

Vladimir Kobelev

Durability of Springs

Second Edition



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Foreword by Thomas Muhr

Technical springs are well-known machine components, which can be reversibly deformed under load and cyclical or oscillating forces. Springs transform kinetic energy into potential energy, store energy, and feed it back nearly without loss into the system when relieved. To use these features for optimized applications, two essential issues have to be considered: the characteristic of the used material as well as an adapted and optimized spring design for the application. Spring steel alloys have the matching material characteristic for the springs, which are mostly highly stressed. Furthermore, the well-calculated shape of the spring allows fulfilling the technical requirements and characteristics like stiffness, fatigue life, and more. Different spring designs are classified according to their shape as well as their type of load stresses, which gives in most cases also the basic understanding for their technical calculation. Due to their multiple technical characteristics and functions, springs are still nearly irreplaceable components in any new and modern machine concept, in planes, ships, buildings, trains, or automobiles. To fulfill all those high demands, standards, and specifications, accurate calculation methods are required, with an approach for all important physical effects for springs.

The purpose of the present script is to explain the mechanical and physical properties of specific steel alloy springs and to present supplementary analytical calculation methods based on already existing and summarized calculation models. Approaches for characteristic spring data like weight and package, lifetime and crack growth, creeping and relaxation rate as well as transverse vibrations and natural frequencies are shown for specific spring shapes. The script contains calculations for helical springs, disc springs, wave springs, and thin-walled rods with a semi-opened cross section. Due to the analytical approach of all calculation models, ambitious development engineers and design engineers get a helpful review and overview of existing and supplementary calculation methods for springs.

Professor Vladimir Kobelev was born in Rostov-on-Don, Russian Federation. He studied Physical Engineering at the Moscow Institute of Physics and Technology. After his Ph.D. at the Department of Aerophysics and Space Research (FAKI), he habilitated at the University of Siegen, Scientific-Technical Faculty.

Today, Prof. Kobelev is a lecturer and APL professor at the University of Siegen in the subject area of Mechanical Engineering.

In his industrial career, Prof. Kobelev is an employee at Mubea, a successful automotive supplier located near Cologne, Germany. In the Corporate Engineering Department, Prof. Kobelev is responsible for the development of calculation methods and physical modeling of Mubea components.

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Foreword by Wolfgang Hermann

One of the oldest elements in machines is the technical spring. Their applications are as varied as the developers' ideas. Whilst working, the components are mostly concealed, almost invisible, and are seldom noticed at all. However, this construction part is not to be underestimated. It does its job, as a safety element in brakes, or as a comfort element in the chassis. No motor could run without a valve spring. No lock could be opened or closed without a spring. These are just some exemplary applications of the hidden helpers. At first glance, springs appear simply trivial and ubiquitous. However, on closer examination, it must be admitted that there is far more behind the spring than most of us realize. The demands on the component are increasing more and more. Whilst in the past the simple relationship between force and distance, Hooke's law, was sufficient, today complex regulations about the load and environmental conditions, durability, and weight reduction have become standard. Successful research has been carried out for many years in the field of springs. Much of the knowledge collected has been included in this book. In this work, developers have the opportunity to gain detailed knowledge of springs. Professor Dr. Kobelev has provided a comprehensive high-level insight into the world of spring development, and thus created a solid basis for the design and engineering of springs. The relationship between the physics of the material and the mechanical load on the part is explained.

I would like to wish readers success in their involvement with this fascinating topic:

Durability of Springs.

Wolfgang Hermann
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German Spring Industry (VDFI),
Hagen, Germany

Introduction

Aims and Methods of the Book

The integral parts of many mechanical systems are elastic elements, or springs (Juvinal, Marshek 2017, Chap. 12). The springs make possible to maintain a tension or a force in a mechanical system, to absorb the shocks and to reduce the vibrations. The high-loaded spring elements in modern industrial equipment and transportation must survive a very high number of cycles with high mean stress as well as high amplitude stress. These springs are manufactured of qualitative wires and by means of distinctive mechanical and heat treatment processes. The standard designations of different spring steels are summarized in (ASTM DS67A 2002).

The spring is the widespread resilient element, which is used in the industrial machinery and automotive systems:

- coil tension springs and/or torsion springs in disc or drum brakes, locks or locking or blocking systems,
- torsion or bending springs for belt tensioners, safety belts and for load compensation in clutch pedals,
- disc springs with or without slots for use in clutches, bearings and for load pre-tensioning,
- leaf springs for chassis suspension,
- helical, or coil, spring for reducing impact events in passenger cars, some heavy trucks and railroad cars
- coil springs for nozzle holders, in transmissions, as valve springs, injection regulators or as vibration dampers in clutches and brake cylinders, as diesel fuel pumps, valve trains, brakes, seats, doors and control elements.

Helical springs are formed by wrapping wire or rod of uniform cross-section around a cylinder. A fixed distance between the successive coils of a spring is maintained, so that the axis of the wire forms a helix. The standard design procedures for helical springs are [SAE, 1996] and [DIN EN 2012, 2013, 2015]. The efficient design procedures for spring elements are based on the modern simulation

and optimization methods. Achieved with these methods, the reduction of weight of the suspension springs causes the decrease of the unsprung mass of the axle. This reduction has a positive influence on the comfort, traction, and steering properties of the car. The development of modern passenger cars has highlighted also the trend towards reduced package space for suspension components in order to maximize package space for occupants and loads. Such requirements lead to reduction in spring dimensions and wire cross-sections. Springs can be found also in high-precision testing devices, where springs play the role of energy harvesters.

One of the most important applications of the highly loaded springs is the valve train for internal combustion engines. Valve springs in combustion engines ensure an enforced contact of all moving valve train components during the valve lift to the maximum engine speed. Assuming an annual production of 100 million cars having roughly 20 valve springs per engine, one gets a rough number of 2000 million valve springs produced. In Europe and North America, the valve spring is produced mainly from high-tensile wire alloyed with the elements chrome and vanadium. The extremely high oscillating stresses on the surface of the wire achieve the peak values up to 2000 MPa. The requirements on failure rate must be below 1.5% for engine operation test. Hereby, they are subjected to extreme vibration stresses and must endure up to $3 \cdot 10^8$ cycles without failure (Muhr 1992).

There are some common characteristics of high-quality and properly designed springs:

- the high homogeneity of stresses on the surface of the spring, thus, the absence of stress-concentrators;
- the considerable residual stresses, which, if properly induced, significantly prolong the operation life;
- high sensitivity to imperfections, flaws, inclusions, and corrosion;
- high amount of specific stored elastic energy.

To fulfil the above requirements, the spring industry developed the specialized materials and sophisticated manufacturing processes.

The springs are mostly produced from oil-tempered steel wire, which is wire formed by drawing a hot rolled steel rod through a drawing die, and oil tempering the resultant wire. Oil-tempering is a term of art identifying a process generally involving heating the wire to austenitization temperatures, quenching it in oil, tempering it by heating it, and recoiling it. This sequence of manufacturing steps increases the ultimate stress of the material. However, the ductility of the austenitized material reduces. The material behaves almost elastically up to the moment of breakage. The influences of both effects, namely, demand to increase the ultimate stress and decrease of ductility, on the fatigue life of material, are conflicting.

The oil-quenched and tempered low-alloy chrome-silicon spring steel wire achieve the strengths of more than 2600 MPa. Even higher strengths can be achieved with chrome-silicon-vanadium alloyed steels for valve applications. The wire is peeled or ground before drawing and subjected to non-destructive crack testing after annealing. At the same time, high strength requires high purity and

surface quality of the semi-finished product, good formability in cold-forming steels and corrosion resistance. High purity also improves ductility while maintaining the same static strength, which is particularly important for cold-formed springs. The latest phase of materials engineering development in spring steels was therefore very much characterized by efforts to achieve very good degrees of purity and surface qualities on the one hand and high-strength steels with the best possible ductility on the other. A metallurgy specially adapted to high-tensile spring steel wires enables a very good degree of purity (Hagiwara et al. 1991; Kawahara et al. 1992; Wiemer 1998). Several publications are devoted to improving wire properties and surface conditions (O'Malley, Hayes 1990; Postma 1993). These also include developments in the direction of thermomechanical forming, improvement of toughness properties for a given strength, considerations on the most suitable heat treatment equipment and the special manufacturing and treatment processes for semi-finished spring products made of stainless materials (Lehnert 1997; Illgner 1995; Schmitz-Cohnen 1994).

The setting reduces the relaxation and improves the creep behavior of springs at operating temperature. It is well-known that the setting influences the static residual stresses in the spring and changes the cyclic fatigue properties of the spring. The cold-setting and heat-setting procedures are used in modern spring manufacturing. Heat setting designates the production step of time-depending loading of spring at elevated temperatures. The main physical process during spring setting is the creep of the material.

Shot peening is another mechanical surface treatment that is used to improve spring performance. The local plastic deformation on the wire surface occurs, which leads to an enhancement and strengthening of a properly machined surface. Shot peening considerably increases the fatigue life of springs.

Consequently, the springs are highly sophisticated elements of modern machinery with the highest fatigue life for relative intensive cyclic stresses. To achieve the required fatigue life, the manufacturing of springs possesses several specific procedures. The production of modern springs includes some distinct techniques, which make the springs stand out above common machine components. It is worth mentioning that occasionally occurring fatigue failure of springs can cause the damage of the complete machinery components. In this case, the failure provokes costs, incompatible even to the price of the highest quality spring.

Structure of the Book

The book reviews the advanced theory of elastic elements from the point of classical mechanics. The book investigates the important problems, which are essential for the clarification of manufacturing and performance of spring elements made of steel. The elements of creep, plasticity and fatigue serve as the building blocks of physical background. What all considered problems have in common is that they are solved in a closed form.

The content of the book is organized in three parts. The first part studies the springs from the design viewpoint. The spring elements of machines and vehicles could be roughly divided into helical, leaf, disk, and twist beams. The models of each family of elastic elements and their optimization are illuminated in the first part of the book, which covers Chaps. 1–5. The second part elucidates the basic processes of spring manufacturing. Plastic cold work is the principal process of production. The coiling and presetting are described in Chaps. 6 and 7. The third part of the manuscript estimates the operational life of the springs. This part covers the creep and fatigue phenomena, which usually confine the springs' lifecycle. This part consists of Chaps. 8–12.

The first part begins with the establishing and valuation of spring design. The optimization of springs is studied in Chap. 1. The helical springs are the typical energy storing elements of valve trains in automotive engines, in the suspensions of passenger cars and railway carriages and in mechanical engineering. The design formulas for linear helical springs with an inconstant wire diameter and with a variable mean diameter of spring are presented. Based on these formulas, the optimization of a spring for given spring rate and strength of the wire is performed. The basic design principles for the leaf springs are also briefly discussed.

Chapter 2 presents analytical solutions for the torsion problem of an incomplete torus with circular and non-circular cross-sections. The hollow cross-sections of the form demonstrate a closed form of analytical solution. The solution is useful for the analysis and design of helical springs with non-circular wires. The torsion problems for straight cylinders with circular and elliptical cross-sections allow the well-known closed form solutions, which are necessary in the next chapters. Two principal load applications, namely the axial force and the axial moment, are analyzed.

Chapter 3 explains a powerful method for the simplification of helical spring equations. Instead of the full treatment of the helical spiral wire, the deformation of the virtual middle line is studied. The virtual middle line possesses the effective extension, torsion, and bending stiffness. It behaves as an initially straight elastic rod or column. This simplification allows uncomplicated solutions of several practically important problems. For explanation, the load dependence of transverse vibrations for helical springs and the transformation of transversal vibration to buckling mode are addressed. The lateral buckling of the spring is considered in the framework of dynamic stability as the limit case for the vibration analysis.

For proper accounting of dynamic effects, the models of flexible springs with massive wire are required. In some cases, such as when the spring is uniform, analytical models for dynamics and buckling can be developed. However, in typical springs, only the central turns are uniform; the ends are often not (for example, having a varying helix angle or cross-section). Thus, obtaining analytical models in this case can be very difficult if possible. A variety of theories to describe the dynamic behavior of helical springs, which involves the interaction of bending (flexural), torsion and longitudinal waves, can be found in the literature. Alongside this, various approximate methods are employed to determine the fundamental

frequencies of the vibrations of springs. One can roughly divide the methods used to determine the fundamental frequencies into three groups:

- analysis methods, based on the concept of an equivalent column;
- exact analysis methods, based on the theory of spatially curved bars;
- numerical methods, based on finite-element formulation for spatially curved bars.

The governing equations for the transverse vibrations of the axially loaded linear helical springs are developed. The method is based on the traditional concept of an equivalent column. We display the effect of axial load on the fundamental frequency of transverse vibrations. The fundamental natural frequency of the transverse vibrations of the spring depends on the variable length of the spring. If the number of active coils remains constant, the frequency with the shortened length of the spring gradually reduces. Finally, when the frequency nullifies, the side buckling spring by divergence mode occurs. Notably, those progressive springs possess the opposite behavior. The number of active coils in compressed progressive springs reduces and the transversal frequency increases. That is why the progressive springs rarely become transverse unstable.

Combination of extensional, compressional and torque loads frequently causes the deformation of spring elements. These loads lead sometimes to spatial buckling if the elements. The stability of helical springs under the combined tension, compression and torsion is elucidated.

Disk springs (also known as Belleville washers) are studied in Chap. 4. The disk springs deliver the examples of high loaded elements of machinery. The coned-disk spring, Belleville spring or cupped spring washer, or Belleville washers are typically used as springs, or to apply a pre-load or flexible quality to a bolted joint or bearing. These springs are the typical energy storing elements of valve trains in clutches and the automatic transmissions of cars. Disk springs are generally manufactured from spring steel and can be subjected to static loads, rarely alternating loads, and dynamic loads. Disk springs must satisfy the severe fatigue life and creep requirements. The basic properties of Belleville washers include high fatigue life, better space utilization, low creep tendency, and high load capacity with a small spring deflection. From a mechanical viewpoint, the disk springs are shallow conical rings that are subjected to axial loads. Normally, the ring thickness is constant and the applied load is evenly distributed over the upper inside edge and lower outside edge. In this chapter, the equations of the equilibrium of disk springs of thin and moderate variable thicknesses are obtained. Variational principles for conical shells are used for the derivation. Simplification is based on the common deformation hypotheses. The closed form analytical solutions of thin and thick truncated conical shells are achieved.

In Chap. 4, disk wave springs are also analyzed. Both linear and non-linear disk wave springs are discussed.

The analysis of thin-walled rods with semi-opened cross-sections is performed in Chap. 5. An essential characteristic for this class of thin-walled beam-like structures is their closed but flattened profile. In the present book, an intermediate class of

thin-walled beam cross-sections is studied. The cross-section of the beam is closed, but the shape of the cross-section is elongated and curved. The walls, which form the cross-section, are nearly equidistant. The unusual shape of semi-opened thin-walled beams allows the efficient optimization due to the wide variability of shapes. The automotive application of thin-walled rods with a semi-opened cross-section is studied later in Chap. 5. The principle application of the theory of semi-opened thin-walled beams is the twist beam of the semi-solid trail arm axle. The analytical expressions for the effective torsion stiffness and effective bending stiffness of the twist beam in terms of section properties of the twist beam with a semi-opened cross-section are derived. Based on the stiffness coefficients of the twist beam, the roll rate, chamber, and lateral rigidity of the suspension are derived.

Mechanical problems arising during the manufacturing of helical springs are examined in the second part, which comprises Chaps. 6 and 7. In Chap. 6, we analyze the coiling of helical springs. For this purpose, we study the plasticization process and the appearance of residual stress. It is well known that the excessive stresses during the coiling of helical springs could lead to the breakage of the rod. Moreover, the high level of residual stress in the formed helical spring reduces considerably its fatigue life. For the practical estimation of residual and coiling stresses in the helical springs, the analytical formulas are necessary. In this chapter, the analytical solution of the problem of the elastic-plastic deformation of the cylindrical bar under combined bending and torsion moments is found for a special nonlinear stress-strain law. The obtained solution allows the analysis of the active stresses during the combined bending and twist. Additionally, the residual stresses in the bar after spring-back are also derived in closed analytical form. The obtained results match the reported measured values. The developed method does not require numerical simulation and is perfectly suited for the programming of coiling machines, estimation of loads during manufacturing of cold-wounded helical springs and for dimensioning and wear calculation of coiling tools.

In Chap. 7, the calculation of presetting for helical springs is developed. The method is based on the deformational formulation of plasticity theory and common kinematic hypotheses. From the mathematical viewpoint, the governing equations of presetting are somewhat analogous to the equations of the coiling process. Two principal types of the helical springs—the compression springs and the torsion springs—are studied. For the first type (axial compression or tension springs), the spring wire is twisted. The basic approach neglects the pitch and curvature of the coil, substituting the helical wire by the straight cylindrical bar. The elastic-plastic torsion of the straight bar with the circular cross-section is examined. The analysis is based on St. Venant's hypothesis. For the second type (torsion helical springs), the helical wire is in the state of flexure. The model analyses the delayed presetting, which is accompanied by considerable creep.

The final, third, part encompasses the lifecycle of the elastic elements; the high static stresses lead to the residual distortion. The sag loss leads to the gradual reduction of spring forces with the resulting disfunction, if the operational length of the spring persists. The breakage of spring due to static creep is the extremely seldom event. The moderate cyclic load is accompanied by some creep and cyclic

sag loss. The severe cyclic loading causes sooner or later to the fatigue destruction of spring. These two origins of possible damages of the elastic elements are summarized in Chaps. 8–12.

The understanding of the long-time behavior of springs under high static load is essential for their correct design. The creep and relaxation of springs is the subject of Chap. 8. Stress analysis for creep has a history in engineering mechanics driven by the requests of design for elevated temperatures. In solid mechanics, creep implies the tendency of materials under the action of external mechanical stresses to deform gradually or enduringly. At stress below the yield strength a slow inelastic deformation take place. In the spring branch, this is called creep when a spring under constant load loses length, and it is called relaxation when a spring under constant compression loses load. The creep and relaxation rates depend on the temperature, the stress in the metal, the yield strength, and the time. Increased temperature, stress and time significantly increase the creep and relaxation rates. Especially, the temperature and stress have the major influence. The precise creep description is essentially important for the correct dimensioning of springs. Finally, Chap. 8 demonstrates the evaluation of creep constants in a wire twist experiment. Further, the exact analytical expressions for torsion and bending creep of rods with the common constitutive models are derived. One of the common creep constitutive models is the Norton-Bailey Law which gives a power law relationship between minimum creep rate and (constant) stress. The power law can be found in high temperature design and creep numerical codes. Other habitual creep laws are exponential and Garofalo laws, which more adequately depict the stress dependence in a wide range of work stresses. For all these laws, we derive the analytical formulas for creep, which is caused by steady or oscillating loading. In Chap. 8, the generalized expression for the creep law is studied. The new expression is based on the experimental data and unifies the primary, secondary, and tertiary regions of the creep curve. The relaxation functions for bending and torsion depend only on the maximal stress in the cross-section, which occurs on the outer surface of the coil. Finally, we explicate the temperature dependence for the creep of spring materials.

Chapters 9 and 10 briefly look into the fatigue effects of springs. For the beginning, we employ the deterministic approach. The presented results pronounce the averaged fatigue characteristics and evaluate the stress levels, for which the majority of springs fail. We speak in this case about median S-N lines. The durability of the spring under oscillating loads is the subject of Chap. 9. The traditional methods of fatigue design are based on the acquisition of numerous experimental data in cyclic tests, data structuring and extraction of empirical formulas. The method for the analysis of crack growth under repeated load is reviewed in Chap. 9. The expressions for spring length over the number of cycles are derived in terms of the higher transcendental function. The proposed method starts from the micromechanically inspired effects of crack propagation, explains the history of crack spreading and finally delivers the stress-life curves.

Several effects on fatigue life, principally the effects of stress ratio and multi-axiality, are discussed in Chap. 10. The presented solutions are used for the estimation of the fatigue life of springs for asymmetric harmonic stressing with the

substantial mean stress. We try to unify diverse traditional methods into one unified Bergmann-Walker formula. Different settings of two fitting parameters into the unified criterion result in the common fatigue criteria.

It is remarkable that the high-quality springs can be distinguished from the low-quality products principally by the scattering ranges. The evaluation of scattering requires the statistical methods, which is the subject of the next chapters. The statistical effects on fatigue life are discussed further in Chaps. 11 and 12. We study the probability descriptions for the fatigue limit of heterogeneously stressed structural elements. The proposed approach for the stress gradient sensitivity of fatigue life is based on the “weakest link” concept. This method is applicable to the exceptionally brittle materials, which fails immediately after the rupture of the first constitutive element. The weakest link approach is applied to calculate the number of cycles to complete destruction under different probability levels. The effect of fluctuating stresses on fatigue life of springs is combined with the influence of heterogeneous stress distribution (stress gradient) over the cross-section of the wire and time-varying stresses. The stress field is inhomogeneous over the cross-section of the wire of the spring. The stress distribution is uniquely defined by the ratio of the diameter of the wire to the diameter of the spring body. The calculated lifetimes are compared with the lifetimes of helical springs subjected to cyclic load.

Chapter 12 examines the stochastic effects on the fatigue life of springs. The stochastic crack propagation is typical for the regions of low stress level and high cycle fatigue. The deviation and branching of cracks are caused by the high inhomogeneity of the polycrystalline structure at the micro level. For the low amplitude of stress, the crack extension pro cycle is less than the typical size of the inhomogeneities. The stochastic differential equation for the travelling crack is derived. The stochastic equation is similar to the equation of the enforced Brownian motion. The methods are based on the unified fatigue laws. These laws lead to analytical solutions for crack length upon the mean value and range of cyclic variation of the stress intensity factor. In this Chapter we demonstrate the closed form expressions for the number of cycles to failure as the function of the initial size of the crack.

Target Audience of the Book

This book was written as an accompanying script for the courses “Applied mechanics of the automobile”, “Automotive engineering, Chassis, II, III”, “Structural optimization in automotive engineering” and “Powertrain modelling and optimization”, delivered by the author in University of Siegen, North Rhine-Westphalia, Germany, since 2001.

This book is recommended first and foremost for engineers dealing with spring design and development, graduated from automotive or mechanical engineering courses in technical high schools, or in other higher engineering schools. The researchers, working on elastic elements and energy harvesting equipment, will also find a broad-spectrum review of the basics of spring methodology.

The actual book demonstrates powerful methods for the analysis of the elastic elements made of steel alloys. The metal springs for the automotive industry are in the focus of interest.

It is well known that the industry explores the design of spring elements and perpetually develops the quality of spring materials. The advance of the new materials arises in the factories of the material suppliers. The tasks of spring manufacturing companies consist in the optimal application of the existing and newly developed material types. The technologically advanced method consists in the target-oriented evaluation of the mechanical properties and the subsequent design of the springs, which makes full use of the measured material characteristics. Thus, the comprehensive unfolding of the improved materials is only possible if their essential properties are rigorously acquired. The design and manufacturing must completely exploit all available capabilities of the semi-finished products.

Enormous number of papers was written on this and related subjects. This does not mean that the science of mechanics and the strength of springs has become completely useless. Rather, in order to elucidate the widespread possibilities, it is necessary to have a thorough understanding of the mechanics and the metal disciplines. Therefore, a good deal of this book is devoted to these subjects.

The precise methods for the design of different types of springs are summarized in the dedicated standards. The industrial research developed the reliable methods for the estimation of fatigue life and creep effects. This book has no aim to replace the established methods of design and pragmatic methods for the evaluation of the operational life of springs. The aim of this book is to explain qualitatively the mechanical behavior of the spring as a unique elastic element, owing to some very specific properties. We try to oversee the widespread and fragmented landscape of a spring from a single viewpoint of classic mechanics. We try to compare the different methods for the evaluation of operational life and point out the most effective and general methods. There are many experimental values, which currently remain the subject of speculation. Possibly these values will be acquired in the future studies. Most of the presented methods are acknowledged and are applicable for other heavy-loaded structural elements as well.

Some words on the solution methods. There exist several acknowledged, commercial finite-element codes, for example, (ANSYS, 2020) and (ABAQUS, 2020). However, for the modelling of technical systems, analytical solutions often offer important advantages. First, we see the transparency of closed form solutions. Because analytical solutions are displayed as mathematical expressions, they provide an understandable sight of how variables and relations between variables affect the result. Second, the proficiency: algorithms and models expressed with analytical solutions are often more effective than the corresponding numerical applications. For example, to compute the solution of an ordinary differential equation for different values of its parametric inputs, it is often faster, more accurate, and more suitable to assess an analytical solution than to integrate it numerically. Third, numerical solutions are sometimes extremely abundant. The main reason is that sometimes we either don't have an analytical approach, or that the analytical solution is too slow and instead of computing for hours and getting an exact

solution, we rather compute for seconds and get a good approximation. Finally, numerical solutions very rarely can contribute to the proofs of new ideas. For that reason, the treatment of material in this book resolves the studied tasks to closed form solutions in the form of mathematical expressions.

The content of the present book is logically related to the work “Design and Analysis of Composite Structures for Automotive Applications” of the same author. The latter book extends the actual book and covers the subject of composite materials. The manuscript (Kobelev 2019a) investigates the distinctive features of composites, such as their anisotropy, inhomogeneity, load direction dependence, stress-coupling and stacking capabilities.

The first edition of this book (Kobelev 2018) was revised to represent the actual tendencies and the state of the art of the spring’ modeling. This book highlights the mechanics of the elastic elements made of steel alloys with focus on the metal springs for the automotive industry. The industry and scientific organizations study intensively the foundations of the design of spring elements and permanently improve the mechanical properties of spring materials. The development responsibilities of spring manufacturing companies involve the optimal application of the existing material types. Thus, the task entails in the target-oriented evaluation of the mechanical properties and the subsequent design of the springs, which makes full use of the attainable material characteristics.

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Chapter 1

Principles of Spring Design



Abstract The calculation formulas for linear helical springs with an inconstant wire diameter and with a variable mean diameter of spring are presented. Based on these formulas the optimization of spring for given spring rate and strength of the wire is performed. The design principles for optimal leaf springs are briefly presented. This chapter is the entry section of first part, which studies the springs from the viewpoint of design.

1.1 Compression, Extension and Torque of Helical Springs

1.1.1 Forces and Moments in Helical Springs

Helical, or coil springs, studied in the beginning of this chapter, are formed by wrapping wire or rod of the certain cross-section (Leiseder 1997). Compression springs resist when a squeezing force is applied to the ends. Tension springs possess loops and resist the extension force that pulls them apart. The torsion springs resist twisting, when the end coils are rotated around the axis of the spring in different directions. The mean coil diameter of the spring is D .

All coil springs can be wound in either a left-hand or a right-hand direction. A left-hand wound spring will spiral in the same direction as a left hand threaded screw. A right-hand wound spring will spiral in the same direction as a right-hand threaded screw. For certainty, the right-hand wound springs are considered, as being more popular.

We take as a reference frame cylindrical polar coordinate system (r, θ, z) . The axis z of the cylindrical polar coordinate system is aligned with the axis of the cylinder (Fig. 1.1). The mean coil radius of helix $R = D/2$ and the cross-section can differ along the helix. The angle $\alpha = \alpha(\theta)$ designates the inclination of the helix with any plane perpendicular to the axis of the coil (pitch angle, lead angle). The pitch angle and pitch can vary along the helix. If pitch is maintained, so that the axis of the wire forms a helix. If the radius of helix is fixed, appears the cylindrical spring. Finally, the standard helical spring outcomes, if the cross-section does not alter along the

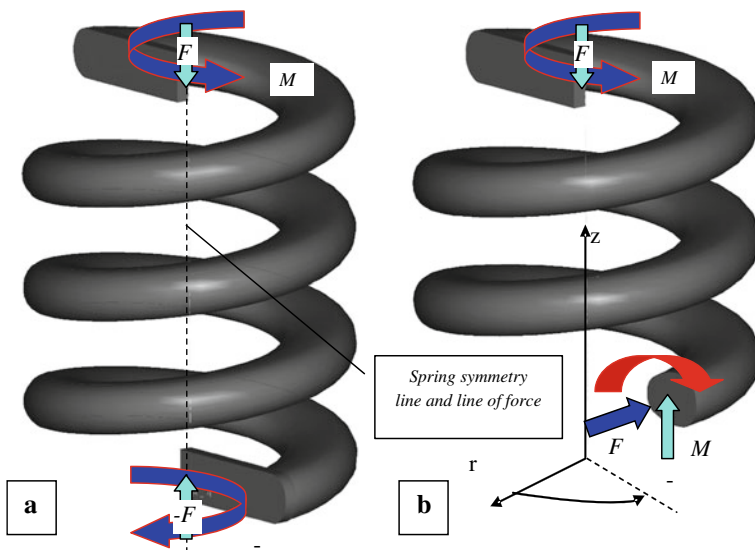


Fig. 1.1 A part of helical spring, subjected to axial loading F and axial torque M_θ

helical spiral. For this type of the spring, the calculation formulas are standard (SAE 1997, Chap. 5, Design of Helical Springs), (EN 13906-1:2013-11).

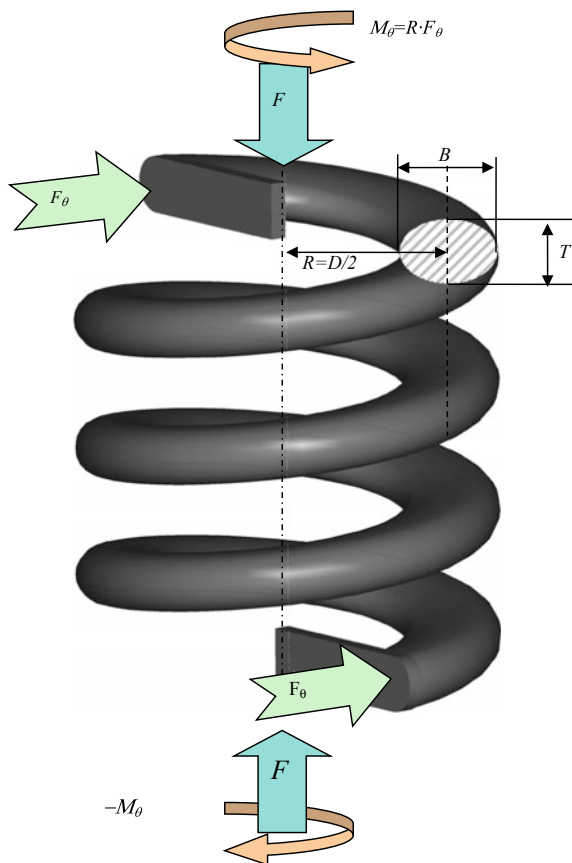
Helical springs store elastic energy by means of torsion and bending of wire (Timoshenko 1948, Chap. VII, Sect. 54) (Fig. 1.2). The elastic energy of the spring is stored its active coils. The compression spring have fewer active coils, if it has closed and squared ends, closed and ground ends, or double closed ends. Ignoring at first other certain complexities of spring technology, we write basic relations for an analysis of the spring. For brevity, we study only the elastic deformation in the active coils. This simplified analysis considers spring ends as “plain”, so that only active coils n_a are considered. The polar angle along the active part of the helical spring is θ :

$$\theta = 0.. \Theta_a, \quad \Theta_a = 2\pi n_a.$$

Consider the middle region of the spring, where the transition effects due to the end bars disappear. The lower spring end is fixed. Due to the symmetry of the spring, all sections of the spring are deformed identically.

Tension and compression springs store elastic energy principally through torsion of wire. The force F acts ideally along the axis z of the cylindrical coordinate system, such that the line of force coincides with the symmetry axis of the spring. The torsion of helically coiled wire occurs about the axis of the wire by forces applied to the ends of springs. Figure 1.3 shows a view of the spring wire at the cut section. The couple M acting in the vertical plane has been shown as a vector. The couple is equal to the product of the axial force F with the arm R to the force of a couple:

Fig. 1.2 Section of helical cylindrical spring, loaded by force F and torque M_θ



$$M = F R. \quad (1.1a)$$

The couple (1.1a) resolves into two components, $M \sin \alpha$ and $M \cos \alpha$, which lie in planes, which are tangential and normal to the helix, respectively (Fig. 1.3a). The couple $M \sin \alpha$ tends to cause bending of the spring wire in the plane, normal to the coil plane. The couple $M \cos \alpha$ causes twisting along the tangential to helix curve.

Torsion springs store elastic energy mainly through bending of wire in the coil plane. The bending of helically coiled wire occurs by terminal moments applied to the ends of springs. The terminal moments twist the coil tighter or looser. The torque

$$M_\theta = F_\theta R, \quad (1.1b)$$

on the upper spring acts clockwise, if we look on the spring from above. The circumferential force F_θ pushes towards the wire, applying normal stresses in the wire cross-section. Thus, the torque causes the equal bending moment in each section of the wire.

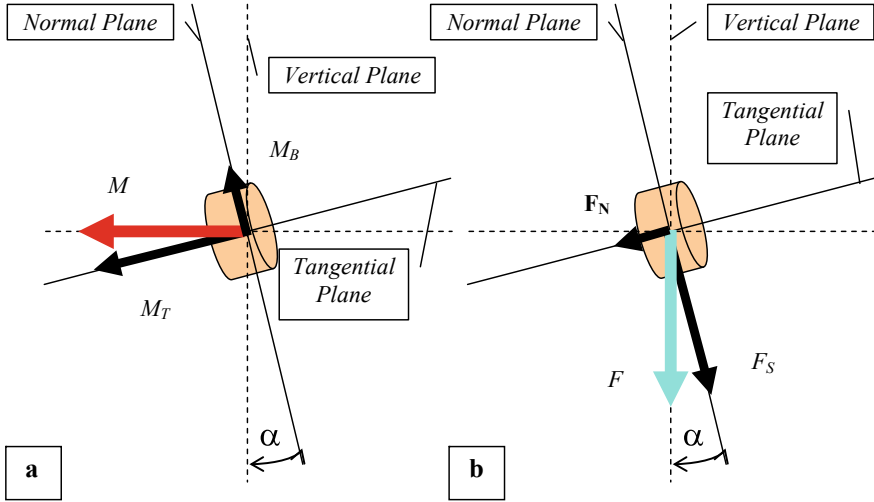


Fig. 1.3 Resolution of moments and forces into normal and tangential components

The springs withstand recurrently to the combination of both loads. In this case, simultaneously act the axial force F and the axial torque M_θ . The magnitudes of the bending and twisting couples in the wire are equal respectively to:

$$m_B = M \sin \alpha - M_\theta \cos \alpha, m_T = M \cos \alpha + M_\theta \sin \alpha. \quad (1.2)$$

Figure 1.3b shows a similar resolution of force F into components, F_N and F_S , which are causing normal and shearing components of stress, respectively, on the cut section:

$$F_N = F \sin \alpha, \quad F_S = F \cos \alpha. \quad (1.3)$$

1.1.2 Elastic Energy of Helical Spring

As already mentioned, the conventional design formulas (SAE 1997, Chap. 5, Design of Helical Springs), (EN 13906-1:2013-11) calculate the cylindrical helical springs with the circular wire and the constant pitch. Several applications require the helical springs with the alternating mean diameter of the spring body as well the varying wire diameter along its length. The calculation of such springs needs the generalization of the customary design formulas. Besides this, the formulas are intended to the optimization purposes.

For the element of length, the following expression is valid:

$$dl = \frac{Dd\theta}{2 \cos \alpha}. \quad (1.4)$$

The volume of wire with variable cross-section is given by the integral of the cross-sectional area $A(\theta)$ over the wire length. The total length of the wire L_W and the mass of the spring m depend upon two functions $D = D(\theta)$, $A = A(\theta)$. With material density is ρ_m , these values are correspondingly:

$$L_W = \int_0^{\Theta_a} dl, \quad m = \rho_m \int_0^{\Theta_a} A dl. \quad (1.5)$$

To derive the stiffness coefficients, we start with the linear spring loaded from its free state L_0 by simultaneously acting axial force F and axial torque M_θ . The stored elastic energy of the spring reads:

$$U_e = \frac{F^2}{2c} + \frac{FM_\theta}{c_{\theta F}} + \frac{M_\theta^2}{2c_\theta}. \quad (1.6)$$

The coefficients of Eq. (1.6) are the compression spring rate c , the compression-twist springs rate $c_{\theta F}$ and the twist springs rate c_θ .

