

MECHANICS OF ELASTIC STABILITY

L.M. Kachanov

Delamination Buckling of Composite Materials

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DELAMINATION BUCKLING OF COMPOSITE MATERIALS

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Delamination Buckling of Composite Materials

by

L.M. KACHANOV

Brookline, Massachusetts, U.S.A.



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PREFACE OF THE EDITOR

It is with great pleasure that I present to the public the masterful treatise of Professor L.M. Kachanov on delamination buckling of composite materials. As the author has pointed out so rightly, an application of composite materials becomes more and more common in many branches of engineering practice. Therefore, it is essential that designers are aware of possible failure modes of such materials and understand their mechanisms. One of these failure modes is delamination buckling, and Professor L.M. Kachanov presents its theory in this book in his usual ingenious way.

Thanks are due to Mrs. G. Keszeli, who devotedly and with much expertise did the wordprocessing of the manuscript. I also owe Mr. H.G. van Dorssen, Publisher in Kluwer Academic Publishers, Dordrecht, The Netherlands, many thanks for his never failing support of the production of the book. May it be received warmly by the academic and engineering community.

Waterloo, Ontario, Canada, March 1988

H. Leipholz

PREFACE OF THE AUTHOR

Considerable advances have been made lately in manufacturing light and high strength fiber-reinforced materials (such as graphite-epoxy and boron-epoxy composites). Due to their high specific strength, fiber-reinforced composites are widely used in thin-walled aircraft structures.

These materials work well under tension; in the zones of *compression*, however, delamination buckling can occur (most often, in the surface layer). This phenomenon significantly reduces resistance of structural elements to compression; *growth* of the delamination zone may lead to failure of the entire structure.

The problems of delamination buckling can be both linear and nonlinear (snap delamination buckling is an example of the latter).

This work is a brief introduction to the subject. A general approach to delamination buckling and some special problems are presented.

The author wishes to express his thanks to Professor H.H.E. Leipholz for many valuable remarks on the manuscript.

Brookline, MA., U.S.A., 1987

L.M. Kachanov

Chapter 1

INTRODUCTION

1.1 On Delamination of Laminated Composites

(a) *Fiber-Reinforced Composites*

Considerable technological advances in the production of high-strength fibers (graphite, boron, etc.) have led to a wide use of light high-strength composite materials (graphite-epoxy, boron-epoxy, etc.). It is expedient, to make thin-walled composite rods, plates, and shells from such materials.

Plates can be made by bonding a set of unidirectional thin fiber layers, Fig.1.1. Such plates are orthotropic, as a rule.

A random short-fiber composite is shown in Fig.1.2. Fiber-reinforced composites are widely used in thin-walled aircraft structures because of their specific high strength. For example, the graphite-epoxy composite is characterized by a unidirectional tensile strength of

$$1.4 \text{ GPa}$$

while the density is

$$1.6 \text{ Mg/m}^3 .$$

For comparison, we may take a steel (steel 4340) whose corresponding properties are identified by values like

$$1.2 \text{ GPa}$$

and

$$7.8 \text{ Mg/m}^3 .$$

1. INTRODUCTION

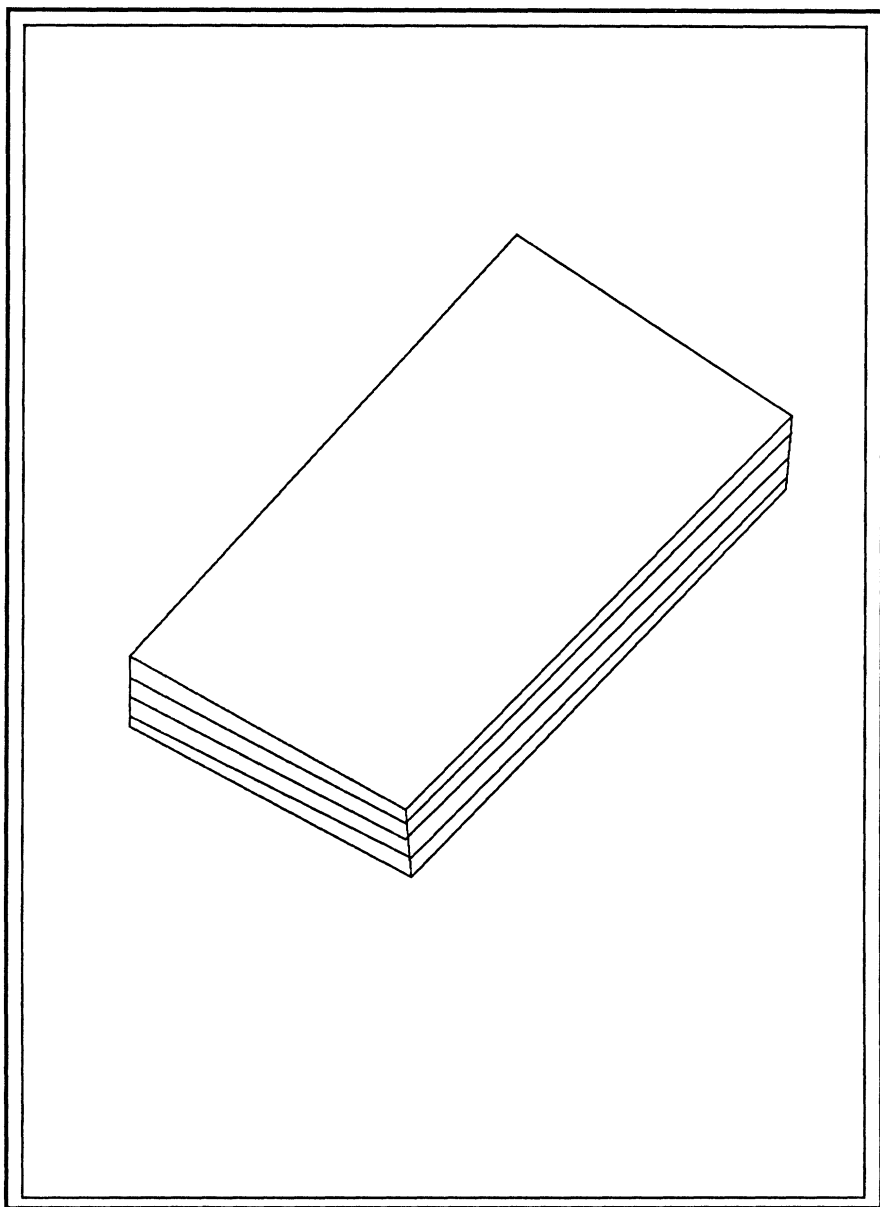


Figure 1.1

1.1. On Delamination of Laminated Composites

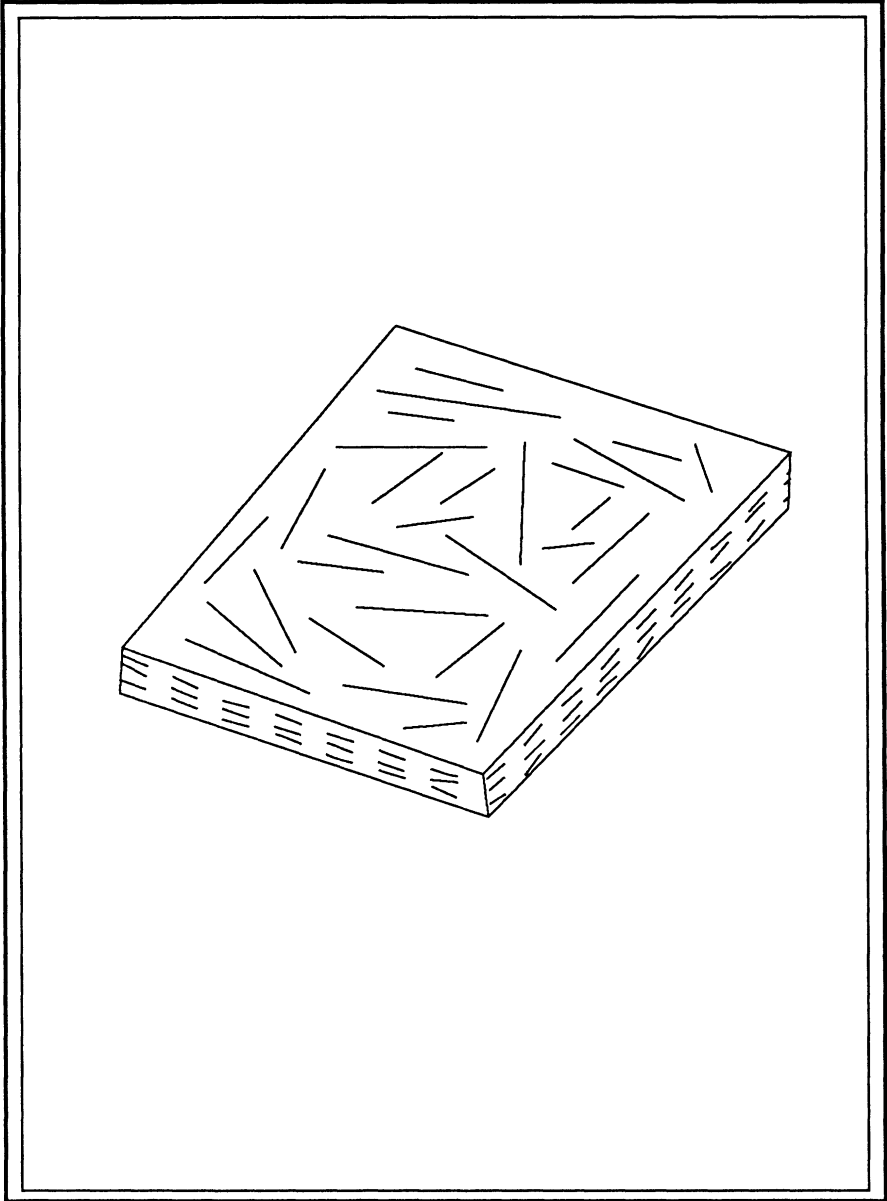


Figure 1.2

1. INTRODUCTION

It is characteristic for laminated plastic material to possess a fairly low bonding. Therefore, low-velocity impacts and defects in manufacturing lead to local delamination.

(b) *Linear Problems of Delamination Buckling*

Delamination can significantly reduce the compressive strength and stiffness of the laminate. Local delamination can be considered as a crack in the bond. Under buckling there appears a high interlaminar stress at the crack edge that leads to a spreading of the crack.

Delamination growth can lead to structural instability.

As it is the surface layer that in most cases is damaged, it is natural to consider delamination buckling of a thin surface layer.

Note that in the case of high compressive stress delamination buckling can occur without initial delamination; and in this case buckling is accompanied by an instantaneous fracture of the interlaminar bond.

The problem of buckling of a thin laminated surface layer (its thickness is denoted by h , Fig.1.3) can be considered as a classical linear problem of buckling of a strip with fixed ends. Using the well-known solution of the problem of buckling of a rod with fixed ends and turning to the case of the cylindrical bending of a strip (when $\varepsilon_z = 0$) we obtain the critical stress

$$\sigma_c = \frac{\pi^2 E}{3(1 - \nu^2)} \left(\frac{h}{\ell} \right)^2 ,$$

where E is the Young's modulus and ν is Poisson's ratio.

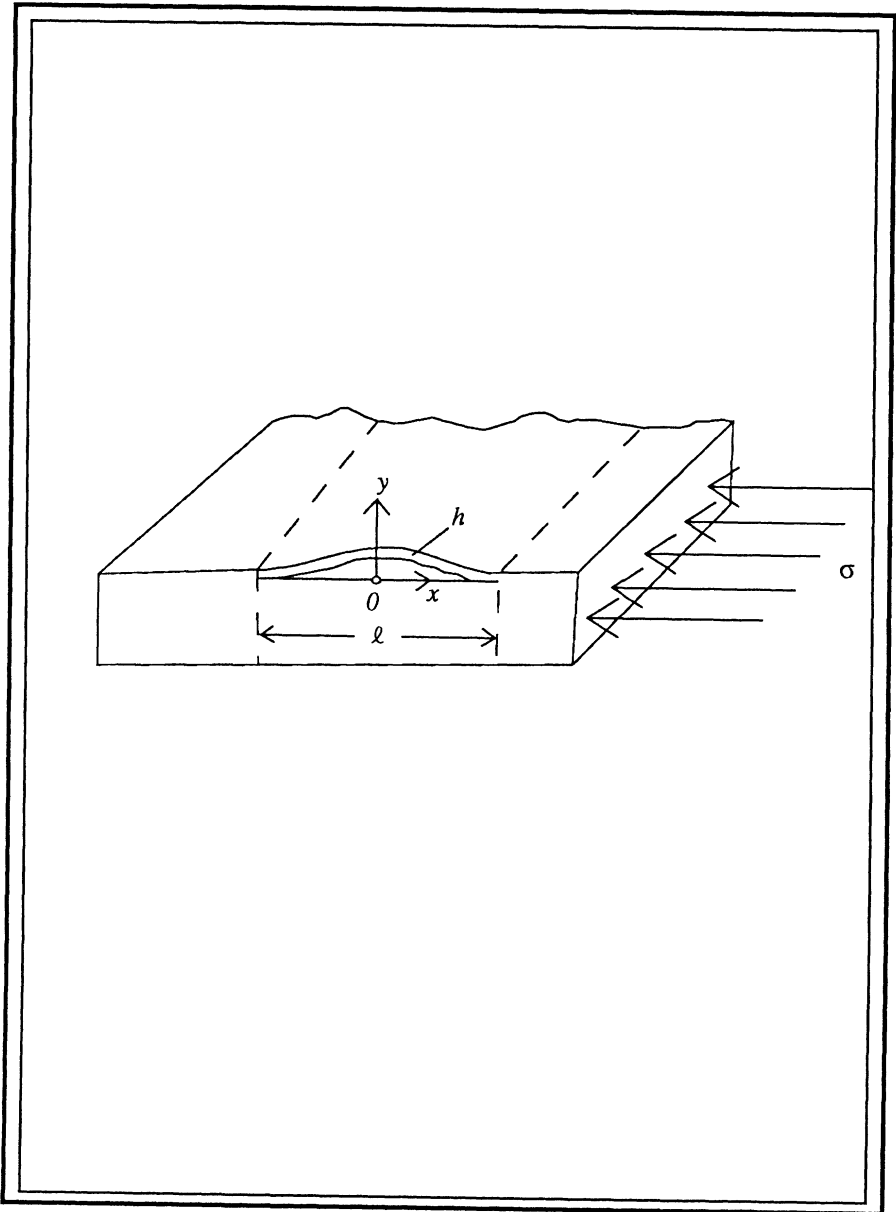


Figure 1.3

1. INTRODUCTION

The buckling deflection of the strip is

$$w = \frac{l}{2} w_0 \left(1 + \cos \frac{2\pi x}{l} \right) ,$$

where w_0 is the deflection at the middle of the strip, i.e. at $x=0$; the value w_0 can be found from the condition that the length contraction due to bending

$$\Delta l_2 = \frac{l}{2} \int_{-l/2}^{l/2} \left(\frac{dw}{dx} \right)^2 dx$$

is equal to the length contraction due to compression before buckling

$$\Delta l_1 = \varepsilon_c l ; \quad \varepsilon_c = \frac{l}{E} \sigma_c .$$

In other words, the length does not change in buckling. Thus,

$$w_0^2 = \frac{4}{\pi^2} \varepsilon_c l^2 .$$

Some other linear delamination buckling problems (for example, the buckling of a circular or elliptical delamination layer of a compressed plate) can be considered by methods of the theory of elastic stability.

1.1. On Delamination of Laminated Composites

Note that local zones of high compressive stress can appear in a stretched plate. So, in the plate with a circular hole, Fig.1.4 (a), at points C , we have the compressive stress $-p$, where p is the tensile stress at infinity. In the middle of the crack, Fig.1.4 (b), the compressive stress is also equal to $-p$. At the mentioned points, we can expect to find local delamination buckling.

(c) *Snap Buckling Problems*

In the linear problems discussed above the equilibrium form after buckling is close to the equilibrium form before buckling. But in some cases transition to a new equilibrium state occurs by a snap, and the new form differs essentially from the initial one. A simple example of snap delamination buckling of a circular ring (tube) under external pressure is shown in Fig.1.5. Under the critical pressure p_c the internal layer snaps in. This problem will be analyzed subsequently (sect.2.4).

(d) *Basic Concepts of Mechanics of Brittle Fracture*

As was mentioned above, delamination can be considered as a crack in the bond. From this point of view spreading of delamination can be interpreted as a crack growth. Hence, it is expedient to use here methods of mechanics of brittle fracture. Consider briefly its basic concepts for the case of plane problem: If a crack grows, i.e. its length ℓ , Fig.1.6, obtains an increment $d\ell$, the potential energy Π of the body decreases; so, there is an energy release. The potential energy Π is equal to the difference between elastic strain energy U of the body and the work A of the external load, i.e.

$$\Pi = U - A .$$

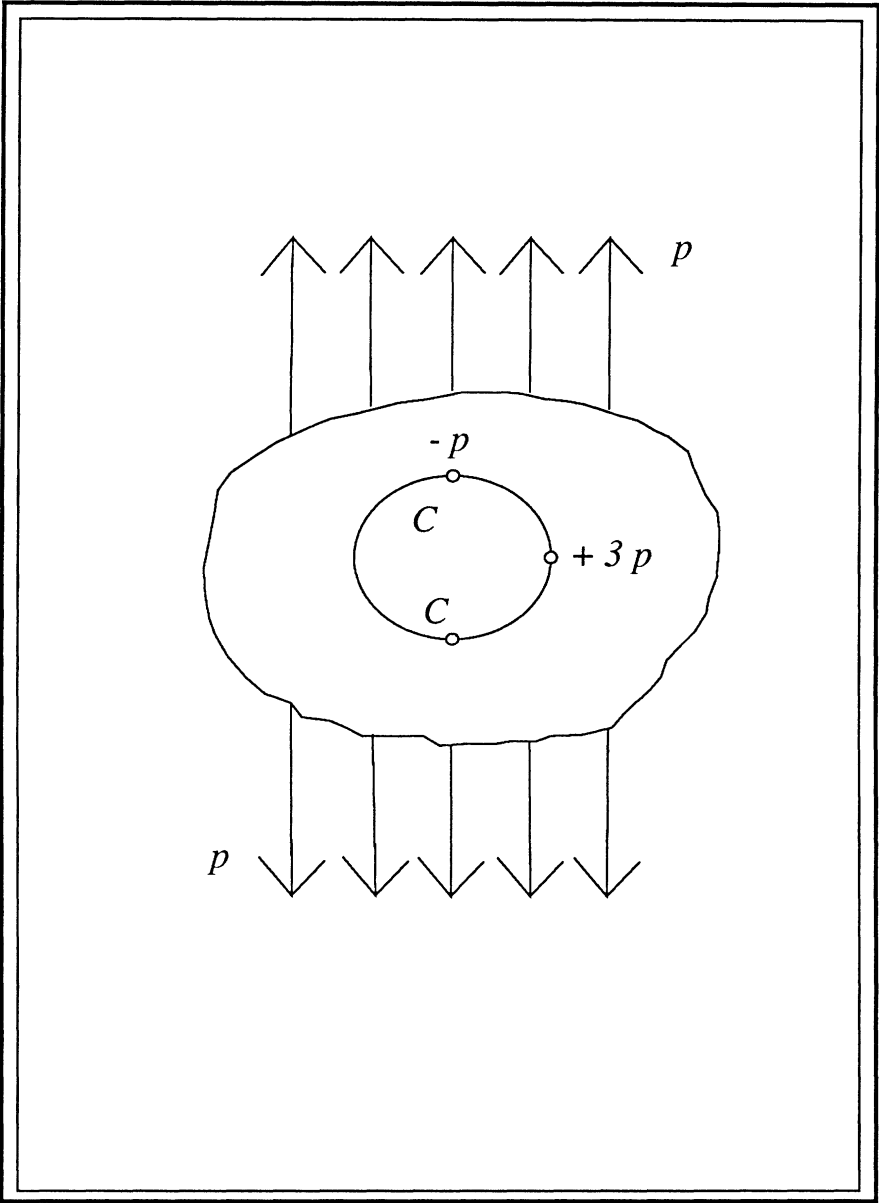


Figure 1.4 (a)

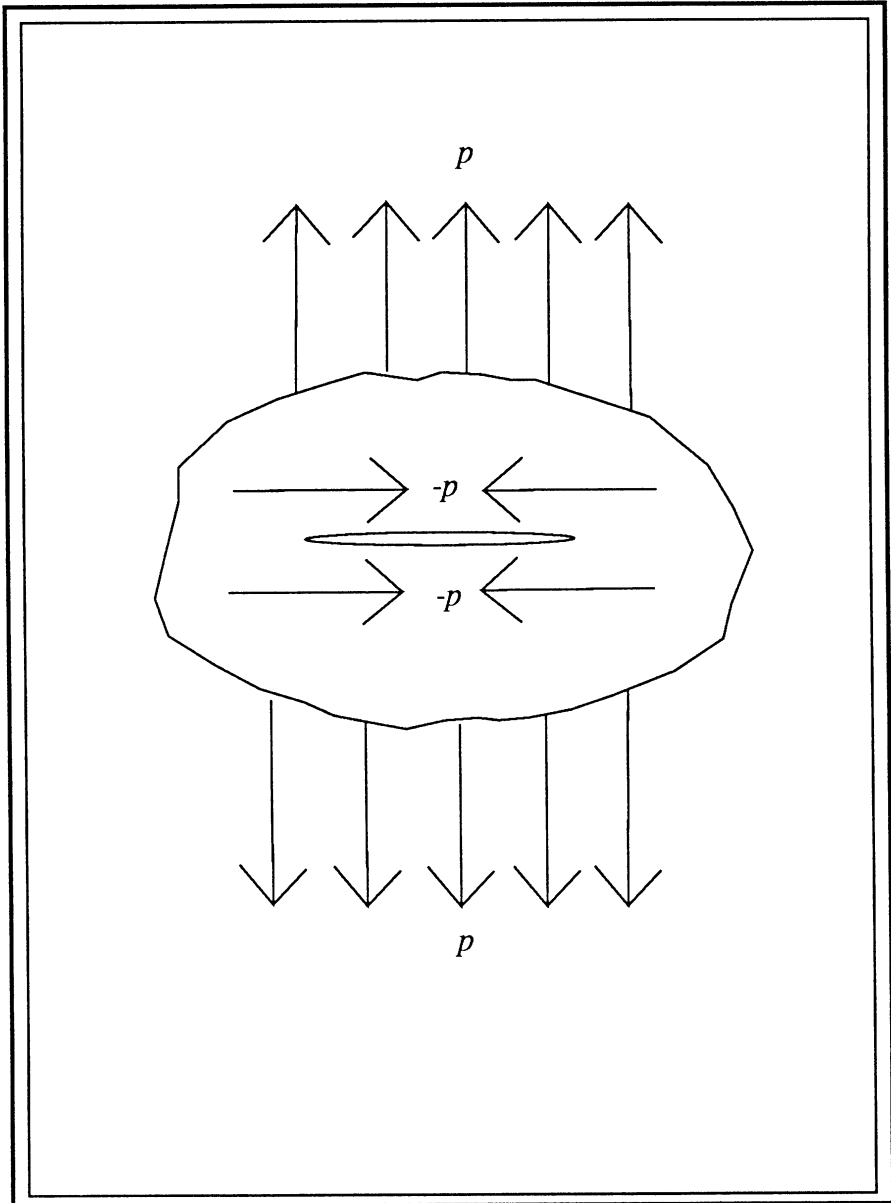


Figure 1.4 (b)

