamma}$.

Let us find the bounds for the values of λ . Introducing a new independent variable ζ by the formula

$$\zeta = \int_{-h/2}^z \frac{dz}{\overline{\mu}(z, s)},$$

one can transform (104) to the form (see, for example, Hartman (1964) for details)

$$\frac{d^2 Z}{d\zeta^2} + \lambda^2 \overline{\mu}(z, s)^2 Z = 0, \quad \left. \frac{dZ}{d\zeta} \right|_{\zeta=0, L} = 0 \quad (107)$$

Here

$$L = L(s) \equiv \int_{-h/2}^{h/2} \frac{dz}{\bar{\mu}(z, s)}$$

Substituting $\zeta = \zeta/L$, one can transform the spectral problem (107) to the canonical form

$$\frac{d^2 Z}{d\zeta^2} + \lambda^2 L(s)^2 \bar{\mu}(z, s)^2 Z = 0, \quad \left. \frac{dZ}{d\zeta} \right|_{\zeta=0,1} = 0 \quad (108)$$

The following theorem exists (Collatz, 1963):

Theorem. *If one has two eigen-value problems*

$$\frac{d^2 Z}{d\zeta^2} + \lambda^2 f_1 Z = 0, \quad \frac{d^2 Z}{d\zeta^2} + \lambda^2 f_2 Z = 0, \quad \left. \frac{dZ}{d\zeta} \right|_{\zeta=0,1} = 0 \quad (109)$$

with two functions $f_1(\zeta)$ and $f_2(\zeta)$ so that $f_1 \leq f_2$, then the following inequality holds true $\lambda_1 \geq \lambda_2$. Here λ_1 and λ_2 are the eigen-values corresponding to the functions $f_1(\zeta)$ and $f_2(\zeta)$, respectively.

Applying this theorem to Eq. (108) and using the inequalities

$$\mu_{\min}(s) \leq \bar{\mu}(z, s) \leq \mu_{\max}(s),$$

we obtain the lower and upper bounds of λ

$$\frac{\pi}{L(s)\mu_{\max}(s)} \leq \lambda(s) \leq \frac{\pi}{L(s)\mu_{\min}(s)} \quad (110)$$

For homogeneous plates $\mu_{\min} = \mu_{\max} = \bar{\mu}(s)$, $L(s) = h/\mu(s)$, and both bounds coincide with each other.

7.4 Quasi-static Behavior of a Symmetric Orthotropic Plate

Let us consider the quasi-static deformations of a symmetric orthotropic plate. In this case Eqs (15) are split into two parts again: the in-plane problem for tangential displacements u_1 and u_2 , and the bending problem for the w , φ_1 and φ_2 , respectively.

Let us assume Cartesian coordinates. The constitutive equations for a symmetric orthotropic plate can be given as follows

$$\begin{aligned}
 \bar{T}_1 &= s(\bar{A}_{11} + 2\bar{A}_{12} + \bar{A}_{22})\bar{\mu}_1 + s(\bar{A}_{11} - \bar{A}_{22})\bar{\mu}_2, \\
 \bar{T}_2 &= s(\bar{A}_{11} - \bar{A}_{22})\bar{\mu}_1 + s(\bar{A}_{11} + 2\bar{A}_{12} + \bar{A}_{22})\bar{\mu}_2, \\
 \bar{T}_{12} &= 2s\bar{A}_{44}\bar{\mu}_{12}, \\
 \bar{T}_{1n} &= s(\bar{T}_1 + \bar{T}_2)\bar{\gamma}_1, \\
 \bar{T}_{2n} &= s(\bar{T}_1 - \bar{T}_2)\bar{\gamma}_2, \\
 \bar{M}_1 &= s(\bar{C}_{33} - 2\bar{C}_{34} + \bar{C}_{44})\bar{\kappa}_1 + s(\bar{C}_{33} - \bar{C}_{44})\bar{\kappa}_2, \\
 \bar{M}_2 &= s(\bar{C}_{33} + \bar{C}_{44})\bar{\kappa}_1 + s(\bar{C}_{33} + 2\bar{C}_{34} + \bar{C}_{44})\bar{\kappa}_2, \\
 \bar{M}_{12} &= s\bar{C}_{22}(\bar{\kappa}_{12} + \bar{\kappa}_{21})
 \end{aligned} \tag{111}$$

With the geometrical relations (19) Eqs (15)_{1,2} are reduced to the form

$$\begin{aligned}
 s(\bar{A}_{11} + 2\bar{A}_{12} + \bar{A}_{22})\bar{u}_{1,11} + s(\bar{A}_{11} - \bar{A}_{22})\bar{u}_{2,21} \\
 + s\bar{A}_{44}(\bar{u}_{1,21} + \bar{u}_{2,11}) + \bar{q}_1 &= 0, \\
 s(\bar{A}_{11} - \bar{A}_{22})\bar{u}_{1,12} + s(\bar{A}_{11} - 2\bar{A}_{12} + \bar{A}_{22})\bar{u}_{2,22} + \\
 + s\bar{A}_{44}(\bar{u}_{1,22} + \bar{u}_{2,12}) + \bar{q}_2 &= 0
 \end{aligned} \tag{112}$$

Equation (15)₃ has the following form

$$(\bar{T}_1 + \bar{T}_2)\bar{w}_{,11} + (\bar{T}_1 - \bar{T}_2)\bar{w}_{,22} + (\bar{T}_1 + \bar{T}_2)\bar{\varphi}_{1,1} + (\bar{T}_1 - \bar{T}_2)\bar{\varphi}_{2,2} + \bar{q}_n/s = 0 \tag{113}$$

Equations (15)_{4,5} result in

$$\begin{aligned}
 s(\bar{C}_{33} - 2\bar{C}_{34} + \bar{C}_{44})\bar{\varphi}_{1,11} + s(\bar{C}_{33} - \bar{C}_{44})\bar{\varphi}_{2,21} \\
 + s\bar{C}_{22}(\bar{\varphi}_{1,22} + \bar{\varphi}_{2,12}) - s(\bar{T}_1 + \bar{T}_2)\bar{\varphi}_1 - s(\bar{T}_1 + \bar{T}_2)\bar{w}_{,1} + \bar{m}_1 &= 0, \\
 s(\bar{C}_{33} + 2\bar{C}_{34} + \bar{C}_{44})\bar{\varphi}_{2,22} + s(\bar{C}_{33} - \bar{C}_{44})\bar{\varphi}_{1,12} \\
 + s\bar{C}_{22}(\bar{\varphi}_{1,21} + \bar{\varphi}_{2,11}) - s(\bar{T}_1 - \bar{T}_2)\bar{\varphi}_2 - s(\bar{T}_1 - \bar{T}_2)\bar{w}_{,2} + \bar{m}_2 &= 0
 \end{aligned} \tag{114}$$

Let us eliminate the functions $\bar{\varphi}_1$ and $\bar{\varphi}_2$ from Eqs (113) and (114). For brevity, let us assume that $\bar{m}_1 = 0$, $\bar{m}_2 = 0$. By using operator notations, Eqs (114) may be rewritten in the form

$$L_{11}\bar{\varphi}_1 + L_{12}\bar{\varphi}_2 = b_1\bar{w}, \quad L_{21}\bar{\varphi}_1 + L_{22}\bar{\varphi}_2 = b_2\bar{w}, \tag{115}$$

where

$$\begin{aligned}
 L_{11} &= ss(\overline{C}_{33} - 2\overline{C}_{34} + \overline{C}_{44})\partial_1^2 + s\overline{C}_{22}\partial_2^2 - s\overline{\Gamma}_1 - s\overline{\Gamma}_2, \\
 L_{22} &= s(\overline{C}_{33} + 2\overline{C}_{34} + \overline{C}_{44})\partial_2^2 + s\overline{C}_{22}\partial_1^2 - s\overline{\Gamma}_1 + s\overline{\Gamma}_2, \\
 L_{21} &= L_{12} = (C_{33} + C_{22} - C_{44})\partial_1\partial_2, \\
 b_1 &= s(\overline{\Gamma}_1 + \overline{\Gamma}_2)\partial_1, \\
 b_2 &= s(\overline{\Gamma}_1 - \overline{\Gamma}_2)\partial_2, \\
 \partial_\alpha(\dots) &\equiv (\dots)_{,\alpha}, \quad \alpha = 1, 2
 \end{aligned}$$

Applying the same solution technique as in the elastic case, finally, the bending equation has the form

$$s(\overline{C}_{33} + \overline{C}_{44})\Delta\Delta\overline{w} = \partial_1\overline{m}_1 + \partial_2\overline{m}_2 - \frac{\overline{C}_{33} + \overline{C}_{44}}{\overline{\Gamma}_1}\Delta\overline{q}_n + \overline{q}_n \quad (116)$$

If $m_1 = m_2 = 0$ and $\overline{\Gamma}_1 \rightarrow \infty$ one gets the viscoelastic Kirchhoff's plate equation

$$s\overline{D}\Delta\Delta\overline{w} = \overline{q}_n$$

with the bending stiffness $\overline{D} = \overline{C}_{33} + \overline{C}_{44}$.

7.5 Examples of Effective Stiffness Relaxation Functions

Homogeneous plate. The simplest test for the correctness of the estimated stiffness properties is the homogeneous isotropic plate. The basic geometrical property is the thickness h . The plate is symmetrical with respect to the mid-plane. All material properties are constant over the thickness, that means they do not depend on the thickness coordinate.

For the sake of simplicity, at first let us consider the case $\nu(t) = \text{const.}$ That means that the following relations hold true: $E(t) = 2\mu(t)(1+\nu)$. The non-zero components of the classical relaxation tensors are

$$\begin{aligned}
 A_{11}(t) &= \frac{E(t)h}{2(1-\nu)}, \quad A_{22}(t) = \frac{E(t)h}{2(1+\nu)} = \mu(t)h, \\
 C_{33}(t) &= \frac{E(t)h^3}{24(1-\nu)}, \quad C_{22}(t) = \frac{E(t)h^3}{24(1+\nu)} = \frac{\mu(t)h^3}{12}
 \end{aligned}$$

Thus, the bending stiffness results in

$$D(t) = \frac{E(t)h^3}{12(1-\nu^2)}$$

The density and the rotatory inertia coefficient are

$$\rho = \rho_0 h, \quad \Theta = \frac{\rho_0 h^3}{12} \quad (117)$$

The transverse shear relaxation function follows from Eq. (103). The solution of Eq. (104) with $\bar{\mu} = \bar{\mu}(s)$ is given by $\cos \lambda z = 0$. It yields to the smallest eigenvalue $\lambda = \pi/h$ which does not depend on s . Finally, one obtains

$$\Gamma(t) = \lambda^2 C_{22} = \frac{\pi^2}{h^2} \frac{\mu(t) h^3}{12} = \frac{\pi^2}{12} \mu(t) h \quad (118)$$

It is evident that in the case of homogeneous viscoelastic plates with constant Poisson's ratio one gets the same relations for the effective stiffness tensors as in the case of elastic plates (Altenbach, 2000a,b). There is only one difference - they are now functions of t .

At second, let us consider the general case $\nu = \nu(t)$. Now D is reconstructed from

$$\bar{D} = \frac{\bar{E} h^3}{12(1 - \bar{\nu}^2)}$$

as follows

$$D(t) = \int_{-\infty}^t \frac{E(t - \tau) h^3}{12[1 - \nu^2(\tau)]} d\tau$$

Using the initial value theorem that $f(0) = \lim_{s \rightarrow \infty} s \bar{f}(s)$ and the final value theorem that $\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} s \bar{f}(s)$ we establish

$$D(0) = \frac{E(0) h^3}{12(1 - \nu_0^2)}, \quad D(\infty) = \frac{E(\infty) h^3}{12(1 - \nu_\infty^2)},$$

where

$$\nu_0 = \frac{E(0)}{2\mu(0)} - 1, \quad \nu_\infty = \frac{E(\infty)}{2\mu(\infty)} - 1$$

ν_∞ may be considered as the Poisson's ratio in the relaxed state, while ν_0 is the Poisson's ratio in the initial state. For the sake of simplicity we use the notation $f(\infty) = \lim_{t \rightarrow \infty} f(t)$.

As an example let us consider the relaxation functions following from the standard linear viscoelastic model (Haupt, 2002)

$$E(t) = E_\infty + (E_0 - E_\infty) e^{-t/\tau_E}, \quad \mu(t) = \mu_\infty + (\mu_0 - \mu_\infty) e^{-t/\tau_\mu}, \quad (119)$$

where E_∞ and E_0 are the equilibrium and the short-time Young's moduli ($E_\infty < E_0$), while μ_∞ and μ_0 are the equilibrium and the short-time shear

moduli ($\mu_\infty < \mu_0$), respectively. τ_E and τ_μ are the relaxation time at tension and shear, respectively. Using the Laplace transforms

$$\overline{E} = \frac{E_\infty}{s} + \frac{E_0 - E_\infty}{s + \tau_E}, \quad \overline{\mu} = \frac{\mu_\infty}{s} + \frac{\mu_0 - \mu_\infty}{s + \tau_\mu}$$

we obtain that

$$\overline{\nu} = \frac{(s + \tau_\mu)(E_\infty \tau_E + E_0 s)}{2(s + \tau_E)(\mu_\infty \tau_\mu + \mu_0 s)} - 1$$

The assumption $\overline{\nu} = \text{const}$ is fulfilled if and only if $E_\infty/E_0 = \mu_\infty/\mu_0$ and $\tau_E = \tau_\mu$. Finally, one gets the expression for the Laplace transform of the bending stiffness

$$\overline{D} = \frac{\overline{\mu}^2 h^3}{3(4\overline{\mu} - \overline{E})} = \frac{(\mu_\infty + \mu_0 \tau_\mu s)^2 h^3}{s^2 (s \tau_\mu + 1)^2 \left[12 \frac{\mu_\infty + \mu_0 \tau_\mu s}{s (s \tau_\mu + 1)} - 3 \frac{E_\infty + E_0 \tau_E s}{s (s \tau_E + 1)} \right]}$$

To avoid an awkward expression we do not present the expression for $D(t)$ corresponding to latter equation. Here we have the following relations

$$D(0) = \frac{E_0 h^3}{12(1 - \nu_0^2)}, \quad D(\infty) = \frac{E_\infty h^3}{12(1 - \nu_\infty^2)},$$

where

$$\nu_0 = \frac{E_0}{2\mu_0} - 1, \quad \nu_\infty = \frac{E_\infty}{2\mu_\infty} - 1$$

The dependence of the dimensionless bending stiffness on time is presented in Fig. 14 (solid line). Here the following values are assumed $\nu_0 = 0.2$, $\nu_\infty = 0.4$, $\mu_\infty = \mu_0/2$, $\tau_\mu = \tau_E$. We also present the two curves of $D(t)$ in the case of constant Poisson's ratios which are equal to 0.1, 0.4 (Fig. 14, dashed lines). Note that in the case $\nu \neq \text{const}$ $D(t)$ is a non-monotonous function of t , while $D(t)$ is the monotonous decreasing function for constant Poisson's ratio.

Functionally graded material. In this paragraph we consider small deformations of an FGM plate made of a viscoelastic polymer foam. For the panel made of a porous polymer foam the distribution of the pores over the thickness can be inhomogeneous (see, for example, Fig. 13). Let us introduce h as the thickness of the panel, ρ_s as the density of the bulk material and ρ_p as the minimum value of the density of the foam. For the description of the symmetric distribution of the porosity we assume the

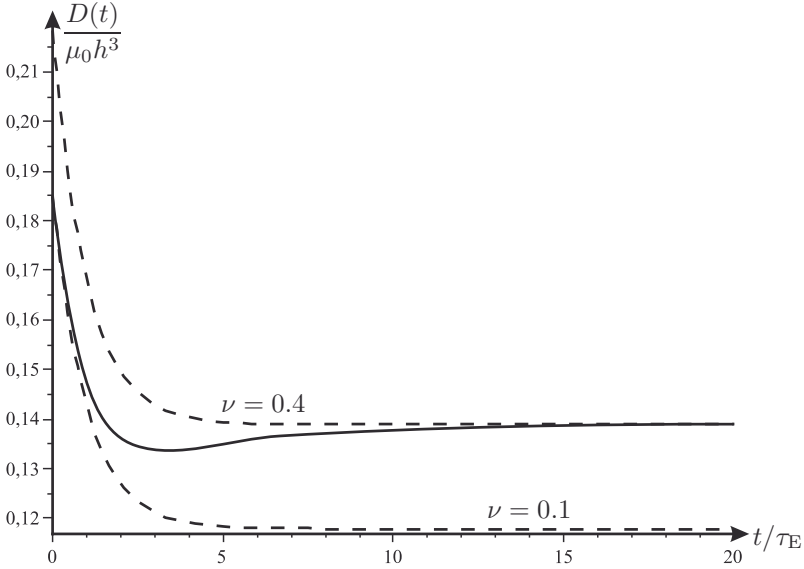


Figure 14. Dimensionless bending stiffness in dependence on time for constant Poisson's ratio (dashed lines) and in the general case (solid line)

power law (Altenbach and Eremeyev, 2008a)

$$V(z) = \alpha + (1 - \alpha) \left| \frac{2z}{h} \right|^n, \quad (120)$$

where $\alpha = \rho_p/\rho_s$ is the minimal relative density, n is the power. $n = 0$ corresponds to the homogeneous plate described in the previous paragraph.

The properties of the foam strongly depend on the porosity and the cell structure. For the polymer foam (Gibson and Ashby, 1997) the modification of the standard linear viscoelastic solid is proposed. For the open-cell foam the constitutive law has the form

$$\dot{\sigma} + \tau_E \sigma = C_1 V(z)^2 [E_\infty \tau_E \varepsilon + E_0 \dot{\varepsilon}], \quad (121)$$

while for the closed-cell foam the constitutive equation has the form

$$\dot{\sigma} + \tau_E \sigma = C_2 [\phi^2 V(z)^2 + (1 - \phi) V(z)] [E_\infty \tau_E \varepsilon + E_0 \dot{\varepsilon}] \quad (122)$$

Here $C_1 \approx 1$, $C_2 \approx 1$, ϕ describes the relative volume of the solid polymer concentrated near the cell ribs. Usually, $\phi = 0.6 \dots 0.7$. E_∞ , E_0 , τ_E are material constants of the polymer used in manufacturing of the foam.

From Eqs (121), (122) one can see that the corresponding relaxation function is given by the relation

$$E = E(z, t) = E(t)\kappa(z), \quad (123)$$

where $E(t)$ is defined by Eq. (119), while

$$\kappa(z) = C_1 V(z)^2$$

for open-cell foam and

$$\kappa(z) = C_2 [\phi^2 V(z)^2 + (1 - \phi)V(z)]$$

for closed-cell foam, respectively. Similar to (123) the following relation can be established for the shear relaxation function

$$\mu = \mu(z, t) = \mu(t)m(z) \quad (124)$$

Equations (123) and (124) have the meaning that the viscoelastic properties of the foam, for example, the time of relaxation, do not depend on the porosity distribution. Note that representations (123) and (124) are only rough estimates for spatial nonhomogeneous foams.

Using experimental data presented in Ashby et al. (2000); Gibson and Ashby (1997) one can assume $\nu = \text{const}$. In this case we obtain that A_{11} , A_{22} , C_{33} , C_{22} are related to

$$A_{11} = \frac{1 + \nu}{1 - \nu} A_{22}, \quad C_{33} = \frac{1 + \nu}{1 - \nu} C_{22}, \quad (125)$$

For the open-cell foam A_{22} and C_{22} are given by

$$\begin{aligned} A_{22} &= h \left[\alpha^2 + \frac{2\alpha(1-\alpha)}{n+1} + \frac{(1-\alpha)^2}{2n+1} \right] \mu(t), \\ C_{22} &= \frac{h^3}{12} \left[\alpha^2 + \frac{6\alpha(1-\alpha)}{n+3} + \frac{3(1-\alpha)^2}{2n+3} \right] \mu(t), \end{aligned} \quad (126)$$

while for the closed-cell foam by

$$\begin{aligned} A_{22} &= h \left\{ \phi^2 \left[\alpha^2 + \frac{2\alpha(1-\alpha)}{n+1} + \frac{(1-\alpha)^2}{2n+1} \right] \right. \\ &\quad \left. + (1-\phi) \left[\alpha + \frac{1-\alpha}{n+1} \right] \right\} \mu(t), \\ C_{22} &= \frac{h^3}{12} \left\{ \phi^2 \left[\alpha^2 + \frac{6\alpha(1-\alpha)}{n+3} + \frac{3(1-\alpha)^2}{2n+3} \right] \right. \\ &\quad \left. + (1-\phi) \left[\alpha + \frac{3(1-\alpha)}{n+3} \right] \right\} \mu(t) \end{aligned} \quad (127)$$

Here we assume that $C_1 = 1$, $C_2 = 1$, and that ϕ does not depend on z .

From Eqs (126), (127) it is easy to see that the classical relaxation functions differ only by factors from the shear relaxation function. Note that one can easily extend Eqs (121), (122) to the case of general constitutive equations (91) or (94). Thus, using assumption that $\nu = \text{const}$, one can calculate the classical effective stiffness relaxation functions for general viscoelastic constitutive equations multiplying the shear relaxation function with the corresponding factor similar to Eqs (126), (127). In the more general situation with $\nu = \nu(t)$ or taking into account other viscoelastic phenomena, for example, the filtration of a fluid in the saturated foam, the effective stiffness relaxation functions may be more complex than for the pure solid polymer discussed here.

Finally, we should mention that in the case of constant Poisson's ratio and with the assumption (124) the determination of the effective in-plane, bending and transverse shear stiffness tensors of a symmetric FGM viscoelastic plate made of a polymer foam can be realized by the same method as for elastic plates discussed in Altenbach (2000a,b); Altenbach and Eremeyev (2008a). The relaxation functions for viscoelastic FGM plates can be found from the values of the corresponding effective stiffness of an elastic FGM plate by multiplication with the normalized shear relaxation function of the polymer solid.

7.6 Bending of a Viscoelastic Plate

Homogeneous plate. Let us assume the plate bending with $\mathbf{m} = \mathbf{0}$. From Eq. (116) we obtain

$$s\overline{D}\Delta\Delta\overline{w} = \overline{q}_n - \frac{\overline{D}}{\overline{T}_1}\Delta\overline{q}_n \quad (128)$$

where $\overline{D} = \overline{E}h^3/12(1 - \overline{\nu}^2)$ is Laplace transform of the bending stiffness relaxation function, $\overline{T}_1 = \pi^2\overline{\mu}h/12$ is Laplace transform of the shear stiffness relaxation function, $\overline{w} = \overline{\mathbf{u}} \cdot \mathbf{n}$ is the Laplace transform of the plate deflection, $\overline{q}_n = \overline{\mathbf{q}} \cdot \mathbf{n}$ is Laplace transform of the transverse load, respectively. Using Eq. (106) Eq. (128) can be transformed to

$$s\overline{D}\Delta\Delta\overline{w} = \overline{q}_n - \frac{2h^2}{\pi^2(1 - \overline{\nu})}\Delta\overline{q}_n \quad (129)$$

Let us assume that $x_1 \in [0, a]$, $x_2 \in [0, b]$, where a and b are the length and the width of the plate, respectively. Considering a sinusoidal load

$$q_n = Q(t) \sin \frac{\pi x_1}{a} \sin \frac{\pi x_2}{b} \quad (130)$$

and the boundary conditions (7), then

$$\bar{q}_n = \bar{Q}(s) \sin \frac{\pi x_1}{a} \sin \frac{\pi x_2}{b}$$

and the solution of Eq. (129) is given by

$$\bar{w} = \bar{w}_{\max}(s) \sin \frac{\pi x_1}{a} \sin \frac{\pi x_2}{b}, \quad (131)$$

where

$$\bar{w}_{\max}(s) = \frac{\bar{Q}(s)}{s\bar{D}\eta^4} \left[1 - \frac{2h^2\eta^2}{\pi^2(1-\bar{\nu})} \right], \quad \eta^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$

is the Laplace transform of maximal deflection.

Using the theorems on the initial and the final values of the Laplace transforms, we obtain that the initial value of the maximal deflection is given by formula

$$w_{\max}(0) = \frac{Q(0)}{D(0)\eta^4} \left[1 - \frac{2h^2\eta^2}{\pi^2(1-\nu_0)} \right],$$

while the relaxed maximal deflexion results in

$$w_{\max}(\infty) = \frac{Q(\infty)}{D(\infty)\eta^4} \left[1 - \frac{2h^2\eta^2}{\pi^2(1-\nu_\infty)} \right]$$

Let us consider Eqs (119) and the step function $Q(t)$

$$Q(t) = Q_0 H(t),$$

where $H(t)$ is the Heaviside's function. Then we obtain that

$$\bar{w}_{\max}(s) = Q_0 \frac{12(1-\bar{\nu}^2)}{\bar{E}h^3\eta^4} \left[1 - \frac{2h^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right)}{1-\bar{\nu}} \right]$$

For the constant Poisson's ratio one gets

$$\begin{aligned} w_{\max}(t) &= Q_0 \frac{12(1-\nu^2)}{h^3\eta^4} \left[1 - \frac{2h^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right)}{1-\nu} \right] \\ &\times \left[\frac{1}{E_0} + \left(\frac{1}{E_0} - \frac{1}{E_\infty} \right) e^{-tE_0/(E_\infty\tau_E)} \right] \end{aligned}$$

The dependence of the dimensionless maximal deflexion on time is presented in Fig. 15 (solid line). Here we again assume that $\nu_0 = 0.2$, $\nu_\infty = 0.4$, $\mu_\infty = \mu_0/2$, $\tau_\mu = \tau_E$. We also present the two curves of $w_{\max}(t)$ in the case of constant Poisson's ratios (Fig. 15, $\nu = 0.1, 0.4$, dashed lines).

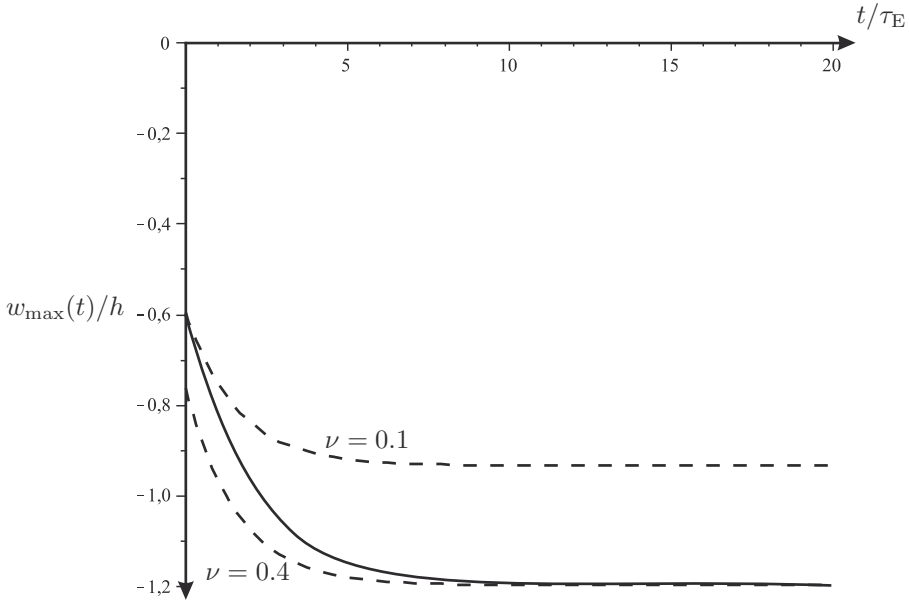


Figure 15. The dimensionless maximal deflexion in dependence on time for constant Poisson's ratio (dashed lines) and in the general case (solid line)

FGM plate. Considering the symmetry of the material properties with respect to the mid-plane one gets a decoupling of the in-plane state and the plate state. Let us assume again the plate bending with $\mathbf{m} = \mathbf{0}$. Using the method presented in Altenbach and Eremeyev (2008a) and the Laplace transform, one can reduce Eq. (116) to

$$s\overline{D}_{\text{eff}}\Delta\Delta\overline{w} = \overline{q}_n - \frac{\overline{D}_{\text{eff}}}{\overline{I}}\Delta\overline{q}_n, \quad (132)$$

where $\overline{D}_{\text{eff}} = \overline{C}_{22} + \overline{C}_{33}$ is Laplace transform of the effective bending stiffness relaxation function. Note that here

$$s\overline{D}_{\text{eff}} = D_{\text{eff}}^0 \overline{\mu}(s)$$

with $D_{\text{eff}}^0 = (C_{22} + C_{33})/\mu(t)$.

To analyze the influence of the transverse shear stiffness on the deflection of the plate let us consider the bending of a rectangular plate made of a functionally graded material. Using the assumption that $\nu = \text{const}$ and

the Eqs (102), (103), and (124) are valid, we can rewrite Eq. (116) in the following form

$$s\overline{D}_{\text{eff}}\Delta\Delta\overline{w} = \overline{q}_n - \frac{2}{\lambda^2(1-\nu)}\Delta\overline{q}_n \quad (133)$$

Introducing dimensionless variables by the formulas

$$W = h^{-1}w, \quad X_1 = h^{-1}x_1, \quad X_2 = h^{-1}x_2, \quad X_1 \in \left[0, \frac{a}{h}\right], \quad X_2 \in \left[0, \frac{b}{h}\right],$$

Eq. (133) transforms to

$$s\overline{\mu}(s)\Delta\Delta\overline{W} = Q - \frac{2}{1-\nu}\frac{1}{\lambda^2h^2}\Delta Q \quad (134)$$

Here

$$\Delta = \frac{\partial^2}{\partial X_1^2} + \frac{\partial^2}{\partial X_2^2}, \quad Q = \frac{\overline{q}_n h^3}{D_{\text{eff}}^0}$$

Let us consider again a sinusoidal load (130) and the boundary conditions (7). Then the solution of Eq. (134) is given by

$$\overline{W} = \frac{K}{\eta^4 h^4} \frac{Q}{s\overline{\mu}(s)} \sin \frac{\pi h X_1}{a} \sin \frac{\pi h X_2}{b}, \quad K = 1 + \frac{2\eta^2}{1-\nu} \frac{1}{\lambda^2} \quad (135)$$

Using the Kirchhoff's plate theory $K = K_K \equiv 1$ and using the homogeneous plate modeled in the sense of Mindlin's plate theory

$$K = K_M \equiv 1 + \frac{2\eta}{1-\nu} \frac{1}{\pi^2}$$

Using bounds (110) for the FGM plate we obtain the inequalities

$$1 + \frac{2\eta}{1-\nu} \frac{L^2 m_{\min}^2}{\pi^2 h^2} \leq K \leq 1 + \frac{2\eta}{1-\nu} \frac{L^2 m_{\max}^2}{\pi^2 h^2}$$

The influence of the shear stiffness on the deflection of the elastic FGM plate is given in Altenbach and Eremeyev (2008a). For the viscoelastic plate both the qualitative and the quantitative influence of the shear stiffness is the same as in Altenbach and Eremeyev (2008a).

For example, let us consider an open-cell foam and following values $\nu = 0.3$, $a = b$, $h = 0.05a$, $\alpha = 0.9$. Using the calculation of Altenbach and Eremeyev (2008a) we obtain the following values of λ : $\lambda = 0.83/h$ for $n = 2$, $\lambda = 0.82/h$ for $n = 5$. The corresponding values of the factor K are given by

$$K_M \approx 1.014, \quad K \approx 1.20 \quad (n = 2), \quad K \approx 1.21 \quad (n = 5)$$

That means that for the functionally graded plates the influence of transverse shear stiffness may be significant. For elastic FGM plates as well as for the cases of other types of boundary conditions the influence of the structure of the viscoelastic plate on the deflection may be greater than for the used simple support type boundary conditions.

8 Plate Theory Deduced from the Cosserat Continuum

The experimental data by Lakes (1986, 1991, 1995) say that some types of foams may be considered as a Cosserat continuum. This continuum model was suggested by Cosserat and Cosserat (1909) and is presented, for example, in Nowacki (1986); Eringen (1999). Since the paper of Ericksen and Truesdell (1958) there have been many publications on the application of the Cosserat approach to the plate and shell theories, see the review by Altenbach et al. (2009) among others. In this section we briefly discuss the derivation of the linear plate equations reducing the three-dimensional equilibrium equations of the Cosserat continuum to the two-dimensional equations as is presented in Altenbach and Eremeyev (2009d).