We derive these coefficients for an arbitrary non-cylindrical helical spring with an arbitrary variable cross-section. The size of the cross-section is assumed much smaller than the radius of coil. For derivation we use Castigliano's method (Teodorescu 2013). The elastic energy stored in the spring (1.6) is the integral of twist and bending energy of wire over its length L_W :

$$2U_e = \int_{L_W} \left[\frac{m_T^2}{GI_T} + \frac{m_B^2}{EI} \right] dl. \quad (1.7)$$

We substitute (1.2) for the torsion moment m_T and bending moment m_B in (1.7). The elastic energy density of wire pro unit of its length is:

$$\begin{aligned} \frac{m_T^2}{GI_T} + \frac{m_B^2}{EI} = & M^2 \left(\frac{\cos^2 \alpha}{GI_T} + \frac{\sin^2 \alpha}{EI} \right) + 2MM_\theta \cos \alpha \sin \alpha \left(\frac{1}{GI_T} - \frac{1}{EI} \right) \\ & + M_\theta^2 \left(\frac{\sin^2 \alpha}{GI_T} + \frac{\cos^2 \alpha}{EI} \right). \end{aligned} \quad (1.8)$$

Using (1.4) in (1.8), reduces the formula for the stored energy to:

$$2U_e = F^2 \int_{L_w} \frac{D^2}{4} \left(\frac{\cos^2 \alpha}{GI_T} + \frac{\sin^2 \alpha}{EI} \right) dl + 2FM_\theta \int_{L_w} \frac{D}{2} \cos \alpha \sin \alpha \left(\frac{1}{GI_T} - \frac{1}{EI} \right) dl \\ + M_\theta^2 \int_{L_w} \left(\frac{\sin^2 \alpha}{GI_T} + \frac{\cos^2 \alpha}{EI} \right) dl.$$

According to Castigliano's method, the second derivatives of the stored energy U_e with respect to F and M_θ provide the spring rates. The compression spring rate c , the compression-twist springs rate $c_{\theta F}$ and the twist springs rate c_θ of an arbitrary non-cylindrical helical spring are:

$$\left\{ \begin{array}{l} \frac{1}{c} = \frac{\partial^2 U_e}{\partial F^2} = \int_{L_w} \frac{D^2}{4} \left(\frac{\cos^2 \alpha}{GI_T} + \frac{\sin^2 \alpha}{EI} \right) dl, \\ \frac{1}{c_{\theta F}} = \frac{\partial^2 U_e}{\partial F \partial M_\theta} = \int_{L_w} \frac{D}{2} \cos \alpha \sin \alpha \left(\frac{1}{GI_T} - \frac{1}{EI} \right) dl, \\ \frac{1}{c_\theta} = \frac{\partial^2 U_e}{\partial M_\theta^2} = \int_{L_w} \left(\frac{\sin^2 \alpha}{GI_T} + \frac{\cos^2 \alpha}{EI} \right) dl, \end{array} \right. \quad (1.9)$$

for the given design variables:

$$GI_T = GI_T(\theta), EI = EI(\theta), D = D(\theta), \alpha = \alpha(\theta).$$

The twist spring rate c_θ is equal to the moment M_θ , which causes the twist angle of spring of one radian in the absence of the axial force. The compression-twist springs rate $c_{\theta F}$ describes the moment M_θ , which causes the unit axial travel of the spring in the absence of the axial force. The unit axial travel is one millimeter or one meter in SI system. Notably, that the value $c_{\theta F}$ is equal to the force, which causes the same axial travel in the absence of the twist moment.

1.1.3 Compression and Twist Spring Rates

When the distance between coils is small, the angle α is very small. For small pitch, following relations are valid:

$$\cos \alpha \cong 1, \sin \alpha \cong 0. dl \cong Dd\theta/2.$$

In this case, the spring is called a closed-coiled spring. With (1.3), the force F_N is also negligible and F_S will approximately equal to F . As follows from (1.2), the torque moment m_T of the wire is nearly equal to M . This means, that for closed-coiled

spring, the axial force leads only to the torsion of the wire. Torque of the spring body leads to the bending of wire. The twist and bending moments of wire are thus equal to:

$$m_T = FR. \quad m_B = M_\theta.$$

According to (1.5), the mass of the spring is:

$$m = \frac{1}{2} \rho_m \int_0^{\Theta_a} A D d\theta. \quad (1.10)$$

From (1.9), we get for compression (or extension) and twist spring rates:

$$\frac{1}{c} = \int_0^{\Theta_a} \frac{1}{GI_T} \left(\frac{D}{2} \right)^3 d\theta, \quad \frac{1}{c_\theta} = \int_0^{\Theta_a} \frac{1}{EI} \frac{D}{2} d\theta. \quad (1.11)$$

The compression-twist springs rate $c_{\theta F}$ of the closed-coiled spring vanishes.

The above formulas are valid for an arbitrary shape of the wire cross-section. For the springs with elliptic and rectangular shapes of the cross-section the geometric characteristics depend upon the orientation of axes B, T . The height of the cross-section t is measured in the direction of the axis. The equations for spring rates and masses of the springs with circular and non-circular cross-sections of wire are given in Tables 1.1 and 1.2.

Equations (1.10) and (1.11) shorten for the springs with the circular cross-section. The formulas for spring rates of the springs with a circular cross-section and the variable wire diameter $d = d(\theta)$ yield after the substitution of the values for GI_T

Table 1.1 Spring rates and masses of the linear springs with number of active coils n and a non-circular wire cross-section

	Spring rate	Mass
$\begin{cases} A = konst \\ D = konst \end{cases}$	$c = \frac{4GI_T}{\pi n D^3}$	$m = \pi n A \rho_m D$
$\begin{cases} A = konst \\ D \neq konst \end{cases}$	$c = \frac{8G}{I_T^{-1} \int_0^{2\pi n} D^3(\varphi) d\varphi}$	$m = \frac{1}{2} A \rho_m \int_0^{2\pi n} D(\varphi) d\varphi$
$\begin{cases} A \neq konst \\ D = konst \end{cases}$	$c = \frac{8G}{D^3 \int_0^{2\pi n} I_T^{-1}(\varphi) d\varphi}$	$m = \frac{1}{2} \rho_m D \int_0^{2\pi n} A(\varphi) d\varphi$
$\begin{cases} A \neq konst \\ D \neq konst \end{cases}$	$c = \frac{8G}{\int_0^{2\pi n} D^3(\varphi) I_T^{-1}(\varphi) d\varphi}$	$m = \frac{1}{2} \rho_m \int_0^{2\pi n} A(\varphi) D(\varphi) d\varphi$

Table 1.2 Spring rates and masses of the linear springs with number of active coils n and with the circular wire cross-section

	Spring rate	Mass
$\begin{cases} A = konst \\ D = konst \end{cases}$	$c = \frac{Gd^4}{8nD^3}$	$m = \frac{1}{4}\pi^2 nd^2 \rho_m D$
$\begin{cases} A = konst \\ D \neq konst \end{cases}$	$c = \frac{\pi G d^4}{4 \int_0^{2\pi n} D^3(\varphi) d\varphi}$	$m = \frac{1}{8}\pi d^2 \rho_m \int_0^{2\pi n} D(\varphi) d\varphi$
$\begin{cases} A = konst \\ D \neq konst \end{cases}$	$c = \frac{\pi G}{4D^3 \int_0^{2\pi n} d^{-4}(\varphi) d\varphi}$	$m = \frac{1}{8}\pi \rho D \int_0^{2\pi n} d^2(\varphi) d\varphi$
$\begin{cases} A \neq konst \\ D \neq konst \end{cases}$	$c = \frac{\pi G}{4 \int_0^{2\pi n} D^3(\varphi) d^{-4}(\varphi) d\varphi}$	$m = \frac{1}{8}\pi \rho_m \int_0^{2\pi n} d^2(\varphi) D(\varphi) d\varphi$

and EI into Eq. (1.11). For the springs with the circular cross-section the expression for mass and for spring rates are:

$$m = \frac{1}{8}\pi \rho_m \int_0^{\Theta_a} d^2 D d\theta. \quad (1.12)$$

$$c = \pi G \left(4 \int_0^{\Theta_a} \frac{D^3 d\theta}{d^4} \right)^{-1}, \quad c_\theta = \pi E \left(32 \int_0^{\Theta_a} \frac{D d\theta}{d^4} \right)^{-1}. \quad (1.13)$$

The study of the optimization problem is based on the formulas for the spring rates Eq. (1.13) and for mass (1.12).

1.1.4 Diameter Alteration Due to Simultaneous Compression and Torque

The axial compression of the spring (spring travel in the direction of force on the upper spring end) is (Ponomarev et al. 1956):

$$s = s_F + s_M, \quad (1.14)$$

$$s_F = \frac{\pi F D^3 n}{4 \cos \alpha} \left(\frac{\cos^2 \alpha}{GI_T} + \frac{\sin^2 \alpha}{EI} \right), \quad s_M = \frac{\pi M_\theta D^2 n}{2} \left(\frac{1}{GI_T} - \frac{1}{EI} \right) \sin \alpha. \quad (1.15)$$

The values s_F and s_M represent the spring travel in cause of compression force and torque respectively. The spring shortens under the action of compression force. Similarly, the length of the right wound spring under the action of clockwise moment on its upper end reduces.

The twist angle of the upper spring end with respect to the axis of the spring reads:

$$\theta = \theta_F + \theta_M, \quad (1.16)$$

$$\theta_F = \frac{\pi F D^2 n}{2} \cdot \left(\frac{1}{G I_T} - \frac{1}{E I} \right) \sin \alpha, \quad \theta_M = \frac{\pi M_\theta D n}{\cos \alpha} \cdot \left(\frac{\cos^2 \alpha}{G I_T} + \frac{\sin^2 \alpha}{E I} \right). \quad (1.17)$$

The values θ_F , θ_M symbolize the reduction of the spring angle in cause of compression force and torque respectively. The positive direction is clockwise. The coil number of spring decreases under the action of compression force. Analogously, the coil number of the right wound spring decreases under the action of clockwise moment on its upper end.

The mean coil diameter of spring increases under the action of axial force and axial moment:

$$\Delta D = \Delta D_F + \Delta D_M, \quad (1.18)$$

$$\Delta D_F = F D^3 \sin \alpha \cdot \left(\frac{1}{2 G I_T} - \frac{\cos 2\alpha}{4 E I \cos^2 \alpha} \right), \quad (1.19)$$

$$\Delta D_M = \frac{M_\theta D^2}{2 \cos \alpha} \cdot \left(\frac{2 \sin^2 \alpha}{G I_T} + \frac{\cos 2\alpha}{E I} \right). \quad (1.20)$$

The diameter of the right wound spring increases under the action of compression force. In other words, the diameter of the compression spring expands when such compression spring has deflected. The diameter of the right wound spring expands under the action of clockwise moment on its upper end. If the moment acts in the opposite direction, such that the circumferential force pulls the wire, the spring's body is being tightened.

1.2 Design Formulas for Compression-Extension Springs

1.2.1 Stiffness and Stored Energy of Cylindrical Helical Springs

Consider the helical spring with a circular wire. In this chapter, we use for the design purposes the conventional formulas (EN 13906 2013, 2014; Meissner et al. 2015). The design variables for helical springs are

$$d, D, n_a, L_0,$$

where n_a is a number of active coils, d is the diameter of wire, D is the mean coil diameter, L_0 is a free length of the spring. The external and internal diameters correspondingly are:

$$D_e = D + d, D_i = D - d.$$

The volume and the mass of the spring material of a cylindrical spring with constant, round cross-section is given by:

$$V = \frac{\pi^2 d^2 D n_a}{4}, \quad m = \rho_m V. \quad (1.21)$$

where ρ_m is a density of spring material.

Consider for definiteness a compression spring with a free length L_0 . Solid length L_s is the height at which the coils of the compressed spring close up (SAE 1997, Chap. 3, Cold wound helical and spiral springs). The stroke of a compression spring is the spring travel from released length L_{rel} to compressed length L_{comp} :

$$s = L_{rel} - L_{comp}. \quad (1.22)$$

For the compression spring is valid:

$$L_0 > L_{rel} > L_{comp} > L_s > 0.$$

The axial spring stiffness, or spring rate, is the force required to produce a unit deflection. In the simplest case, the force-deflection characteristic is approximately linear and can be calculated from the geometry and shear modulus G of the spring material. In this case, Eq. (1.11) leads to the traditional formula for the cylindrical spring made of the round wire with the constant diameter:

$$c = \frac{Gd^4}{8D^3n_a}. \quad (1.23)$$

Using Eq. (1.23), the spring loads at lengths L_{comp} , L_{rel} and L_s are correspondingly:

$$F_{min} = c \cdot (L_0 - L_{rel}), \quad F_{max} = c \cdot (L_0 - L_{comp}), \quad F_s = c \cdot (L_0 - L_s).$$

The energy capacity of the linear compression spring is expressed as:

$$U_e = \frac{c}{2} \left[(L_{comp} - L_0)^2 - (L_{rel} - L_0)^2 \right] = \frac{1}{2c} (F_{max}^2 - F_{min}^2). \quad (1.24)$$

The work of applied forces on the spring travel is:

$$U_f = \frac{F_{min} + F_{max}}{2} s.$$

The tension spring is handled in the same way. The stroke of a tension spring is the spring travel from released length L_{rel} to extended length L_{ext} :

$$s = L_{ext} - L_{rel}.$$

Clearly, the lengths for tension spring shuffle:

$$0 < L_s < L_0 < L_{rel} < L_{ext}.$$

1.2.2 Stresses in Spring Wire

The pitch can be neglected $\alpha = 0$ for the fairly accurate stress evaluation. One turn, or coil, of an undeformed helical spring becomes a torus, generated by rotating the cross-section about the z axis of the cylindrical coordinate system (Fig. 1.4). For the stress calculations, the torus is considered as incomplete. The two ends of the turn are

Fig. 1.4 Incomplete torus within the cylindrical coordinate system

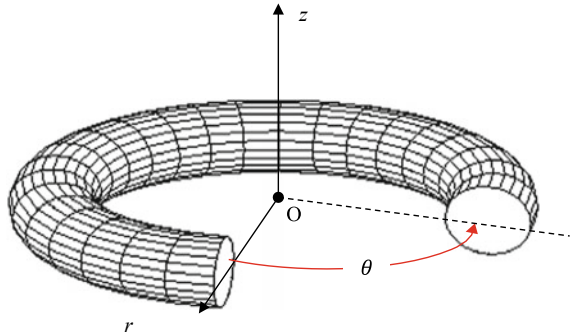


Table 1.3 Stiffness of wires with different cross-sections

Cross-section of wire	Area and moments of inertia	Torsion constant
Circular, diameter d	$\begin{cases} A = \frac{\pi d^2}{4}, \\ I = I_r = \frac{\pi d^4}{64} \end{cases}$	$I_T = \frac{\pi d^4}{32}$
Squared, side size a	$\begin{cases} A = a^2, \\ I = I_r = \frac{a^4}{12} \end{cases}$	$I_T = 0.141.. a^4$
Rectangular, T —hight B —width	$\begin{cases} A = BT, \\ I = \frac{BT^3}{12}, \\ I_r = \frac{B^3T}{12} \end{cases}$	$I_T = \begin{cases} \text{if } T > B : \xi_1 \left(\frac{T}{B}\right) TB^3 \\ \text{if } T < B : \xi_1 \left(\frac{B}{T}\right) BT^3 \end{cases}$
Elliptic T —hight B —width	$\begin{cases} A = \frac{\pi}{4} BT \\ I = \frac{\pi BT^3}{64}, \\ I_r = \frac{\pi B^3T}{64} \end{cases}$	$I_T = \frac{\pi B^3T^3}{16(B^2+T^2)},$

$$\xi_1(x) = \frac{1}{3} \left(1 - \frac{192}{\pi^3 x} \sum_{k=1,3,5..}^{\infty} \frac{1}{k^5} \tanh \frac{k\pi x}{2} \right) \approx \frac{1}{3} \left(1 - \frac{0.63..}{x} + \frac{0.052..}{x^5} \right)$$

not joined. They carry equal in magnitude and opposite shear stress distributions with resultant F . The line of action of the resultant force F is coincident due to symmetry with the z axis. Any segment of the coil is therefore in equilibrium, because on the ends act two opposite axial forces F with the same magnitude. Nonzero components of the shear stress in cylindrical coordinates are $\tau_{r\theta}$, $\tau_{\theta z}$. Again, due to symmetry, these components are independent of θ .

One must differ the basic and corrected shear stress in the spring. The basic stress is obtained by dividing the torsion moment in the wire m_T : by its section modulus W_T in torsion (Tables 1.3 and 1.4). Thus, the basic stress is (SAE 1997, Chap. 5, Design of Helical Springs):

$$\tau_b = \frac{m_T}{W_T} = \frac{8DF}{\pi d^3}. \quad (1.25)$$

The corrected stress τ_c is calculated by multiplying the basic stress τ_b by the correction factor

$$k_\tau = k_\tau(w), \quad (1.26)$$

such that

$$\tau_c = k_\tau \tau_b. \quad (1.27)$$

Table 1.4 Section modules of wires with different cross-sections

Cross-section of wire	Bending section modules W_b, W_{br}	Twist section modulus W_t
Circular, diameter d	$W_b = W_{br} = \frac{\pi d^3}{32}$,	$W_t = \frac{\pi d^3}{16}$.
Squared, side size a	$W_b = W_{br} = \frac{a^4}{6}$,	$W_t = 0.208..a^3$.
Rectangular, T —hight B —width	$W_b = \frac{BT^2}{6}, W_{br} = \frac{TB^2}{6}$,	$W_t = \begin{cases} \text{if } T > B : \xi_2(\frac{T}{B})TB^2, \\ \text{if } T < B : \xi_2(\frac{B}{T})BT^2. \end{cases}$
Elliptic T —hight B —width	$W_b = \frac{\pi BT^2}{32}, W_{br} = \frac{\pi B^2T}{32}$,	$W_t = \frac{\pi}{16} \min(B^2T, BT^2)$
$\xi_2(x) = 1 - \frac{8}{\pi^2} \sum_{k=1,3,5..}^{\infty} \frac{1}{k^2 \cosh \frac{k\pi x}{2}} \approx \frac{1+x^2}{0.35+x^2} \xi_1(x)$		

The correction factor k_τ depends upon the ratio of mean coil diameter to wire diameter:

$$w = D/d. \quad (1.28)$$

The ratio w known as the spring index. A low index indicates a tightly wound spring (a relatively large wire size wound around a relatively small diameter mandrel giving a high rate). The correction factor $k_\tau(w)$ accounts for stress concentration due to curvature of the spring as well as direct shear.

The governing equations for the closed-coiled helical spring were developed using semi-inverse StVenant's method by Michell (1899). Unfortunately, the closed form of StVenant's solutions, which is well known in the theory of torsion of straight circular or elliptic rods, does not exist for curved rods in terms of elementary functions. The approximate solutions for rectangular and circular cross-sections were delivered by Wahl (1929) and Göhner (1932a, b). The solutions for helical springs with circular cross-sections in terms of series of appropriate Legendre functions in toroidal coordinates were found by Freiberger (1949) and Henrici (1955).

The exact solution according to (Herichi 1955) expresses the correction factor k_τ as an infinite series:

$$k_1(w) = 1 + \frac{5}{4w} + \frac{7}{8w^2} + \frac{15}{256w^3} + \dots \quad (1.29)$$

For springs made of wire with circular cross section, the traditional correction factors for the given coiling ratio were determined by empirically established

formulas (SAE 1997, Table 5.2) or (EN 13906-1:2013-11 2013). The following formulas apply to the standard correction factors on the spring index:

$$k_2(w) = \frac{w + 0,5}{w - 0,75}, \quad (\text{Bergstrasser EN 13906-1 : 2013-11}), \quad (1.30)$$

$$k_3(w) = \frac{4w - 1}{4w - 4} + \frac{0,615}{w}. \quad (\text{Wahl 1929}). \quad (1.31)$$

Shear stresses over the cross-sections and on the outer surfaces of wires in helical springs will be considered more detailed in Chap. 2. In this chapter, we use for the design purposes the standard formulas (1.30) and (1.31) (Meissner et al. 2015).

An important remark considers the effects of stress concentrations. Properly designed springs possess no notches and stress concentrators, because of very high negative impact on fatigue life. If the cyclic stresses are low, only microscopic plastic deformation occurs in each load cycle. The high cycle fatigue is typical for the most types of springs in automotive and industrial applications of spring elements. The calculations are based on the empirical stress factors. These factors average the stress over the surface of the wire. The factors consider the possibility of defect in the less stressed regions. They estimate the admissible level of stresses more correctly, than the standard stress factors. We explain in Chap. 10 the application of the empirical stress factors for fatigue calculations.

The springs usually have a moderate, rather natural stress concentration. The explanation is the following. Equation (1.32) estimates the for fatigue relevant stresses based on the ideal stresses. The ideal stresses neglect the escalation of stress on the inside of wire, which faces the midline of helix. This effect could be treated as the stress concentration factor. This specific factor is distinct for helical springs.

There are several further methods to consider the influence of stress concentration on the fatigue of structural elements (see Chaps. 10 and 12). The approximate methods are based on stress-strain conversion rules. The methods were widely applied for the cyclically loaded elements of structures:

- (1) the simplest linear rule, which assumes that strain concentration factor is equal to theoretical stress concentration factor;
- (2) The rule (Neuber 1961a, b) relates the product of stress and strain concentration factors to the hypothetical stress concentration factor;
- (3) Stowell-Hardrath-Ohman expression of stress concentration factor is a function of speculative stress concentration factor and elasticity moduli (Wundt 1972);
- (4) “Equivalent Strain Energy Density” approach (Molski and Glinka 1981). This approach assumes, that the strain energy density in the notch root is related to the energy density due to nominal stress and strain by a definite factor.

The difference between the predictions for the cited approaches could be significant, if the considerable plastic deformation in each load cycle occurs. Experimental and numerical investigations were performed for verification of the cited rules. Examination (Adibi-Asl and Seshadri 2009) demonstrates that the Neuber rule predicts

an upper bound on maximum local strains, the linear rule provides a lower bound and the “Equivalent Strain Energy Density” method estimates the fatigue life in the middle region. Because the significant plastic deformation is not typical for industrial and automotive spring applications, we follow further only the traditional approach.

1.2.3 Fatigue Life and Damage Accumulation Criteria

If the spring is to operate a definite, prescribed number of times through a deflection s , it must be designed so that the material does not fail in fatigue. For design purposes, the simple fatigue criterion for compression spring design is usually assumed (Spring Design Manual 1996):

$$\left(\frac{\tau_{max}}{\tau_w} + \frac{\tau_a}{\tau_e} \right) S_f \leq 1, \quad (1.32)$$

where

$$\tau_{min} = k_\tau F_{min} / W_T,$$

$$\tau_{max} = k_\tau F_{max} / W_T,$$

$$R_\sigma = \frac{\tau_{min}}{\tau_{max}} = \frac{F_{min}}{F_{max}} < 1 \text{ is the stress ratio of cyclic load}$$

$$\tau_m = (\tau_{max} + \tau_{min})/2 \text{ is the mean stress per cycle,}$$

$$\tau_a = (\tau_{max} - \tau_{min})/2 \text{ is the stress amplitude,}$$

$$\tau_w \text{ is the ultimate shear strength of material,}$$

$$\tau_e \text{ is a endurance limit for completely reversed stress,}$$

$$S_f \text{ is a factor for safety.}$$

The safety factor S_f for simplicity is assumed to be 1.

Both τ_w and τ_e usually vary with wire diameter, as discussed in Chap. 10, Sect. 10.6.1. As we can see, both τ_w and τ_e attain maximum values for a small wire diameter.

The fatigue life of springs is also habitually based on the damage evaluation from the Smith, Watson, Topper rule (Smith et al. 1970; Landgraf 1973) or Walker rule (Walker 1970).

According to Smith, Watson, Topper rule, the governing parameter for damage characterization is a product of total strain range and maximum stress. For discussion regarding applicability of Smith, Watson, Topper rule for automotive applications see (Fuchs et al. 1977). During spring deformation, the wire undergoes torsion, where the pure shear stresses predominate. Applying this approach to shear deformation, the $\mathbf{P}_{SWT}$ (Smith, Watson, Topper parameter) transforms to:

$$p_{SWT.\tau} = \sqrt{G\gamma_a \tau_{max}}.$$

Here

$$\gamma_a = \tau_a / G$$

is the shear strain amplitude.

The damage parameter is plotted versus number of reversals, so that damage per range between two reversals is a function of damage parameter. The accepted damage for a selected material during fatigue life of the spring is characterized by the condition:

$$p_{SWT.\tau} \equiv \sqrt{\tau_a \tau_{max}} \leq p_{SWT.0}. \quad (1.33)$$

The experimentally acquired constant $p_{SWT.0}$ depends on material properties and accepted damage level for application under consideration (see Chaps. 9 and 10).

The fatigue behavior of the springs depends highly upon the surface treatment, mainly the shot peened layer on the surface. The highly inhomogeneous stresses in the shot peening layer are responsible for the crack arrest due to the compression stresses. The simulation methods must adequately describe the stress origin and depth variation of shot peening stresses. Cold formed springs also preserve another kind of residual stress due to the coiling. The influence of residual stresses on damage accumulation must be accounted for in fatigue calculations. The fatigue life of springs will be discussed in details in Chaps. 9, 10 and 11. The mechanical properties of spring materials were comprehensively discussed in Yamada (2007). The discussion of fatigue life evaluation of springs continues in Chaps. 9, 10 and 12.

1.3 Helical Springs of Minimal Mass

1.3.1 Restricted Optimization Problem

The designer of springs usually quests the springs of minimum weight for the definite strength and energy capacity limitations. The mean coil diameter of the spring body and the wire diameter depend of the polar angle along the spring wire. These two functions (Fig. 1.5) are the design variables:

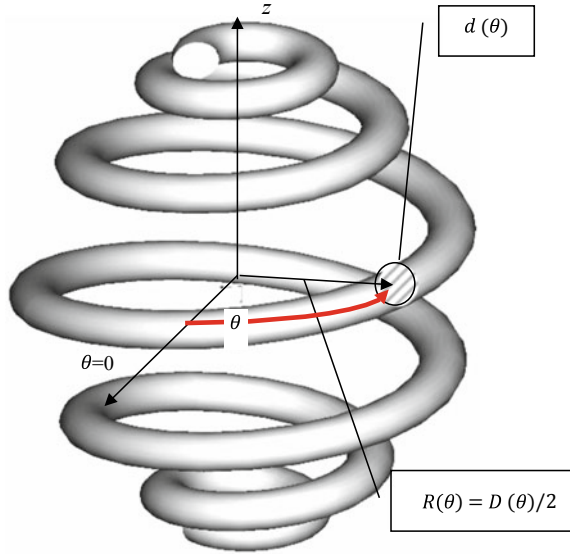
$$D = D(\theta), \quad d = d(\theta).$$

For the analytical treatment, we constrain ourselves to the following optimization problem:

Minimize the mass of the spring:

$$m \rightarrow \min_{D,d},$$

Fig. 1.5 Helical spring with variable wire diameter and non-cylindrical form



assuming the spring rate is equal to a given positive constant c^* :

$$c(D, d) = c^* \quad (1.34)$$

and the forces at installed height F_{min} and full stroke F_{max} are prescribed, the fatigue conditions (1.32), (1.33) fulfilled, and the ideal stress at full stroke is limited:

$$\frac{F_s}{W_T} \leq \tau_w, \quad F_s = c \cdot (L_0 - L_s).$$

1.3.2 Optimization of Helical Springs for Maximal Stress

Consider at first the practically important case of the non-cylindrical springs with variable circular cross-section: the stress at solid height must be less than τ_w to protect the spring from inadvertent damage. This restriction, applied on the basic shear stress at solid height:

$$\tau(d) \equiv \frac{8F_s D}{\pi d^3} \leq \tau_w.$$

This inequality could be expressed in terms of wire diameter:

$$d(\theta) \geq d_1(\theta).$$

The optimal diameter of wire:

$$d_1 \equiv \sqrt[3]{\frac{8F_s D}{\pi \tau_w}} \quad (1.35)$$

is the solution of algebraic equation $\tau = \tau_w$ with respect to $d(\theta)$.

We use the general formula for spring rate (1.13), which is valid for the changeable diameter of the spring body and the spring wire: $D = D(\theta)$, $d = d(\theta)$. Taking into account, that $d(\theta) \geq d_1(\theta)$, the inequality for the spring rate reads:

$$c(D, d) = \pi G \left(4 \int_0^{\Theta_a} \frac{D^3}{d^4} d\theta \right)^{-1} \geq \pi G \left(4 \int_0^{\Theta_a} \frac{D^3}{d_1^4} d\theta \right)^{-1}.$$

We substitute the expression (1.35) for the optimal diameter of wire into this inequality and express the stiffness requirement (1.39) as:

$$c^* = c(D, d) \geq 4\pi G \left(\frac{F_s}{\pi \tau_w} \right)^{4/3} \left(\int_0^{\Theta_a} D^{5/3} d\theta \right)^{-1}. \quad (1.36)$$

Similarly, we substitute the expression (1.35) for the optimal diameter of wire the expression (1.33) for the mass of spring. After this substitution, it follows the second inequality for the mass:

$$m \geq \frac{1}{2} \pi \rho_m \left(\frac{F_s}{\pi \tau_w} \right)^{2/3} \left(\int_0^{\Theta_a} D^{5/3} d\theta \right). \quad (1.37)$$

The inequalities of the same sign can be multiplied. The multiplication of the inequalities (1.36) and (1.37) results in a final lower boundary for spring mass:

$$m \cdot c^* \geq \frac{2\rho_m G F_s^2}{\tau_w^2}. \quad (1.38)$$

This important inequality establishes the exact lower boundary for the mass of spring of an arbitrary variable shape and an arbitrary variable circular cross-section, which satisfy the stress condition at solid length:

$$m \geq m_1 = \frac{2\rho_m G F_s^2}{\tau_w^2 c^*}. \quad (1.39)$$

The maximal stored elastic energy per unit volume (volume energy density) is equal to $\tau_w^2/2G$. The ultimate stored elastic energy per unit mass (mass energy density) is:

$$\tilde{U}_{e.1} = \tau_w^2 / 2\rho_m G.$$

The inequality (1.39) indicates the mass in terms of specific elastic energy density in the spring:

$$m \geq m_1 = \frac{F_s^2}{\tilde{U}_{e.1} c^*}. \quad (1.40)$$

1.3.3 Design for Fatigue Life

The spring is to operate a definite number of cycles through a deflection s measured as additional compression from L_0 . The application of a similar optimization procedure, as applied above, for the fatigue condition (1.32) leads to optimal wire diameter:

$$d_2 \equiv \sqrt[3]{\frac{8DF_{max}}{\pi} \cdot \left(\frac{1}{\tau_w} + \frac{1-R_\sigma}{2\tau_e} \right)}. \quad (1.41)$$

Using Eq. (1.41), we get in this case the lower boundary for the mass:

$$m \geq m_2 = \frac{2\rho_m G F_{max}^2}{c^*} \cdot \left(\frac{1}{\tau_w} + \frac{1-R_\sigma}{2\tau_e} \right)^2. \quad (1.42)$$

Consistently with the Eq. (1.40), the optimal mass expresses in terms of specific energy density:

$$m \geq m_2 = \frac{F_{max}^2}{\tilde{U}_{e.2} c^*}, \quad \tilde{U}_{e.2} = \frac{1}{2\rho_m G \cdot \left(\frac{1}{\tau_w} + \frac{1-R_\sigma}{2\tau_e} \right)^2}. \quad (1.43)$$

These expressions determine the optimal spring, acceptable from the viewpoint of fatigue life criterion (1.32).

If the fatigue calculation uses Smith, Watson, Topper rule (1.33), then the optimal wire diameter is:

$$d_3 \equiv \sqrt[3]{\frac{8DF_{max}}{\pi p_{SWT.0}} \sqrt{\frac{1-R_\sigma}{2}}}. \quad (1.44)$$

The optimal mass of spring satisfies with Eq. (1.44) the condition:

$$m \geq m_3 = \frac{2\rho_m G F_{max}^2}{c^*} \frac{1-R_\sigma}{2p_{SWT.0}^2}. \quad (1.45)$$

The optimal mass displays the lower limit for springs, which designed only for fatigue life:

$$m \geq m_3 = \frac{F_{max}^2}{\tilde{U}_{e,3} c^*}, \quad \tilde{U}_{e,3} = \frac{p_{SWT.0}^2}{\rho_m G (1 - R_\sigma)}. \quad (1.46)$$

The values $\tilde{U}_{e,1}$, $\tilde{U}_{e,2}$, $\tilde{U}_{e,3}$ express the mass energy density J/kg . These values are significant material constants for spring design. The smallest value signifies the attainable specific energy density of the material, if all three conditions must be simultaneously fulfilled:

$$\tilde{U}_e = \min(\tilde{U}_{e,1}, \tilde{U}_{e,2}, \tilde{U}_{e,3}). \quad (1.47)$$

1.3.4 Spring Quality Parameter for Helical Springs

Combining the optimization results (1.35), (1.41) and (1.44), we obtain the expression for the optimal wire diameter:

$$d_{opt} = \sqrt[3]{\frac{8D}{\pi}} Q_p \quad (1.48)$$

The absolute lowest mass of the spring follows from (1.39), (1.42) and (1.45):

$$m_{opt} = \frac{2\rho_m G}{c^*} Q_p^2. \quad (1.49)$$

Here

$$Q_p = \max \left[\frac{F_s}{\tau_w}, F_{max} \cdot \left(\frac{1}{\tau_w} + \frac{1 - R_\sigma}{2\tau_e} \right), \frac{F_{max}}{p_{SWT.0}} \sqrt{\frac{1 - R_\sigma}{2}} \right] \quad (1.50)$$

is a spring quality parameter, which accounts for different fatigue and endurance limits.

The optimal wire diameter (1.48) was determined from the stress condition. It was proved, that the optimal wire diameter guarantees the lowest possible mass of the spring (1.49). This mass depends on the ultimate allowable stress, the workloads and the spring stiffness.

The usual additional requirements for the spring design include, among others, several practically important requirements:

- the outer diameter of the spring $D + d$ is to be no greater then D_m ;
- the total stored elastic energy in the spring is prescribed;

- the certain natural frequency of the spring ω is limited;
- the assembly volume of the helical spring $\pi D_m^2 L_0$ is limited;
- the volume of the helical spring $\pi D_m^2 L_s$ is limited.

Investigations were conducted into minimization of spring volume and weight, using other technological requirements. The full mathematical treatment of the design cases is too complex for analytical treatment and requires the application of numerical optimization methods. For details, see a survey (Kruzelecki 1990).

The optimization of the springs that subjected to axial torque is performed analogously. Because the wire is in the state of bending, the optimization results are similar to the discussed in the next section.

1.4 Semi-elliptic Longitudinal and Transverse Leaf Springs of Minimal Mass

1.4.1 Rectangular Cross-Section

Another example of bending-dominated spring is the leaf spring. This type of spring locates, for example, the solid drive axle in the Hotchkiss drive that used in light trucks (Gillespie 1992). For optimization of this type of springs, the leaf spring is considered. The leaf spring is a simply supported beam of length L_0 . For brevity, the deformation of the spring is assumed to be small and we make no difference between the free L_0 and deformed L lengths of the spring. The eyes are provided for attaching the spring with the car body. The spring is fixed to the wheel axle by means of central clamp. Chamber is the amount of bend that is given to the spring from the central line, passing through the eyes. The wheel exercises the force F on the spring and support reactions at the two eyes of the spring come from the carriage. The force F leads to the bending moment in the cross-section of the spring:

$$m_B = m_B(x), \quad (1.51)$$

which depends on the position x along the axis of the spring. For example, the bending moment in the cross-section of the simply supported spring with the force F in its middle section reads:

$$m_B(x) = F \begin{cases} L/2 + x, & -L/2 \leq x < 0, \\ L/2 - x, & 0 \leq x \leq L/2. \end{cases} \quad (1.52)$$

The wheel-guided transverse leaf spring axle possesses a slightly different moment distribution over the spring length (WO 2008125076 A1). In this application the distance between the force appliance points is l_f . The bending moment along the spring length reads:

$$m_B(x) = F \begin{cases} L/2 + x, & -L/2 \leq x < -l_f/2, \\ (L - l_f)/2, & -l_f/2 \leq x \leq l_f/2, \\ L/2 - x, & l_f/2 \leq x \leq L/2. \end{cases} \quad (1.53)$$

The stored elastic energy with the modulus in tension or bending E is given by:

$$U_e = \frac{1}{2} \int_L \frac{m_B^2}{EI} dx = \frac{F^2}{2c}. \quad (1.54)$$

This expression delivers the general expression for spring rate of leaf spring helical, non-cylindrical springs:

$$\frac{1}{c} = \frac{1}{F^2} \int_L \frac{m_B^2}{EI} dx. \quad (1.55)$$

The volume of material with variable cross-section is given by the integral of the cross-sectional area $A = A(x)$ over the spring length, so that the mass of the spring is:

$$m = \rho_m \int_L A dx. \quad (1.56)$$

The stress at solid height must be less than the ultimate tensile strength $\sigma_w = R_m$ to protect the spring from inadvertent damage. This restriction, applied on the bending stress:

$$\sigma \equiv \frac{m_B}{W_B} \leq \sigma_w. \quad (1.57)$$

For the springs with a rectangular cross-section with axes of the cross-section

$$B = B(x), \quad T = T(x) \quad (1.58)$$

the geometric characteristics of cross-section are the following (Tables 1.3 and 1.4):

$$W_B = \frac{T^2 B}{6}, \quad I = \frac{BT^3}{12}, \quad A = BT. \quad (1.59)$$

Minimize the mass of the spring (1.56) assuming the spring rate

$$c = c^* \quad (1.60)$$

and the force F are prescribed and the stress restriction (1.57) is fulfilled.