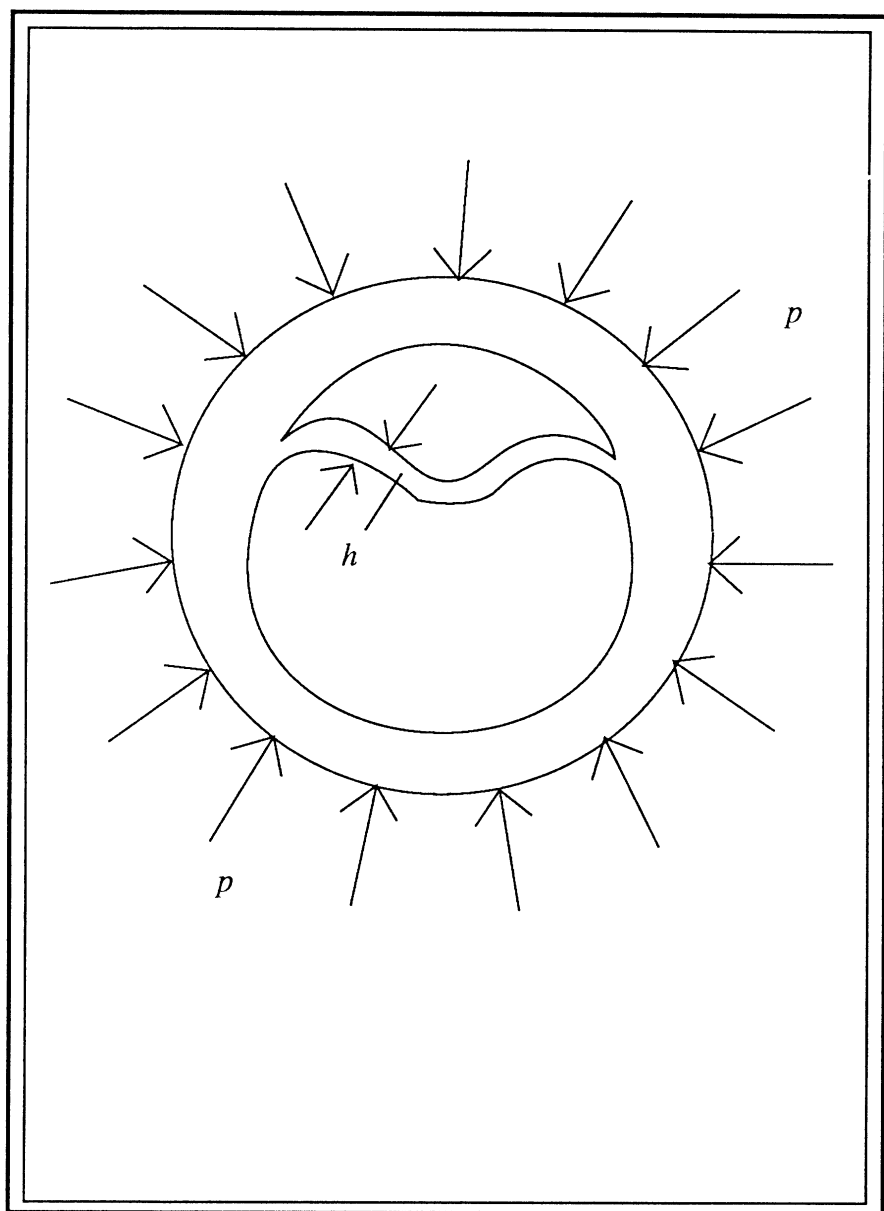


Figure 1.5

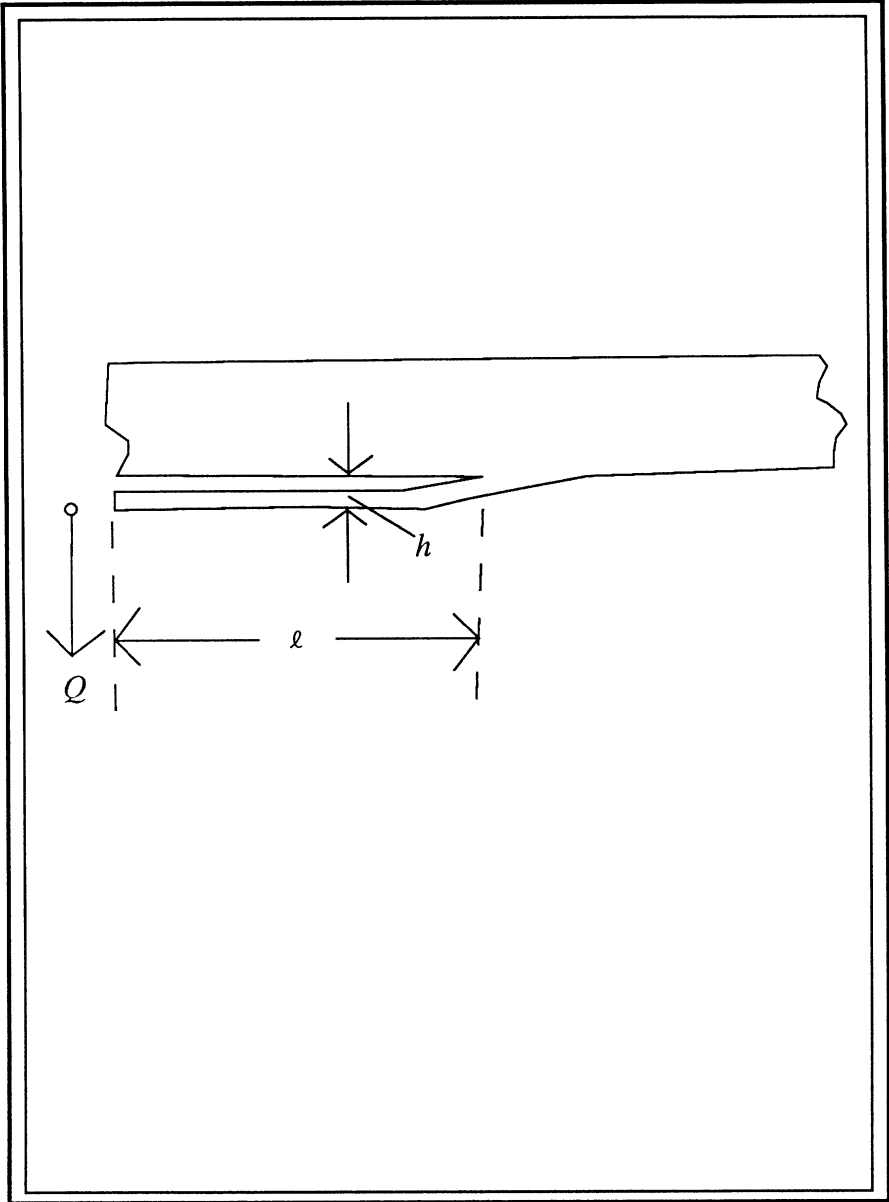


Figure 1.6

1. INTRODUCTION

The stress state in the vicinity of the crack tip is characterized by the singularity of the type

$$k/\sqrt{r} \quad (r \rightarrow 0)$$

where r is the distance from the crack tip, and k is the stress intensity factor. The latter depends on the form of the body and the form of the crack, on the load, acting on the body, and on conditions of its fixation.

According to Griffith's theory of brittle fracture, the released energy flows into the crack tip and is expended there on fracture. Let us denote by dW the work of fracture corresponding to the increment $d\ell$; note that

$$dW = 2\gamma d\ell ,$$

where γ is the surface energy (energy required to form a unit of a new surface in the body).

The Griffith's energy criterion has the form

$$d\Pi = dW = 2\gamma d\ell . \quad (1.1)$$

It is well-known [1] that the released energy is equal to

$$J = - \frac{\partial \Pi}{\partial \ell} . \quad (1.2)$$

J can be represented by a path independent integral (J - integral). When the released energy reaches the critical value J_c essential for the formation of a new surface, the crack starts growing. Thus, the condition of crack propagation (i.e. - delamination in our problem) is

$$J = J_c . \quad (1.3)$$

1.1. On Delamination of Laminated Composites

If the thickness h of a laminated strip is small as compared to its length, it is appropriate to use the so-called beam approach, first used by Obreimov [2]. Then, the strip can be considered as a thin rod and the potential energy can be calculated by applying the elementary theory of bending.

Consider a simple example of delamination of a strip (length ℓ thickness h) under the action of the force Q (per unit width), Fig.1.6. The work A is

$$A = Qq ,$$

where

$$q = \frac{Q \ell^3}{3 D}$$

is the end displacement and $D = \frac{Eh^3}{12(1-\nu^2)}$ is the flexural rigidity.

The strain energy per unit width is

$$U = \frac{1}{2} Qq .$$

The total potential energy is

$$\Pi = U - A = - \frac{Q^2 \ell^3}{6D} .$$

The released energy is

$$J = - \frac{\partial \Pi}{\partial \ell} = \frac{Q^2 \ell^2}{2D} .$$

The condition of delamination growth is therefore

$$\frac{Q^2 \ell^2}{2D} = J_c .$$

The material constant J_c is determined by special tests.

1.2 Delamination of a Fiber-Glass Tube Due to Residual Stresses

(a) *Statement of the Problem*

In the process of production of fiber-glass tubes there appear tensile radial stresses in the tube's wall. These stresses cause delamination along a cylindrical surface. Sometimes the delamination area is large. Consider [3] the condition of delamination following the Griffith's theory of fracture.

The tube is thin-walled. The ratio of the thickness $2h=b-a$ to the mean radius $c = \frac{1}{2}(b+a)$ is less than $0.1 \sim 0.2$; b and a are the external and internal radii respectively.

(b) *Solution*

It is natural to assume that delamination takes place along the surface where the stress σ_r is maximal. Denote the maximum by p and assume that it is reached at the mean radius c ; this condition is not essential for the analysis but it simplifies the latter. Delamination is assumed to be sufficiently long in the axial direction; hence, we can consider the problem as a plane one.

We will apply here the method of superposition usually employed in the theory of cracks. Instead of considering a crack in the stress field σ_r we will consider opening of a crack by stresses p applied to the crack's edges, Fig.1.7. Superposing this state and the basic state of the tube, we receive the actual state of the delaminated tube.

The ratio of the thickness h to the crack length $2c\theta$ is small; hence, we can use the beam approach.

The part of the tube above the crack ("arch" 1) is stretched under the action of pressure p . As a result of this, there

1.2. Delamination of a Fiber-Glass Tube Due to Residual Stresses

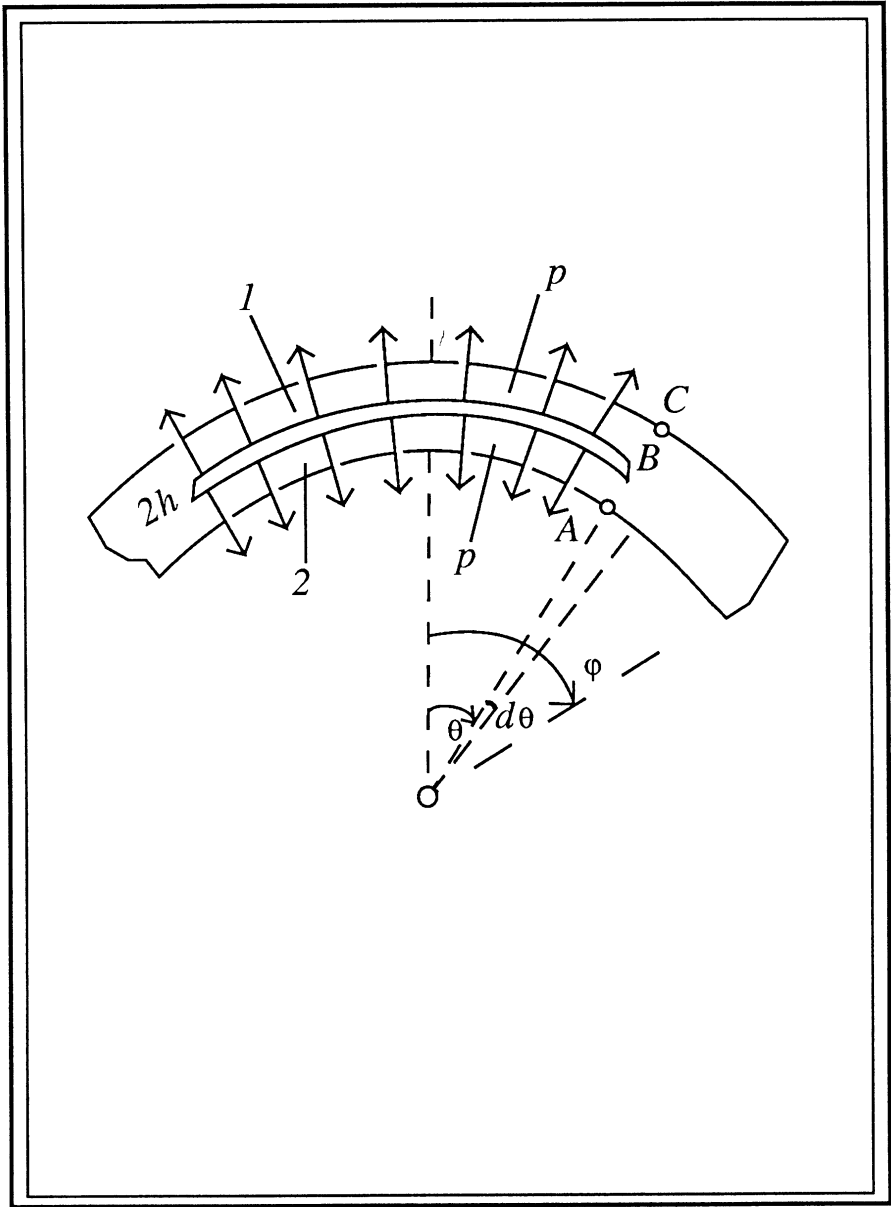


Figure 1.7

1. INTRODUCTION

develops a tensile force $N=pc$ in the section BC. The part below the crack is compressed which results in the appearance of compressive force $-N$ in the section AB. These forces form the moment Nh which bends the tube and changes the slope of the section ABC. Caused by this change, there appears the moment M which bends arches 1, 2.

The part $\varphi > \theta$ is bent by the moment $Nh - 2M$.

The moment M is determined by the condition of compatibility of deformations of the arches 1,2 and the remaining part of the section of the tube.

Let us denote the strain energy of the arches 1,2 by U_1, U_2 and by V the strain energy of the remaining part of the section.

Since the mean radii of the arches 1,2 are close to each other we can assume that $U_1 \approx U_2 = U$ and that the mean radius of each arch is equal to c . The normal stress in the arch is

$$\sigma_{\varphi} = \frac{N}{h} - \frac{My}{I} , \quad (1.4)$$

where the distance y is reckoned from the neutral line, and $I=1.h^3/12$ is the corresponding moment of inertia of the arch (per unit of width).

Calculating, we find

$$U = \frac{c\theta}{2Eh} \left(p^2 c^2 + \frac{12M^2}{h^2} \right) ; \quad V = \frac{c(\pi - \theta)}{2Eh^3} (pch - 2M)^2 . \quad (1.5)$$

According to Castigliano's theorem,

$$\frac{\partial}{\partial M} (2U + V) = 0 .$$

1.2. Delamination of a Fiber-Glass Tube Due to Residual Stresses

Thus, we obtain

$$M = \frac{pch}{2} \frac{\pi - \theta}{\pi + 3\theta} .$$

Let the incremental crack growth be $cd\theta$. The external forces (pressure p) will do work dA . Strain energy of the tube $\bar{U} = 2U + V$ will increase by $d\bar{U}$. Besides, fracture of the element $cd\theta$ requires the work dW .

The energy balance is

$$dA = d\bar{U} + dW .$$

According to Clapeyron's theorem,

$$A = 2\bar{U} .$$

The fracture work per unit of width is

$$W = 2\gamma c\theta .$$

Griffith's criterion of fracture leads to the relation

$$\frac{\partial}{\partial c\theta} (\bar{U} - A) = 2\gamma . \quad (1.6)$$

Calculation yields the critical stress

$$\frac{p_*}{p_0} = 1 + 3 \frac{\theta}{\pi} , \quad (1.7)$$

where

$$p_0 = \frac{1}{c} \left(\frac{1}{2} E h \gamma \right)^{1/2} .$$

1. INTRODUCTION

Thus, the crack in the tube is stable; it increases with the increase of stress p . The crack starts forming when $p \geq p_0$.

The constant γ can be determined by special tests on tearing strips of laminated material.

Such tests, for example, were made by Gurney and Hunt [11].

Chapter 2

DELAMINATION BUCKLING

2.1 Two Types of Delamination Buckling

(a) *Linear Problems*

Delamination of a thin strip (thickness h) from a uniaxially compressed plate is shown in Fig.2.1 (a).

Delamination of a strip from a concave surface in the case of bending is shown in Fig.2.1. (b).

In the first case (type I) the transition to a new equilibrium state close to the initial one takes place.

In the second case (type II), there is a snap through to a new equilibrium state not close to the initial one.

Now, consider in detail an example of the first case. The strip, Fig.2.2 is subjected to the uniaxial compressive stress σ in conditions of plane strain ($\varepsilon_z = 0$). The critical stress is

$$\sigma_c = \frac{\pi^2 E}{3(1 - \nu^2)} \left(\frac{h}{l} \right)^2 . \quad (2.1)$$

The corresponding critical strain is

$$\varepsilon_c = \frac{1 - \nu^2}{E} \sigma_c . \quad (2.2)$$

The buckling deflection is given by

$$w = \frac{l}{2} w_0 \left(1 + \cos \frac{2\pi x}{l} \right) , \quad (2.3)$$

2. DELAMINATION BUCKLING

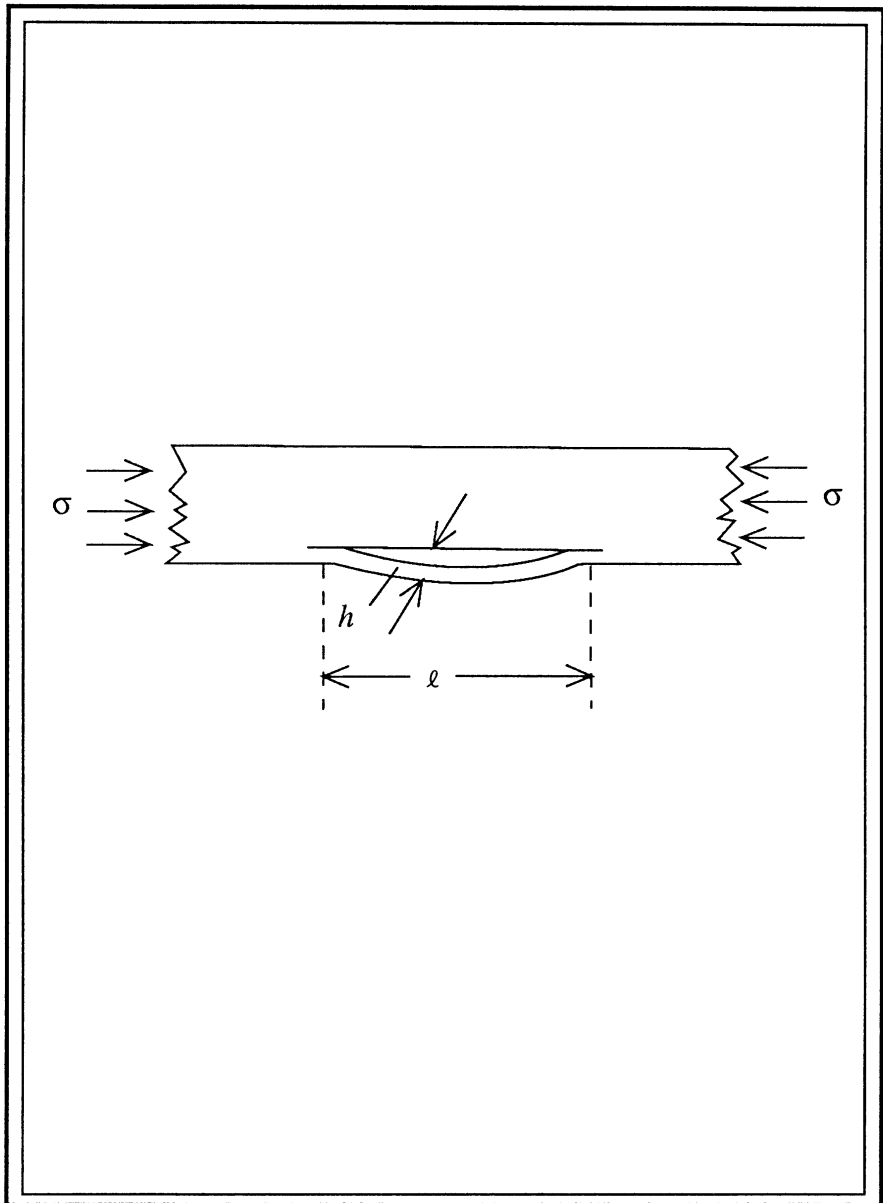


Figure 2.1 (a)