8.1 Two-dimensional Governing Equations

Here we use a linear variant of the plate theory based on the 6-parameters or micropolar shells theory presented in books by Libai and Simmonds (1998); Chróścielewski et al. (2004); Eremeyev and Zubov (2008). Let us introduce the basic equations for micropolar plates based on the direct approach. We consider a deformable plane surface in the reference (undeformed) state. Each material point of this surface is an infinitesimal rigid body with 3 translational and 3 rotational degrees of freedom. The balances of momentum and moment of momentum are formulated as follows

$$\begin{aligned} \mathbf{F}_s^* &\equiv \int_{\mathcal{M}_*} \mathbf{q} \, dA + \int_{\mathcal{C}_*} \mathbf{t}_s \, ds = \mathbf{0}, \\ \mathbf{M}_s^* &\equiv \int_{\mathcal{M}_*} (\mathbf{x} \times \mathbf{q} + \mathbf{c}) \, dA + \int_{\mathcal{C}_*} (\mathbf{x} \times \mathbf{t}_s + \mathbf{m}_s) \, ds = \mathbf{0}, \end{aligned} \quad (136)$$

where $\mathbf{q}$ and $\mathbf{c}$ are the vectors of surface loads (forces and moments), $\mathbf{x}$ is the position vector of the plane surface $\mathcal{M}$, $\times$ the cross product, $\mathbf{t}_s$ and $\mathbf{m}_s$ the surface analogues of the stress vector and the couple stress vector, respectively. From Eqs (136) we obtain the local form of the balances of momentum and moment of momentum as well as the static boundary condi-

tions. In the case of sufficient smooth fields the local equilibrium equations can be stated as follows

$$\nabla \cdot \mathbf{T} + \mathbf{q} = \mathbf{0}, \quad \nabla \cdot \mathbf{M} + \mathbf{T}_\times + \mathbf{c} = \mathbf{0} \quad (137)$$

Here the surface (plane) nabla operator ∇ is given by $\nabla = \mathbf{i}_\alpha \frac{\partial}{\partial x_\alpha}$ with the Cartesian coordinates x_α , and $\mathbf{i}_\alpha$ are the unit base vectors. The Greek indices take the values 1, 2. The tensors $\mathbf{T}$ and $\mathbf{M}$ denote the surface stress and couple stress tensors, respectively. They are related to $\mathbf{t}_s$ and $\mathbf{m}_s$ by the Cauchy's theorem: $\boldsymbol{\nu} \cdot \mathbf{T} = \mathbf{t}_s$ and $\boldsymbol{\nu} \cdot \mathbf{M} = \mathbf{m}_s$, where $\boldsymbol{\nu}$ is the outward normal vector to $\mathcal{C}$. $\mathbf{T}$ and $\mathbf{M}$ take the form

$$\mathbf{T} = T_{\alpha\beta} \mathbf{i}_\alpha \mathbf{i}_\beta + T_{\alpha 3} \mathbf{i}_\alpha \mathbf{n}, \quad \mathbf{M} = M_{\alpha\beta} \mathbf{i}_\alpha \mathbf{i}_\beta + M_{\alpha 3} \mathbf{i}_\alpha \mathbf{n} \quad (\alpha, \beta = 1, 2) \quad (138)$$

Let us introduce the linear strain measures

$$\mathbf{e} = \nabla \mathbf{v} + \mathbf{A} \times \boldsymbol{\theta}, \quad \mathbf{k} = \nabla \boldsymbol{\theta}, \quad (139)$$

where $\mathbf{A} \equiv \mathbf{I} - \mathbf{n} \otimes \mathbf{n}$ is the first metric tensor, and $\mathbf{n}$ the unit normal to $\mathcal{M}$. Applying the methodology presented in Libai and Simmonds (1998); Chróscielewski et al. (2004) one can show that the linear strain measures are work-conjugated to the stress measures $\mathbf{T}$ and $\mathbf{M}$.

For the isotropic plate the surface strain energy W can be assumed to be

$$\begin{aligned} 2W = & \alpha_1 \text{tr}^2 \mathbf{e}_S + \alpha_2 \text{tr} \mathbf{e}_S^2 + \alpha_3 \text{tr} (\mathbf{e}_S \cdot \mathbf{e}_S^T) + \alpha_4 \mathbf{n} \cdot \mathbf{e}^T \cdot \mathbf{e} \cdot \mathbf{n} \\ & + \beta_1 \text{tr}^2 \mathbf{k}_S + \beta_2 \text{tr} \mathbf{k}_S^2 + \beta_3 \text{tr} (\mathbf{k}_S \cdot \mathbf{k}_S^T) + \beta_4 \mathbf{n} \cdot \mathbf{k}^T \cdot \mathbf{k} \cdot \mathbf{n} \end{aligned} \quad (140)$$

(Chróscielewski et al., 2004; Eremeyev and Pietraszkiewicz, 2006; Eremeyev and Zubov, 2008). Here $\mathbf{e}_S = \mathbf{e} \cdot \mathbf{A}$, $\mathbf{k}_S = \mathbf{k} \cdot \mathbf{A}$, and α_i, β_i are the elastic stiffness parameters, $i = 1, 2, 3, 4$. The constitutive equations are

$$\mathbf{T} \equiv \frac{\partial W}{\partial \mathbf{e}}, \quad \mathbf{M} \equiv \frac{\partial W}{\partial \mathbf{k}}$$

The surface strain energy W must be positive definite, from which it follows (Eremeyev and Zubov, 2007), that

$$\begin{aligned} 2\alpha_1 + \alpha_2 + \alpha_3 &> 0, & \alpha_2 + \alpha_3 &> 0, & \alpha_3 - \alpha_2 &> 0, & \alpha_4 &> 0, \\ 2\beta_1 + \beta_2 + \beta_3 &> 0, & \beta_2 + \beta_3 &> 0, & \beta_3 - \beta_2 &> 0, & \beta_4 &> 0. \end{aligned} \quad (141)$$

Note that for an isotropic three-dimensional micropolar solid we have only 6 elastic moduli while the micropolar plate theory contains 8 elastic stiffness parameters. The increase of the number of parameters can be explained as the minimum of two reasons: reduced symmetry of the constitutive equations and the reduction of three-dimensional equations to two-dimensional ones.

8.2 Reduction of the Three-dimensional Micropolar Equations

A plate-like body as shown in Fig. 16 is assumed. Let us introduce the balance equations of the micropolar elasticity, see Eringen (1999). The equilibrium conditions of any part of a micropolar body occupying the volume $V_* \subset V$ consist of the following relations

$$\begin{aligned} \mathbf{F}^* &\equiv \int_{V_*} \rho \mathbf{f} dV + \int_{S_*} \mathbf{t} dA = \mathbf{0}, \\ \mathbf{M}^* &\equiv \int_{V_*} \rho (\mathbf{r} \times \mathbf{f} + \mathbf{l}) dV + \int_{S_*} (\mathbf{r} \times \mathbf{t} + \mathbf{m}) dA = \mathbf{0}, \end{aligned} \quad (142)$$

where $\mathbf{f}$ and $\mathbf{l}$ are the mass forces and mass couples vectors, respectively, ρ the density, $\mathbf{r}$ the position vector, $S_* = \partial V_*$, $\mathbf{t}$ and $\mathbf{m}$ the stress and couple stress vectors, respectively. $\mathbf{F}^*$ and $\mathbf{M}^*$ are the total force and the total couple acting on V_* , respectively. Hence, for any part of the micropolar body Eq. (142)₁ states that the vector of the total force is zero, while Eq. (142)₂ states that the vector of the total moment is zero. With the relations $\mathbf{n} \cdot \boldsymbol{\sigma} = \mathbf{t}$, $\mathbf{n} \cdot \boldsymbol{\mu} = \mathbf{m}$, the local equilibrium equations are

$$\tilde{\nabla} \cdot \boldsymbol{\sigma} + \rho \mathbf{f} = \mathbf{0}, \quad \tilde{\nabla} \cdot \boldsymbol{\mu} + \boldsymbol{\sigma}_\times + \rho \mathbf{l} = \mathbf{0} \quad (143)$$

Now the nabla operator $\tilde{\nabla}$ is a three-dimensional operator, ρ the density, and $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ are the stress and couple stress tensors, respectively. $\boldsymbol{\sigma}_\times$ is the vectorial invariant of a second-order tensor $\boldsymbol{\sigma}$.

The small strains of the micropolar continuum are usually presented by the displacement vector $\mathbf{u}$ and the vector of microrotation $\boldsymbol{\vartheta}$. The linear strain measures, i.e. the stretch tensor $\mathbf{E}$ and the wryness tensor $\mathbf{K}$, are given by the relations

$$\mathbf{E} = \tilde{\nabla} \mathbf{u} + \boldsymbol{\vartheta} \times \mathbf{I}, \quad \mathbf{K} = \tilde{\nabla} \boldsymbol{\vartheta} \quad (144)$$

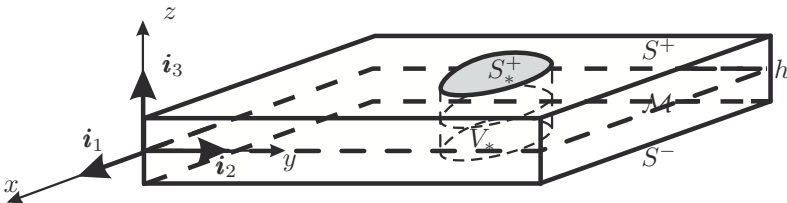


Figure 16. Plate-like body

Finally, the isotropic elastic constitutive equations are

$$\boldsymbol{\sigma} = \lambda \mathbf{I} \text{tr} \mathbf{E} + \mu \mathbf{E}^T + (\mu + \kappa) \mathbf{K}, \quad \boldsymbol{\mu} = \alpha \mathbf{I} \text{tr} \mathbf{K} + \beta \mathbf{K}^T + \gamma \mathbf{K} \quad (145)$$

The classical isotropic elasticity can be deduced setting $\kappa = \alpha = \beta = \gamma = 0$. In this case the stress tensor $\boldsymbol{\sigma}$ is a symmetric tensor. In addition, $\boldsymbol{\mu}$ and $\mathbf{l}$ vanish.

The integration procedure is performed as follows. Let us consider that our plate-like body occupies a volume with one dimension which is significantly smaller in comparison to the other two. The coordinate z denotes this special direction and h is the plate thickness which takes the values $-h/2 \leq z \leq h/2$. The boundary conditions of the upper (+) and lower (−) plate surfaces can be given

$$\mathbf{n}^\pm \cdot \boldsymbol{\sigma} = \mathbf{t}^\pm, \quad \mathbf{n}^\pm \cdot \boldsymbol{\mu} = \mathbf{m}^\pm, \quad (146)$$

where $\mathbf{t}^\pm$ and $\mathbf{m}^\pm$ are the surface loads (forces and moments) and $\mathbf{n}^\pm = \mathbf{i}_3$.

The main idea of the reduction procedure is the application of the three-dimensional equilibrium conditions (142) to any volume V_* of the plate-like body and the transformation of the results to the two-dimensional case as in Eqs (136). Following Altenbach and Eremeyev (2009d) we transform Eqs (142) to the relations

$$\begin{aligned} \mathbf{F}^* &= \int_{\mathcal{M}_*} \mathbf{q} \, dA + \int_{\mathcal{C}_*} \boldsymbol{\nu} \cdot \langle \boldsymbol{\sigma} \rangle \, ds = \mathbf{0}, \\ \mathbf{M}^* &= \int_{\mathcal{M}_*} [\mathbf{x} \times \mathbf{q} + \mathbf{c}] \, dA \\ &+ \int_{\mathcal{C}_*} [\boldsymbol{\nu} \cdot \langle \boldsymbol{\mu} \rangle - \boldsymbol{\nu} \cdot \langle z \boldsymbol{\sigma} \times \mathbf{i}_3 \rangle - \boldsymbol{\nu} \cdot \langle \boldsymbol{\sigma} \rangle \times \mathbf{x}] \, dA = \mathbf{0}, \end{aligned} \quad (147)$$

where the following notations are introduced

$$\langle (\dots) \rangle = \int_{-h/2}^{h/2} (\dots) \, dz,$$

$$\mathbf{q} = \langle \rho \mathbf{f} \rangle + \mathbf{t}^+ + \mathbf{t}^-, \quad \mathbf{c} = \langle \rho \mathbf{l} \rangle + \mathbf{m}^+ + \mathbf{m}^- + \mathbf{i}_3 \times \langle \rho z \mathbf{f} \rangle + \frac{h}{2} \mathbf{i}_3 \times (\mathbf{t}^+ - \mathbf{t}^-)$$

The comparison of Eqs (147) and (136) leads to the determination of the stress resultants tensors by the following relations

$$\mathbf{T} = \langle \mathbf{A} \cdot \boldsymbol{\sigma} \rangle, \quad \mathbf{M} = \langle \mathbf{A} \cdot \boldsymbol{\mu} \rangle - \langle \mathbf{A} \cdot z \boldsymbol{\sigma} \times \mathbf{i}_3 \rangle \quad (148)$$

From Eq. (148)₂ it follows that the components $M_{\alpha 3}$ depend only on the couple stress tensor $\boldsymbol{\mu}$. Indeed, $\mathbf{M} \cdot \mathbf{i}_3 = \langle \mathbf{A} \cdot \boldsymbol{\mu} \cdot \mathbf{i}_3 \rangle$.

To establish the relations with the vectors $\mathbf{u}$ and $\boldsymbol{\vartheta}$ used in the three-dimensional theory and their analogues $\mathbf{v}$ and $\boldsymbol{\theta}$ in the two-dimensional theory, we apply the following approximation of $\mathbf{u}$ and $\boldsymbol{\vartheta}$, see Altenbach and Eremeyev (2009d) for details,

$$\mathbf{u}(x, y, z) = \mathbf{v}(x, y) - z\boldsymbol{\phi}(x, y), \quad \boldsymbol{\vartheta} = \boldsymbol{\phi}(x, y) \times \mathbf{i}_3 + \vartheta_3(x, y)\mathbf{i}_3 = \boldsymbol{\theta}, \quad (149)$$

where $\boldsymbol{\phi} \cdot \mathbf{i}_3 = 0$. This means that the couple stress tensor $\boldsymbol{\mu}$ does not depend on z , while the stress tensor $\boldsymbol{\sigma}$ depends on z linearly as in Eringen (1999). But the approximation (149) is more restrictive than the in Eringen (1999) applied one. As a result the effective stiffness parameters can be estimated

$$\begin{aligned} \alpha_1 &= \tilde{\lambda}h, \quad \tilde{\lambda} \equiv \frac{\lambda(2\mu + \kappa)}{\lambda + 2\mu + \kappa}, \quad \alpha_2 = \mu h, \\ \alpha_3 &= (\mu + \kappa)h, \quad \alpha_4 = (\mu + \kappa)h, \\ \beta_1 &= \alpha h - \mu \frac{h^3}{12}, \quad \beta_2 = \beta h - \tilde{\lambda} \frac{h^3}{12}, \\ \beta_3 &= \gamma h + (2\mu + \kappa + \tilde{\lambda}) \frac{h^3}{12}, \quad \beta_4 = \gamma h \end{aligned}$$

The in-plane stiffness parameters $\alpha_1, \alpha_2, \alpha_3$, and the transverse shear stiffness α_4 depend linearly on h . The dependence of $\beta_i, i = 1, \dots, 4$ on h is more complicated. The parameters β_i have linear asymptotes when h tends to zero: $\beta_i \sim h$. The considered case differs from the cases of Kirchhoff's and Reissner's plates when $\beta_i \sim h^3$.

Introducing the engineering constants

$$G = \frac{2\mu + \kappa}{2}, \quad \nu = \frac{\lambda}{2\lambda + 2\mu + \kappa}, \quad l_b^2 = \frac{\gamma}{2(2\mu + \kappa)},$$

where G is the shear modulus, ν the Poisson's ratio, l_b the characteristic length under bending, see Eringen (1999), we obtain the expression

$$D = \frac{Gh^3}{12(1 - \nu)} \left[1 + 2 \frac{l_b^2}{h^2} \right] \quad (150)$$

The dependence $D/\tilde{D}$ on h is given in Fig. 17, where

$$\tilde{D} = \frac{Gh^3}{12(1 - \nu)}$$

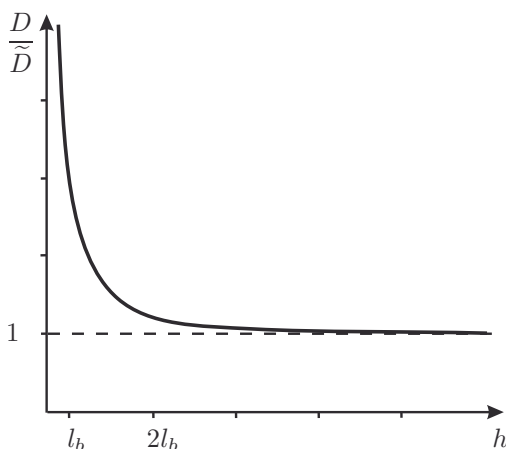


Figure 17. Dimensionless bending stiffness $D/\tilde{D}$ vs. the dimensionless thickness h/l_b

is the value of the bending stiffness used by Eringen (1999). It can be seen that the micropolar properties are negligible if $h > 2l_b$.

The values of the elastic stiffness parameters for two porous materials are given in Tables 4 and 5. Here we used the experimental data presented by Lakes (1986, 1991, 1995). The index * denotes the case of the material without the micropolar properties, i.e. if $\kappa = \alpha = \beta = \gamma = 0$.

To calculate the elastic stiffness we used the approximation (149). Let us note that in the literature on the shell theory there are many different approaches to derive the shell equations from the 3D elasticity which leads to different values of the stiffness parameters, see Libai and Simmonds (1998); Chróścielewski et al. (2004) among others. For the Cosserat continuum the derivation of the shell theories are given in many publications, see the review in Altenbach et al. (2009). This means that the elastic stiffness parameters presented above can be considered as some estimates. With other words, these parameters show the influence of the micropolar properties. In particular, Eq. (150) demonstrates the size-effect, which is well-known in the micropolar elasticity, see Lakes (1986, 1991, 1995); Eringen (1999). The analysis of Eq. (150) shows that the influence of the micropolar properties is essential if the thickness h of the plate has the same value as the characteristic length of the microstructure of the material.

In this section we present the alternative approach to model plates within the framework of the six-parametric (or the micropolar) linear theory of

Table 4. Effective stiffness of a plate made of PU foam (h has the dimension m)

Elastic constants	Foam, PU	Foam, PU *
$\alpha_1, \text{N/m } 10^6$	$0.165h$	$0.165h$
$\alpha_2, \text{N/m } 10^6$	$1.001h$	$104h$
$\alpha_3, \text{N/m } 10^6$	$1.199h$	$1.1h$
$\alpha_4, \text{N/m } 10^6$	$1.199h$	$1.1h$
$\beta_1, \text{N}\cdot\text{m } 10^6$	$-2.6 \cdot 10^{-6}h - 0.083h^3$	$-0.092h^3$
$\beta_2, \text{N}\cdot\text{m } 10^6$	$-10^{-4}h - 0.014h^3$	$-0.014h^3$
$\beta_3, \text{N}\cdot\text{m } 10^6$	$1.1 \cdot 10^{-4}h + 0.197h^3$	$0.197h^3$
$\beta_4, \text{N}\cdot\text{m } 10^6$	$1.1 \cdot 10^{-4}h$	0

plates with two vector fields of translations and of rotations as the independent kinematic variables. Within the proposed theory one may take into account an external surface drilling moment. The influence of the micropolar properties on the stiffness parameters of the plate is illustrated for two foams.

9 Summary

A plate theory based on the direct approach method in the theory of shells for foams and FGMs is introduced. The basic items of the theory are related to

- the formulation of all balances of a deformable directed surface (á priori two-dimensional equation) and
- the specific constitutive equation for the elastic and the viscoelastic material behavior.

For each specific type of plates the identification of the unknown material parameters is performed. It is clearly shown that, based on the assumption of linear elastic or viscoelastic behavior, the identification of different materials with homogeneous and non-homogeneous properties in the thickness direction can be realized. It is helpful that considered theory includes elements of symmetry groups for two-dimensional objects and the concept of effective properties. Simple static, quasi-static and dynamic problems are

Table 5. Effective stiffness of a plate made of PS foam (h has the dimension m)

Elastic constants	Foam, PS	Foam, PS *
$\alpha_1, \text{N/m } 10^6$	$138.67h$	$138.67h$
$\alpha_2, \text{N/m } 10^6$	$99.84h$	$104h$
$\alpha_3, \text{N/m } 10^6$	$108.16h$	$104h$
$\alpha_4, \text{N/m } 10^6$	$108.16h$	$104h$
$\beta_1, \text{N}\cdot\text{m } 10^6$	$-6.7 \cdot 10^{-6}h - 8.3h^3$	$-8.67h^3$
$\beta_2, \text{N}\cdot\text{m } 10^6$	$-2.5 \cdot 10^{-5}h + 11.6h^3$	$-11.5h^3$
$\beta_3, \text{N}\cdot\text{m } 10^6$	$4.5 \cdot 10^{-5}h + 28.9h^3$	$28.8h^3$
$\beta_4, \text{N}\cdot\text{m } 10^6$	$4.5 \cdot 10^{-5}h$	0

solved and the solutions are compared with a Kirchhoff-type, a Mindlin-type and a Reissner-type theory. We present the analysis of bending and free oscillations of foams or FGM plates based on the generalized equations of motion and take into account not only the transverse shear stiffness but also the rotatory inertia. We have shown that both parameters play an important role. In particular, the non-zero rotatory inertia allows us to describe the high-oscillated solutions. We established the bounds for the natural frequencies of FGM plates. The presented approach to model FGM plates within the framework of a 5-parametric theory of plates has an advantage with respect to theories of sandwich or laminated plates. The results on the theory of elastic plates are extended on the case of viscoelastic behaviour. We formulated 2D constitutive equations for a viscoelastic plate and analyzed the effective relaxation functions which are depended on the viscoelastic properties of a bulk material and on the structure of a plate. Finally we considered more general 6-parametric theory of plates (the theory of micropolar plates) and discuss the influence of the micropolar properties of foams on the effective stiffness parameters.

Acknowledgements

The research work was supported by the DFG grant AL 341/33-1 and by the RFBR grant 09-01-00459.

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Plasticity Theory of Porous and Powder Metals

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Abstract The present chapter mainly deals with several non-standard issues in the theory of plasticity for traditional metals as well as porous and powder metals. The flow formulation is adopted throughout the chapter. All the material models considered are rigid plastic, i.e. the elastic portion of the strain rate tensor is neglected. Assuming a rigid/perfectly plastic material model, it is shown that the velocity fields adjacent to surfaces of maximum friction must be describable by non-differentiable functions where the equivalent strain rate approaches infinity. This result is extended to the double-shearing model. Qualitative behavior of solutions based on various models of pressure-dependent plasticity is considered by means of problems permitting closed-form solutions. In particular, such features of the solutions as non-uniqueness, non-existence, singularity and rigid zones are emphasized. One possible application of the aforementioned singular solutions to describing intensive plastic deformation in a narrow layer near friction surfaces is shortly discussed.

1 Introduction

There are a great number of textbooks and monographs on the classical theory of plasticity and its applications (for example, Hill (1950); Avitzur (1980); Druyanov and Nepershin (1994); Kamenjarzh (1996); Chakrabarty (2006) among others). This theory belongs to the class of flow theories of plasticity whose constitutive equations involve strain rates rather than strains. The present chapter concerns with rigid plastic solids, i.e. the elastic portion of the strain rate tensor is neglected. Theories of plasticity for porous and powder materials are reduced to the classical theory of plasticity at specific values of input parameters. Such theories can be divided into two groups. The theories that belong to one of these groups include the associated flow rule as a constitutive equation. A typical area of applications

of such theories is forming processes in powder metallurgy (Druyanov, 1993). The theories that belong to the other group do not include the associated flow rule but are based on various relations between stresses and strain rates (Ishlinskii, 1954; Ostrowska-Maciejewska and Harris, 1990; Spencer, 1982; Harris and Grekova, 2005). A typical area of applications of such theories is the deformation of granular materials, though they are also applicable to some traditional metals (Spitzig et al., 1976).

In contrast to the aforementioned textbooks and monographs, the present chapter reviews non-standard features of solution behavior for the classical theory of plasticity and some of its generalizations for porous and powder materials. The starting point is the singular solution behavior in the vicinity of maximum friction surfaces for the classical theory of plasticity. In particular, the equivalent strain rate (quadratic invariant of the strain rate tensor) obeys an inverse square root rule in the form

$$\xi_{eq} = \frac{D}{\sqrt{s}} + o\left(\frac{1}{\sqrt{s}}\right), \quad s \rightarrow 0 \quad (1.1)$$

where s is the normal distance from the maximum friction surface and D is the strain rate intensity factor. This name for D has been introduced in Alexandrov and Richmond (2001a). It is seen from (1.1) that $\xi_{eq} \rightarrow \infty$ as $s \rightarrow 0$. For the first time, this feature of rigid perfectly plastic solutions has been mentioned in Sokolovskii (1956) for plane strain deformation. Further results have been obtained by Druyanov and Alexandrov (1992); Alexandrov (1995); Alexandrov and Richmond (2001a). Alexandrov and Lyamina (2002); Alexandrov (2005) have extended (1.1) to materials obeying the double-shearing model (Spencer, 1982). It is well known that cracks can produce stress singularities in linear elastic materials. One common type of stress singularity is $\sigma \propto K/\sqrt{r}$ where r is the distance from the crack tip and K is the stress intensity factor. The stress intensity factor approach is very effective in structural design. In particular, a popular fracture criterion is formulated in the form $K = K_c$ where K_c is a critical value of the stress intensity factor, a material constant (for example, Kanninen and Popelar (1985)). This type of singularity is somehow analogous to (1.1). Actually, it has been shown in Alexandrov (2001) that the analogy is quite complete. Therefore, it is reasonable to expect that the concept of the strain rate intensity factor can play a similar role in the mechanics of plastic deformation as the concept of the stress intensity factor in the mechanics of cracks. In particular, it is known that material properties in a narrow layer near frictional interfaces are very different from those in the bulk after metalworking processes (Dutton et al., 1998; Aukrust and LaZghab, 2000; Moylan et al., 2003). The strain rate intensity factor can be used to describe such

experimental observations (Alexandrov et al., 2003, 2009; Lyamina et al., 2007). In addition to the aforementioned singularity in solution behavior, the present chapter concerns with such qualitative features of solutions as non-uniqueness, non-existence, discontinuities and rigid zones. In this case however a number of closed-form solutions, rather than a general theory, are given and analyzed.

2 Fundamentals of the Theory of Plasticity

2.1 Rigid Perfectly/Plastic Solids

The classical theory of plasticity for metals is based on a pressure-independent yield criterion and its associated flow rule (Hill, 1950). The yield criterion is a postulated mathematical expression of the states of stress that will induce yielding. Stress invariants only can be involved in this expression for isotropic materials. Most popular stress invariants are the principal stresses, σ_1 , σ_2 and σ_3 , and the linear, σ , quadratic, σ_{eq} , and cubic, I_3 , invariants. The linear invariant is also called the hydrostatic stress and the quadratic invariant is the equivalent stress. The linear, quadratic and cubic invariants are defined as

$$\sigma = \frac{\sigma_{\alpha\alpha} + \sigma_{\beta\beta} + \sigma_{\gamma\gamma}}{3} \quad (1)$$

$$\sigma_{eq} = \sqrt{\frac{3}{2} \left(\tau_{\alpha\alpha}^2 + \tau_{\beta\beta}^2 + \tau_{\gamma\gamma}^2 + 2\tau_{\alpha\beta}^2 + 2\tau_{\beta\gamma}^2 + 2\tau_{\gamma\alpha}^2 \right)} \quad (2)$$

$$I_3 = \begin{vmatrix} \tau_{\alpha\alpha} & \tau_{\alpha\beta} & \tau_{\alpha\gamma} \\ \tau_{\beta\alpha} & \tau_{\beta\beta} & \tau_{\beta\gamma} \\ \tau_{\gamma\alpha} & \tau_{\gamma\beta} & \tau_{\gamma\gamma} \end{vmatrix} \quad (3)$$

Here $\sigma_{\alpha\alpha}$, $\sigma_{\beta\beta}$ and $\sigma_{\gamma\gamma}$ are the normal components of the stress tensor, $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$, $\sigma_{\beta\gamma} = \sigma_{\gamma\beta}$ and $\sigma_{\gamma\alpha} = \sigma_{\alpha\gamma}$ are the shear components of the stress tensor in an arbitrary curvilinear orthogonal coordinate system $\alpha\beta\gamma$. Also, $\tau_{\alpha\alpha}$, $\tau_{\beta\beta}$, $\tau_{\gamma\gamma}$, $\tau_{\alpha\beta}$, $\tau_{\beta\gamma}$, and $\tau_{\gamma\alpha}$ are the deviatoric stress components defined by

$$\tau_{\alpha\alpha} = \sigma_{\alpha\alpha} - \sigma, \tau_{\beta\beta} = \sigma_{\beta\beta} - \sigma, \tau_{\gamma\gamma} = \sigma_{\gamma\gamma} - \sigma, \tau_{\alpha\beta} = \sigma_{\alpha\beta}, \tau_{\beta\gamma} = \sigma_{\beta\gamma}, \tau_{\gamma\alpha} = \sigma_{\gamma\alpha} \quad (4)$$

Note that the stress components introduced above are the "physical" components, i.e. the dyadic representation of the stress tensor $\mathbf{T}$ is

$$\begin{aligned} \mathbf{T} = & \sigma_{\alpha\alpha} \mathbf{e}_\alpha \mathbf{e}_\alpha + \sigma_{\beta\beta} \mathbf{e}_\beta \mathbf{e}_\beta + \sigma_{\gamma\gamma} \mathbf{e}_\gamma \mathbf{e}_\gamma + \sigma_{\alpha\beta} \mathbf{e}_\alpha \mathbf{e}_\beta + \sigma_{\beta\alpha} \mathbf{e}_\beta \mathbf{e}_\alpha + \\ & + \sigma_{\beta\gamma} \mathbf{e}_\beta \mathbf{e}_\gamma + \sigma_{\gamma\beta} \mathbf{e}_\gamma \mathbf{e}_\beta + \sigma_{\gamma\alpha} \mathbf{e}_\gamma \mathbf{e}_\alpha + \sigma_{\alpha\gamma} \mathbf{e}_\alpha \mathbf{e}_\gamma \end{aligned} \quad (5)$$

where $\mathbf{e}_\alpha$, $\mathbf{e}_\beta$ and $\mathbf{e}_\gamma$ are the unit vectors tangential to the coordinate curves.

A general form of the yield criterion for isotropic pressure-independent materials in terms of the principal stresses is

$$F(\sigma_1 - \sigma_2, \sigma_2 - \sigma_3, \sigma_3 - \sigma_1) = 0 \quad (6)$$

where F is supposed to be convex. An alternative form of the general yield criterion is

$$\Phi(\sigma_{eq}, I_3) = 0 \quad (7)$$

where Φ is convex as well.

Von Mises' yield criterion can be written in the following alternative forms

$$\sigma_{eq} = \sigma_0 \quad (8)$$

or

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_0^2 \quad (9)$$

where σ_0 is the yield stress in tension, a material constant for rigid perfectly/plastic solids. It is obvious from (8) and (9) that the von Mises yield criterion is independent of the hydrostatic stress. It is also seen from (8) that the von Mises yield criterion is independent of I_3 . The representation of the von Mises yield criterion in the form of (9) is convenient for its geometric interpretation which is called the yield surface. The π - plane in the principal stress space is defined by the equation $\sigma = 0$. Since the von Mises yield criterion is pressure-independent, it is sufficient to consider the yield locus on an arbitrary plane, Ω - plane, parallel to the π - plane. It follows from (9) that the yield locus is a circle whose radius is $R = \sqrt{2/3}\sigma_0$ (see Fig. 1).

It follows from (8) and (9) that the von Mises yield criterion depends on one parameter, the yield stress in tension. Therefore, once the value of the yield stress in tension has been determined from experiment, all other yield stresses can be calculated from (8) or (9). In particular, it follows from (8) that the yield stresses in tension and compression are equivalent. In order to find the shear yield stress, it is necessary to recall that the state of stress in this case is

$$\sigma_1 = -\sigma_2 = \tau, \quad \sigma_3 = 0 \quad (10)$$

At yielding, $\tau = \tau_0$ where τ_0 is the shear yield stress. Substituting (10) at yielding into (9) gives

$$\tau_0 = \frac{\sigma_0}{\sqrt{3}} \quad (11)$$

In passing, it is worth noting that (11) is a theoretical relation, i.e. it is not necessary that the experimental shear yield stress satisfies (11).

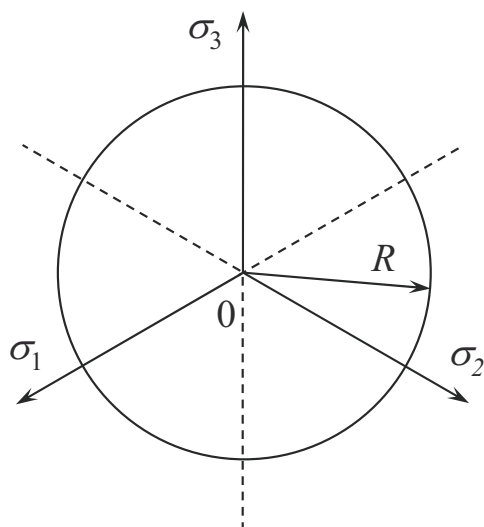


Figure 1. Von Mises yield locus

Tresca's yield criterion for three-dimensional flow is usually written in terms of the principal stresses since its representation in terms of σ_{eq} and I_3 is very cumbersome. Moreover, in contrast to the von Mises yield criterion, this yield criterion is singular and is given by six expressions

$$\left. \begin{aligned} \sigma_1 - \sigma_2 &\leq \sigma_0 \\ \sigma_2 - \sigma_3 &\leq \sigma_0 \\ \sigma_3 - \sigma_1 &\leq \sigma_0 \\ \sigma_2 - \sigma_1 &\leq \sigma_0 \\ \sigma_3 - \sigma_2 &\leq \sigma_0 \\ \sigma_1 - \sigma_3 &\leq \sigma_0 \end{aligned} \right\} \quad (12)$$

Plastic yielding occurs if the equality holds in one or two inequalities shown in (12). The regime of flow corresponding to the former case is sometimes referred to as simple flow and the regime corresponding to the latter case as full flow. Obviously, the equality cannot hold in three or more inequalities in (12). The yield locus on the Ω - plane is shown in Fig. 2. Substituting (10) at yielding into (12) gives

$$\tau_0 = \frac{\sigma_0}{2} \quad (13)$$

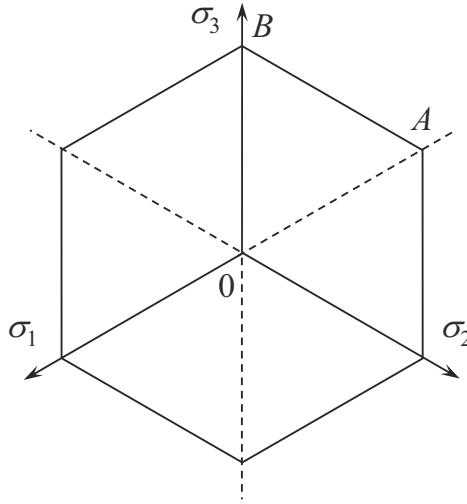


Figure 2. Tresca yield locus

It follows from (11) and (13) that the shear yield stress predicted by the Tresca yield criterion is lower than that predicted by the von Mises yield criterion, assuming that the same magnitude of the yield stress in tension is chosen for both. The latter is a typical assumption and the value of R in Fig. 1 is equal to the length of OA in Fig. 2 in this case. Von Mises and Tresca loci satisfying this assumption are shown together in Fig. 3. However, experimental data can be available for the shear yield stress, for example from the torsion test. In this case it is natural to use the same value of this yield stress in both von Mises and Tresca yield criteria. Then, it follows from (11) and (13) that the yield stress in tension predicted by Tresca yield criterion is larger than that predicted by von Mises yield criterion (see Fig. 4).