The following estimation for the height of the cross-section follows from (1.57):

$$T(x) \geq T_o(x), \quad T_o(x) = \sqrt{\frac{6m_B}{B\sigma_w}}. \quad (1.61)$$

The lowest mass possesses the uniform stress beam, which is known as a parabolic spring. For the given moment distribution, the profile height of the uniform stress beam is $T_o(x)$. The substitution of the height from (1.61) into Eq. (1.55) leads to inequality for the spring rate:

$$\frac{1}{c^*} \leq \frac{1}{F^2} \int_L \frac{M_B^2}{\frac{EB}{12} \left(\frac{6m_B}{b\sigma_w} \right)^{3/2}} dx = \frac{1}{F^2 \frac{E}{12} \left(\frac{6}{\sigma_w} \right)^{3/2}} \int_L \sqrt{Bm_B} dx. \quad (1.62)$$

Similarly, the substitution (1.61) into the expression for the spring mass delivers:

$$\rho_m \int_L \sqrt{\frac{6Bm}{\sigma_w}} dx \leq \rho_m \int_L BT dx = m. \quad (1.63)$$

The multiplication of inequalities (1.62) and (1.63) leads to an inequality:

$$\frac{\sqrt{6}}{c^* \sqrt{\sigma_w}} \rho_m \int_L \sqrt{Bm_B} dx \leq \frac{m}{F^2 \frac{E}{12} \left(\frac{6}{\sigma_w} \right)^{3/2}} \int_L \sqrt{Bm_B} dx. \quad (1.64)$$

The integrals of the positive function $\sqrt{Bm_B}$ are there on the both sides of the inequality (1.64) and could be shortened. From the inequality (1.64) the exact lower mass of the spring with rectangular cross-section reads:

$$m \geq \frac{3E\rho_m}{c^*} \frac{F^2}{\sigma_w^2}. \quad (1.65)$$

The elastic potential energy per unit volume (specific volume energy density) is equal to $\sigma_w^2/2E$. The elastic potential energy per unit mass (specific mass energy density) is:

$$\tilde{U}_e = \sigma_w^2/2\rho_m E.$$

This value is the material constant in state of normal stress. The reference mechanical properties for the customary spring materials are displayed in Table 1.5. Based on these values, Fig. 1.6 displays the specific volume energy density and specific mass energy density of the common spring materials. The analogous method is applicable for the springs, made of composite materials. Because the composite materials

Table 1.5 Selected materials for springs

	Designation	EN	Material characterization	Density	Application temperature	Strength	Moduli		Mass density	Volume density
							G	E		
				ρ_m kg/dm ³	°C	σ_w MPa	GPa	GPa	$\frac{\sigma_w^2}{2pE}$ J/kg	$\frac{\sigma_w^2}{2E}$ J/dm ³
1	1.4301/X5CrNi1810, Spring steel V2A	10088-3	Corrosion resistance	7.9	250	500..700	68	180	126	1000
2	1.4310/X10CrNi18-8, Spring steel V2A	10270-3	High corrosion resistance	7.9	200	500..750	70	185	155	1230
3	1.4401/X5CrNiMo171-12, Spring steel V4A	10270-3	Corrosion resistant, good relaxation, non-magnetic	8.0	300	500..700	68	180	125	1000
4	1.4436/X5CrNiMo17133, Spring steel V4A	10088-3	Good corrosion resistance, slightly magnetic	7.98	300	670	68	180	156	1250
5	1.4539/X1NiCrMoCuN25-20, Spring steel V4A	10088	Heavy corrosion conditions, non-magnetic	8	300	530..730	68	180	137	1100
6	1.4568/X7CrNiAl17-7, Spring steel V4A	10270-3	Low relaxation, high fatigue strength	7.81	350	900..1100	73	195	328	2560
7	1.4571/X6CrNiMoTi17, Spring steel V4A	10270-3	Corrosion resistant, higher strength	7.95	300	500..700	68	185	124	973

(continued)

Table 1.5 (continued)

	Designation	EN	Material characterization	Density	Application temperature	Strength	Moduli		Mass density	Volume density
							G	E		
				ρ_m kg/dm ³	°C	σ_w MPa	GPa	GPa	$\frac{\sigma_w^2}{2pE}$ J/kg	$\frac{\sigma_w^2}{2E}$ J/dm ³
8	2.4610/NiMo16Cr16Ti, Hastelloy C4	–	In very corrosive atmosphere, non-magnetic	8.64	450	700..900	76	210	176	1520
9	2.4632/NiCr20CO18Ti, Nimonic 90	–	Corrosion resistant against most gases	8.2	500	1200	83	213	412	3380
10	2.4669/NiCr15Fe7TiAl, Inconel X750	–	High temperature, non-magnetic	8.2	600	980	76	213	274	2250
11	CW101C/CuBe2, Copper beryllium alloy	12166	Corrosion-resistant, anti-magnetic, spark-free	8.3	–200..80	1150	47	120	663	5510
12	CW452K/CuSn6, Spring bronze	12166	Nonmagnetic, solderable, weldable, corrosion resistant	8.8	–200..60	340..500	42	115	87	767
13	CW507L/CuZn36, Brass wire	12166	Non-magnetic	8.45	–200..60	300..560	39	110	198	840
14	CoNiCrFe, Duratherm	–	High temperature	8.45	600	1100..1880	85..95	205..225	619	5230

(continued)

Table 1.5 (continued)

Designation	EN	Material characterization	Density	Application temperature	Strength	Moduli		Mass density	Volume density
						G	E		
			ρ_m kg/dm ³	°C	σ_w MPa	GPa	GPa	$\frac{\sigma_w^2}{2pE}$ J/kg	$\frac{\sigma_w^2}{2E}$ J/dm ³
15	EN 10270-1, Type DH	Spring steel wire All common springs, high static and medium dynamic stress	7.95	80	2400..3500	81.5	206	2747	21800
16	EN 10270-1, Type SH	Spring steel wire All common springs, high static and medium dynamic stress	7.95	80	1900-2940	81.5	206	1908	15200
17	EN 10270-2/VDC Valve spring wire	For high continuous vibration stress	7.95	80	1250..2100	79.5	206	989	7860
18	EN 10270-2/VDSiCr Valve spring wire	High dynamic load over 100C, good relaxation properties	7.95	120	1670..2100	79.5	206	1221	9710

(continued)

Table 1.5 (continued)

Designation	EN	Material characterization	Density	Application temperature	Strength	Moduli		Mass density	Volume density
			ρ_m kg/dm ³		σ_w MPa	G GPa	E GPa	$\frac{\sigma_w^2}{2pE}$ J/kg	$\frac{\sigma_w^2}{2E}$ J/dm ³
19 TiAl6V4, Titanium alloy	–	Insensitivity to cold, heat and corrosion	4.43	–200...300 °C	1286	39	111	1794	7950

- 1—Material data sheet 1.4301/ X5CrNi1810, HSM Stahl- und Metallhandel GmbH, www.hsm-stahl.de
- 2—Material data sheet 1.4310 / X10CrNi18-8, HSM Stahl- und Metallhandel GmbH, www.hsm-stahl.de
- 3—Material data sheet 1.4401/ X5CrNiMo171-12-2, HSM Stahl- und Metallhandel GmbH, www.hsm-stahl.de
- 4—Material data sheet 1.4436, DEUTSCHE EDELSTAHLWERKE GMBH
- 5—Material data sheet 1.4539 AISI 904 L, Metalcor GmbH, www.metalcor.de
- 6—Material data sheet 1.4568 17-7 PH®, Metalcor GmbH, www.metalcor.de
- 7—Material data sheet Acidur 4571, DEUTSCHE EDELSTAHLWERKE GMBH
- 8—Material data sheet HASTELLOY® C-4, Zapp Materials Engineering GmbH, specialtymaterials@zapp.com
- 9—Material data sheet 2.4969 ALLOY 90, Metalcor GmbH, www.metalcor.de
- 10—Material data sheet 2.4669 ALLOY X-750, Metalcor GmbH, www.metalcor.de
- 11—Material data sheet 2.1247 CuBe2, Metalcor GmbH, www.metalcor.de
- 12—Material data sheet CW452K, Seeberger GmbH & Co. KG, www.seeberger.net
- 13—Material data sheet VACUUMSCHMELZE GmbH & Co. KG, <http://www.vacuumschmelze.com>
- 14—Material data sheet DH, Hartgezogener Federstahldraht, <https://metalprice.metalleschmidt.de/de>
- 15—Material data sheet SH, Hartgezogener Federstahldraht, <https://metalprice.metalleschmidt.de/de>
- 16—Material data sheet VDC, Ölschlussvergüteter Federstahldraht, <https://metalprice.metalleschmidt.de/de>
- 17—Material data sheet VDSiC, Ölschlussvergüteter Federstahldraht, <https://metalprice.metalleschmidt.de/de>
- 18—Material data sheet VDSiC, Ölschlussvergüteter Federstahldraht, https://www.huh.de/fileadmin/dateien/pdfs/Materialdatenblaetter/huh_Materialdatenblatt_Titan.pdf
- 19—Material data sheet TiAl6V4, https://www.huh.de/fileadmin/dateien/pdfs/Materialdatenblaetter/huh_Materialdatenblatt_Titan.pdf

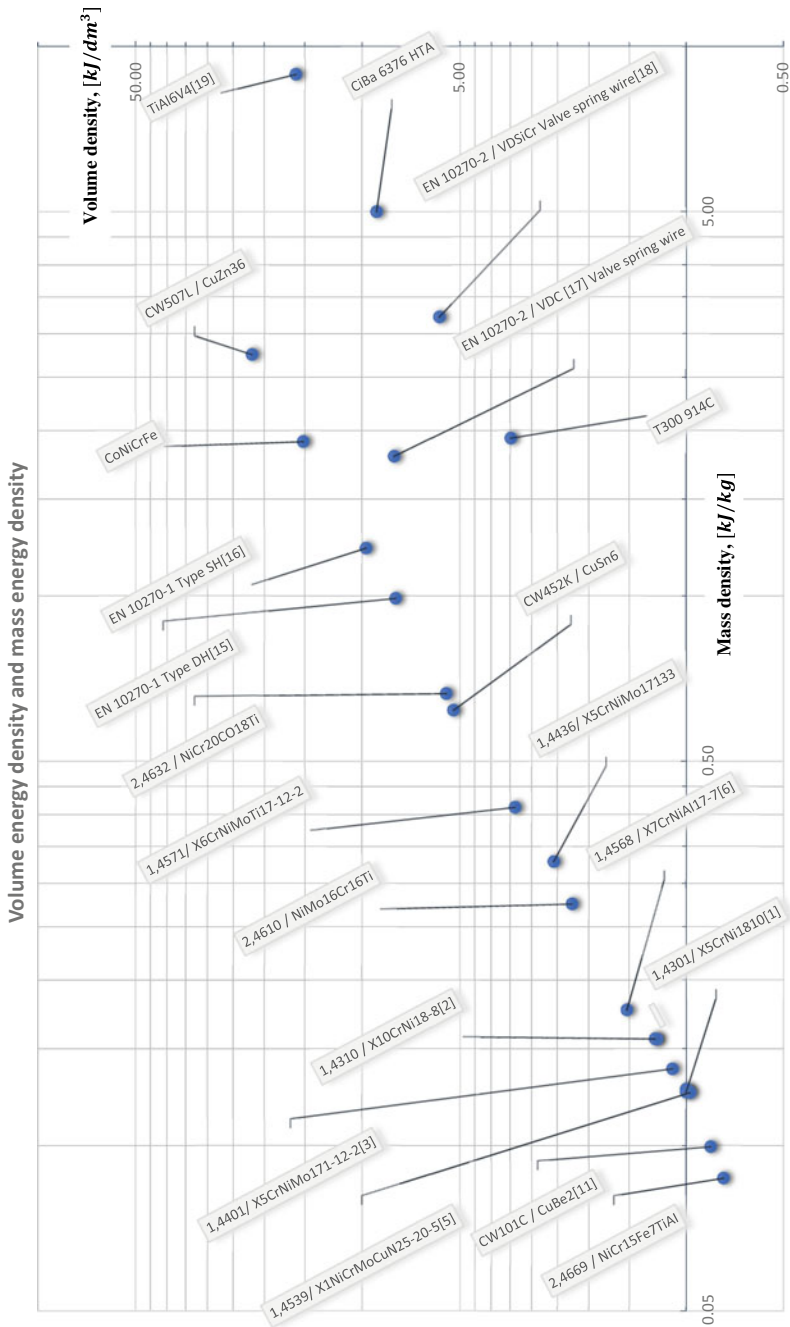


Fig. 1.6 Volume energy density and mass energy density of the common spring materials from Table 1.5

are anisotropic, the evaluation of the specific mass energy density depend on the orientation of fibres (Kobeleev 2019).

The inequality (1.65) indicates the lowest mass in terms of stored elastic energy:

$$m \geq 3F^2/2\tilde{U}_e c^*.$$

1.4.2 Circular Cross-Section

For the springs with a circular cross-section with the diameter $d = d(x)$ the geometric characteristics of cross-section are the following (Tables 1.3 and 1.4):

$$W_B = \frac{\pi d^3}{32}, I = \frac{\pi d^4}{64}, A = \frac{\pi d^2}{4}.$$

From (1.57) the estimation for the diameter of the cross-section follows as:

$$d \geq \sqrt[3]{\frac{32m_B}{\pi\sigma_w}}. \quad (1.66)$$

The substitution of the wire diameter from (1.66) into Eq. (1.68) leads to inequality for the spring rate:

$$\frac{64\pi^{1/3}\sigma_w^{4/3}}{32^{4/3}E} \int_L m_B^{2/3} dx \geq \frac{F^2}{c^*}. \quad (1.67)$$

The substitution (1.66) into the expression for the spring mass reads:

$$m \geq \frac{\rho_m \pi}{4} \int_L \left(\frac{32m_B}{\pi\sigma_w} \right)^{2/3} dx. \quad (1.68)$$

The multiplication of inequalities (1.62) and (1.63) leads to an inequality for the spring mass with the circular cross-section:

$$m \geq \frac{4E\rho_m}{c^*} \frac{F^2}{\sigma_w^2}. \quad (1.69)$$

The inequality (1.69) specifies the minimal mass in terms of specific elastic energy density in the spring:

$$m \geq 2F^2/\tilde{U}_e c^*.$$

The lowest mass of the circular wire is 1/3 more than the lowest mass of the rectangular wire. This relation is valid for all fully-stress-designed springs for an arbitrary moment M_B along the spring axes, if the restrictions (1.59) and (1.57) are satisfied.

The same estimations are valid for other springs, which are subjected to bending loads, for example for the twisted helical springs.

1.5 Multi-material Design of Springs

The seminal paper (Ashby and Bréchet 2003) explored the designing of hybrid materials, giving emphasis to the selection of components, their shape, and their scale. The new design variables were introduced to expand the design space and create the new “hybrid materials” with specific property profiles. A paper (Wargnier et al. 2014) that proposed a multi-material design procedure could be considered as a continuation of the primary effort.

In this section we study a question of multi-material design for springs. Namely, consider a number of helical springs made of different materials that are assembled parallel and in series. Each spring assembly acts as a single spring. The springs are in series if they are connected at their ends and the forces in every spring are the same. The sum of reciprocal spring rates is equal to the spring rate of serial assembly. Similarly, the springs are in parallel if they are connected side-by-side and each spring has an equal travel. The parallel assembly spring rate is the sum of spring rates for single springs.

We examine the following question: what is the minimal mass of a serial and a parallel spring assembly if the total force and the assembly spring rate are prescribed? For certainty we explore the design problem for helical springs.

Minimize the mass of the spring assembly:

$$m = \sum_{i=1}^N m_i \rightarrow \min. \quad (1.70)$$

assuming the assembly spring rate is equal to a given positive constant c^* , Eq. (1.60), the forces at installed height F_1 and full stroke F_2 are prescribed, the fatigue conditions (1.32), (1.33) fulfilled, and the basic stress at full stroke for each material

$$\tau_b \leq \tau_{w,i}, \quad i = 1, \dots, N$$

are limited. Here N is the number of materials with the corresponding admissible working stresses $\tau_{w,i}$, shear modules G_i and densities ρ_i .

For the parallel assemblies the spring travel in each spring is

$$s = F_i/c_i \equiv F/c^*, \quad F = \sum_{i=1}^N F_i, \quad (1.71)$$

where F_i is the spring force and c_i is the spring rate of the spring with number i . From these equations follows that the total spring rate of the parallel assembly c^* is the sum of spring rates of all springs c_i :

$$c^* = \sum_{i=1}^N c_i.$$

According to Eq. (1.68), the mass of the spring made of material with number i is restricted from below:

$$m_i \geq m_{i,opt} = \frac{2\rho_i G_i F_i^2}{\tau_{w,i}^2 c_i} = \frac{2\rho_i G_i F_i s}{\tau_{w,i}^2}, \quad (1.72)$$

Applying the estimation (1.71) for each spring and calculating the total mass of parallel assembled spring, the following exact boundary for the mass is stated:

$$m = \sum_{i=1}^N m_i \geq \sum_{i=1}^N m_{i,opt} = \sum_{i=1}^N \frac{2\rho_i G_i F_i s}{\tau_{w,i}^2} \quad (1.73)$$

From Eqs. (1.71) and (1.73) follows the inequality:

$$\sum_{i=1}^N \frac{2\rho_i G_i F_i s}{\tau_{w,i}^2} \geq F s \min_{1 \leq i \leq N} \frac{2\rho_i G_i}{\tau_{w,i}^2} = \frac{F^2}{c^*} \min_{1 \leq i \leq N} \frac{2\rho_i G_i}{\tau_{w,i}^2}. \quad (1.74)$$

The combination of Eqs. (1.73) and (1.74) yields:

$$m \geq \frac{F^2}{c^*} \min_{1 \leq i \leq N} \frac{2\rho_i G_i}{\tau_{w,i}^2}. \quad (1.75)$$

The estimation (1.75) proves, that the minimal mass of the springs in parallel possesses the single spring with the uppermost specific elastic energy density of material.

For the springs in series, the force in each spring is equal to F , and the spring travel in each spring is

$$s_i = F/c_i, \quad s = \sum_{i=1}^N s_i = F/c^*. \quad (1.76)$$

These equations show that the reciprocal spring rate of the serial assembly c^* is the sum of reciprocal spring rates of each spring in the assembly c_i :

$$\frac{1}{c^*} = \sum_{i=1}^N \frac{1}{c_i}.$$

The mass of each spring in the serial assembly is:

$$m_i \geq m_{i,opt} = \frac{2\rho_i G_i F^2}{\tau_{w,i}^2 c_i} = \frac{2\rho_i G_i F s_i}{\tau_{w,i}^2}. \quad (1.77)$$

Applying the estimation (1.77) for each spring, we state the exact lower boundary for the total mass of the serial assembly:

$$m = \sum_{i=1}^N m_i \geq \sum_{i=1}^N m_{i,opt} = F \sum_{i=1}^N \frac{2\rho_i G_i s_i}{\tau_{w,i}^2}. \quad (1.78)$$

Using Eq. (1.76), from Eq. (1.78) follows:

$$F \sum_{i=1}^N \frac{2\rho_i G_i s_i}{\tau_{w,i}^2} \geq F s \min_{1 \leq i \leq N} \frac{2\rho_i G_i}{\tau_{w,i}^2} = \frac{F^2}{c^*} \min_{1 \leq i \leq N} \frac{2\rho_i G_i}{\tau_{w,i}^2}. \quad (1.79)$$

From Eqs. (1.78) and (1.79) follows the desired inequality:

$$m \geq \frac{F^2}{c^*} \min_{1 \leq i \leq N} \frac{2\rho_i G_i}{\tau_{w,i}^2} \quad (1.80)$$

Both lower boundaries (1.75) and (1.80) guarantee the equal lowest mass of the assembly irrespectively of the assembly type. The minimal mass of each multi-material assembly is more or equal the mass of the single spring with the highest specific elastic energy density of material.

1.6 Conclusions

It was proved, that the optimal wire shape, determined from the certain equal stress condition, guarantees the lowest possible mass of the spring. This mass depends only on the ultimate allowable stress for the spring material, the load at full stroke and the spring stiffness. This is an important milestone for comparison of different spring designs and spring materials. As the density and shear module are almost the same for all spring steels, the spring quality parameter can serve as the benchmarking property for spring design.

The minimal mass of the springs in parallel or in series assemblies possesses the single spring with highest specific elastic energy density of material.

The detailed design of springs is performed with numerical methods. The theoretical background and actual methodologies for numerical analysis of springs supplies (Shimoseki et al. 2003). The cited book explains the design examples, calculated by finite-element software and their comparison with experiments on real springs.

1.7 Summary of Principal Results

The following expressions are displayed in the current chapter:

- the compression spring rate, the compression-twist springs rate and the twist springs rate of an arbitrary non-cylindrical helical spring;
- the formulas for the reduction of the spring angle in cause of compression force and torque;
- the simple expressions for the shear stresses over the cross-sections and on the outer surfaces of wires in helical springs;
- the simple fatigue criterion for compression spring design;
- the exact lower boundary for the mass of helical spring of an arbitrary variable shape and an arbitrary variable circular cross-section, which satisfy the stress condition;
- the exact lower mass of the leaf spring with the rectangular cross-section;
- the exact lower mass of the leaf spring with the circular cross-section.

It was demonstrated, that:

- the lowest mass of the circular wire is 1/3 more that the lowest mass of the rectangular wire;
- the equal lowest mass of the multi-material assembly does not depend on the assembly type (serial or parallel);
- the minimal mass of each multi-material assembly is more or equal the mass of the single spring with the highest specific elastic energy density of material.

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Chapter 2

Stress Distributions Over Cross-Section of Wires



Abstract The stress distribution over the cross-section wire of helical springs is examined in this chapter. For simplification, the pitch of the helical spring is neglected and the traditional representation of one coil as an incomplete torus is used. This model generalizes the StVenant's torsion problem of an elliptical straight rod accounting the curvature rod. The closed form solution for the torsion problem of an incomplete torus is discussed. This Chapter is the next section of first part, which studies the springs from viewpoint of design.

2.1 Warping Function

In the present chapter, we consider the helical springs with the elliptical, circular and the ovate cross-sections of the spring wire. The force acts ideally along the axis z of the cylindrical coordinate system, such that the line of force coincides with the symmetry axis of the spring. The radius of the spring helix is $R = D/2$, where the mean diameter of the spring is D . A view of the spring wire at the cut section has been taken from the right and was shown at Fig. 1.3, Chap. 1. The couple $M = F R$, acting in the vertical plane, has been shown as a vector. The couple was resolved into two components, which lie in planes which are tangential and normal to the helix, respectively (Fig. 1.3a). The distance between coils is assumed as small. For the closed-coiled spring, the torque moment of the wire is nearly equal to M . The axial force leads only to the torsion of the wire. Thus, both tension and compression helical springs store elastic energy principally through torsion of wire. The component of couple, which tends to cause bending of the isolated portion of the spring wire, is neglected.

The isolated portion of the spring wire is the incomplete torus. For this incomplete torus we search the analytical solution for the torsion problem. If the curvature of the torus vanishes, the isolated portion of the spring wire turns into an ovate straight solid or hollow rod. StVenant examined the torsion problem for an ovate straight solid or hollow rod. The torsion of straight cylinders with circular and elliptical cross-sections was reviewed by Sneddon and Berry (1958) and de Veubecke (1979).

For the helical spring the curvature of the wire could not be neglected (Michell 1899). For this purpose, consider an elastic torus, generated by rotating the cross-section about the z axis of the cylindrical coordinate system. This torus represents one coil of the helical spring with a negligible pitch (“closed-coiled spring”). The cross-section Ω considered, until explicitly stated otherwise, to be simply connected. The torus is assumed to be incomplete, i.e. the two ends of the turn are not joined.

We examine now the vector displacement field:

$$\langle U(r, \theta, z), V(r, \theta, z), W(r, \theta, z) \rangle$$

in reference cylindrical coordinate system (r, θ, z) .

For the displacement field, which arises in the helical spring, the axial displacement W linearly depends upon the polar angle θ , while the radial displacement U vanishes. The section will not remain planar during the deformation. Each section is deformed, however, in precisely the same way, so we may take the components of displacements of the form:

$$\langle U, V, W \rangle = \langle 0, r\psi(r, z), \theta \cdot B_s \rangle, \quad (2.1)$$

where $\psi(r, z)$ is the warping function. For the torus, which embodies a single complete coil, $0 < \theta < 2\pi$. The torus turns during the infinitesimal deformation into a helix with the tiny constant pitch. The two ends of coil separate from each other by the relative axial displacement:

$$W|_{\theta=2\pi} - W|_{\theta=0} = 2\pi \cdot B_s. \quad (2.2)$$

Worthwhile mentioned, that the helical spring deforms exactly as an elastic body with a giant screw dislocation. The edge of the dislocation coincides with the axis of the helix. From the physical viewpoint, the constant B_s is equal to the magnitude of the Burgers vector of this giant screw dislocation (Hirth and Lothe 1991). The magnitude of the Burgers vector B_s is proportional to the displacement of one coil and defines the pitch change due to the axial compression or extension of the helical spring.

According to (2.1), the only non-zero components of the strain tensor in the cylindrical coordinate system in the case of screw dislocation are:

$$\gamma_{r\theta} = r \frac{\partial}{\partial r} \left(\frac{V}{r} \right), \gamma_{\theta z} = \frac{\partial V}{\partial z} + \frac{1}{r} \frac{\partial W}{\partial \theta}. \quad (2.3)$$

After the substitution of representations for displacements from (2.1) into (2.3) the two components of strain tensor reduce to:

$$\gamma_{r\theta} = r \frac{\partial \psi}{\partial r}, \gamma_{\theta z} = r \frac{\partial \psi}{\partial z} + \frac{B_s}{r}. \quad (2.4)$$

The components (2.4) are the same for all cross-sections, being independent of θ . The magnitude of the Burgers vector of screw dislocation B_s in (2.2) depends linearly upon the applied axial force F . The dependency of constant of screw dislocation upon the force will be determined in the next section.

Next, we derive the equilibrium equations in term of the warping. For this purpose, consider equilibrium of the linear-elastic, isotropic body in the absence of any volume forces. With Hook's law and strains in form (2.4), the shear stresses in terms of the warping function $\psi = \psi(r, z)$ will be:

$$\tau_{r\theta} = Gr \frac{\partial \psi}{\partial r}, \tau_{\theta z} = Gr \frac{\partial \psi}{\partial z} + \frac{G}{r} B_s. \quad (2.5)$$

In cylindrical coordinate system, the axial component of the equilibrium equation in the volume of the torus reads:

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial \tau_{\theta z}}{\partial z} + 2 \frac{\tau_{r\theta}}{r} = 0. \quad (2.6)$$

There are two other components of the equilibrium equation in other directions, but they are satisfied by Eq. (2.5) identically. After the substitution of (2.5) in (2.6) and division by r , Eq. (2.6) reduces to:

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} + \frac{3}{r} \frac{\partial \psi}{\partial r} = 0. \quad (2.7)$$

In terms of Laplace operator

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2},$$

the equilibrium Eq. (2.7) takes the form

$$\left(\Delta - \frac{1}{r^2} \right) (r\psi) = 0. \quad (2.8)$$

In cylindrical coordinate system, the normal vector to the mantle of the torus is $\langle n_r, 0, n_z \rangle$. The component n_θ disappears for the toroidal surface. The mantle of the torus, i.e. the boundary of torus without bases, is stress-free. This boundary condition requires the vanishing of the shear stress in the normal direction to the surface of the torus:

$$\tau_n = n_r \tau_{r\theta} + n_z \tau_{\theta z} = 0. \quad (2.9)$$

The substitution of Eq. (2.5) in Eq. (2.9) delivers the boundary condition:

$$\tau_n = n_r Gr \frac{\partial \psi}{\partial r} + n_z \left(Gr \frac{\partial \psi}{\partial z} + \frac{B_s}{r} \right) = Gr \frac{\partial \psi}{\partial n} + n_z \frac{B_s}{r} = 0. \quad (2.10)$$

In Eq. (2.10), we use the formula: $\partial/\partial n = n_r \partial/\partial r + n_z \partial/\partial z$.

2.2 Prandtl Stress Function

Alternatively, the torsion problem could be resolved using the Prandtl stress function $\phi = \phi(r, z)$.

The shear stresses in terms of the stress function are:

$$\tau_{r\theta} = -\frac{1}{r^2} \frac{\partial \phi}{\partial z}, \quad (2.11)$$

$$\tau_{z\theta} = \frac{1}{r^2} \frac{\partial \phi}{\partial r}. \quad (2.12)$$

The equilibrium equation in the volume of the coil (2.6) is satisfied by (2.11) and (2.12) identically. The substitution of Eqs. (2.11) and (2.12) in the boundary condition (2.9) leads to the boundary condition in terms of the stress function:

$$\tau_n = -\frac{n_r}{r^2} \frac{\partial \phi}{\partial z} + \frac{n_z}{r^2} \frac{\partial \phi}{\partial r} = \frac{1}{r^2} \frac{\partial \phi}{\partial s} = 0. \quad (2.13)$$

The cross-section Ω is assumed to be simply-connected with the boundary Γ . The upper and lower sections of curve Γ are $Z_+(r)$, $Z_-(r)$ (Fig. 2.1).

As follows from Eq. (2.13) the stress function is constant on the contour Γ :

$$\phi = \phi_0 \text{ on } \Gamma. \quad (2.14)$$

The value of the constant ϕ_0 does not influence the value of stresses. This constant it is assumed to be zero:

$$\phi = 0 \text{ on } \Gamma. \quad (2.15)$$

The tangential derivative expresses in terms of the partial derivatives as:

$$\partial/\partial s = -n_r \partial/\partial z + n_z \partial/\partial r.$$

As mentioned above, the equilibrium equations are identically satisfied by (2.11) and (2.12) via stress function. Notably, that the compatibility conditions should be fulfilled. The easiest way is to use the compatibility conditions in terms of stresses. These conditions are known as Beltrami-Michell equations (Sneddon and Berry 1958; de Veubecke 1979). For the stress state, which is given by Eqs. (2.11),

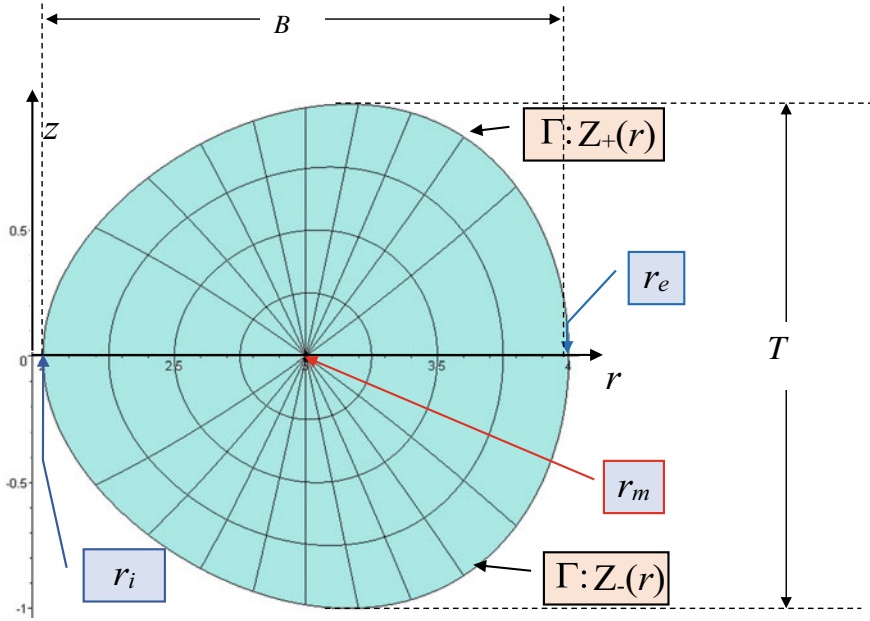


Fig. 2.1 Ovate cross-section. The contour lines of the Prandtl stress function are shown

(2.12), the Beltrami-Michell equation in cylindrical coordinates assumes the form:

$$\left(\Delta - \frac{4}{r^2}\right)\tau_{r\theta} = 0, \quad \left(\Delta - \frac{4}{r^2}\right)\tau_{z\theta} = 0 \quad (2.16)$$

Substituting (2.11) in the first Beltrami-Michell equation (2.16), we obtain equation for the stress function:

$$\frac{\partial}{\partial z} \left(\frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \phi}{\partial r^2} - \frac{3}{r} \frac{\partial \phi}{\partial r} \right) = 0. \quad (2.17)$$

Equation (2.17) results in:

$$\left(\Delta - \frac{4}{r^2}\right)\phi = -\frac{2G}{r^2}B_s. \quad (2.18)$$

The substitution of (2.12) in the second Beltrami-Michell equation (2.16) leads to the same governing Eq. (2.18).

Generally saying, the resultant force in the cross-section is a vector. As only two components of the stress tensor remain, only two components of force vector could exist. The resultant forces in the axial and radial direction are given by integration of the shear stresses $\tau_{z\theta}$ and $\tau_{r\theta}$ over the cross-section Ω of the torus, respectively:

$$F = \int_{\Omega} \tau_{z\theta} d\Omega. \quad (2.19)$$

$$F_r = \int_{\Omega} \tau_{r\theta} d\Omega. \quad (2.20)$$

The radial component of resultant force F_r vanishes:

$$\begin{aligned} F_r &= -G \int_{\Omega} \frac{1}{r^2} \frac{\partial \phi}{\partial z} d\Omega = -G \int_{r_i}^{r_e} \frac{1}{r^2} \left[\int_{Z_-(r)}^{Z_+(r)} \frac{\partial \phi}{\partial z} dz \right] dr = \\ &= -G \int_{r_i}^{r_e} \frac{1}{r^2} [\phi(z_+) - \phi(z_-)] dr = 0. \end{aligned} \quad (2.21)$$

The boundary condition (2.5) was applied in Eq. (2.21). The only remaining component of force is the force F in the direction of axis z .

The resultant moments in the cross-section are treated similarly. Consider the radial component of the moment vector with respect to the point O at the origin of the coordinate system. The only two non-zero components of stress tensor $\tau_{z\theta}$, $\tau_{r\theta}$ result in the moment vector. The moment vector is directed normally to the considered cross-section and causes a twist of cross-section. The resultant value of torsional moment about point O is equal to

$$M_r = \int_{\Omega} (\tau_{z\theta} r - \tau_{r\theta} z) d\Omega = G \int_{r_i}^{r_e} \frac{1}{r^2} \left[\int_{Z_-(r)}^{Z_+(r)} \left(\frac{\partial \phi}{\partial r} r + \frac{\partial \phi}{\partial z} z \right) dz \right] dr. \quad (2.22)$$

The torsional moment M_r vanishes for all cross-sections, which are symmetrical about plane $z = 0$, such that $Z_+(r) = -Z_-(r)$ (Pilgram 1913). Apparently, two remaining components of the resultant momentum vector with respect to the origin of coordinate system disappear because of the cyclic symmetry of stress state. Thus, the resultant force F_r and resultant value of moment M_r are both equal to zero for the considered stress-strain state. This demonstrates, that the resultant force vector for each cross-section has actually only one non-zero component F and its line of action is aligned with the axis z of the coordinate system.

The elasticity of the axially loaded spring is characterized by means of the spring rate c_w of one complete turn of the helix. The spring rate can be specified through magnitude of the Burgers vector from (2.3). For the linearly elastic spring, the magnitude of the Burgers vector B_s depends linearly upon the axial resultant force F . The spring rate c_w of one coil is, by definition, the force that causes a unit displacement in the axial direction

$$c_w = \frac{F}{W|_{\theta=2\pi} - W|_{\theta=0}} = \frac{F}{2\pi B_s}. \quad (2.23)$$

2.3 Shear Stresses on Surface of Elliptic and Circular Wires

According to Eq. (1.28), the spring index designates the ratio of mean coil diameter to wire diameter

$$w = R/r \equiv D/d.$$

For the springs with infinite radius of helix R , the spring index infinitely increases and the curvature vanishes. In this case, the stress distribution coincides with the stress field of the twisted straight wire. Thus, the spring with the infinite spring index possesses the constant stress over the surface and this stress is equal to the corresponding basic stress.

The stresses on the surface of spring with the finite spring index vary over the surface of the spring. For the spring with the circular cross-section, the maximum shear stress in a helical spring occurs on the inner face of the spring coils. Thus, the curvature of the helical spring results the shear stresses on the inner faces, that are higher than the basic stress.

However, if the cross-section is elliptical, the maximum of the shearing stress moves to the minor axis of the ellipse. Thus, for the flattened elliptical section the maximum locates in the middle of the upper and lower surfaces of wire. For the moderately flattened elliptical section the maximum of the stress is between the inner and the upper faces of the spring in the upper half of the cross-section. In the lower half, the point of maximum reflects with respect to the symmetry axis of the cross-section. Clear, that the stress distribution depends on the curvature of wire and with higher curvature the point of maximum moves to the inner face.

The easiest way to evaluate the shear stress on the outer surface of the spring wire is to follow the acknowledged results. Göhner (1932a, b) provided the formulas for stresses for the spring with the elliptic wire cross-section. The axes of the elliptic cross-section are T and B (Fig. 2.1). The local coordinate system of the cross-section originates in the center of ellipse. The shear stresses were presented as Taylor series:

$$\tau_{r\theta} = \tau_{r\theta}^{(1)} + \tau_{r\theta}^{(2)} + \dots, \quad \tau_{z\theta} = \tau_{z\theta}^{(1)} + \tau_{z\theta}^{(2)} + \dots \quad (2.24)$$

In Cartesian coordinates x , y of ellipse, the first two coefficients of Taylor series in (2.24) are (Göhner 1932a, b; Biezeno and Grammel 1971):

$$\tau_{r\theta}^{(1)} = -G\theta \frac{2T^2 y}{T^2 + B^2}, \quad (2.25)$$

$$\tau_{r\theta}^{(2)} = -G\theta \frac{2T^2(2T^2 + 3B^2)xy}{(T^2 + B^2)(T^2 + 3B^2)R}, \quad (2.26)$$

$$\tau_{z\theta}^{(1)} = G\theta \frac{2T^2x}{T^2 + B^2}, \quad (2.27)$$

$$\tau_{z\theta}^{(2)} = G\theta \frac{B^2[(16T^2 + 12B^2)x^2 - 12T^2y^2 + 3T^2B^2]}{4(T^2 + B^2)(T^2 + 3B^2)R}. \quad (2.28)$$

In the above formulas θ is the torsion angle pro length unit. The torsion angle θ relates to the axial displacement W with Eq. (2.1).

The functions $\tau_{r\theta}^{(1)}$ and $\tau_{z\theta}^{(1)}$ are the first degree Taylor polynomials, and the functions $\tau_{r\theta}^{(2)}$, $\tau_{z\theta}^{(2)}$ are the second degree Taylor polynomials of Cartesian coordinates x , y .