2.1. Two Types of Delamination Buckling

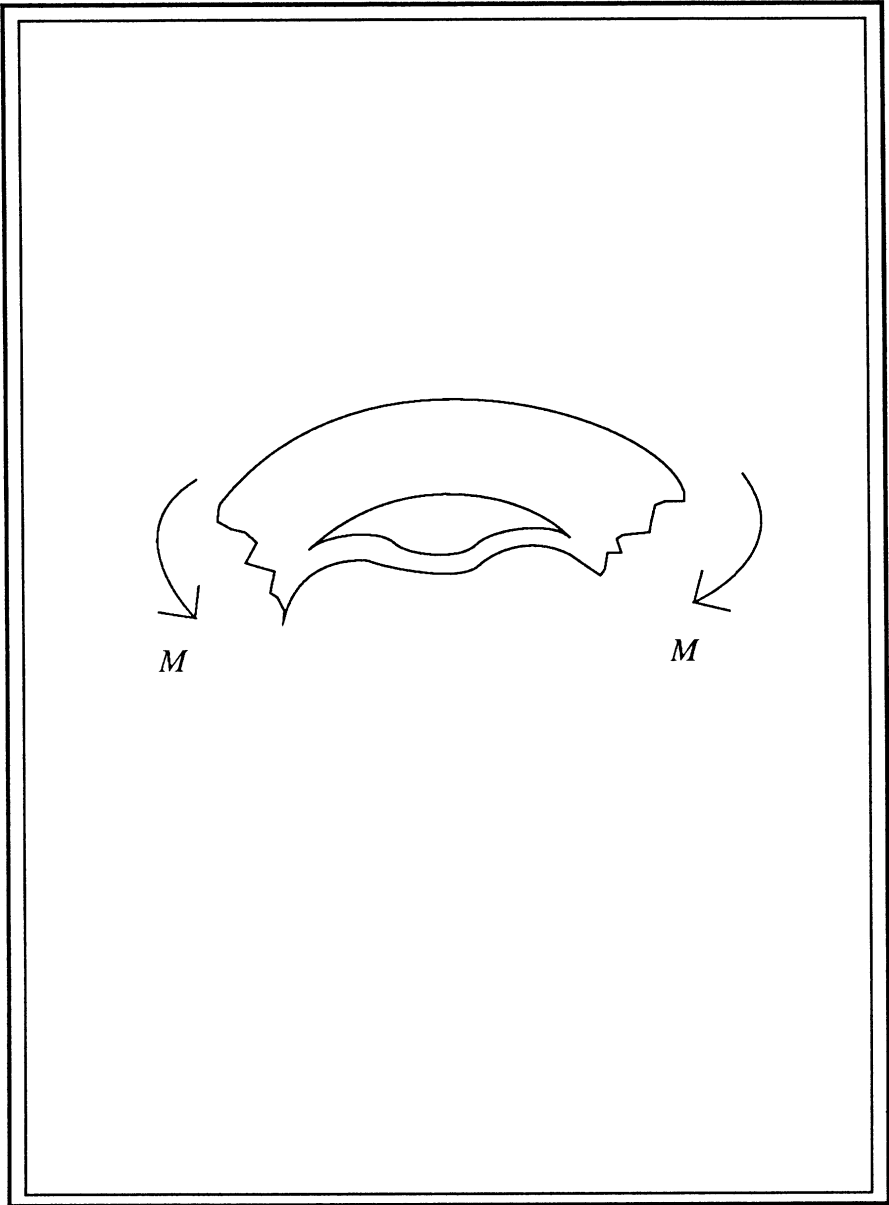


Figure 2.1 (b)

2. DELAMINATION BUCKLING

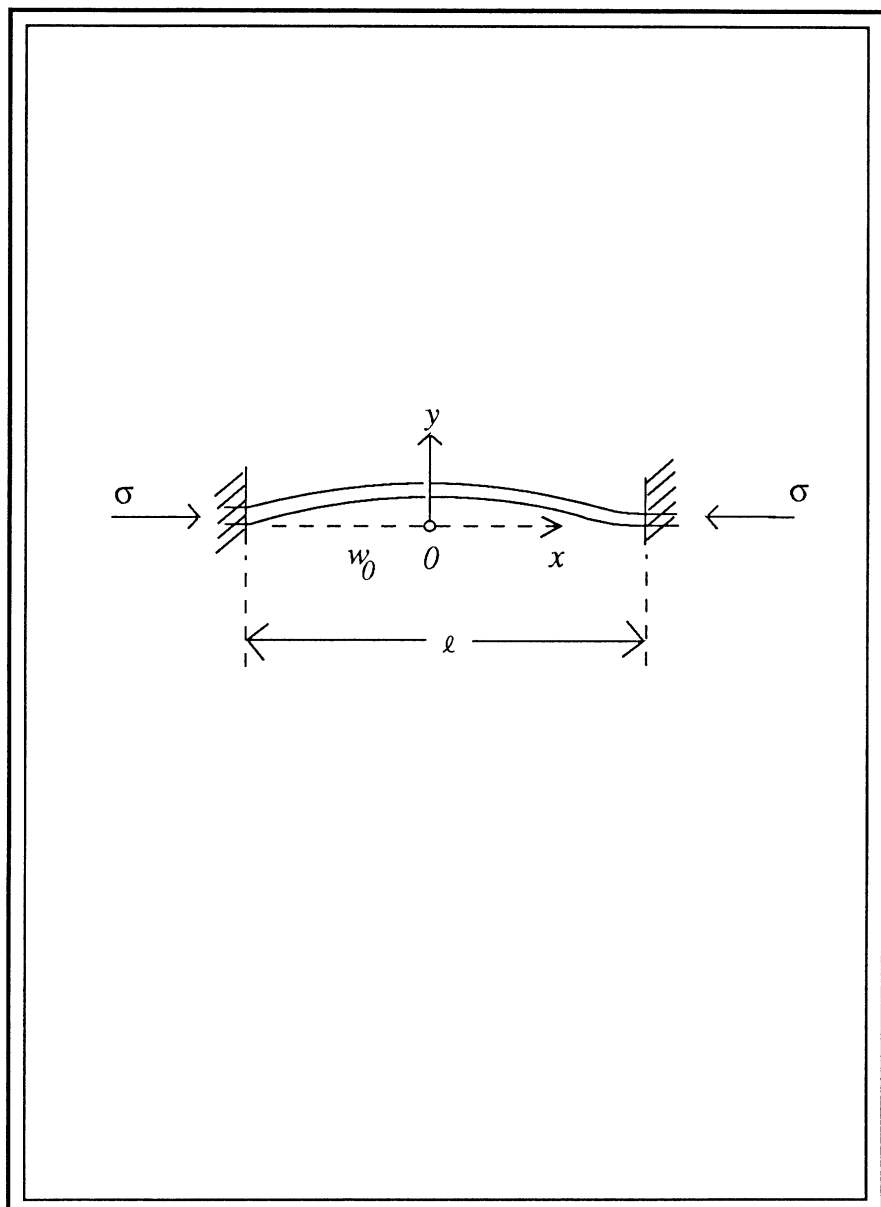


Figure 2.2

2.1. Two Types of Delamination Buckling

where w_0 is the deflection at $x = 0$. The value w_0 is determined by the condition that the length ℓ does not change in buckling.

The contraction $\Delta \ell_2$ due to bending (in buckling) is

$$\Delta \ell_2 = \frac{1}{2} \int_{-\ell/2}^{\ell/2} \left(\frac{dw}{dx} \right)^2 dx .$$

The contraction $\Delta \ell_1$ due to compression (before buckling) is

$$\Delta \ell_1 = \varepsilon_c \ell .$$

Equating $\Delta \ell_1 = \Delta \ell_2$ and calculating, we obtain

$$w_0^2 = \frac{4}{\pi^2} \varepsilon_c \ell^2 . \quad (2.4)$$

The solution of the problem of buckling of a compressed rod with one fixed end (in nonlinear formulation see, for example, [6]) shows that for a load P even slightly exceeding the critical load P_c the deflection of the rod can be large.

For example, if $\frac{P}{P_c} = 1.01$ the deflection of the free end is equal to 0.18ℓ , where ℓ is the length of the rod.

Note that this conclusion is generally also true for other boundary conditions at $x = 0, x = \ell$. In advanced post buckling equilibrium, the deflection is large as compared to the thickness h . Therefore, the post buckling behaviour of plates should be analyzed using the nonlinear equations of bending of flexible plates derived by von Karman (see [6]). Here, we shall use the linear approach.

Now, let us estimate the strain energy of bending as compared to the energy of compression. Consider a strip of a unit width in conditions of post buckling equilibrium, Fig.

2. DELAMINATION BUCKLING

2.3; P is the load per unit width. The deflection w is assumed to be small.

The energy U_1 of compression of an element $\Delta \ell$ is

$$U_1 = \frac{P^2}{2EF} \Delta \ell,$$

where F is the cross-sectional area of the strip.

The bending moment is equal to Pw . The bending energy of the same element is

$$U_2 = \frac{(Pw)^2}{2D}.$$

Hence

$$\frac{U_2}{U_1} = 12 (1 - \nu^2) \left(\frac{w}{h}\right)^2.$$

If $w \geq h$, we have

$$U_2 \gg U_1.$$

So, in the post buckling equilibrium, it is possible to neglect the energy of compression as compared to the energy of bending. This conclusion is especially true for the case of snap buckling when the final equilibrium form differs essentially from the initial one.

Now, let us consider the general case of buckling of a surface layer of the plate subjected to uniaxial compression. We assume that because of previous damage there is a delamination in the central part of the plate Fig.2.4.

The buckling deflection can be represented in the form

$$w = w_0 f(x, y); \quad f(0, 0) = 1.$$

2.1. Two Types of Delamination Buckling

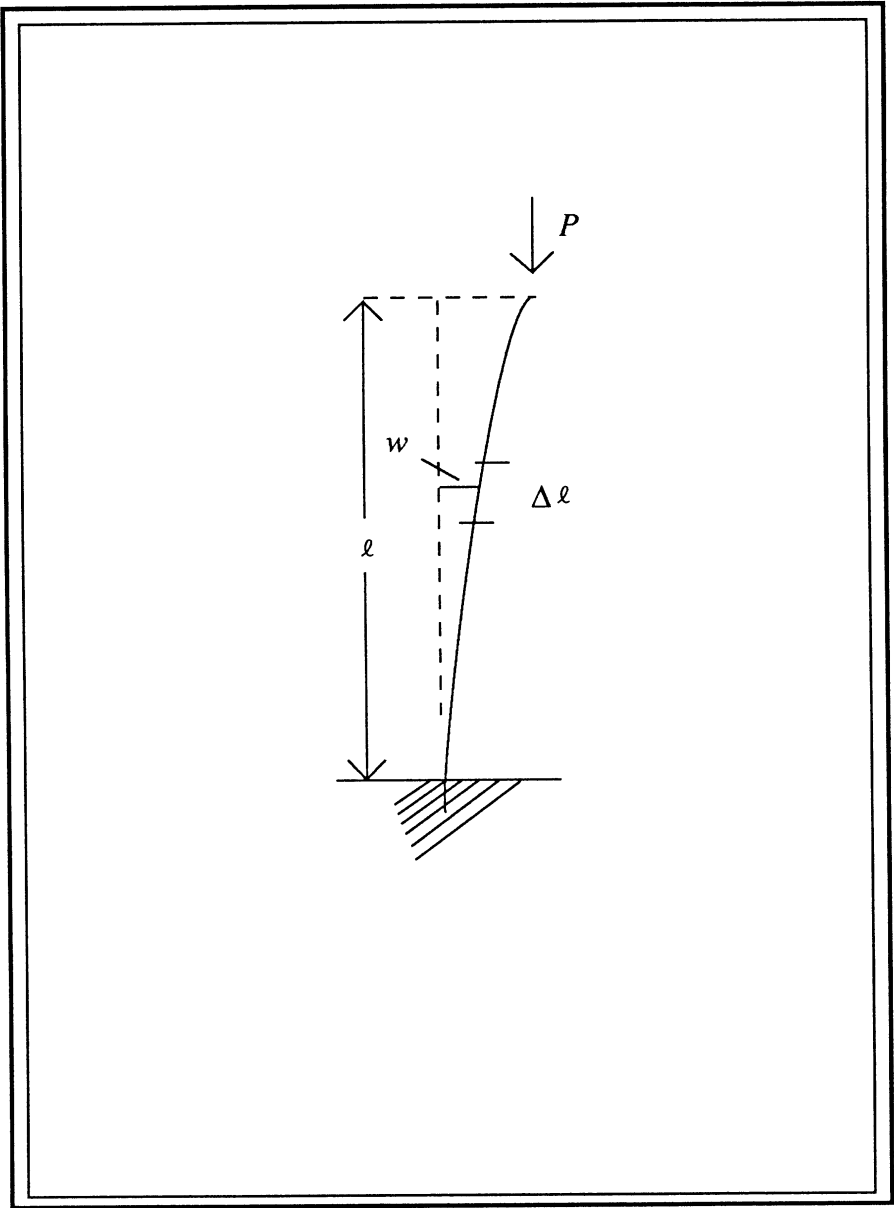


Figure 2.3

2. DELAMINATION BUCKLING

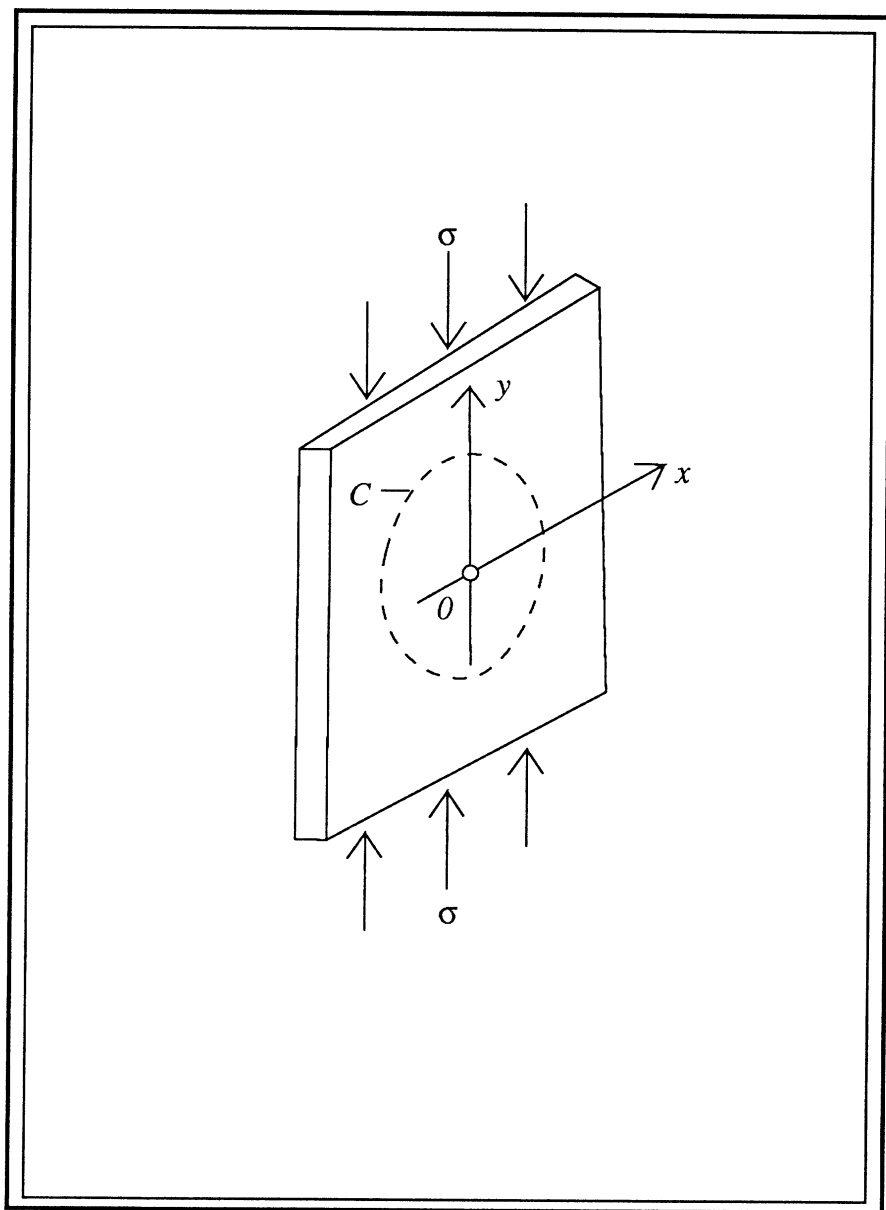


Figure 2.4

2.1. Two Types of Delamination Buckling

On the contour C , $f(x,y) = 0$.

The amplitude w_0 is determined by comparison of contraction $\Delta \ell_1 = \frac{\sigma}{E} \ell$, where ℓ is the characteristic dimension of the delaminated area in the direction of compression, with the bending contraction

$$\Delta \ell_2 = \frac{1}{2} \int_{-\ell/2}^{\ell/2} \left(\frac{dw}{d\eta} \right)^2 d\eta ,$$

where η is the coordinate in the direction of the characteristic dimension ℓ .

Assuming that

$$\eta = \ell \xi ,$$

and specifying the corresponding value of the second coordinate, we obtain

$$\Delta \ell_2 \sim w_0^2 / \ell .$$

By using the notation

$$\Delta \ell \sim f(\ell) ,$$

we mean that

$$\Delta \ell = c f(\ell) ,$$

where c is some positive constant.

The bending strain energy of the buckling isotropic layer is given by

$$U_2 = \frac{1}{2} D \iint F dx dy , \quad (2.6)$$

where

$$F = (\Delta w)^2 - 2(1 - \nu)(w_{xx} w_{yy} - w_{xy}^2) .$$

2. DELAMINATION BUCKLING

Here, $\Delta w = w_{xx} + w_{yy}$ is the Laplace operator; F is a homogeneous quadratic form involving derivatives of the second order.

Introducing dimensionless variables, we find that

$$U_2 \sim \sigma h^3 . \quad (2.7)$$

Now, we consider a classical linear problem of buckling of a thin plate fixed along the contour C . There are different methods of solution to this problem.

If the contour C has a complex form, the corresponding variational formulation and Ritz' method may be used. Specifically, the finite element method is effective.

(b) *Nonlinear Problems. Snap Buckling*

In the case of snap buckling from a concave surface S it is possible to neglect the compression of the buckling layer, and the value w_0 does not depend on the compressive stress σ .

The bending energy is given by

$$U_2 = \frac{1}{2} D \iint [(\Delta \kappa_1)^2 + (\Delta \kappa_2)^2 + 2\nu \Delta \kappa_1 \Delta \kappa_2] dS , \quad (2.8)$$

where $\Delta \kappa_1, \Delta \kappa_2$ are the changes of the principal normal curvatures.

It is easy to see from (2.8) that for the second type we have the estimation

$$U_2 \sim E h^3 . \quad (2.9)$$

In the case of snap buckling it is necessary to solve a difficult nonlinear problem.

2.1. Two Types of Delamination Buckling

As an example, a specific case of snap buckling of a circular ring under external pressure will be considered in sect.2.4.

For the general case of problems of type II we can recommend the use of approximate geometric theory of buckling of thin elastic shells developed by Pogorelov [7].

In this theory it is assumed that after buckling the shell becomes isometric to its initial form. For example, a mirror image of a part of the initial form of the shell is isometric to the initial one; the final (buckled) form of equilibrium is, then, not a smooth surface: it consists of a smooth part S and a rib C , Fig.2.5 (the reflected part of the surface is indicated by a dashed line).

The bending strain energy of the buckled layer is

$$U_2 \approx \bar{U}_2 + U'_2 ,$$

where

$$\bar{U}_2 \approx \frac{1}{2} D \int [(\Delta \kappa_1)^2 + (\Delta \kappa_2)^2 + 2 \nu \Delta \kappa_1 \Delta \kappa_2] dS$$

is the strain energy of the smooth part, and

$$U'_2 = \int u' dC$$

is the strain energy of the rib and u' is the bending strain energy per unit length of the rib. For very thin shells, u' can be calculated as a certain limiting value (see [7]). It should be noted that, from the energy criterion viewpoint, the preferred isometric transformation corresponds to a minimal strain energy of the buckled layer.