For smooth yield surfaces such as the von Mises yield surface (see Fig. 1) or the faces of the Tresca yield surface (see Fig. 2) the associated flow rule has the following form

$$\begin{aligned} \xi_{\alpha\alpha} &= \lambda_0 \frac{\partial \Phi}{\partial \sigma_{\alpha\alpha}}, \quad \xi_{\beta\beta} = \lambda_0 \frac{\partial \Phi}{\partial \sigma_{\beta\beta}}, \quad \xi_{\gamma\gamma} = \lambda_0 \frac{\partial \Phi}{\partial \sigma_{\gamma\gamma}}, \\ \xi_{\alpha\beta} &= \lambda_0 \frac{\partial \Phi}{\partial \sigma_{\alpha\beta}}, \quad \xi_{\beta\gamma} = \lambda_0 \frac{\partial \Phi}{\partial \sigma_{\beta\gamma}}, \quad \xi_{\gamma\alpha} = \lambda_0 \frac{\partial \Phi}{\partial \sigma_{\gamma\alpha}} \end{aligned} \quad (14)$$

where $\xi_{\alpha\alpha}$, $\xi_{\beta\beta}$, $\xi_{\gamma\gamma}$, $\xi_{\alpha\beta}$, $\xi_{\beta\gamma}$, and $\xi_{\gamma\alpha}$ are the components of the strain rate

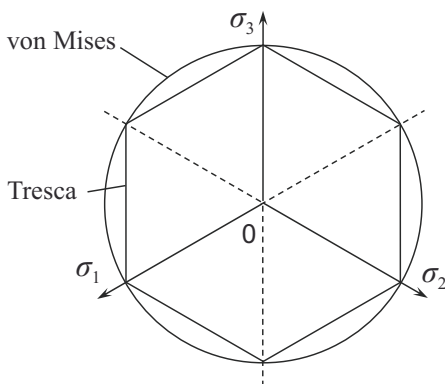


Figure 3. Von Mises and Tresca yield loci based on the assumption that the yield stress in tension is determined from experiment

tensor in the coordinate system $\alpha\beta\gamma$, the function Φ has been introduced in (7) and λ_0 is a non-negative multiplier whose value is determined from the solution to boundary value problems. In passing, it is worth noting that because of the relations $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$, $\sigma_{\beta\gamma} = \sigma_{\gamma\beta}$ and $\sigma_{\gamma\alpha} = \sigma_{\alpha\gamma}$ each of these pairs of the shear stress components is usually combined in a single term in the representation of the yield criterion (see, for example, (2) and (8)). However, each of the aforementioned stress components should be considered separately in (14). For example, substituting (2) and (8) into (14) gives the associated flow rule for the von Mises yield criterion in the form

$$\begin{aligned} \xi_{\alpha\alpha} &= \lambda\tau_{\alpha\alpha}, \quad \xi_{\beta\beta} = \lambda\tau_{\beta\beta}, \quad \xi_{\gamma\gamma} = \lambda\tau_{\gamma\gamma}, \\ \xi_{\alpha\beta} &= \xi_{\beta\alpha} = \lambda\tau_{\alpha\beta}, \quad \xi_{\beta\gamma} = \xi_{\gamma\beta} = \lambda\tau_{\beta\gamma}, \quad \xi_{\gamma\alpha} = \xi_{\alpha\gamma} = \lambda\tau_{\alpha\gamma}, \end{aligned} \quad (15)$$

where $\lambda = 3\lambda_0/2\sigma_0$ can be considered as a new unknown instead of λ_0 . Since $\lambda_0 \geq 0$, $\lambda \geq 0$ as well. Moreover, $\lambda = 0$ if and only if $\lambda_0 = 0$. It is seen from (14) and (15) that no plastic flow occurs if $\lambda = 0$. The equivalent strain rate is defined by

$$\xi_{eq} = \sqrt{\frac{2}{3} \left(\xi_{\alpha\alpha}^2 + \xi_{\beta\beta}^2 + \xi_{\gamma\gamma}^2 + 2\xi_{\alpha\beta}^2 + 2\xi_{\beta\gamma}^2 + 2\xi_{\gamma\alpha}^2 \right)} \quad (16)$$

Using (2), (8), (15) and (16) it is possible to express λ as

$$\lambda = \frac{3}{2} \frac{\xi_{eq}}{\sigma_0} \quad (17)$$

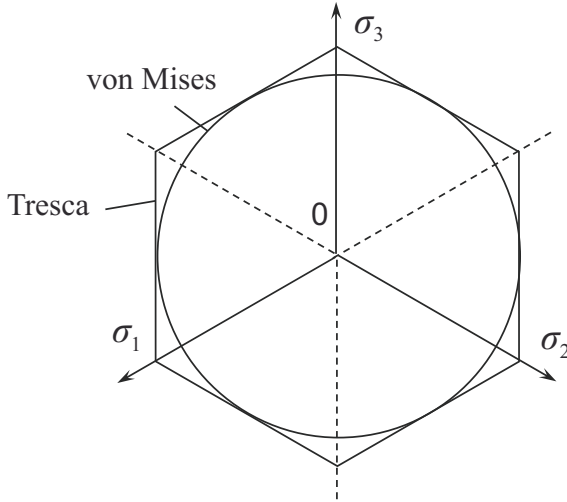


Figure 4. Von Mises and Tresca yield loci based on the assumption that the shear yield stress is determined from experiment

Then, Eqs (15) transform to

$$\begin{aligned} \xi_{\alpha\alpha} &= \frac{3\xi_{eq}}{2\sigma_0}\tau_{\alpha\alpha}, \quad \xi_{\beta\beta} = \frac{3\xi_{eq}}{2\sigma_0}\tau_{\beta\beta}, \quad \xi_{\gamma\gamma} = \frac{3\xi_{eq}}{2\sigma_0}\tau_{\gamma\gamma}, \\ \xi_{\alpha\beta} &= \frac{3\xi_{eq}}{2\sigma_0}\tau_{\alpha\beta}, \quad \xi_{\beta\gamma} = \frac{3\xi_{eq}}{2\sigma_0}\tau_{\beta\gamma}, \quad \xi_{\alpha\gamma} = \frac{3\xi_{eq}}{2\sigma_0}\tau_{\alpha\gamma} \end{aligned} \quad (18)$$

These equations include both the von Mises yield criterion and its associated flow rule. The incompressibility equation in terms of the normal strain rate components has the form

$$\xi_{\alpha\alpha} + \xi_{\beta\beta} + \xi_{\gamma\gamma} = 0 \quad (19)$$

In particular, this equation immediately follows from (15) and the identity $\tau_{\alpha\alpha} + \tau_{\beta\beta} + \tau_{\gamma\gamma} = 0$.

The associated flow rule can be written in terms of the principal strain rates, ξ_1 , ξ_2 and ξ_3 . To this end it is sufficient to replace Φ with F introduced in (6) in the first three equations of (14) and differentiate with respect to the principal stresses. The three relations between the principal stresses and the principal strain rates obtained should be supplemented by the condition that the principal directions of the stress tensor coincide with the principal directions of the strain rate tensor.

The derivatives involved in (14) do not exist at singular points of the yield surface, for example points A and B in Fig. 2. When the singularity is formed by two smooth surfaces, say $\Phi_1(\sigma_{eq}, I_3) = 0$ and $\Phi_2(\sigma_{eq}, I_3) = 0$, a commonly adopted generalization of the associated flow rule (14) is

$$\begin{aligned}\xi_{\alpha\alpha} &= \lambda_1 \frac{\partial \Phi_1}{\partial \sigma_{\alpha\alpha}} + \lambda_2 \frac{\partial \Phi_2}{\partial \sigma_{\alpha\alpha}}, \quad \xi_{\beta\beta} = \lambda_1 \frac{\partial \Phi_1}{\partial \sigma_{\beta\beta}} + \lambda_2 \frac{\partial \Phi_2}{\partial \sigma_{\beta\beta}}, \\ \xi_{\gamma\gamma} &= \lambda_1 \frac{\partial \Phi_1}{\partial \sigma_{\gamma\gamma}} + \lambda_2 \frac{\partial \Phi_2}{\partial \sigma_{\gamma\gamma}}, \quad \xi_{\alpha\beta} = \lambda_1 \frac{\partial \Phi_1}{\partial \sigma_{\alpha\beta}} + \lambda_2 \frac{\partial \Phi_2}{\partial \sigma_{\alpha\beta}}, \\ \xi_{\beta\gamma} &= \lambda_1 \frac{\partial \Phi_1}{\partial \sigma_{\beta\gamma}} + \lambda_2 \frac{\partial \Phi_2}{\partial \sigma_{\beta\gamma}}, \quad \xi_{\gamma\alpha} = \lambda_1 \frac{\partial \Phi_1}{\partial \sigma_{\gamma\alpha}} + \lambda_2 \frac{\partial \Phi_2}{\partial \sigma_{\gamma\alpha}}\end{aligned}\quad (20)$$

where $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$. The first three equations of (20) can be used to express the principal strain rates if $F_1(\sigma_1 - \sigma_2, \sigma_2 - \sigma_3, \sigma_3 - \sigma_1) = 0$ and $F_2(\sigma_1 - \sigma_2, \sigma_2 - \sigma_3, \sigma_3 - \sigma_1) = 0$ are given. To this end it is just necessary to replace Φ_1 with F_1 , Φ_2 with F_2 and differentiate with respect to the principal stresses. For example, the regime of flow corresponding to point A (see Fig. 2) occurs if the following equations of the system (12) are satisfied

$$F_1 \equiv \sigma_3 - \sigma_1 - \sigma_0 = 0, \quad F_2 \equiv \sigma_2 - \sigma_1 - \sigma_0 = 0 \quad (21)$$

Then,

$$\xi_1 = -\lambda_1 - \lambda_2, \quad \xi_2 = \lambda_2, \quad \xi_3 = \lambda_1 \quad (22)$$

It immediately follows from these equations that $\xi_1 < 0$, $\xi_2 > 0$ and $\xi_3 > 0$. Point B (see Fig. 2) can be considered analogously. In this case

$$F_1 \equiv \sigma_3 - \sigma_1 - \sigma_0 = 0, \quad F_2 \equiv \sigma_3 - \sigma_2 - \sigma_0 = 0 \quad (23)$$

and

$$\xi_1 = -\lambda_1, \quad \xi_2 = -\lambda_2, \quad \xi_3 = \lambda_1 + \lambda_2 \quad (24)$$

Thus $\xi_1 < 0$, $\xi_2 < 0$ and $\xi_3 > 0$. An essential difference between the regimes of flow corresponding to points A and B is that the principal strain rate of the greatest magnitude is negative at point A and is positive at point B . Substituting (22) and (24) into (19) shows that the incompressibility equation is indeed satisfied.

The models of rigid perfectly/plastic solids are rate- and history-independent. The former means that the speed of loading has no effect on the load required to deform the material. The latter means that the solution is completely determined by the current configuration and is independent of the history of deformation.

2.2 Rigid Plastic Hardening Solids

Plastic strain has an effect on the yield criterion. To account for this effect various rules of isotropic and kinematic hardening are adopted. The further discussion is restricted to isotropic hardening of the material whose initial yield criterion is that of von Mises. In this case the yield criterion after any amount of deformation has the form

$$\sigma_{eq} = \sigma_0 \Omega(\varepsilon_{eq}) \quad (25)$$

where the equivalent strain is defined by the following equation

$$\varepsilon_{eq} = \int_0^t \xi_{eq} dt \quad (26)$$

where t is the time and the integral is taken over the actual strain path. The initial stage corresponds to $t = 0$. Then, it follows from (26) that it is assumed that $\varepsilon_{eq} = 0$ at the initial stage. The model under consideration is history-dependent but is rate-independent. In particular, the latter means that the time can be replaced with any time-like parameter. The function $\Omega(\varepsilon_{eq})$ involved in (25) satisfies the conditions

$$\Omega(0) = 1, \quad \frac{d\Omega}{d\varepsilon_{eq}} > 0 \quad (27)$$

It follows from (27) that the yield criterion (25) reduces to (8) at the initial instant. A geometric interpretation of (25) and (27) is that the value of R (see Fig. 1) increases with the equivalent plastic strain.

Various empirical representations of the function $\Omega(\varepsilon_{eq})$ are available in the literature. An excellent review of such representations is provided in Rees (2006). Some of them are given below. The Ludwik power law is

$$\Omega(\varepsilon_{eq}) = 1 + \left(\frac{\varepsilon_{eq}}{\varepsilon_0} \right)^n \quad (28)$$

where n and ε_0 are empirical constants and n is usually called the strain hardening exponent. Since $n < 1$ for real materials, a disadvantage of the representation (28) is that $d\Omega/d\varepsilon_{eq} \rightarrow \infty$ as $\varepsilon_{eq} \rightarrow 0$. This may cause a difficulty with a numerical treatment of boundary value problems involving the constitutive equation (28). The Swift power law is free of this drawback and is given by

$$\Omega(\varepsilon_{eq}) = \left(1 + \frac{\varepsilon_{eq}}{\varepsilon_0} \right)^n \quad (29)$$

Both Ludwik's and Swift's laws assume that $\sigma_{eq} \rightarrow \infty$ as $\varepsilon_{eq} \rightarrow \infty$, i.e. the yield locus becomes infinitely large ($R \rightarrow \infty$ in Fig. 1) as $\varepsilon_{eq} \rightarrow \infty$. Voce's exponential law predicts qualitatively different behavior of σ_{eq} as $\varepsilon_{eq} \rightarrow \infty$. In particular, it can be written in the form

$$\Omega(\varepsilon_{eq}) = \frac{\sigma_s}{\sigma_0} - \left(\frac{\sigma_s}{\sigma_0} - 1 \right) \exp \left(-\frac{\varepsilon_{eq}}{\varepsilon_0} \right) \quad (30)$$

where σ_s is a saturation stress, i.e. $\sigma_{eq} \rightarrow \sigma_s$ as $\varepsilon_{eq} \rightarrow \infty$.

The associated flow rule leads to Eqs (15) with λ being an arbitrary unknown function. This function can be excluded by means of (2), (16) and (25) to give

$$\begin{aligned} \xi_{\alpha\alpha} &= \frac{3\xi_{eq}}{2\sigma_0\Omega(\varepsilon_{eq})} \tau_{\alpha\alpha}, \quad \xi_{\beta\beta} = \frac{3\xi_{eq}}{2\sigma_0\Omega(\varepsilon_{eq})} \tau_{\beta\beta}, \quad \xi_{\gamma\gamma} = \frac{3\xi_{eq}}{2\sigma_0\Omega(\varepsilon_{eq})} \tau_{\gamma\gamma}, \\ \xi_{\alpha\beta} &= \frac{3\xi_{eq}}{2\sigma_0\Omega(\varepsilon_{eq})} \tau_{\alpha\beta}, \quad \xi_{\beta\gamma} = \frac{3\xi_{eq}}{2\sigma_0\Omega(\varepsilon_{eq})} \tau_{\beta\gamma}, \quad \xi_{\alpha\gamma} = \frac{3\xi_{eq}}{2\sigma_0\Omega(\varepsilon_{eq})} \tau_{\alpha\gamma} \end{aligned} \quad (31)$$

This representation of the associated flow rule contains the yield criterion as well.

2.3 Rigid Viscoplastic Solids

Viscoplastic models are used to take into account rate effects. Such models are of special importance for deformation processes at elevated temperature and high strain rate. They are also adopted for describing superplastic deformation. The size of the yield surface for viscoplastic materials depends on the equivalent strain rate. The further discussion is restricted to materials whose yield criterion at $\xi_{eq} = 0$ is that of von Mises. The yield criterion at any strain rate can be written in the form

$$\sigma_{eq} = \sigma_0 \Lambda(\xi_{eq}) \quad (32)$$

where the function $\Lambda(\xi_{eq})$ satisfies the conditions

$$\Lambda(0) = 1, \quad \frac{d\Lambda}{d\xi_{eq}} > 0 \quad (33)$$

The simplest and most popular function is the following power law

$$\Lambda(\xi_{eq}) = 1 + \left(\frac{\xi_{eq}}{\xi_0} \right)^m \quad (34)$$

where m and ξ_0 are empirical constants. The Bingham model is obtained from (34) at $m = 1$. The size of the yield surface described by (34) increases

with no limit as $\xi_{eq} \rightarrow \infty$. However, there are observations that demonstrate that the magnitude of the equivalent stress approaches a limit as the equivalent strain rate increases (Sinczak et al., 1992). For such materials, a reasonable assumption is that $\Lambda(\xi_{eq}) \rightarrow \sigma_s/\sigma_0$ as $\xi_{eq} \rightarrow \infty$ where σ_s is a saturation stress, i.e. $\sigma_{eq} \rightarrow \sigma_s$ as $\xi_{eq} \rightarrow \infty$. A typical function of this class is

$$\Lambda(\xi_{eq}) = \frac{\sigma_s}{\sigma_0} - \left(\frac{\sigma_s}{\sigma_0} - 1 \right) \exp\left(-\frac{\xi_{eq}}{\xi_0}\right) \quad (35)$$

The associated flow rule leads to Eqs (15) with λ being an arbitrary unknown function. This function can be excluded by means of (2), (16) and (32) to give

$$\begin{aligned} \xi_{\alpha\alpha} &= \frac{3\xi_{eq}}{2\sigma_0\Lambda(\xi_{eq})}\tau_{\alpha\alpha}, \quad \xi_{\beta\beta} = \frac{3\xi_{eq}}{2\sigma_0\Lambda(\xi_{eq})}\tau_{\beta\beta}, \quad \xi_{\gamma\gamma} = \frac{3\xi_{eq}}{2\sigma_0\Lambda(\xi_{eq})}\tau_{\gamma\gamma}, \\ \xi_{\alpha\beta} &= \frac{3\xi_{eq}}{2\sigma_0\Lambda(\xi_{eq})}\tau_{\alpha\beta}, \quad \xi_{\beta\gamma} = \frac{3\xi_{eq}}{2\sigma_0\Lambda(\xi_{eq})}\tau_{\beta\gamma}, \quad \xi_{\alpha\gamma} = \frac{3\xi_{eq}}{2\sigma_0\Lambda(\xi_{eq})}\tau_{\alpha\gamma}, \end{aligned} \quad (36)$$

This representation of the associated flow rule contains the yield criterion as well.

The viscoplastic models under consideration are history-independent but are rate-dependent. The latter means that the speed of loading has an effect on the magnitude of the load required to deform the material.

There are a great number of empirical relations for σ_{eq} that involve both $\Omega(\varepsilon_{eq})$ introduced in (25) and $\Lambda(\xi_{eq})$ introduced in (32). A review of such relations can be found in Gronostajski (2000). Such models are history- and rate-dependent.

2.4 Maximum Friction Law and Singular Velocity Fields (Rigid Perfectly/Plastic Material)

The maximum friction law for rigid perfectly/plastic materials obeying the von Mises yield criterion (8) states that

$$\tau_f = \tau_0 \quad (37)$$

at sliding. Here τ_f is the friction stress and τ_0 is related to the yield stress in tension by (11). A surface where the friction law (37) is satisfied is called the maximum friction surface. As a rule, velocity fields adjacent to maximum friction surfaces are singular in the sense that the equivalent strain rate and, therefore, some components of the strain rate tensor approach infinity in the vicinity of such surfaces (Alexandrov and Richmond, 2001a). The easiest way to demonstrate it is to consider axisymmetric flow (see Fig.

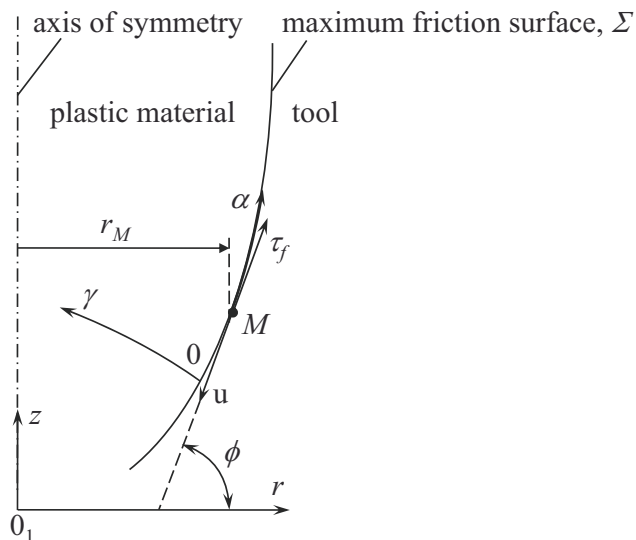


Figure 5. Axisymmetric flow near the maximum friction surface

5). Introduce a cylindrical coordinate system $r\theta z$ with its z -axis coinciding with the axis of symmetry of the flow and another orthogonal curvilinear coordinate system by the relations $\alpha = \alpha(r, z)$, $\beta \equiv \theta$ and $\gamma = \gamma(r, z)$. The solution is independent of θ (or β). Let Σ be a tool surface where the condition (37) is satisfied. Assume that the regime of sliding occurs and consider Eqs (2) and (8) at an arbitrary point, M , on that surface. It is always possible to choose such a coordinate system that its α -line coincides with the friction surface (this surface is represented by a curve on planes $\beta = \text{constant}$). Then, it follows from (37) that $|\tau_{\alpha\gamma}| = \tau_0$ at point M . Substituting this value of $\tau_{\alpha\gamma}$ into (2) and (8) gives

$$\tau_{\alpha\alpha} = \tau_{\beta\beta} = \tau_{\gamma\gamma} = \tau_{\alpha\beta} = \tau_{\beta\gamma} = 0 \quad (38)$$

If the equivalent strain rate is finite, Eqs (18) and (38) result in

$$\xi_{\alpha\alpha} = \xi_{\beta\beta} = \xi_{\gamma\gamma} = \xi_{\alpha\beta} = \xi_{\beta\gamma} = 0 \quad (39)$$

An advantage of the axisymmetric flow under consideration is that the circumferential strain rate, $\xi_{\alpha\alpha} \equiv \xi_{\theta\theta} = u_r/r$, is defined by the radial velocity, u_r , and not by derivatives of velocity components. The regime of sliding requires that the magnitude of the velocity vector, u , does not vanish at

the friction surface (the tool can be regarded as motionless with no loss of generality). Then (see Fig. 5),

$$\xi_{\beta\beta} = -\frac{u \cos \phi}{r_M} \quad (40)$$

This contradicts (39) and, therefore, the supposition that the equivalent strain rate is finite at point M , unless $\phi = \pi/2$. Thus, in general, $\xi_{eq} \rightarrow \infty$ as $\gamma \rightarrow 0$ (see Fig. 5). In order to find the dependence of the equivalent strain rate on coordinates in the vicinity of maximum friction surfaces, it is necessary to complete an asymptotic analysis of the constitutive equations supplemented with the equilibrium equations. It will be done for general three-dimensional flow.

The equilibrium equations in a Cartesian coordinate system, $x_1 \equiv \alpha$, $x_2 \equiv \beta$, $x_3 \equiv \gamma$, are

$$\begin{aligned} \frac{\partial \sigma}{\partial x_1} + \frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{13}}{\partial x_3} &= 0, \\ \frac{\partial \sigma}{\partial x_2} + \frac{\partial \tau_{22}}{\partial x_2} + \frac{\partial \tau_{21}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_3} &= 0, \\ \frac{\partial \sigma}{\partial x_3} + \frac{\partial \tau_{33}}{\partial x_3} + \frac{\partial \tau_{31}}{\partial x_1} + \frac{\partial \tau_{32}}{\partial x_2} &= 0 \end{aligned} \quad (41)$$

Let Σ be a tool surface where the condition (37) is satisfied, and consider Eqs (8), (15) and (41) at an arbitrary point, M , on that surface. The tool is regarded as fixed. The local Cartesian coordinate system is taken to be situated at M with its x_3 -axis directed along the normal to Σ , away from the rigid tool and towards the plastic material (see Fig. 6). The x_1 -axis directed along the velocity vector, $\mathbf{u}$, at M , and the x_2 -axis is perpendicular to both x_1 and x_3 forming a right-hand system. Because the friction stress is always directed opposite to the velocity vector the x_2 -axis coincides with one of the principal directions of the stress tensor. The principal directions in the x_1x_3 plane at point M make angles $\pm\pi/4$ with the x_1 -axis as shown in Fig. 7. In the coordinate system chosen, the components of the deviator stress tensor at point M are determined from (2), (8) and (37) as

$$\tau_{13} = \tau_0, \quad \tau_{11} = \tau_{22} = \tau_{33} = \tau_{12} = \tau_{23} = 0 \quad (42)$$

Equations (15) combined with (42) give $\xi_{11} = \xi_{22} = \xi_{33} = \xi_{12} = \xi_{23} = 0$, if $\lambda < \infty$. It has been already shown by considering axisymmetric flow that it is not always possible. Therefore, the further analysis is restricted to the case

$$\lambda \rightarrow \infty \quad \text{as} \quad x_3 \rightarrow 0 \quad (43)$$

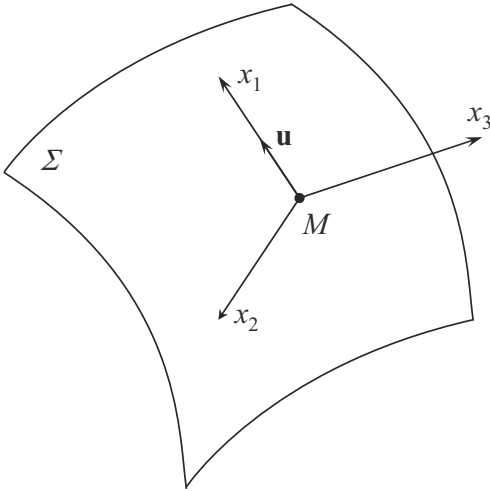


Figure 6. Local Cartesian coordinate system near point M on the maximum friction surface

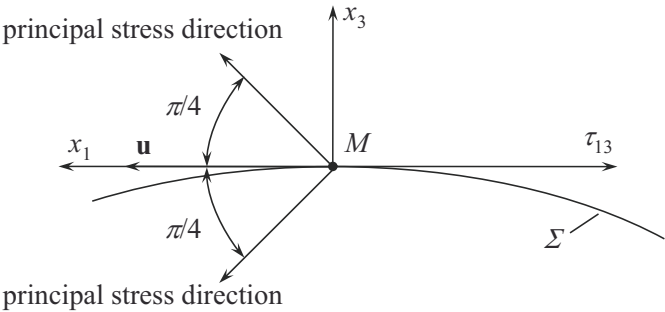


Figure 7. x_1x_3 coordinate surface and principal stress directions at point M on maximum friction surface

Note that there are many analytic solutions in which this condition is satisfied on a surface of maximum friction (Spencer, 1965; Shield, 1955; Hill, 1950; Alexandrov and Lyamina, 2008). It follows from Eqs (15) and (42) that $\xi_{13} = \lambda\tau_0$. Then, this equation and (43) lead to

$$\xi_{13} \rightarrow \infty \quad \text{as} \quad x_3 \rightarrow 0 \tag{44}$$

In Cartesian coordinates

$$\begin{aligned} \xi_{11} &= \frac{\partial u_1}{\partial x_1}, \quad \xi_{22} = \frac{\partial u_2}{\partial x_2}, \quad \xi_{33} = \frac{\partial u_3}{\partial x_3}, \\ \xi_{12} &= \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right), \quad \xi_{13} = \frac{1}{2} \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right), \quad \xi_{23} = \frac{1}{2} \left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \right) \end{aligned} \quad (45)$$

where u_1 , u_2 and u_3 are the components of the velocity vector relative to these coordinates. It is natural to assume that velocity and its in-surface derivatives are bounded at point M . Then, it follows from Eqs (44) and (45) that

$$\frac{\partial u_1}{\partial x_3} \rightarrow \infty \quad \text{as} \quad x_3 \rightarrow 0 \quad (46)$$

Thus, expression of the velocity component u_1 must involve non-differentiable functions.

The condition (44) may be rewritten in the form

$$\xi_{13} = E_0 x_3^{-\omega} + o(x_3^{-\omega}) \quad \text{as} \quad x_3 \rightarrow 0 \quad (47)$$

where

$$E_0 > 0 \quad \text{and} \quad \omega > 0 \quad (48)$$

Substituting (47) into (45) and integrating gives

$$u_1 = u_{10} + O(x_3^{1-\omega}) \quad \text{as} \quad x_3 \rightarrow 0 \quad (49)$$

where $u_{10} = O(1)$ as $x_3 \rightarrow 0$. It follows from Eq. (49) and the assumption that the velocity is bounded that

$$1 - \omega > 0 \quad (50)$$

Then combining Eqs (48) and (50) results in

$$0 < \omega < 1 \quad (51)$$

Equations (45) and the assumption that the in-surface derivatives of the velocity are bounded at M lead to

$$\xi_{11} = O(x_3^{\omega_{11}}) \quad \text{and} \quad \xi_{22} = O(x_3^{\omega_{22}}) \quad \text{as} \quad x_3 \rightarrow 0 \quad (52)$$

where $\omega_{11} \geq 0$ and $\omega_{22} \geq 0$. Then the incompressibility equation (19) combined with (52) gives

$$\xi_{33} = O(x_3^{\omega_{33}}) \quad \text{as} \quad x_3 \rightarrow 0 \quad (53)$$

where $\omega_{33} \geq 0$. The x_2 -axis coincides with a principal direction of the stress tensor (see Figs 6 and 7) and, because of coaxiality, with a principal direction of the strain rate tensor. Therefore,

$$\xi_{12} = O(x_3^{\omega_{12}}) \quad \text{and} \quad \xi_{23} = O(x_3^{\omega_{23}}) \quad \text{as} \quad x_3 \rightarrow 0 \quad (54)$$

where $\omega_{12} > 0$ and $\omega_{23} > 0$. Thus, all components of the strain rate tensor are bounded except ξ_{13} . It has been already shown by considering axisymmetric flow that there are processes in which at least one of these components is not zero at $x_3 = 0$. It follows from Eqs (52) and (54) that it is possible if and only if

$$\omega_{11} = 0 \quad \text{and/or} \quad \omega_{22} = 0 \quad (55)$$

Then, from (52)

$$\xi = O(1) \quad \text{as} \quad x_3 \rightarrow 0 \quad (56)$$

where ξ stands for the strain rate component (or components) for which the condition (56) is satisfied.