The octahedral shear stress for the elliptical wire with the same order of precision reads:

$$\tau_{oct}(\rho, \varphi) = \sqrt{\left(\tau_{r\theta}^{(1)} + \tau_{r\theta}^{(2)}\right)^2 + \left(\tau_{z\theta}^{(1)} + \tau_{z\theta}^{(2)}\right)^2} \quad (2.29)$$

Equation (2.29) delivers the stress distribution also for the circular wire as its limit case. In this case, the axes of the ellipse turn into the sole wire diameter:

$$T = B = d.$$

The polar coordinates of the circular cross-section are:

$$x = \rho \cos \varphi, y = \rho \sin \varphi, 0 \leq \rho \leq r = \frac{d}{2}, 0 \leq \varphi < 2\pi.$$

For the circular wire of radius r and for the given spring index w , the octahedral shear stress $\tau_{oct}(\rho, \varphi)$ yields from Eq. (2.29). After the substitution of expressions for polar coordinates in Eq. (2.29), the octahedral shear stress $\tau_{oct}(\rho, \varphi)$ reads (Kobelev 2017):

$$\tau_{oct}(\rho, \varphi) = \frac{\tau_b}{32r^2w} \sqrt{c_0 + c_1 \cos \varphi + c_2 \cos^2 \varphi}, \quad (2.30)$$

$$c_0 = 9R^4 - 72R^2\rho^2 + 1024w^2R^2\rho^2 + 144\rho^4, \quad (2.31)$$

$$c_1 = 192\rho R^3w + 1792\rho^3R, \quad (2.32)$$

$$c_2 = 240R^2\rho^2 + 640\rho^4. \quad (2.33)$$

In (2.30), the octahedral shear stress on the surface of the straight circular bar is:

$$\tau_b = \sqrt{\tau_{r\theta}^{(1)^2} + \tau_{z\theta}^{(1)^2}} = G\theta R. \quad (2.34)$$

Stress τ_b is the “basic”, or “uncorrected stress, Eq. (1.25)” (SAE 1997, Chap. 5, Design of Helical Springs). The basic stress is equal to the shear stress on the surface of the straight circular wire, which is twisted by the same torque. The “basic stress” is obtained by dividing the torsion moment acting on the wire, by the section modulus in torsion the shear stress on the surface of the circular wire. The basic stress represents the mean value of the shear stress on the surface of the wire and does not depend on the spring index w . Specifically, the basic stress is the octahedral stress for the limit case of infinitely high spring index $w = \infty$:

$$\lim_{w \rightarrow \infty} \tau_{oct}(\rho, \varphi) = \tau_b.$$

In this case the stress equalizes over the surface of the wire. Consequently, the basic stress is equal to the shear stress on the surface of the straight cylindrical wire, loaded by the same torsion moment as the wire of the helical spring.

Using Eq. (2.30), we introduce the stress correction function (Kobeleev 2017):

$$k_0(w, \varphi) = \frac{\sqrt{81 + 1024w^2 + 1984w \cos \varphi + 880 \cos^2 \varphi}}{32w}. \quad (2.35)$$

The shear stresses on the surface of the wire $\tau_s(w, \varphi) \stackrel{\text{def}}{=} \tau_{oct}(\rho = r, \varphi)$ depend solely on polar angle φ and spring index:

$$\tau_s(w, \varphi) = k_0(w, \varphi) \tau_b, \quad 0 \leq \varphi < 2\pi. \quad (2.36)$$

The shear stress $\tau_s(w, \varphi)$ is the function of the spring index w and of the polar angle φ . The maximal value $\tau_c(w)$ is the corrected stress on the surface of the wire, Eq. (1.27):

$$\tau_c(w) = \max_{0 \leq \varphi < 2\pi} \tau_s(w, \varphi). \quad (2.37)$$

Consider several springs with the same wire diameter but with different spring indices. Let the torque in the wire remains the same for all springs. The value θ of the torsion angle pro length unit remains equal for all springs with different spring indices. The basic stress does not alter in this case, but the variation of stress over the surface reduces with the increasing spring index w . In the limit case of infinite spring index the variation fades away. Correspondingly Eq. (2.36) turns into the basic stress σ_b :

$$\lim_{w \rightarrow \infty} \tau_s(w, \varphi) = \lim_{w \rightarrow \infty} \frac{\sigma_{id}}{32w} \sqrt{81 + 1024w^2 + 1984w \cos \varphi + 880 \cos^2 \varphi} = \tau_b. \quad (2.38)$$

The maximum shear stress in a helical spring and is equal in the employed approximation to the value:

$$\tau_c(w) = k_4(w)\tau_b, \quad (2.39)$$

where

$$k_4(w) = \frac{\sqrt{961 + 1984w + 1024w^2}}{32w}. \quad (2.40)$$

The coefficient $k_4(w)$ before τ_b is the stress correction factor. This coefficient is the function of spring index w only. The advantage of newly developed correction function (2.32) is its dependence on the polar angle φ on the surface of the wire. This function allows to account the distribution of defects on the surface of the wire. The distribution of defects on the surface is necessary for the statistical evaluation of failures in course of cyclic loading and evaluation of the failure probabilities.

2.4 Shear Stresses on Surface of Ovate Wire

In this Section we find the solution of the equilibrium equation for an ovate wire. The solution is based on separation of the variables for the stress function. The representation of the stress function is assumed as a sum

$$\phi(r, z) = \phi_1(r) + \phi_2(z) \quad (2.41)$$

of the auxiliary functions $\phi_1(r)$, $\phi_2(z)$, satisfying the ordinary differential equations

$$\frac{d^2\phi_2}{dz^2} = C_s, \quad \frac{d^2\phi_1}{dr^2} - \frac{3}{r} \frac{d\phi_1}{dr} + 2GB_{sep} = -C_s \quad (2.42)$$

with the separation constant C_s .

The general solutions of the auxiliary differential Eq. (2.42) are

$$\phi_1(r) = \frac{C_s + 2GB_s}{4}r^2 + \frac{C_{11}}{4}r^4 + C_{12}, \quad \phi_2(z) = \frac{1}{2}C_{sep}z^2 + C_{13}z + C_{14},$$

where C_{11} , C_{12} , C_{13} , C_{14} —unknown integration constants. The representation (2.41) delivers the stress function as a polynomial

$$\phi(r, z) = \frac{C_s + 2GB_s}{4}r^2 + \frac{C_{11}}{4}r^4 + C_{12} + \frac{1}{2}C_s z^2 + C_{13}z + C_{14}. \quad (2.43)$$

Substituting (2.43) in (2.11) and (2.12) provides the following expressions for shear stresses:

$$\tau_{r\theta} = -\frac{C_s z + C_{13}}{r^2}, \tau_{z\theta} = \left(\frac{C_s}{2} + G B_s\right) \frac{1}{r} + C_{11} r. \quad (2.44)$$

Secondly, the solution of the equation for the warping function (2.8) is based on separation of the variables

$$\psi(r, z) = z \psi_1(r), \quad (2.45)$$

where the auxiliary function $\psi_1(r)$ satisfies the differential equation

$$\frac{d^2 \psi_1}{dr^2} + \frac{3}{r} \frac{d \psi_1}{dr} = 0. \quad (2.46)$$

The solution of the ordinary differential Eq. (2.46) leads to the following representation of the solution for Eq. (2.8):

$$\psi(r, z) = C_{21} z + C_{22} \frac{z}{r^2}. \quad (2.47)$$

The integration constants C_{21} , C_{22} are not independent and are linked to the integration constants from (2.43). Substitution of (2.47) into the expression for shear stresses (2.5) results in the following representation

$$\tau_{r\theta} = -\frac{2GC_{22}z}{r^2}, \tau_{z\theta} = G(C_{22} + B_s) \frac{1}{r} + GC_{21} r. \quad (2.48)$$

Equating two different representations of shear stresses (2.44) and (2.48), we obtain the algebraic relations between integration constants from (2.43) and (2.47):

$$C_{21} = C_{11}/G, C_{22} = C_s/(2G), C_{13} = 0.$$

The integration constants for a given contour could be obtained from the boundary condition (2.5). The inverse method consists of searching for the shape of the contour, which satisfies the boundary condition (2.5). To find the equation of the unknown contour, which satisfies the boundary Eq. (2.5), we equate the polynomial (2.43) to zero. Assuming the contour to be symmetrical with respect to the axis $z = 0$ and factorizing the polynomial in (2.43), we obtain the equation of the “quasi-elliptical” contour in a form (Fig. 2.1) (Kobelev 2002):

$$z^2 = Z_0^2 (r_e^2 - r^2) (r^2 - r_i^2), \quad (2.49)$$

The integration constants could be expressed through the intensity of screw dislocation B_s and geometrical characteristics of the cross-section. The value

$$T = Z_0(r_e^2 - r_i^2) \quad (2.50a)$$

is the maximum height of the profile at the pivot point

$$r_m = \sqrt{(r_e^2 + r_i^2)/2}. \quad (2.50b)$$

The solution of torsion problem is also given by the warping function (2.47), satisfying the equilibrium Eq. (2.6) and the boundary conditions (2.8):

$$\psi(r, z) = \frac{B_s z}{r^2} \frac{4T^2 r^2 + (r_e^2 - r_i^2)^2}{(r_e^2 - r_i^2)^2 + 2T^2(r_e^2 + r_i^2)}. \quad (2.51)$$

The separation constant C_s expresses through the magnitude of the Burgers vector B_s as:

$$C_s = -\frac{(r_i + r_e)^2(r_e - r_i)^2}{(r_e^2 - r_i^2)^2 + 2T^2(r_e^2 + r_i^2)} 2B_s G.$$

Substitution of integration constants in the expression (2.43) leads to the final expression of the stress function

$$\phi(r, z) = \frac{B_s G (r_e^2 - r_i^2)^2}{(r_e^2 - r_i^2)^2 + 2T^2(r_e^2 + r_i^2)} \cdot \left[z^2 - \frac{T^2}{(r_e^2 - r_i^2)^2} (r_e^2 - r^2)(r^2 - r_i^2) \right]. \quad (2.52)$$

2.5 Quasi-elliptical Cross-Section

According to (2.49), the ovate, of “quasi-elliptical” form of cross-section: Ω for incomplete torus is given by the equations:

$$Z_+(r) = Z_0 \sqrt{(r_e^2 - r^2)(r^2 - r_i^2)}, \quad Z_-(r) = -Z_0 \sqrt{(r_e^2 - r^2)(r^2 - r_i^2)} \quad (2.53)$$

Herein r_i and r_e are respectively the inner and outer radii of torus.

The volume of one coil (volume of the complete torus) with the cross-section (2.53) is:

$$V_1 = 2\pi \int_{\Omega} r d\Omega = 2\pi \int_{r_i}^{r_e} \left[\int_{Z_-(r)}^{Z_+(r)} dz \right] r dr = \frac{\pi^2}{4} T (r_e^2 - r_i^2). \quad (2.54)$$

The substitution of (2.51) into (2.4) results in the components of the shear stresses:

$$\tau_{\theta r} = \frac{(r_i + r_e)^2(r_e - r_i)^2}{(r_e^2 - r_i^2)^2 + 2T^2(r_e^2 + r_i^2)} \frac{2B_s G z}{r^2}, \quad (2.55)$$

$$\tau_{z\theta} = \frac{-2r^2 + r_i^2 + r_e^2}{(r_e^2 - r_i^2)^2 + 2T^2(r_e^2 + r_i^2)} \frac{2B_s G T^2}{r}. \quad (2.56)$$

The octahedral shear stress reads:

$$\tau_{oct} = \sqrt{\tau_{z\theta}^2 + \tau_{\theta r}^2}. \quad (2.57)$$

The stresses on the outer surface of the wire depend on the radii r_i , r_e and the height T of the profile. Figure 2.2 displays the influence of the heights on the octahedral shear stress for the fixed radii. The “stars” mark the points of maximum of the octahedral shear stress. If the shape is flattened along the spring axis, the maximal stress occurs on the inner face of the wire. If the wire has same dimensions in both directions, the maximum region splits in two narrow strips near the inner face. For the gradually flattened along the radius ovate shape, the maximum moves to the top of the cross-section.

Figure 2.3 draws the stresses on the surfaces of the ovate wires with the same profile height and width, but with the different distance of pivot points (2.50b) from the axis of the helix. One again, the maximum region splits with the rising r_m . The stress distributes more evenly over the inner face of the wire. This effect reduces the maximal octahedral stress for the given spring rate and is frequently imposed for the optimization of the fatigue life of the springs.

The force in the axial direction is given by integration of the shear stress $\tau_{z\theta}$ over the cross-section Ω of the torus:

$$F = \int_{\Omega} \tau_{z\theta} d\Omega = \int_{r_i}^{r_e} \left[\int_{Z_-(r)}^{Z_+(r)} \tau_{z\theta}(r, z) dz \right] dr. \quad (2.58)$$

The use of expression (2.56) in (2.58) delivers the relationship between the magnitude of the Burgers vector of the giant screw dislocation and the axial force F :

$$F = \frac{GT^3\pi}{2(r_e + r_i)} \frac{(r_e - r_i)^3}{(r_e^2 - r_i^2)^2 + 2T^2(r_e^2 + r_i^2)} B_s. \quad (2.59)$$

The spring rate of one coil, according to (2.13) is given by:

$$c_w = \frac{F}{2\pi B_s} = \frac{GT^3}{4(r_e + r_i)} \frac{(r_e - r_i)^3}{(r_e^2 - r_i^2)^2 + 2T^2(r_e^2 + r_i^2)}. \quad (2.60)$$

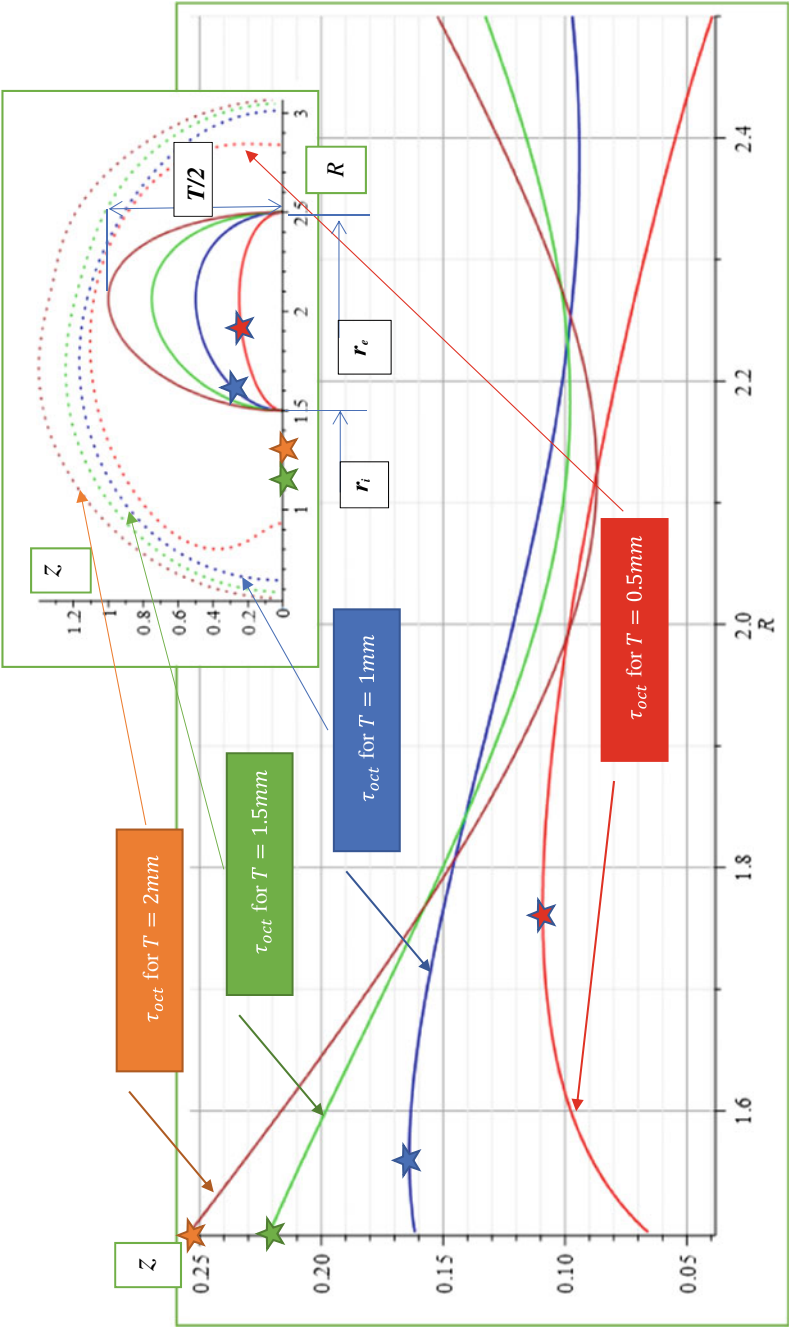


Fig. 2.2 Shearing stress τ_{oct} on the surface of wires with different heights. The contours with equal radii and different heights are shown in the upper sketch together with the polar diagrams for octahedral stresses on the wire surface

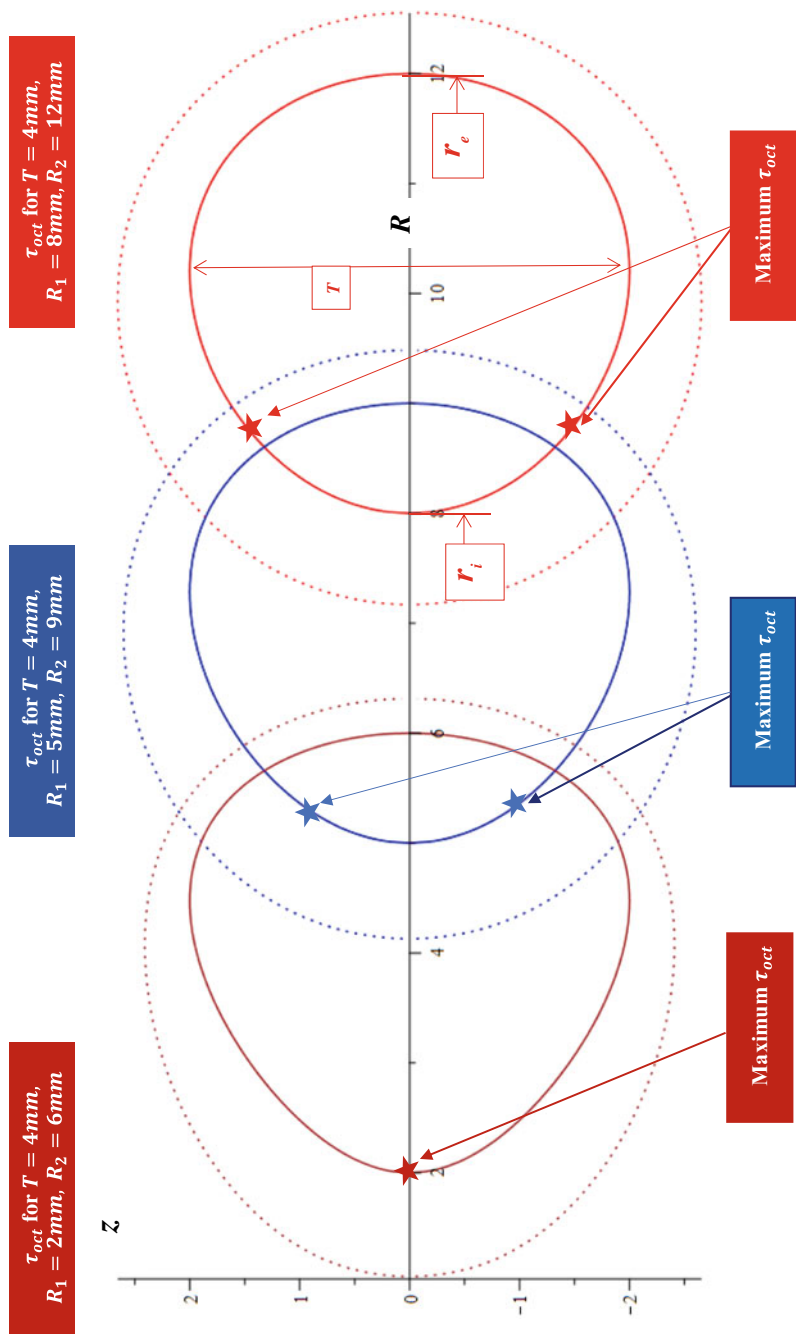


Fig. 2.3 Shearing stress on the surface of wired with different radii and the same height of cross-sections

After the replacement of the integration constant through the expression (2.58), the shear stresses from (2.55), (2.56) assume the form:

$$\tau_{r\theta} = \frac{4F}{\pi T^3} \frac{(r_i + r_e)^3}{r_e - r_i} \frac{z}{r^2}, \quad (2.61)$$

$$\tau_{z\theta} = \frac{4F}{\pi T} \frac{(r_i + r_e)}{(r_e - r_i)^3} \frac{r_i^2 + r_e^2 - 2r^2}{r}. \quad (2.62)$$

The resultant moment (2.12) about the point O at the origin of the coordinate system for the stress fields (2.61) and (2.62) disappears:

$$\begin{aligned} M_r &= \int_{r_i}^{r_e} \left[\int_{z_-(r)}^{z_+(r)} (\tau_{z\theta} r - \tau_{r\theta} z) dz \right] dr = \\ &= \frac{8F}{3\pi (r_e^2 - r_i^2)^4} \int_{r_i}^{r_e} \sqrt{(r^2 - r_i^2)(r_e^2 - r^2)} \cdot \frac{5r^4 - 2r^2(r_e^2 + r_i^2) + r_i^2 r_e^2}{r^2} dr = 0. \end{aligned} \quad (2.63)$$

This proves that the shear stresses over a cross-section Ω are statically equivalent to the axial resultant force F with the line of action at the axis z of helix.

One can rewrite the expressions (2.54) and (2.60), introducing the inner diameter D_i , the outer diameter D_e , the mean diameter D of the spring and the width of wire $B = r_e - r_i$:

$$r_i = \frac{D_i}{2} = \frac{D}{2} - \frac{B}{2}, r_e = \frac{D_e}{2} = \frac{D}{2} + \frac{B}{2}. \quad (2.64)$$

The expressions of the volume of one coil (2.54) and the spring rate of one coil (2.60) in terms of mean diameter and mean radius assume the form

$$V_1 = \frac{\pi^2}{4} D B T, \quad (2.65)$$

$$c_w = \frac{T^3 B^3}{T^2 D^2 + D^2 B^2 + T^2 B^2} \frac{G}{4D}. \quad (2.66)$$

The spring rate of the spring with n_a active coils is c_w/n_a .

The values (2.65), (2.66) are useful in engineering applications for evaluation of weight and consequently, the fundamental frequencies of coil springs. The knowledge of fundamental frequencies is essential for simulation of dynamical behavior of the mechanical systems containing the springs as their parts Muhr (1993) and Köhler (1998).

2.6 Hollow Ovate Wire

Helical springs with hollow cross-section are formed by wrapping tube of uniform cross-section around a cylinder. One turn, or coil, of an undeformed helical spring becomes a hollow torus, generated by rotating the voided cross-section about the z axis of the cylindrical coordinate system. We obtain the closed form of analytical solution for the torsion problem of the incomplete torus with the hollow, or tubular, cross-section.

Consider the hollow ovate profile (Fig. 2.4) with the equation of the outer boundary Γ of the cross-section

$$Z_+(r) = Z_0 \sqrt{(r_e^2 - r^2)(r^2 - r_i^2)}, \quad Z_-(r) = -Z_0 \sqrt{(r_e^2 - r^2)(r^2 - r_i^2)} \quad (2.66)$$

and the inner boundary γ of the cross-section

$$z_+(r) = \sqrt{Z_0^2(r_e^2 - r^2)(r^2 - r_i^2) - \kappa^2}, \quad z_-(r) = -\sqrt{Z_0^2(r_e^2 - r^2)(r^2 - r_i^2) - \kappa^2}. \quad (2.67)$$

The coefficient κ varies in the interval

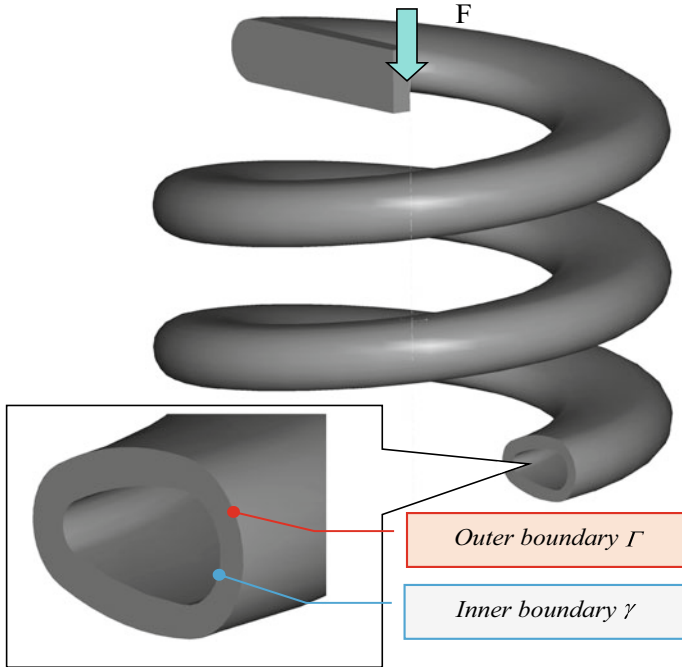


Fig. 2.4 A section of the spring with the hollow profile

$$0 \leq \kappa \leq \frac{1}{2} Z_0 (r_e^2 - r_i^2). \quad (2.68)$$

The shape of the hole can be rewritten in the form similar to the outer curve:

$$z_+(r) = Z_0 \sqrt{(\rho_2^2 - r^2)(r^2 - \rho_1^2)}, \quad z_-(r) = -Z_0 \sqrt{(\rho_2^2 - r^2)(r^2 - \rho_1^2)}, \quad (2.69)$$

where ρ_1, ρ_2 are the real positive roots of the equation:

$$Z_0^2 (r_e^2 - r^2)(r^2 - r_i^2) = \kappa^2.$$

These roots are

$$\rho_1 = \frac{1}{\sqrt{2}} \sqrt{r_i^2 + r_e^2 - \sqrt{(r_i^2 - r_e^2)^2 - 4(\kappa/Z_0)^2}}, \quad (2.70)$$

$$\rho_2 = \frac{1}{\sqrt{2}} \sqrt{r_i^2 + r_e^2 + \sqrt{(r_i^2 - r_e^2)^2 - 4(\kappa/Z_0)^2}}. \quad (2.71)$$

The stress function for the hollow profile satisfies the boundary conditions on both boundaries of the profile:

$$\phi = 0 \text{ on } \Gamma, \quad (2.72)$$

$$\phi = \text{const on } \gamma. \quad (2.73)$$

The stress function $\phi(r, z)$ from Eq. (2.52) satisfies the equilibrium equation over the region Ω . The isolines of stress function $\phi(r, z)$ match with the curves, given by the expression (2.67) with the parameter κ . Therefore, stress function (2.52) is constant along the inner boundary γ . This proves, that the solution of the problem for the hollow cross-section delivers once again the stress function $\phi(r, z)$.

Performing a double integration (2.54) one gets the expression for volume of one hollow coil:

$$V_1 = \frac{\pi^2 D}{4} (BT - B_i T_i). \quad (2.74)$$

Here

$T_i = Z_0(\rho_2^2 - \rho_1^2)$ is the height of the ovate opening,

$B_i = \rho_1 - \rho_2$ is the width of the ovate opening.

The spring rate for one hollow coil is given by the formula:

$$c_w = \left(\frac{T^3 B^3}{T^2 D^2 + D^2 B^2 + T^2 B^2} - \frac{T_i^3 B_i^3}{T_i^2 D^2 + D^2 B_i^2 + T^2 B_i^2} \right) \frac{G}{4D}. \quad (2.75)$$

2.7 Dislocation Character of Helical Springs Deformation

2.7.1 Screw and Edge Dislocations

In the solid-state physics, the screw dislocation is a form of a line defect in which the defect occurs when the planes of atoms in the crystal lattice trace a helical path around the dislocation line (Hull and Bacon 2011). The Burgers vector of a screw dislocation is parallel to the line of the dislocation, also to the axis of the helix. As it was already mentioned, the axial compression or extension of the helical spring leads to the deformation of the material in form of a screw giant dislocation. The line of the screw dislocation coincides with the axis of the spring. The magnitude of the Burgers vector is equal to the axial displacement of one coil. According to Eq. (2.1), the elastic medium of the spring wire deforms to single surface helicoid (Fig. 2.5). The case of axial compression of the spring, which causes the “giant” screw dislocation in the elastic helix, was already studied in Sect. 2.1. Interesting to note, that if the spring loading causes the plastic deformation of its material, the single giant dislocation of the whole spring splits into a huge number of microscopic edge dislocations of the material. Each microscopic dislocation possesses the Burgers vector of one distance between the atoms in crystal. The sum of all these microscopic Burgers vectors is equal of the giant Burgers vector due to plastic deformation of the helical spring.

In solid-state physics, the edge dislocation formed by inserting one or more extra half-planes of atoms. The Burgers vector of an edge dislocation is normal to the line

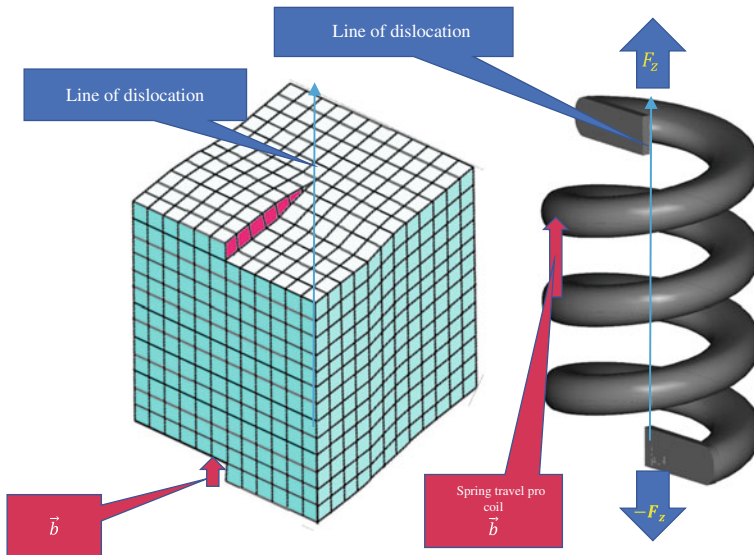


Fig. 2.5 Extension of the helical spring as a screw dislocation

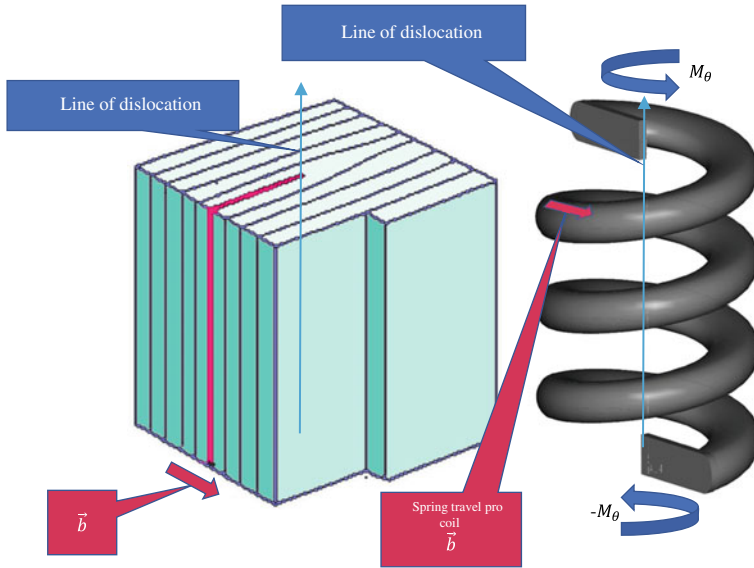


Fig. 2.6 Twist of the helical spring as an edge dislocation

of the dislocation. Similarly, the torsion M_θ of the helix leads to the deformation of spring material, which is equivalent to the giant edge dislocation. The line of dislocation is once again the helical axis. In this case, the magnitude of the Burgers vector is equal to the opening of one coil due to action of torsional moment, which is oriented in the direction of the spring axis (Fig. 2.6). The magnitude of Burgers vector is equal to the diameter change of helical spring. Two forces, which enforce the increase of the diameter of the incomplete torus, lead to the same deformation of elastic medium, as the insertion of the extra matter in the gap of the torus. We examine below the case of torsion of the spring, which causes the “giant” edge dislocation in the elastic helix. Once again, if the spring undergoes the plastic deformation, this giant edge dislocation splits into a massive number of the microscopic edge dislocations. These microscopic edge dislocations accompany the plastic deformation of the material. The sum of the Burgers vectors of microscopic edge dislocations is equal to the Burgers vector of the giant edge dislocation in course of the torsional plastic deformation of the spring.

In general case, both the axial force F and the axial moment M_θ act on the helix simultaneously. Because of axial symmetry of the spring, the dislocation line coincides with the axis of helix. The axial component of the Burgers vector is equal to the axial displacement per one coil. The radial component of the Burgers vector is equal to the opening of the gap of an incomplete torus due to the twist of helix. Thus, Burgers vector of the giant dislocation lies at an arbitrary angle to the dislocation line. The giant dislocation has a mixed edge and screw character (Nabarro 1967).