2. DELAMINATION BUCKLING

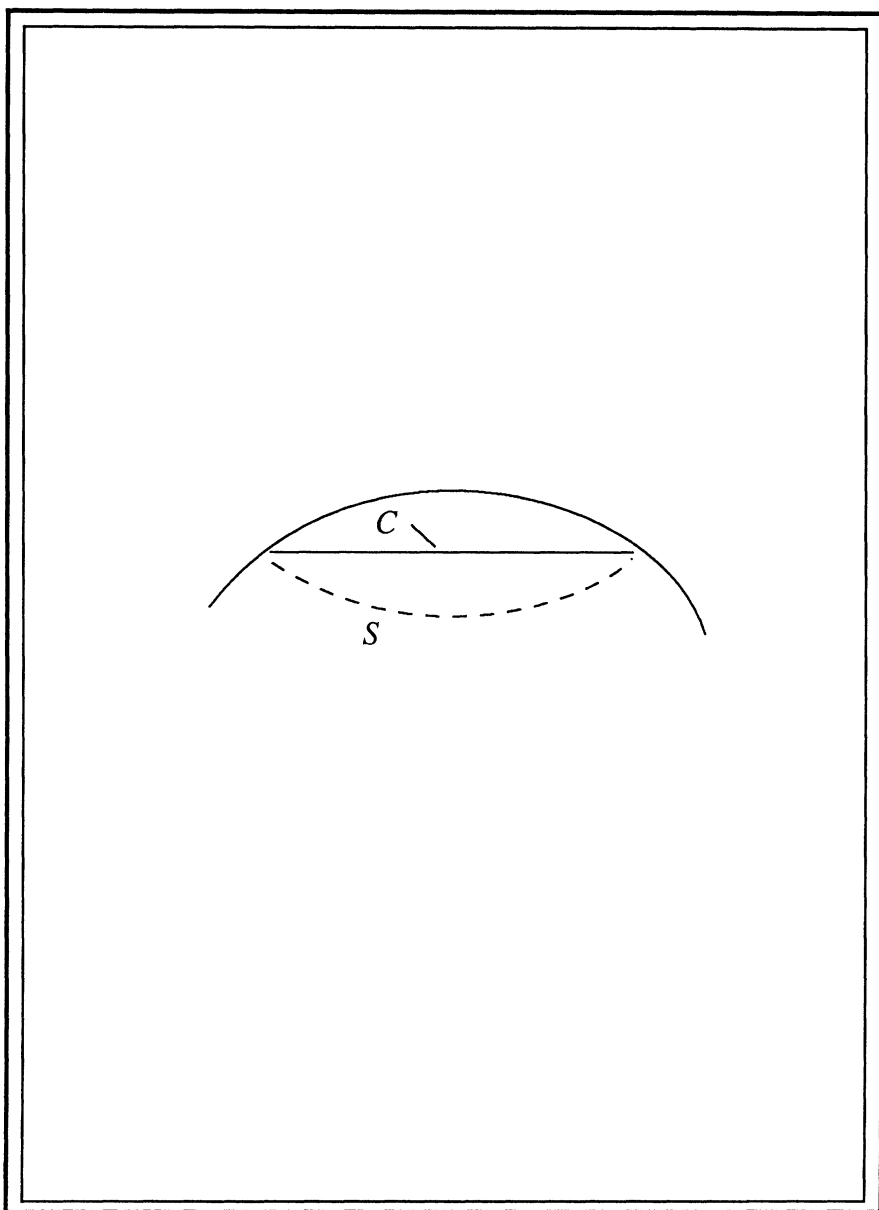


Figure 2.5

Note, also, that the formation of the rib can be accompanied by plastic deformation (i.e. by "plastic hinges") and even by fracture. These effects can be taken into account by an appropriate definition of u' .

2.2 Energy Criterion

(a) *General Remarks*

In the problems of type I, the transition to a new equilibrium state is a quasistatic process. If an initial delamination is present, we have, as mentioned above, a classical linear problem of elastic stability.

If there is no initial delamination, the loss of stability is accompanied by a simultaneous fracture of the interlaminar bond. In this case, the work of fracture should be taken into account.

For problems of type II, the transition to a new equilibrium state occurs by a snap. The complete analysis of this dynamic process is very difficult and, besides, requires analyzing perturbations of various natures.

Under statical loading, the difference $\Delta\kappa$ between the increment of work of the external load and increments of the internal energy and of fracture work transforms into kinetic energy.

The initial equilibrium state is stable if

$$\Delta\kappa < 0. \quad (2.10)$$

We shall use the energy criterion comparing the energy of the initial state to the energy of the final state (after buckling).

2. DELAMINATION BUCKLING

The energy U_1 of compression is accumulated in the layer before it buckles. After buckling, the layer gets the elastic energy U_2 . It is possible to assume that U_2 is mostly the energy of bending.

The fracture work is

$$W = 2\gamma S \quad (2.11)$$

where S is the area of fracture, and the material constant γ is the so-called surface energy.

We can assume that in buckling a transition to a lower energy level takes place, i.e.

$$U_1 > U_2 + W.$$

It can be expected that the lower bound of the dangerous stress is determined by the equality

$$U_1 = U_2 + W. \quad (2.12)$$

The analysis (in the case $W = 0$) has shown (see [6]) that this criterion leads to an approximate value of the critical load.

Note, that generally, in the transition mentioned above, we might deal with an energy barrier ("energy threshold"). This threshold is associated with various imperfections. Below, we shall assume that there is no energy threshold.

(b) *Basic Relations for a Buckling Layer*

Introducing the values U_1 , U_2 and W into the energy relation (2.12), we obtain

in the case of type I

$$\frac{(1-\nu)\sigma^2}{E} Sh - \bar{k}_1 \sigma h^3 - 2\gamma S = 0 \quad , \quad (2.13)$$

in the case of type II

$$\frac{(1-\nu)\sigma^2}{E} Sh - \bar{k}_2 Eh^3 - 2\gamma S = 0 \quad , \quad (2.14)$$

where $\bar{k}_1, \bar{k}_2$ are some positive coefficients depending on the buckling form $f(x,y)$. Note that $S \sim \ell^2$.

The equations (2.13) and (2.14) are quadratic with respect to σ/E ; solving them, we find correspondingly

in the case of type I

$$\frac{\sigma}{E} = k_1 \delta^2 [1 + (1 + \kappa_1 \delta^{-5})^{1/2}] \quad , \quad (2.15)$$

in the case of type II

$$\frac{\sigma}{E} = \sqrt{2k_2} \delta (1 + \kappa_2 \delta^{-3})^{1/2} \quad , \quad (2.16)$$

where k_1, k_2 are positive coefficients, $\delta = \frac{h}{\ell}$ and

$$\kappa_1 = \frac{2\gamma}{E \ell (1-\nu) k_1^2} \quad , \quad \kappa_2 = \frac{\gamma}{E \ell (1-\nu) \bar{k}_2^2} \quad .$$

The dependence of σ/E on δ is shown in Fig. 2.6. It is obvious that if values κ_1, κ_2 are specified, we have a minimal delamination stress σ_l and corresponding to it a layer δ_l ("weak layer"), which out of all the layers is most prone to delamination.

2. DELAMINATION BUCKLING

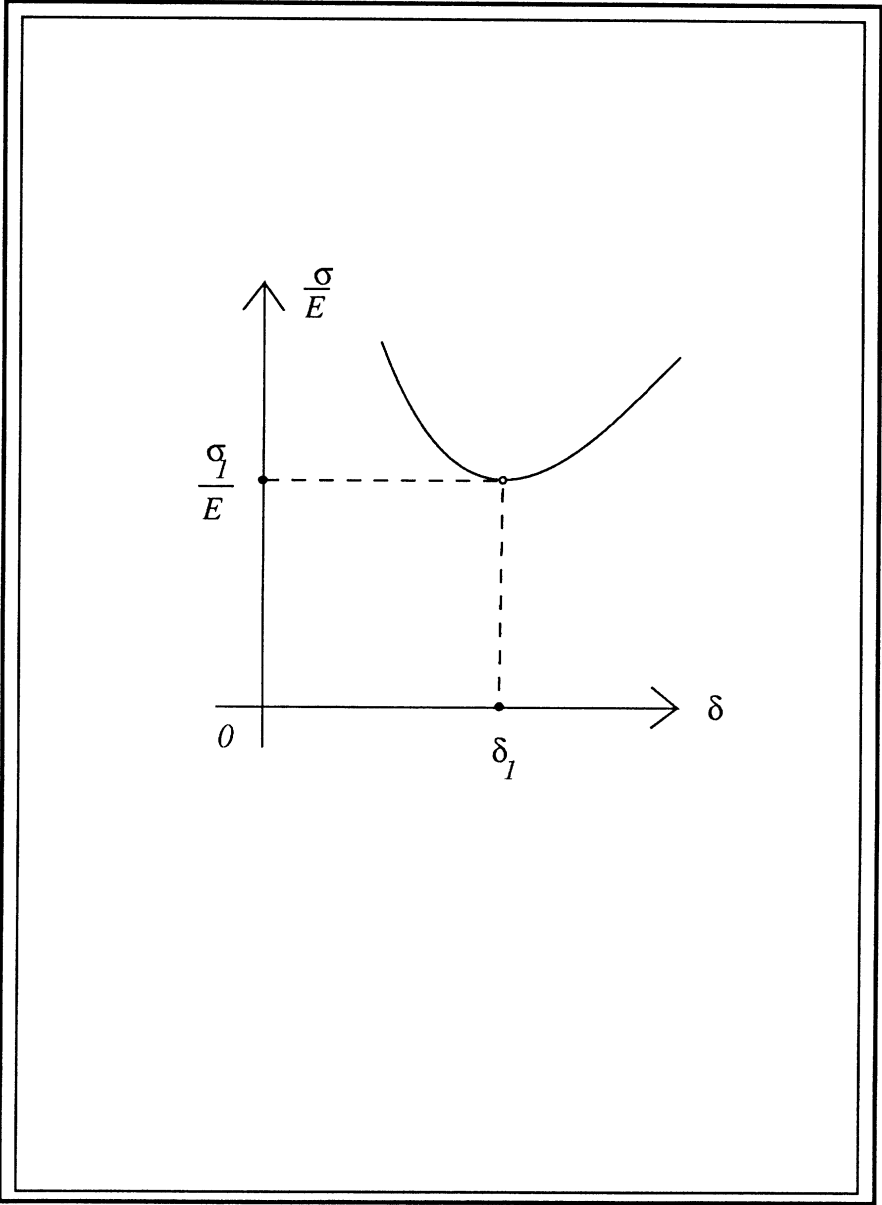


Figure 2.6

Now, it is not difficult to find these extremal values:

in the case of type I

$$\delta_I = \left(\frac{\kappa_1}{8}\right)^{1/5}; \quad \frac{\sigma_I}{E} = 2 \left(\frac{\kappa_1^2}{2}\right)^{1/5} k_1, \quad (2.17)$$

in the case of type II

$$\delta_I = \left(\frac{\kappa_2}{2}\right)^{1/3}; \quad \frac{\sigma_I}{E} = \kappa_2^{1/3} 1.94 (k_2)^{1/2}. \quad (2.18)$$

Since $\ell \leq L$, where L is the characteristic body dimension in the direction of delamination, there exists a lower bound of the stress, σ_I . This bound is found from (2.17), (2.18) if $\ell = L$.

If there is an initial delamination of area S_0 preceding the loading, the fracture work will be

$$W = 2\gamma(S - S_0).$$

Here, all basic relations will be identical, but instead of γ it is necessary to introduce $\gamma(1 - S_0/S)$. In this case, the critical load decreases.

(c) Note

We have considered delamination buckling of the plate under compression, Fig.2.1 (a). In the case of pure bending of the plate analogous to the case shown in Fig.2.1 (b), the analysis of delamination buckling is practically the same, but note that stress σ now signifies the maximal compressive stress in bending.

2. DELAMINATION BUCKLING

2.3 Axisymmetric Compression of a Plate

(a) *Statement of the Problem*

Consider an isotropic laminated plate subjected to the uniform compressive stress σ , Fig.2.7.

Assume that delamination buckling over the circle of radius a occurs when the stress σ reaches the critical value σ_c . It is obvious that, here, the form of buckling is the same as that of a uniformly compressed circular plate with the clamped edge $r = a$. On this contour,

$$w = \frac{dw}{dr} = 0 \quad , \quad (2.19)$$

where w is the buckling deflection. According to the well-known solution, this deflection is given (see [6]) by

$$w = w_0 [I_0(r\alpha) - I_0(a\alpha)] \quad , \quad (2.20)$$

where I_0 is the Bessel function of order zero, and

$$\alpha = 3.83 \frac{1}{a} \quad .$$

Note that,

$$\frac{dw}{dr} = -\alpha w_0 I_1(r\alpha) \quad ,$$

where I_1 is the Bessel function of order one.

The contraction of the radial dimension due to buckling is equal to

$$\Delta \ell_2 = \frac{1}{2} \int_0^a \left(\frac{dw}{dr} \right)^2 dr = k w_0^2 \frac{1}{a} \quad ,$$

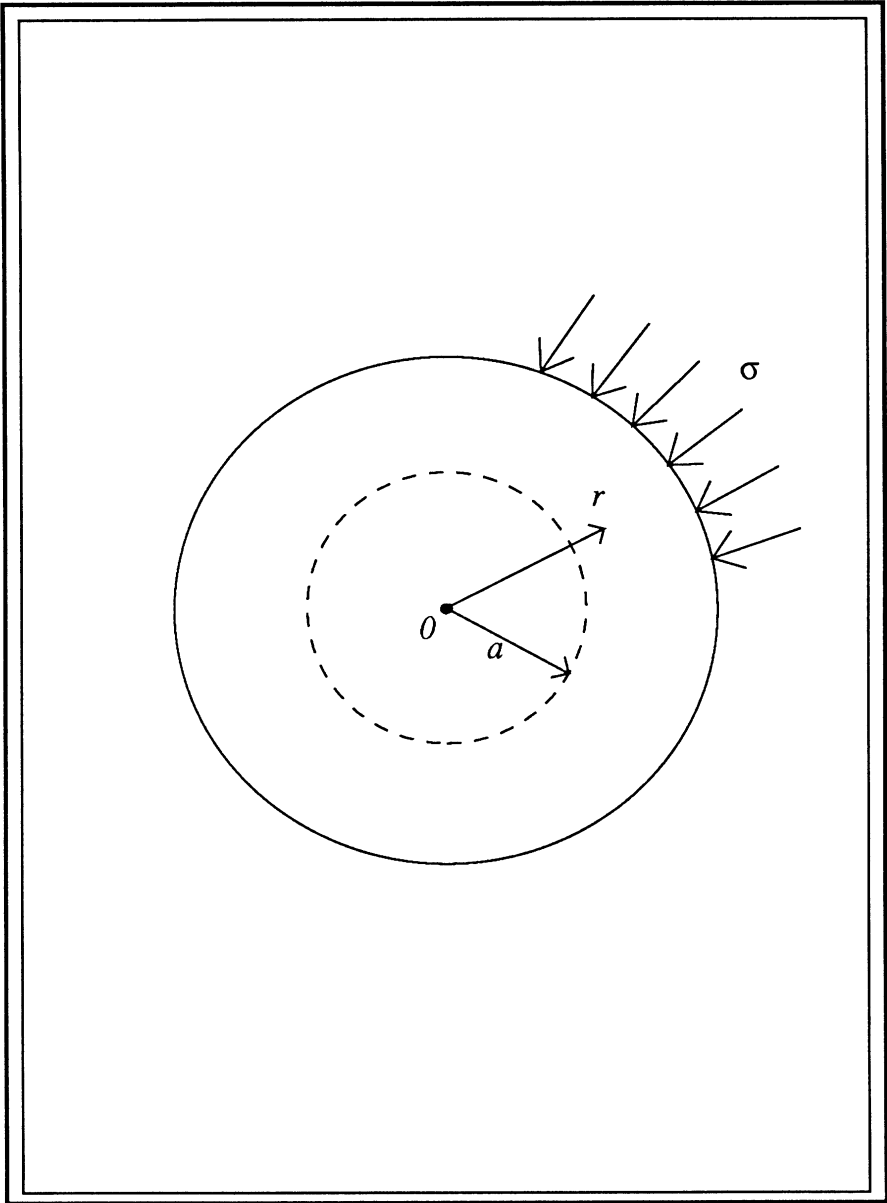


Figure 2.7

2. DELAMINATION BUCKLING

where

$$k = 3.83 \int_0^{3.83} I_1^2(x) dx = 2.44 \quad .$$

Since $\Delta \ell_I = \frac{1-\nu}{E} \sigma a$, we obtain

$$w_0^2 = \frac{1-\nu}{kE} \sigma a^2 \quad .$$

It is not difficult to find the elastic strain energy of bending

$$U_2 = K \sigma h^3 \quad ,$$

where

$$K = \frac{\pi}{12(1+\nu)k} \int_0^1 \left[\left(\frac{d\bar{w}}{dx} \right)^2 + \frac{2\nu}{x} \frac{d\bar{w}}{dx} \frac{d^2\bar{w}}{dx^2} + \frac{1}{x^2} \left(\frac{d\bar{w}}{dx} \right)^2 \right] x dx \quad .$$

Here, it is assumed that

$$\bar{w} = I_0(3.83x) - I_0(3.83) \quad .$$

Introducing $\bar{w}$ in K , and performing (after some simple transformation) a numerical integration, we find

$$K = \frac{1.83}{1+\nu} \quad .$$

Furthermore, we obtain

$$U_I = \pi^2 \frac{1-\nu}{E} \sigma^2 a^2 h \quad .$$

2.3 Axisymmetric Compression of a Plate

The work of fracture is

$$W = 2\gamma\pi a^2 .$$

Now, it is not difficult to find the coefficients in the formula (2.15):

$$k_I = \frac{K}{2\pi(1-\nu)} ; \quad \kappa_I = \frac{2\gamma}{EK^2a(1-\nu)} .$$

(c) *Concluding Remarks*

Above, delamination buckling of an isotropic thin layer was considered. It is not difficult to extend the linear scheme to the case of both a transversely isotropic layer and an orthotropic one.

Buckling of a close to the surface delaminated elliptic (in plane) layer under uniaxial compression has been recently analyzed in the paper of Chai and Babcock [12] by the Rayleigh-Ritz method. They also considered the post buckling behaviour.

Buckling of a thin sublaminar in a quasi-isotropic composite is studied in the work of Shivakumar and Whitcomb [13]. They use the finite-element method.

Delamination buckling of the two-layer axisymmetric model of a circular plate with initial penny-shaped delamination has been analyzed by Bottega and Maewal [14] on the basis of von Karman's geometrically nonlinear plate theory.

They have also used the same two-layer model for the analysis of delamination buckling under dynamic loading [15].

Note that one-dimensional modelling of delamination buckling is analyzed in detail by Chai, Babcock and Knauss [16].

2. DELAMINATION BUCKLING

Buckling of a short-fiber composite with a mid-plane delamination under uniaxial compression has recently been studied by Wang, Zahlan and Suemasu [17]. They used the linear equation of buckling. The solution was obtained by means of the Rayleigh-Ritz method and by finite element technique.

They have carried out a series of experiments on the subject.

2.4 Snap Delamination Buckling of a Ring under External Pressure

(a) *Statement of the Problem*

Let us consider a circular fiber-glass ring under uniform external pressure p , Fig.2.8. The layer next to the internal surface $r = a$ of the ring is subjected to the compressive stress σ ; it can be found by applying the solution of the well-known Lamé problem.

If the stress σ is high, snap buckling of the internal layer takes place. This phenomenon is explained by a relatively low strength of the interlaminar bond.

Let the thickness of the buckled layer be h and let delamination spread over the arc $2\varphi_l$. Because of symmetry, we can consider one half $0 \leq \varphi \leq \varphi_l$ of the arc.

The elastic strain energy of the compressed arc per unit width is

$$U_I = \frac{\sigma^2}{2E_*} a \varphi_l h ,$$

where E_* is the effective Young's modulus in the circumferential direction.

2.4 Snap Delamination Buckling of a Ring under External Pressure

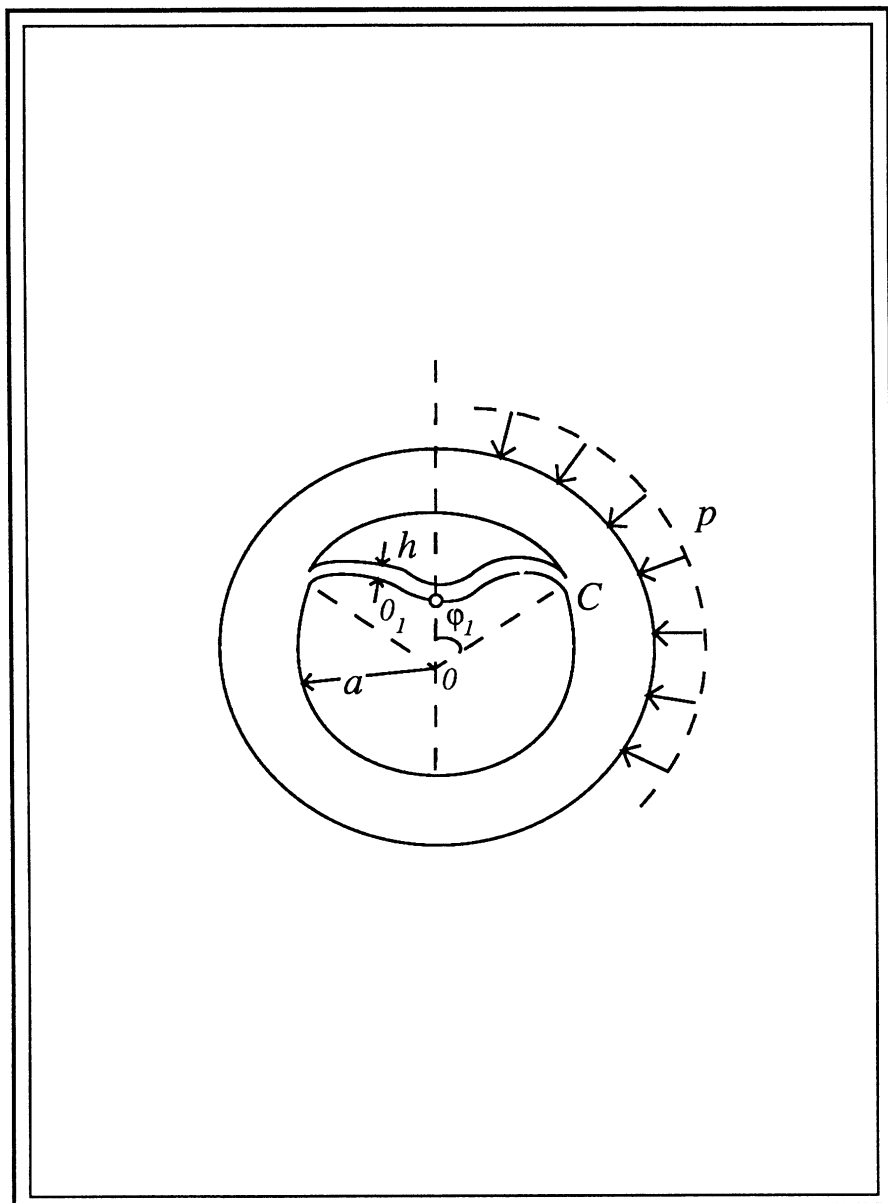


Figure 2.8

2. DELAMINATION BUCKLING

According to the energy criterion, the condition for delamination buckling has the form

$$U_2 + W \leq U_1 , \quad (2.21)$$

where U_2 is the strain energy of the buckling layer and

$$W + 2\gamma a \varphi_1$$

is the work of fracture.