Equations (42) can be rewritten in the form

$$\begin{aligned} \tau_{13} &= \tau_0 + o(1), \tau_{11} = O(x_3^{\mu_{11}}), \tau_{22} = O(x_3^{\mu_{22}}), \\ \tau_{33} &= O(x_3^{\mu_{33}}), \tau_{12} = O(x_3^{\mu_{12}}), \tau_{23} = O(x_3^{\mu_{23}}), \\ \mu_{11} &> 0, \mu_{22} > 0, \mu_{33} > 0, \mu_{12} > 0, \mu_{23} > 0 \end{aligned} \quad (57)$$

as $x_3 \rightarrow 0$. It follows from (47), the first equation of (57) and (15) that

$$\lambda = O(x_3^{-\omega}) \quad \text{as} \quad x_3 \rightarrow 0 \quad (58)$$

The equation of the associated flow rule (15) corresponding to the strain rate component ξ introduced in (56) has the form $\xi = \lambda \tau$ where τ is the component of the stress deviator tensor corresponding to the strain rate component ξ . Combining this equation and (58) results in

$$\tau = O(x_3^{\omega}) \quad \text{as} \quad x_3 \rightarrow 0 \quad (59)$$

The first equation of (57) can be rewritten in the form

$$\tau_{13} = \tau_0 + O(x_3^{\mu_{13}}) \quad \text{as} \quad x_3 \rightarrow 0 \quad (60)$$

with $\mu_{13} > 0$. It is natural to assume that the stresses and their in-surface derivatives are bounded at $x_3 = 0$. In this case, because the derivatives of τ_{23} and τ_{13} with respect to x_3 are involved in the equilibrium equations (41), the following inequalities hold

$$\mu_{13} \geq 1 \quad \text{and} \quad \mu_{23} \geq 1 \quad (61)$$

Since all components of the strain rate tensor are bounded except ξ_{13} (see the conclusion after Eq. (54)), it follows from the associated flow rule (15) and (58) that

$$\mu_{11} \geq \omega, \mu_{22} \geq \omega, \mu_{33} \geq \omega, \mu_{12} \geq \omega, \quad \text{and} \quad \mu_{23} \geq \omega \quad (62)$$

Then, substituting (59) and (60) into the yield criterion (8) leads to

$$O(x_3^{2\omega}) = O(x_3^{\mu_{13}}) \quad \text{as} \quad x_3 \rightarrow 0 \quad (63)$$

It is necessary to point out that even if a few components of the stress deviator tensor satisfy (59) the left-hand side of Eq. (63) cannot vanish because all terms of the order $x_3^{2\omega}$ are positive. It follows from (63) that

$$\mu_{13} = 2\omega \quad (64)$$

or taking into account Eqs (51) and (61)

$$\frac{1}{2} \leq \omega < 1 \quad (65)$$

Assume that $1/2 < \omega < 1$. Then taking into account (57), (60) and (64) the first equation of the system (41) can be rewritten in the form

$$\frac{\partial \sigma}{\partial x_1} + O(x_3^{\mu_{11}}) + O(x_3^{\mu_{12}}) + O(x_3^{2\omega-1}) = 0 \quad (66)$$

It is necessary to examine the cases

$$\mu_{12} = 2\omega - 1, \quad (67)$$

$$\mu_{11} = 2\omega - 1, \quad (68)$$

$$\frac{\partial \sigma}{\partial x_1} = O(x_3^{2\omega-1}) \quad \text{as} \quad x_3 \rightarrow 0 \quad (69)$$

Using (62), Eqs (67) and (68) lead to $2\omega - 1 \geq \omega$ which contradicts the assumption that $1/2 < \omega < 1$. So, suppose case (69) is true. Then, the third equation of the system (41) with the use of (57), (60) and (64) gives

$$O(x_3^{2\omega-2}) + O(x_3^{\mu_{33}-1}) + O(x_3^{2\omega}) + O(x_3^{\mu_{23}}) = 0 \quad \text{as} \quad x_3 \rightarrow 0 \quad (70)$$

It is necessary to examine the cases

$$\mu_{33} = 2\omega - 1, \quad (71)$$

$$\mu_{23} = 2(\omega - 1) \quad (72)$$

It follows from (61) and (72) that in this case $\omega \geq 3/2$ which contradicts (65). Equation (71) contradicts the assumption $1/2 < \omega < 1$ as follows from (62). So, the equilibrium equations (41) cannot be satisfied if $1/2 < \omega < 1$. Therefore, it is necessary to put

$$\omega = \frac{1}{2} \quad (73)$$

Then it follows from (47) and (16) that

$$\xi_{eq} = \frac{D}{\sqrt{x_3}} + o\left(\frac{1}{\sqrt{x_3}}\right) \quad \text{as } x_3 \rightarrow 0 \quad (74)$$

It is obvious that this representation of the equivalent strain rate coincides with (1.1) and D is the strain rate intensity factor.

2.5 Maximum Friction Law and Other Models of Pressure-independent Plasticity

The maximum friction law in the case of rigid plastic hardening and viscoplastic materials has the form

$$\tau_f = \frac{\sigma_0}{\sqrt{3}} \Omega(\varepsilon_{eq}) \quad \text{and} \quad \tau_f = \frac{\sigma_0}{\sqrt{3}} \Lambda(\xi_{eq}) \quad (75)$$

respectively. It follows from (11), (25), (27), (32) and (33) that the law (75) reduces to (37) at $\varepsilon_{eq} = 0$ for rigid plastic hardening materials and at $\xi_{eq} = 0$ for viscoplastic materials. Consider a boundary value problem involving the maximum friction law for rigid perfectly/plastic solids, rigid plastic hardening solids, and viscoplastic solids. The statements of the boundary value problem, including the constitutive equations and the friction law, for the rigid plastic hardening and viscoplastic solids reduce to that for the rigid perfectly/plastic solids as $d\Omega/d\varepsilon_{eq} \rightarrow 0$ for any ε_{eq} or $d\Lambda/d\xi_{eq} \rightarrow 0$ for any ξ_{eq} . Nevertheless, the solution behavior essentially depends on the model chosen, independently of how close the derivatives are to zero. In particular, the models with no saturation stress lead to the regime of sticking at maximum friction surfaces and the velocity field is not singular in the vicinity of such surfaces (Alexandrov et al., 2000; Alexandrov and Alexandrova, 2000a,b). It may be advantageous to adopt rigid plastic hardening and viscoplastic models that possess a smooth transition of qualitative behavior between solutions for these models and the rigid perfectly/plastic model. It is only possible if a saturation stress is involved in the constitutive equations. No general theory of this kind has been developed yet. Nevertheless,

a number of analytic solutions show that the regime of sliding can occur in the case of rigid plastic hardening and viscoplastic models with a saturation stress and that the velocity field in such solutions can be singular (Alexandrov and Richmond, 2001b; Alexandrov and Mishuris, 2007; Alexandrov et al., 2008a).

3 Plasticity Theory for Porous and Powder Metals Based on the Associated Flow Rule

3.1 Preliminaries

A representative element of porous or powder material consists of two major phases, namely the solid phase and pores (see Fig. 8^a). In order

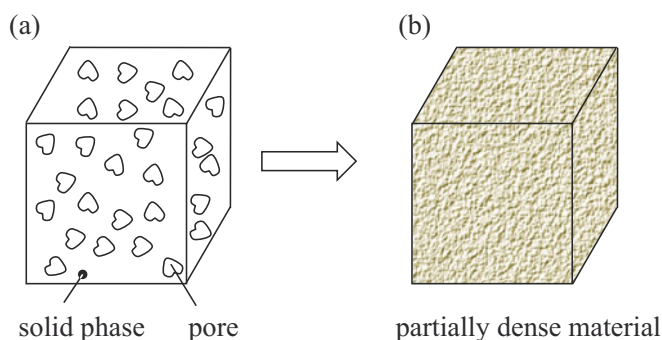


Figure 8. Representative elements of porous or powder materials

to apply continuum mechanics models to the aggregate, the representative element shown in Fig. 8^a is replaced with another representative element (see Fig. 8^b) with a homogeneous distribution of partially dense material whose relative density ρ is defined by

$$\rho = \frac{V_s}{V} \quad (76)$$

where V is the volume of the representative element and V_s is the volume of its part occupied by the solid phase. It is indeed assumed here that the representative volume is of infinitesimal size such that the distribution of the density is uniform. In passing, it is worth noting that ρ is a non-dimensional quantity varying in the theoretical range $0 < \rho \leq 1$ and $\rho = 1$ for fully dense (pore free) materials.

The mass of any volume is determined by

$$M = \rho_s \int_V \rho dV \quad (77)$$

where ρ_s is the density of the solid phase. It has been assumed that ρ_s is constant. Since the mass of any material volume is constant, $dM/dt = 0$. Substituting (77) into this equation gives

$$\int_V \frac{\partial \rho}{\partial t} dV + \int_S \rho u_n dS = 0 \quad (78)$$

where S is the surface of V and u_n is the velocity component normal to S , measured outwards (see Fig. 9). Using the divergence theorem Eq. (78)

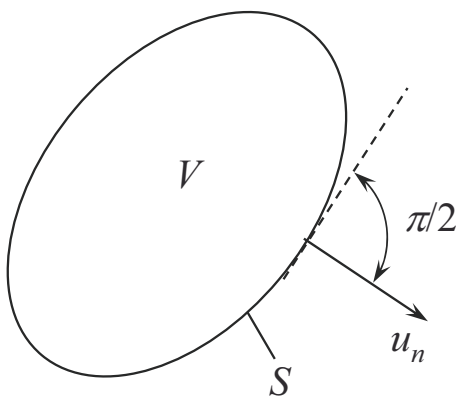


Figure 9. Material volume and the velocity component normal to its surface

can be transformed to

$$\int_V \frac{\partial \rho}{\partial t} dV + \int_V \operatorname{div}(\rho \mathbf{u}) dV = \int_V \left[\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) \right] dV = 0 \quad (79)$$

Since V is an arbitrary volume, it follows from (79) that

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0 \quad (80)$$

This is the continuity equation. It can be rewritten in terms of the derivative of ρ along the actual strain path, $d\rho/dt$. It is known from the vector analysis that $\text{div}(\rho\mathbf{u}) = \rho \text{div}(\mathbf{u}) + \mathbf{u} \cdot \text{grad}\rho$. Substituting this equation into (80) gives

$$\frac{\partial\rho}{\partial t} + \rho \text{div}\mathbf{u} + \mathbf{u} \cdot \text{grad}\rho = 0 \quad (81)$$

By definition,

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \mathbf{u} \cdot \text{grad}\rho \quad (82)$$

Combining (81) and (82) leads to

$$\frac{d\rho}{dt} + \rho \text{div}\mathbf{u} = 0 \quad (83)$$

Conventional theories of plasticity for pore free materials (Section 2) assume that plastic deformation has no effect of the density of material. Therefore, $\rho = \rho_s$ is constant and Eq. (83) results in the incompressibility equation in the form of $\text{div}\mathbf{u} = 0$, which is indeed equivalent to (19).

Still another popular representation of the continuity equation, in addition to (80) and (83), is

$$\frac{d\rho}{dt} + \rho\xi_\nu = 0 \quad (84)$$

where ξ_ν is the volumetric strain rate (the first invariant of the strain rate tensor) defined by

$$\xi_\nu = \text{div}\mathbf{u} = \xi_{\alpha\alpha} + \xi_{\beta\beta} + \xi_{\gamma\gamma} \quad (85)$$

It is sometimes convenient to use the porosity, ϑ , instead of the relative density. These two quantities are related to each other by

$$\rho = 1 - \vartheta \quad (86)$$

Thus the porosity can vary in the range $0 \leq \vartheta < 1$ and $\vartheta = 0$ for fully dense materials. Equations (81) and (83) can be rewritten in terms of the porosity as

$$\frac{\partial\vartheta}{\partial t} - (1 - \vartheta) \text{div}\mathbf{u} + \mathbf{u} \cdot \text{grad}\vartheta = 0 \quad \text{and} \quad \frac{d\vartheta}{dt} - (1 - \vartheta) \text{div}\mathbf{u} = 0 \quad (87)$$

An essential feature of porous and powder metals is that plastic deformation of such materials is usually accompanied by a change in the porosity that cannot be ignored. This change strongly depends on the hydrostatic stress. Therefore, in order to take this feature into account, it is necessary to include the porosity and the hydrostatic stress in the constitutive equations considered in Section 2. For material models based on the associated

flow rule it is only possible by assuming that the yield criterion depends on both. The associated flow rule is represented by Eqs (14) (or (20)) where the function Φ (or Φ_1 and Φ_2) may now depend on three invariants of the stress tensor, say σ , σ_{eq} and I_3 .

3.2 Yield Criteria and the Associated Flow Rule for Porous and Powder Materials

Both von Mises and Tresca yield criteria introduced in Eqs (8) and (12), respectively, have been generalized on porous and powder materials (Green, 1972; Shima and Oyane, 1976; Gurson, 1977; Doraivelu et al., 1984; Druyanov, 1993). The further discussion is restricted to yield criteria representing an extension of the von Mises criterion for fully dense materials to porous and powder materials. In particular, a possible effect of the third invariant of the stress tensor on yielding behavior is ignored. For this reason a convenient geometric representation of yield criteria (yield surfaces) can be provided in the space of the first and second invariants of the stress tensor rather than on the Ω -plane, as in the case of pressure-independent materials (see Figs 1 to 4).

The simplest yield criterion for porous and powder materials has the form (Druyanov, 1993)

$$\left. \begin{aligned} \sigma_{eq} &\leq \sigma_a = \sigma_0 \chi(\rho) \\ |\sigma| &\leq p_a = \sigma_0 p(\rho) \end{aligned} \right\} \quad (88)$$

where σ_0 is the yield stress in tension of the solid phase, σ_a is the yield stress in tension of the aggregate, p_a is the yield stress of the aggregate under hydrostatic pressure, $\chi(\rho)$ and $p(\rho)$ are prescribed functions of the relative density. The yield surface corresponding to the yield criterion (88) is shown in Fig. 10. The von Mises yield surface for the solid phase is depicted by a dashed line. General requirements imposed on the functions $\chi(\rho)$ and $p(\rho)$ are

$$\frac{d\chi}{d\rho} > 0, \quad \frac{dp}{d\rho} > 0, \quad \chi(1) = 1, \quad \lim_{\rho \rightarrow 1} p(\rho) = \infty \quad (89)$$

It follows from these relations that the yield criterion (88) approaches (8) as $\rho \rightarrow 1$.

Plastic yielding occurs if the equality holds in one or two equations in (88). The case $\sigma_{eq} = \sigma_a$ and $|\sigma| < p_a$ corresponds to straight line AB in Fig. 10 (the end points are excluded). Applying the associated flow rule (14) to the yield criterion $\Phi(\sigma, \sigma_{eq}, I_3) \equiv \sigma_{eq} - \sigma_a = 0$ leads to Eqs (15). Therefore, at any point of the plastic zone the relations between the

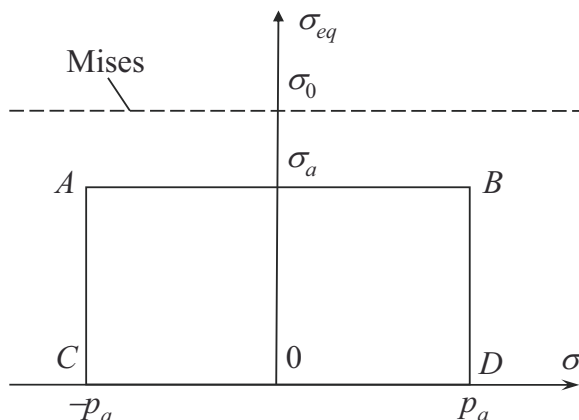


Figure 10. Geometric interpretation of yield criterion (25)

components of the stress and strain rate tensors for the yield criterion under consideration are the same as for the von Mises yield criterion. In particular, the incompressibility equation (19) is satisfied. Then, it follows from (83) that the relative density does not change at this point. This coincidence of the constitutive equations at one point does not result in the equivalent boundary value problems for von Mises material and material obeying the yield criterion (88) even if the case under consideration is valid for all the points of the plastic zone. For, the distribution of the relative density is in general not uniform because of the initial condition and/or the history of deformation. Therefore, the equivalent stress is not constant through the plastic zone, as follows from (88). Moreover, it is unknown in advance whether or not the state of stress satisfies the condition $|\sigma| < p_a$. Therefore, it is necessary to verify this condition in course of solving the boundary value problem.

The case $\sigma_{eq} < \sigma_a$ and $-\sigma = p_a$ corresponds to straight line AC in Fig. 10 (point A is excluded). Applying the associated flow rule (14) to the yield criterion

$$\Phi(\sigma, \sigma_{eq}, I_3) \equiv -\sigma - p_a = 0 \quad (90)$$

gives

$$\xi_{\alpha\alpha} = \xi_{\beta\beta} = \xi_{\gamma\gamma} = -\frac{\lambda_0}{3}, \quad \xi_{\alpha\beta} = \xi_{\beta\gamma} = \xi_{\gamma\alpha} = 0 \quad (91)$$

It is obvious from these relations that the state of strain rate in the case under consideration corresponds to hydrostatic contraction. However, the

state of stress corresponds to hydrostatic pressure at point C only and not at other points of line AC (see Fig. 10). This fact can be considered as a drawback of the yield criterion (88), though most likely this regime of flow does not occur in deformation processes of practical importance. In passing, it is worth noting that the associated flow rule must be applied to the yield criterion in the form of (90) though the equation $-\sigma = p_a$ is equivalent to $\sigma + p_a = 0$ as well. The rule is that in general $\Phi \leq 0$ and $\Phi = 0$ at yielding. No state of stress resulting in $\Phi > 0$ can exist. Substituting the state of stress $\sigma = 0$, which can certainly exist, into (90) shows that $\Phi = -p_a < 0$. On the other hand, the definition for Φ in the form $\Phi(\sigma, \sigma_{eq}, I_3) \equiv \sigma + p_a = 0$ would result in $\Phi = p_a > 0$ at $\sigma = 0$. This comment is relevant to all yield criteria.

The case $\sigma_{eq} < \sigma_a$ and $\sigma = p_a$ corresponds to straight line BD in Fig. 10 (point B is excluded). This case is even less significant for applications than the previous one and, moreover, can be treated in a similar manner. Therefore, it is not considered here in details.

Probably, the most important case for practical applications is $-\sigma = p_a$ and $\sigma_{eq} = \sigma_a$. It corresponds to point A (see Fig. 10). Because it is a singular point of the yield surface it is necessary to adopt the associated flow rule in the form of (20). In particular, it is possible to put $\Phi_1(\sigma, \sigma_{eq}, I_3) \equiv \sigma_{eq} - \sigma_a = 0$ and $\Phi_2(\sigma, \sigma_{eq}, I_3) \equiv -\sigma - p_a = 0$. Then,

$$\begin{aligned} \xi_{\alpha\alpha} &= \frac{3}{2} \frac{\lambda_1}{\sigma_{eq}} \tau_{\alpha\alpha} - \frac{\lambda_2}{3}, \quad \xi_{\beta\beta} = \frac{3}{2} \frac{\lambda_1}{\sigma_{eq}} \tau_{\beta\beta} - \frac{\lambda_2}{3}, \quad \xi_{\gamma\gamma} = \frac{3}{2} \frac{\lambda_1}{\sigma_{eq}} \tau_{\gamma\gamma} - \frac{\lambda_2}{3}, \\ \xi_{\alpha\beta} &= \frac{3}{2} \frac{\lambda_1}{\sigma_{eq}} \tau_{\alpha\beta}, \quad \xi_{\beta\gamma} = \frac{3}{2} \frac{\lambda_1}{\sigma_{eq}} \tau_{\beta\gamma}, \quad \xi_{\gamma\alpha} = \frac{3}{2} \frac{\lambda_1}{\sigma_{eq}} \tau_{\gamma\alpha} \end{aligned} \quad (92)$$

Using the identity $\tau_{\alpha\alpha} + \tau_{\beta\beta} + \tau_{\gamma\gamma} = 0$ and the definition for the volumetric strain rate given in Eq. (85) it is possible to exclude λ_1 in the first three equations of (92) to arrive at

$$\lambda_2 = -\xi_\nu \quad (93)$$

Note that the volumetric strain rate is involved in the continuity equation (84). Introduce the second invariant of the strain rate tensor in the form

$$\begin{aligned} \eta_{eq} &= \left(\frac{2}{3}\right)^{1/2} \left[\left(\xi_{\alpha\alpha} - \frac{\xi_\nu}{3}\right)^2 + \left(\xi_{\beta\beta} - \frac{\xi_\nu}{3}\right)^2 + \right. \\ &\quad \left. + \left(\xi_{\gamma\gamma} - \frac{\xi_\nu}{3}\right)^2 + 2\xi_{\alpha\beta}^2 + 2\xi_{\beta\gamma}^2 + 2\xi_{\gamma\alpha}^2 \right]^{1/2} \end{aligned} \quad (94)$$

Note that η_{eq} for incompressible materials coincides with ξ_{eq} introduced in (16). Substituting (93) into (92) and using the definitions for σ_{eq} and η_{eq} , Eqs (2) and (94), gives

$$\lambda_1 = \eta_{eq} \quad (95)$$

Since $\sigma_{eq} = \sigma_a$ at point A (see Fig. 10), it follows from (92), (93) and (95) that

$$\begin{aligned} \xi_{\alpha\alpha} &= \frac{3}{2} \frac{\eta_{eq}}{\sigma_a} \tau_{\alpha\alpha} + \frac{\xi_\nu}{3}, \quad \xi_{\beta\beta} = \frac{3}{2} \frac{\eta_{eq}}{\sigma_a} \tau_{\beta\beta} + \frac{\xi_\nu}{3}, \quad \xi_{\gamma\gamma} = \frac{3}{2} \frac{\eta_{eq}}{\sigma_a} \tau_{\gamma\gamma} + \frac{\xi_\nu}{3}, \\ \xi_{\alpha\beta} &= \frac{3}{2} \frac{\eta_{eq}}{\sigma_a} \tau_{\alpha\beta}, \quad \xi_{\beta\gamma} = \frac{3}{2} \frac{\eta_{eq}}{\sigma_a} \tau_{\beta\gamma}, \quad \xi_{\gamma\alpha} = \frac{3}{2} \frac{\eta_{eq}}{\sigma_a} \tau_{\gamma\alpha} \end{aligned} \quad (96)$$

These equations reduce to (18) for incompressible materials.

The regime of flow $\sigma = p_a$ and $\sigma_{eq} = \sigma_a$ (point B in Fig. 10) is very similar to the previous case and is less important for applications in powder metallurgy because the hydrostatic stress is positive. Therefore, it is not considered here in details.

Another yield criterion taking into account the hydrostatic stress is (Druyanov, 1993)

$$\frac{\sigma_{eq}}{\sigma_a} + \frac{|\sigma|}{p_a} = 1 \quad (97)$$

The corresponding yield surface is shown in Fig. 11. It has three singular

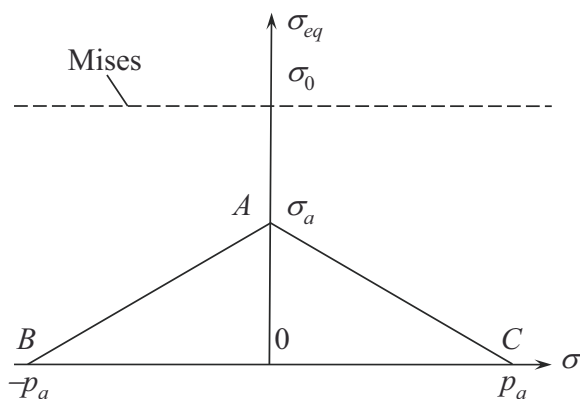


Figure 11. Geometric interpretation of yield criterion (97)

points, A , B and C . These points are not so important for practical applications and, therefore, the further analysis is restricted to the regime of

flow corresponding to AB . In this case the yield criterion (97) reduces to

$$\Phi(\sigma, \sigma_{eq}, I_3) \equiv \frac{\sigma_{eq}}{\sigma_a} - \frac{\sigma}{p_a} - 1 = 0, \quad 0 > \sigma > -p_a \quad (98)$$

Applying the associated flow rule in the form of (14) to (98) gives

$$\begin{aligned} \xi_{\alpha\alpha} &= \lambda \left(\tau_{\alpha\alpha} - \frac{2}{9} \frac{\sigma_a}{p_a} \sigma_{eq} \right), \xi_{\beta\beta} = \lambda \left(\tau_{\beta\beta} - \frac{2}{9} \frac{\sigma_a}{p_a} \sigma_{eq} \right), \\ \xi_{\gamma\gamma} &= \lambda \left(\tau_{\gamma\gamma} - \frac{2}{9} \frac{\sigma_a}{p_a} \sigma_{eq} \right), \xi_{\alpha\beta} = \lambda \tau_{\alpha\beta}, \xi_{\beta\gamma} = \lambda \tau_{\beta\gamma}, \xi_{\gamma\alpha} = \lambda \tau_{\gamma\alpha}, \end{aligned} \quad (99)$$

$$\lambda = \frac{3\lambda_0}{2\sigma_a\sigma_{eq}}$$

The volumetric strain rate can be found from these equations in the form

$$\xi_\nu = -\frac{2\lambda\sigma_a\sigma_{eq}}{3p_a} \quad (100)$$

Also, using (99), (100) and the definition (94) the value of η_{eq} is determined as

$$\eta_{eq} = \frac{2\lambda\sigma_{eq}}{3} \quad (101)$$

It follows from (100) and (101) that the ratio ξ_ν/η_{eq} is independent of the state of stress and is given by $\xi_\nu/\eta_{eq} = -\sigma_a/p_a$. It is indeed a consequence of a linear relation between σ_{eq} and σ in (98). Equations (99) reduce to (18) for incompressible materials.

The regime corresponding to AC (see Fig. 11) can be treated in a similar manner. Therefore, it is not considered here in details.

A popular smooth yield criterion is given by Green (1972)

$$\Phi(\sigma, \sigma_{eq}, I_3) \equiv \left(\frac{\sigma_{eq}}{\sigma_a} \right)^2 + \left(\frac{\sigma}{p_a} \right)^2 - 1 = 0 \quad (102)$$

The corresponding yield surface is shown in Fig. 12. It follows from (102) that it is an ellipse in the space used in this figure. Substituting (102) into (14) gives

$$\begin{aligned} \xi_{\alpha\alpha} &= \lambda \left(\tau_{\alpha\alpha} + \frac{2}{9} \frac{\sigma_a^2}{p_a^2} \sigma \right), \xi_{\beta\beta} = \lambda \left(\tau_{\beta\beta} + \frac{2}{9} \frac{\sigma_a^2}{p_a^2} \sigma \right), \\ \xi_{\gamma\gamma} &= \lambda \left(\tau_{\gamma\gamma} + \frac{2}{9} \frac{\sigma_a^2}{p_a^2} \sigma \right), \xi_{\alpha\beta} = \lambda \tau_{\alpha\beta}, \xi_{\beta\gamma} = \lambda \tau_{\beta\gamma}, \xi_{\gamma\alpha} = \lambda \tau_{\gamma\alpha}, \end{aligned} \quad (103)$$

$$\lambda = \frac{3\lambda_0}{\sigma_a^2}$$

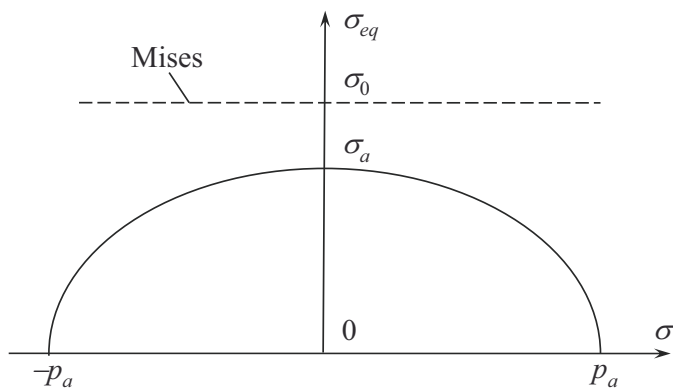


Figure 12. Geometric interpretation of yield criterion (102)

The volumetric strain rate can be found from these equations in the form

$$\xi_\nu = \frac{2\lambda\sigma_a^2}{3p_a^2}\sigma \quad (104)$$

It follows from this equation that $\xi_\nu < 0$ for $\sigma < 0$, $\xi_\nu > 0$ for $\sigma > 0$ and the incompressibility Eq. (19) is satisfied for $\sigma = 0$. Excluding σ in (103) by means of (104) and using the definition (94) it is possible to verify by inspection that (101) found from the yield criterion (98) gives η_{eq} in the case under consideration as well. Then, it follows from (101) and (104) that

$$\frac{\xi_\nu}{\eta_{eq}} = \frac{\sigma}{\sigma_{eq}} \frac{\sigma_a^2}{p_a^2} \quad (105)$$

It follows from (102) that the equivalent stress varies in the range $0 \leq \sigma_{eq} \leq \sigma_a$ and the hydrostatic stress in the range $-p_a \leq \sigma \leq p_a$. Then, it can be immediately found from Eq. (105) that $-\infty < \xi_\nu/\eta_{eq} < \infty$.

3.3 Additional Remarks on the Yield Criteria

The yield criteria considered are symmetric relative to the hydrostatic stress. Accordingly, the yield surfaces are symmetric relative to the axis $\sigma = 0$ (see Figs 10 to 12). A more general description of material response is obtained by shifting the yield surfaces along the σ -axis (see Fig. 13). These yield surfaces have an axis of symmetry $\sigma = \sigma_m$. A further generalization is to use yield surfaces that have no axis of symmetry parallel to the σ_{eq} -axis (see, for example Bier and Hartmann (2006)).

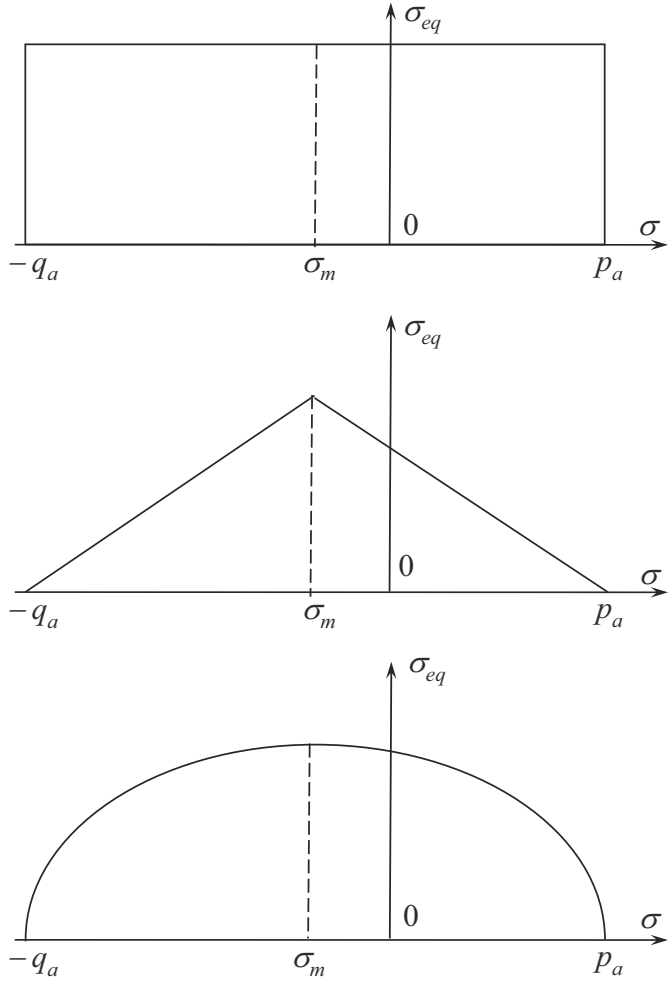


Figure 13. Yield surfaces shifted along the σ -axis

3.4 Simple Analytic Example

Consider instantaneous flow of an infinite hollow cylinder of porous material on a rigid fibre subject to uniform pressure p_b on its external radius (see Fig. 14). The external radius of the cylinder is b and its internal radius as well as the radius of the fibre is a . The distribution of the porosity is

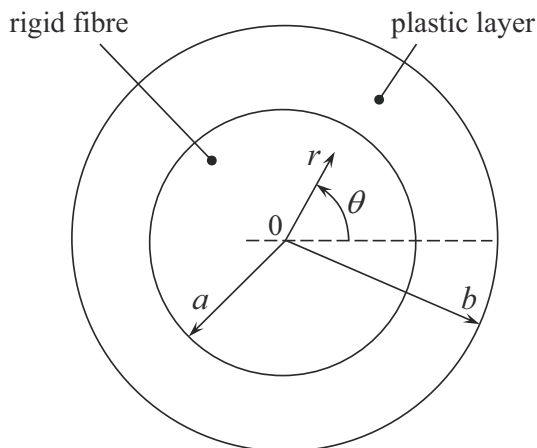


Figure 14. Contraction of a porous layer on a rigid fiber—notation

uniform. Therefore, σ_a and p_a are independent of space coordinates. It is convenient to introduce a cylindrical polar coordinate system $r\theta z$ ($r \equiv \alpha$, $\theta \equiv \beta$, $z \equiv \gamma$) with its z -axis coinciding with the axis of symmetry of the cylinder. Symmetry in the circumferential direction and plane strain deformation dictate that

$$\xi_{zz} = \xi_{r\theta} = \xi_{zr} = \xi_{\theta z} = 0, \quad \tau_{r\theta} = \tau_{zr} = \tau_{\theta z} = 0 \quad (106)$$

The boundary conditions are

$$u_r = 0 \quad (107)$$

at $r = a$ (on the surface of the rigid fibre) and

$$u_r = -u_0 \quad (108)$$

at $r = b$. The value of $u_0 > 0$ is not essential for rate independent models. The pressure at $r = b$ required to deform the plastic layer should be found from the solution. The only non-trivial equilibrium equation is

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \quad (109)$$

Assume that the yield criterion (88) is valid. It is seen from (91) that the regime of flow corresponding to (90) contradicts (106). It is also obvious that

no isochoric flow is possible for the problem under consideration. Therefore, it is necessary to assume that the regime of flow corresponds to point *A* of the yield surface (see Fig. 10). Then, it follows from (92) and (106) that

$$\xi_{zz} = \frac{3}{2} \frac{\lambda_1}{\sigma_{eq}} \tau_{zz} - \frac{\lambda_2}{3} = 0 \quad (110)$$

Since $\lambda_1 > 0$ and $\lambda_2 > 0$, it follows from this equation that

$$\tau_{zz} > 0 \quad (111)$$

Using (110) Eqs (92) for ξ_{rr} and $\xi_{\theta\theta}$ transform to

$$\xi_{rr} = \lambda (\tau_{rr} - \tau_{zz}), \quad \xi_{\theta\theta} = \lambda (\tau_{\theta\theta} - \tau_{zz}), \quad \lambda = \frac{3}{2} \frac{\lambda_1}{\sigma_{eq}} \quad (112)$$

In the case under consideration $\xi_{rr} = \partial u_r / \partial r$ and $\xi_{\theta\theta} = u_r / r$. Therefore, Eqs (112) result in

$$\frac{\partial u_r}{\partial r} = \frac{u_r}{r} \frac{(\tau_{rr} - \tau_{zz})}{(\tau_{\theta\theta} - \tau_{zz})} \quad (113)$$

The unique solution to this equation satisfying the boundary condition (107) is $u_r = 0$, unless $\tau_{\theta\theta} = \tau_{zz}$ at $r = a$. The latter condition is however true, as follows from (112) because $\xi_{\theta\theta} = u_r / r = 0$ at $r = a$. Therefore, it is necessary to find the distribution of the stress components for integrating (113).