2.7.2 Torsion of Helical Spring

We examine in this section the torsion of the helical spring, which causes the “macroscopic” edge dislocation in the elastic helix from the viewpoint of elasticity theory. The torsion was already studied from the perspective of beam theory in Chap. 1. However, the application of the beam theory is insufficient to the correct evaluation of the stress growth on the inner face of the wire. Consequently, for low spring index more accurate solutions are necessary for the precise stress analysis. If pitch vanishes, the torsion of the spring causes the bending of the wire (Michell 1899). We follow the method of solution (Timoshenko and Goodier 1951, Chap. 13, Art. 131) and consider the symmetric elastic stress distribution of an elastic toroidal sector subjected to pure bending under the effect of moments M_θ applied to the free ends. In the case of bending of wire, the first and second Beltrami-Michell equation (2.16) are identically satisfied. The third to sixth Beltrami-Michell equations for four appearing stresses $\sigma_r(r, z)$, $\sigma_\theta(r, z)$, $\sigma_z(r, z)$, $\tau_{rz}(r, z)$ read:

$$\Delta\sigma_r - \frac{2}{r^2}(\sigma_r - \sigma_\theta) + \frac{1}{1+\nu} \frac{\partial^2 \mathbf{I}(\boldsymbol{\sigma})}{\partial r^2} = 0, \quad \mathbf{I}(\boldsymbol{\sigma}) = \sigma_r + \sigma_\theta + \sigma_z, \quad (2.76)$$

$$\Delta\sigma_\theta + \frac{2}{r^2}(\sigma_r - \sigma_\theta) + \frac{1}{1+\nu} \frac{\partial \mathbf{I}(\boldsymbol{\sigma})}{r \partial r} = 0, \quad (2.77)$$

$$\Delta\sigma_z + \frac{1}{1+\nu} \frac{\partial^2 \mathbf{I}(\boldsymbol{\sigma})}{\partial z^2} = 0, \quad (2.78)$$

$$\Delta\tau_{rz} - \frac{\tau_{rz}}{r^2} + \frac{1}{1+\nu} \frac{\partial^2 \mathbf{I}(\boldsymbol{\sigma})}{\partial r \partial z} = 0. \quad (2.79)$$

The approximate solution of Eqs. (2.76)...(2.79) was delivered in the above cited works for the circular cross-section. We search the solution for the rectangular wire, as shown on (Fig. 2.7). The height of section is T , and its width is $B = r_e - r_i$. The middle diameter of the helix is $D = 2R = r_e + r_i$ and the spring index is $w = R/B$. The surfaces of the wire are stress-free. The conditions on the free surfaces assure the independence of the stresses upon coordinate z : $\sigma_r(r)$, $\sigma_\theta(r)$, $\tau_{rz}(r)$ and $\sigma_z(r)$. The problem (2.76)...(2.79) reduces to the ordinary differential equations:

$$\frac{1}{r} \frac{d\sigma_r}{dr} \left(r \frac{d\sigma_r}{dr} \right) - \frac{2}{r^2}(\sigma_r - \sigma_\theta) + \frac{1}{1+\nu} \frac{\partial^2(\sigma_\theta + \sigma_r)}{\partial r^2} = 0, \quad (2.80)$$

$$\frac{1}{r} \frac{d\sigma_\theta}{dr} \left(r \frac{d\sigma_\theta}{dr} \right) + \frac{2}{r^2}(\sigma_r - \sigma_\theta) + \frac{1}{1+\nu} \frac{\partial(\sigma_\theta + \sigma_r)}{r \partial r} = 0, \quad (2.81)$$

$$\frac{1}{r} \frac{d\sigma_z}{dr} \left(r \frac{d\sigma_z}{dr} \right) = 0, \quad (2.82)$$

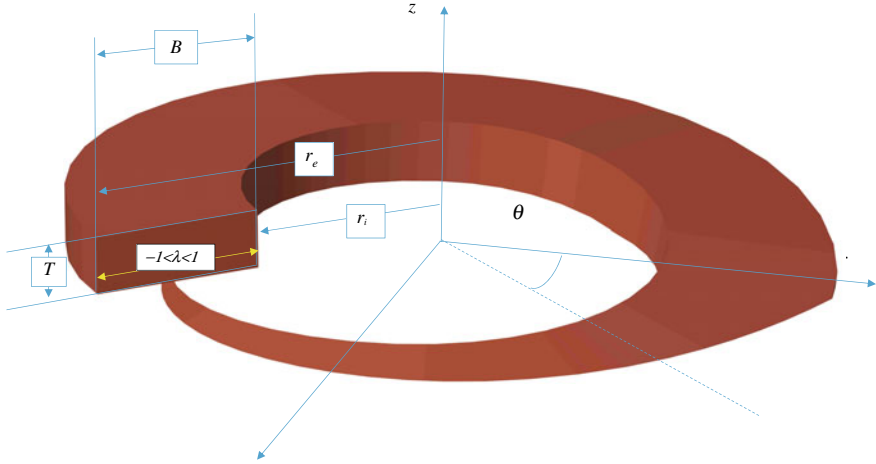


Fig. 2.7 Helical spring with the rectangular wire

$$\frac{1}{r} \frac{d\tau_{rz}}{dr} \left(r \frac{d\tau_{rz}}{dr} \right) - \frac{\tau_{rz}}{r^2} = 0. \quad (2.83)$$

The solution of Eqs. (2.80)...(2.83) reads:

$$\begin{aligned} \sigma_r &= \frac{c_1 r^2 + c_2 \ln(r) r^2 + c_3 r^4 + c_4}{r^2}, \\ \sigma_\theta &= c_1 - c_4 r^2 + c_2 \frac{(1 + \nu) \ln(r) + 1}{1 + \nu} + \frac{c_6}{1 + \nu} - \frac{c_3}{r^2}, \\ \tau_{rz} &= c_7 r + \frac{c_8}{r}, \quad \sigma_z = c_6 \ln(r) + c_5. \end{aligned} \quad (2.84)$$

With the boundary conditions on the free surface the integration constants of (2.84) yield to:

$$\begin{aligned} c_1 &= \frac{\ln(r_i) r_e^2 - \ln(r_e) r_i^2}{\ln(r_e) - \ln(r_i)} c_4 - \frac{\ln(r_i) r_i^2 - \ln(r_e) r_e^2}{r_i^2 r_e^2 (\ln(r_i) - \ln(r_e))} c_3, \\ c_2 &= \frac{r_e^2 - r_i^2}{\ln(r_i) - \ln(r_e)} c_4 + \frac{r_e^2 - r_i^2}{r_i^2 r_e^2 [\ln(r_e) - \ln(r_i)]} c_3, \\ c_3 &= c_4 = c_5 = c_6 = c_7 = c_8 = 0. \end{aligned} \quad (2.85)$$

Two integration constants c_3, c_4 should be determined from the load of the beam sector. The total force over the cross-section F_θ disappears. The bending moment is equal to the given torsion moment M_θ of the helical spring:

$$F_\theta = T \int_{r_i}^{r_e} \sigma_\theta dr = 0, \quad M_\theta = T \int_{r_i}^{r_e} \sigma_\theta r dr. \quad (2.86)$$

The linear algebraic equation (2.86) determine the constants c_3, c_4 . The constants c_1, c_2 . result from (2.85). The final formulas are bulky and we provide only the simpler expression the tangential stress in the case $v = 0$:

$$\sigma_\theta = \frac{4M_\theta}{Tr^2} \cdot \frac{\ln(r_i)r_i^2(r^2 + r_e^2) - \ln(r_e)r_e^2(r^2 + r_i^2) + r^2(r_e^2 - r_i^2)(1 + \ln(r))}{4r_e^2r_i^2[\ln(r_e) - \ln(r_i)]^2 - (r_e - r_i)^2(r_e + r_i)^2}. \quad (2.87)$$

The next step is to determine the deformation of the wire and the Burgers constant for the edge dislocation. For derivation we use Castigliano's method (Teodorescu 2013). The elastic energy stored in the spring (1.6) is the integral of bending energy of wire over its length L_W . With the expression for σ_θ , the elastic strain energy of one coil in the Timoshenko-Michell theory is:

$$U_M = T \int_{r_i}^{r_e} \frac{\sigma_\theta^2}{2E} 2\pi r dr. \quad (2.88)$$

The spring rate pro one coil according to the Timoshenko-Michell theory is:

$$c_M = \left(\frac{d^2 U_M}{dM_\theta^2} \right)^{-1}. \quad (2.89)$$

The Bernoulli beam theory provides the expression for the tangential stress:

$$\sigma_\theta = \frac{12M_\theta}{TB^3} (r - R). \quad (2.90)$$

For comparison, on Fig. 2.8 we show the stresses in the cross-section of the wire for different spring indices. Two graphs display the Bernoulli function (2.89) in blue color and the Timoshenko-Michell function in red. The parameter λ runs in the interval $-1 \leq \lambda \leq 1$. This parameter is used instead of radius: $r = r_i + B \cdot (\lambda + 1)/2$. With the increasing spring index both stress distributions become indistinguishable.

With (2.87) the elastic strain energy of one coil and the spring rate in Bernoulli theory read:

$$U_B = \frac{12\pi M_\theta^2 w^3}{R^2 T E}, \quad c_B = \left(\frac{d^2 U_B}{dM_\theta^2} \right)^{-1} = \frac{R^2 T E}{24\pi w^3}. \quad (2.91)$$

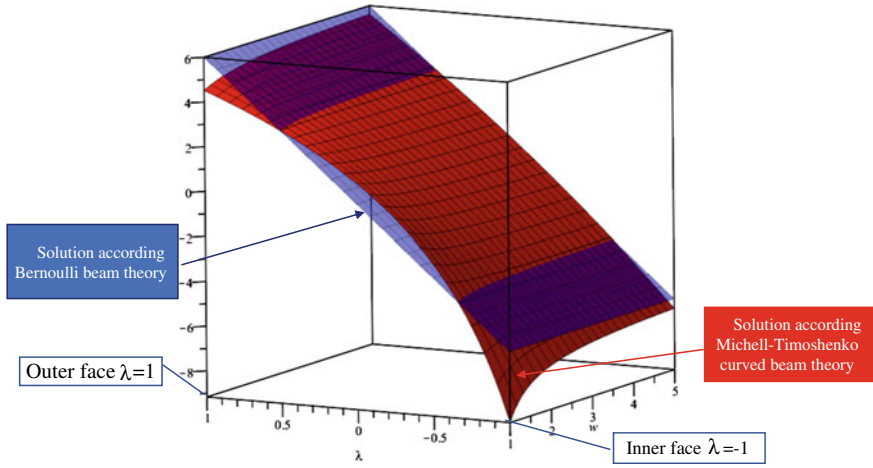


Fig. 2.8 Tangential stress $\sigma_\theta(r)$ as the function of spring index w

If the axial moment M_θ applies to the spring, in the elastic material appears the edge dislocation. The magnitudes Burgers vector of the edge dislocation for both approaches will be correspondingly to $M_\theta R/c_M$ and $M_\theta R/c_B$. These values correspond to the deflections of the cross-sections of one full coil under the action of torque M_θ .

The ratio of stored elastic energies U_M/U_B is shown on Fig. 2.9. Ratios of stored elastic energy according to Timoshenko-Michell and Bernoulli theories for different

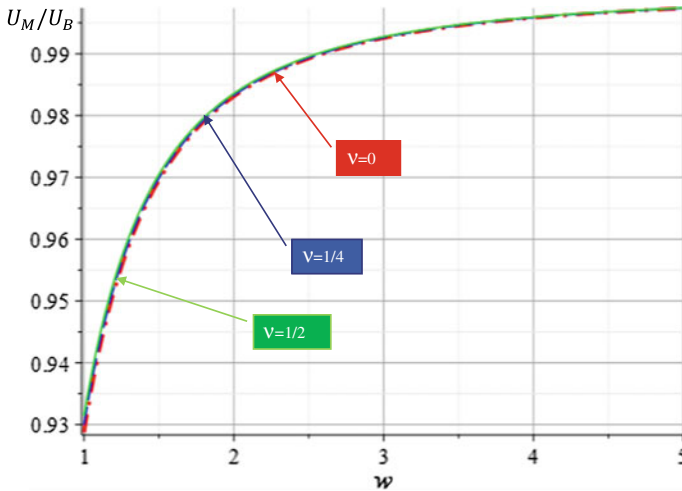


Fig. 2.9 Ratios of stored elastic energy according to Timoshenko-Michell and Bernoulli theories for different Poisson ratios as the functions of spring index $w = R/B$

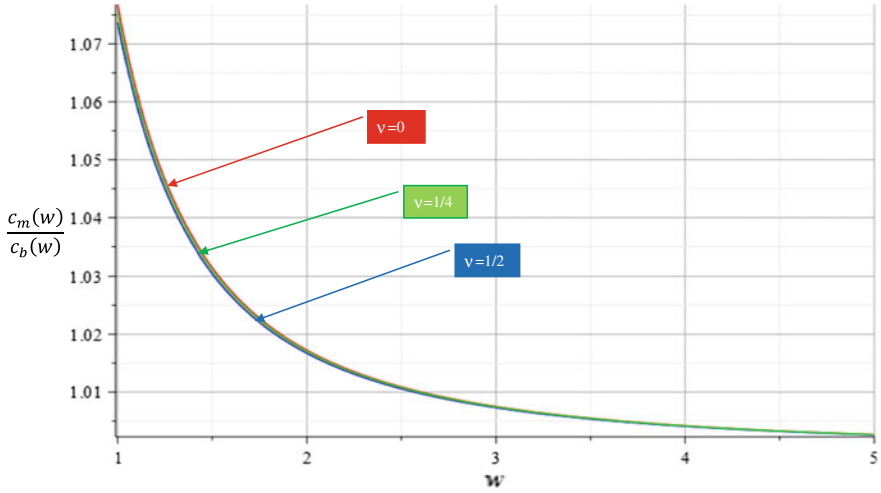


Fig. 2.10 Ratios of the spring rates for different Poisson coefficients according to Michell-Timoshenko curved beam theory to Bernoulli-beam theory spring rate for rectangular wire as the function of spring index w

Poisson ratios are drawn as the functions of spring index $w = R/B$. Evidently, that the Poisson coefficient has no valuable influence on the stored energy. From the stored energies result the spring rates pro coil. Figure 2.10 demonstrates the ratios of the spring rates according to Michell-Timoshenko curved beam theory to Bernoulli-beam theory spring rate. The functions are shown for different Poisson coefficients for rectangular wire as the function of spring index w . One again, the influence of the Poisson ratio is almost negligible. The asymptotic expansions of the ratios of stored elastic energy and spring rates according to Timoshenko-Michell and Bernoulli theories read:

$$\begin{aligned} \frac{U_M}{U_B} &= 1 - \frac{(2+v)(2+3v)}{60(1+v)^2} \cdot \frac{1}{w^2} + o\left(\frac{1}{w^2}\right), \\ \frac{c_M}{c_B} &= 1 + \frac{(2+v)(2+3v)}{60(1+v)^2} \cdot \frac{1}{w^2} + o\left(\frac{1}{w^2}\right). \end{aligned} \quad (2.92)$$

The stress correction factors could be defined in the similar manner, as for the axially compressed springs. For this purpose we compare the tangential stresses according to the exact theory (2.87) and the stresses in the bending beam (2.90), as shown of Fig. 2.8. The first correction factor is the ratio of the to Michell-Timoshenko stress to the Bernoulli stress on the inner surface $r = r_i$. This stress correction factor is denoted k_{Mi} and it is always greater than 1. The second correction factor is the ratio of both stresses on the outer surface $r = r_e$. This stress correction factor is denoted k_{Me} . This value is less than 1 for all Poisson ratios and spring indices. Instead of providing the bulky formulas for both coefficients, we list only the asymptotic expansions for

both factors:

$$k_{Mi} = 1 + \frac{\nu + 2}{6(1 + \nu)} \cdot \frac{1}{w} + \frac{4\nu + 7}{60(1 + \nu)} \cdot \frac{1}{w^2} + \frac{16\nu^2 + 29\nu + 16}{360(1 + \nu)^2} \cdot \frac{1}{w^3} + \dots \quad (2.93)$$

$$k_{Me} = 1 - \frac{\nu + 2}{6(1 + \nu)} \cdot \frac{1}{w} + \frac{4\nu + 7}{60(1 + \nu)} \cdot \frac{1}{w^2} - \frac{16\nu^2 + 29\nu + 16}{360(1 + \nu)^2} \cdot \frac{1}{w^3} + \dots \quad (2.94)$$

The only difference between the expressions (2.93) and (2.94) is the alternation of signs in the series (2.94). The graphs of the stress correction factor $k_{Mi}(w)$ and $k_{Me}(w)$ for different Poisson coefficients are demonstrated on Fig. 2.11. The bounds of both factors are:

$$1 < k_{Mi} \leq \frac{16 - 36 \ln 3}{27(\ln 3)^2 - 48}, \quad 1 > k_{Me} \geq \frac{-16 + 4 \ln 3}{27(\ln 3)^2 - 48}. \quad (2.95)$$

With these formulas we finalize the investigation of the elastic stresses in the helical springs.

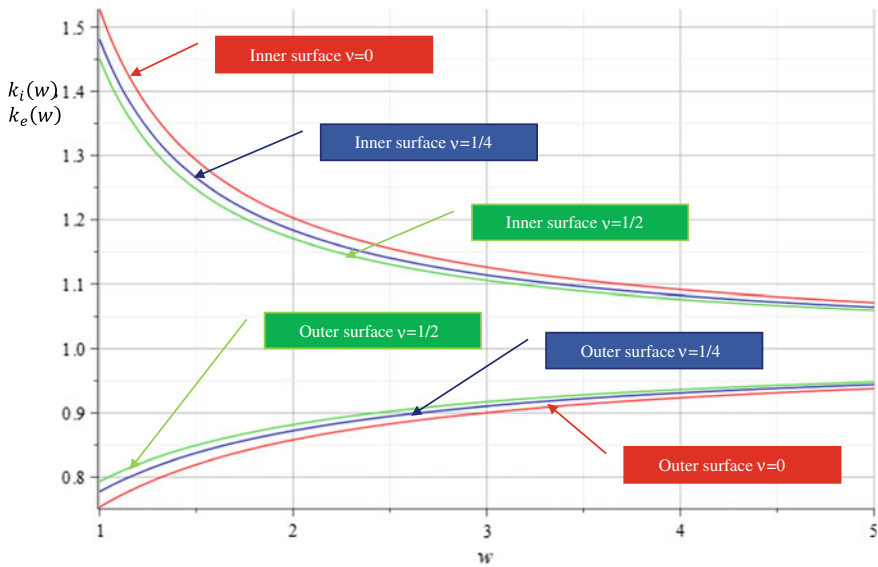


Fig. 2.11 Correction factors for tangential stress $\sigma_\theta(r)$ on the outer and inner surface of wire as the function of spring index w for different Poisson coefficients

2.8 Conclusions

For coil springs made of circular wires the shear stresses are distributed unevenly over the wire circumference. A better utilization of the material is achieved using the equalizing the shear stress on the surface of the wire. The idea of coil springs with non-circular, ovate wire cross-sections with a smooth stress over its surface was first mentioned by Fuchs (1959). The initial cross-section of the wire, based on rough approximations, was improved by several studies, Nagaya (1986), Matsumoto (1988). Advanced applications require the wires with ovate, optimized cross-sections. The shape optimization of the cross-section was exclusively based on finite element methods and numerical optimization algorithms.

One significant appliance of the valve springs with ovate wire is the valvetrain for internal combustion engines (Muhr 1992). Valve springs in combustion engines guarantee a forced link of all moving valve train components during the valve lift to the maximum engine speed. Currently a large amount of all valve springs for valve-trains are manufactured using wire with noncircular cross-section. This type of valve spring is produced mainly from high-tensile wire alloyed with the elements chrome and vanadium. This environment makes the exact stress analysis of springs with non-circular cross-section a practically important problem. The stress distributions for arbitrary cross-sections and shape optimization were studied with the application of numerical methods of finite and boundary elements (Sato et al. 1969). The closed form solution of torsion problem for torus can serve as a probe for numerical algorithms for calculation and optimization of helical springs, primarily valve springs (Kamiya and Kita 1990).

As known from solid-state physic, the line defects are forms of crystallographic defects. There are two forms of these defects, known as edge dislocations and screw dislocations. The difference between screw and edge dislocations is the following. The screw dislocation occurs when the planes of atoms in the crystal lattice trace a helical path around the dislocation line. The equivalent deformation of the elastic helix is its extension of compression due to the axial force. In its turn, the edge dislocation occurs when an extra half-plane of atoms exists in the middle of the crystal lattice, or the helical spring or the incomplete torus distort under the action of forces that increase the distance between the cross-section of the torus. The action of these forces is statically equivalent to the moment which cause the bending the wire. This moment is equal to the torque along the axis of the torus or the helix.

If both the axial force and the axial moment act on the helix simultaneously, the Burgers vector lies at an arbitrary angle to the dislocation line and the dislocation is a combination of edge and screw types. The study of dislocations is based on the Michell equations for the axially symmetric problems of elasticity.

2.9 Summary of Principal Results

- Stresses on the surface of spring with different shapes of the cross-section of wire.
- Stresses for ovate, elliptical, and circular wires of solid and hollow cross-sections.
- Spring rate and volume of wire for the ovate wire.
- For the circular and elliptic wires, the solutions are expressed as the power series of spring index.
- Analogy between screw and edge dislocations and compression and torque deformation of helical spring.

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Chapter 3

“Equivalent Columns” for Helical Springs



Abstract In this chapter, the helical spring is substituted by a flexible rod that is located along the axis of helix. This rod possesses the same mechanical features, as the spring itself. Its bending, torsion and compression stiffness are equal to the corresponding stiffness of the helical spring. This rod is known as an “equivalent column” of the helical spring. The “equivalent column” equations are considerable easier to handle than the original equations of the helical spring. The integral spring properties, as an axial and transversal stiffness, buckling loads, fundamental frequencies could be directly determined using the “equivalent column” equations. In contrast, the local properties, like stresses in the wire or contact forces, could be evaluated only with the more complicated equations of the helical elastic rod. In this chapter, the stability and transversal vibrations of the spring are studied from the unified point of view, which is based on the “equivalent column” concept. Buckling refers to the loss of stability up to the sudden and violent failure of straight bars or beams under the action of pressure forces, whose line of action is the column axis. This concept is applied for the stability of helical springs. An alternative approach method is based on the dynamic criterion for the spring stability. The equations for transverse (lateral) vibrations of the compressed coil springs were derived. This solution expresses the fundamental natural frequency of the transverse vibrations of the column as the function of the axial force, as well as the variable length of the spring. This chapter is the third section of first part, which studies the springs from the design viewpoint.

3.1 Static Stability Criteria of Helical Springs

A coil spring is a special form of spatially curved column. The center of each cross-section is located on a helix. The helix is a curve that winds around with a constant slope of the surface of a cylinder. In the mechanical engineering, the helical spring is commonly modeled as a massless, frequency independent stiffness element. However, for a typical suspension spring, these assumptions are only valid in the quasi-static case or at low frequencies. At higher frequencies, the influence of the internal resonances of the spring increases and thus a detailed model is required.

Moreover, the valve springs of internal combustion engines are affected by periodical forces with frequencies, that comparable with the spring natural frequency. The massless spring model is not applicable for such excitations.

An exact analysis of vibrations and stability based on the theory of spatially curved bars is relative complicated and difficult for engineering applications. The simulation models deliver high precision for the vibration frequencies and buckling loads. However, the numerical models are not well suited for the primary design purposes. Hence, in most engineering applications the traditional concept of an equivalent column is applied for the stability and vibration analysis. For simplification of the basic equations, the spring is substituted by an equivalent column (Collins et al. 2010, Chap. 14). It is the concept of equivalent column that forms the background for spring calculation in the industry (Encyclopedia of Spring Design 2013; Helical Springs 1974). The axis of the equivalent bar coincides with the hypothetical central line of the spring. Instead of describing the displacement of the points of the spiral, the displacement of the point mass at the same height of the centerline is used. Such a column must account for compressibility of axis and shear effects. The averaged axial stiffness and transverse stiffness of the helical spring are equal to the corresponding axial stiffness and transverse stiffness of initially straight hypothetical beam that substitutes the real helical wire.

The stiffness of the equivalent length of the column and with a circular cross-section was calculated by Grammel (1924), Biezeno and Koch (1925), Haringx (1948) or Dick (1942). There is a considerable discrepancy between the cited above and Ponomarev's solutions (Ponomarev 1948). This discrepancy results from different estimation of the shear effect. This problem was investigated in detail by Ziegler (1982). He showed that for helical springs a so-called modified approach is the proper way of taking shear, or transverse, force into account, whereas Biezeno and Koch applied Engesser's approach.

The models of equivalent columns were also applied for study of buckling under combined compression and torsion (Ziegler and Huber 1950), and (Satoh et al. (1988).

A more exact equivalent column for buckling of helical springs was introduced in (Kruzelecki nad Zyczkowski 1990). It accounts for the pitch angle and possible buckling in two planes. Non-linear compression rigidity, local bending and shear rigidities as well as lower bounds for the mean values of these rigidities were established.

The concept of an equivalent column was adopted by Skoczniński and Skrzypek (1992) in order to examine the bifurcation buckling of S-shaped bellows. The overall buckling of axially compressed bellows (axial force, internal pressure) in the presence of pre-buckling nonlinearities is investigated. Instability within the class of axisymmetric deformations (snap-through) was also discussed on the basis of the finite deflections and rotations theory. The geometrical limitation of large elastic deflections of the bellows is taken into account. Curves of axial buckling force versus number of segments were presented for various types of bellows segments.

The linearized disturbance equations governing the static buckling behavior of circular-bar helical springs subjected to combined compression and twist were solved in Chassie et al. (1997) numerically using the transfer matrix method to produce buckling design charts. The effects of the number of turns of the spring and the angle

of twist were investigated for clamped–clamped ends and the results were compared to existing theories of instability.

In the article (Tabarrok and Xiong 1992) the governing equations for the static buckling of spatial rods were derived by considering perturbations about the critical state. These equations involve the curvature and twist of the rod which in general were different from those of the initial, i.e. unloaded, geometry. An incremental procedure is outlined for updating the rod's geometry up to the critical state. Other generalities incorporated into the analysis include the effect of the initial bending moments and transverse forces as well as the axial loads. Based on the theory outlined it is shown how an exact stiffness matrix and an approximate geometric stiffness matrix may be developed for a curved and twisted rod.

By (Lee et al. 2009) the dynamic characteristics of the shape memory alloy helical spring proposed to be used in the semi-active suspension platform were considered. The component mounted on the platform generates periodic oscillation due to parametric excitation, such as rotation of an eccentric mass. In this way, the induced vibration of the platform could load the suspension spring in both axial and lateral directions. In the cited paper, the spring constant of the helical spring in transverse deformation was first derived by employing the first theorem of Castigliano's. The derived spring constant was then used to define the equivalent flexural stiffness of the spring.

The equivalent rod was applied for calculations of natural lateral vibrations of springs (Michalczyk 2015). The model allows one to calculate natural frequencies of the clamped–clamped spring. It was shown that models based on the equivalent beam concept are easier to apply than the models treating the spring as the spatially curved rod and these models provide sufficient accuracy for practical calculations.

3.2 Static “Equivalent Column” Equations

The simplest way to demonstrate the essence of the dynamic stability method consists in its application to the traditional concept of the equivalent column.

On the coil spring and the hypothetical column act the equal force and moment. Under these conditions, the axial s and lateral s_Q displacement of the coil correspond the displacements of the hypothetical beam. Therefore, the hypothetical beam is also referred to as equivalent beam. An equivalent beam deforms exactly like the coil spring under the same load (Fig. 3.1).

Well known, that the Bernoulli's hypothesis for the equivalent beams as used by Grammel or Dick leads to considerable overestimation of flexural stiffness of the spring. The Timoshenko-type model is an extension of the Euler–Bernoulli model by considering two additional effects: shearing force effect and rotary motion effect. Consequently, the equations governing the helical spring to the equivalent beam have to be studied in the framework of Timoshenko-type beam theory for the equivalent column.

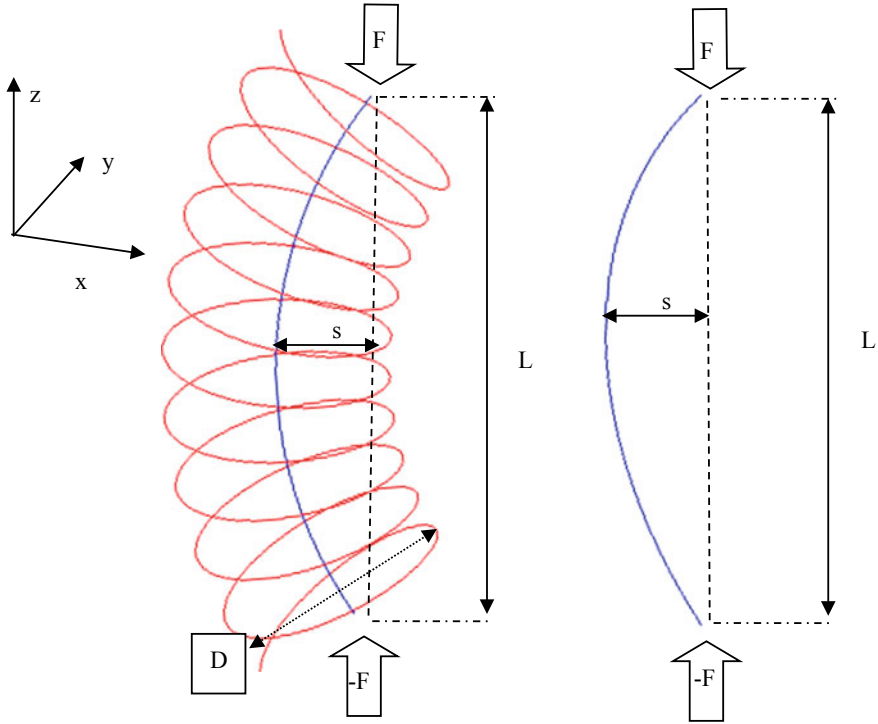


Fig. 3.1 Helical spring (red) and its “equivalent column” (blue)

The centerline of the equivalent to helical spring column is of variable length L and coincides with the z axis.

In static case, the equivalent column is subjected to a static axial load. If the spring is loaded with the lateral force F and the bending moment m_B , the original straight axis of the equivalent column turns to be curved. The angle of inclination of the bent axis can be found by means of the derivation of the lateral movement in a linear approximation:

$$\varphi_Q = \frac{ds_Q}{dz}.$$

In any beam except one subject to pure bending only, a displacement due to the shear stress occurs. The solution to the vibration problem requires this displacement to be considered. This total transversal displacement is the sum of two partial displacements:

$$s_Q = s_b + s_s. \quad (3.1)$$

In this equation the following notations for transverse displacements are used:

s_s is the displacement caused by the transverse force,

s_b is the displacement caused by bending moment.

The static equations of the equivalent column are:

$$M_B = \langle EI_B \rangle \frac{d^2(s_Q - s_s)}{dz^2} + F \cdot s_Q, \quad (3.2)$$

$$Q = \langle GS \rangle \frac{ds_s}{dz} - F \frac{d(s_Q - s_s)}{dz}, \quad (3.3)$$

$$Q + \frac{dM_B}{dz} = m_B, \quad (3.4)$$

$$\frac{dQ}{dz} = f_Q. \quad (3.5)$$

Here

Q is the shear, or transverse, force

M_B is bending moment

m_B is the external torque per unit length (in y-direction),

f_Q is the external load in the transverse direction per unit length (in the x-direction).

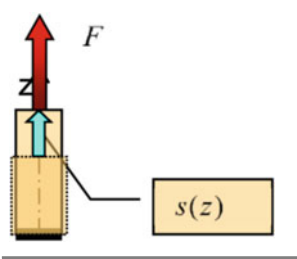
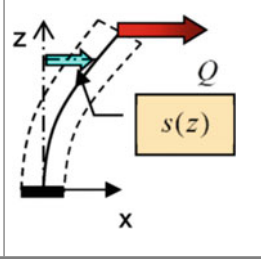
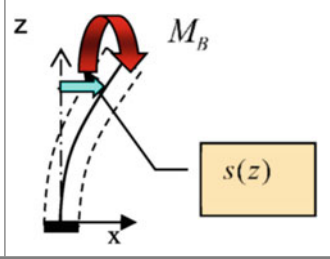
The static deflection of a straight column with the equivalent bending stiffness $\langle EI_B \rangle$, and shear stiffness $\langle GS \rangle$ under the action of the torque m_B and external force f_Q is described by the Eqs. (3.2)–(3.5).

The first derivation of the equivalent shear, bending and axial compression stiffness was performed in the cited papers of Grammel and Dick. The formulas of (Ponomarev and Andreeva 1980) are reasonably general and are valid for an arbitrary wire of noncircular cross-section. The shear, bending and axial compression rigidities of beam, which is equivalent to the helical coil spring, are presented in Table 3.1. The brackets express the fact, that the formulas represent the averaged stiffness of the hypothetical equivalent beam. Mass per unit length $\langle m \rangle$ is a mass related to the unit length of the coil axis of the spring. The stiffness of spring wire in the case of bending in normal direction EI is assumed to be different from the stiffness of wire EI_r by the bending in the direction of binormal. A is the area of the wire cross-section, GI_T is the twist stiffness of wire with respect to its axis, d is the diameter of round wire, D is the mean diameter of coil, n is the number of active

Table. 3.1 Effective rigidities and mass of the cylindrical helical spring for shear, bending and compression (I, I_r, I_T in the Table 1.3)

	Arbitrary wire	Round wire	Unit
Shear stiffness (GS)	$\frac{8EI_r L}{\pi n_a D^3}$	$\frac{Ed^4 L}{8n_a D^3}$	N
Bending stiffness (EI_B)	$\frac{2EIL}{\pi n_a D \left(1 + \frac{EI_r}{GI_T}\right)}$	$\frac{Ed^4 L}{32(2+\nu)n_a D}$	Nm ²
Axial stiffness (ES) $\equiv c_F L$	$\frac{4GI_T}{\pi n_a D^3} L$	$\frac{Gd^4}{8n_a D^3} L$	N
Mass per unit length (m)	$\frac{\rho_m \pi A n_a D}{L}$	$\frac{\rho_m \pi^2 d^2 n_a D}{4L}$	$\frac{\text{kg}}{\text{m}}$

Table 3.2 Deformation of an “equivalent column” under the action of axial load, transverse force and bending moment

		
Axial load F	Shear load Q	Bending moment M_B
$s(z) = \frac{F}{ES}z$	$s(z) = \frac{Q}{6(EI_B)}z^2(3L - z)$	$s(z) = \frac{M_B}{2(EI_B)}z^2$

coils, ρ_m is the density of the material. The geometric properties of various cross-sections of wires are given in Tables 1.3 and 1.4. Typical deformations of the spring, considered as an “equivalent column”, under the action of axial and transverse forces and a bending moment are shown in Table 3.2.

3.3 Dynamic “Equivalent Column” Equations

Consider now the equations of the equivalent column in dynamic case. Damping of material is not accounted. The factor that affects the lateral vibration of the beam, neglected in Euler–Bernoulli’s model, is the fact that each section of the beam rotates slightly in addition to its lateral motion when the beam deflects. The influence of the beam section rotation is considered through the moment of inertia, which modifies the equation of moments acting on an infinitesimal beam element. The equivalent mass per length element of the hypothetical beam matches the mass spiral spring, distributed over the axis of the spring. The bending moment m_B is the product of static moment of coil

$$\frac{1}{8}\langle m \rangle D^2$$

and the angular acceleration.

$$\frac{\partial^3 s_Q}{\partial z \partial t^2},$$

,

such that the bending moment reads:

$$m_B = \langle m \rangle \frac{D^2}{8} \frac{\partial^3 s_Q}{\partial z \partial t^2}. \quad (3.6)$$

Accordingly, the transverse force is the mass of the coil multiplied by its acceleration:

$$f_Q = \langle m \rangle \frac{\partial^2 s_Q}{\partial t^2}. \quad (3.7)$$

Substitution of (3.6) and (3.7) into the Eqs. (3.2) to (3.5) delivers the corresponding equations with partial derivatives:

$$M_B = \langle EI_B \rangle \frac{\partial^2 (s_Q - s_s)}{\partial z^2} + F \cdot s_Q \quad (3.8)$$

$$Q = \langle GS \rangle \frac{\partial s_s}{\partial z} - F \frac{\partial (s_Q - s_s)}{\partial z} \quad (3.9)$$

$$Q + \frac{\partial M_B}{\partial z} = \langle m \rangle \frac{D^2}{8} \frac{\partial^3 s_Q}{\partial z \partial t^2} \quad (3.10)$$

$$\frac{\partial Q}{\partial z} = \langle m \rangle \frac{\partial^2 s_Q}{\partial t^2}. \quad (3.11)$$

The unknowns of four partial differential Eqs. (3.8), (3.9), (3.10), and (3.11) are the functions:

$$M_B, Q, s_Q, s_s.$$

In terms of s_Q, s_s the Eqs. (3.8), (3.9), (3.10), and (3.11) reduce to partial differential equations:

$$\begin{aligned} \langle GS \rangle \frac{\partial s_s}{\partial z} - F \frac{\partial (s_Q - s_s)}{\partial z} + \frac{\partial}{\partial z} \left[\langle EI_B \rangle \frac{\partial^2 (s_Q - s_s)}{\partial z^2} + F \cdot s_Q \right] &= \langle m \rangle \frac{D^2}{8} \frac{\partial^3 s_Q}{\partial z \partial t^2}; \\ \frac{\partial}{\partial z} \left[\langle GS \rangle \frac{\partial s_s}{\partial z} - F \frac{\partial (s_Q - s_s)}{\partial z} \right] &= \langle m \rangle \frac{\partial^2 s_Q}{\partial t^2}. \end{aligned}$$

The transverse vibration is represented by two coupled differential equations of second order in time. These equations could be reduced to one differential equation for the entire lateral deformation as an unknown function $s_Q(z, t)$. To account the boundary conditions the expressions for bending moment and transverse force as functions of total lateral deformation of hypothetical axis are required. From Eqs. (3.8) to (3.11) follows:

$$\begin{aligned} \frac{\partial s_s}{\partial z} &= \frac{1}{\langle GS \rangle + F} \left(Q + F \frac{\partial s_Q}{\partial z} \right), \\ M_B &= \frac{\langle EI_B \rangle \langle GS \rangle}{\langle GS \rangle + F} \frac{\partial^2 s_Q}{\partial z^2} - \frac{\langle EI_B \rangle \langle m \rangle}{\langle GS \rangle + F} \frac{\partial^2 s_Q}{\partial t^2} + F \cdot s_Q, \end{aligned}$$

$$Q = \left[\langle m \rangle \frac{D^2}{8} - \frac{\langle EI_B \rangle \langle m \rangle}{\langle GS \rangle + F} \right] \frac{\partial^3 s_Q}{\partial z \partial t^2} - \frac{\langle EI_B \rangle \langle GS \rangle}{\langle GS \rangle + F} \frac{\partial^3 s_Q}{\partial z^3} - F \cdot \frac{\partial s_Q}{\partial z}.$$

These equations also include as required, only the total lateral deformation s_Q as an unknown.

After grouping the terms the basic equation for the transverse vibrations of axially pre-stressed coil spring reads:

$$\frac{\langle EI_B \rangle \langle GS \rangle}{\langle GS \rangle + F} \frac{\partial^4 s_Q}{\partial z^4} - \langle m \rangle \left(\frac{\langle EI_B \rangle}{\langle GS \rangle + F} + \frac{D^2}{8} \right) \frac{\partial^4 s_Q}{\partial z^2 \partial t^2} + \langle m \rangle \frac{\partial^2 s_Q}{\partial t^2} + F \frac{\partial^2 s_Q}{\partial z^2} = 0. \quad (3.12)$$

With these equations, the problem of transverse vibration of helical spring reduces to the problem of vibrations of Timoshenko-type beam. Equation (3.12) is a linear partial differential equation of fourth order in coordinate and second order in time.

The boundary values are determined by the static conditions on the spring ends. There are two boundary conditions at each end (Table 3.3).

In the paper (Majkut 2009) was shown, that the solution form of the vibration differential equation of Timoshenko-type beam depends on the examined vibration frequency. The change of the solution form occurs when the frequency crosses a specific value. This value is known from literature as the cut-off frequency (Chan et al. 2002) and (Stephen and Puchegger 2006). This phenomenon could be immediately studied for problem under consideration. Namely, displacement, caused by the transverse force, is accounted at the spring ends. For the simplification of the mathematical treatise the boundary conditions for moment-free, hinged ends are investigated:

$$s_Q|_{z=0} = 0, \\ \left[\frac{EI_B GS}{GS + F} \frac{\partial^2 s_Q}{\partial z^2} - \frac{EI_B m}{GS + F} \frac{\partial^2 s_Q}{\partial t^2} + F \cdot s_Q \right]_{z=0} = 0,$$

Table. 3.3 Conditions at the ends of the spring in terms of transverse force, moment and total transversal displacement

Type of clamping		Condition 1		Condition 2	
1. Fixed	Clamped end deflection	Zero deflection	$s_Q = 0$	Zero slope	$s'_Q = 0$
2. Free	Free end transverse force	Zero force	$Q = 0$	Zero moment	$M_B = 0$
3. Hinged	Moment-free end (ball joint)	Zero deflection	$s_Q = 0$	Zero moment	$M_B = 0$
3. Sliding	Lateral force-free, sliding end	Zero slope	$s'_Q = 0$	Zero force	$Q = 0$

$$s_Q|_{z=L} = 0, \\ \left[\frac{EI_B GS}{GS + F} \frac{\partial^2 s_Q}{\partial z^2} - \frac{EI_B m}{GS + F} \frac{\partial^2 s_Q}{\partial t^2} + F \cdot s_Q \right]_{z=L} = 0.$$

The following initial values for displacement and velocity are assumed:

$$s_Q(z, t = 0) = s_o(z), \quad (3.13a)$$

$$\dot{s}_Q(z, t = 0) = v_o(z) \dot{s}_Q(z, t = 0) = v_o(z) \text{ for } 0 < z < L, \quad (3.13b)$$

$$s_o|_{z=0} = s_o|_{z=L} = 0, \quad (3.13c)$$

$$v_o|_{z=0} = v_o|_{z=L} = 0. \quad (3.13d)$$

Here $\dot{s}_o(z)$ is the initial transverse deflection and $v_o(z)$ the velocity in the time $t = 0$.