(b) *Equilibrium Form after Snap Buckling*

This form is determined by the solution of the nonlinear Bernoulli equation of bending

$$E_* I \left(\frac{1}{\rho} + \frac{1}{a} \right) = M , \quad (2.22)$$

where M is the bending moment (per unit width), $I = h^3/12$ is the moment of inertia and $1/\rho$ is the curvature of the elastic line. Note that

$$\frac{1}{\rho} = \frac{d\theta}{ds} , \quad (2.23)$$

where θ is the angle formed by the elastic line with the x -axis, and s is the arc length.

The bending energy is

$$U_2 = \frac{E_* I}{2} \int_0^{a\varphi_1} \left(\frac{d\theta}{ds} + \frac{1}{a} \right)^2 ds . \quad (2.24)$$

It is well-known that the solution of the Bernoulli equation can be represented by means of elliptic integrals.

2.4 Snap Delamination Buckling of a Ring under External Pressure

The form of equilibrium has a point of inflection.

We introduce a new function ψ by the relation

$$\sin \frac{\theta}{2} = m \sin \psi , \quad (2.25)$$

where $0 \leq m < 1$ is an arbitrary number. It is not difficult to see that

$$\frac{d\theta}{dS} = 2 km \cos \psi .$$

Here,

$$k = \left(\frac{P}{EI} \right)^{1/2} , \quad (2.26)$$

where P is the force (per unit width) at the initial point O_I .

Now, the solution of the Bernoulli equation can be written in the form (see, for example, [8]),

$$ks = F(\psi) - F(\psi_0) , \quad (2.27)$$

where

$$F(\psi) = \int_0^\psi \frac{d\psi}{\sqrt{1 - m^2 \sin^2 \psi}}$$

is the elliptic integral of the first kind; m is the modulus of the elliptic integral, ψ_0 is the value of ψ at $s = 0$ (point O_I).

Since $\theta = 0$ when $s = 0$, we obtain $\psi_0 = 0$.

Let X, Y be the coordinate system originating at the point O_I . Integrating the relations

$$\frac{dX}{ds} = \cos \theta , \quad \frac{dY}{ds} = \sin \theta ,$$

2. DELAMINATION BUCKLING

we find the coordinates of the elastic line

$$\begin{aligned} X &= \frac{2}{k} E(\psi) - s, \\ Y &= -\frac{2m}{k} (\cos \psi - 1), \end{aligned} \quad (2.28)$$

where $E(\psi)$ is the elliptic integral of the second kind.

At the final point C , we have $s = a\varphi_1$, $\theta = -\varphi_1$, $X = a \sin \varphi_1$, and the variable ψ has some value ψ_1 .

Substituting these values into the equations (2.25) - (2.28), and eliminating φ_1 and k , we find the relation

$$[2E(\psi_1) - F(\psi_1)] \cos \psi_1 - (1 - m^2 \sin^2 \psi_1)^{1/2} \sin \psi_1 = 0, \quad (2.29)$$

which describes the dependence of ψ_1 on k .

From the obtained solution, we calculate the bending strain energy

$$U_2 = \frac{E_* a h}{6} \left(\frac{h}{a}\right)^2 \varphi_1 g(m), \quad (2.30)$$

where

$$g(m) = \frac{ka}{\varphi_1} [E(\psi_1) - (1 - m^2) F(\psi_1)] + \frac{1}{\varphi_1} \sin^{-1}(m \sin \psi_1) + \frac{1}{4}.$$

The function $g(m)$ depends solely on m . The table of the function $g(m)$ is given below:

m	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
g(m)	2.52	2.53	2.54	2.55	2.56	2.58	2.61	2.68	2.88

2.4 Snap Delamination Buckling of a Ring under External Pressure

The function $g(m)$ increases very slowly.

For most values of m (from $m = 0$ to $m = 0.6$, which corresponds to the angles $\varphi_1 = 0$ and $\varphi_1 \approx 33^\circ$) the function $g(m)$ is almost constant.

Substituting U_1 , U_2 and W into the energy criterion (2.21) with the equality sign, we obtain the critical stress [4]

$$\frac{\sigma_c}{E_*} = \left[\frac{1}{3} \left(\frac{h}{a} \right)^2 g(m) + \frac{4\gamma}{E_* h} \right]^{1/2}. \quad (2.31)$$

$g(m)$ being approximately constant, it is expedient to substitute it by its lower bound $g(0)$. Then,

$$\frac{\sigma_c}{E_*} = 0.916 \left[\left(\frac{h}{a} \right)^2 + \kappa \left(\frac{h}{a} \right)^1 \right], \quad (2.32)$$

where

$$\kappa = \frac{4.76\gamma}{E_* a}.$$

The dependence (2.32) is shown in Fig.2.9. There is a stress minimum

$$\left(\frac{\sigma_c}{E_*} \right)_{\min} = 1.26 \kappa^{1/3} \quad (2.33)$$

at the point

$$\left(\frac{h}{a} \right)_0 = \left(\frac{\kappa}{2} \right)^{1/3}. \quad (2.34)$$

Hence, in this special problem, we see again that there exists a "weak layer" which is most prone to delamination.

2. DELAMINATION BUCKLING

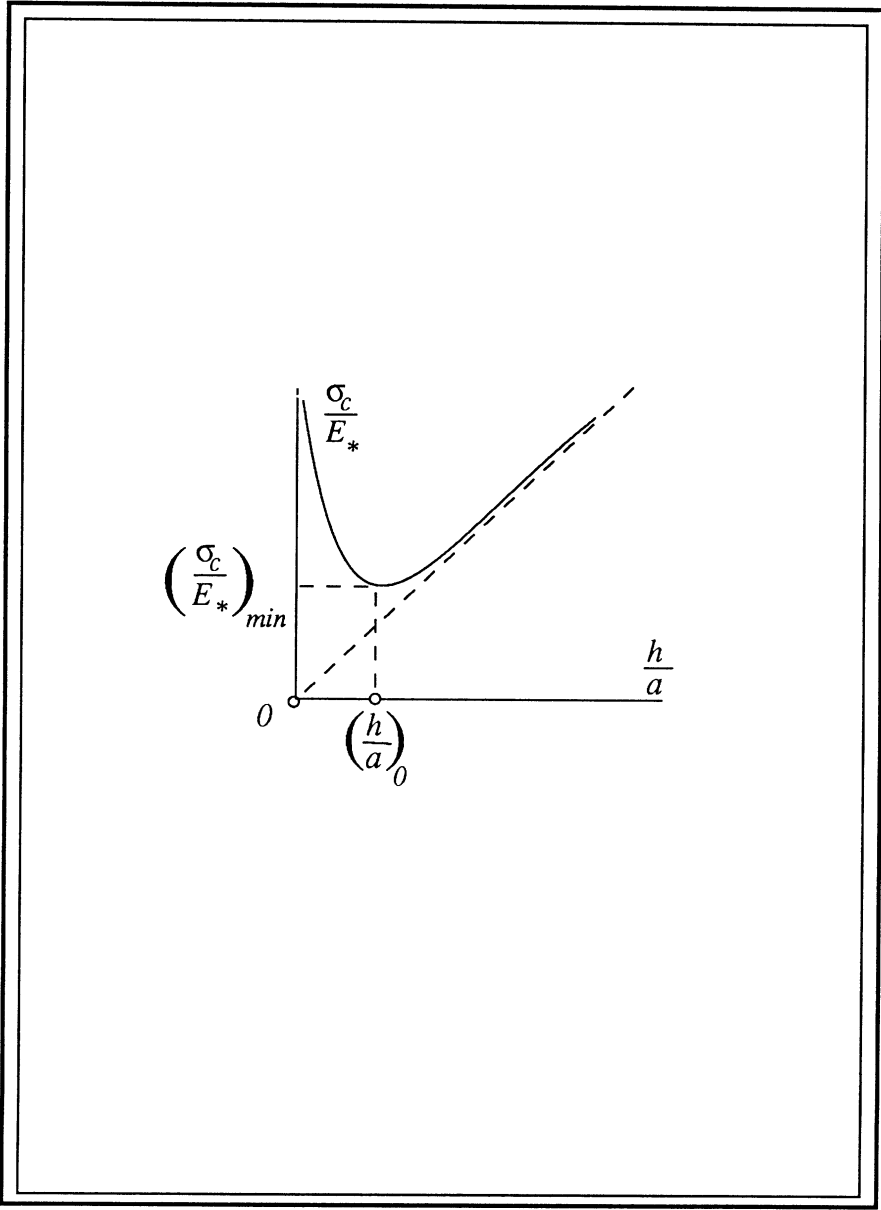


Figure 2.9

(c) *Experimental Data*

The solution described above was compared with experimental data in Bugakov's work [9]. Fiber-glass rings were subjected to uniform external pressure. The strength constant γ was determined by special tests on tearing off a strip of the ring.

The lower bound of dangerous stress given by the formula (2.29) agrees with that given by the experimental data; the tests confirm that there exists a weak delamination layer.

Thus, the above considered theoretical scheme has a reasonable agreement with some experimental data. Note, in conclusion, that, using the given above solution, it is not difficult to analyze the snap buckling in the case of initial delamination.

2.5 Uniaxial Compression of a Plate

(a) *Basic Equations*

Consider an isotropic composite plate under uniaxial compressive stress p , Fig.2.10. The plate has an initial near surface delamination area Ω , Fig.2.10, bounded by the contour C .

When the load reaches the critical value p_{cr} , the near surface laminate buckles. If the initial delamination is a strip $|x| \leq a$, the bulged laminate has a cylindrical shape, Fig.2.11.

We consider here several typical shapes of Ω : circle, ellipse [Fig.2.12 (a)], rectangle [Fig.2.12 (b)].

2. DELAMINATION BUCKLING

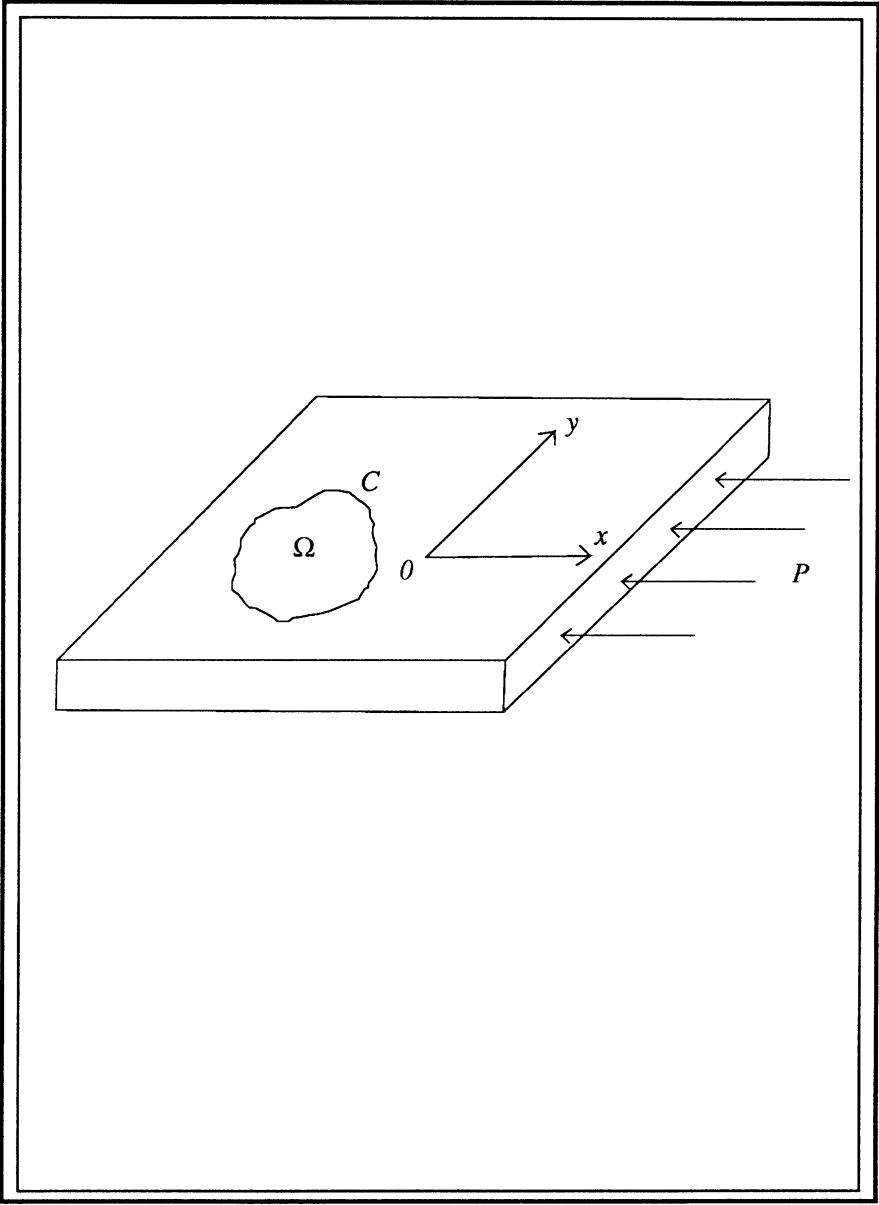


Figure 2.10

2.5 Uniaxial Compression of a Plate

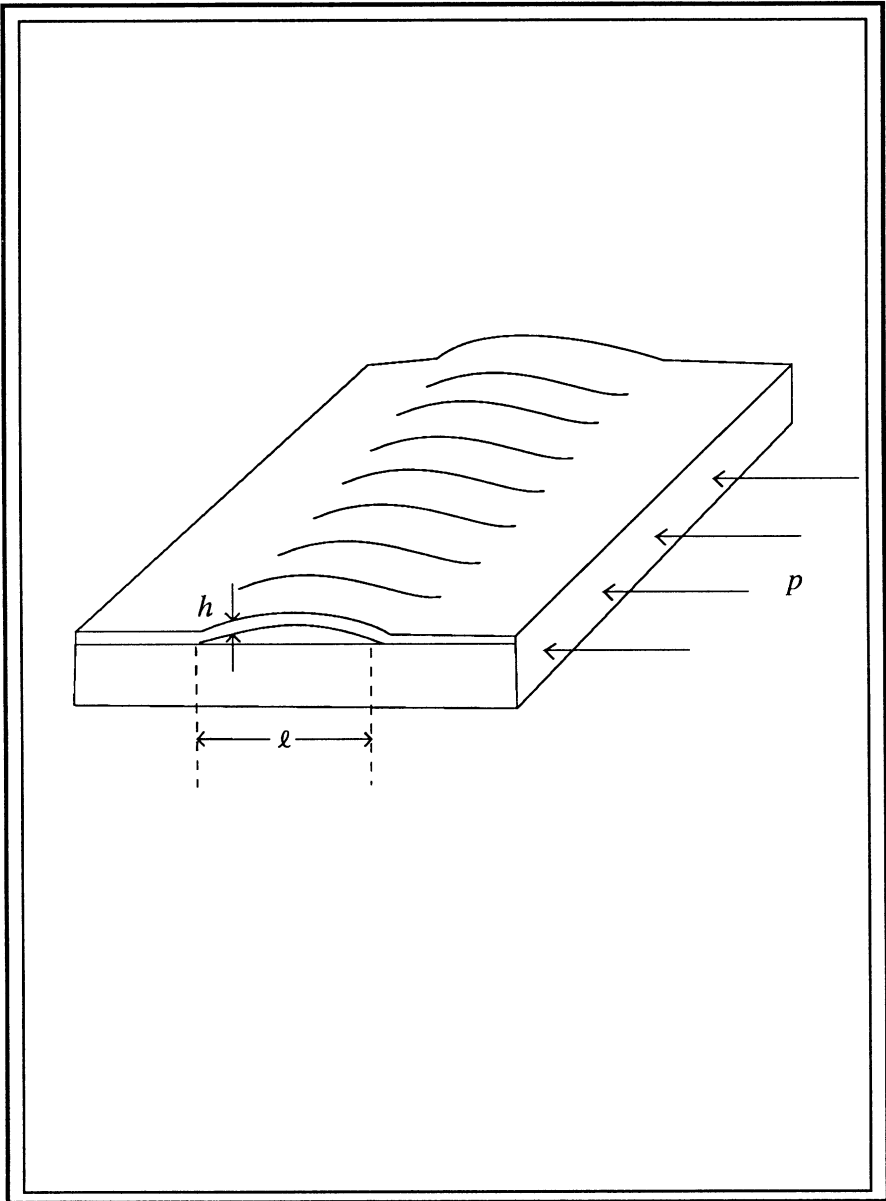


Figure 2.11

2. DELAMINATION BUCKLING

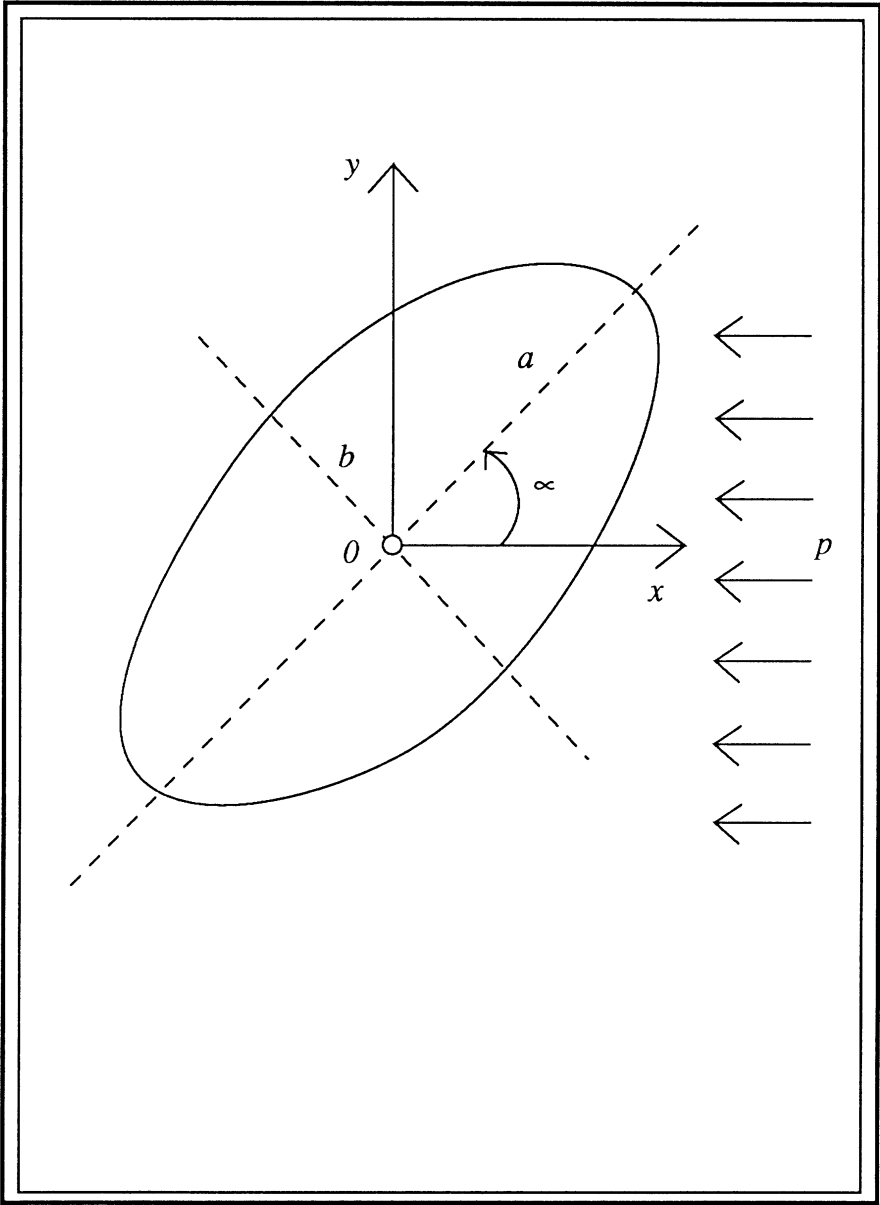


Figure 2.12 (a)

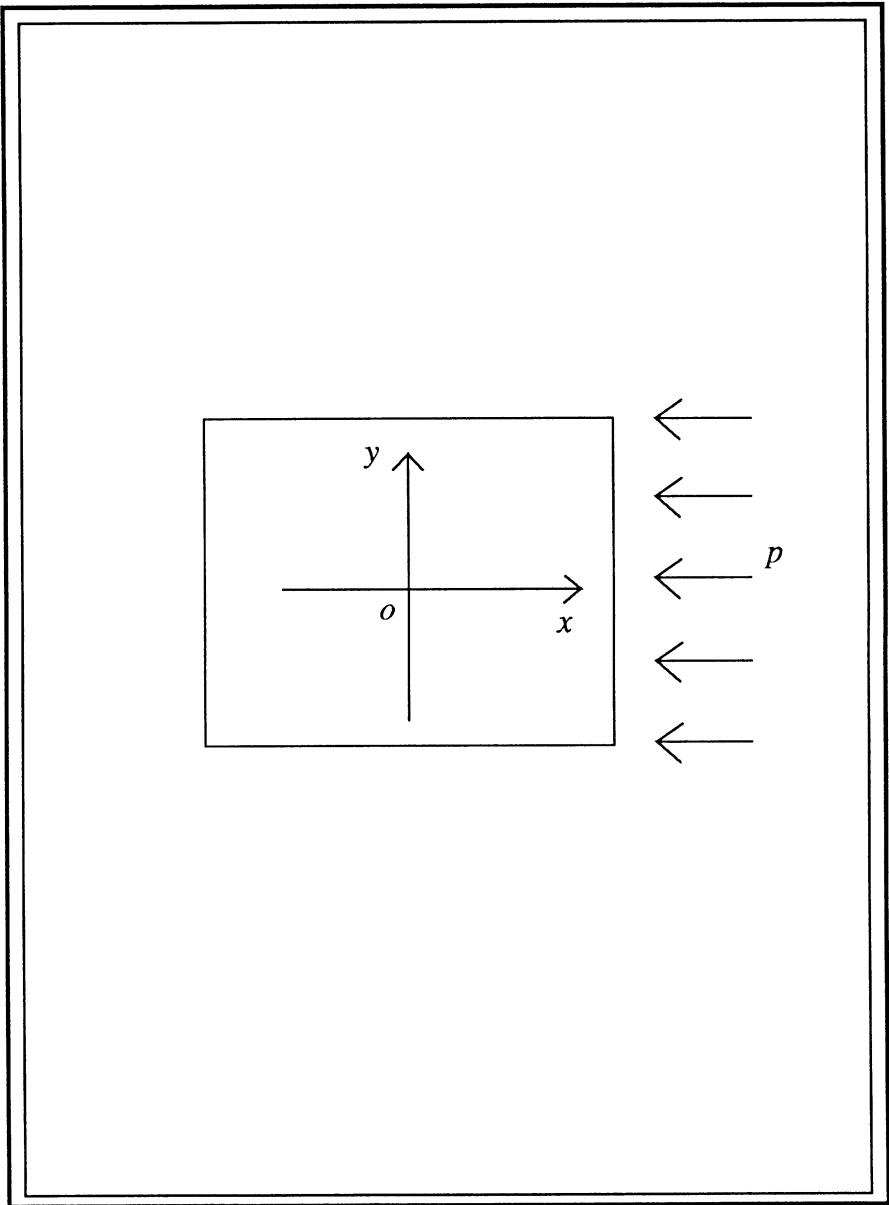


Figure 2.12(b)

2. DELAMINATION BUCKLING

The buckling deflection $w(x,y)$ is assumed to be small as compared to the thickness of the delaminated laminate. Then, the buckling problem can be analyzed in a linear approximation. Thus, in the case of uniaxial compression in x -direction, the buckling deflection $w(x,y)$ satisfies the following differential equation (see, for example, [6])

$$\frac{D}{h} \nabla^4 w + p \frac{\partial^2 w}{\partial x^2} = 0 \quad , \quad (2.35)$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

is the flexural stiffness of the buckled laminate, h - its thickness, ν - Poisson's ratio and $\nabla^4 w$ is the biharmonic operator

$$\nabla^4 w = \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \quad .$$

Transition from the bulged surface to the plane shape is smooth; therefore, we have the following boundary conditions on the contour C :

$$\begin{aligned} w &= 0 \\ \frac{\partial w}{\partial n} &= 0 \end{aligned} \quad (2.36)$$

where n is a normal to the contour C .