The equivalent stress at point *A* (see Fig. 10) satisfies the relation $\sigma_{eq} = \sigma_a$. Using the identity

$$\tau_{rr} + \tau_{\theta\theta} + \tau_{zz} = 0 \quad (114)$$

and the definition (2) this relation can be transformed to

$$3 (\tau_{rr}^2 + \tau_{zz}^2 + \tau_{rr} \tau_{zz}) = \sigma_a^2 \quad (115)$$

This equation is satisfied by the following substitution

$$\tau_{zz} = \frac{2}{3} \sigma_a \sin \varphi, \quad \tau_{rr} = \frac{2}{3} \sigma_a \cos \left(\varphi + \frac{\pi}{6} \right) \quad (116)$$

Then, it follows from (114) that

$$\tau_{\theta\theta} = -\frac{2}{3} \sigma_a \left[\sin \varphi + \cos \left(\varphi + \frac{\pi}{6} \right) \right] \quad (117)$$

Equations (111) and (116) show that

$$0 < \varphi < \pi \quad (118)$$

Substituting (116) and (117) into (109) and taking into account that $\sigma = -p_a$ is independent of r gives

$$\sin\left(\varphi + \frac{\pi}{6}\right) \frac{\partial\varphi}{\partial r} = \frac{\sqrt{3} \cos \varphi}{r} \quad (119)$$

The solution to this equation is

$$\frac{r}{a} = \exp\left(\frac{\varphi - \varphi_a}{2\sqrt{3}}\right) \sqrt{\frac{\cos \varphi_a}{\cos \varphi}} \quad (120)$$

where φ_a is the value of φ at $r = a$. It has been already mentioned before that $\tau_{\theta\theta} - \tau_{zz} = 0$ at $r = a$. Therefore, it follows from (116) and (117) that

$$-2 \sin \varphi_a = \cos\left(\varphi_a + \frac{\pi}{6}\right) \quad (121)$$

The solution to this equation in the range (118) is

$$\varphi_a = \frac{5\pi}{6} \quad (122)$$

Expanding the right hand side of (120) in a series in the vicinity of $\varphi = \varphi_a$ gives

$$\frac{r}{a} = 1 + \frac{1}{3}(\varphi - \varphi_a)^2 + o\left[(\varphi - \varphi_a)^2\right], \quad \varphi \rightarrow \varphi_a \quad (123)$$

Substituting (116) and (117) into (113) leads to

$$\frac{\partial u_r}{u_r \partial r} = -\frac{1}{r} \frac{(\cos \varphi - \sqrt{3} \sin \varphi)}{(\cos \varphi + \sqrt{3} \sin \varphi)} \quad (124)$$

Expanding the right hand side of this equation in a series in the vicinity of $\varphi = \varphi_a$ (or $r = a$) gives with the use of (123)

$$\frac{\partial u_r}{u_r \partial r} = -\frac{\sqrt{3}}{2a(\varphi - \varphi_a)} + o\left(\frac{1}{\varphi - \varphi_a}\right) = \mp \frac{1}{2\sqrt{a}\sqrt{r-a}} + o\left(\frac{1}{\sqrt{r-a}}\right) \quad (125)$$

as $\varphi \rightarrow \varphi_a$ (or $r \rightarrow a$). Integrating (125) leads to

$$\ln \frac{u_r}{U_0} = \mp \frac{\sqrt{r-a}}{\sqrt{a}} + o(\sqrt{r-a}), \quad r \rightarrow a \quad (126)$$

where U_0 is a constant of integration. It is seen from this representation of the solution in the vicinity of the surface $r = a$ that the boundary condition (107) is not compatible with the assumptions made, unless $u_r = 0$

everywhere. The latter however has no physical sense for porous materials. In order to illustrate that this feature of the boundary value problem is not caused by the simplified yield criterion adopted, the derivation based on the yield criterion (98) is provided below.

It follows from (106) and (99) that

$$\frac{2\sigma_a^2}{9p_a^2}\sigma = -\tau_{zz}, \quad \xi_{rr} = \lambda(\tau_{rr} - \tau_{zz}), \quad \xi_{\theta\theta} = \lambda(\tau_{\theta\theta} - \tau_{zz}) \quad (127)$$

Taking into account (114) and expressing the strain rate components in terms of the radial velocity the last two equations of (127) give

$$\frac{\partial u_r}{\partial r} = \frac{u_r}{r} \frac{(2\tau_{rr} + \tau_{\theta\theta})}{(2\tau_{\theta\theta} + \tau_{rr})} \quad (128)$$

The shear components of the stress tensor involved in the yield criterion (102) vanish. Therefore, excluding σ in this criterion by means of the first equation of (127) and, then, τ_{zz} by means of (114) leads to

$$\tau_{rr}^2 + \tau_{\theta\theta}^2 + 2h\tau_{rr}\tau_{\theta\theta} = g^2 \quad (129)$$

where

$$h = \frac{2\sigma_a^2 + 27p_a^2}{4\sigma_a^2 + 27p_a^2} \leq 1, \quad g = \sigma_a \sqrt{\frac{2}{3}} \left(2 + \frac{27p_a^2}{2\sigma_a^2} \right)^{-1/2} \quad (130)$$

Equation (129) is satisfied by the following substitution

$$\frac{\tau_{rr}}{g} = \frac{\cos \varphi}{\sqrt{1-h^2}}, \quad \frac{\tau_{\theta\theta}}{g} = \sin \varphi - \frac{h \cos \varphi}{\sqrt{1-h^2}} \quad (131)$$

Then, using (114) and (127) the hydrostatic stress is given by

$$\frac{\sigma}{g} = \frac{(2h-1)}{3(1-h)} \left(\sqrt{\frac{1-h}{1+h}} \cos \varphi + \sin \varphi \right) \quad (132)$$

It has been taken into account here that the coefficient of σ in (127) is expressible in terms of h by means of (130) as

$$\frac{9p_a^2}{2\sigma_a^2} = \frac{2h-1}{3(h-1)} \quad (133)$$

Substituting (131) and (132) into the equilibrium equation (109) and using (133) gives

$$r \left[\frac{(2h-1)}{3(1-h)} \left(\cos \varphi - \sqrt{\frac{1-h}{1+h}} \sin \varphi \right) - \frac{\sin \varphi}{\sqrt{1-h^2}} \right] \frac{\partial \varphi}{\partial r} + \sqrt{\frac{1+h}{1-h}} \cos \varphi - \sin \varphi = 0 \quad (134)$$

Since $\partial u_r / \partial r = (\partial u_r / \partial \varphi) (\partial \varphi / \partial r)$, Eq. (128) can be combined with (131) and (134) to arrive at

$$\frac{\partial u_r}{u_r \partial \varphi} = \frac{[2-h+\sqrt{1-h^2} \tan \varphi] \left[\frac{(2h-1)}{3(1-h)} \left(1 - \sqrt{\frac{1-h}{1+h}} \tan \varphi \right) - \frac{\tan \varphi}{\sqrt{1-h^2}} \right]}{[1-2h+2\sqrt{1-h^2} \tan \varphi] \left(\tan \varphi - \sqrt{\frac{1+h}{1-h}} \right)} \quad (135)$$

It follows from the structure of this equation that a necessary condition to satisfy (107) is that the denominator in the right hand side of (135) vanishes at $r = a$. Let φ_a be the value of φ at $r = a$. Then, the aforementioned condition is satisfied if

$$\tan \varphi_a = \frac{2h-1}{2\sqrt{1-h^2}} \quad (136)$$

or

$$\tan \varphi_a = \sqrt{\frac{1+h}{1-h}} \quad (137)$$

Suppose that case (137) is true. Expanding the right hand side of (135) in a series in the vicinity of $\varphi = \varphi_a$ gives

$$\frac{1}{u_r} \frac{\partial u_r}{\partial \varphi} = -\frac{1}{(1-h)(\tan \varphi - \tan \varphi_a)} = -\frac{1}{2(\varphi - \varphi_a)} + o\left(\frac{1}{\varphi - \varphi_a}\right), \quad \varphi \rightarrow \varphi_a \quad (138)$$

Integrating this equation leads to

$$u_r = \frac{U_0}{\sqrt{\varphi - \varphi_a}} + o\left(\frac{1}{\sqrt{\varphi - \varphi_a}}\right), \quad \varphi \rightarrow \varphi_a \quad (139)$$

where U_0 is a constant of integration. It follows from this representation of the solution in the vicinity of $r = a$ that the boundary condition (107) is not satisfied by any choice of U_0 .

Suppose that case (136) is true. It is possible to verify by inspection that the right hand side of (135) reduces to 0/0 at this value of φ_a . Expanding

it in a series in the vicinity of φ_a gives

$$\frac{1}{u_r} \frac{\partial u_r}{\partial \varphi} = \frac{1}{3} \sqrt{\frac{1+h}{1-h}} + o(1), \quad \varphi \rightarrow \varphi_a \quad (140)$$

Integrating this equation leads to

$$\ln \frac{u_r}{U_0} = \frac{1}{3} \sqrt{\frac{1+h}{1-h}} (\varphi - \varphi_a) + o(\varphi - \varphi_a), \quad \varphi \rightarrow \varphi_a \quad (141)$$

It follows from this representation of the solution in the vicinity of $r = a$ that the boundary condition (107) is not satisfied by any choice of U_0 .

The analysis completed shows that the only solution to equation (124) (or (135)) satisfying the boundary condition (107) is $u_r = 0$, i.e. there is no plastic flow. It of course contradicts the conception of porous plastic material since, according to this solution, the material should support any large value of the pressure at $r = b$. A possible way to resolve this contradiction is to assume that the normal velocity is discontinuous at $r = a$ at the initial instant. To the best of authors knowledge, no exact solution of this kind is available in the literature, though an approximate solution for compaction of powder material in a closed container shows such a feature (Alexandrov and Druyanov, 1990).

4 Plasticity Theory for Porous and Powder Metals Based on Non-associated Flow Rules

4.1 Stress Equations

There is no commonly accepted model of pressure-dependent plasticity based on a non-associated flow rule, though most of such models reduce to classical plasticity (see Section 2) at a specific set of parameters. Nevertheless, the Coulomb-Mohr yield criterion is usually assumed to hold independently of the relation between stress and strain-rate adopted. This criterion has the following form in terms of the principal components of stress

$$\left. \begin{aligned} \sigma_1 - \sigma_2 + (\sigma_1 + \sigma_2) \sin \phi &\leq 2k \cos \phi \\ \sigma_2 - \sigma_3 + (\sigma_2 + \sigma_3) \sin \phi &\leq 2k \cos \phi \\ \sigma_3 - \sigma_1 + (\sigma_3 + \sigma_1) \sin \phi &\leq 2k \cos \phi \\ \sigma_2 - \sigma_1 + (\sigma_2 + \sigma_1) \sin \phi &\leq 2k \cos \phi \\ \sigma_3 - \sigma_2 + (\sigma_3 + \sigma_2) \sin \phi &\leq 2k \cos \phi \\ \sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3) \sin \phi &\leq 2k \cos \phi \end{aligned} \right\} \quad (142)$$

where k is the cohesion and ϕ is the angle of internal friction. It is assumed that k and ϕ are constants. The yield surface in principal stress space is

a right hexagonal pyramid equally inclined to the σ_1 -, σ_2 -, σ_3 - axes, with its vertex at the point $\sigma_1 = \sigma_2 = \sigma_3 = k \cos \phi$. The yield criterion (142) is a generalization of Tresca's yield criterion (12), and reduces to Tresca criterion when $\phi = 0$. Plastic yielding occurs if the equality holds in one or two inequalities shown in (142).

In the case of plane strain deformation it is always possible to order the principal stresses such that $\sigma_1 > \sigma_2 > \sigma_3$ where the principal stress axes which correspond to σ_1 and σ_3 lie in the plane of flow. Then, the yield criterion (142) reduces to

$$\sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3) \sin \phi = 2k \cos \phi \quad (143)$$

Choose an orthogonal curvilinear system of coordinates $\alpha\beta\gamma$ with its γ - lines being straight parallel lines orthogonal to the plane of flow. Then, $\sigma_2 \equiv \sigma_{\gamma\gamma}$. Let ψ be the angle which the axis corresponding to σ_1 makes with the tangent to α - lines, measured from the tangent anti-clockwise (see Fig. 15). It is convenient to introduce the quantities

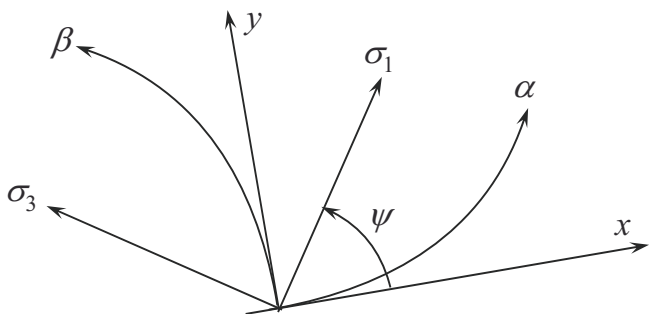


Figure 15. Coordinate systems, $\alpha\beta$ and xy , and principal stress directions in plane strain deformation

$$s = -\frac{1}{2}(\sigma_1 + \sigma_3), \quad q = \frac{1}{2}(\sigma_1 - \sigma_3) \geq 0 \quad (144)$$

Because σ_1 and σ_3 are stress invariants, s and q are stress invariants as well. In particular, using the transformation equations for stress components in

a plane gives

$$\begin{aligned}\sigma_{\alpha\alpha} &= \frac{1}{2}(\sigma_1 + \sigma_3) + \frac{1}{2}(\sigma_1 - \sigma_3)\cos 2\psi, \\ \sigma_{\beta\beta} &= \frac{1}{2}(\sigma_1 + \sigma_3) - \frac{1}{2}(\sigma_1 - \sigma_3)\cos 2\psi, \\ \sigma_{\alpha\beta} &= \frac{1}{2}(\sigma_1 - \sigma_3)\sin 2\psi\end{aligned}\quad (145)$$

Then, it immediately follows from (144) and (145) that

$$s = -\frac{1}{2}(\sigma_{\alpha\alpha} + \sigma_{\beta\beta}), \quad q = \frac{1}{2}\sqrt{(\sigma_{\alpha\alpha} - \sigma_{\beta\beta})^2 + 4\sigma_{\alpha\beta}^2} \quad (146)$$

Also, Eqs (145) with the use of (143) and (144) can be transformed to

$$\begin{aligned}\sigma_{\alpha\alpha} &= k \cot \phi + q \frac{(\cos 2\psi \sin \phi - 1)}{\sin \phi}, \\ \sigma_{\beta\beta} &= k \cot \phi - q \frac{(\cos 2\psi \sin \phi + 1)}{\sin \phi}, \\ \sigma_{\alpha\beta} &= q \sin 2\psi\end{aligned}\quad (147)$$

The representation of the stress components in this form satisfies the yield criterion (143). There are two unknowns in (147), q and ψ . The equations for these quantities should be derived from the equilibrium equations.

The equilibrium equations in Cartesian coordinates ($\alpha \equiv x$ and $\beta \equiv y$) are

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \quad (148)$$

Substituting (147) into (148) gives

$$\begin{aligned}\frac{(\cos 2\psi \sin \phi - 1)}{\sin \phi} \frac{\partial q}{\partial x} - 2q \sin 2\psi \frac{\partial \psi}{\partial x} + \sin 2\psi \frac{\partial q}{\partial y} + 2q \cos 2\psi \frac{\partial \psi}{\partial y} &= 0, \\ -\frac{(\cos 2\psi \sin \phi + 1)}{\sin \phi} \frac{\partial q}{\partial y} + 2q \sin 2\psi \frac{\partial \psi}{\partial y} + \sin 2\psi \frac{\partial q}{\partial x} + 2q \cos 2\psi \frac{\partial \psi}{\partial x} &= 0\end{aligned}\quad (149)$$

The Cartesian coordinates can be rotated such that the x -axis coincides with the principal axis of the stress tensor corresponding to σ_1 at a generic point M (see Fig. 16). Then, $\psi = 0$ at this point (see Fig. 15) and Eqs (149) become

$$\frac{(\sin \phi - 1)}{\sin \phi} \frac{\partial q}{\partial x} + 2q \frac{\partial \psi}{\partial y} = 0, \quad -\frac{(\sin \phi + 1)}{\sin \phi} \frac{\partial q}{\partial y} + 2q \frac{\partial \psi}{\partial x} = 0 \quad (150)$$

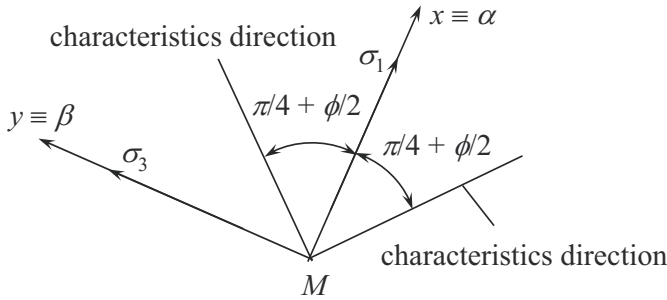


Figure 16. Local Cartesian coordinates, principal stress directions and stress characteristics directions at a generic point

The characteristic directions are determined from these equations by

$$\begin{vmatrix} 1 - \operatorname{cosec} \phi & 0 & 0 & 2q \\ 0 & -(1 + \operatorname{cosec} \phi) & 2q & 0 \\ 1 & dy/dx & 0 & 0 \\ 0 & 0 & 1 & dy/dx \end{vmatrix} = 0 \quad (151)$$

The solution to this equation is

$$\left(\frac{dy}{dx}\right)^2 = \frac{1 + \sin \phi}{1 - \sin \phi} = \tan^2 \left(\frac{\pi}{4} + \frac{\phi}{2}\right) \quad (152)$$

or

$$\frac{dy}{dx} = \pm \tan \left(\frac{\pi}{4} + \frac{\phi}{2}\right) \quad (153)$$

Thus the system of equations (150) (and (149) as well) is hyperbolic and its characteristics are inclined to the major principal stress direction at angles $\pm (\pi/4 + \phi/2)$ (see Fig. 16).

Obviously, Eqs (147) can be combined with the equilibrium equations in any orthogonal coordinate system. The specific choice of the coordinate system affects the form of Eqs (149) but not the orientation of the characteristics directions relative to the principal axes of the stress tensor.

4.2 Kinematic Theories

There are a great number of kinematic theories for pressure-dependent materials. One of the theories that accounts for plastic compressibility of materials has been considered in the previous Section. This Section concerns

with several theories that include the equation of incompressibility, Eq. (19). The discussion is restricted to plane strain deformation such that the distribution of stress is covered by (147) and (149). In particular, the characteristic directions for stress are given by (153). Equation (19) reduces to

$$\xi_{\alpha\alpha} + \xi_{\beta\beta} = 0 \quad (154)$$

In total, there are four essential unknowns, ψ , q and two components of the velocity vector. Since three equations have already been introduced, (154) in which the strain rate components should be expressed in terms of the velocity components and (149), the only difference between the theories of the class considered lies in the last equation. This last equation should connect stresses and velocities. A brief review of three models (the coaxial model, the double-shearing model, and the double-slip and rotation model) is given below.

4.3 The Coaxial Model

In the case of the coaxial model (Ishlinskii, 1954; Ostrowska-Maciejewska and Harris, 1990) the angle ψ introduced for stresses (see Fig. 15) also determines the orientation of the major principal strain rate relative to α -lines. Therefore, the system of equations (149) and (154) is supplemented by

$$(\xi_{\alpha\alpha} - \xi_{\beta\beta}) \sin 2\psi - 2\xi_{\alpha\beta} \cos 2\psi = 0 \quad (155)$$

Using the Cartesian coordinate system shown in Fig. 16, i.e. taking $\psi = 0$ at point M , Eqs (154) and (155) become

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0, \quad \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} = 0 \quad (156)$$

at point M . The characteristic directions are determined from these equations by

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & dy/dx & 0 & 0 \\ 0 & 0 & 1 & dy/dx \end{vmatrix} = 0 \quad (157)$$

The solution to this equation is $(dy/dx)^2 = 1$ or

$$\frac{dy}{dx} = \pm 1 \quad (158)$$

Thus the system of equations (156) (as well (154) and (155), indeed) is hyperbolic and its characteristics are inclined to the major principal stress

direction at angles $\pm\pi/4$. Comparing (153) and (158) shows that the characteristic directions for stress and velocity do not coincide. The characteristic directions at a generic point M are illustrated in Fig. 17.

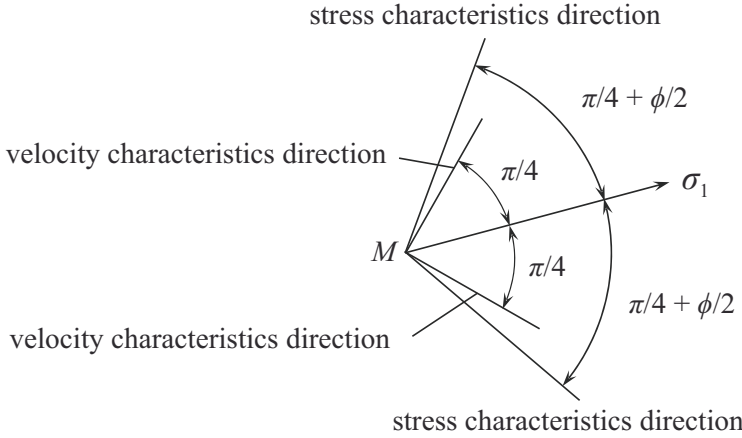


Figure 17. Major principal stress direction, stress and velocity characteristics directions (coaxial model)

4.4 The Double-shearing Model

According to (153) any plastic domain is covered by a net of characteristic curves, μ and ν . A motion consists of a shearing deformation along the μ - lines if, in a coordinate system in which the μ - line direction in the neighborhood of a generic particle M is fixed, the relative velocity of two adjacent particles near M is directed in the direction of the μ - line if the particles do not lie on the same μ - line, and is zero if they do lie on the same μ - line. The definition for the shearing deformation along the ν - lines can be introduced in a similar manner. The double-shearing model is based on the hypothesis that the deformation consists of two simultaneous superposed shearing deformations on the μ - and ν - lines respectively (Spencer, 1964, 1982). Under this assumption the equation that connects stresses and velocities is

$$2\xi_{\alpha\beta} \cos 2\psi - (\xi_{\alpha\alpha} - \xi_{\beta\beta}) \sin 2\psi + 2 \sin \varphi \left(\omega_{\alpha\beta} + \dot{\psi} \right) = 0 \quad (159)$$

where $\omega_{\alpha\beta}$ is the spin component relative to the $\alpha\beta$ coordinates and $\dot{\psi}$ is the convected derivative of ψ . These quantities are defined by

$$\begin{aligned}\omega_{\alpha\beta} &= \frac{1}{2H_\alpha H_\beta} \left[\frac{\partial(H_\alpha u_\alpha)}{\partial\beta} - \frac{\partial(H_\beta u_\beta)}{\partial\alpha} \right], \\ \dot{\psi} &= \frac{\partial\psi}{\partial t} + \frac{u_\alpha}{H_\alpha} \frac{\partial\psi}{\partial\alpha} + \frac{u_\beta}{H_\beta} \frac{\partial\psi}{\partial\beta}\end{aligned}\quad (160)$$

where $\partial\psi/\partial t$ is the time derivative at a point which is fixed relative to the $\alpha\beta$ coordinates, u_α and u_β are the velocity components relative to the $\alpha\beta$ coordinates, and H_α and H_β are the Lamé coefficients of the α - and β -lines, respectively. It is seen from (159) that the constitutive equations of the double-shearing model involve a rate of stress. Using the Cartesian coordinate system shown in Fig. 16, i.e. taking $\psi = 0$ at point M , Eqs (154) and (159), with the use of (160), become

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0,$$

$$\begin{aligned}(1 - \sin\phi) \frac{\partial u_y}{\partial x} + (1 + \sin\phi) \frac{\partial u_x}{\partial y} + 2u_x \sin\phi \frac{\partial\psi}{\partial x} + \\ + 2u_y \sin\phi \frac{\partial\psi}{\partial y} + 2\sin\phi \frac{\partial\psi}{\partial t} = 0\end{aligned}\quad (161)$$

at point M . Since the derivatives $\partial\psi/\partial x$ and $\partial\psi/\partial y$ are involved in (161), it is in general necessary to consider the system of equations consisting of (149) or (150) and (161) to determine the type of the system. In particular, assuming that the axes x and y coincide with the principal stress directions at point M Eqs (150) and (161) should be used. The characteristic directions are then determined from the equation

$$\begin{vmatrix} w_1 & 0 & 0 & 2q & 0 & 0 & 0 & 0 \\ 0 & w_2 & 2q & 0 & 0 & 0 & 0 & 0 \\ 1 & dy/dx & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & dy/dx & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & dy/dx & 0 & 0 \\ 0 & 0 & u_x & u_y & 0 & -w_2/2 & -w_1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & dy/dx \end{vmatrix} = 0 \quad (162)$$

where $w_1 = 1 - \operatorname{cosec} \phi$ and $w_2 = -(1 + \operatorname{cosec} \phi)$. It is obvious from the structure of this determinant that Eq. (162) can be represented in the form

$$\begin{vmatrix} w_1 & 0 & 0 & 2q \\ 0 & w_2 & 2q & 0 \\ 1 & dy/dx & 0 & 0 \\ 0 & 0 & 1 & dy/dx \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & dy/dx & 0 & 0 \\ 0 & -w_2/2 & -w_1/2 & 0 \\ 0 & 0 & 1 & dy/dx \end{vmatrix} = 0 \quad (163)$$

The first multiplier in (163) coincides with (151). Therefore, (153) satisfies (163) as well and these characteristic directions can be called the stress characteristics directions. The velocity characteristics directions are then obtained from

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & dy/dx & 0 & 0 \\ 0 & -w_2/2 & -w_1/2 & 0 \\ 0 & 0 & 1 & dy/dx \end{vmatrix} = 0 \quad (164)$$

The solution to this equation leads to (152). Therefore, the velocity characteristics directions coincide with the stress characteristics directions. This feature of the constitutive equations of the double-shearing model is an advantage of the model, as compared to the coaxial model.

4.5 The Double-slip and Rotation Model

This model is based on the hypothesis that the stress-rate term in the double-shearing model ($\dot{\psi}$ in Eq. (159)) is replaced by an intrinsic spin, Ω , due to grain rotation (Harris and Grekova, 2005). Therefore, Eq. (159) becomes

$$2\xi_{\alpha\beta} \cos 2\psi - (\xi_{\alpha\alpha} - \xi_{\beta\beta}) \sin 2\psi + 2 \sin \varphi (\omega_{\alpha\beta} + \Omega) = 0 \quad (165)$$

In general Ω is an unknown variable which is governed by the equation of rotational motion. To the best of author's knowledge, no specific equation for Ω has been yet proposed. Therefore, the further discussion is restricted to the case where Ω is a constant. Using the Cartesian coordinate system shown in Fig. 16, i.e. taking $\psi = 0$ at point M , Eqs (154) and (165), with the use of (160), become

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0, \quad (1 - \sin \phi) \frac{\partial u_y}{\partial x} + (1 + \sin \phi) \frac{\partial u_x}{\partial y} + 2\Omega \sin \phi = 0 \quad (166)$$

at point M . This system involves no derivative of ψ and q . Therefore, it can be treated independently of (150). In particular, it follows from (166) that the velocity characteristics directions are determined by (164). Thus, as in

the case of the double-shearing model, the stress and velocity characteristics directions coincide in the double-slip and rotation model.

5 Qualitative Behavior of Plastic Solutions for Porous and Powder Metals in the Vicinity of Frictional Interfaces

5.1 Preliminaries

The objective of this Section is to compare, by means of a problem permitting a closed-form solution, qualitative behavior of solutions based on three models of pressure-dependent plasticity, the coaxial model, the double-shearing model, and the double-slip and rotation model (Alexandrov and Harris, 2006). For completeness, the corresponding solution for classical pressure-independent plasticity is also presented. The constitutive equations of each model reduce to classical plasticity at specific values of input parameters. Nevertheless, the solution behavior essentially depends on the model chosen, independently of how close the input parameters are to these specific values. The source of such solution behavior is the maximum friction law. Such features of the solutions as non-uniqueness, non-existence and singularity are emphasized. Note that a qualitative feature of solution behavior for two models of pressure-dependent plasticity based on the associated flow rule has been considered in Section 3 but this feature is not caused by the maximum friction law.

5.2 Statement of the Problem

Consider an infinite circular hollow cylinder of internal initial radius a_0 and external initial radius b_0 subject to the system of loading consisting of normal and tangential stresses on its internal radius. Due to this system of loading, the cylinder is both expanded/contracted and twisted. It is convenient to introduce a cylindrical polar coordinate system $r\theta z$ ($\alpha \equiv r$, $\beta \equiv \theta$, $\gamma \equiv z$) with its z -axis coinciding with the axis of symmetry of the cylinder. The state of strain is plane and, therefore, σ_{zz} is a principal stress. The current internal and external radii will be denoted by a and b , respectively. The rate of expansion/contraction of the internal radius will be denoted by $\dot{a} \equiv da/dt$. For rate-independent materials, it is not an essential parameter. The external radius is fixed against rotation. Therefore,

$$u_r = \dot{a} \quad (167)$$

at $r = a$ and

$$u_\theta = 0 \quad (168)$$

at $r = b$. Here u_r and u_θ are the radial and circumferential velocities, respectively. One of the stress boundary conditions is

$$\sigma_{rr} = -P_a < 0 \quad (169)$$

at $r = a$. P_a is given, but its value is not essential for understanding the general features of the solution. The final boundary condition depends on the regime of friction at $r = a$, whether the material is sliding or sticking at the inner boundary. In the case of sliding, the value of the shear stress at $r = a$ is defined by a frictional law and the maximum friction law is adopted. Its analytical representation depends upon the constitutive equations chosen. In the case of pressure-independent plasticity this law is given by (37). However, for plane strain deformation of a material obeying the coaxial model, the double-shearing model, or the double-slip and rotation model it is most conveniently formulated in terms of the angle ψ introduced in the previous Section (see Fig. 15). Without a loss of generality, it is possible to assume that $\sigma_{r\theta} > 0$. Then, according to (147),

$$\frac{\pi}{2} > \psi > 0 \quad (170)$$

The mathematical meaning of the maximum friction law for the material models describable by hyperbolic equations is that the friction surface coincides with an envelope of characteristics such that the solution cannot be extended beyond this surface. In the case of the double-shearing model and the double-slip and rotation model, the characteristics for stress and velocity coincide (see the previous Section) and, according to (153), are inclined to the r -axis at

$$\chi = \psi \pm \left(\frac{\pi}{4} + \frac{\phi}{2} \right) \quad (171)$$

In the problem under consideration, the friction surface makes an angle $\pi/2$ with the r -axis. Putting $\chi = \pi/2$ in (171) gives

$$\psi = \frac{\pi}{2} \mp \left(\frac{\pi}{4} + \frac{\phi}{2} \right) \quad (172)$$

at the friction surface. The lower sign in (172) contradicts (170). Therefore, the frictional boundary condition at sliding for the double-shearing and the double-slip and rotation models is

$$\psi = \psi_w = \frac{\pi}{4} - \frac{\phi}{2} \quad (173)$$

at $r = a$. In the case of the coaxial model, the orientation of stress characteristics is given by (153) and the orientation of velocity characteristics

by (158). Therefore, it is possible to formulate two conditions at $r = a$ corresponding to the maximum friction law at sliding, one is given by (173) and the other by

$$\psi = \psi_w = \frac{\pi}{4} \quad (174)$$

Which of these two boundary conditions at sliding will actually occur follows from the structure of the solution. In Figs 18^a to 18^c the directions of the major principal stress, velocity characteristics and stress characteristics are shown at a generic point of the friction surface. It is possible to distinguish three qualitatively different cases: (i) $\psi > \pi/4$, this case is shown in Fig. 18^a; (ii) $\psi < \pi/4 - \phi/2$, this case is shown in Fig. 18^b; and (iii) $\pi/4 > \psi > \pi/4 - \phi/2$, this case is shown in Fig. 18^c. It is seen from Fig. 18^a that only a velocity characteristic envelope can coincide with the friction surface. Therefore, in this case, condition (174) holds. It follows from (147) that for large ψ , but within range (170), $\sigma_{\theta\theta}$ is significantly larger than σ_{rr} . Therefore, it is reasonable to expect that the case (i) corresponds to expansion. It is seen from Fig. 18^b that only a stress characteristic envelope can coincide with the friction surface. Therefore, in this case, condition (173) holds. It follows from (147) that σ_{rr} is significantly larger than $\sigma_{\theta\theta}$. Therefore, it is reasonable to expect that the case (ii) corresponds to contraction. Finally, for the case depicted in Fig. 18^c, either condition (173) or (174) may represent the maximum friction law. Case (iii) may correspond to either expansion or contraction. The correspondence between the cases (i)-(iii), the two regimes of friction (sticking or sliding), and the two types of deformation (expansion or contraction) will be precisely determined from the solution to the problem.

In the case of sticking, the circumferential velocity is prescribed,

$$u_\theta = -u_t \quad (175)$$

at $r = a$, for all models. The assumption $\sigma_{r\theta} > 0$ requires $u_t > 0$. The quantity u_t can be regarded as the circumferential velocity of points of the tool surface assuming that the tool is an expanding/contacting and rotating rod inserted into the hole of the cylinder. The rate of rod expansion/contraction is indeed equal to $\dot{a}$.