The zero boundary conditions for the displacement simplify the zero boundary conditions for moment at the end section:

$$s_Q|_{z=0} = 0, \quad \frac{\langle EI_B \rangle \langle GS \rangle}{\langle GS \rangle + F} s_Q''|_{z=0} = 0, \quad (3.14a)$$

$$s_Q|_{z=L} = 0, \quad \frac{\langle EI_B \rangle \langle GS \rangle}{\langle GS \rangle + F} s_Q''|_{z=L} = 0. \quad (3.14b)$$

Therefore, the zero boundary conditions for moment at the end section require the secondary derivative at the end to vanish:

$$s_Q''|_{z=0} = s_Q''|_{z=L} = 0.$$

3.4 Natural Frequency of Transverse Vibrations

The transverse vibration is represented by a differential equation of fourth order in place and second order in time. The Fourier method of variable separation is employed to find function $s_Q(z, t)$ satisfying the Eq. (3.12). It is assumed that the function can be presented in the form of a product of a function dependent on the spatial coordinate and a function dependent on time. The solution of the model Eq. (3.12) without damping could be obtained by separation of variables:

$$s_Q(z, t) = Z(z)T(t). \quad (3.15)$$

After separation of variables the Eq. (3.12) reads:

$$\begin{aligned} \frac{\langle EI_B \rangle \langle GS \rangle}{\langle GS \rangle + F} Z^{IV}(z) T(t) - \langle m_t \rangle \left(\frac{\langle EI_B \rangle}{\langle GS \rangle + F} + \frac{D^2}{8} \right) Z''(z) \dot{T}(t) + \\ + \langle m \rangle Z(z) \ddot{T}(t) + F Z''(z) T(t) = 0 \end{aligned} \quad (3.16)$$

with the common notation for ordinary derivatives:

$$\dot{T} = \frac{dT}{dt}, \quad Z' = \frac{dZ}{dz}.$$

After separation of variables in (3.16), the following condition established:

$$\frac{1}{\langle m \rangle} \frac{\frac{\langle EI_B \rangle \langle GS \rangle}{\langle GS \rangle + F} Z^{IV}(z) + F Z''(z)}{\left(\frac{\langle EI_B \rangle}{\langle GS \rangle + F} + \frac{D^2}{8} \right) Z''(z) - Z(z)} = - \frac{\ddot{T}}{T(t)} = \omega^2 \quad (3.17)$$

This condition can be satisfied only for a certain constant ω^2 .

Equation (3.17) is separated into two ordinary differential equations:

(a) The eigenvalue problem:

$$\frac{\langle EI_B \rangle \langle GS \rangle}{\langle GS \rangle + F} Z^{IV}(z) + F Z''(z) = \omega^2 \langle m \rangle \left[\left(\frac{\langle EI_B \rangle}{\langle GS \rangle + F} + \frac{D^2}{8} \right) Z''(z) - Z(z) \right] \quad (3.18)$$

(b) The evolution equation:

$$\ddot{T} + \omega^2 T = 0. \quad (3.19)$$

For solution of the eigenvalue problem (a) the representation is used:

$$Z(z) = C \exp(\Lambda z).$$

With this substitution, Eq. (3.18) leads to:

$$\left\{ \Lambda^4 \frac{\langle EI_B \rangle \langle GS \rangle}{\langle GS \rangle + F} + \Lambda^2 F - \omega^2 \langle m \rangle \left[\left(\frac{\langle EI_B \rangle}{\langle GS \rangle + F} + \frac{D^2}{8} \right) \Lambda^2 - 1 \right] \right\} \cdot C \exp(\Lambda z) = 0.$$

Since

$$\exp(\Lambda z) > 0,$$

this algebraic equation can be satisfied only if:

$$\Lambda^4 \frac{\langle EI_B \rangle \langle GS \rangle}{\langle GS \rangle + F} + \Lambda^2 F = \omega^2 \langle m \rangle \left[\left(\frac{\langle EI_B \rangle}{\langle GS \rangle + F} + \frac{D^2}{8} \right) \Lambda^2 - 1 \right]. \quad (3.20)$$

The quartic equation (3.20) has four solutions:

$$\Lambda = \pm i\beta_1, \quad \Lambda = \pm i\beta_2.$$

The solution to the problem (a) is the linear combination of the partial solutions:

$$Z(z) = c_1 \sin(\beta_1 z) + c_2 \cos(\beta_1 z) + c_3 \sin(\beta_2 z) + c_4 \cos(\beta_2 z). \quad (3.21)$$

On both sides the conditions of hinged ends are used. Displacement, caused by the transverse force, is supposedly omitted at the spring ends. The conditions for bending moment at the hinged ends

$$M_B|_{z=0} = 0, \quad M_B|_{z=L} = 0$$

in terms of functions $Z(z)$, $T(t)$ transform to:

$$\frac{\langle EI_B \rangle \langle GS \rangle}{\langle GS \rangle + F} T(t) Z''|_{z=0} = 0, \quad \frac{\langle EI_B \rangle \langle GS \rangle}{\langle GS \rangle + F} T(t) Z''|_{z=L} = 0.$$

The boundary conditions for the hinged end read in this case:

$$Z(0) = Z''(0) = Z(L) = Z''(L) = 0. \quad (3.22)$$

This simplification will be omitted in the final paragraph.

The use of (3.21) into (3.22) gives the following system of homogeneous equations:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -\beta_1^2 & 0 & -\beta_2^2 \\ \sin\beta_1 L & \cos\beta_1 L & \sin\beta_2 L & \cos\beta_2 L \\ -\beta_1^2 \sin\beta_1 L & -\beta_1^2 \cos\beta_1 L & -\beta_2^2 \sin\beta_2 L & -\beta_2^2 \cos\beta_2 L \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (3.23)$$

The homogeneous system of Eqs. (3.23) only leads to non-trivial solutions if its determinant is zero:

$$(\beta_1^2 - \beta_2^2)^2 \sin(\beta_1 L) \sin(\beta_2 L) = 0. \quad (3.24)$$

The condition (3.18) is satisfied if:

$$\beta_{1,2} = \lambda_N \equiv \pi \frac{N}{L} \text{ with } N = 1, 2, 3... \quad (3.25)$$

The circular natural frequencies resulting to be:

$$\omega_N^2 = \frac{\left(\frac{\pi N}{L}\right)^2 \left[\left(\frac{\pi N}{L}\right)^2 \frac{\langle EI_B \rangle \langle GS \rangle}{\langle GS \rangle + F} - F \right]}{\langle m \rangle \left[\left(\frac{\langle EI_B \rangle}{\langle GS \rangle + F} + \frac{D^2}{8} \right) \left(\frac{\pi N}{L}\right)^2 + 1 \right]} \text{ with } N = 1, 2, 3... \quad (3.26)$$

The corresponding mode shapes:

$$Z_N(z) = c_1 \sin \lambda_N z.$$

Two dimensionless parameters are introduced: the slenderness ratio of the free spring as:

$$\xi = \frac{L_0}{D}$$

and relative dimensionless length as.

$$\mu = \frac{L}{L_0}.$$

The slenderness ratio of the deformed spring is equal to: $L/D = \xi \mu$.

The dimensionless deflection μ characterizes the degree of compression of the spring. For the free length of the spring.

$$\mu = 1.$$

The flat state of spring corresponds to.

$$\mu = 0.$$

In this—although pure imaginary—state the spring is completely flattened. Though, the real springs cannot be compressed to this state due to coils clash.

The axial spring force is a function of the actual spring length L . The axial spring force can be also expressed as the function of dimensionless deflection:

$$F = c_F(L_0 - L) \equiv \langle ES \rangle (\mu^{-1} - 1). \quad (3.27)$$

After inserting the related rigidities in (3.26) the natural frequency ω_n could be obtained as a function of the relative dimensionless length μ and slenderness ratio ξ . Natural frequency is the function of the deflection dimensionless μ and slenderness ratio ξ (Kobeleev 2014):

$$\omega_N = \frac{d\xi^2 N}{n_a L_0^2} \sqrt{\frac{4G\mu}{\rho_m} \sqrt{\frac{\xi^2(2+\nu)(\mu-1)(2\mu\nu+\mu+1) + (1+\nu)^2\pi^2 N^2}{8\xi^2(2\mu\nu+\mu+1)\mu^2 + \pi^2 N^2(6\mu+9\mu\nu+2+2\mu\nu^2+\nu)}}} \quad (3.28)$$

This equation explicitly demonstrates that the fundamental natural frequency of the transverse vibrations depends on the current compressed length of the spring, which is specified by its relative dimensionless length μ .

The natural frequency ω_N^0 of the free, unloaded spring follows from the formula (3.28), when the dimensionless deflection is $\mu = 1$. For the spring in its free length the natural frequency is equal to (Kobelev 2014):

$$\omega_N^0 = \omega_N|_{\mu=1} = \frac{d\xi^2 N}{n_a L_0^2} \sqrt{\frac{2G}{\rho_m} \frac{(1+\nu)\pi^2 N^2}{8\xi^2 + \pi^2 N^2(4+\nu)}}.$$

The relative fundamental frequency of transverse oscillations:

$$\Omega_N(\mu) = \omega_N / \omega_N^0$$

is of order N introduced. This formula accounts slenderness as a parameter, fixed for any given spring.

3.5 Stability Conditions and Buckling of Spring

Stability conditions could be derived as the expressions of degeneration of natural frequency of transversal vibrations. Namely, if the relative fundamental frequency $\Omega_N(\mu)$ nullifies, the lateral buckling of the spring with the natural form N occurs. The condition of vanishing of fundamental relative frequency:

$$\Omega_N(\mu) = 0 \quad (3.29)$$

determines the critical spring compression $\mu^*(N)$. At that moment the degeneration of transversal vibration to buckling mode occurs and the spring loses its stability by divergence mode. The matching mode shape with $N = 1$ corresponds to the buckling of the spring with moment-free, simply supported, hinged ends. The mode $N = 2$ matches approximately to the buckling of the spring with clamped ends.

Substitution of the expression for frequency (3.28) into the condition (3.29) leads to the following quadratic equation for critical dimensionless deflection μ as the function of slenderness ratio of the spring ξ :

$$\xi^2(2+\nu)(\mu-1)(2\mu\nu+\mu+1) + (1+\nu)^2\pi^2 N^2 = 0. \quad (3.30)$$

For each mode shape the quadratic Eq. (3.30) has two roots (Kobelev 2014):

$$\mu_+^*(N) = \frac{\nu}{2\nu + 1} + \frac{(\nu + 1)\sqrt{(\nu + 2)\xi^2 - (2\nu + 1)\pi^2 N^2}}{(2\nu + 1)\xi\sqrt{\nu + 2}} \quad (3.31)$$

$$\mu_-^*(N) = \frac{\nu}{2\nu + 1} - \frac{(\nu + 1)\sqrt{(\nu + 2)\xi^2 - (2\nu + 1)\pi^2 N^2}}{(2\nu + 1)\xi\sqrt{\nu + 2}} \quad (3.32)$$

If the spring is compressed from its free length:

$$\mu = 1,$$

the buckling occurs when μ reaching the critical deflection $\mu_+^*(N)$ Eq. (3.31) from above.

Contrarily, if the spring is unloaded from its flat state:

$$\mu = 0,$$

the buckling occurs at the state of the critical deflection $\mu_-^*(N)$ Eq. (3.32) from below. The first and second axial Euler-Grammel buckling shapes are shown on the Fig. 3.2. Figure 3.3 illustrates the first $\mu_\pm^*(1)$ and second $\mu_\pm^*(2)$ critical compressions that correspond to first and second buckling shapes.

There is the unstable region between these two critical states (3.31), (3.32):

$$\mu_-^*(N) < \mu < \mu_+^*(N).$$

These two critical states (3.31), (3.32) exist only if the expression under square root is positive. This happens, if the following condition is satisfied:

$$\xi \geq \xi^*(N) = \pi N \sqrt{\frac{2\nu + 1}{\nu + 2}}$$

Otherwise, if

$$\xi < \xi^*(N) = \pi N \sqrt{\frac{2\nu + 1}{\nu + 2}}$$

the spring is always stable over the compression range.

In this case, no buckling of the spring occurs and the spring remains straight over the full compression range:

$$0 < \mu < 1.$$

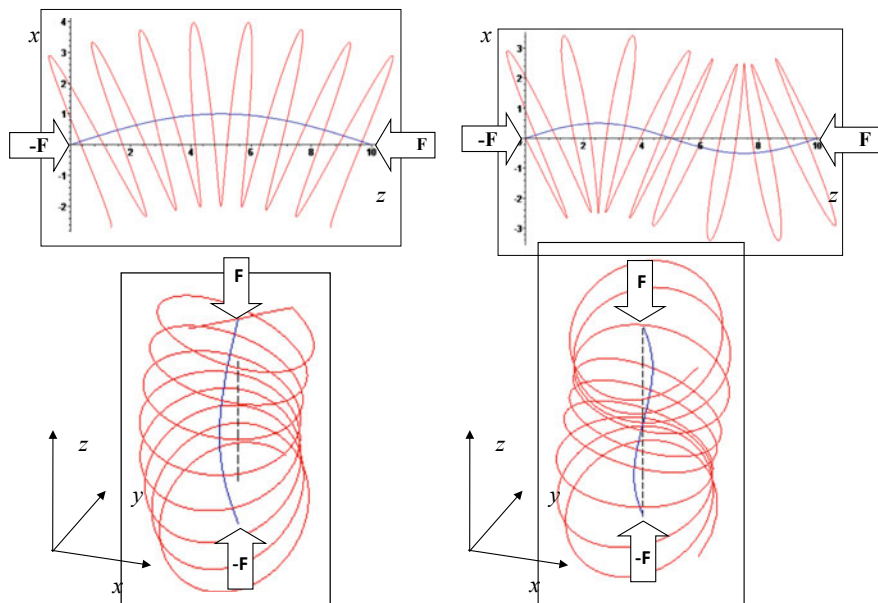


Fig. 3.2 First (left) and second (right) Euler-Grammel axial buckling shapes of simply supported helical spring

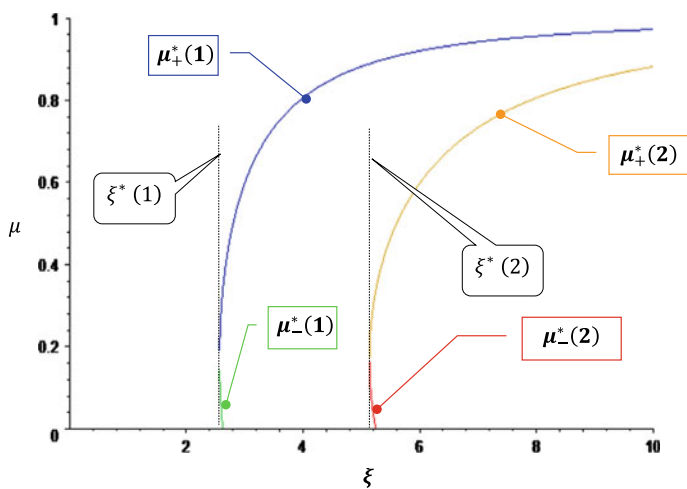


Fig. 3.3 Critical deflection $\mu_{\pm}^*(N)$, $N = 1, 2$ as a function of the degree of slenderness ξ

The first and second relative natural frequencies of hinged linear spring are shown in Figs. 3.4 and 3.5. The resonant frequency decreases with the degree of compression μ . The shorter the spring, the lower will be usually the first natural, or fundamental, frequency.

On the Fig. 3.4 the contours plot of the first fundamental frequency is drawn. The highest relative natural frequency is 1 and this value is achieved at

$$\mu = 1.$$

With the successive compression of spring the relative natural frequency gradually reduces. The shorter the spring, the lower will be the fundamental frequency. In this region, the force action line of the spring drifts from the center line of the spring. On the left region of the plot:

$$\xi < \xi^*(1)$$

the relative natural frequency gradually reduces until the spring flattens. On the right region of the plot:

$$\xi > \xi^*(1)$$

the relative natural frequency sharply reduces to zero and the spring buckles, when its critical length is achieved:

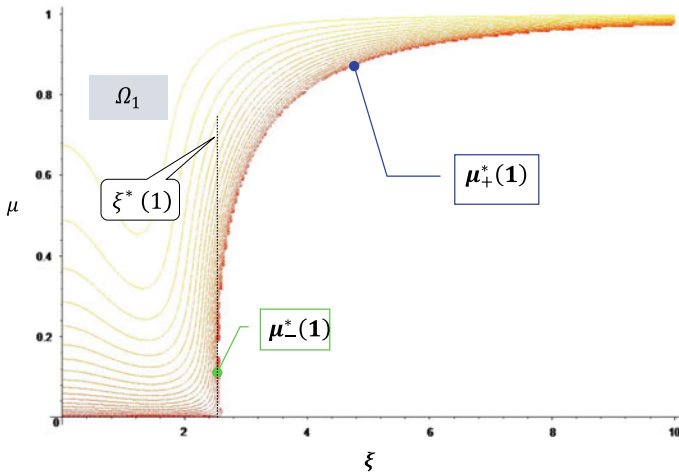


Fig. 3.4 Contour plot of fundamental frequency $\Omega_1 = \omega_1/\omega_1^0$ as a function of deflection μ and slenderness ξ

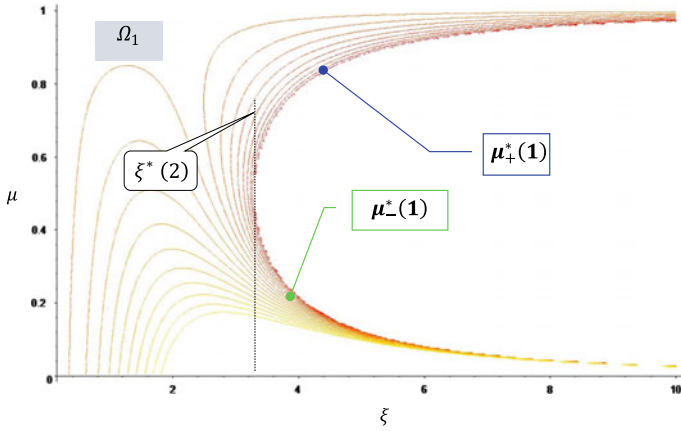


Fig. 3.5 Contour plot of the second natural frequency $\Omega_2 = \omega_2/\omega_2^0$ as a function of deflection μ and slenderness ξ

$$\mu = \mu_+^*(1).$$

Finally, the spring loses its stability on the boundary and lateral buckling of spring occurs. The explanation of data on Fig. 3.5 for the second fundamental frequency is similar.

The spring behaves similarly being unloaded from its flattened shape. Consider the spring in its flattened state. The spring is not stable in this position for all values of the slenderness ratio. Namely, in the tiny region:

$$\xi^*(1) \leq \xi < \xi_{cr}$$

the flattened spring remains stable. Here ξ_{cr} is the solution of the equation.

$$\mu_-^*(N) = \frac{\nu}{2\nu + 1} - \frac{(\nu + 1)\sqrt{(\nu + 2)\xi_{cr}^2 - (2\nu + 1)\pi^2 N^2}}{(2\nu + 1)\xi_{cr}\sqrt{\nu + 2}} = 0.$$

Stating expanding from its flattened position, the frequency of spring drops rapidly and the spring buckles when:

$$\mu = \mu_-^*(1).$$

For all other values of

$$\xi \geq \xi_{cr}$$

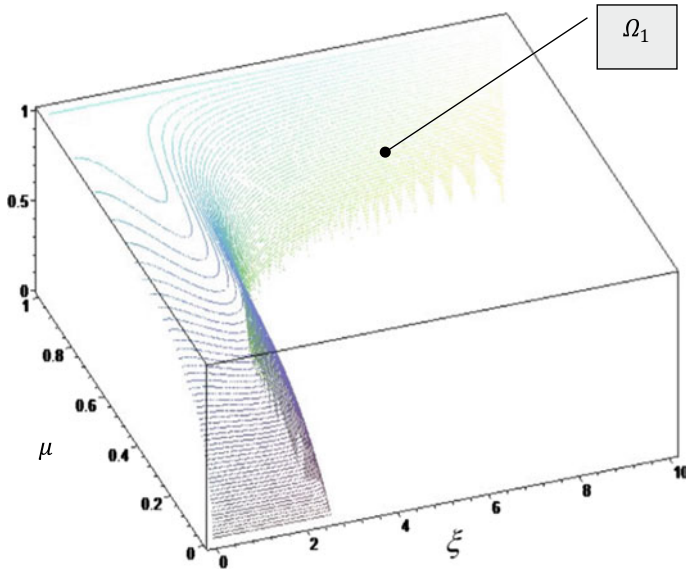


Fig. 3.6 The fundamental frequency $\Omega_1 = \omega_1/\omega_1^0$ as a function of deflection μ and slenderness ξ

the spring is unstable and immediately snaps out being released from its flat state. The flattened form of the spring is also predominantly unstable. The perspective plots of fundamental frequencies are shown respectively on Figs. 3.6 and 3.7. The non-covered regions on these graphs correspond to unstable states of the springs, where the frequencies possess non-vanishing imaginary values.

3.6 Buckling of Twisted, Compressed, and Tensioned Helical Spring

3.6.1 *Instability of Twisted Helical Spring*

The counterpart for the Euler buckling problem is the Greenhill's problem, which studies the forming of a loop in an elastic bar under torsion. The torsion spring can wrinkle in a similar way. We study the buckling of the helical spring, twisted by couples M_T applied at its ends alone (Greenhill's problem for helical spring). The ends of the rod are assumed to be attached to the supports by ideal spherical hinges and are free to rotate in any directions. The twisting couple retains its initial direction during buckling. Introduction of new unknown variables reduces the problem to the exactly solvable auxiliary problem with constant coefficients, such that the critical buckling moment of an equivalent column with an arbitrary shape allows also the exact closed form solution.

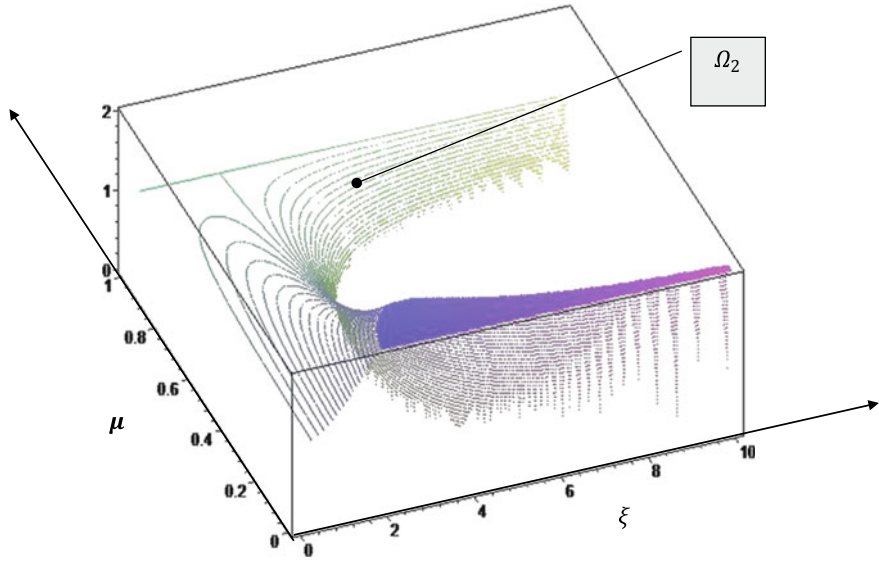


Fig. 3.7 The fundamental frequency $\Omega_2 = \omega_2/\omega_2^0$ as a function of deflection μ and slenderness ξ

Consider an equivalent column of the length L_0 , twisted by couples applied at its ends alone (Greenhill's problem for an equivalent column). The twisting couple M_T retains its initial direction during buckling (Bazant and Cedolin 2010). The effective bending stiffness of the helical spring $\langle EI_B \rangle$ is the same for both directions. For simplicity, we neglect the shear stiffness of the spring.

Consider at first an equivalent column with the constant bending stiffness along the length of the spring (Table 3.1):

$$\langle EI_B \rangle = \frac{Ed^4 L_0}{32(2 + \nu)n_a D}.$$

The Greenhill's twist buckling equations are given by:

$$\langle EI_B \rangle \frac{d^2 u}{dz^2} = M_T \frac{dv}{dz}, \quad \langle EI_B \rangle \frac{d^2 v}{dz^2} = -M_T \frac{du}{dz}, \quad (3.33)$$

here $u(z)$, $v(z)$ are the deflections of the equivalent column in directions of axes x , y respectively. The ends of the rod $z = 0$ and $z = L_0$ are assumed to be attached to the supports by ideal spherical hinges and are free to rotate in any directions:

$$u(0) = v(0) = u(L_0) = v(L_0) = 0. \quad (3.34)$$

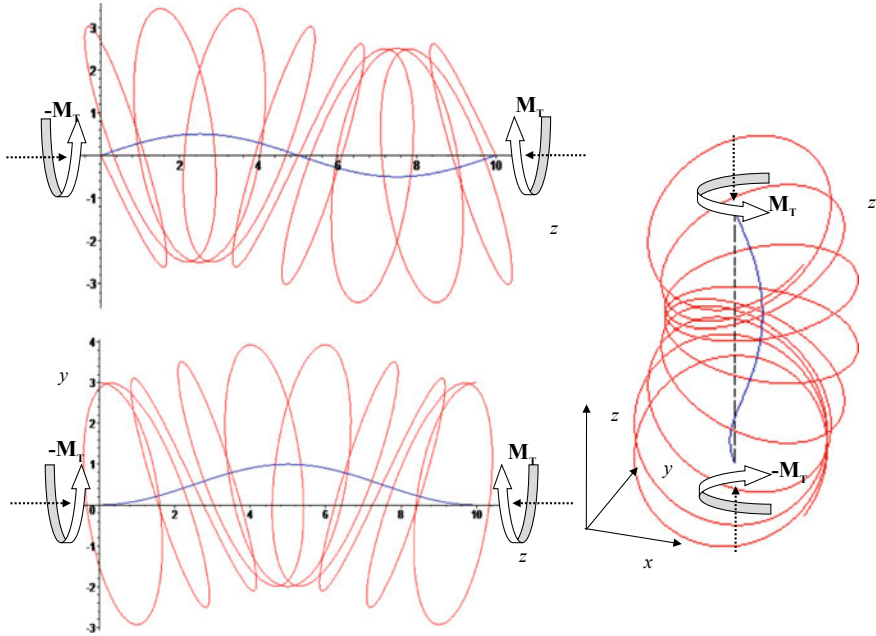


Fig. 3.8 First twist buckling shape (Greenhill) of the simply supported helical spring

The integration of the Eqs. (3.33) delivers new equations with two unknown constants c_1, c_2 :

$$\langle EI_B \rangle \frac{du}{dz} = M_T v + c_1, \quad \langle EI_B \rangle \frac{dv}{dz} = -M_T u + c_2.$$

With (3.34) the first buckling shape as follows (Fig. 3.8):

$$u(z) = \sin(2\pi z/L_0), \quad v(z) = 1 - \cos(2\pi z/L_0), \quad c_1 = c_2 = 0.$$

The critical buckling torque for Greenhill's problem is given by :

$$M_T^* = \frac{2\pi \langle EI_B \rangle}{L_0} = \frac{\pi E d^4}{16(2 + \nu) n_a D}. \quad (3.35)$$

At second, consider the spring with a variable bending stiffness along it axis. For example, the coil diameter or wire diameter vary along the axis of the spring. Introduction of the independent variable ξ :

$$z(\xi) = \int_0^\xi \frac{dt}{F(t)}, \quad F(t) > 0.$$

In the new axes the problem (3.34)–(3.35) transforms to:

$$F(\xi)EI_B \frac{dU}{d\xi} = M_T V + c_1, \quad F(\xi)EI_B \frac{dV}{d\xi} = -M_T U + c_2. \quad (3.36)$$

$$U(0) = V(0) = U(L) = V(L) = 0 \quad (3.37)$$

with

$$u(z) = u\left(\int_0^\xi \frac{dt}{F(t)}\right) = U(\xi), \quad v(z) = v\left(\int_0^\xi \frac{dt}{F(t)}\right) = V(\xi).$$

The functions $U(\xi)$ $V(\xi)$ are the deflections of an equivalent column with a variable bending stiffness along its length in directions of axes x , y respectively. The value L is the solution of the algebraic equation

$$L_0 = \int_0^L \frac{d\xi}{F(\xi)} \quad (3.38)$$

Eigenvalue problem (3.36)–(3.37) describes the Greenhill's problem with a variable effective bending stiffness of the “equivalent column” over the length:

$$\langle EI_B(t) \rangle = \langle EI_B \rangle \dot{F}(t), 0 \leq t \leq L.$$

The critical eigenvalues of the problems (3.36)–(3.37) and (3.33)–(3.34) match and are equal to the value (3.35). Substitution of (3.38) in the expression (3.35) delivers the analytical expression for critical torque (Kobelev 2016):

$$M_T^{**} = 2\pi \left(\int_0^L \frac{dt}{\langle EI_B \rangle F(t)} \right)^{-1} \equiv 2\pi \left(\int_0^L \frac{dt}{\langle EI_B(t) \rangle} \right)^{-1} \quad (3.39)$$

This function delivers the critical load of Greenhill's problem for a twisted helical spring with a variable effective bending stiffness of the length L . Other clamping conditions could be examined (Kobelev 2017).

3.6.2 *Instability of Helical Springs Under Torque and Axial Force*

Consider a geometrically perfect helical spring supported on two spherical hinges, loaded by axial compressive force F and torque M_T . Both loads are assumed to keep their direction during buckling. Adding the expressions for moments due to axial force $(-Fu, -Fv)$ to the Greenhill's twist buckling Eqs. (3.33), we get the buckling equations for the helical springs under torque and axial force:

$$\langle EI_B \rangle \frac{d^2 u}{dz^2} = M_T \frac{dv}{dz} - Fu, \quad \langle EI_B \rangle \frac{d^2 v}{dz^2} = -M_T \frac{du}{dz} - Fv, \quad (3.40)$$

here $u(z)$, $v(z)$ are the deflections of the equivalent column in directions of axes x , y respectively. The tolerance of the solution could be improved, if we consider the value L as the already compressed length of the helical spring under the action of the axial force F , i.e. $L = L_0 - F/c$.

The ends of the compressed spring $z = 0$ and $z = L$ are assumed to be attached to the supports by ideal spherical hinges and are free to rotate in any directions, Eq. (3.34). We seek the general solution of Eqs. (3.40) in the form:

$$u(z) = A \exp(i\lambda z), \quad v(z) = B \exp(i\lambda z). \quad (3.41)$$

The substitution of (3.41) into (3.40) leads to a system of two homogeneous linear algebraic equations:

$$\begin{bmatrix} F - \langle EI_B \rangle \lambda^2 & i\lambda M_T \\ -i\lambda M_T & F - \langle EI_B \rangle \lambda^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (3.42)$$

Nontrivial solution of (3.42) is possible, only if the determinant vanishes:

$$\begin{vmatrix} F - \langle EI_B \rangle \lambda^2 & i\lambda M_T \\ -i\lambda M_T & F - \langle EI_B \rangle \lambda^2 \end{vmatrix} = 0. \quad (3.43)$$

For positive torque M_T , two possible solutions of the characteristic equation are:

$$\lambda_{1,2} = \frac{-M_T \pm \sqrt{M_T^2 + 4\langle EI_B \rangle F}}{2\langle EI_B \rangle}. \quad (3.44)$$

The general solution is the real or imaginary part of

$$u(z) = A_1 e^{i\lambda_1 z} + A_2 e^{i\lambda_2 z}, \quad v(z) = B_1 e^{i\lambda_1 z} + B_2 e^{i\lambda_2 z}. \quad (3.45)$$

The boundary conditions Eq. (3.34) reduce with the solution (3.45) to:

$$A_1 + A_2 = B_1 + B_2 = e^{i\lambda_1 L} A_1 + e^{i\lambda_2 L} A_2 = e^{i\lambda_1 L} B_1 + e^{i\lambda_2 L} B_2 = 0. \quad (3.46)$$

Four equations (3.46) possess the nontrivial solution only if $e^{i\lambda_1 L} = e^{i\lambda_2 L}$. In this case:

$$\lambda_1 = \lambda_2 + \frac{2k\pi}{L}, k = 1, 2, \dots \quad (3.47)$$

The smallest possible buckling load Eq. (3.47) corresponds to $k = 1$:

$$\frac{-M_T + \sqrt{M_T^2 + 4\langle EI_B \rangle F}}{2\langle EI_B \rangle} = \frac{-M_T - \sqrt{M_T^2 + 4\langle EI_B \rangle F}}{2\langle EI_B \rangle} + \frac{2\pi}{L}. \quad (3.48)$$

As defined in Sect. 3.4, the slenderness ratio of the free spring is $\xi = L_0/D$ and relative dimensionless length is $\mu = L/L_0$. Equation (3.48) results in the buckling condition for simultaneous axial compression and torsion of helical spring:

$$\frac{F}{F^*} + \left(\frac{M}{M_T^*} \right)^2 = 1, \quad (3.49)$$

where for the cylindrical spring with the constant wire diameter are:

$$F^* = \langle EI_B \rangle \left(\frac{\pi}{L} \right)^2 \equiv \langle EI_B \rangle \left(\frac{\pi}{L_0 - \frac{F^*}{c}} \right)^2 = \frac{Ed^4}{32D^2n_a} \cdot \frac{\xi + \sqrt{\xi^2 - \xi^{*2}}}{1 + \nu}, \quad (3.50)$$

$$M_T^* = 2\pi \frac{\langle EI_B \rangle}{L} \equiv \frac{\pi Ed^4}{16(2 + \nu)Dn_a}. \quad (3.51)$$

F^* is the Euler's critical load for compression,

M_T^* is the Greenhill's critical twist buckling moment, Eq.(3.35). The Greenhill's critical twist buckling moment is independent upon slenderness ratio. The dimensionless value in the expression for Euler's critical compressive load Eq. (3.50) is:

$$\xi^* = \pi \sqrt{2(1 + \nu)/(2 + \nu)}. \quad (3.52)$$

The value ξ^* in Eq. (3.52) denotes the minimal slenderness ratio for instability, which is similar to discussed in Sect. 3.5. If the slenderness is less than ξ^* , i.e. $\xi < \xi^*$, no Euler buckling is possible.

The relation between both critical loads reads:

$$\frac{M_T^*}{F^*} = \frac{2D}{\pi} \mu^* \xi, \quad (3.56)$$

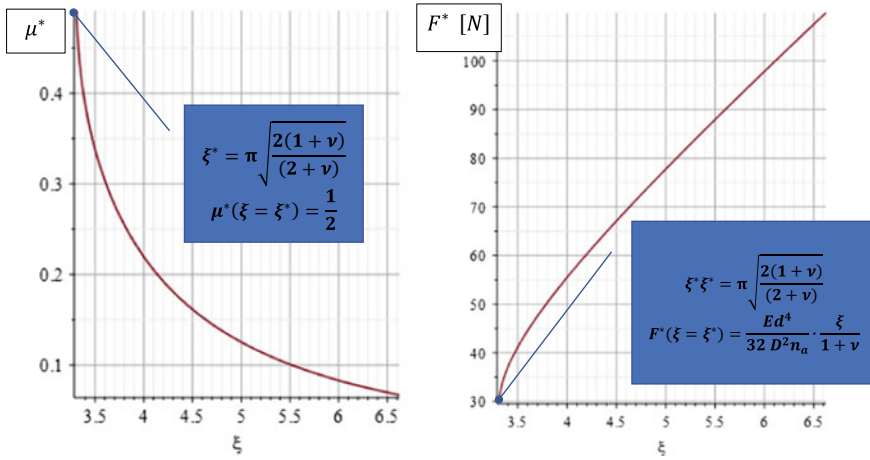


Fig. 3.9 Dependences of the relative dimensionless length μ^* and Euler critical buckling load F^* as functions of slenderness ratio ξ . $\xi^* = \pi\sqrt{2(1+\nu)/(2+\nu)}$ is the minimal slenderness ratio for instability

where is the character dimensionless length:

$$2\mu^* = 1 - \sqrt{1 - \frac{\xi^{*2}}{\xi^2}}. \quad (3.57)$$

The graphs of the critical buckling loads are shown on the following figures. The following values were used for calculations:

$$E = 200 \text{ GPa}, \nu = 0.25, n_a = 10, D = 30 \text{ mm}, L_0 = 120 \text{ mm}, d = 2 \dots 3 \text{ mm}.$$

Figure 3.9 displays the functions of the relative dimensionless length μ^* and Euler critical buckling load F^* slenderness ratio. On Fig. 3.10 the pure Greenhill's critical twist buckling moment M_T^* Eq. (3.51) and pure compressive Euler's critical load F^* Eq. (3.50) are drawn as the functions of wire diameter d .

Figure 3.11 illustrates the simultaneous buckling of twisted and compressed helical spring as described in Eq. (3.49). The functions of the Greenhill's critical twist buckling moment M_T^* and the Euler's critical load for compression F^* are shown for three wire diameters d and other equal geometrical parameters.