2.5 Uniaxial Compression of a Plate

It is convenient sometimes to use the energy formulation of the buckling problem. The buckling deflection w satisfies the variational equation

$$\delta(U - A) = 0 \quad (2.37)$$

that follows from the principle of virtual displacements. U is the potential energy of bending of the buckled laminate,

$$U = \frac{1}{2} D \iint_{\Omega} [(\nabla^2 w)^2 - (1 - \nu) L(w)] dx dy, \quad (2.38)$$

where ∇^2 is the Laplace operator,

$$L(w) = 2 \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right],$$

and A is the work of the external load on the buckling displacement. Equation (2.37) implies that the total potential energy of the system $U - A \equiv \Pi$ has an extremum in the buckled form of equilibrium:

$$\Pi = \text{extr.}$$

The critical load p_{cr} is the eigenvalue of the problem. It is well known that it is extremal in the actual buckled state. Note that the Ritz' method constitutes an efficient approach to the variational problem. Because of the extremal property of the critical load p_{cr} (see[6]), even a rough Ritz' approximation may lead to a relatively accurate value of p_{cr} .

Several examples are considered.

2. DELAMINATION BUCKLING

(b) *Delamination along a strip*

In this case, $w = w(x)$, and the differential equation (2.35) yields

$$\frac{D}{h} \frac{d^4 w}{dx^4} + P \frac{d^2 w}{dx^2} = 0 \quad .$$

The boundary conditions are

$$w=0 \quad , \quad \frac{dw}{dx}=0 \quad \text{at} \quad x = \pm \frac{\ell}{2} \quad .$$

The critical stress for this problem was determined in sect.1.1 as

$$p_{cr} = \frac{\pi^2 E}{3(1 - \nu^2)} \left(\frac{h}{\ell} \right)^2 \quad . \quad (2.39)$$

The buckling deflection is

$$w = \frac{l}{2} w_0 \left(1 - \cos \frac{2\pi x}{\ell} \right) \quad , \quad (2.40)$$

where the buckling amplitude w_0 is given by

$$w_0^2 = \frac{4}{\pi^2} \ell^2 \varepsilon_{cr} \quad (2.41)$$

(c) *Circular Delamination*

In this case it is convenient to write the differential equation (2.35) in polar coordinates and seek the solution by separation of variables. This problem was solved by Shivakumar and

2.5 Uniaxial Compression of a Plate

Whitcomb [13] by the finite elements technique and the Ritz' approach.

They considered, also, a more complex problem of an elliptical delamination, Fig.2.12 (a), in an orthotropic plate.

Using the Ritz' method, they seek the solution in the form ($\alpha = 0$) :

$$w = \left[1 - \left(\frac{x}{a} \right)^2 - \left(\frac{y}{b} \right)^2 \right] (C_1 + C_2 x^2 + C_3 y^2) ,$$

where a, b are semiaxes of the ellipse and C_1, C_2, C_3 are arbitrary constants (Ritz' coefficients) to be found from the variational equations.

(d) *Square Shape of Delamination*

The initial delamination has the form of the square

$$|x| \leq a, \quad |y| \leq a .$$

In this case, the work of the external load is

$$A = \frac{1}{2} p a h \int_0^a \left(\frac{\partial w}{\partial x} \right)^2 dx .$$

In the first approximation, the buckling deflection can be taken in the simple form

$$w = C_0 (x - a)^2 (y - a)^2 .$$

In the next approximation,

$$w = (x - a)^2 (y - a)^2 [C_1 + C_2 x^2 + C_3 y^2] .$$

2. DELAMINATION BUCKLING

(e) *The Case of an Orthotropic Plate*

In this case, instead of (2.35), we have [6] the differential equation

$$D_{11} \frac{\partial^4 w}{\partial x^4} + 2D_{12} \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} + hp \frac{\partial^2 w}{\partial x^2} = 0 , \quad (2.42)$$

where the D 's are the flexural stiffness constants for the orthotropic laminate.

In the variational equation (2.37), the expression for the potential energy has a form more complex than (2.38), namely,

$$U = \frac{1}{2} \iint_{\Omega} \left[D_{11} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + 2D_{12} \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + D_{22} \left(\frac{\partial^2 w}{\partial y^2} \right)^2 \right] dx dy .$$

The first Ritz' approximations given above can be used in this case as well since the geometrical conditions (2.36) on the contour C are satisfied, as before.

Chapter 3

DELAMINATION GROWTH

3.1 Buckling and Delamination Growth

(a) *Postcritical Deformation*

When the compressional stress p exceeds the critical stress p_{cr} , both the deflection and the delamination increase. Because of the geometrical nonlinearity of the postcritical deflection it is difficult to analyze the postcritical behaviour of the buckled near surface laminate. So, simple models are preferable for the analysis of this problem.

For instance, Bottega and Maewall [14] use a model of a two-layer circular disc with initial penny-shaped delamination. The disc is loaded by axisymmetric in plane compression (Fig.3.1); the same problem in the dynamic formulation was considered in [15].

Another simple model is considered by Chai, Babcock and Knauss [16]. They analyze the process of delamination of a thin strip under uniaxial compression.

Here, we analyze a simple case when there is an initial delamination prior to loading; we use the approach developed in [16]. Let us consider a thick plate under uniaxial compressive stress Fig.3.2 (a), (b). Near the surface there is an initial delamination of length ℓ (dashed line).

It is more convenient to consider the strain ε_0 of the plate than the corresponding stress. When the plate is loaded, the elastic strains are:

$$\varepsilon_x = -\varepsilon_0, \quad \varepsilon_z = \nu \varepsilon_0.$$

3. DELAMINATION GROWTH

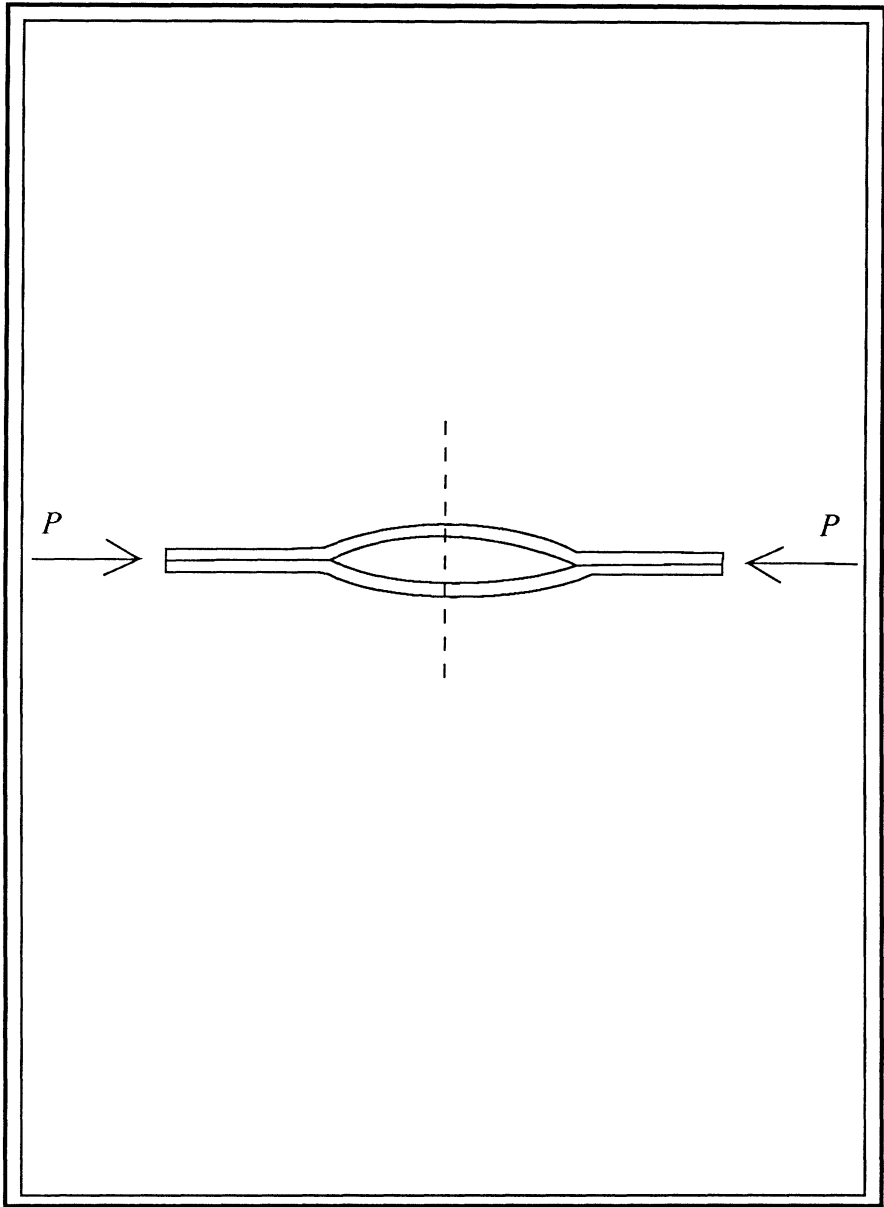


Figure 3.1

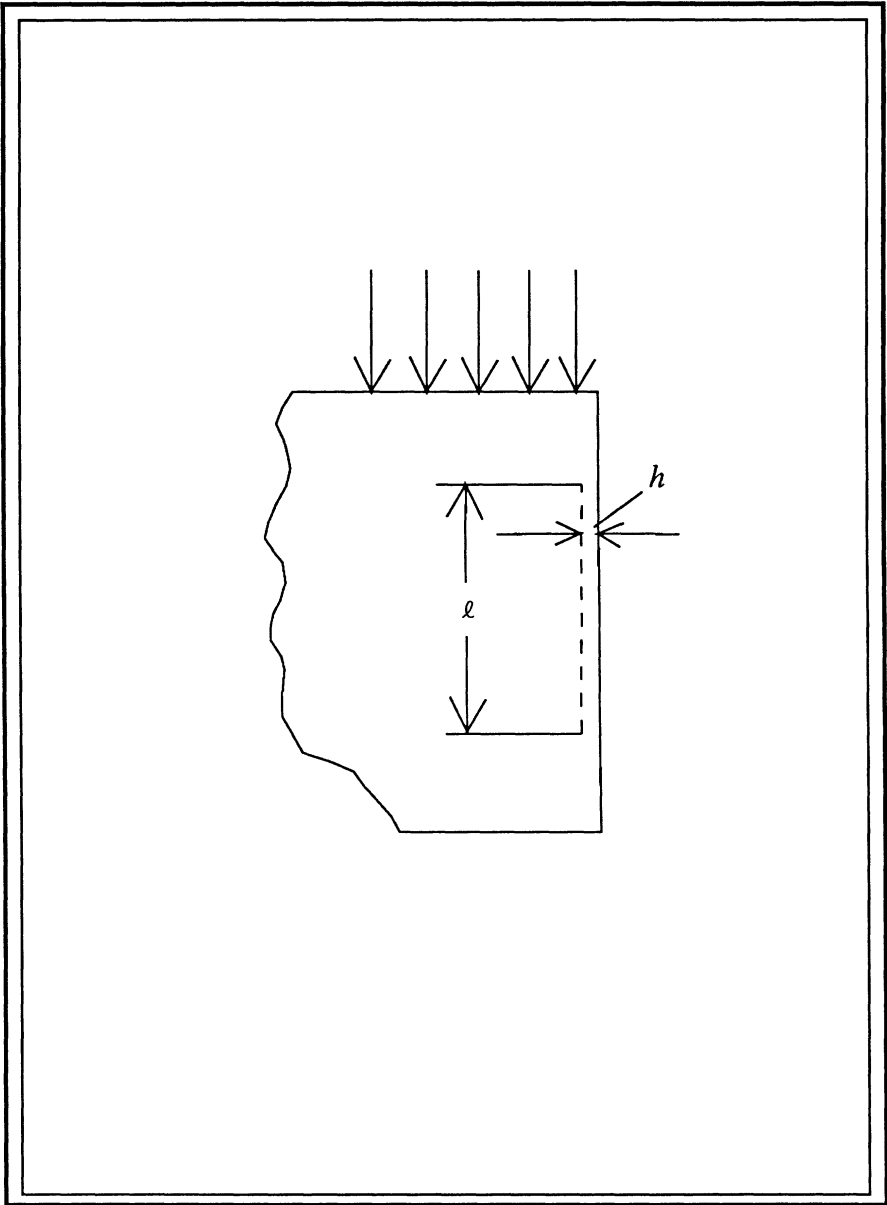


Figure 3.2 (a)

3. DELAMINATION GROWTH

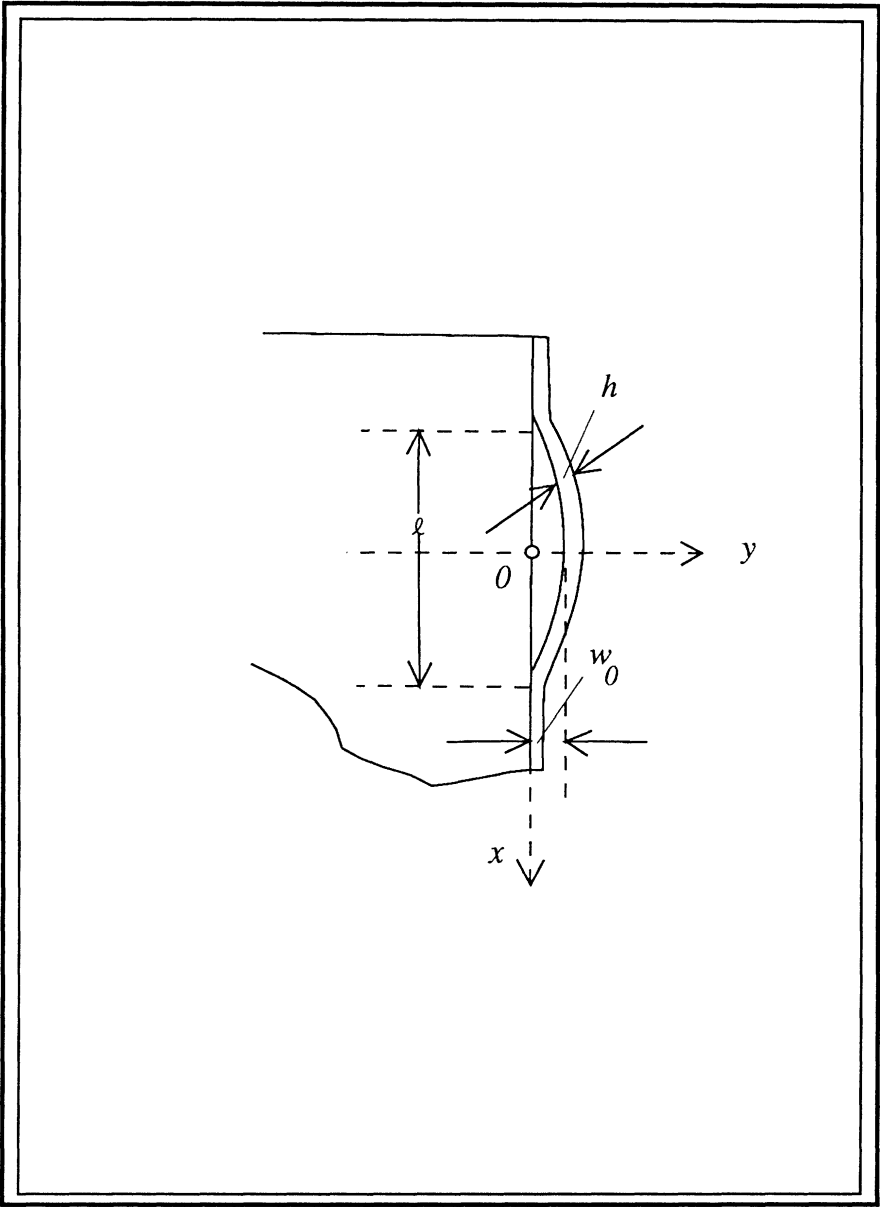


Figure 3.2 (b)

3.1. Buckling and Delamination Growth

When the compressive strain reaches the critical value ε_{cr} the layer near the surface buckles. Using the solution of Euler's classical problem for a column under compression (see sect.1.1), we have the following expressions for ε_{cr} :

$$\varepsilon_{cr} = \frac{\sigma_{cr}}{E} = \frac{\pi^3}{3(1 - \nu^2)} \left(\frac{h}{\ell} \right)^2 , \quad (3.1)$$

and for the buckling displacement:

$$y = \frac{l}{2} w_0 \left(1 + \cos \frac{2\pi x}{\ell} \right) , \quad (3.2)$$

where w_0 is the buckling amplitude determined by the condition that the length contraction due to bending is equal to the length contraction due to compression; the stress in the buckled strip is the same as the buckling stress. These conditions lead to the expression

$$(\varepsilon_0 - \varepsilon_{cr}) (1 - \nu^2) \ell = \int_{-\ell/2}^{\ell/2} \frac{1}{2} \left(\frac{dy}{dx} \right)^2 dx .$$

Calculations of sect.1.1 yield

$$w_0^2 = (\varepsilon_0 - \varepsilon_{cr}) (1 - \nu^2) \left(\frac{2\ell}{\pi} \right)^2 .$$

Taking $\varepsilon_0 = 0$ and changing the sign of ε_{cr} , we obtain the formula (2.4) without the factor $(1 - \nu^2)$ (the latter is present because of the large width of the laminated strip in the z -direction).

Prior to buckling ($\varepsilon_0 < \varepsilon_{cr}$) the elastic potential energy of the strip per unit width is

3. DELAMINATION GROWTH

$$U_0 = \frac{l}{2} E h \ell \varepsilon_0^2 \quad . \quad (3.3)$$

If the deflection is relatively small, we can assume that the postcritical equilibrium shape $y(x)$ is analogous to the shape in the critical state.

The potential energy of the buckled strip consists of the membrane energy and the bending energy:

$$U = \frac{l}{2} E h \ell \left[\varepsilon_{cr}^2 (1 - \nu^2) + \nu^2 \varepsilon_0^2 \right] + \int_{-\ell/2}^{\ell/2} \frac{E h^3}{24(1 - \nu^2)} \left(\frac{d^2 y}{dx^2} \right)^2 dx \quad . \quad (3.4)$$

Prior to buckling, $\varepsilon_0 < \varepsilon_{cr}$, and the potential energy is equal to U_0 .

When $\varepsilon_0 = \varepsilon_{cr}$, buckling occurs; at $\varepsilon_0 > \varepsilon_{cr}$, the strip is in a postcritical state, and the potential energy is given by (3.4).

Let us introduce the dimensionless "load"

$$\frac{\varepsilon_0}{\varepsilon_{cr}}$$

and the normalized potential energy

$$U/U_{cr} \quad ,$$

where

3.1. Buckling and Delamination Growth

$$U_{cr} = \frac{1}{2} E h \ell \varepsilon_{cr}^2 = U_0 \Big|_{\varepsilon_0 = \varepsilon_{cr}} .$$

Hence,

$$U_0 / U_{cr} = \left(\frac{\varepsilon_0}{\varepsilon_{cr}} \right)^2 . \quad (3.5)$$

Thus, the potential energy U_0 is a quadratic function of the load; it is shown in Fig.3.3 by the dashed line.