The models under consideration are history-independent. Hence, to understand general features it is sufficient to consider the solutions at one instant, i.e. for a specific geometry. In particular, b and a may be treated as independent variables, though they are in fact related to each other. This relation should be used only if the process of deformation for a specific specimen needs to be studied. An exception is the transition between the sticking regime and sliding regime where a finite time interval should be

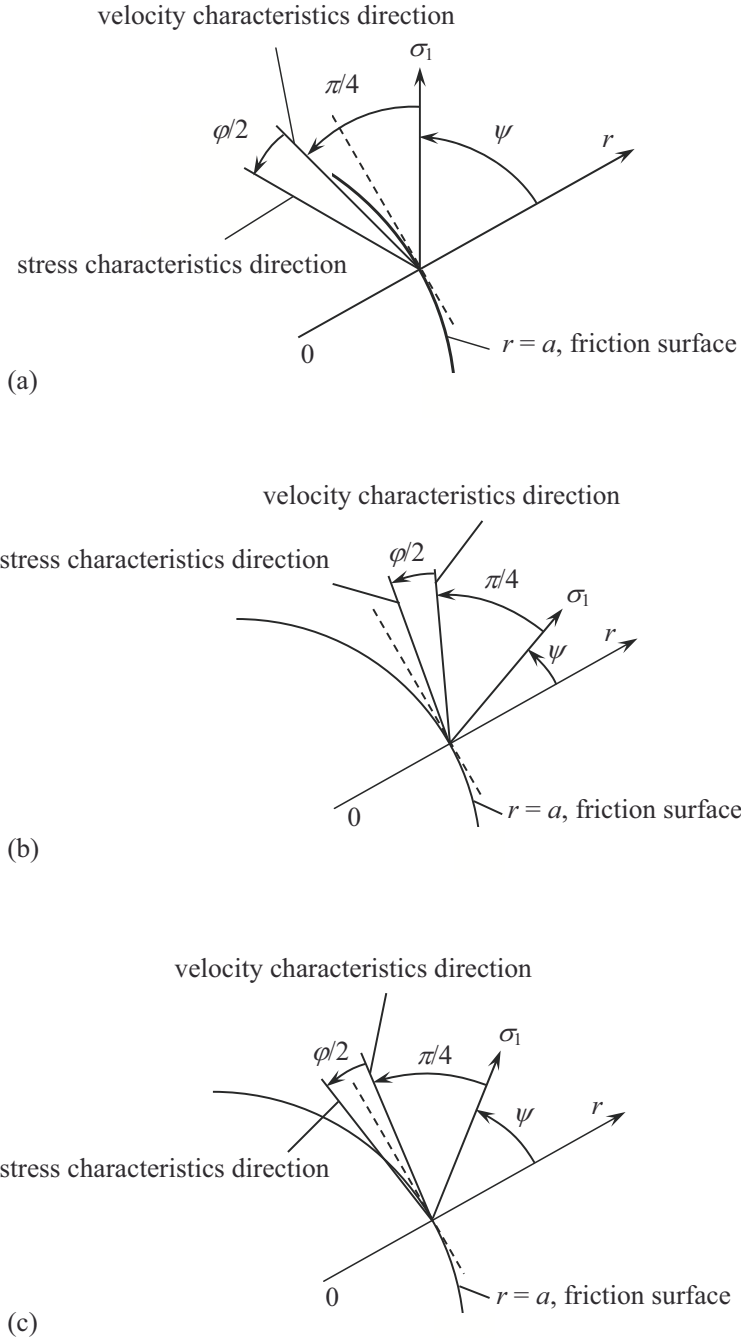


Figure 18. Illustration of the maximum friction law in cases (i), (ii) and (iii)

considered. However, even in such cases the specific value of b is not essential. In what follows, the solutions are mainly understood as instantaneous solutions for given a and b .

5.3 Solution for Stresses

Since the solution is independent of θ and z , the equilibrium equations in the cylindrical coordinates are

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \quad \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} = 0 \quad (176)$$

The general solution to the second equation of this system is

$$\frac{\sigma_{r\theta}}{k} = \frac{R^2}{r^2} \quad (177)$$

where R is an arbitrary function of a . Combining (147) and (177) leads to

$$\frac{\sigma_{rr}}{k} = \cot \phi - \frac{R^2 (1 - \cos 2\psi \sin \phi)}{r^2 \sin 2\psi \sin \phi}, \quad \frac{\sigma_{\theta\theta}}{k} = \cot \phi - \frac{R^2 (1 + \cos 2\psi \sin \phi)}{r^2 \sin 2\psi \sin \phi} \quad (178)$$

Substituting (177) and (178) into the first equation of system (176) gives the following equation for ψ :

$$\frac{\partial \psi}{\partial r} = -\frac{\sin 2\psi}{r (\cos 2\psi - \sin \phi)} \quad (179)$$

This equation can be immediately integrated to give

$$\frac{r}{a} = \frac{\cos \psi_a \tan^m \psi_a}{\cos \psi \tan^m \psi}, \quad m = \frac{1 - \sin \phi}{2} \quad (180)$$

where ψ_a is the value of ψ at $r = a$ and is a function of a . Equation (180) determines ψ as a function of r and a in implicit form. Substituting ψ in Eq. (178) gives, along with (177), the solution for stresses in terms of r and a depending on two functions, $R(a)$ and $\psi_a(a)$. These functions are to be found from the boundary conditions.

For later convenience, the value of ψ at $r = b$, ψ_b , is introduced here. It follows from (180) that

$$\frac{b}{a} = \frac{\cos \psi_a \tan^m \psi_a}{\cos \psi_b \tan^m \psi_b} \quad (181)$$

5.4 Solutions for Velocities

The incompressibility equation (154) is involved in all the models under consideration. Since the solution is independent of θ and z , this equation takes the form $\partial u_r / \partial r + u_r / r = 0$. Its solution satisfying the boundary condition (167) is

$$u_r = \frac{\dot{a}a}{r} \quad (182)$$

The coaxial model. For the problem under consideration, Eq. (155) becomes

$$\sin 2\psi \left(\frac{\partial u_r}{\partial r} - \frac{u_r}{r} \right) - \cos 2\psi \left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) = 0 \quad (183)$$

Substituting (182) into this equation and, then, using (179) and (180) Eq. (183) can be transformed to

$$\frac{\partial u_\theta}{\partial \psi} + u_\theta \frac{(\cos 2\psi - \sin \phi)}{\sin 2\psi} = \frac{2\dot{a}(\cos 2\psi - \sin \phi) \cos \psi \tan^m \psi}{\cos \psi_a \tan^m \psi_a \cos 2\psi} \quad (184)$$

The solution to this equation satisfying the boundary condition (168) is

$$\frac{u_\theta}{\dot{a}} = \frac{2}{\cos \psi_a \tan^m \psi_a \cos \psi \tan^m \psi} \int_{\psi_b}^{\psi} \frac{(\cos 2\chi - \sin \phi) \tan^{2m} \chi \cos^2 \chi}{\cos 2\chi} d\chi \quad (185)$$

The double-shearing model. For the problem under consideration, Eq. (159), with the use of (160), becomes

$$\begin{aligned} \sin 2\psi \left(\frac{\partial u_r}{\partial r} - \frac{u_r}{r} \right) - \cos 2\psi \left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) + \\ + \sin \phi \left(\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} - 2 \frac{\partial \psi}{\partial t} - 2u_r \frac{\partial \psi}{\partial r} \right) = 0 \end{aligned} \quad (186)$$

Using (179), (180) and (182) this equation can be transformed to

$$\begin{aligned} \sin 2\psi \frac{\partial u_\theta}{\partial \psi} + (\cos 2\psi + \sin \phi) u_\theta - \frac{2a\dot{a} \sin 2\psi}{r} - \\ - 2\dot{a}r \sin \phi \frac{\partial \psi}{\partial a} + \frac{2a\dot{a} \sin \phi \sin 2\psi}{r(\cos 2\psi - \sin \phi)} = 0 \end{aligned} \quad (187)$$

Using (180) it is possible to find the derivative

$$\frac{\partial \psi}{\partial a} = \frac{\sin 2\psi}{(\cos 2\psi - \sin \phi)} \left[\frac{1}{a} + \frac{(\cos 2\psi_a - \sin \phi)}{\sin 2\psi_a} \frac{d\psi_a}{da} \right] \quad (188)$$

Substituting (180) and (188) into (187) gives

$$\begin{aligned} & \frac{\partial u_\theta}{\partial \psi} + \frac{(\cos 2\psi + \sin \phi)}{\sin 2\psi} u_\theta = \\ &= \frac{2\dot{a}}{\cos \psi_a \tan^m \psi_a (\cos 2\psi - \sin \phi)} \frac{\sin \phi \left[\sin 2\psi_a + a (\cos 2\psi_a - \sin \phi) \frac{d\psi_a}{da} \right]}{2 \tan^{\sin \phi} \psi_a \cos \psi \tan^m \psi} + \\ &+ \frac{2\dot{a}}{\cos \psi_a \tan^m \psi_a (\cos 2\psi - \sin \phi)} \cos \psi \tan^m \psi (\cos 2\psi - 2 \sin \phi) \quad (189) \end{aligned}$$

The solution to this equation satisfying the boundary condition (168) is

$$\begin{aligned} \frac{u_\theta}{\dot{a}} &= \frac{\tan^m \psi}{\cos \psi_a \tan^{m+\sin \phi} \psi_a \sin \psi} \times \\ &\times \left\{ \int_{\psi_b}^{\psi} \frac{\sin \phi \left[\sin 2\psi_a + a (\cos 2\psi_a - \sin \phi) \frac{d\psi_a}{da} \right] \tan^{\sin \phi} \chi}{(\cos 2\chi - \sin \phi)} d\chi + \right. \\ &\quad \left. + \int_{\psi_b}^{\psi} \frac{\tan^{\sin \phi} \psi_a \sin 2\chi (\cos 2\chi - 2 \sin \phi)}{(\cos 2\chi - \sin \phi)} d\chi \right\} \quad (190) \end{aligned}$$

The double-slip and rotation model. For the problem under consideration, Eq. (165), with the use of (160), becomes

$$\sin 2\psi \left(\frac{\partial u_r}{\partial r} - \frac{u_r}{r} \right) - \cos 2\psi \left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) + \sin \phi \left(\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} - 2\Omega \right) = 0 \quad (191)$$

Using (179), (180) and (182) this equation can be transformed to

$$\frac{\partial u_\theta}{\partial \psi} + \frac{(\cos 2\psi + \sin \phi)}{\sin 2\psi} u_\theta = 2\dot{a} \left(\frac{\Omega a \cos \psi_a \tan^m \psi_a \sin \phi}{\dot{a} \sin 2\psi \cos \psi \tan^m \psi} + \frac{\cos \psi \tan^m \psi}{\cos \psi_a \tan^m \psi_a} \right) \quad (192)$$

The solution to this equation satisfying the boundary condition (168) is

$$\begin{aligned} \frac{u_\theta}{\dot{a}} &= \frac{\tan^m \psi}{\sin \psi} \times \\ &\times \left[\frac{(\cos 2\psi_b - \cos 2\psi)}{2 \cos \psi_a \tan^m \psi_a} + \frac{\Omega a}{\dot{a}} \cos \psi_a \tan^m \psi_a \sin \phi \int_{\psi_b}^{\psi} \frac{d\chi}{\cos^2 \chi \tan^{2m} \chi} \right] \quad (193) \end{aligned}$$

5.5 Frictional Boundary Condition

The solutions obtained contain two arbitrary functions of a , namely R and ψ_a . There are two boundary conditions to determine these functions, Eq. (169) and the friction boundary condition. Once $\psi_a(a)$ has been found, $R(a)$ can be determined from Eqs (169) and (178) with no difficulty. Therefore, the key point is the friction boundary condition from which $\psi_a(a)$ is to be found. In the case of sliding, $\psi_a = \psi_w$ and the latter quantity is known due to equations (173) or (174). However, the sliding regime occurs if and only if the shear stress at the friction surface attains a certain level. It will be shown that the solutions based on the models chosen are qualitatively different because of this frictional condition.

Let u_0 denote u_t under the sticking condition and u_{sl} under the sliding condition. In the case of sticking, u_0 is prescribed and ψ_a may be found from the solution. In the case of sliding, $\psi_a = \psi_w$ is prescribed according to the frictional law, Eqs (173) or (174), and $u_0 \equiv u_{sl}$ should be found from the solution.

The coaxial model. Equations (181) and (185) show that the solution depends on the ratio b/a rather than on b and a separately. Solution (185) gives

$$\frac{u_0}{\dot{a}} = \frac{2}{\cos^2 \psi_a \tan^{2m} \psi_a} \int_{\psi_a}^{\psi_b} \frac{(\cos 2\chi - \sin \phi) \tan^{2m} \chi \cos^2 \chi}{\cos 2\chi} d\chi \quad (194)$$

It should be noted that the integral in (194) is divergent if the point $\psi = \pi/4$ lies between the points $\psi = \psi_a$ and $\psi = \psi_b$. Therefore, the points $\psi = \psi_a$ and $\psi = \psi_b$ lie on one side of the point $\psi = \pi/4$ and the sub-intervals $0 \leq \psi \leq \pi/4$ and $\pi/4 \leq \psi \leq \pi/2$ should be considered separately. The second sub-interval corresponds to case (i), the first to cases (ii) and (iii) introduced before (see sub-section 5.2). The sub-interval $0 \leq \psi \leq \pi/4$ may be further sub-divided as follows. If $\psi_a < \pi/4$ and $\psi_b < \pi/4$, typical behavior of the integrand in (194) and the function r/a defined by Eq. (180) is illustrated in Fig. 19^a. It is seen from this figure that the function r/a attains its minimum value at $\psi = \pi/4 - \phi/2$. Therefore, the interval $0 \leq \psi \leq \pi/4$ may be divided into sub-intervals, $0 \leq \psi \leq \pi/4 - \phi/2$ and $\pi/4 - \phi/2 \leq \psi \leq \pi/4$, and both $\psi = \psi_a$ and $\psi = \psi_b$ must lie within one and only one of these sub-intervals. For, if $\psi = \pi/4 - \phi/2$ were between $\psi = \psi_a$ and $\psi = \psi_b$, the function r/a would attain its minimum value between the points $\psi = \psi_a$ and $\psi = \psi_b$ whereas, by definition, its minimum value is at $\psi = \psi_a$ where $r/a = 1$. This further division into sub-intervals separates

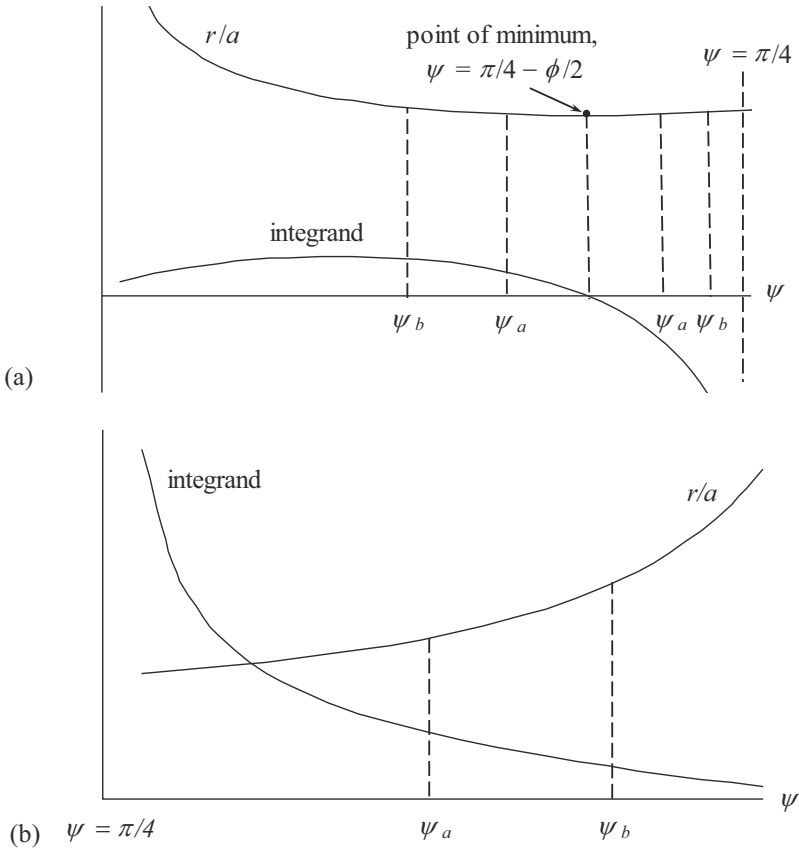


Figure 19. (a) Coaxial model: r/a and integrand of (194) as functions of ψ , $0 \leq \psi \leq \pi/4$, (b) Coaxial model: r/a and integrand of (194) as functions of $\pi/4 \leq \psi \leq \pi/2$

case (ii) from case (iii).

Now consider each of the sub-intervals separately. Since $b > a$, the order of the points $\psi = \psi_a$ and $\psi = \psi_b$ within each sub-interval must be such as shown in Fig. 19^a. Combining the sign of the integrand, the order of the points $\psi = \psi_a$ and $\psi = \psi_b$, and the structure of Eq. (194), it may be seen that the right-hand side of (194) is negative in each sub-interval. Since $u_0 > 0$, a solution exists if and only if $\dot{a} < 0$. In other words, the interval $0 \leq \psi \leq \pi/4$ corresponds to the contraction of the cylinder. This confirms

the preliminary conclusion concerning the case (ii) and also clarifies the situation with case (iii). It is seen from Figs 18^b, 18^c and 19^a that sliding satisfying condition (173) may occur in both cases (ii) and (iii). However, in case (iii), where $\pi/4 - \phi/2 \leq \psi \leq \pi/4$, the integral in (194) may approach infinity allowing arbitrarily large values of u_t . But the curve for r/a depicted in Fig. 19^a shows that b/a is small for any ψ_a and ψ_b within the interval $[\pi/4 - \phi/2, \pi/4]$. The critical velocity $u_{cr}/\dot{a}$ at which the sticking regime is replaced with the sliding regime is determined from (194) at $\psi_a = \psi_w$. A typical dependence of $u_{cr}/\dot{a}$ on b/a calculated at $\phi = \pi/6$ is illustrated in Fig. 20. The dashed curve corresponds to the regime (iii) and the solid

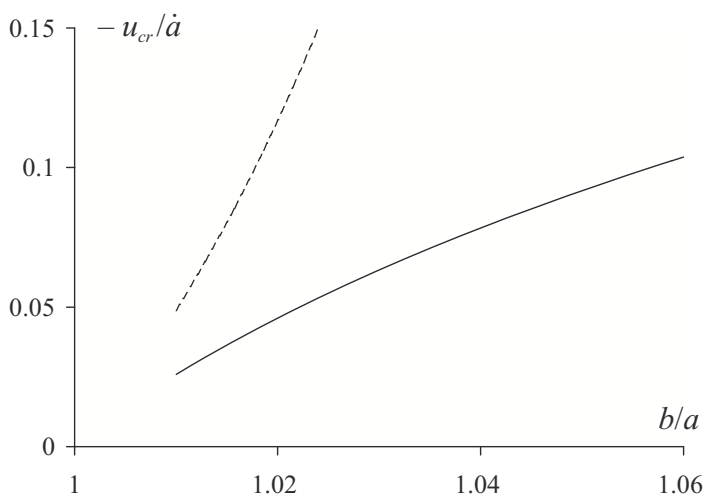


Figure 20. Coaxial model: transition between sticking and sliding

curve to the regime (ii). It is seen from this figure that the critical velocity in the regime (ii) is lower than the critical velocity in the regime (iii) in the entire range of b/a . Therefore, sliding starts when the critical velocity in the case (ii) is reached and thus the case (iii) never occurs.

In the case (i), combining the sign of the integrand, the order of the points $\psi = \psi_a$ and $\psi = \psi_b$ (see Fig. 19^b), and the structure of Eq. (194), it is possible to see that its right-hand side is positive. Since $u_0 > 0$, a solution exists if and only if $\dot{a} > 0$. Hence the interval $\pi/4 \leq \psi \leq \pi/2$ corresponds to the expansion of the cylinder. This confirms the preliminary conclusion concerning the case (i). In this process of the expansion combined with twist, it is seen from Figs 18^a and 19^b that sliding satisfying condition (174) might occur according to the frictional law. However, since the integral in

(194) is divergent if $\psi_a = \pi/4$, sticking is the only possible friction regime in this case, which makes a significant difference from the process of the contraction combined with twist.

The double-shearing model. In this case solution (190) gives

$$\begin{aligned} \frac{u_0}{\dot{a}} = & \frac{2}{\sin 2\psi_a \tan^{\sin \phi} \psi_a} \\ & \times \left\{ \int_{\psi_a}^{\psi_b} \frac{\sin \phi \left[\sin 2\psi_a + a (\cos 2\psi_a - \sin \phi) \frac{d\psi_a}{da} \right] \tan^{\sin \phi} \chi}{(\cos 2\chi - \sin \phi)} d\chi \right. \\ & \left. + \int_{\psi_a}^{\psi_b} \frac{\tan^{\sin \phi} \psi_a \sin 2\chi (\cos 2\chi - 2 \sin \phi)}{(\cos 2\chi - \sin \phi)} d\chi \right\} \quad (195) \end{aligned}$$

At sliding $\psi_a = \psi_w$ is prescribed according to (173) and $u_0 \equiv u_{sl}$ should be found from the solution. For the sticking condition, Eq. (195) is an ordinary differential equation for ψ_a and a difficulty with this equation is that there is no initial condition for ψ_a , unless sliding occurs at the beginning of the process. In the latter case, $\psi_a = \psi_w = \pi/4 - \phi/2 = \text{const}$ and, therefore, $d\psi_a/dt = 0$. Substituting these conditions into (195) and eliminating m using (180) gives

$$\frac{u_{sl}}{\dot{a}} = \frac{2}{m_1 \cos \phi} \int_{\psi_a}^{\psi_b} \frac{[\sin \phi \cos \phi \tan^{\sin \phi} \psi + m_1 \sin 2\chi (\cos 2\chi - 2 \sin \phi)]}{(\cos 2\chi - \sin \phi)} d\chi \quad (196)$$

Here and in what follows

$$m_1 = \left(\frac{\cos \phi}{1 + \sin \phi} \right)^{\sin \phi}$$

It follows from (181) and (196) that in the case of the double-shearing model the solution at sliding depends on the ratio b/a rather than on b and a separately. Note that it is not true in general, since $d\psi_a/dt$ may not vanish under the sticking condition and, therefore, Eq. (195) may involve a . In such cases, it is more convenient to adopt b/a and a as the independent variables instead of b and a .

It is easy to see that the denominator and the numerator of the integrand

in (196) vanish at $\chi = \psi_w$. Applying l'Hospital rule it is possible to obtain

$$\lim_{\chi \rightarrow \psi_w} \left[\frac{\sin \phi \cos \phi \tan^{\sin \phi} \chi + m_1 \sin 2\chi (\cos 2\chi - 2 \sin \phi)}{(\cos 2\chi - \sin \phi)} \right] = m_1 \cos \phi \quad (197)$$

Therefore, Eq. (196) where ψ_b should be eliminated using (181) determines the finite value of u_{sl} . The kinematic condition for sliding to occur is $u_t > u_{sl}$. The critical value of the velocity u_t at which transition between the sliding and sticking regimes may occur, u_{cr} , is determined by the equation $u_{cr} = u_{sl}$ and has been found at $\phi = \pi/6$ (see Figure 21). The dashed curve

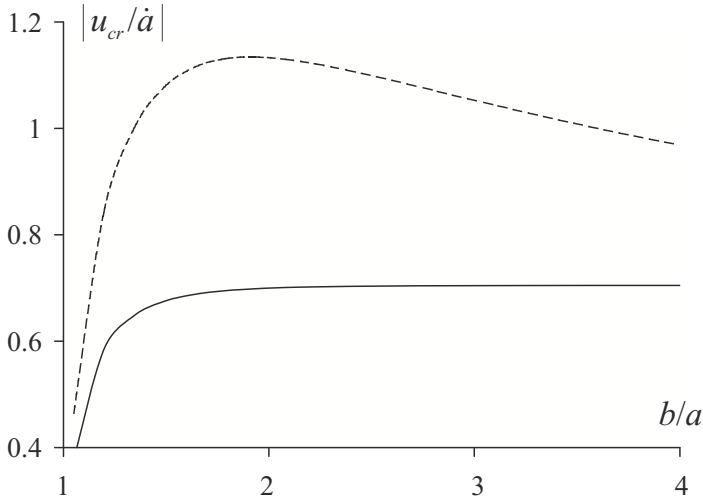


Figure 21. Double-shearing model: transition between sticking and sliding

corresponds to the expansion, $\pi/6 - \phi/2 \leq \psi < \pi/2$, and the solid curve to the contraction, $0 < \psi \leq \pi/4 - \phi/2$.

Assume that $u_t > u_{cr}$ at the beginning of the process when $a = a_0$, $u_t = u_{cr}$ at $a = a_{cr}$ and then $u_t < u_{cr}$. In such a process, the regime of sticking must replace the regime of sliding at $a = a_{cr}$. Then Eq. (195) considered as the differential equation for $\psi_a(a)$ may be solved for $a \geq a_{cr}$ in the case of the expansion or $a \leq a_{cr}$ in the case of the contraction together with the boundary condition

$$\psi_a = \psi_w \quad \text{at} \quad a = a_{cr} \quad (198)$$

As in the case of Eq. (196), the integrand in Eq. (195) reduces to the expression $0/0$ at $\psi_a = \psi_w$ and $a = a_{cr}$. However, in contrast to (196), it is

a function of two variables, χ and a for $a \geq a_{cr}$ in the case of the expansion or $a \leq a_{cr}$ in the case of the contraction. To find the asymptotic behavior of $\psi_a(a)$ in the vicinity of the point $\psi_a = \psi_w$ and $a = a_{cr}$, the functions involved in the numerator and the denominator of the integrand in (195) can be expanded in a series with respect to χ around $\chi = \psi_w$. The result is

$$\begin{aligned} \tan^{\sin \phi} \chi &= m_1 [1 + 2(\chi - \psi_w) \tan \phi] + o(\chi - \psi_w), \quad \chi \rightarrow \psi_w \\ \sin 2\chi (\cos 2\chi - 2 \sin \phi) &= -\cos \phi \sin \phi - 2(\chi - \psi_w) + o(\chi - \psi_w), \quad \chi \rightarrow \psi_w \\ \cos 2\chi - \sin \phi &= -2(\chi - \psi_w) \cos \phi + o(\chi - \psi_w), \quad \chi \rightarrow \psi_w \end{aligned} \quad (199)$$

Neglecting terms of order of $o(\chi - \psi_w)$ as $\chi \rightarrow \psi_w$ Eq. (195), using (199) in the vicinity of the point $a = a_{cr}$, may be rewritten in the form

$$\frac{u_t}{\dot{a}} = \frac{2}{m_1 \cos \phi} \int_{\psi_a}^{\psi_w + \psi_\delta} \left[w_1(a) + \frac{w_2(a)}{(\chi - \psi_w)} \right] d\chi + I_1(a) \quad \text{as } \chi \rightarrow \psi_w \quad (200)$$

where

$$\begin{aligned} w_1(a) &= m_1 \left\{ \cos \phi + 2 \tan \phi \frac{d\psi_a}{da} \left(\cos \phi + a_{cr} \sin \phi \frac{d\psi_a}{da} \right) (a - a_{cr}) \right\} + \\ &\quad + o(a - a_{cr}), \quad a \rightarrow a_{cr} \\ w_2(a) &= a_{cr} m_1 \sin \phi \left(\frac{d\psi_a}{da} \right)^2 (a - a_{cr}) + o(a - a_{cr}), \quad a \rightarrow a_{cr} \end{aligned} \quad (201)$$

In (201), the derivatives are understood to be calculated at $a = a_{cr}$ (or $\psi_a = \psi_w$). Also, $\psi_\delta \ll 1$ is a constant such that $\psi_w \leq \psi_a \leq \psi_w + \psi_\delta$ and $I_1(a)$ is a non-singular function of a resulting from (195) after integration from $\psi_w + \psi_\delta$ to ψ_b . After integration Eq. (200) takes the form

$$\frac{u_t}{\dot{a}} = -\frac{2w_2(a)}{m_1 \cos \phi} \ln |\psi_a - \psi_w| + I_2(a) \quad (202)$$

where $I_2(a)$ is the non-singular term consisting of $I_1(a)$ and the non-singular term obtained after integration in (200). It follows from (201) that the singular term in (202) approaches zero for any function $\psi_a(a)$ with a bounded derivative at $\psi_a = \psi_w$. Therefore, its contribution to the magnitude of u_t is negligible. However, some derivatives of u_t may be controlled by the singular term near the point $\psi_a = \psi_w$. In fact, assuming that $d\psi_a/da \neq 0$ at $\psi_a = \psi_w$ and differentiating (202) using (201) gives

$$\frac{d}{da} \left(\frac{u_t}{\dot{a}} \right) = -2a_{cr} \tan \phi \left(\frac{d\psi_a}{da} \right)^2 \ln (\psi_a - \psi_w) \quad (203)$$

to leading order. To avoid infinity in this equation, it is necessary to put $d\psi_a/da = 0$ at $\psi_a = \psi_w$. The latter condition is sufficient to get a bounded value for the second derivative of $u_t/\dot{a}$. However, the condition that the third derivative is bounded requires that $d^2\psi_a/da^2 = 0$ at $\psi_a = \psi_w$. This process can be continued to the derivative of any order. The same result can be obtained directly from (195). The denominator of the integrand is independent of a . Therefore, as $\psi_a \rightarrow \psi_w$, the derivative of any order contains a singular term in the form

$$\int_{\psi_a}^{\psi_b} \frac{J(a)}{(\cos 2\chi - \sin \phi)} d\chi \quad (204)$$

where $J(a)$ may include derivatives of ψ_a as $\psi_a \rightarrow \psi_w$. In the original expression (195), the function $J(a) \rightarrow 0$ as $a \rightarrow a_{cr}$ (or $\psi_a \rightarrow \psi_w$) leading to a converging integral independently of the value of $d\psi_a/dt$ at $\psi_a = \psi_w$. This condition is however not satisfied when Eq. (204) is the singular term involved in a derivative of $u_t/\dot{a}$ and it is necessary to put $d^n\psi_a/da^n = 0$ at $\psi_a = \psi_w$ for obtaining bounded values of $d^n(u_t/\dot{a})/da^n$ and $d^{n+1}(u_t/\dot{a})/da^{n+1}$. Therefore, it must be concluded that transition from sliding to sticking with a continuous function $\psi_a(a)$ is possible if and only if the function $u_t/\dot{a}$ satisfies certain restrictions. In particular, its derivatives of order n and higher do not exist (i.e. they approach infinity) at $\psi_a = \psi_w$. The specific number n is determined by the behavior of the function $\psi_a(a)$ in the vicinity of the point $a = a_{cr}$. The larger the value of n , the higher the order of the derivative of $\psi_a(a)$ vanishing at $a = a_{cr}$ must be. On the other hand, $u_t/\dot{a}$ is an independent function determined by the motion of the tool. Since this motion may be incompatible with the aforementioned behavior of the function $\psi_a(a)$, the solution for transition between the sliding and sticking regimes may not exist. On the other hand, it is possible to show that the solution at $u_t = u_{sl}$ is not unique. To this end, a solution under the sticking condition is found below assuming that $u_t = u_{sl}$. It is sufficient to consider the initial instance when $a = a_{cr}$. In (195), u_0 should be replaced with u_{sl} . Then, this equation can be treated in the following manner. Prescribe any value of ψ_a within the interval $(\pi/4 - \phi/2, \pi/2)$ in the case of the expansion or within the interval $(0, \pi/4 - \phi/2)$ in the case of the contraction. Equation (195) should be solved for $ad\psi_a/da$. If such a solution exists, the corresponding value of ψ_a can be used as the boundary condition to solve (195) and by the choice of this value the sticking regime occurs. The existence of the solution has been verified by its actual calculation at $\phi = \pi/6$ and several values of b/a .

Thus in the case of the double-shearing model there are two types of non-uniqueness where $u_t = u_{sl}$: (i) one can obtain the solution either under the sliding condition or under the sticking condition, (ii) in the case of sticking, it is possible to get an infinite number of solutions by selecting different couples ψ_a and $ad\psi_a/da$ satisfying (195) at $a = a_{cr}$. If $u_t \neq u_{sl}$, the solution is not unique because there is no initial condition for Eq. (195) considered as the ordinary differential equation for ψ_a , which is somehow similar to the case (ii) when $u_t = u_{sl}$.

The double-slip and rotation model. In this case Eqs (181) and (193) show that the solution depends on the ratio b/a and a . Equation (193) combined with the frictional boundary condition gives

$$\frac{u_0}{\dot{a}} = \frac{(\cos 2\psi_a - \cos 2\psi_b)}{\sin 2\psi_a} + \frac{\Omega a}{\dot{a}} \cot \psi_a \tan^{2m} \psi_a \sin \phi \int_{\psi_a}^{\psi_b} \frac{d\chi}{\cos^2 \chi \tan^{2m} \chi} \quad (205)$$

It is seen from this equation that the integral is not singular at $\psi_a = \psi_w$, in contrast to both the double-shearing and coaxial models. Therefore, for any given values of Ω and $\dot{a}$, specific values of u_{cr} and $u_{cr} < \infty$ can be found from (205) at $\psi_a = \psi_w$. If $u_t < u_{cr}$, the unique solution at sticking exists and if $u_t > u_{cr}$ the unique solution at sliding exists. At $u_t = u_{cr}$, transition between the regimes occurs such that ψ_a is a continuous function of a .