3.6.3 Instability of Tension Spring

The other type of instability occurs when a gradually increasing tensile force is applied to the ends of a pre-stressed helical spring (Andreeva 1962). An elegant

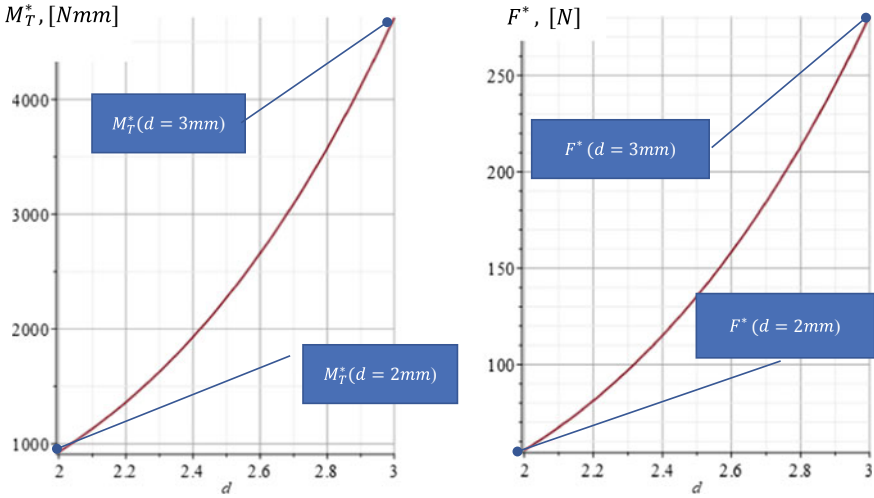


Fig. 3.10 Pure Greenhill's critical twist buckling moment M_T^* and pure Euler's critical load for compression F^* as the function of wire diameter d

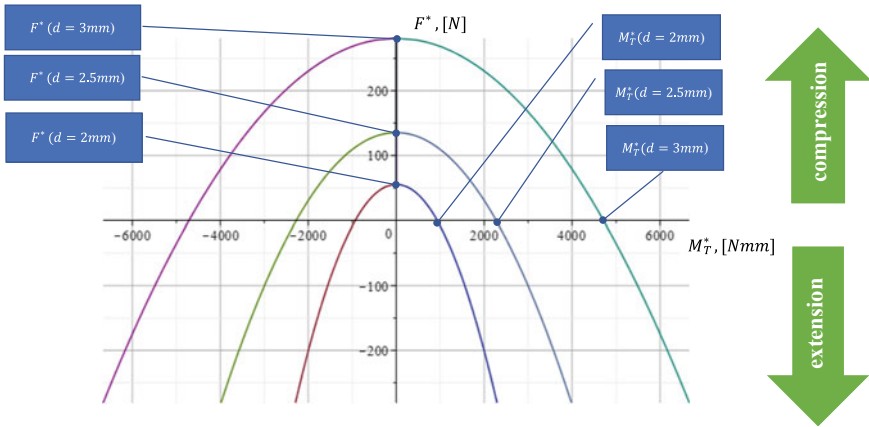


Fig. 3.11 Simultaneous buckling of twisted and compressed helical spring. Greenhill's critical twist buckling moment M_T^* and Euler's critical load for compression F^* are shown for three wire diameters d and other equal geometrical parameters

explanation of this curious effect was provided by Ponomarev and Andreeva (1980). The instability occurs for the springs with sufficiently large ratios of radius to pitch and twist to bending rigidity. Experimentally observed, that the end-to-end distance suffers a series of discontinuous stretching evolutions. This phenomenon was discovered also somewhat later by Kessler and Rabin (2003).

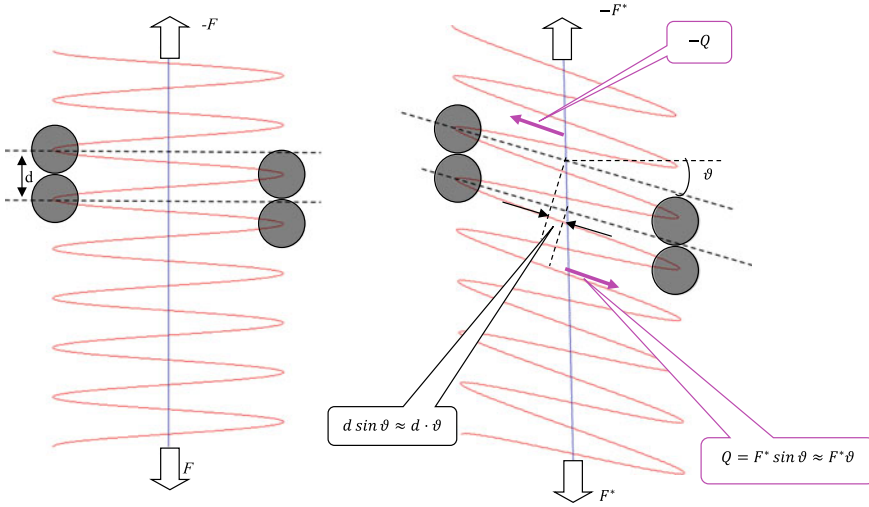


Fig. 3.12 Coil-gliding instability of an extensional helical spring. The coils slide resulting shear force

Consider a closed coiled helical spring. The coils of the spring are positioned in planes, normal to the axis of the spring. When the closed coiled helical is tensioned, the sudden inclination of coils happens at certain tensile load. The inclined coils lay no more in the normal plane to the spring axis (Fig. 3.12). The inclination angle between the new coils plane and the normal plane is ϑ . The first instability occurs when the first coil abruptly inclines. The other coils follow to incline with the slightly increasing tension load. Due to pre-stress, each coil remains in contact to its neighboring coils. The inclined coils slide over the upper surface of other coils, such that their centers shift in the inclination plane. The deformation of the shifted coils is the pure shear deformation under the action of the transverse force Q . The shear angle is exactly the inclination angle. The shear angle of an “equivalent column” is given by:

$$\vartheta = Q / \langle GS \rangle, \quad (3.58)$$

where

$$\langle GS \rangle = \frac{Ed^4L}{8n_a D^3} \quad (3.59)$$

is the effective shear stiffness of the helical spring (Table 3.1). On the other hand, the transverse force Q is the projection of the axial tensile force F^* on the inclined plane and for small angle ϑ is equal to:

$$Q = F^* \sin \vartheta \approx F^* \vartheta. \quad (3.60)$$

For closed coiled spring the coils lie in contact to each other and the spring length is $L = n_d d$. The substitution of (3.58), (3.59) into the Eq. (3.60) results in the critical force of the gliding instability:

$$F^* = \langle GS \rangle = \frac{Ed^5}{8D^3} \quad (3.61)$$

3.7 Spatial Models for Dynamic Behavior of Helical Springs

In the spatial models do not use the reduction to the hypothetical elastic axis of the spring and equivalent stiffness calculations. The variety of spatial models is explained by the difficulty of the exact equations in the framework of spatial rod theory and by significant complexity with their analytical solution.

Wittrick (1996) derived a set of 12 linear coupled partial differential equations for a uniform helical spring based on the Timoshenko beam theory. Costello (1975) presented a work on the significance of torsional oscillations on the radial expansion of helical springs.

Guido et al. (1978) applied Timoshenko theory for the determination of the transverse frequencies of cylindrical helical springs for different values of slenderness and relative compression. Springs have been considered as having damped ends and lacking end coils. Under these boundary conditions the spring deflection curves result from the superposition of two deflection curves with different wave length and characterized by bending and shear displacements in phase or 180° out of phase. The static behavior of a spatial bar of an elastic and isotropic material under arbitrary distributed loads having a non-circular helical axis and cross-section supported elastically by single and/or continuous supports was studied in (Haktanir 1995) by the stiffness matrix method based on the complementary functions approach. By considering the geometrical compatibility conditions together with the constitutive equations and equations of equilibrium, a set of 12 first-order differential equations having variable coefficients was obtained for spatial elements.

Yildirim (1997, 2002) conducted a series of studies to compute the eigenvalues of helical springs of arbitrary shape.

Lee and Thompson (2001) used the dynamic stiffness method to calculate the natural frequencies of helical springs and compared the results with those of the transfer matrix and the finite element method. The equation of free wave motion in a helical spring, derived from Timoshenko beam theory and Frenet formulae, has been used to obtain the dynamic stiffness matrix. The natural frequencies were calculated in the cited Article from this matrix, after applying suitable boundary conditions, by using the Wittrick-Williams method. By computing the axial and transverse transfer stiffness it has been shown how the spring becomes much stiffer at high frequencies, compared to the static stiffness. The effect of the helix angle on three different

transition frequencies has been investigated. This effect was adapted by the addition of a static preload.

Becker et al. (2002) investigated the effect of static axial compression upon the natural frequencies of helical springs by the transfer matrix method. The linear disturbance equations governing the resonant frequencies of a helical spring subjected to a static axial compressive load were solved numerically using the transfer matrix method for clamped ends and circular cross-section to produce frequency design charts. The effect of varying the number of turns of the spring was investigated, and in the limit of large numbers of turns, our results validate earlier work on the vibration of helical compression springs in which the helix was modeled as an elastic beam with rigidities corresponding to those of unclosed circular rings.

The pseudospectral method was applied by Lee (2007) to the free vibration analysis of cylindrical helical springs. The displacements and the rotations were approximated by the series expansions of Chebyshev polynomials and the governing equations were collocated. Numerical examples were provided for fixed–fixed, free–free, fixed–free and hinged–hinged boundary conditions.

Taktak et al. (2008) presented a finite element for the dynamic analysis of the cylindrical isotropic helical spring. The hybrid-mixed formulation was used to compute the stiffness matrix. A simple approach was used to calculate the mass matrix. These matrices were used for solving the dynamic equation of the spring to calculate natural frequencies and the dynamic response of a simple or an assembled spring for different types of cross-section.

A numerical solution was presented in Ayadi and Hadj-Taïeb (2008) to describe wave propagations in axially impacted helical springs. The governing equations for such problem were two coupled hyperbolic, partial differential equations of second order. The axial and rotational strains and velocities were considered as principal dependent variables. Since the governing equations were non-linear, the solution of the system of equations can be obtained only by some approximate numerical simulation. The finite element method was applied for the discrete formulation of the mathematical equations leading to a non-linear system of equations solved by an iterative Gauss substitution method.

In the paper of (Sorokin 2009), the validity ranges of alternative theories were assessed by comparison of the location of the dispersion curves and a rigorous asymptotic analysis of the exact dispersion equation with two small parameters. It allows for the identification of significant regimes of linear wave motion in a helical spring. In each of these regimes, simple formulae for wave numbers were obtained by the dominant balance method.

An analytical study for free vibration of naturally curved and twisted beams with uniform cross-sectional shapes was carried in Yu et al. (2010) out using spatial curved beam theory based on the Washizu’s static model. In the governing equations of motion of the beams, all displacement functions and the generalized warping coordinate were defined at the centroid axis and the effects of rotary inertia, transverse shear deformations and torsion-related warping were included in the proposed model. Explicit analytical expressions were derived for the vibrating mode shapes of a

curved, bending-torsional-shearing coupled beam under clamped–clamped boundary condition.

The paper (Leamy 2010) presents an efficient intrinsic finite element approach for modeling and analyzing the forced dynamic response of helical springs. The finite element treatment employs intrinsic curvature (and strain) interpolation vice rotation (and displacement) interpolation, and thus can accurately and efficiently represent initially curved and twisted beams with a sparse number of elements. The governing equations of motion contain nonlinearities necessary for large curvatures. In addition, a constitutive model was developed which captures coupling due to non-zero initial curvature and strain. The method was employed to efficiently study dynamically-loaded helical springs.

The governing equations and the associated natural boundary conditions of a pre-twisted helical beam with non-circular cross-sections have been derived in Leung (2010) from differential geometry and variational principles. For isotropic materials, the formulation was identical to the existing literature for helical beam with circular cross-sections. Explicit analytical expressions that give the vibrating mode shapes were derived in Yun and Hao (2011) by rigorous application of the Muller root search method was used to determine the natural frequencies. The free vibration analysis of cylindrical helical springs with noncircular cross-sections was carried out by means of an analytical study. In the governing equations of motion of a spring, all displacement functions and a generalized warping coordinate were defined at the centroidal principal axis. The effects of the rotational inertia, axial and shear deformations, including torsion-related warping deformations, were also considered in the formulations.

The paper (Frikha et al. 2011) proposes a physical analysis of the effect of axial load on the propagation of elastic wave in helical beams. The model was based on the equations of motion of loaded helical Timoshenko beams. The dimensionless for beams of circular cross-section and the number of parameters governing the problem was reduced to four (helix angle, helix index, Poisson coefficient, and axial strain) were derived and a parametric study was conducted. The outcome of loading was shown to be different in high, medium, and low frequency ranges.

The modeling of non-uniform springs were considered in Renno and Mace (2012). The uniform part of helical springs was modeled using the wave and finite element method since a helical spring can be regarded as a curved waveguide. This model was obtained by post-processing the finite element model of a single straight or curved beam element using periodic structure theory.

The paper (Hamza et al. 2013) discusses the vibrations of a coil in helical compression spring which was excited axially. The government equations form the system of four hyperbolic partial differential equations of first order with unknown variables, which were angular and axial deformations and velocities. The numerical resolution was based on the conservative finite difference scheme of Lax-Wendroff. The impedance method was applied to calculate the frequency spectrum. The spring was excited by a sinusoidal axial velocity at its end. The results obtained by using this method were used to analyze the evolution in time of deformations and velocities in different sections. The paper (Hamza et al. 2013) studies the resonance of axially

excited helical compression springs. The mathematical formulation of the dynamic behavior of the springs was composed of a system of four partial differential equations of first order hyperbolic type, which were the equations of momentum and the laws of constitution. The variables were angular and axial deformations and velocities. To calculate the frequency spectrum and to study the natural frequency response the impedance method was applied.

In the article (Yildirim 2012) a set of twelve linearized disturbance dynamic equations in canonical form was derived systematically and in a comprehensive manner based on the first order shear deformation theory to study the buckling and vibration analysis of helical coil springs made of isotropic linear materials. Those complete equations comprise the axial and shear deformation effects together with rotatory inertia effects. The special case of these equations corresponds also to the equations for straight and circular rods.

Principally, all cited models for transverse vibrations of the compressed coil springs could be used for study of buckling behavior of the springs as well.

Namely, the fundamental natural frequency of the transverse vibrations of the column is the function of the conservative axial force, as well as the variable length of the spring. If damping exists, the fundamental frequency was a complex number, which depends on the compressed length of the spring.

The criterion of a static equilibrium state is considered using perturbation method (Godoy 1999). A perturbation is introduced in the form of a vibration about the static equilibrium. In general, the study of a nonlinear dynamic response is required. The initial linear dynamic responses were indicators of the possible nonlinear dynamic behavior of the perturbed system. This leads to stability in the local sense.

3.8 Conclusions

The local dynamic criterion may be stated as follows: An equilibrium state is stable if, for small vibrations about such a state, all the frequencies of vibration are real.

Thus, the measure of the dynamics of the perturbed system is carried out by considering the frequencies of vibration. If at least one frequency of vibration is zero, we say that the equilibrium state is critical. In this case stability can only be evaluated using nonlinear vibrations. Thus, is the real part of fundamental natural frequency turns to be to zero, is the lateral buckling of the spring occurs. The loss of spring stability occurs by divergence.

If the loading of spring is nonconservative, the loss of stability may not show up by the system going into dynamic equilibrium state but by going into unbounded motion. If at least one frequency is imaginary, we say that the equilibrium state turns to be unstable. To cover this possibility the dynamic behavior of the system must be considered because stability is essentially a dynamic concept (Bolotin 1964).

The results of the actual study demonstrate the behavior of natural frequency of spring during its compression from free, undeformed state. The linear spring reduces its frequency with the gradually compression of the spring. The shorter the spring

in these peculiar regions, the lower will be the fundamental frequency. Is the fundamental natural frequency of transverse oscillations turns to be to zero, is the lateral buckling of the spring occurs. The condition of vanishing for the frequency of lateral vibration delivers the criterion of lateral stability if the spring. The static stability criterion corresponds to the standard static stability criterion (DIN EN 13906-2:2013 2013). The main advantage of the applied method consists in its accordance to the standard static formulation. This method delivers the known results for resonance frequency of transverse vibration the undeformed and compressed linear spring.

The previous consideration uses the model of linear spring without coil contacts. The nonlinear, progressive spring behaves differently. The coils of nonlinear spring come in contact and turn to be inactive. For the length of the spring the length of active coils must be considered. The resonant frequency of nonlinear progressive springs predominantly increases with the degree of compression. The shorter the spring, the higher will be usually the fundamental frequency. Nevertheless, even for progressive springs could exist the peculiar regions where the frequency reduces. Thus, the sufficiently thin progressive spring can suffer lateral buckling as well. For accurate investigation of these phenomena the numerical methods are applicable due to complexity of governing equations.

3.9 Summary of Principal Results

- The deformation of central line of the spring characterizes the deformation of helical wire of spring.
- This central line is considered as the “equivalent column”, which possesses closely the same bending, extension, and torsional stiffness as the helical spring itself.
- The static and dynamic equations of the “equivalent column” are derived.
- The closed form solutions for stability and vibration problems of helical springs are found.
- The stability of helical springs under compression and torsion loads is investigated.
- The closed form solutions for critical axial compression force and torque are derived.

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Chapter 4

Disk Springs



Abstract In the current chapter, we examine the disk spring using the models of thin and moderately thick isotropic shells. The equations developed here are based on common assumptions and are simple enough to be applied to the applied analysis. The analysis of thin-walled disk springs could be performed using basic spreadsheet tools, removing the need to perform an onerous finite element analysis. Further, the theory of linear and progressive disk wave springs is presented. This chapter is the fourth section of first part, which studies the design and modelling of springs.

4.1 Thick Shell Model for Disk Springs

4.1.1 Mechanical Models of Elastic Disk Springs

Disk spring (Belleville spring, disk spring washer), is well known for its configuration to nonlinearly withstand a large force with minimum deflection while storing a large amount of energy in relation to the space occupied. Load–displacement formula for a disk spring or a non-slotted disk spring was first developed by Timoshenko and Woinowsky-Krieger (1957), Almen and Laszlo (1936). The cited theory is based on experimental observations according to which the cross-section of the spring merely rotates about a center point (assumed to be on the middle line of the cross-section] without undergoing an appreciable deflection. The results of Almen and Laszlo agreed with Timoshenko in regarding the radial stresses as negligible and succeeded in calculating tangential stresses and displacements of a disk spring subjected to an axial load uniformly distributed on inner and outer edges. Based on this theory, the dimensions used to define the geometries refer to the middle line of the disk spring cross-section. The equations provided in DIN EN 16,984:2017–02 (2017) are sufficiently accurate for evaluating relative flat disk springs of moderate thickness. According to these equations, the deformation behavior of the disk spring is treated as a one-dimensional inversion of a circular ring of rectangular cross-section about an inversion center point. The resulting inverted stress condition is overlaid by a bending stress condition caused by the change in the cone angle resulting from the deflection

(Hengstenberg 1983). The cross-section of the disk spring remains rectangular so that force is always applied at the edges I and III.

Alternatively, Hubner (1982, 1984) obtained the equations of static response of disk springs using the shell theory. In the cited articles the static response of truncated disk springs under central load was obtained numerically using shooting method and the Runge–Kutta method. Axisymmetric conical shells under axial forces were investigated numerically in Hübner and Emmerling (1982). While disk springs are rather flat and thick, the shell theory is applicable for steep and thin shells. A geometrically nonlinear approach leads to the Reissner-Meissner equations, which allow the calculation of large deformations. These two nonlinear second order equations have been integrated by a matrix method.

The expressions for determining the stresses, strains and displacements of a truncated or complete thin conical shell with constant thickness and axisymmetric load distributed or concentrated along the meridian are presented in Tavares (1996).

The analysis of non-linear characteristic of a disk spring continues to interest engineers (Niepage et al. 1987; Niepage 1983). The disk springs with variable thickness were studied in Ferrari (2013), Rosa et al. (1998). A theoretical analysis on Belleville spring with varying thickness was presented in the study (Saini et al. 2007). The expression for linearly varying thickness used by La Rosa et al. (1998) was modified to incorporate a curvature factor and the corresponding load as well as stress equations are derived using the hypothesis of Almen and Laszlo.

In the paper Blom et al. (2009) the field of application of the variable-stiffness concept is extended to three-dimensional conical shells with arbitrary dimensions that can be fabricated using advanced fiber placement machines. Elastic stiffness tailoring of laminated composite panels by allowing the fibers to curve within the plane of the laminae has proven to be beneficial and practical for flat rectangular plate designs.

In this chapter, we investigate the disk springs using the variation methods. The methods consequently apply the kinematic hypothesis and equations of thin and moderately thick shells. The interesting feature of the presented theory is its possibility to calculate the disk springs of a meridional variable thickness and with free and elastically supported edges.

4.1.2 Geometry of Disk Spring in Undeformed State

Consider a truncated conical shell. Its middle surface in the free state is the frustum $\overline{\sigma}$ (Fig. 4.1). The inside radius of middle surface of free spring is r_i , the outside radius is r_e . The ratio of outer radius to inside radius reads:

$$\Delta = \frac{r_e}{r_i} > 0.$$

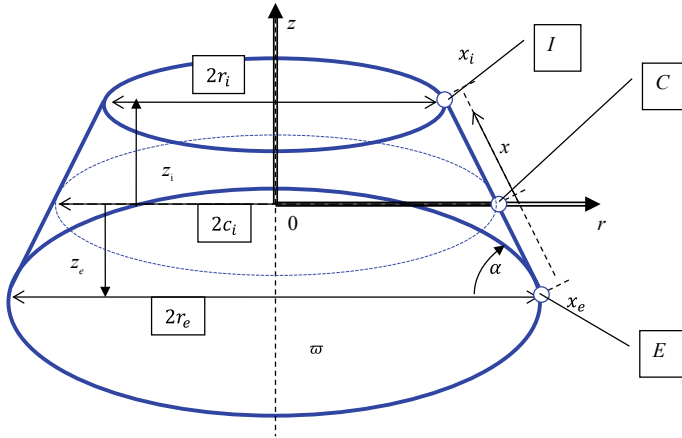


Fig. 4.1 The middle surface of the free disk spring

The slope angle α of the conical shell is constant. The position of the middle surface of the shell relates to a neutral plane C. In cylindrical coordinates (r, θ, z) , the middle surface of a free conical shell respectively is described parametrically:

$$r = c_i - x \cos \alpha, z = -x \sin \alpha. \quad (4.1)$$

The arc length x serves as the meridional coordinate:

$$x_e \leq x \leq x_i.$$

The arc length is measured from a certain inversion point c_i , where

$$x_i = \frac{c_i - r_i}{\cos \alpha} > 0, x_e = \frac{c_i - r_e}{\cos \alpha} < 0. \quad (4.2)$$

The distance from the point with coordinates (x, θ, y) to the axis of the undeformed cone is:

$$\tilde{r} = r + y \sin \alpha = c - x \cos \alpha + y \sin \alpha, \quad (4.3)$$

where y is the normal distance from the point to the middle surface of the shell¹:

$$-T/2 \leq y \leq T/2.$$

The thickness of the spring depends on the arc length $T = T(x)$.

From (4.1) and (4.2) follows:

¹For the material thickness the symbol T is used, because symbol t is reserved for time.

$$r_i = c_i - x_i \cos(\alpha), r_e = c_i - x_e \cos(\alpha), dr = \cos(\alpha) dx. \quad (4.4)$$

The heights of the inner and outer edges of the middle surface of the free shell are correspondingly (Fig. 4.1):

$$z_i = x_i \sin \alpha > 0, z_e = x_e \sin \alpha < 0. \quad (4.5a)$$

The total height of the middle surface of the unloaded disk spring is:

$$h = z_i - z_e = (\Delta - 1) r_i \tan \alpha. \quad (4.5b)$$

Correspondingly, the principal radii of curvature r_1 and r_2 for the unloaded conical shell are:

$$\frac{1}{r_1} = \frac{\sin \alpha}{r}, \frac{1}{r_2} = 0. \quad (4.6)$$

The width of the middle surface of the disk spring in its free state is:

$$b = r_e - r_i = (\Delta - 1) \cdot r_i.$$

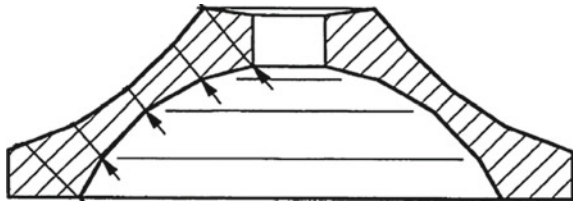
4.1.3 Mass of Disk Spring with Variable Material Thickness

The applied method is valid for an arbitrary thickness distribution over the frustum. We consider the variable thickness along the meridian: $T = T(x)$. Particularly, the thickness function could be represented as follows:

$$T = T_e \cdot \left(\frac{x - x_i}{x_e - x_i} \right)^{\tilde{p}} + T_i \cdot \left(\frac{x_e - x}{x_e - x_i} \right)^{\tilde{p}}, \tilde{p} > 0, T_e = T(x_e), T_i = T(x_i). \quad (4.7a)$$

If $\tilde{p} \geq 1$, the thickness in the middle of frustum ϖ reduces. Otherwise, if $1 > \tilde{p} > 0$ the thickness in the middle increases. The mass of the disk spring could be reduced using the convex or concave variable thickness (Fig. 4.2) (Kobelev and Hesselmann 2002)

Fig. 4.2 The disk spring with the variable thickness along the meridian (Kobelev and Hesselmann 2002)



2002). For the optimal design, the parameters T_e , T_i , $\tilde{p}$ serve as the design variables. If $\tilde{p} = 1$, the thickness is the linear function over meridian:

$$T = T_e \cdot \frac{x - x_i}{x_e - x_i} + T_i \cdot \frac{x_e - x}{x_e - x_i}. \quad (4.7b)$$

The average thickness of material in this case is $T^* = (T_i + T_e)/2$.

If $\tilde{p} = 0$, the thickness of material is constant and the cross-section of disk is rectangular. The mass of the disk spring with an axisymmetric variable thickness is:

$$m = \pi \rho_m \int_{x_e}^{x_i} T(x) r dx. \quad (4.8a)$$

For the thickness function (4.7a), the mass of the spring is equal:

$$m = \frac{\pi \rho_m \sqrt{h^2 + b^2}}{(p+1)(p+2)} [T_e \cdot (r_e p + r_e + r_i) + T_i \cdot (r_i p + r_e + r_i)]. \quad (4.8b)$$

For the linear thickness function (4.7b), the mass of the spring is equal:

$$m = \frac{\pi \rho_m \sqrt{h^2 + b^2}}{6} [T_e \cdot (2r_e + r_i) + T_i \cdot (r_e + 2r_i)]. \quad (4.8c)$$

Currently, the mainstream disk springs are manufactured with the constant thickness. Finally, for the constant thickness, the (4.8b) reduces to:

$$m = \pi \rho_m (r_i + r_e) \sqrt{h^2 + b^2} T. \quad (4.8d)$$

4.1.4 Load-Caused Alteration of Strain and Curvature

The main hypothesis of the model is that the slope angle ψ of deformed conical shells changes, but also remains constant over the meridian (Kobelev 2016). Namely, the point C is the inversion center point for the cross-section of the conical shell. The generatrice rotate about the point C, but remain straight and their lengths do not alter. The points of the middle surface, located on the neutral plane C, do not deflect. This hypothesis is essentially the same, as the hypothesis of Timoshenko and Almen and Laszlo. Analogous to the giant dislocation, which accompanies the distortion of helical springs, the deformation of disk springs is the sum of a giant twist and a giant wedge Volterra disclinations (Nabarro 1967). The middle surface of the shell in the deformed state is again a frustum Ω , as shown above on (Fig. 4.3).

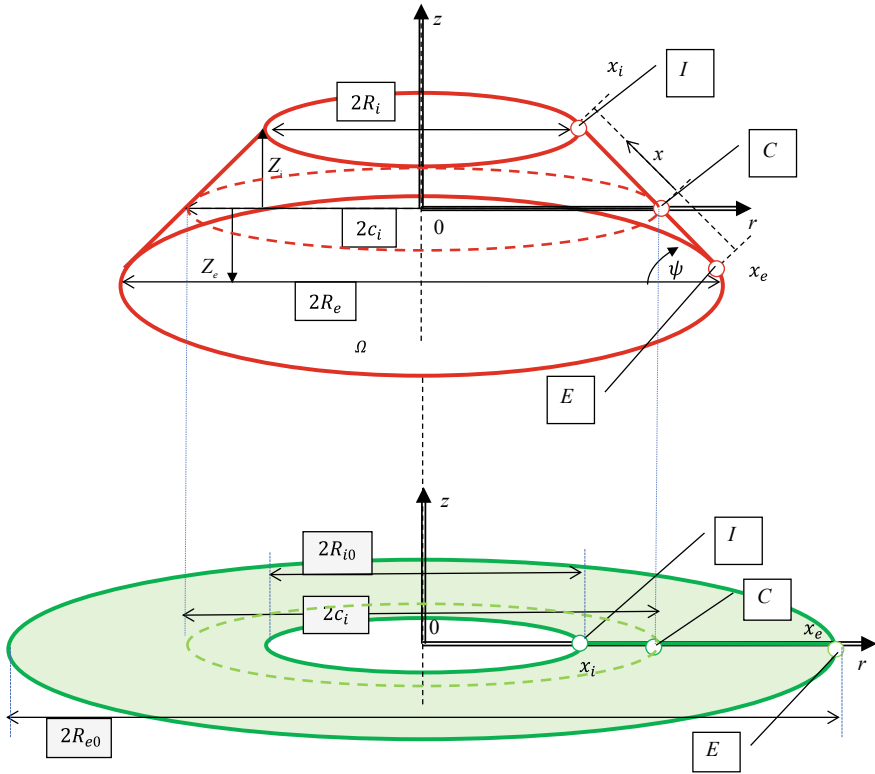


Fig. 4.3 The middle surfaces of the deformed and completely flattened disk spring

In cylindrical coordinates (r, θ, z) , the middle surface Ω is given by the parametric equations:

$$R = c_i - x \cos \psi, \quad Z = -x \sin \psi. \quad (4.9)$$

The distance from the point of the thick-walled cone with coordinates (x, θ, y) to the axis of the deformed cone is:

$$\tilde{R} = R + y \sin \psi = c_i - x \cos \psi + y \sin \psi. \quad (4.10)$$

Correspondingly, the principal radii of curvature R_1 and R_2 of the middle surface Ω are:

$$\frac{1}{R_1} = \frac{\sin \psi}{r}, \quad \frac{1}{R_2} = 0. \quad (4.11)$$

The inner and outer radii of Ω are equal to:

$$R_i = c_i - x_i \cos \psi, R_e = c_i - x_e \cos \psi. \quad (4.12)$$

The distances from the plane C to the inner and outer edges of surface Ω read:

$$Z_i = x_i \sin \psi, Z_e = x_e \sin \psi. \quad (4.13)$$

The height and width of middle surface in the deformed state Ω are correspondingly:

$$H = Z_i - Z_e = r_i \cdot \frac{\sin \psi}{\cos \alpha} \cdot (\Delta - 1), \quad (4.14)$$

$$B = R_i - R_e = r_i \cdot \frac{\cos \psi}{\cos \alpha} \cdot (\Delta - 1). \quad (4.15)$$

Upon specializing the strain displacement relations to the case of a conical shell with no transverse shear deformations the following expression for the circumferential mid-surface strain:

$$\varepsilon_1 = \frac{R - r}{r} = \frac{\cos \alpha - \cos \psi}{c_i - x \cos \alpha} x. \quad (4.16)$$

The circumferential curvature change is:

$$\kappa_1 = \frac{1}{R_1} - \frac{1}{r_1} = \frac{\sin \psi - \sin \alpha}{r}. \quad (4.17)$$

The slope angle vanishes in the flattened state, as shown below on (Fig. 4.3). With Eq. (4.2), the inner and outer radii of the plane circular strip will be equal nearly to:

$$R_{i0} = c_i - x_i, R_{e0} = c_i - x_e.$$

4.1.5 Disk Springs of Moderate Material Thickness

Considering the thickness of the material, the expressions for height and width must be corrected (Fig. 4.4). The total height and width of the disk spring in its free state from upper inside edge to lower outside are respectively:

$$\tilde{h} = h + T^* \cos \alpha = (\Delta - 1) \cdot r_i \tan \alpha + T^* \cos \alpha, \quad (4.18a)$$

$$\tilde{b} = b - T^* \sin \alpha = (\Delta - 1) \cdot r_i - T^* \sin \alpha. \quad (4.18b)$$

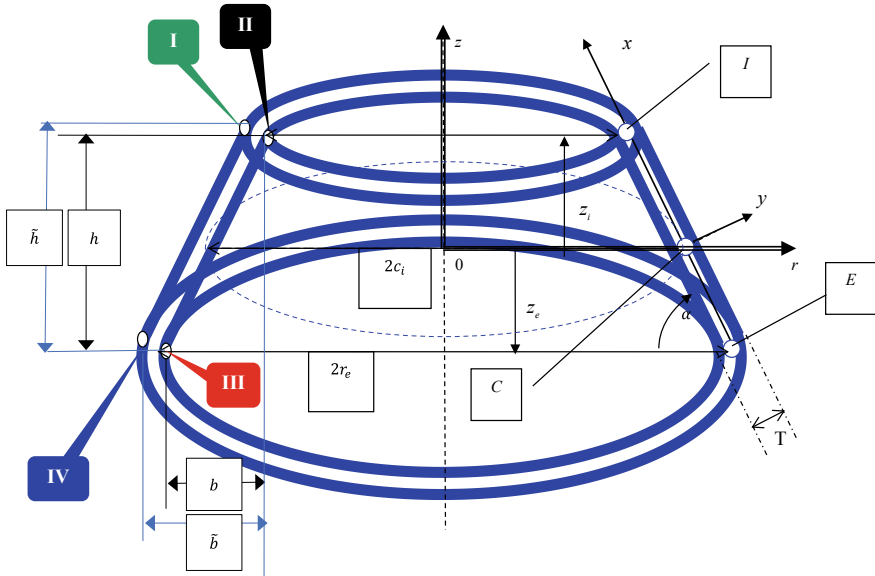


Fig. 4.4 The solid representation the free disk spring

The total height and width of the deformed disk spring from upper inside edge to lower outside are correspondingly:

$$\tilde{H} = H + T^* \cos \psi = r_i \cdot \frac{\sin \psi}{\cos \alpha} \cdot (\Delta - 1) + T^* \cos \psi, \quad (4.18c)$$

$$\tilde{B} = B - T^* \sin \psi = r_i \cdot \frac{\cos \psi}{\cos \alpha} \cdot (\Delta - 1) - T^* \sin \psi. \quad (4.18d)$$

The spring travel on the middle surface is equal to:

$$s = h - H = (\Delta - 1) \cdot r_i \cdot \frac{\sin \alpha - \sin \psi}{\cos \alpha}. \quad (4.19)$$

The spring travel from upper inside edge to lower outside is:

$$\tilde{s} = \tilde{h} - \tilde{H} = (\Delta - 1) \cdot r_i \cdot \frac{\sin \alpha - \sin \psi}{\cos \alpha} + T^* \cdot (\sin \psi - \sin \alpha). \quad (4.20)$$

For the constant thickness $T^* = T$. The influence of rounding on the corners could be also accounted without difficulty.

4.2 Disk Springs of Moderate Thickness

4.2.1 Deformation of Thick Conical Shell

The meridional and circumferential direct stresses relate to strains by means of relations (Marsden and Hughes 1994):

$$\sigma_1 = \frac{E}{(1+\nu)(1-2\nu)} ((1-\nu) \cdot E_1 + \nu \cdot (E_2 + E_3)), \quad (4.21a)$$

$$\sigma_2 = \frac{E}{(1+\nu)(1-2\nu)} ((1-\nu) \cdot E_2 + \nu \cdot (E_3 + E_1)), \quad (4.21b)$$

$$\sigma_3 = \frac{E}{(1+\nu)(1-2\nu)} ((1-\nu) \cdot E_3 + \nu \cdot (E_1 + E_2)). \quad (4.21c)$$

In Eqs. (4.21a), (4.21b) and (4.21c) we use the following notations for the principal strains:

- E_1 the circumferential strain,
- E_2 the meridional strain and
- E_3 the strain normal to surface of the shell.

All shear strains and shear stresses nullify due to the symmetry conditions. The hypothesis assumes the absence of the stress in the meridional and normal directions respectively:

$$\sigma_2 = 0, \sigma_3 = 0.$$

From these conditions the meridional strain E_2 and strain normal to surface of the shell E_3 depend solely on the circumferential strain E_1 :

$$E_2 = -\nu E_1, E_3 = -\nu E_1. \quad (4.22)$$

The strain displacement relations (4.16) and (4.17) lead to the following expressions for the strains in the solid elastic cone with the thickness $T(x)$:

$$E_1 = \varepsilon_1 + y\kappa_1 = \frac{(\cos \alpha - \cos \psi) \cdot x + (\sin \psi - \sin \alpha) \cdot y}{c_i - x \cos \alpha}, -\frac{T}{2} \leq y \leq \frac{T}{2}. \quad (4.23)$$

4.2.2 Variation Method for Thick Shell Models of Disk Springs

The total potential energy

$$\Pi = U_e + U_f \quad (4.24)$$

is the sum of the elastic strain energy U_e , stored in the deformed body and the potential energy U_f of the applied forces:

$$U_f = \int_{\alpha}^{\psi} M d\psi, \quad M = F_Z B - F_R H. \quad (4.5.25)$$

The shell is stressed by the forces in the direction of the rotation axis of the shell or by the forces in radial direction. Normally, the applied load is evenly distributed over the upper inside edge and lower outside edge (Fig. 4.5). The total axial force acting on the inner edge “I” is F_z and the total radial force is F_R .