In the normalized variables, formula (3.4) takes the form

$$\frac{U}{U_{cr}} = (1 - \nu^2) \left[2 \frac{\varepsilon_0}{\varepsilon_{cr}} - 1 + \frac{\nu^2}{1 - \nu^2} \left(\frac{\varepsilon_0}{\varepsilon_{cr}} \right)^2 \right] . \quad (3.6)$$

For $\varepsilon_0 / \varepsilon_{cr} \geq 1$, this dependence is shown in Fig.3.3 by a solid line (for the case $\nu = 0.3$).

The total potential energy released in the process of delamination may result in a further interlaminar fracture, i.e. in a further delamination.

(b) *Criterion of Fracture; Delamination Growth*

Interlaminar fracture can be considered as a crack propagation. Therefore, it is reasonable to use the concepts of fracture mechanics for the analysis of the delamination growth.

Let us compare the energy release rate with the rate of fracture work in the process of delamination.

The energy release rate J is calculated for the condition that the length of delamination changes from ℓ to $\ell + \Delta \ell$.

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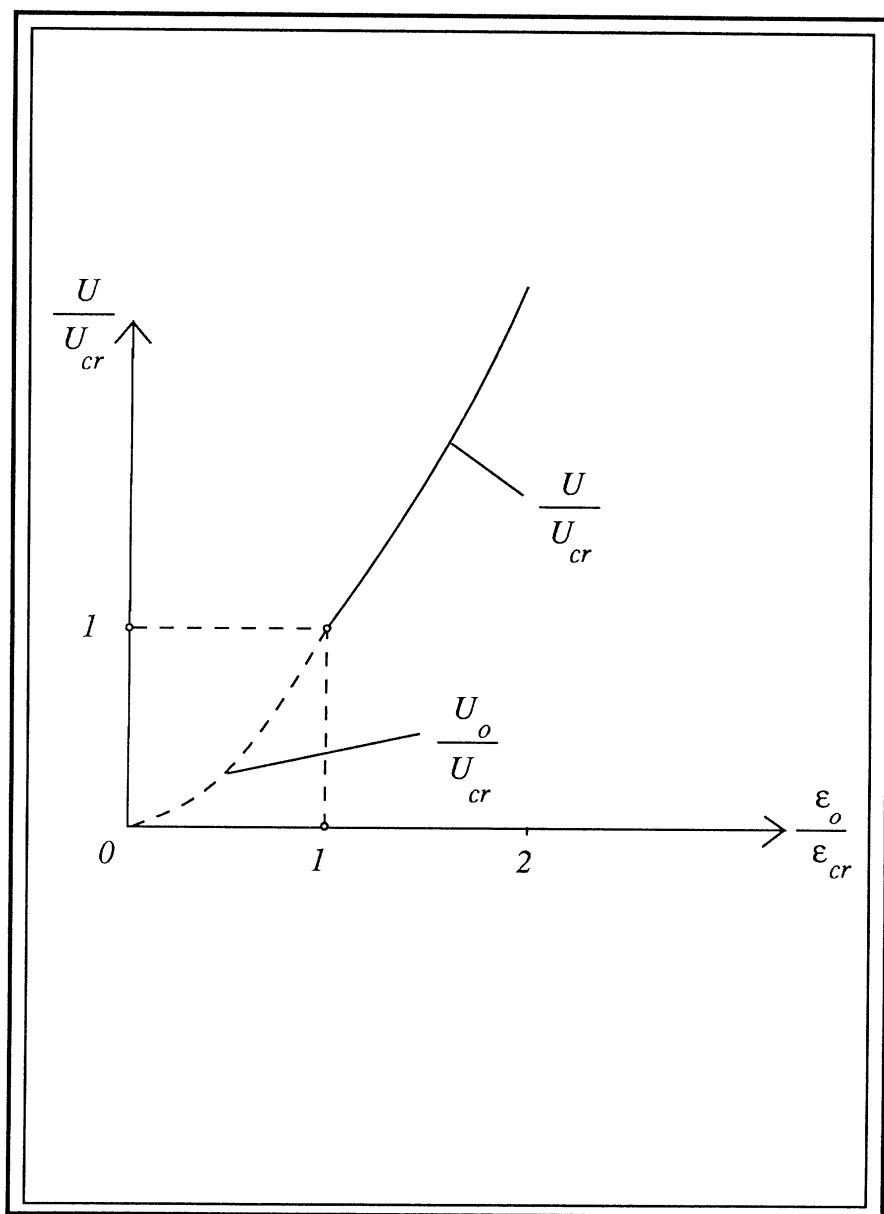


Figure 3.3

Thus

$$J = \lim_{\Delta \ell \rightarrow 0} \left\{ U(\ell) + \frac{1}{2} E \varepsilon_0^2 h \Delta \ell - U(\ell + \Delta \ell) \right\} \frac{1}{\Delta \ell} .$$

The second term in the brackets corresponds to the increment of work of the external compressive stresses. Calculation yields

$$J = \frac{1}{2} E h (1 - \nu^2) (\varepsilon_0 - \varepsilon_{cr}) (\varepsilon_0 + \varepsilon_{cr}) . \quad (3.7)$$

The stresses at the ends of the delamination ("delamination crack tips") promote further splitting only after buckling has occurred.

Therefore, the question of delamination growth is of interest only for $\varepsilon_0 > \varepsilon_{cr}$.

Only in this case the energy release rate is positive.

Let us introduce the value Γ_0 that is defined as the energy required to produce a unit of new delamination. Note that a unit of delamination generates two units of new fracture surface so that $\Gamma_0 = 2\gamma_0$, where γ_0 is the specific surface energy.

The relation (3.6) can be rewritten in the form

$$J = \frac{1}{2} E h (1 - \nu^2) \varepsilon_{cr}^2 \left(\frac{\varepsilon_0}{\varepsilon_{cr}} - 1 \right) \left(\frac{\varepsilon_0}{\varepsilon_{cr}} + 3 \right) . \quad (3.8)$$

3. DELAMINATION GROWTH

According to (3.1),

$$\varepsilon_{cr} = \frac{\pi^2}{3(1-\nu^2)} \left(\frac{h}{\ell} \right)^2.$$

Hence,

$$J_* = \left(\frac{\varepsilon_0}{\varepsilon_{cr}} - 1 \right) \left(\frac{\varepsilon_0}{\varepsilon_{cr}} + 3 \right) \left(\frac{h}{\ell} \right)^4, \quad (3.9)$$

where the dimensionless energy release rate

$$J_* = \frac{18}{\pi^4} \frac{J}{Eh}$$

is introduced; note that $\ell > \ell_0$ where ℓ_0 is the initial delamination.

The factor

$$\left(\varepsilon_0 - 1 \right) \left(\frac{\varepsilon_0}{\varepsilon_{cr}} + 3 \right)$$

characterizes the level of load. The dependence (3.9) is shown in Fig.3.4 for different values of

$$\frac{\varepsilon_0}{\varepsilon_{cr}}.$$

Note that

$$\varepsilon_{0,1} < \varepsilon_{0,2} < \varepsilon_{0,3}.$$

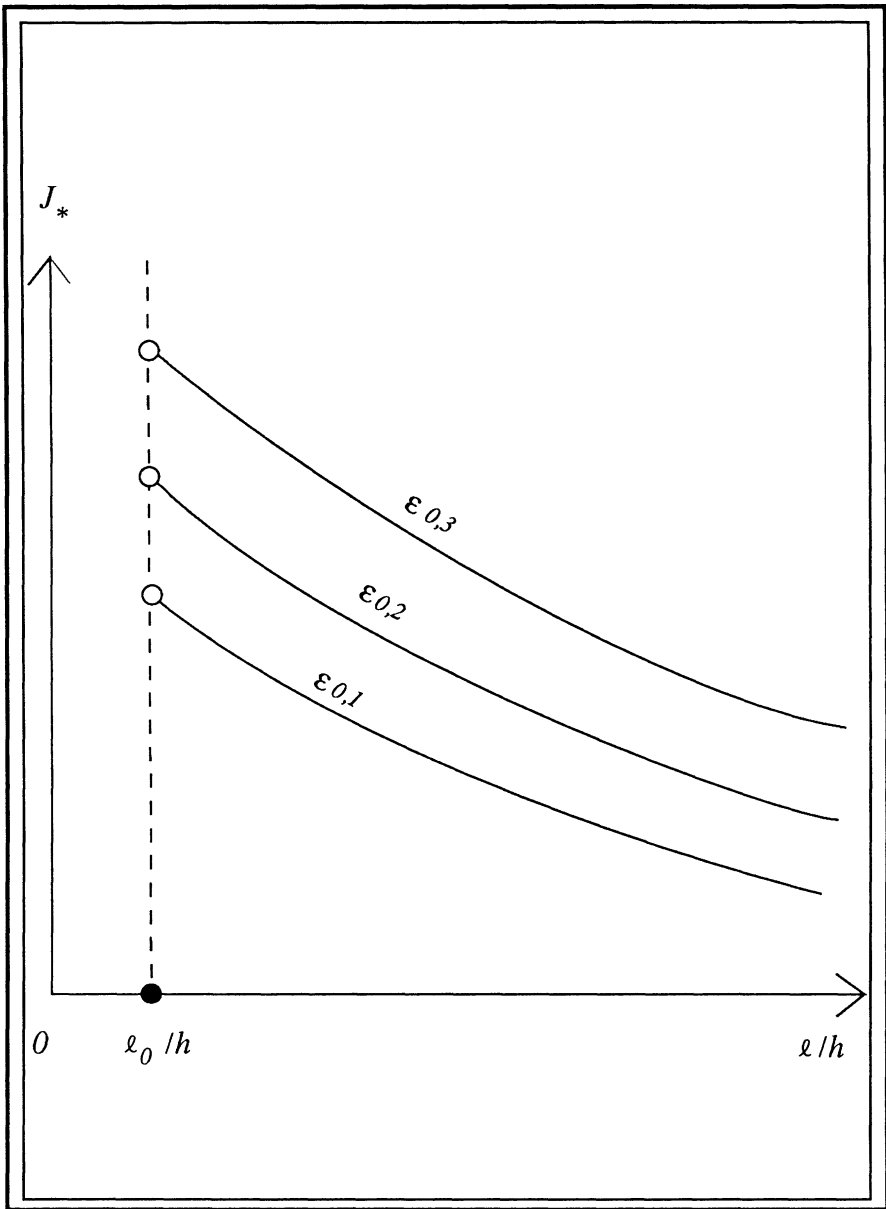


Figure 3.4

3. DELAMINATION GROWTH

3.2 Time Effects

Mechanical behaviour of fiber composites changes with time. Here, we will consider only their deformation, but will not consider such time effects as aging and environmental degradation.

When a composite is subjected to the constant stress σ , strain increases with time (creep); and when it is under constant strain ϵ stress decreases with time (relaxation).

These phenomena can be illustrated by a simple Maxwell's model consisting of an elastic element (E - elastic modulus) and a viscous element (μ - coefficient of viscosity), Fig.3.5. More complex time effects can be described by combining elastic and viscous elements with various values of constants E and μ . Therefore, the mechanical behaviour of polymers is termed viscoelastic.

(a) *Linear Viscoelasticity*

The mechanical behaviour of polymers is generally described by linear differential or integral relations. Although the linear theory is widely used and convenient in application, it, certainly, is only an approximation, it does not precisely describe the behaviour of materials.

The general form of the linear stress-strain relation has been given by L.Boltzmann and V.Volterra. This relation is based on the principle of superposition and on the introduction of some function ("hereditary coefficient") which characterizes the influence of the previous deformation on the successive one.

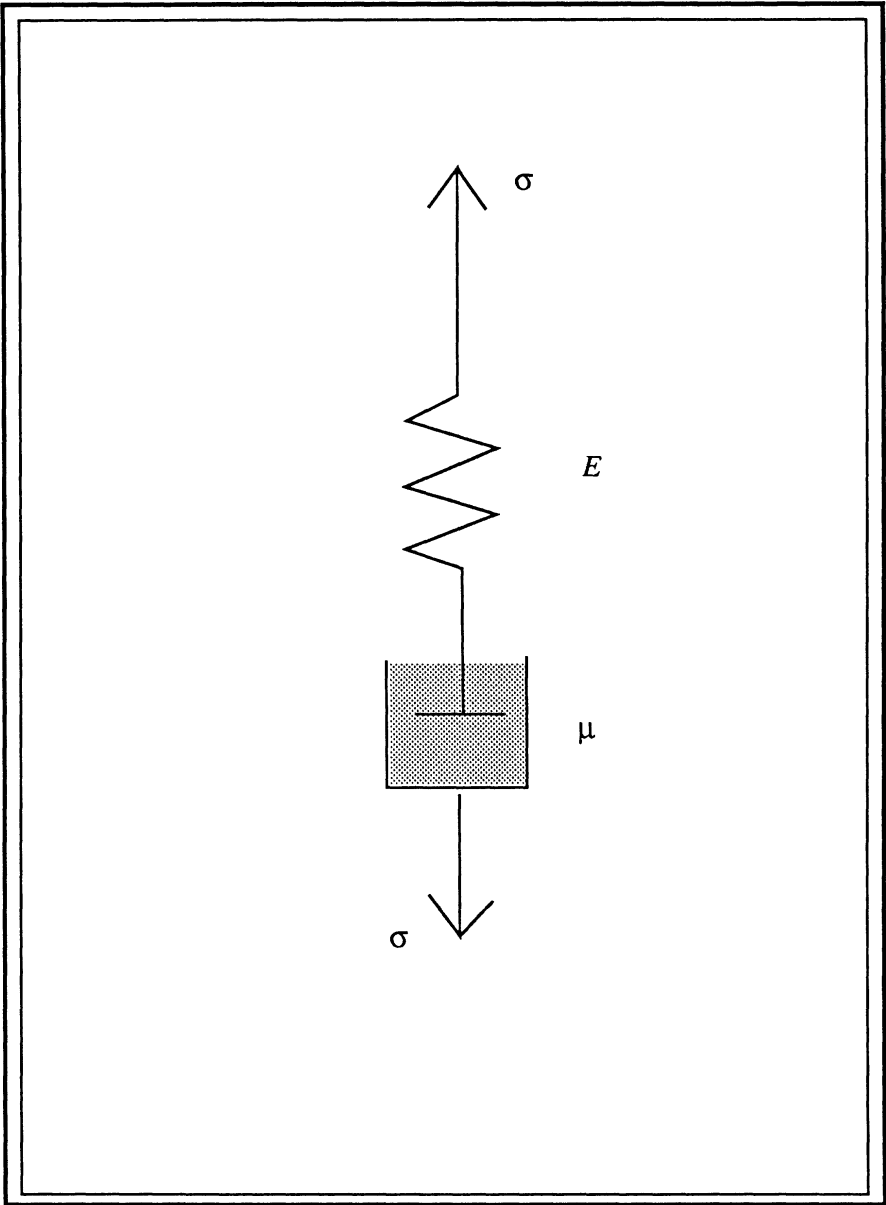


Figure 3.5

3. DELAMINATION GROWTH

Boltzmann's integral relation in the case of uniaxial tension is

$$\varepsilon(t) = \frac{\sigma(t)}{E} \int_0^t \varphi(t - \tau) \sigma(\tau) d\tau, \quad (3.10)$$

where E is Young's modulus, t the time.

The first term represents the instantaneous elastic deformation.

The kernel $\varphi(t - \tau)$ is a decreasing function (that corresponds to the "fading memory").

The relation (3.10) with respect to stress σ is a Volterra integral equation.

Solving this equation, we obtain

$$\sigma(t) = E\varepsilon(t) - \int_0^t \psi(t - \tau) \varepsilon(\tau) d\tau, \quad (3.11)$$

where $\psi(t - \tau)$ is the resolvent of the kernel $\varphi(t - \tau)$. Different materials are described by appropriate choices of the kernel $\varphi(t - \tau)$ [or $\psi(t - \tau)$].

Let us take

$$\psi(t - \tau) = \frac{E}{t_0} \exp\left(-\frac{t - \tau}{t_0}\right), \quad (3.12)$$

where t_0 is some constant ("time of relaxation").

Differentiating the equation (3.11) with respect to time, and eliminating the integral term, we obtain Maxwell's relation

$$E \frac{d\varepsilon}{dt} = \frac{d\sigma}{dt} + \frac{\sigma}{t_0} \quad (3.13)$$

which describes the behaviour of the model shown in Fig.3.5.

In the case of relaxation we have

$$\varepsilon(t) = \text{const.} = \varepsilon_0 \quad \text{for } t \geq 0.$$

Then, the equation (3.11) becomes

$$\sigma(t) = \varepsilon_0 \left[E - \int_0^t \psi(t - \tau) d\tau \right]. \quad (3.14)$$

This relation describes the relaxation curve.

Note that Boltzmann's theory is extended to the case of multiaxial stress state.

On the right-hand side of the equations (3.10) and (3.11), a linear integral operation is used; we can rewrite these equations in operator form as

$$\sigma = \tilde{E} \varepsilon, \quad (3.15)$$

$$\varepsilon = \tilde{E}^{-1} \sigma, \quad (3.16)$$

where $\tilde{E}$ and $\tilde{E}^{-1}$ are corresponding linear integral operators.

So, the viscoelastic law is analogous to Hooke's law with modulus E substituting by operator $\tilde{E}$.

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(b) *Correspondence Principle*

Consider the equilibrium of an elastic body under specified surface tractions which are constant in time.

Let us denote the stress field in the body by σ_{ij} ($i, j = 1, 2, 3$), the strain field by ϵ_{ij} and the displacement field by u_i . Consider now a *linearly viscoelastic body* of the form identical to the elastic one and under identical boundary conditions.

It is not difficult to see that the stress field in the viscoelastic body will be the same as in the elastic one.

The fields of strain and displacement in the viscoelastic body can be found by substituting the elastic constants by the corresponding linear integral operators in the "elastic solution".

Note that the corresponding principle holds true also for a body with a slow growing crack.

(c) *Buckling of a Viscoelastic Strip*

Consider a simple case of buckling of a thin viscoelastic strip, Fig.2.2, with clamped ends which is subjected to a constant compressive load. Let the material of the strip obey Maxwell's relation (3.13).

According to Bernoulli's assumption of plane sections in bending, the axial strain rate is

$$\dot{\epsilon} = \dot{\kappa} z \quad (3.17)$$

where

$$\dot{\kappa} = -\dot{w}''$$

3.2. Time Effects

is the curvature rate of the strip axis and $w = w(x, t)$ is the deflection in buckling. The distance z is reckoned from the strip axis. Here, primes denote differentiation with respect to coordinate x , dots - with respect to time t .

Substituting $\dot{\varepsilon}$ into (3.13), multiplying it by z and integrating over the cross-sectional area, we obtain

$$EI \dot{w}'' + \frac{I}{t_0} \dot{M} + M = 0, \quad (3.18)$$

where I is the moment of inertia of the cross-sectional area, and M is the bending moment in buckling.

To simplify the calculations, consider first the buckling of the strip $|x| \leq \ell/2$ with pinned ends under a compressive load P .

In this case, the bending moment is

$$M = Pw. \quad (P = \text{const}).$$

Thus, the equation (3.18) becomes

$$EI \dot{w}'' + \frac{P}{t_0} \dot{w} + Pw = 0. \quad (3.19)$$

Seeking the solution in the form

$$w = T(t) X(x),$$

and using the boundary conditions

$$w = 0 \quad \text{at} \quad x = \pm \ell/2,$$

3. DELAMINATION GROWTH

we obtain

$$X = \cos \frac{\pi x}{\ell}$$

and

$$T = C \exp \left(\frac{P}{P_c - P} \frac{t}{t_0} \right) ,$$

where C is an arbitrary constant, and

$$P_c = \frac{\pi^2 EI}{\ell^2}$$

is the critical load for the elastic strip.

If the initial deflection at $x = 0$ is equal to w_0 , we obtain

$$T = w_0 \exp \left(\frac{P}{P_c - P} \frac{t}{t_0} \right) . \quad (3.20)$$

It is obvious that $P \leq P_c$.

Thus, the deflection grows exponentially with time. If the load approaches the critical one, the rate of deflection

$$\dot{w} \rightarrow \infty .$$

Now, when we consider a strip with clamped ends, we note that in this case the elastic line consists of four equivalent parts of a sinusoid. Therefore, in the formula (3.20), it is necessary to assume that

$$P_c = \frac{4 \pi^2 EI}{\ell^2} .$$

In conclusion, we must say that above given scheme can be extended to the general case of the behaviour of viscoelastic material, the one described by the Boltzman's hereditary equation (3.10).

3.3 Criteria of Visco-Brittle Fracture

(a) *Stress and Displacement Fields at the Crack Tip*

Let us consider a crack in a body under tension in the case of two-dimensional deformation, Fig.3.6. The stress field in the vicinity of the crack tip in a viscoelastic body is identical to that in an elastic one, and it has an identical singularity

$$1/\sqrt{r} \ ,$$

where r is the distance from the crack tip.

But, with time, displacements in the viscoelastic body increase and so does the crack opening; it can be found by using the correspondence principle.