5.6 Solution for Pressure-independent Plasticity

The general system of equations is given in Section 2. In particular, the yield criterion (143) coincides with (9) and, therefore, with (8) at $\phi = 0$ and $k = \sigma_0/\sqrt{3}$ since $\sigma_2 = (\sigma_1 + \sigma_3)/2$ in the case of plane strain deformation of the pressure-independent material. The yield criterion is satisfied by the standard substitution

$$\sigma_{rr} - \sigma_{\theta\theta} = 2k \cos 2\psi, \quad \sigma_{r\theta} = k \sin 2\psi \quad (206)$$

Then, the associated flow rule (15) results in (183). Moreover, (186) and (191) reduce to (183) at $\phi = 0$. Even though the co-axiality condition is satisfied in pressure-independent plasticity, an essential difference from the co-axial model is that in the case under consideration the characteristics for stresses and velocities coincide, which is a property of the double-shearing and double-slip and rotation models. The maximum friction law (173) reduces to (174) at $\phi = 0$.

As before, the distribution of the shear stress is given by (177) and the radial velocity by (182). Using these solutions and (206), the solution to

(183) satisfying the boundary condition (168) can be found in the form

$$\frac{u_\theta}{\dot{a}} = \pm \frac{ra}{R^2} \left(\frac{\sqrt{b^4 - R^4}}{b^2} - \frac{\sqrt{r^4 - R^4}}{r^2} \right) \quad (207)$$

Here and in what follows the upper sign corresponds to the contraction, $0 < \psi < \pi/4$, and the lower sign to the expansion, $\pi/4 \leq \psi < \pi/2$.

In the case of sliding, it follows from (174), (177) and (206) that $R = a$. Substituting this equation into (207) at $r = a$ gives

$$\frac{u_{sl}}{\mp \dot{a}} = \sqrt{1 - \left(\frac{a}{b}\right)^4} \quad (208)$$

In the case of sticking, it follows from (175) and (207) that

$$\frac{u_t}{\mp \dot{a}} = \frac{a^2}{R^2} \left(\frac{\sqrt{b^4 - R^4}}{b^2} - \frac{\sqrt{a^4 - R^4}}{a^2} \right) \quad (209)$$

This equation determines R at given a and b . In particular, it follows from (209) that

$$0 < R \leq a \quad (210)$$

At given a and b , the right-hand side of (209) is a monotonically increasing function of R in the interval $0 < R \leq a$. Therefore, within this interval the maximum is attained at $R = a$. Hence taking into account (210), it is possible to conclude that the solution under the sticking condition does not exist for $u_t > u_{cr}$ where u_{cr} is determined from Eq. (209) at $R = a$. Comparing with (208) shows that $u_{sl} = u_{cr}$. Therefore, in the case of classical plasticity, both regimes, sliding and sticking, may occur and the solution is unique for both processes, the expansion and contraction.

5.7 Singularity in Velocity Fields

In the case of sliding, the velocity fields in the solutions based the double-shearing model, the double-slip and rotation model and the classical plasticity model are singular. To show this it is necessary to find the shear strain rate. In the problem under consideration it is defined by

$$2\xi_{r\theta} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \quad (211)$$

The first term may be written in the form

$$\frac{\partial u_\theta}{\partial r} = \frac{\partial u_\theta}{\partial \psi} \frac{\partial \psi}{\partial r} \quad (212)$$

It follows from (189), using (197), that for the double-shearing model the derivative $\partial u_\theta / \partial \psi$ is bounded and, in general, is not zero at $\psi = \psi_w$. The same conclusion can be drawn from (192) for the double-slip and rotation model. On the other hand, it follows from (179) that $\partial \psi / \partial r \rightarrow \infty$ as $\psi \rightarrow \psi_w$. Therefore, it is clear from (211) and (212) that $\xi_{r\theta} \rightarrow \infty$ as $\psi \rightarrow \psi_w$ and thus the velocity field is singular. In the vicinity of the point $\psi = \psi_w$ (or $r = a$), Eq. (179) can be rewritten in the form

$$\frac{\partial \psi}{\partial r} = \frac{1}{2a(\psi - \psi_w)} \quad (213)$$

to leading order. The solution to Eq. (213) satisfying the condition $\psi = \psi_w$ at $r = a$ is

$$\psi - \psi_w = \pm \frac{\sqrt{r-a}}{\sqrt{a}} + o(\sqrt{r-a}), \quad r \rightarrow a \quad (214)$$

Substituting (214) into (213) gives

$$\frac{\partial \psi}{\partial r} = O\left(\frac{1}{\sqrt{r-a}}\right), \quad r \rightarrow a \quad (215)$$

Then, it follows from (211) and (212) that

$$\xi_{r\theta} = O\left(\frac{1}{\sqrt{r-a}}\right), \quad r \rightarrow a \quad (216)$$

Since $\xi_{r\theta} \rightarrow \infty$ as $r \rightarrow a$, its magnitude controls the behavior of the equivalent strain rate in the vicinity of the friction surface

$$\xi_{eq} = O\left(\frac{1}{\sqrt{r-a}}\right), \quad r \rightarrow a \quad (217)$$

This is a general property of the equations of classical plasticity (see Eq. (74)). In particular, in the problem under consideration, (217) can be obtained from (207) by inspection. Alexandrov and Lyamina (2002); Alexandrov (2005) have found the behavior (217) in the general case of plane-strain and axisymmetric deformation of materials obeying the double-shearing model.

In the case of the co-axial model, Eq. (184) shows that $\partial u_\theta / \partial \psi$ is proportional to $(\cos 2\psi - \sin \phi)$. Therefore, the product on the right in Eq. (212) is bounded at the friction surface, as follows from Eq. (179) and, therefore, (217) does not hold for this model.

Other comparative studies of solution behavior in the vicinity of maximum friction surfaces for the coaxial and double-shearing models have been completed in Alexandrov (2003); Alexandrov and Lyamina (2005).

A boundary value problem similar to that considered in this Section has been solved for the model based on the yield criterion (88) and its associated flow rule in Alexandrov et al. (2008b). In particular, it has been found that the velocity field satisfies (217).

6 Applications

The area of applicability of this or that model is rather conditional on the specific area of application. Therefore, it is advantageous that the solutions based on various models that reduce to the same model at specific values of input parameters show the same qualitative behavior everywhere. The structure of the closed-form solutions considered includes a number of qualitative features, such as the existence of rigid zones, the transition between the regimes of sticking and sliding, the singularity in the velocity field, non-existence of the solution, and non-uniqueness of the solution. It is, therefore, ideal in helping to select this or that generalization of the classical theory of plasticity for specific applications. For instant, experimental evidence of pressure-dependency of traditional metals, which can be treated as incompressible materials, can be found in Spitzig et al. (1976); Kao et al. (1990). As follows from the comparative study in Section 5, among the pressure-dependent models considered the double slip and rotation model is most suitable for such materials, if their pressure-dependency should be taken into account. The aforementioned closed-form solutions are also useful for verifying numerical codes, which is a necessary step before their use in practical applications (for example, Roberts et al. (1992)).

A possible area of applications of the singular velocity fields near maximum friction surfaces, Eq. (1.1), is the description of the formation of a thin layer in the vicinity of frictional interfaces where material properties are very different from the properties in the bulk in manufacturing processes (Dutton et al., 1998; Aukrust and LaZghab, 2000; Moylan et al., 2003; Trunina and Kokovkhin, 2008). In particular, the structure of material in a narrow layer near the friction surface after plane-strain extrusion of a brass specimen is shown in Figs 22 and 23 (Alexandrov et al., 2009). It is seen from Fig. 22 that the layer where material properties are very different from the properties in the bulk is quite thin (about $100\ \mu\text{m}$). A theoretical experimental approach for determining the thickness of such layers has been proposed in Alexandrov et al. (2009). The approach is based on the strain rate intensity factor. Moreover, this layer can be further divided into sub-layers in which various features of the material structure can be seen at a finer resolution (see Fig. 23). Probably, an approach similar to that proposed in Alexandrov et al. (2009) can be developed for determining

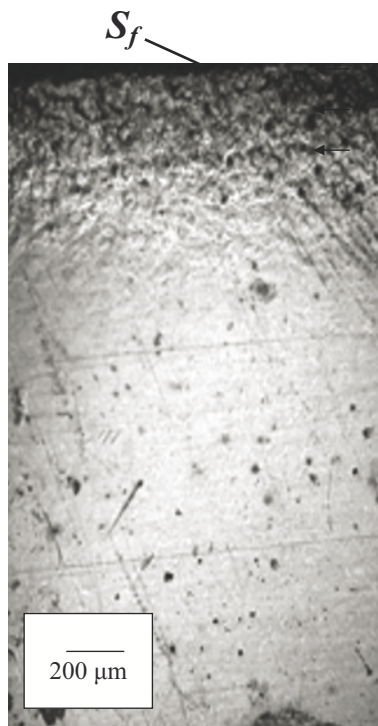


Figure 22. Layer of intensive plastic deformation in the vicinity of the frictional interface, S_f , after plane strain extrusion of a brass specimen

the thickness of each of these sub-layers. Other approaches based on the strain rate intensity factor for predicting this or that feature of deformation processes in a narrow layer near frictional interfaces have been proposed in Alexandrov et al. (2003); Alexandrov and Lyamina (2006, 2007); Lyamina et al. (2007). In order to apply these approaches, it is necessary to calculate the strain rate intensity factor. It is impossible by means of commercial numerical codes because of the singularity in the velocity field. A review of the strain rate intensity factors available from closed-form solutions is provided in Alexandrov (2009). A numerical method for determining the strain rate intensity factor in plane strain problems for rigid perfectly/plastic solids has been proposed in Lyamina (2007). This method cannot be extended to other modes of deformation or other material models because it is based on

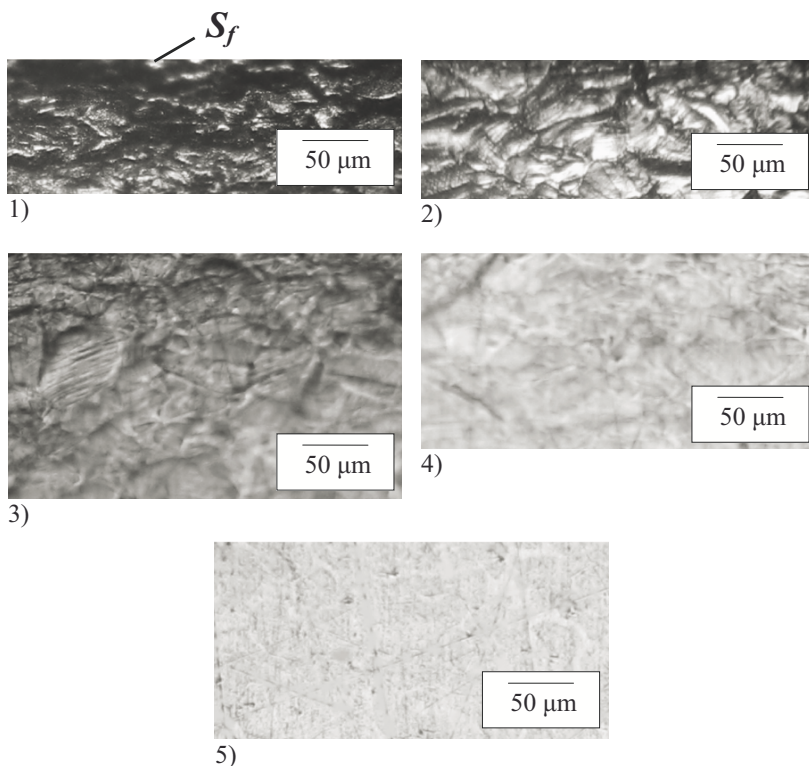


Figure 23. Structure of the layer shown in Fig. 22 at a finer resolution

specific properties of the system of equations for plane-strain deformation of rigid perfectly/plastic solids.

The upper bound method is a convenient tool for finding approximate velocity fields and evaluating the load required to deform the material (Avitzur, 1980; Kamenjarzh, 1996). In utilizing this method, the velocity fields are typically approximated by a class of assumed functions. This is true even when finite element methods are used. The assumed functions can have a large effect of the accuracy of the result. Therefore, it is advantageous to select these functions (kinematically admissible velocity fields) such that (1.1) is satisfied, even though it is not a requirement of the upper bound theorem. It has been demonstrated in Alexandrov et al. (2001). Kinematically admissible velocity fields that account for (1.1) have been

also used in Alexandrov (1999); Alexandrov and Kocak (2008); Alexandrov and Richmond (2000) among others.

Acknowledgment - The research described was partly supported by grants 08-01-00700 (RFBR, Russia) and NSH-134.2008.1 (Russia).

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Impact of Cellular Materials

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Abstract Cellular materials are used as impact energy absorbers due to their large densification strain at the plateau stress during the plastic compression. For a cellular rod struck by a rigid object, the critical impact velocity is determined. If the impact velocity is higher than the critical impact velocity, the elastic wave will be followed by plastic shock waves. Plastic shock waves and shock arrest are investigated analytically for longitudinal impacts. Shock behaviors are characterized for material design purpose and will be used for impact protection.

1 Introduction

Cellular materials, such as metal foams, are used as impact energy absorbers in crash and blast protection due to their unique constitutive behavior, e.g., Ashby et al. (2000). Three stages can be identified in the stress-strain curve of the uniaxial compression of cellular materials, as shown schematically by the solid line in Fig. 1.

- Stage I: For a closed-cell cellular material, deformation is in the form of bending of the cell walls and edges and is, in general, reversible. At the end of this stage some cells suffer collapse. This may be due to elastic buckling, plastic deformation or fracture.
- Stage II: The almost constant compressive stress appears in a wide range of strain. Buckling and plastic collapse occurs successively until all cells are collapsed. The deformation in this stage is unrecoverable.
- Stage III: Cell walls and edges contact each other and are crushed; giving rise to a steeply rising stress.

For impact analysis, Reid and Peng (1997) firstly treated cellular materials subject to uniaxial compression using a simplified rigid, perfectly-plastic, locking (RPPL) model as shown by the dashed lines in Fig. 1. The constitutive behavior in stage - I is simplified as rigid. Stage - II is treated as perfect plastic with the yielding plateau stress σ_{pl} . The second stage ends with the

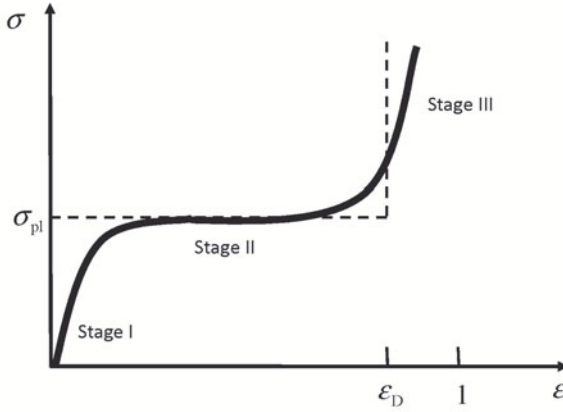


Figure 1. Constitutive behavior of a typical cellular material. The solid line shows schematically the relation between stress σ and strain ε . The dashed line shows the idealization of the stress-strain curve in the rigid, perfect-plastic, locking (RPPL) model. In stage - I, the material is treated as rigid. In stage - II, the material is in perfect plasticity with yield stress σ_{pl} . In stage - III, the material is again treated as rigid, with the densification strain ε_D

locking (densification) strain ε_D . Stage - III is again idealized as rigid. Tan et al. (2002, 2005a,b) used a RPPL idealization of foam materials to study the inertia effects and shock enhancement in uniaxial dynamic compression. Radford et al. (2005) used RPPL model to study the shock behavior in a metal foam projectiles. Harrigan et al. (2009) compared RPPL with other analytical approaches on shock wave based models.

Alporas aluminium foam developed by the Shinko Wire Company has a plateau stress σ_{pl} in the range of 1.3 ~ 1.7 MPa, the Young's modulus E in the range of 0.4 ~ 1.0 GPa, and densification strain ε_D in the range of 0.7 ~ 0.82 (Ashby et al., 2000). The strain at the end of first stage is of the order σ_{pl}/E , which is around 0.2%; compared with the densification strain ε_D it is reasonable to ignore this part of deformation and treat the material in stage I as rigid, as shown in Fig. 1.

To keep the mathematics and models simple, let us only consider uniaxial compressive stress and strain. In the following, impact of a one-dimensional cellular material as shown in Fig. 2 will be studied. For simplicity, the RPPL model will be adopted.

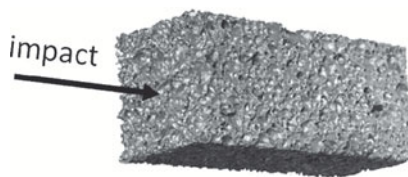


Figure 2. Impact on a cellular rod. Planar shock wave will be generated from the impact end, and propagates along the impact direction

2 Wave Propagation in a Cellular Rod

To design the crush and blast protection using cellular materials, we need to understand the wave behaviors in those materials under different impact velocities. Lopatnikov et al. (2004) considered in the general case of four significantly different possibilities of high-velocity impact on a cellular material that separated by three characteristic velocities: sound velocity of the constituent material, sound velocity of the cellular material and sound velocity during the plateau and up to the complete densification of the foam.

After impacted by a striking object, as shown in Fig. 2, there are two kinds of waves generated in the cellular rod, first an elastic wave, then followed by a plastic shock wave. However for the generation of the plastic shock wave, there is a minimum requirement for the impact velocity v_m to be higher than a critical value v_m^c . This section finds the critical impact velocity for the generation of plastic shock waves.

The case of a rigid mass impacting a semi-infinite rod, as shown in Fig. 3, is studied. A one-dimensional coordinate system is set along the rod, with

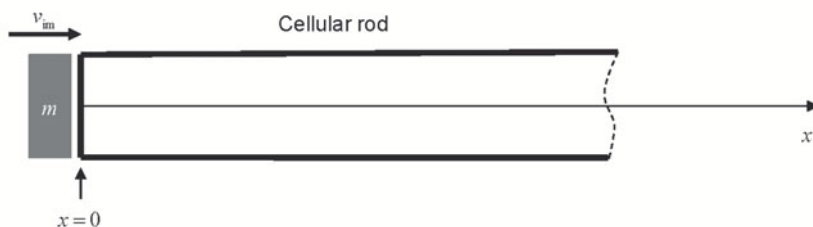


Figure 3. One-dimensional coordinate system on a cellular rod, with impact from a striking object of mass m and velocity v_m

the origin at the impact end. The general one-dimensional wave equation is

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}, \quad (1)$$

where σ and u are the stress and displacement along the x -axis, ρ is the density, and t is time with $t = 0$ corresponds to the moment of the impact commencement.

When the impact level is low, the density of the cellular material is ρ_0 , and the relation between the uniaxial stress σ and strain ε is elastic that takes the form

$$\sigma = E\varepsilon \quad (2)$$

where E is the Young's modulus. The strain ε relates to the axial displacement u through

$$\varepsilon = \frac{\partial u}{\partial x} \quad (3)$$

Substituting the above relations between the stress, strain and displacement into Eq. (1), the displacement wave equation can be rewritten as

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c_{\text{el}}^2} \frac{\partial^2 u}{\partial t^2}, \quad (4)$$

where

$$c_{\text{el}}^2 = E/\rho_0 \quad (5)$$

The solution of the above wave equation can be expressed in a general form as

$$u(x, t) = f_{\text{forward}}(x - c_{\text{el}}t) + f_{\text{backward}}(x + c_{\text{el}}t) \quad (6)$$

which gives the meaning of c_{el} as the elastic wave speed, $f_{\text{forward}}(x - c_{\text{el}}t)$ and $f_{\text{backward}}(x + c_{\text{el}}t)$ are two arbitrary functions representing waves that propagate in the forward and backward directions along the x -axis, respectively. Detailed expressions of $f_{\text{forward}}(x - c_{\text{el}}t)$ and $f_{\text{backward}}(x + c_{\text{el}}t)$ need to be determined by detailed physical conditions.

The displacement waves generated from the impact end $x = 0$ propagate in the positive x direction and are initially elastic, which entails the general wave expression as

$$u(x, t) = H(c_{\text{el}}t - x)f(c_{\text{el}}t - x), \quad (7)$$

where $H(\xi)$ is the Heaviside function. For Alporas metal foam, the Young's modulus E and the density ρ_0 are in the range of $0.4 \sim 1.0$ GPa and $0.2 \sim 0.25 \times 10^3$ kg/m³, respectively (Ashby et al., 2000); the elastic wave speed is,

calculated from Eq. (5), in the range $1.4 \sim 2.2$ km/s. The impact velocity we consider, such as during the car crash accident, is normally on the order of several tens of meters per second, much slower compared with the elastic wave speed in typical cellular protection materials.

Detailed expression of the function f , for the case of a rod struck by a rigid object of impact velocity v_m as shown in Fig. 3, will be explored in the following. The areal mass m of the striking body relates to the total mass of the striking object M_{impact} through the relation

$$m = \frac{M_{\text{impact}}}{A} \quad (8)$$

where A the cross section area of the rod. The impulse-momentum relation applied to the striking mass gives

$$m dv_m = \sigma_m dt, \quad (9)$$

where σ_m is the stress in the striking object. Assuming that the striking object is attached to the rod after impact, i.e., the striking object and the impact end of the rod share the same velocity and stress so that

$$v_m = \frac{\partial u(0, t)}{\partial t}, \quad \sigma_m = E \frac{\partial u(0, t)}{\partial x} \quad (10)$$

With the general expression of $u(x, t)$ in Eq. (7), the velocity v_m and stress σ_m can be expressed using the function f as

$$v_m = c_{\text{el}} f'(c_{\text{el}} t), \quad \sigma_m = -E f'(c_{\text{el}} t) \quad (11)$$

Substitution of the expressions for v_m and σ_m into Eq. (9) generates the differential equation

$$m c_{\text{el}} df'(c_{\text{el}} t) + E f'(c_{\text{el}} t) dt = 0 \quad (12)$$

The initial conditions

$$v_m|_{t=0} = v_m, \quad u(0, 0) = 0 \quad (13)$$

give the initial conditions of function $f(\xi)$ as

$$f(0) = 0, \quad f'(0) = \frac{v_m}{c_{\text{el}}} \quad (14)$$

The solution of the differential equation on $f(\xi)$ with the above initial conditions gives

$$f(\xi) = \frac{m c_{\text{el}} v_m}{E} \left[1 - \exp \left(-\frac{E}{m c_{\text{el}}^2} \xi \right) \right], \quad (15)$$

Using Eq. (7), the general displacement field in the rod is given by

$$u(x, t) = \frac{mv_m}{c_{el}\rho_0} \left[1 - \exp \left(-\frac{E}{mc_{el}^2} (c_{el}t - x) \right) \right] H(c_{el}t - x) \quad (16)$$

Using

$$v(x, t) = \frac{\partial u(x, t)}{\partial t}, \quad (17)$$

the evolution of velocity field is

$$v(x, t) = v_m \exp \left[-\frac{\rho_0}{m} (c_{el}t - x) \right] H(c_{el}t - x), \quad (18)$$

Using

$$\sigma(x, t) = E \frac{\partial u(x, t)}{\partial t}, \quad (19)$$

the evolution of velocity field is

$$\sigma(x, t) = -c_{el}\rho_0 v_m \exp \left[-\frac{\rho_0}{m} (c_{el}t - x) \right] H(c_{el}t - x) \quad (20)$$

The relation between the velocity and stress field is

$$\frac{v(x, t)}{\sigma(x, t)} = \frac{1}{c_{el}\rho_0} \quad (21)$$

Figure 4 shows the time evolution of the stress fields. The vertical axis is the compressive stress nondimensionalized as $-\sigma/c_{el}\rho_0 v_m$. The horizontal axis is the nondimensionalized position $x\rho_0/m$. Stress waves are drawn for three moments of $m/\rho_0 c_{el}$, $2m/\rho_0 c_{el}$ and $3m/\rho_0 c_{el}$. The compressive stresses at the wave front keep the same magnitude of $c_{el}\rho_0 v_m$, which is independent of the mass of the striking object.

At the impact end $x = 0$, the time evolution of the displacement, velocity and stress fields are

$$\begin{aligned} u(0, t) &= \frac{mv_m}{c_{el}\rho_0} \left[1 - \exp \left(-\frac{c_{el}\rho_0}{m} t \right) \right], \\ v(0, t) &= v_m \exp \left(-\frac{c_{el}\rho_0}{m} t \right), \\ \sigma(0, t) &= -c_{el}\rho_0 v_m \exp \left(-\frac{c_{el}\rho_0}{m} t \right), \end{aligned} \quad (22)$$

respectively. At time $t = 0$, the compressive stress at the impact end of the cellular rod is $\rho_0 c_{el} v_m$. When

$$c_{el}\rho_0 v_m > \sigma_{pl}, \quad (23)$$

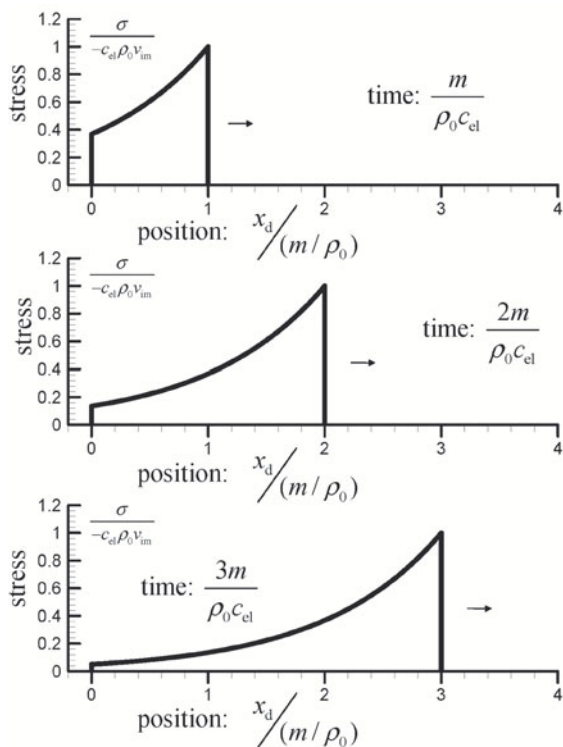


Figure 4. Impact induced compressive stress wave in an elastic rod. The wave propagates along x (rod axis) direction. In the figure, the position is nondimensionalized as $x\rho_0/m$, and the stress is nondimensionalized as $-\sigma/c_{el}\rho_0v_m$. The wave is shown at 3 different times: m/ρ_0c_{el} , $2m/\rho_0c_{el}$ and $3m/\rho_0c_{el}$

the plastic shock wave will follow the elastic wave.

The response of materials and structures to suddenly applied loads, such as shown in Fig. 3, can be quite different from their response when subjected to loads which increases slowly. The critical impact velocity is

$$v_{im}^c = \frac{\sigma_{pl}}{\rho_0 c_{el}} \quad (24)$$

If the impact velocity is slower than v_{im}^c , only an elastic wave will be generated from the impact end. If the impact velocity is higher than v_{im}^c , the elastic wave will be followed by plastic shock waves. A plastic shock wave is

made by a rapid, continuous push, and is characterized by an abrupt, nearly discontinuous change in the mechanical properties of the cellular rod: stress, mass density, particle velocity.

For Alporas metal foam the plateau stress σ_{pl} , density ρ_0 and elastic wave speed c_{el} are in the ranges of $1.3 \sim 1.7$ MPa, $0.2 \sim 0.25 \cdot 10^3$ kg/m³ and $1.4 \sim 2.2$ km/s, respectively. Therefore, the critical impact velocity v_{im}^c is in the range of $2.6 \sim 6.0$ m/s. According to Eq. (21), when the wave stress reaches $-\sigma_{pl}$ the particle velocity is $\sigma_{pl}/(c_{el}\rho_0)$. This particle velocity of the elastic wave is relatively small compared with the impact velocity v_{im} which can be around 30 m/s during cases such as a highway car crashing.

Shock-wave analysis is used in the following sections for high speed impact. We study two cases on the impact of cellular materials: a rigid object striking on a cellular rod with fixed end, and a rigid object striking on a free cellular rod. The kinetic energy of the striking object is transmitted to the rod and absorbed by deformation and damage in the cellular material.

3 Rigid Object Strikes on a Cellular Rod of Fixed End

In crash protection the absorber must absorb the kinetic energy of the moving object without reaching complete densification, and the stress it transmits never exceeds the plateau stress.

Consider the impact of one end of a stationary rod by a rigid striker at a high velocity v_m , as denoted in Fig. 5(a). The rod is homogeneous and made from a RPPL material. After impact a shock wave moves from the impacted end to the opposite fixed end of the rod, as shown in Fig 5(b). The stress ahead of the shock wave is compressive with magnitude σ_{pl} brought up by the fast travelling elastic wave. According to the argument in the previous section particle velocity of the elastic wave is relatively small compared with the impact velocity v_m , therefore, the material ahead of the shock front can be treated as static with zero particle velocity. The material behind the shock front has attained a strain ε_D , its particle velocity is v_d , its density has been raised from the initial value ρ_0 to the densification value ρ_D , and the compressive stress has been raised to σ_d .

3.1 Basic Assumptions

Assumptions are made for analytically investigating the shock behavior in a one-dimensional cellular rod:

1. There is a sharp shock front separating the compressed and undeformed regions of the foam. The micrograph in Fig. 6 clearly shows a

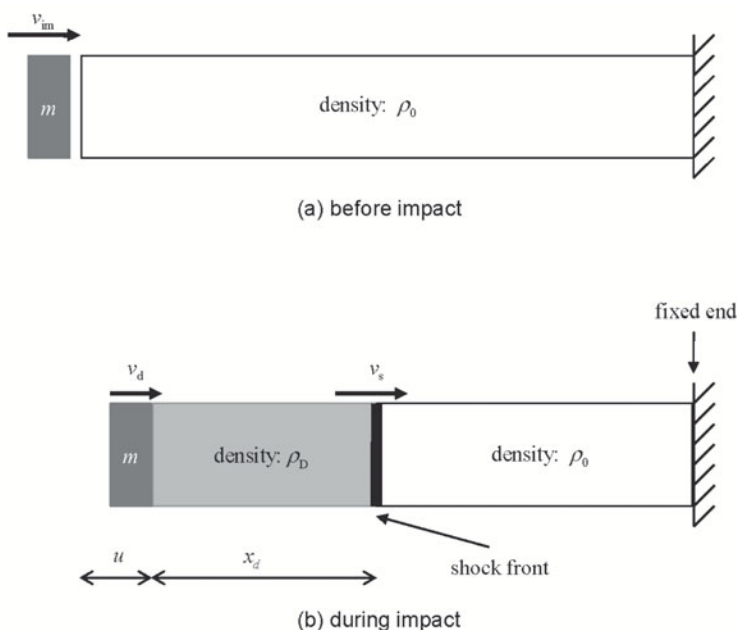


Figure 5. A fixed cellular rod struck by a rigid object. The object has an areal mass m and impact velocity v_m . The cellular rod is separated by the shock front into two portions, the uncondensed portion is in white colour, and the condensed portion is in gray colour. For the uncondensed portion the particle velocity is 0 and the material density is ρ_0 . For the densified portion the particle velocity is v_d and the density is ρ_D . The velocity of shock front is v_s

sharp shock front separating the compressed and undeformed regions of the foam.

2. After the densification, the particle velocities of the densified materials are equal to each other and equal to the velocity of the striking body.
3. The densified layer has a density which is spatially constant.
4. The rate effects that affect the deformation and failure modes of cellular materials (e.g., Calladine and English (1984)) are ignored.

3.2 Shock Wave Analysis

Here we consider the case where the impact speed is lower than the elastic wave speed. Therefore, in the undeformed region ahead of the shock wave,

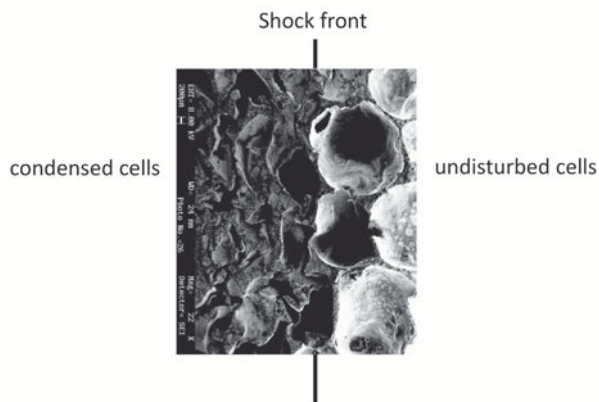


Figure 6. A scanning electron micrograph of the metal foam specimen sectioned along its impact axis. The impact is acting from left to the right (Radford et al., 2005)

the stress is equal to the plateau stress σ_{pl} , while the strain and particle velocity can be ignored.

According to definition, the densification strain ε_D

$$\varepsilon_D = \frac{u}{x_d + u}, \quad (25)$$

where, as shown Fig. 5, u is the displacement of the rigid mass at time t , and x_d is the deformed length of the crushed cellular material.

Conservation of mass across the shock front gives

$$\frac{\rho_0}{\rho_D} = 1 - \varepsilon_D \quad (26)$$

As shown in Fig. 1, the densification strain ε_D in the RPPL model cannot be exactly defined from the measured stress-strain curve. Equation (26) provides a method to determine ε_D experimentally, that is to measure the material density before and after densification, the densification strain ε_D can thus be determined through $\varepsilon_D = 1 - \rho_0/\rho_D$.