(b) *Brittle Fracture of an Elastic Body*

In the case of an elastic body with a crack, Griffith-Irvin's criterion of fracture can be used

When the crack grows in an elastic body, there is an energy release. The released energy flows into the crack tip where it expends on fracture. The released energy is characterized by the well-known path independent integral J . The fracture criterion is

$$J = J_c \ , \quad (3.21)$$

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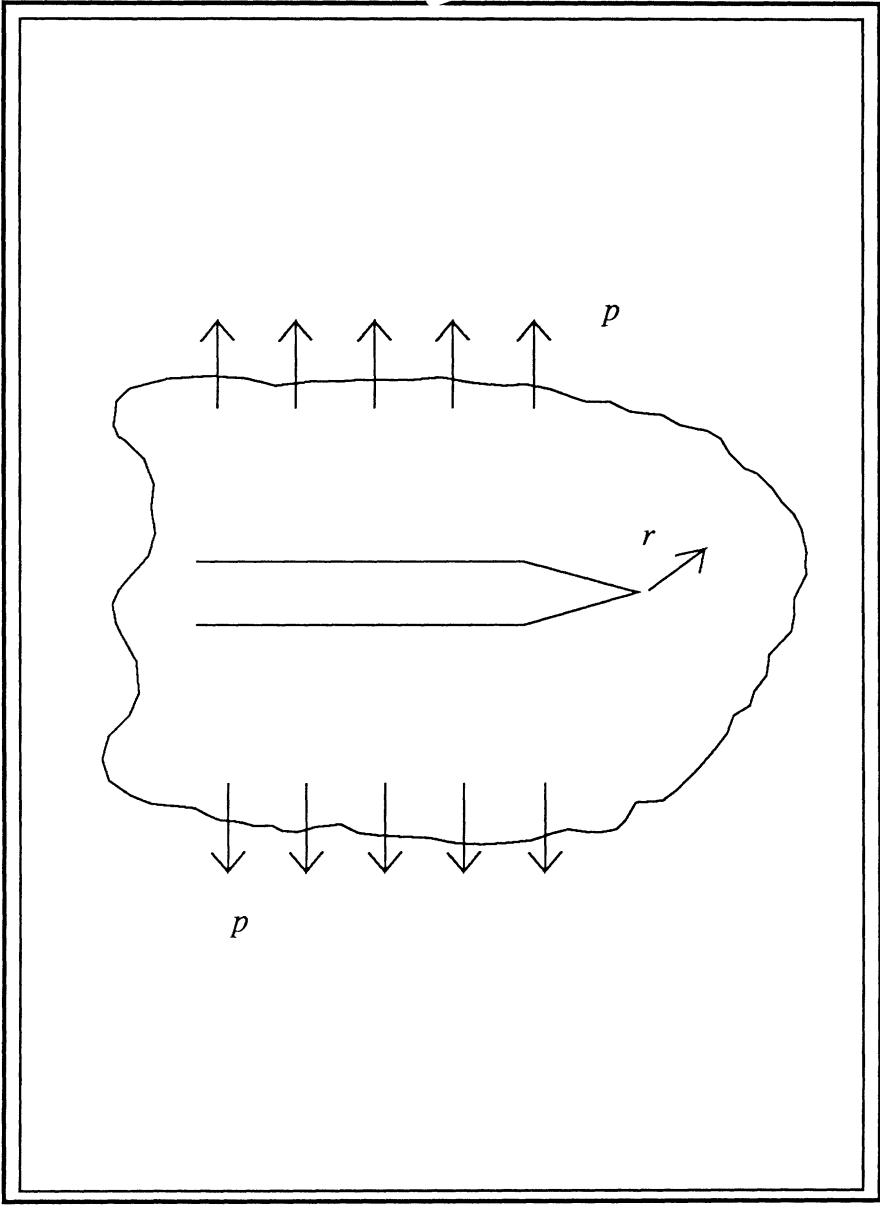


Figure 3.6

where J_c is the critical value that can be considered as a material constant.

(c) *Dissipative Criterion of Fracture*

In the case of a viscoelastic body, it is necessary to introduce a new criterion of fracture.

This criterion can probably be best formulated in terms of the crack opening displacement.

The displacement field in a viscoelastic body depends on the kernel $\varphi(t - \tau)$ and, generally, may be rather complex.

There are several works on the subject (see, for example, Shapery's work [18]).

Here, we shall consider a simple case of Maxwell's medium. For this case, the fracture criterion can be formulated in a relatively simple form. So, we can easily obtain some solutions (mostly qualitative) of delamination growth.

When Maxwell's medium is subjected to a constant load, there appear both elastic deformation and (under the condition of small strain) a slow viscous flow.

The stress field at the crack tip, as mentioned above, is identical to that in the elastic body. The velocity field in Maxwell's medium is analogous to the displacement field in the corresponding elastic medium.

The viscous flow results in dissipation.

Consider now a slow flow in a linearly viscous body.

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Analogously to the total potential energy $\Pi = U - A$ of an elastic body (sect.1.1), we can introduce for the viscous body the total dissipation rate

$$D = \dot{U} - \dot{A} , \quad (3.22)$$

where $\dot{U}$ denotes the dissipation rate of the viscous flow of the body, and $\dot{A}$ denotes the rate of the external load.

When a crack grows, the total dissipation rate D decreases (analogously to the decrease of the total potential energy in the case of an elastic body).

The dissipation "flows" into the crack tip. The crack tip acts as a source of dissipation. Analogously to the path independent integral J for the elastic body, the path independent integral J_* can be introduced [10] for the linearly viscous body.

It is not difficult to prove

$$J_* = - \frac{\partial D}{\partial \ell} . \quad (3.23)$$

In the singular point of the field (crack tip), there is an accumulation of dissipation.

We can assume that this accumulation damages the structure of the material and decreases its strength. Thus, the value J_* can characterize the rate of crack growth $\dot{\ell}_v$.

It is natural to introduce the dissipative criterion of fracture in the form [10]

$$\dot{\ell}_v = c J_* \quad (3.24)$$

where c is a material constant.

(d) *Criterion of Visco-Brittle Fracture*

Consider now a linear viscoelastic body of the Maxwell's type. This medium consists of two phases - elastic and viscous. Therefore, the fracture criterion of the medium must embrace two possible mechanisms of fracture - brittle and viscous.

We assume that the rate of crack growth $\dot{\ell}$ consists of two components - brittle $\dot{\ell}_B$ and viscous $\dot{\ell}_v$, i.e.

$$\dot{\ell} = \dot{\ell}_B + \dot{\ell}_v . \quad (3.25)$$

The viscous component is given by (3.24). The energy release rate is determined by the total velocity $\dot{\ell}$.

According to Griffith-Irvin's criterion,

$$d\Pi = -J d\ell . \quad (3.26)$$

Hence, the brittle component can be considered as proportional to $J\dot{\ell}$, i.e.

$$\dot{\ell}_B = \alpha J \dot{\ell} , \quad (3.27)$$

where α is a constant coefficient.

Since the stress field at the crack tip in the considered viscoelastic body is identical to that in the elastic body, the integral J_* is proportional to the integral J , so

$$\dot{\ell}_v = \beta J , \quad (3.28)$$

where β is a certain constant.

Thus, we obtain

3. DELAMINATION GROWTH

$$\dot{\ell} = \frac{\beta J}{1 - \alpha J} \quad . \quad (3.29)$$

In the case $\alpha J = 1$, we have Griffith-Irvin's criterion of fracture; then

$$J = J_c \quad ,$$

and $\dot{\ell} \rightarrow \infty$. Note that $\alpha = 1/J_c$.

The J integral is a function of the crack parameter ℓ .

Integrating the differential equation (3.29) under the initial condition

$$\ell = \ell_0 \quad \text{at} \quad t = 0 \quad , \quad (3.30)$$

we find the current dimension of the crack.

(e) *Example: Delamination Growth under Compression*

Consider the problem of delamination growth from a long compressed column, Fig.3.7.

Assume that under the compressive stress σ the delamination length is equal to ℓ ; the thickness of the delaminated strip is assumed to be small.

The deflection of the delaminated strip is

$$w = w_0 \sin^2 \frac{\pi x}{\ell} \quad .$$

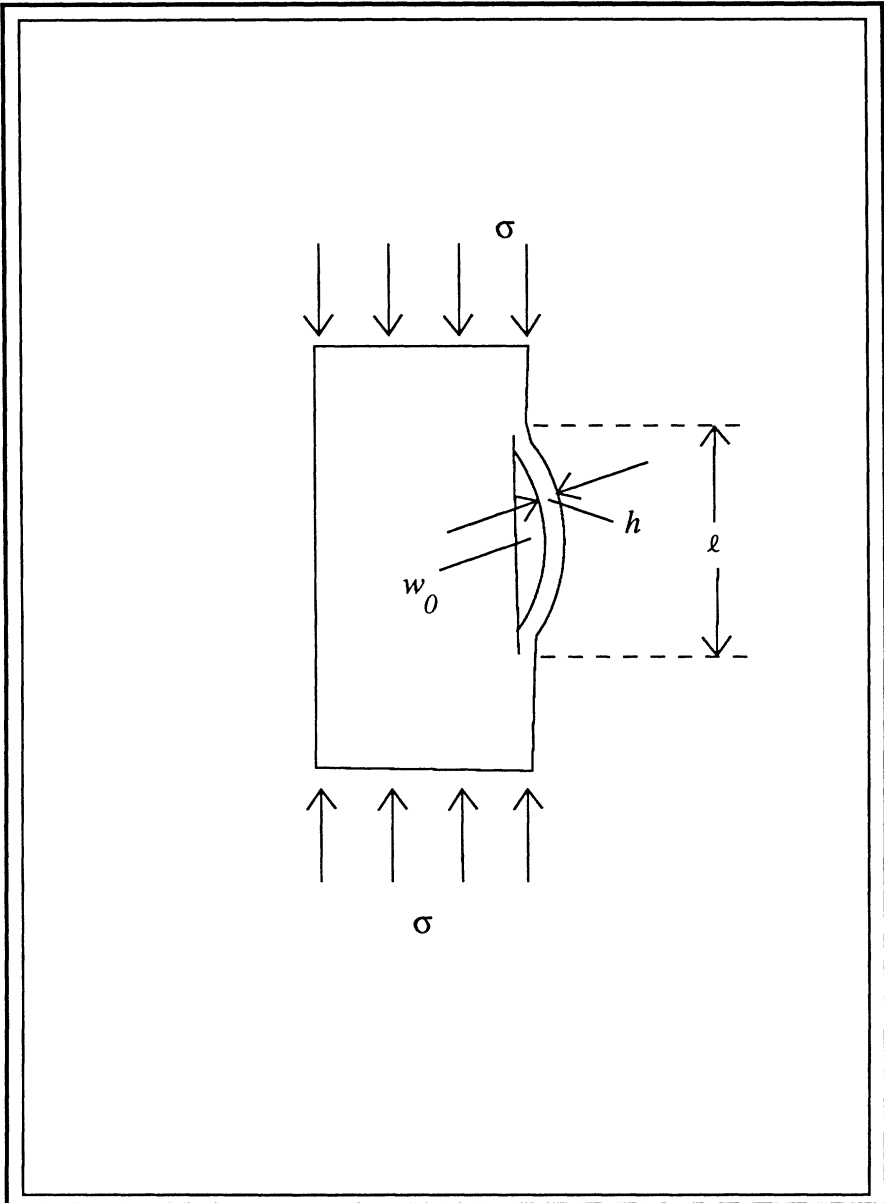


Figure 3.7

3. DELAMINATION GROWTH

Earlier, we have found that

$$w_0^2 = \frac{4}{\pi^2} \frac{\sigma}{E} \ell^2 .$$

It is not difficult to calculate the corresponding bending energy

$$U_2 = \frac{\pi^2 \sigma}{3 \ell} h^3$$

(per unit width of the strip).

The work of the external load (applied to the ends of the strip) is equal to

$$A = \sigma h \Delta \ell_1 ,$$

where

$$\Delta \ell_1 = \frac{\sigma}{E} \ell .$$

Hence,

$$A = \frac{\sigma^2}{E} h \ell .$$

The total potential energy of the strip is

$$\Pi = U_2 - A = \sigma h \left[\frac{\pi^2}{3 \ell} h^2 - \frac{\sigma}{E} \ell \right] .$$

When delamination grows, the energy release is equal to

3.3. Criteria of Visco-Brittle Fracture

$$J = -\frac{\partial \Pi}{\partial \ell} = \sigma h \left[\frac{\sigma}{E} + \frac{\pi^2}{3} \left(\frac{h}{\ell} \right)^2 \right] . \quad (3.31)$$

As mentioned above, Griffith-Irvin's criterion of crack growth can be written in the form

$$J = J_c . \quad (3.32)$$

Let ℓ_0 be the initial value of ℓ ; note that $\ell \geq \ell_0$.

The released energy J is a positive decreasing function of ℓ .

From (3.31) and (3.32) it follows that

$$\frac{h}{\ell} = \frac{\pi}{\sqrt{3}} \left(\frac{E}{\sigma_c - \sigma} \right)^{1/2} , \quad (3.33)$$

where

$$\sigma_c = EJ_c / \sigma h$$

is the critical stress of fracture; if $\sigma \rightarrow \sigma_c$, the length $\ell \rightarrow \infty$. The crack ("delamination") length grows with increasing stress σ . Hence, we have the case of a stable crack.

Consider now delamination growth with time. Integrating the differential equation (3.29), we obtain

$$t = \int_{\ell_0}^{\ell} \frac{1 - \alpha J}{\beta J} d\ell . \quad (3.34)$$

The integrand is a positive increasing function of ℓ . Hence, the time t is a monotonically increasing function of the length ℓ and v.v. .

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Note that the length ℓ also depends on the stress σ as a parameter, i.e.

$$\ell = \ell(t, \sigma) .$$

Besides,

$$\dot{\ell} = \infty \quad \text{for} \quad \sigma = \sigma_c .$$

For $\sigma < \sigma_c$, the derivative $\dot{\ell}$ is positive and bounded. The relationship $\ell = \ell(t)$ is shown in Fig.3.8.

(f) *Concluding Remarks*

Using the same scheme, it is possible to analyse various problems of delamination growth.

In particular, it is not difficult to solve the problem in the case of axisymmetric compression of a circular plate (sect. 2.2), assuming that the energy release is given by the derivative $-\partial\Pi/\partial a$, where a is the radius of the delaminated circle and Π is the potential energy in buckling.

3.4 Wave of Delamination

Sometimes one can observe a delamination of a surface layer in the form of a "bubble" rapidly propagating along the boundary subjected to a uniaxial compression σ , Fig.3.9. Denote the velocity of propagation by a . Assuming that the boundary is long, the velocity can be considered as constant. It can be evaluated in the following simple way. Assume that the "height" of the bubble is small. The energy released per unit time and unit width is

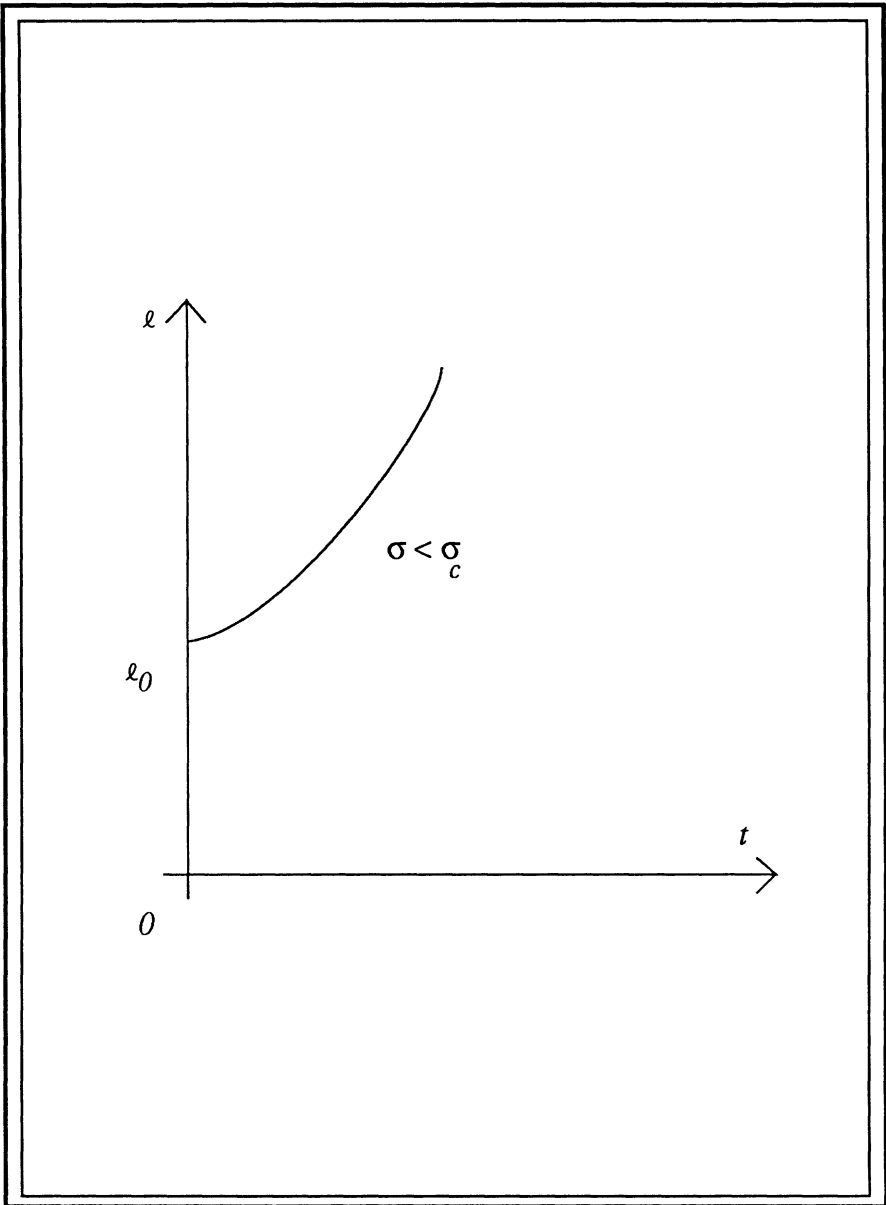


Figure 3.8

3. DELAMINATION GROWTH

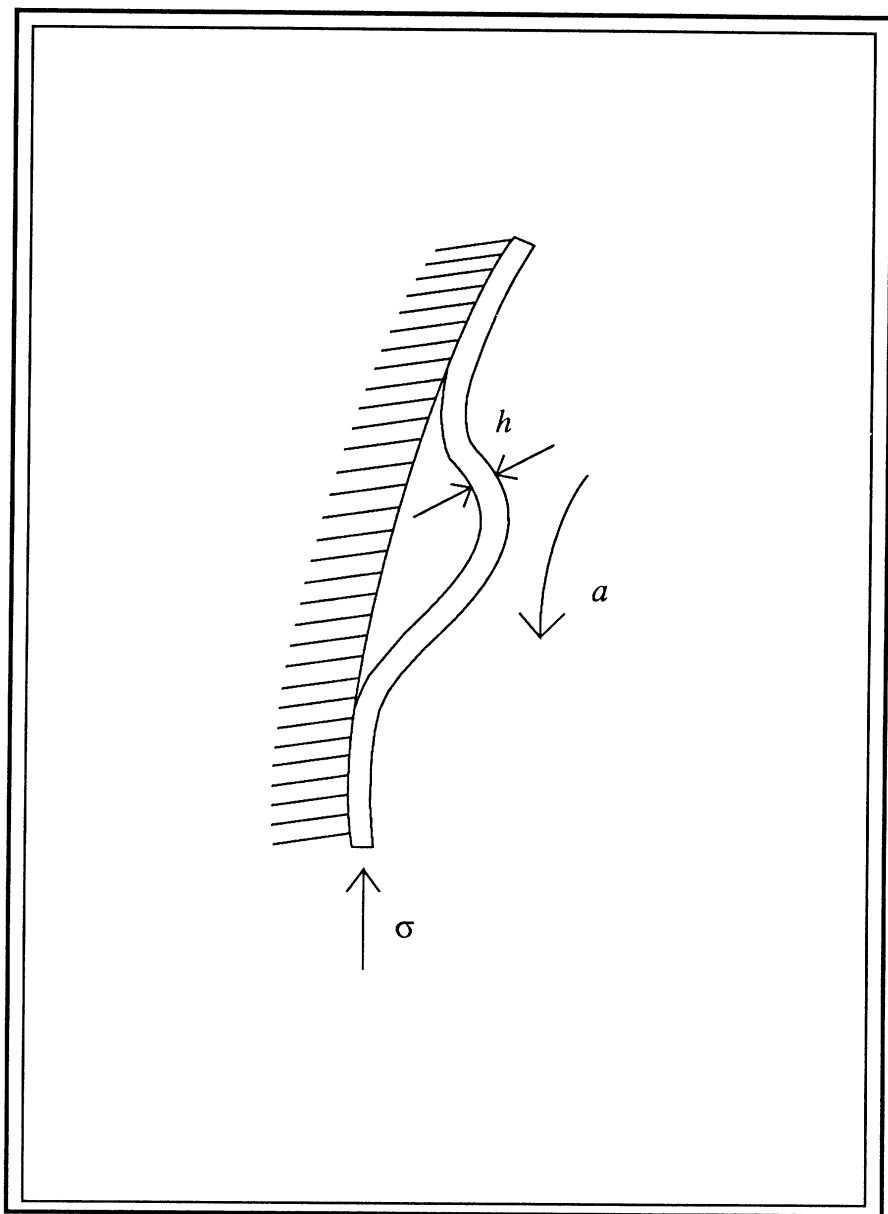


Figure 3.9

3.4. Wave of Delamination

$$\frac{\sigma^2}{2E} ah \quad ,$$

where h is the thickness of the delaminated strip. This energy transforms into the fracture work $2\gamma a$ and the kinetic energy

$$\rho ah \frac{a^2}{2} \quad ,$$

where ρ is the material density.

Thus,

$$\frac{\sigma^2}{2E} ah = 2\gamma a + \rho ah \frac{a^2}{2} \quad .$$

Hence, we obtain

$$a^2 = \frac{1}{\rho} \left(\frac{\sigma^2}{E} - \frac{4\gamma}{h} \right) \quad .$$

The velocity a is relatively small.

It is obvious that the wave of delamination propagates if the surface energy γ is sufficiently small.

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NOTATIONS

U	Elastic Strain Energy
Π	Total Potential Energy
A	Work of External Load
W	Work of Fracture
γ	Surface Energy
w	Buckling Deflection
E	Young's Modulus
t	Time
h	Thickness of Delaminated Layer

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