The cellular material behind the shock front is idealized as rigid. The instantaneous velocity (reducing) of the rigid mass with respect to a stationary frame is the same as the particle velocity of the condensed part, and is denoted as

$$v_d = \frac{du}{dt} \quad (27)$$

The shock velocity v_s is faster than the particles velocity v_d , and this gives the increase of x_d with respect of time,

$$\frac{dx_d}{dt} = v_s - v_d \quad (28)$$

This provides the relation between the shock velocity v_s and the condensed part velocity v_d as

$$v_s = \frac{v_d}{\varepsilon_D}, \quad (29)$$

and

$$\frac{dx_d}{dt} = \frac{1 - \varepsilon_D}{\varepsilon_D} v_d \quad (30)$$

Considering the momentum change of the body consists of the striking object and the cellular material behind the shock front, one has

$$d[(m + \rho_D x_d)v_d] = -\sigma_{pl} dt, \quad (31)$$

where m is the areal mass of the impact object. The above equation gives

$$v_d d[(m + \rho_D x_d)v_d] + \sigma_{pl} \frac{1 - \varepsilon_D}{\varepsilon_D} dx_d = 0,$$

which can be rewritten as

$$\frac{1 - \varepsilon_D}{\sigma_{pl} \varepsilon_D + \rho_0 v_d^2} dv_d^2 + \frac{2}{m + \rho_0 x_d} = 0$$

The above differential equation, together with the initial condition

$$v_d = v_m \quad \text{when} \quad x_d = 0 \quad (32)$$

provides

$$(\rho_0 v_d^2 + \sigma_{pl} \varepsilon_D)(m + \rho_0 x_d)^2 = (\rho_0 v_d^2 + \sigma_{pl} \varepsilon_D)m^2 \quad (33)$$

This further gives the particle velocity behind the shock as

$$v_d = v_C \sqrt{\left[1 + \left(\frac{v_m}{v_C}\right)^2\right] \left(\frac{m}{m + \rho_D x_d}\right)^2 - 1}, \quad (34)$$

where the characteristic velocity v_C is defined as

$$v_C = \sqrt{\frac{\varepsilon_D \sigma_{pl}}{\rho_0}} \quad (35)$$

The evolution of x_d can be solved from the following differential equation

$$\frac{dx_d}{dt} = \frac{1 - \varepsilon_D}{\varepsilon_D} v_C \sqrt{\frac{1 + \tilde{v}^2}{\left(1 + \frac{\rho_D x_d}{m}\right)^2} - 1},$$

where

$$\tilde{v} = \frac{v_m}{v_C} \quad (36)$$

The solution of the above differential equation, together with the initial condition

$$x = 0 \quad \text{at} \quad t = 0, \quad (37)$$

gives

$$\frac{1 - \varepsilon_D}{\varepsilon_D} v_C t + \sqrt{\frac{m^2 \tilde{v}^2}{\rho_D^2} - \frac{2m}{\rho_D} x_d - x_d^2} = \frac{m \tilde{v}}{\rho_D}$$

Solve the equation for x_d , one has the time evolution of the length of the crushed foam as

$$x_d = -\frac{m}{\rho_D} + \sqrt{\left(\frac{m}{\rho_D}\right)^2 - \left(\frac{1 - \varepsilon_D}{\varepsilon_D} v_C t\right)^2 + 2\frac{m}{\rho_D} \frac{1 - \varepsilon_D}{\varepsilon_D} v_{im} t}, \quad (38)$$

which can be written in a nondimensionalized form as

$$\frac{x_d}{m/\rho_D} = \sqrt{1 - \tilde{t}^2 + 2\tilde{v}\tilde{t}} - 1, \quad (39)$$

where the nondimensionalized time $\tilde{t}$ is defined as

$$\tilde{t} = \frac{t}{m\varepsilon_D/\rho_0 v_C} \quad (40)$$

Figure 7 shows the time evolution of the crushed length x_d affected by the impact velocity.

Shock arrest, which means complete stopping of the shock wave, requires

$$\frac{dx_d}{dt} = 0, \quad (41)$$

which gives the arrest distance x_d^{arrest} , or the maximum length of the crushed cellular material, as

$$\frac{x_d^{\text{arrest}}}{m/\rho_D} = \sqrt{1 - \tilde{v}^2} - 1 \quad (42)$$

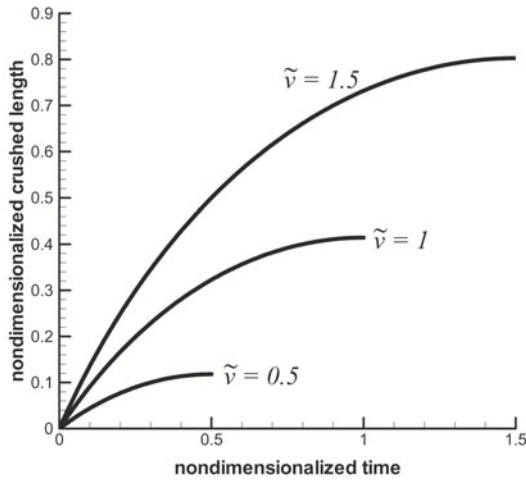


Figure 7. Nondimensionalized crushed-length, $x_d/(m/\rho_D)$, evolves as a function of nondimensionalized time, $t/(\varepsilon_D m/\rho_0 v_C)$, for different impact velocities v_m at $0.5\sqrt{\varepsilon_D \sigma_{pl}/\rho_0}$, $\sqrt{\varepsilon_D \sigma_{pl}/\rho_0}$ and $1.5\sqrt{\varepsilon_D \sigma_{pl}/\rho_0}$, respectively

Figure 8 shows the relation between the arrest distance and the impact velocity.

According to Eq. (39), the time to reach shock arrest is

$$\tilde{t} = \tilde{v}, \quad (43)$$

which means

$$t = \frac{mv_{im}}{\sigma_{pl}} \quad (44)$$

The above relation can also be derived from the momentum-impulse relations during the time period from impact commencement to the shock arrest.

According to Eq. (42), to crush a impact velocity of v_{im} , the mass of the cellular material for energy absorbing, $M_{cellular}$, should be at least

$$\frac{M_{cellular}}{M_{impact}} = \sqrt{1 + \left(\frac{v_{im}}{v_C}\right)^2} - 1, \quad (45)$$

where M_{impact} is the mass of the impact object. For the case of $v_{im} = v_C$, $M_{cellular}/M_{impact} = \sqrt{2} - 1$, i.e., for the complete protection (shock arrest)

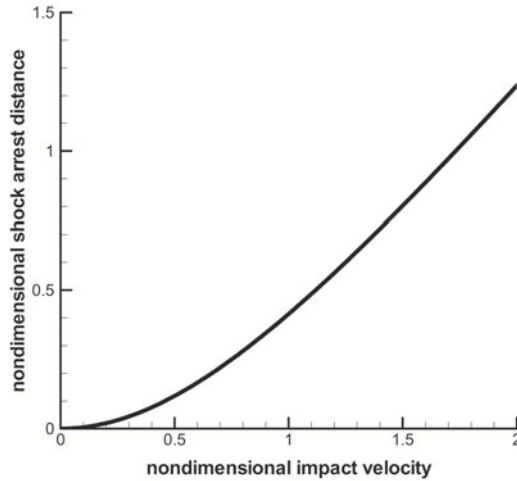


Figure 8. Nondimensionalized arrest distance, $x_d^{\text{arrest}}/(m/\rho_D)$, as a function of the nondimensionalized impact velocity $\tilde{v} = v_m/v_C$

against a impact velocity of v_C , the ratio of the mass of the cellular material for energy absorbing and that of the impact object should be at least 1.4. From Eq. (45), a fixed cellular rod can fully arrest a striking object of mass M_{impact} with impact velocity up to

$$v_C \sqrt{\left(1 + \frac{M_{\text{cellular}}}{M_{\text{impact}}}\right)^2 - 1}$$

For car crash, the impact velocity is about $v_m = 30$ m/s. Table 1 gives the material properties (density, densification strain and plateau stress) for several typical cellular materials. The characteristic velocity is given through Eq. (35).

Stress behind the shock front. For a shock wave passing a cellular material, the shock enhancement effect under high speed impact (>100 m/s) was originally proposed by Reid and Peng (1997). Afterwards, a number of authors also reported this effect for various cellular materials at high impact speeds (Lopatnikov et al., 2003, 2004; Tan et al., 2002, 2005a,b; Radford et al., 2005). For relatively low impact speeds, there is the so-called critical velocity under which shock enhancement is not significant (~ 50 m/s).

Table 1. Material properties for some typical cellular materials

Foam	Density ρ_0 (kg/m ³)	Plateau stress σ_{pl} (MPa)	Densification strain ε_D	Characteristic velocity v_C (m/s)
Polyurethane	34	0.25	0.55	64
Cork	164	1	0.55	58
Alporas metal	200 ~ 250	1.3 ~ 1.7	0.7 ~ 0.82	60.3 ~ 83.4

Stress behind the shock front is derived in the following. Conservation of momentum across the shock front gives the compressive stress behind the shock σ_d as

$$\sigma_d = - \left(\sigma_{pl} + \frac{\rho_0}{\varepsilon_D v_d^2} \right) \quad (46)$$

Using Eq. (33), the stress behind the shock can be written as a function of the length of the crushed cellular rod x_d

$$\sigma_d = -\sigma_{pl} \left[1 + \left(\frac{v_m}{v_C} \right)^2 \right] \left(\frac{m}{m + \rho_D x_d} \right)^2 \quad (47)$$

Using Eq. (38) for x as a function of time t , and the nondimensionalized notation for impact velocity $\tilde{v} = v_m/v_C$ and time $\tilde{t} = t/(m\varepsilon_D/\rho_0 v_C)$, Eq. (47) provides

$$\frac{\sigma_d}{\sigma_{pl}} = - \frac{1 + \tilde{v}^2}{1 + 2\tilde{v}\tilde{t} - \tilde{t}^2} \quad (48)$$

Figure 9 shows the time evolution of the stress behind the shock front affected by impact velocity. In a dimensional form

$$\frac{\sigma_d}{\sigma_{pl}} = - \frac{\sigma_{pl}\varepsilon_D + \rho_0 v_{im}^2}{\varepsilon_D m^2 \sigma_{pl} + 2m v_{im} \rho_0 \sigma_{pl} t - \rho_0 \sigma_{pl} \sigma_{pl} t^2} m^2 \quad (49)$$

At time $t = 0$,

$$\sigma_d = - \left(\sigma_{pl} + \frac{\rho_0 v_{im}^2}{\varepsilon_D} \right), \quad (50)$$

which shows that higher densification strain and lower material density of the cellular rod can reduce the impact stress.

Stress on the striking object. Substitute the expression for x_d as a function of time into Eq. (25), one has

$$u = \frac{m\varepsilon_D}{\rho_0} \left(-1 + \sqrt{1 + 2\tilde{v}\tilde{t} - \tilde{t}^2} \right) \quad (51)$$

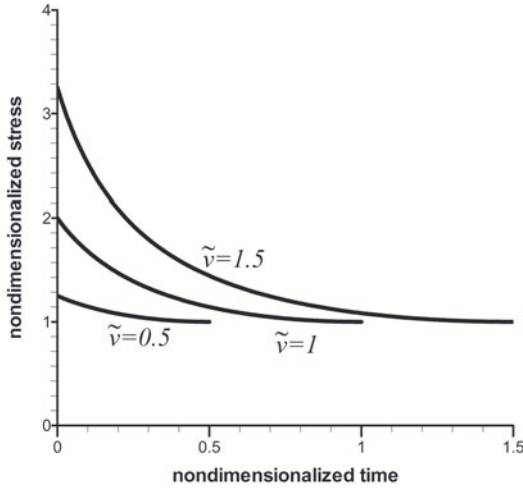


Figure 9. Nondimensionalized stress behind the shock front, σ_d/σ_{pl} , evolves as a function of nondimensionalized, $t/(\varepsilon_D m/\rho_0 v_C)$, for different impact velocities v_m at $0.5\sqrt{\varepsilon_D \sigma_{pl}/\rho_0}$, $\sqrt{\varepsilon_D \sigma_{pl}/\rho_0}$ and $1.5\sqrt{\varepsilon_D \sigma_{pl}/\rho_0}$, respectively

The stress on the mass can be calculated from

$$\sigma_m = m \frac{d^2 u}{dt^2}, \quad (52)$$

which gives

$$\frac{\sigma_m}{\sigma_{pl}} = - \frac{1 + \tilde{v}_m^2}{(1 + 2\tilde{v}_m \tilde{t} - \tilde{t}^2)^{3/2}} \quad (53)$$

In a dimensional form,

$$\frac{\sigma_m}{\sigma_{pl}} = - \frac{1 + \frac{\rho_0 v_{im}^2}{\varepsilon_D \sigma_{pl}}}{\left(1 + 2 \frac{\rho_0}{\varepsilon_D m} v_{im} t - \frac{\rho_0 \sigma_{pl}}{\varepsilon_D m^2} t^2\right)^{3/2}} \quad (54)$$

Figure 10 shows the time evolution of the stress acted on the mass affected by the impact velocity. At time $t = 0$,

$$\sigma_m = - \left(\sigma_{pl} + \frac{\rho_0 v_{im}^2}{\varepsilon_D} \right), \quad (55)$$

which shows that higher densification strain and lower material density of the cellular rod can reduce the impact stress.

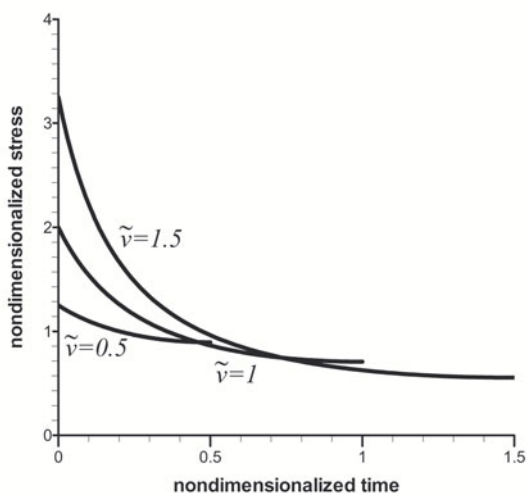


Figure 10. Nondimensionalized stress on the striking object, $-\sigma_m/\sigma_{pl}$, evolves as a function of nondimensionalized, $t/(\varepsilon_D m/\rho_0 v_C)$, for different impact velocities v_m at $0.5\sqrt{\varepsilon_D \sigma_{pl}/\rho_0}$, $\sqrt{\varepsilon_D \sigma_{pl}/\rho_0}$ and $1.5\sqrt{\varepsilon_D \sigma_{pl}/\rho_0}$, respectively

4 Rigid Object Strikes on a Free Cellular Rod

This section investigates the behavior of a cellular rod struck by a rigid object as shown in Fig. 11(a). The rod is homogeneous and made from a RPPL cellular material. The original length of the rod is L . The striking object has areal mass m and is at a high impact velocity v_{im} . After impact a shock wave is generated from the impacted end and moves to the opposite free end of the rod, as shown in Fig. 11(b). The shock front moves at velocity v_s . The material ahead of the shock wave (uncondensed) is treated as rigid that moves at velocity v_0 . The material behind the shock front has attained a strain ε_D and a particle velocity v_d ; its density has been raised from the initial value ρ_0 to the densification value ρ_D , and the compressive stress has been raised to σ_d . The length of the condensed cellular rod is denoted as x_d .

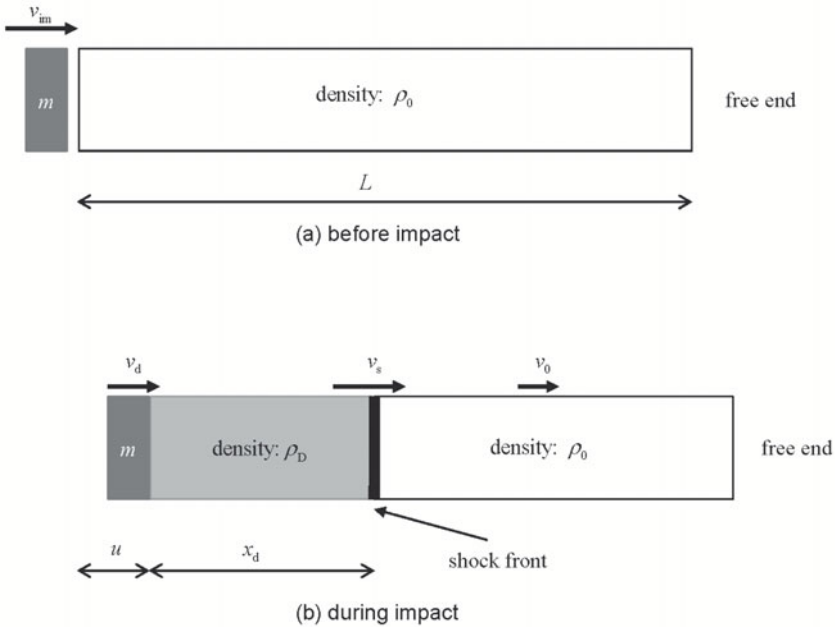


Figure 11. A free stationary cellular-rod struck by a rigid object of areal mass m and impact velocity v_m . Shock wave is generated from the impact end and propagates at a velocity v_s . The shock front separate the rod into two parts, denoted as state “0” (with initial density ρ_0 , zero stress and strain, and particle velocity v_0) and state “d” (with condensed strain ε_D , density ρ_D , and particle velocity v_d). During the impact, the displacement of the striker is u , and the length of the crushed rod is x_d

The usage of the subscripts in this section is in consistent with that in the last section. The capital letter “D” (such as that in ρ_D and ε_D) denotes material constants during the densification, and the small letter “d” (such as that in $x + d$, $v + d$ and σ_d) denotes variables that change during the densification. The subscript “0” includes both material constants (such as ρ_0) and variables (such as v_0) before densification.

The displacement of the striking object is u . The relation between the shock velocity v_s , condensed particle velocity v_d , and the increasing rate of the condensed length dx_d/dt can be written as

$$v_s = \frac{d(x_d + u)}{dt} = \frac{dx_d}{dt} + v_d \quad (56)$$

Conservation of mass across the shock front gives the relation between the material densities before and after densification as $\rho_0 = \rho_D(1 - \varepsilon_D)$.

A shock coordinate system that sits on the shock front can be established as in Fig. 12. In this reference system particles enter into the shock front at

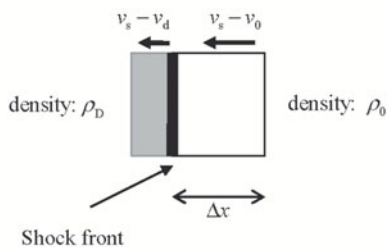


Figure 12. Local coordinate system at the shock front that travels with the shock wave. Viewing in this reference coordinate system, the particles move across the shock front at the speed of $v_s - v_0$ to the left, and comes out of the shock wave front at the speed of $v_s - v_d$

velocity $v_s - v_0$, and come out at velocity $v_s - v_d$. During a short time period Δt , an element of length $\Delta X = (v_s - v_0)\Delta t$ (shown as a white block in Fig. 12) and density ρ_D enters into the shock front. Meanwhile, an element of length $(v_s - v_d)\Delta t$ (shown as a gray block in Fig. 12) and density ρ_D moves out of the shock front. The conservation of mass across the shock front provides

$$\rho_D(v_s - v_d) = \rho_0(v_s - v_0),$$

which gives

$$v_s = \frac{v_d - v_0}{\varepsilon_D} + v_d \quad (57)$$

The mass of the foam behind the shock is $\rho_D x$, and the mass of the foam ahead the shock is $\rho_0 L - \rho_D x$. Conservation of momentum of the whole system that includes both impact object and the cellular rod gives

$$(m + \rho_D x_d)v_d + (\rho_0 L - \rho_D x_d)v_0 = mv_{\text{im}} \quad (58)$$

For the region ahead of the shock wave front, we have

$$\sigma_{\text{pl}} = (\rho_0 L - \rho_D x_d) \frac{dv_0}{dt} \quad (59)$$

From Eq. (56) to (59), a non-dimensional equation system that contains 3 equations for 3 time-varying variables, $\tilde{x}_d$, $\tilde{v}_d$ and $\tilde{v}_0$, can be derived as

$$\begin{aligned} (\tilde{m} + \tilde{x}_d)\tilde{v}_d + (1 - \tilde{x}_d)\tilde{v}_0 &= \tilde{m}, \\ \frac{d\tilde{x}_d}{d\tilde{t}} &= \tilde{v}_d - \tilde{v}_0, \quad \frac{d\tilde{v}_d}{d\tilde{t}} = \frac{1}{\eta(1 - \tilde{x}_d)}, \end{aligned} \quad (60)$$

where the mass coefficient

$$\tilde{m} = \frac{1}{\rho_0 L} m \quad (61)$$

is the ratio of the impact mass versus the foam mass; and the impact coefficient

$$\eta = \frac{\rho_0 v_{\text{im}}^2}{\sigma_{\text{pl}} \varepsilon_{\text{D}}} = \left(\frac{v_{\text{im}}}{v_{\text{C}}} \right)^2 \quad (62)$$

characterizes the impact strength, where the characteristic velocity of the cellular material v_{C} is defined in Eq. (35); the time is expressed in a non-dimensional form that combines the impact velocity v_{im} , the length of the foam L and densification strain ε_{D} as

$$\tilde{t} = \frac{v_{\text{im}}}{L \varepsilon_{\text{D}}} t; \quad (63)$$

and the 3 nondimensionalized time-varying variables ($\tilde{x}_d$, $\tilde{v}_d$ and $\tilde{v}_0$) are defined as

$$\tilde{x}_d = \frac{x_d}{(1 - \varepsilon_{\text{D}})L}, \quad \tilde{v}_d = \frac{v_d}{v_{\text{im}}}, \quad \tilde{v}_0 = \frac{\tilde{v}_0}{v_{\text{im}}} \quad (64)$$

The range for $\tilde{x}_d$ is $[0, 1]$ where $\tilde{x}_d = 0$ corresponds to the commencement for impact, and $\tilde{x}_d = 1$ corresponds to complete densification of the whole foam rod.

As an example, we use Alporas metal foams for the protection from car crashing. For this type of foams, the characteristic v_{C} is in the range 60.3 ~ 83.4 m/s. The impact velocity during the highway car crash is around 30 m/s. For this impact velocity, the impact coefficient η is in the range 0.13 ~ 0.25.

The relation between $\tilde{v}_0$ and $\tilde{x}_d$ is derived in the following. Equations (60) generate

$$\eta[\tilde{m} - \tilde{v}_0(1 + \tilde{m})]d\tilde{v}_0 = \left(\frac{1 + \tilde{m}}{1 - \tilde{x}_d} - 1 \right) d\tilde{x}_d$$

The integration gives

$$\tilde{m}\tilde{v}_0 = \frac{1}{2}\tilde{v}_0^2(1 + \tilde{m}) + \frac{1}{\eta}[\tilde{x}_d + (1 + \tilde{m})\ln(1 - \tilde{x}_d)] = \text{const}$$

With the initial condition

$$\tilde{v}_0 = 0 \quad \text{when} \quad \tilde{x}_d = 0,$$

the above equation generates

$$\frac{1}{2}(1 + \tilde{m})\tilde{v}_0^2 - \tilde{m}\tilde{v}_0 - \frac{1}{\eta}[\tilde{x}_d + (1 + \tilde{m})\ln(1 - \tilde{x}_d)] = 0$$

Solving the above equation for $\tilde{v}_0$ one has

$$\tilde{v}_0 = \frac{1}{1 + \tilde{m}} \left[\tilde{m} - \sqrt{\tilde{m}^2 + \frac{2}{\eta}(1 + \tilde{m})[\tilde{x}_d + (1 + \tilde{m})\ln(1 - \tilde{x}_d)]} \right] \quad (65)$$

The relation between $d\tilde{x}_d/d\tilde{t}$ and $\tilde{x}_d$ is derived in the following. Eq. (60) gives the relation

$$\frac{d\tilde{x}_d}{d\tilde{t}} = \frac{\tilde{m} - (1 - \tilde{m})\tilde{v}_0}{\tilde{m} + \tilde{x}_d} \quad (66)$$

Substitution of the expression for $\tilde{v}_0$ into the above equation generates

$$\frac{d\tilde{x}_d}{d\tilde{t}} = \frac{1}{\tilde{m} + \tilde{x}_d} \sqrt{\tilde{m}^2 + \frac{2}{\eta}(1 + \tilde{m})[\tilde{x}_d + (1 + \tilde{m})\ln(1 - \tilde{x}_d)]} \quad (67)$$

The above equation gives the time evolution of the length of the crushed cellular rod.

Figure 13 shows the time evolution of $\tilde{x}_d$ for several impact coefficients at $\eta = 0.5, 1, 2$ and 4 . When both condensed part and uncondensed part of the cellular rod reach the same particle velocity, i.e.,

$$\tilde{v}_d = \tilde{v}_0, \quad (68)$$

shock wave disappears. This is termed as shock arrest. The arrest distance (nondimensionalized) to the shock wave, which is the non-dimensional length of crushed foam at this moment, is denoted as $\tilde{x}_d^{\text{arrest}}$. Using Eq. (60), the nondimensionalized arrest distance $\tilde{x}_d^{\text{arrest}}$ can be determined from

$$\frac{d\tilde{x}_d}{d\tilde{t}} = 0 \quad (69)$$

Further using Eq. (67), $\tilde{x}_d^{\text{arrest}}$ can be determined as the root of the following equation

$$F(\tilde{x}_d) = \tilde{x}_d + (1 + \tilde{m})\ln(1 - \tilde{x}_d) + \frac{\eta\tilde{m}^2}{2(1 + \tilde{m})} = 0 \quad (70)$$

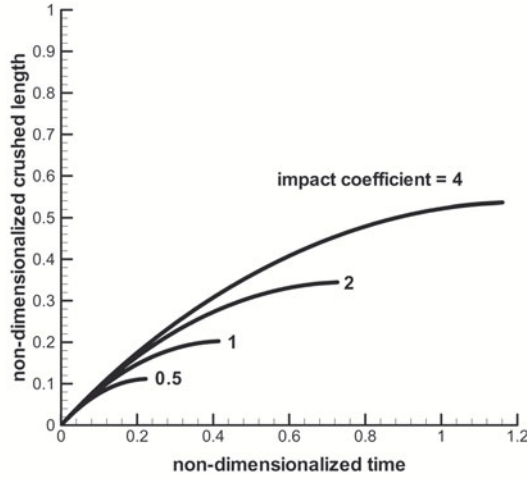


Figure 13. Nondimensionalized length of the crushed cellular rod, $\tilde{x}_d = x/(1 - \varepsilon_D)L$, as a function of nondimensionalized time, $\tilde{t} = v_m t/L\varepsilon_D$, for several impact coefficients η at 0.5, 1, 2 and 4, respectively. The mass coefficients is set at $\tilde{m} = 1$

For the function $F(\tilde{x}_d)$, the derivative is $dF/d\tilde{x}_d = -(\tilde{x}_d + \tilde{m})/(1 - \tilde{x}_d)$. In the range $(0,1)$, $F(0) > 0$, $F(1) < 1$ and $dF/d\tilde{x}_d < 0$. So there exists a root $\tilde{x}_d^{\text{arrest}}$ for the equation $F(\tilde{x}_d) = 0$, which corresponds to the $d\tilde{x}_d/d\tilde{t} = 0$, i.e., the end for the shock wave. Figure 14 shows the relation between the nondimensionalized arrest distance $\tilde{x}_d^{\text{arrest}}$ and the impact coefficient η for several mass coefficients at $\tilde{m} = 0.5, 1$ and 2 .

Stress on the striking object. The stress on the striking object σ_m can be calculated from

$$\sigma_m = m \frac{dv_d}{dt} \quad (71)$$

The non-dimensional form can be written as

$$\frac{\sigma_m}{\sigma_{pl}} = \eta \tilde{m} \frac{d\tilde{v}_d}{d\tilde{t}} \quad (72)$$

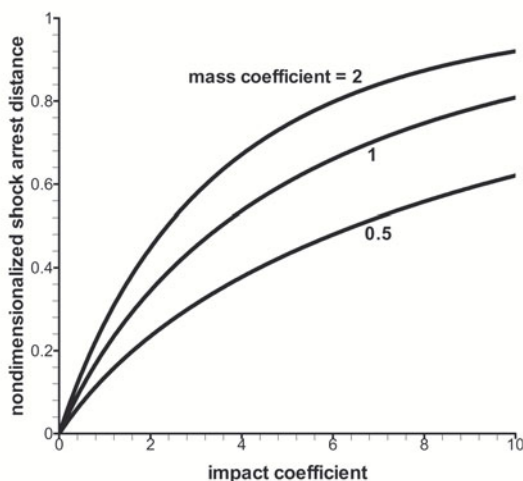


Figure 14. Relation between the nondimensionalized arrest distance $\tilde{x}_d^{\text{arrest}}$ and the impact coefficient η for several mass coefficients $\tilde{m}$: 0.5, 1 and 2, respectively

Since $\tilde{v}_d = (d\tilde{x}_d/d\tilde{t}) + \tilde{v}_0$, substitution of the expression for $d\tilde{x}_d/d\tilde{t}$ and $\tilde{v}_0$ gives

$$\begin{aligned} \tilde{v}_d &= \left(\frac{1}{\tilde{m} + \tilde{x}_d} + \frac{1}{1 + \tilde{m}} \right) \sqrt{\tilde{m}^2 + \frac{2}{\eta}(1 + \tilde{m})[\tilde{x}_d + (1 + \tilde{m})\ln(1 - \tilde{x}_d)]} \\ &+ \frac{\tilde{m}}{1 + \tilde{m}} \end{aligned} \quad (73)$$

The derivative of $\tilde{v}_d$ over time t gives

$$\frac{d\tilde{v}_d}{d\tilde{t}} = -\frac{1}{(\tilde{m} + \tilde{x}_d)^2} \sqrt{\tilde{m}^2 + \frac{2}{\eta}(1 + \tilde{m})[\tilde{x}_d + (1 + \tilde{m})\ln(1 - \tilde{x}_d)]} - \frac{1}{\eta} \frac{1}{\tilde{m} + \tilde{x}_d} \quad (74)$$

Therefore,

$$-\frac{\sigma_m}{\sigma_{pl}} = \frac{\tilde{m}}{\tilde{m} + \tilde{x}_d} + \frac{\tilde{m}}{(\tilde{m} + \tilde{x}_d)^2} \sqrt{\eta^2 \tilde{m}^2 + 2\eta(1 + \tilde{m})[\tilde{x}_d + (1 + \tilde{m})\ln(1 - \tilde{x}_d)]} \quad (75)$$

At the commencement of the impact, $\tilde{x}_d = 0$, the stress on the striking object is $\sigma_m = -(1 - \eta)\sigma_{pl}$. Substitute the expression of the impact coefficient η , the stress on the striking object is

$$\sigma_m = - \left(\sigma_{pl} + \frac{\rho_0 v_{im}^2}{\varepsilon_D} \right)$$

Higher densification strain and lower material density of the cellular rod reduce impact stress.

Stress behind the shock front. Shock wave brings an abrupt raise of the stress at the shock front. During the time period Δt the impulse acting on the mass element ΔX in Fig. 12 is $(-\sigma_d - \sigma_{pl})$, which changes the velocity of the mass element $\rho_0 \Delta x$ from v_0 to v_d . The impulse-momentum equation gives

$$(-\sigma_d - \sigma_{pl})\Delta t = \rho_0 \Delta x (v_d - v_0)$$

Remove Δt from the two sides of the equation, one has

$$-\sigma_d - \sigma_{pl} = \rho_0 (v_s - v_0)(v_d - v_0),$$

which can be further written as

$$-\sigma_{pl} = \sigma_d + \rho_0 \frac{(v_d - v_0)^2}{\varepsilon_D} \quad (76)$$

The stress behind the shock front is compressive, hence the negative sign of σ_d . The nondimensionalization of the above equation gives

$$-\frac{\sigma_d}{\sigma_{pl}} = 1 + \eta(\tilde{v}_d - \tilde{v}_0)^2, \quad (77)$$

which relates to the length change rate of crushed cellular rod as

$$-\frac{\sigma_d}{\sigma_{pl}} = 1 + \eta \left(\frac{d\tilde{x}_d}{d\tilde{t}} \right)^2 \quad (78)$$

Substituting the expression for $d\tilde{x}_d/d\tilde{t}$ gives

$$-\frac{\sigma_d}{\sigma_{pl}} = 1 + \frac{1}{(\tilde{m} + \tilde{x}_d)^2} \{ \eta \tilde{m}^2 + 2(1 + \tilde{m})[\tilde{x}_d + (1 + \tilde{m}) \ln(1 - \tilde{x}_d)] \} \quad (79)$$

At the commencement of the impact, $\tilde{x}_d = 0$, the stress behind the shock front is $\sigma_d = -(1 + \eta)\sigma_{pl}$. This is the same as the stress on the striking object. After impact the two stresses becomes different.

5 Concluding Remarks

Cellular materials are used as impact energy absorbers due to their large densification strain ε_D at the plateau stress σ_{pl} during the plastic compression.

For a cellular rod struck by a object of areal mass m , the critical impact velocity is determined as

$$v_{im}^c = \frac{\sigma_{pl}}{\rho_0 c_{el}},$$

where ρ_0 is the density of the cellular material before. If the impact velocity is slower than the critical impact velocity v_{im}^c , only elastic wave will be generated from the impact end. If the impact velocity is higher than v_{im}^c , the elastic wave will be followed by plastic shock waves.

To fully absorb the kinetic energy of a striking object with mass M_{impact} and impact velocity v_{im} , the mass of the protecting rod should be higher than

$$M_{impact} \left[\sqrt{1 + \left(\frac{v_{im}}{v_c} \right)^2} - 1 \right],$$

where $v_c = \sqrt{\varepsilon_D \sigma_{pl} / \rho_0}$ is the characteristic velocity of the cellular material. A fix-ended cellular rod can fully arrest a striking object of mass M_{impact} with impact velocity up to

$$v_c \sqrt{1 + \left(\frac{M_{cellular}}{M_{impact}} \right)^2} - 1,$$

At the commencement of the impact the stress on the striking object is

$$- \left(\sigma_{pl} + \frac{\rho_0 v_{im}^2}{\varepsilon_D} \right)$$

Higher densification strain and lower material density of the cellular rod can reduce the impact stress.

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