The strain energy $U_e = U_1$ of the elastically deformed cone is:

$$U_1 = \pi G \int_{x_e}^{x_i} \left[\int_{-T(x)/2}^{T(x)/2} \left(E_1^2 + E_2^2 + E_3^2 + \frac{\nu}{1-2\nu} \cdot (E_1 + E_2 + E_3)^2 \right) r dy \right] dx, \quad (4.26)$$

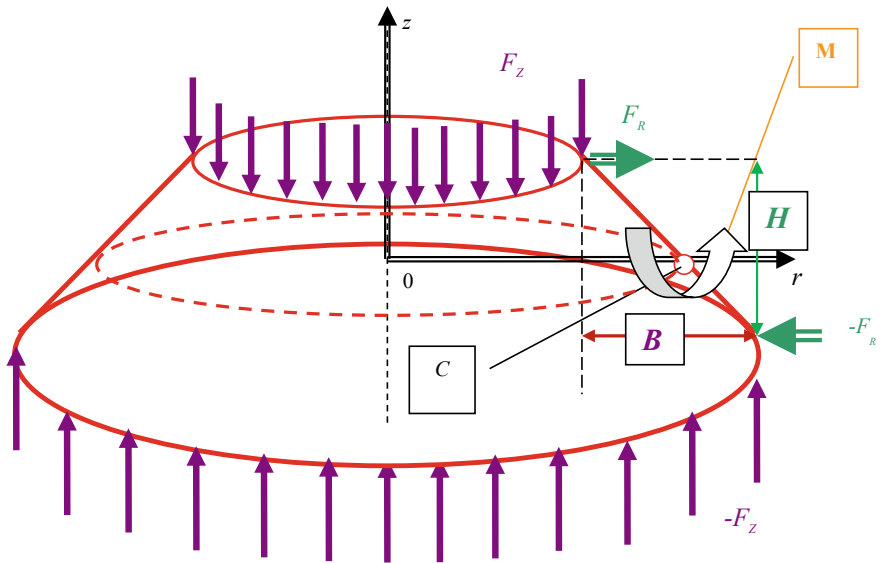


Fig. 4.5 Axial and radial forces on the disk spring

Using the Eqs. (4.21a), (4.21b), (4.21c) and (4.22) the expression (4.26) for the elastic energy for the nonconstant thickness reads:

$$U_1 = \pi E \int_{x_e}^{x_i} \left[\int_{-T(x)/2}^{T(x)/2} E_1^2 r dy \right] dx. \quad (4.27)$$

For the constant thickness, the integration (4.27) over the cross-section delivers:

$$U_1 = \frac{\pi E T^3}{12} \cdot \ln \Delta \cdot \frac{(\sin \psi - \sin \alpha)^2}{\cos \alpha} - \frac{\pi E T}{2} \cdot \frac{(\cos \psi - \cos \alpha)^2}{\cos^2 \alpha} \cdot [(\Delta^2 - 1)r_i^2 + 4(1 - \Delta)r_i c_i + 2c_i^2 \ln \Delta]. \quad (4.28)$$

The position of the neutral plane is determined from the condition of minimum for elastic energy with respect to c_i :

$$\frac{\partial U_1}{\partial c_i} = -\frac{\pi E T}{2} \cdot \frac{(\cos \psi - \cos \alpha)^2}{\cos^2 \alpha} \cdot [4(1 - \Delta)r_i + 4c_i \ln \Delta] = 0. \quad (4.29)$$

Solution of the Eq. (4.29) delivers the radius of the inversion center point C:

$$c_1 = \frac{\Delta - 1}{\ln \Delta} r_i \equiv \frac{D_e - D_i}{2 \ln \left(\frac{D_e}{D_i} \right)}. \quad (4.30)$$

According to the principle of virtual work, the external virtual work is equal to internal virtual work when equilibrated forces and stresses undergo unrelated but consistent displacements and strains. The necessary stationarity condition with respect to ψ delivers the circumferential moment:

$$M_1 = \frac{\partial U_1}{\partial \psi} = E r_i^3 \mu \frac{(\cos \psi - \cos \alpha) \sin \psi}{\cos^3 \alpha} \cdot \left[2(\Delta - 1) - \frac{1}{2}(\Delta^2 - 1) \right] + E r_i^3 \mu^3 \cdot \frac{(\sin \psi - \sin \alpha) \cos \psi}{12 \cos^3 \alpha} \ln \Delta, \mu = \frac{T}{r_i}. \quad (4.31)$$

In the absence of the radial force, the axial force on the shell is:

$$F_{1Z} = \frac{1}{B} \frac{\partial U_1}{\partial \psi} = \frac{M_1}{B}. \quad (4.32)$$

The substitution of (4.26) into (4.28) delivers the central formula for the axial force of disk spring F_{1Z} and the spring rate of disk spring c_{1Z} of the elastic cone:

$$F_{1Z} = F_{1Z}|_{c_i=c_1} = \pi E r_i^2 \cdot \frac{(F_e \mu + F_f \mu^3)}{\cos \psi}, c_{1Z} = \frac{dF_{1Z}/d\psi}{ds/d\psi} \quad (4.33)$$

$$F_e = \frac{2(1 - \Delta) + (1 + \Delta)\ln \Delta}{\ln \Delta} \cdot \frac{(\cos \psi - \cos \alpha)}{\cos^2 \alpha} \sin \psi, \quad (4.34)$$

$$F_f = \frac{\ln \Delta}{6(\Delta - 1)} \cdot (\sin \alpha - \sin \psi) \cos \psi. \quad (4.35)$$

The equivalent radial force F_{1R} , which leads to the same stored elastic energy of the spring, relates to the axial force F_{1Z} on the shell as:

$$F_{1R} = -\frac{1}{H} \frac{\partial U_1}{\partial \psi} = -\frac{M_1}{H} = -F_{1Z} \tan \psi. \quad (4.36)$$

If the applied load is evenly distributed over the upper inside edge and lower outside edge the expressions for force and spring rate must be corrected. Accounting the thickness of the material, the axial force $\tilde{F}_{1Z}$ and spring rate $\tilde{c}_{1Z}$ express:

$$\tilde{F}_{1Z} = \frac{1}{\tilde{B}} \frac{\partial U_1}{\partial \psi} = \frac{M_1}{\tilde{B}} = \frac{B}{\tilde{B}} F_{1Z}, \tilde{c}_{1Z} = \frac{d\tilde{F}_{1Z}/d\psi}{d\tilde{s}/d\psi}.$$

4.2.3 Comparison of Calculation Techniques

The next task is to compare the results, obtained by the variation methods for thin shell and solid cone with the formulas, derived by common stress method.

Firstly, in the total circumferential force in the stress method is derived using the integration of circumferential stress $\sigma_1 = E \cdot E_1$ over the cross-sectional surface:

$$F_{AL} = E \int_{x_e}^{x_i} \left[\int_{-T(x)/2}^{T(x)/2} E_1 dy \right] dx. \quad (4.38)$$

Performing the integration (4.38) over the cross-section with the constant thickness, one gets the total circumferential force:

$$F_{AL} = E \mu r_i \frac{(\cos \psi - \cos \alpha)}{\cos^2 \alpha} [c_i \ln \Delta - r_i (\Delta - 1)]. \quad (4.39)$$

The inversion center radius is determined from the condition of vanishing for circumferential force:

$$F_{AL} = 0.$$

From this condition results the position of the inversion center:

$$c_{AL} = \frac{\Delta - 1}{\ln \Delta} r_i \equiv \frac{D_e - D_i}{2 \ln \left(\frac{D_e}{D_i} \right)}. \quad (4.40)$$

The comparison of (4.30) and (4.40) demonstrates the equality both expressions for inversion center radii, obtained by variation and stress methods.

Secondly, the circumferential moment in the cross-section is the integral of circumferential stress $\sigma_1 = E \cdot \epsilon_1$ with the arm

$$\rho = x \sin(\alpha - \varphi) + y \cos(\alpha - \varphi)$$

over the cross-section. That is, the moment is the integral over the cross-sectional surface:

$$M_{AL} = E \int_{x_e}^{x_i} \left[\int_{-T(x)/2}^{T(x)/2} \rho \epsilon_1 dy \right] dx. \quad (4.41)$$

For the constant material thickness, the integration in (4.41) yields the moment:

$$\begin{aligned} M_{AL} = & E r_i^3 \mu \cdot \frac{(\cos \psi - \cos \alpha) \sin \psi}{\cos^3 \alpha} \cdot \left[2(\Delta - 1) - \frac{1}{2}(\Delta^2 - 1) \right] \\ & + E r_i^3 \mu^3 \cdot \frac{(\sin \psi - \sin \alpha) \cos \psi}{12 \cos^3 \alpha} \cdot \ln \Delta. \end{aligned} \quad (4.42)$$

Remarkably, that both methods lead to the same result and the expressions (4.31) and (4.41) match.

4.3 Statics of Thin Disk Springs

4.3.1 Forces and Moments in Disk Springs

In this Section the expression for the elastic energy is derived using the thin conical shell model. The meridional and circumferential direct forces and moments for thin shell relate to strains and curvature changes by means of relations (Ventsel and Krauthammer 2001, (14.32)):

$$N_1 = \frac{ET}{1 - \nu^2} (\epsilon_1 + \nu \epsilon_2), \quad N_2 = \frac{ET}{1 - \nu^2} \cdot (\epsilon_2 + \nu \epsilon_1), \quad (4.43)$$

$$M_1 = \frac{ET^3}{12(1-\nu^2)} \cdot (\kappa_1 + \nu\kappa_2), M_2 = \frac{ET^3}{12(1-\nu^2)} \cdot (\kappa_2 + \nu\kappa_1). \quad (4.44)$$

The values in Eqs. (4.43) and (4.44) are:

$$\begin{aligned} ET/(1-\nu^2) & \quad \text{the extensional rigidity of shell,} \\ ET^3/12(1-\nu^2) & \quad \text{the flexural rigidity.} \end{aligned}$$

The second hypothesis assumes the absence of the direct stress in the meridional direction:

$$N_2 = 0.$$

The third hypothesis assumes the absence of the moment in the meridional direction:

$$M_2 = 0.$$

From these conditions the meridional strain ε_2 and curvature change κ_2 depend solely on the circumferential strain ε_1 and curvature change κ_1 :

$$\varepsilon_2 = -\nu\varepsilon_1, \quad (4.45)$$

$$\kappa_2 = -\nu\kappa_1. \quad (4.46)$$

Substitution of (4.45) and (4.46) in (4.43) and (4.44) delivers the expressions for the circumferential stress σ_1 and moment M_1 :

$$\sigma_1 = \frac{N_1}{T} = E\varepsilon_1, \quad (4.47)$$

$$M_1 = ET^3\kappa_1/12. \quad (4.48)$$

4.3.2 Strain Energy of Thin Disk Springs

The strain energy of the shell can be determined by means of Kirchhoff's assumptions (Ventsel, Krauthammer, 2001, Eq. (14.51)):

$$U_2 = U_{2e} + U_{2f}, \quad (4.49a)$$

$$U_{2e} = 2\pi \int_{x_e}^{x_i} W_2^e r dx, U_{2f} = 2\pi \int_{x_e}^{x_i} W_2^f r dx. \quad (4.49b)$$

The extensional and flexural elastic energies in Eq. (4.49a) and (4.49b) respectively are (Libai and Simmonds 1988):

$$U_{2e} = \pi E \int_{x_e}^{x_i} T(x) [(\varepsilon_1 + \varepsilon_2)^2 - 2(1 - \nu)\varepsilon_1\varepsilon_2] r dx, \quad (4.50)$$

$$U_{2f} = \frac{\pi E}{12} \int_{x_e}^{x_i} T^3(x) [(\kappa_1 + \kappa_2)^2 - 2(1 - \nu)\kappa_1\kappa_2] r dx. \quad (4.51)$$

Using the formulas (4.45) and (4.46) in (4.50) and (4.51), the extensional and flexural surface densities of elastic energy reduce to:

$$2W_2^e = ET\varepsilon_1^2, \quad 2W_2^f = ET^3\kappa_1^2/12.$$

Accordingly, the extensional and flexural elastic energies are:

$$U_{2e} = \pi E \int_{x_e}^{x_i} r \varepsilon_1^2 T(x) dx, \quad (4.52)$$

$$U_{2f} = \frac{\pi E}{12} \int_{x_e}^{x_i} r \kappa_1^2 T^3(x) dx. \quad (4.53)$$

For the constant thickness, the integration (4.52) and (4.53) reduce the extensional and flexural elastic energy components respectively to:

$$U_{2e} = -\frac{\pi ET}{2} \cdot \frac{(\cos \psi - \cos \alpha)^2}{\cos^2 \alpha} \cdot [(\Delta^2 - 1)r_i^2 + 4(1 - \Delta)r_i c_i + 2c_i^2 \ln \Delta], \quad (4.54)$$

$$U_{2f} = \pi \frac{ET^3}{12} \ln \Delta \frac{(\sin \psi - \sin \alpha)^2}{\cos \alpha}. \quad (4.55)$$

The position of the neutral plane is determined from the requirement of minimum for elastic energy:

$$\frac{\partial U_2}{\partial c} = -\frac{\pi ET}{2} \cdot \frac{(\cos \psi - \cos \alpha)^2}{\cos^2 \alpha} \cdot [4(1 - \Delta)r_i + 4c_i \ln \Delta] = 0. \quad (4.56)$$

Solution of the Eq. (4.56) delivers the radius of the inversion center point C:

$$c_2 = \frac{\Delta - 1}{\ln \Delta} r_i \equiv \frac{D_e - D_i}{2 \ln \left(\frac{D_e}{D_i} \right)}. \quad (4.57)$$

This method delivers the same expression for inversion center radius as (4.30) and (4.40).

Finally, the circumferential moment is:

$$M_2 = \frac{\partial U_2}{\partial \psi} = Er_i^3 \mu \cdot \frac{(\cos \psi - \cos \alpha) \sin \psi}{\cos^3 \alpha} \cdot \left[2(\Delta - 1) - \frac{1}{2}(\Delta^2 - 1) \right] + Er_i^3 \mu^3 \cdot \frac{(\sin \psi - \sin \alpha) \cos \psi}{12 \cos^3 \alpha} \ln \Delta \quad (4.58)$$

Comparison demonstrates the identity of the expressions for moment (4.41) and (4.58). Consequently, it is proved that both methods lead to the same expressions for spring forces and spring rates.

For comparison the finite element simulation using program ANSYS (2020) was performed. For the finite element modeling three models were developed (Kobelev 2016). The finite element simulation with ABAQUS (2020) leads to the similar results.

4.3.3 Almen and Laszlo Method for Thin Disk Springs

The common calculation for the force of disk spring with the constant material thickness T is based on the Almen and Laszlo formulas (Almen and Laszlo 1936):

$$F_{zDIN} = \frac{4E}{1 - \nu^2} \cdot \frac{T^4}{K_1 D_e^2} \cdot \frac{s}{T} \cdot \left[\left(\frac{h}{T} - \frac{s}{T} \right) \cdot \left(\frac{h}{T} - \frac{s}{2T} \right) + 1 \right], c_{zDIN} = \frac{dF_{zDIN}}{ds} \quad (4.59)$$

These formulas are used in European standard (EN 16984:2017-02 2017). For comparison one example of spring calculation was considered. The parameters of the disk spring for the example are:

$$D_e = 110 \text{ mm}; D_i = 90 \text{ mm}; h = 6 \text{ mm}; T = 2 \text{ mm}, E = 200 \text{ GPa}; \nu = 0.3$$

The forces and spring rates for the example were calculated according the Eq. (4.59) for the DIN standard and are shown on the Figs. 4.6 and 4.7 with blue dashed lines. The forces and spring rates for the actual theory were calculated according the Eq. (4.33) are shown on the Figs. 4.6 and 4.7 with solid red lines.

The corner points according to standard (EN 16,984:2017–02 2017) are depicted on the Fig. 4.4. The coordinates of corner points are correspondingly:

$$I : \rho = D_i/2, y = T/2; \quad (4.60a)$$

$$II : \rho = D_i/2, y = -T/2, \quad (4.60b)$$

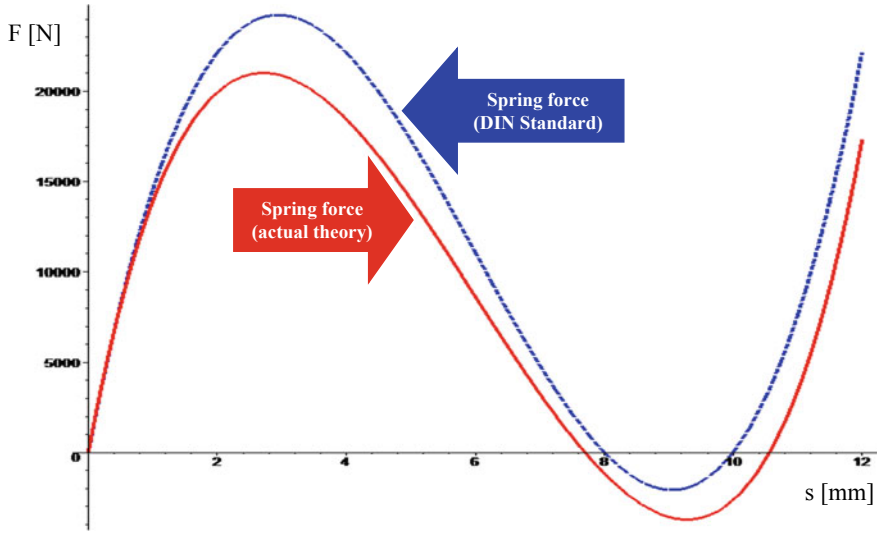


Fig. 4.6 Comparison of axial forces on the disk spring: DIN standard, Eq. (3.19), blue, actual theory, Eq. (4.13), red

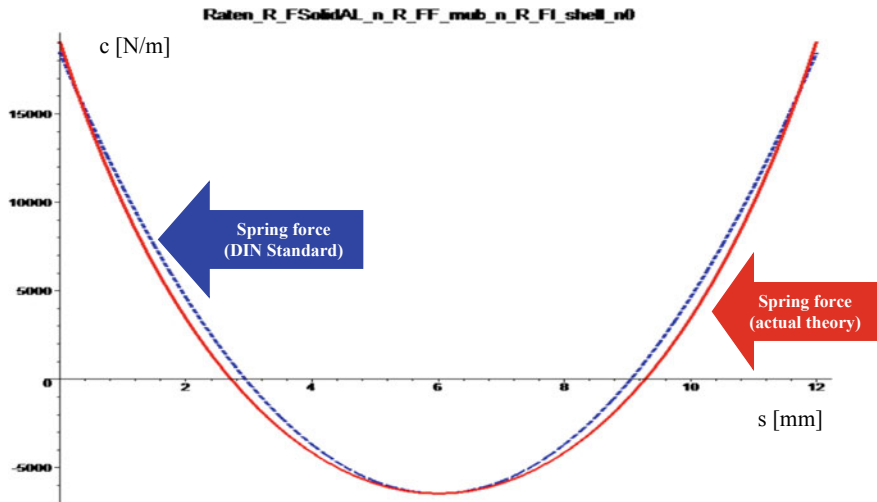


Fig. 4.7 Comparison of spring rates on the disk spring: DIN standard (blue) and actual theory (red)

$$III : \rho = D_e/2, y = -T/2; \quad (4.60c)$$

$$IV : \rho = D_e/2, y = T/2. \quad (4.60d)$$

The stresses in the spring on the corner points I,..IV are pure elastic and are equal to (DIN EN 16984:2017-02 [2017](#)):

$$\sigma_I = -\frac{4E}{1-\nu^2} \cdot \frac{T^2}{K_1 D_e^2} \cdot \frac{s}{T} \cdot \left[K_2 \cdot \left(\frac{h}{T} - \frac{s}{2T} \right) + K_3 \right], \quad (4.61)$$

$$\sigma_{II} = -\frac{4E}{1-\nu^2} \cdot \frac{T^2}{K_1 D_e^2} \cdot \frac{s}{T} \left[K_2 \cdot \left(\frac{h}{T} - \frac{s}{2T} \right) - K_3 \right], \quad (4.62)$$

$$\sigma_{III} = -\frac{4E}{1-\nu^2} \cdot \frac{T^2}{K_1 D_e^2} \cdot \frac{s}{T \Delta} \cdot \left[(K_2 - 2K_3) \cdot \left(\frac{h}{T} - \frac{s}{2T} \right) - K_3 \right], \quad (4.63)$$

$$\sigma_{VI} = -\frac{4E}{1-\nu^2} \cdot \frac{T^2}{K_1 D_e^2} \cdot \frac{s}{T \Delta} \cdot \left[(K_2 - 2K_3) \cdot \left(\frac{h}{T} - \frac{s}{2T} \right) + K_3 \right]. \quad (4.64)$$

The coefficients in the Eqs. (4.59) and (4.63) are DIN EN 16,984:2017–02 ([2017](#)):

$$K_1 = \frac{1}{\pi} \left(\frac{\Delta - 1}{\Delta} \right)^2 \cdot \left(\frac{\Delta + 1}{\Delta - 1} - \frac{2}{\ln \Delta} \right)^{-1}, \quad (4.65)$$

$$K_2 = \frac{6}{\pi \ln \Delta} \cdot \left(\frac{\Delta - 1}{\ln \Delta} - 1 \right), K_3 = \frac{3(\Delta - 1)}{\pi \ln \Delta}. \quad (4.66)$$

For the considered, ideal rectangular form of the spring the value of the coefficient is $K_4 = 1$.

The introduction of the “partial stresses” eases the estimation of effects of the circumferential bending and uniaxial strain. The “partial stresses” eases could be evaluated from DIN EN 16984:2017-02 ([2017](#)):

$$\sigma_{Bi} = -\frac{4E}{1-\nu^2} \cdot \frac{K_3 T^2}{K_1 D_e^2} \cdot \frac{s}{T}, \quad (4.67)$$

$$\sigma_{Be} = -\frac{4E}{1-\nu^2} \cdot \frac{K_3 T^2}{K_1 D_e^2 \Delta} \cdot \frac{s}{T}, \quad (4.68)$$

$$\sigma_{Ti} = -\frac{4E}{1-\nu^2} \cdot \frac{K_2 T^2}{K_1 D_e^2} \cdot \frac{s}{T} \cdot \left(\frac{h}{T} - \frac{s}{2T} \right), \quad (4.69)$$

$$\sigma_{Te} = -\frac{4E}{1-\nu^2} \cdot \frac{(K_2 - 2K_3) T^2}{K_1 D_e^2} \cdot \frac{s}{T \Delta} \cdot \left(\frac{h}{T} - \frac{s}{2T} \right). \quad (4.70)$$

The physical meaning of partial stresses is the following:

σ_{Be} is the stress due to bending on the outer diameter D_e ;

σ_{Bi} is the stress due to bending on the inner diameter D_i ;

σ_{Te} is the stress due to circumferential strain on the outer diameter D_e ;

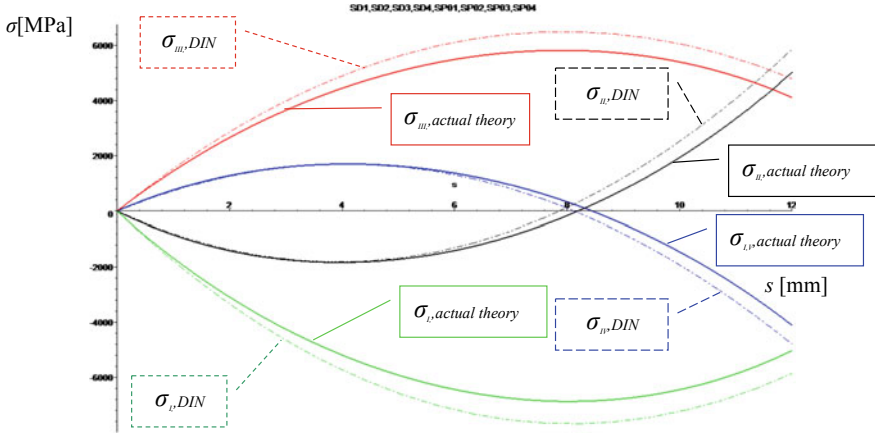


Fig. 4.8 Comparison of corner stresses: DIN standard (blue), actual theory (red)

σ_{Ti} is the stress due to circumferential strain on the inner diameter D_i .

The stresses on the corner points from Eqs. (4.61)–(4.64) relate to the “partial stresses” Eqs. (3.27)–(3.30):

$$\sigma_I = \sigma_{Bi} + \sigma_{Ti}, \sigma_{II} = -\sigma_{Bi} + \sigma_{Ti}, \sigma_{III} = -\sigma_{Be} + \sigma_{Te}, \sigma_{VI} = \sigma_{Be} + \sigma_{Te} \quad (4.68)$$

The stresses on the corner points for the example were calculated according the Eqs. (4.61)–(4.64) for the DIN standard and are shown on the Fig. 4.8 by dashed lines. The stresses for the actual theory were calculated according the Eq. (4.68) are shown on the Fig. 4.8 by solid lines.

4.3.4 Stresses in Disk Springs

The stresses in corner points could be evaluated in the shell theory of disk springs. For this purpose, we use the Eq. (4.1) for stress–strain dependence and Eq. (4.3) for circumferential strain. The expressions for “partial stresses” follow from these formulas after substitution of coordinates for the corner points (4.60a)–(4.60d):

$$\sigma_{Bi} = \frac{E}{1-\nu^2} \cdot \frac{\sin \alpha - \sin \psi}{2} \mu, \sigma_{Be} = \frac{\sigma_{Bi}}{\Delta}, \mu = \frac{T}{r_i}, \quad (4.69)$$

$$\sigma_{Ti} = \frac{E}{1-\nu^2} \cdot \frac{\cos \psi - \cos \alpha}{\cos \alpha} \cdot \frac{1 - \Delta + \ln \Delta}{\ln \Delta}, \sigma_{Te} = \frac{1 - \Delta + \Delta \ln \Delta}{(1 - \Delta + \ln \Delta) \Delta} \sigma_{Ti}. \quad (4.70)$$

With the Eqs. (4.69) and (4.70) for partial stresses, the stresses on the corner points follow from Eq. (4.68). The corresponding travel of the spring provide the Eqs. (4.19) and (4.20).

4.4 Disk Wave Springs

4.4.1 Application Fields of Disk Wave Springs

The disk wave springs are the corrugated springs, which comprise an annular spring band. The spring band describes a corrugation line oscillating about a reference surface over the circumference. Such corrugated springs are known as so-called “axial corrugated springs” or “disk wave springs” for absorbing axial forces in a spring fashion wherein the corrugation line oscillates over the circumference relative to a radial base plane. The radial base plane lies normal to the ring axis. In this case, the spring band is generally closed in a ring shape. Such corrugated springs are furthermore known as so-called “radial corrugated springs” in which the spring band oscillates over the circumference relative to a ring cylinder which lies coaxial to the ring axis. Corrugated springs of this type are often slotted over the circumference. Common corrugated springs have an almost linear spring characteristic with a constant spring state (spring stiffness, Fig. 4.9).

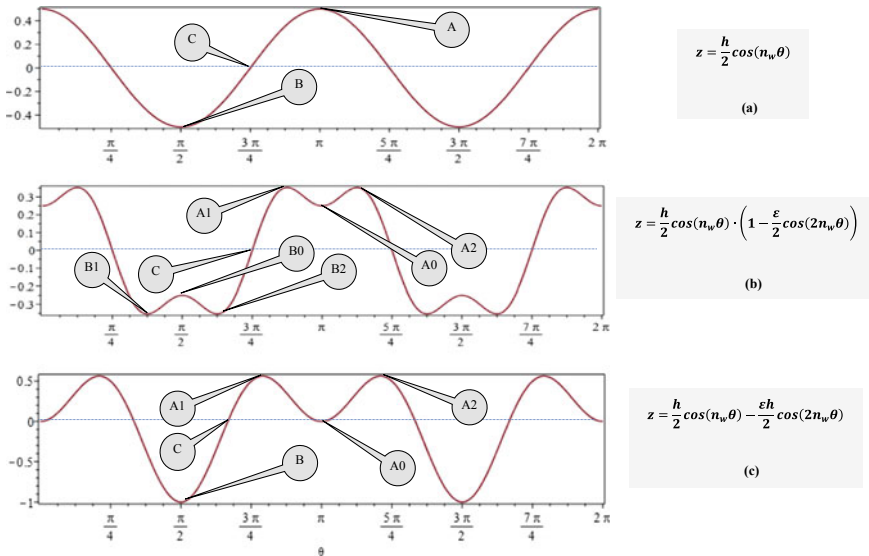


Fig. 4.9 Wave disk springs: **a** Linear **b** Nonlinear symmetric **c** Nonlinear asymmetric

The axial corrugated springs or axial radial corrugated springs are utilized as an elastic compensating disk in valve clearance compensating elements in the valve gear mechanism of internal combustion engines. In this case the valve clearance compensating elements are constructed as multi-part valve spring plates comprising a cup, a plate and an interposed corrugated spring.

A second preferred utilization for axial corrugated springs consists in their application as a clearance compensating and damping element in multiple-disk clutches of automatic gear mechanisms in motor vehicles. In this case, one or a plurality of corrugated springs are inserted at one end of the disk package of the multiple-disk clutches for coupling the gears, that is on the pressure side or the support side of the disk package. In this case, it is also possible to have a layered arrangement of a plurality of corrugated springs in the same orientation or in the opposite orientations.

For particular applications, certain gradually progressive spring characteristics are desirable. With known corrugated springs this has conventionally only been possible by aggregation of a plurality of corrugated springs have different spring characteristics with interposed flat disks.

The gradually progressive corrugated spring comprises a closed annular spring band which describes a corrugation line oscillating about a radial reference plane over the circumference (Kobelev et al. 2003) (Fig. 4.10). The corrugation line has a plurality of maxima of different height and a plurality of minima of different height,

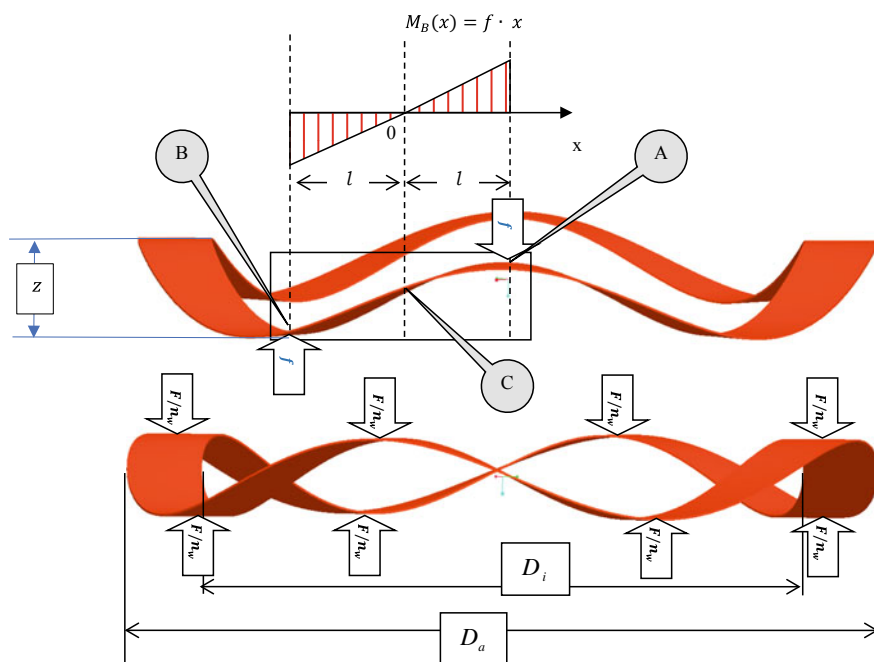


Fig. 4.10 Forces on common linear wave disk spring

such that the corrugation line comprises a plurality of periods over the circumference. Between two adjacent absolute maxima there is a relative minimum, and wherein between two adjacent absolute minima there is a relative maximum. Every second absolute maximum is immediately adjacent to an absolute minimum and vice versa around the circumference.

4.4.2 Design Formulas for Linear Disk Wave Springs

The central surface of the axial corrugated spring is a curved circular surface with the width:

$$B = (D_e - D_i)/2 \equiv r_e - r_i \quad (4.71)$$

and with the mean diameter

$$D = (D_e + D_i)/2 \equiv r_e + r_i. \quad (4.72)$$

The hollow circular surface is represented for simplification by a flat straight bending strip. The middle surface of the disk wave spring is developable ruled surface. The number of waves is an even integer number n_w , such that the length of a wave is:

$$l_w = \pi D / n_w. \quad (4.73)$$

If the height of the middle surface is h , the directrix line of the middle surface in cylindrical coordinates $\{r, \theta, z\}$ reads (Fig. 4.9a):

$$r = \frac{D}{2}, \theta = 0..2\pi, z = \frac{h}{2} \cdot \cos(n_w \theta).$$

The mass of the disk wave spring is:

$$m = m_0 \cdot \varrho,$$

where ϱ is the auxiliary function:

$$\varrho = \frac{4h}{\pi D} \cdot \mathbf{E} \left(\frac{1}{\sqrt{1 + \left(\frac{D}{hn_w} \right)^2}} \right) \cdot \sqrt{1 + \left(\frac{D}{hn_w} \right)^2} \approx 1 + \frac{h^2 n_w^2}{4D^2} - \frac{3h^4 n_w^4}{64D^4},$$

$\mathbf{E}(m)$ is the complete elliptic integral of the second kind (App. B) and

$$m_0 = \pi \rho_m T D B$$

is the mass of the flat disk spring with the same projected area $\pi D B$ and material thickness T .

In the flattened state the projected area of disk wave spring increases in size. The scaling factor is ϱ . Thus, the diameters of flattened strip are: ϱD_e , ϱD_i , ϱD :

For the calculation, the oscillating corrugated shape of disk wave spring is represented by an almost flat bending strip. The arc effects are neglected for stiffness calculation. On the upper and lower tips of waves the disk wave spring contacts to the flat support pistons. Due to the contact reactions, the strip is loaded with pairwise opposite forces. The forces act normally to the strip surface on the upper and lower tips of waves. Overall, the n_w spring forces act perpendicularly downwards and n_w forces perpendicularly upwards. Since the total force at the corrugated spring is F , the partial force acts in one or the other direction at each attachment point reads:

$$f = \frac{F}{n_w}. \quad (4.74)$$

The calculation of the spring rate of the conventional, linear disk wave spring is based on the theory of thin beams. Consider the partial section between two adjacent abutment points (Fig. 4.10). The length of the strip between two adjacent abutment points A and B is:

$$l = \frac{l_w}{2} = \frac{\pi D}{2n_w}. \quad (4.75)$$

The bending moment in the cross-section of the corrugated spring disappears in the point C lying centrally between points A and B. At this point, we place a local coordinate system with the axis x in the circumferential direction of the bending beam. The bending moment as a function of the distance to the point C is:

$$M_B(x) = fx, 0 \leq x \leq l. \quad (4.76)$$

The stored elastic energy of the part portion of the elastic plate between points C and A of elastic shaft using (4.76) reads:

$$W = \int_0^{l/2} \frac{M_B^2}{2EI} dx = \frac{1}{2EI} \int_0^{l/2} (fx)^2 dx = \frac{1}{2EI} \frac{f^2}{3} \left(\frac{l}{2}\right)^3 \quad (4.77)$$

The second derivative of the stored elastic energy (4.77) yields the spring rate of a single subsection:

$$\frac{1}{c_a} = \frac{\partial^2 W}{\partial f^2} = \frac{1}{EI} \frac{l^3}{24} = \frac{1}{EI} \left(\frac{\pi D}{n_w}\right)^3 \frac{1}{192}. \quad (4.78)$$

The total spring rate of the linear wave spring with n_w total waves is:

$$c_w = n_w c_a = \frac{192 n_w^4 E I}{\pi^3 D^3}. \quad (4.79)$$

The surface moment of inertia of the bending beam with the moderate width B and the material thickness T is

$$I = \frac{BT^3}{12} = \frac{D_e - D_i}{2} \frac{T^3}{12}. \quad (4.80)$$

For considerably wide springs the formulas for flexural rigidity of thin shells must be applied (Ventsel and Krauthammer 2001):

$$I = \frac{BT^3}{12(1 - \nu^2)} = \frac{D_e - D_i}{2} \cdot \frac{T^3}{12(1 - \nu^2)}.$$

The spring rate of the linear disk wave spring of a small annular width for small deflections derived with the simplest assumption of an elongated rectangular bending beam. After the substitution (4.80) in (4.79) the spring rate of the conventional, linear disk is:

$$c_w = 16 E B n_w^4 \cdot \left(\frac{T}{\pi D} \right)^3 \equiv \frac{64}{\pi^3} \cdot \frac{D_e - D_i}{(D_e + D_i)^3} \cdot E n_w^4 T^3. \quad (4.81)$$

The stresses in the disk wave spring oscillate along the circumference of the spring. Maximum bending moments in the cross-section of the corrugated spring are located at the extreme points below the force application points:

$$M_B = \pi F \cdot \frac{D_e + D_i}{n_w}. \quad (4.82)$$

The moment attains its maximum M_B exactly below the force application points. The stress calculation with (4.82) lead to the highest bending stresses at these points:

$$\sigma = \frac{Fl}{8W_B} = \frac{6\pi D}{8BT^2} \cdot \frac{F}{n_w} = \frac{3\pi}{4T^2} \cdot \frac{D_e + D_i}{D_e - D_i} \cdot \frac{F}{n_w}.$$

The bending moments disappear in the reversal regions between the force application points.

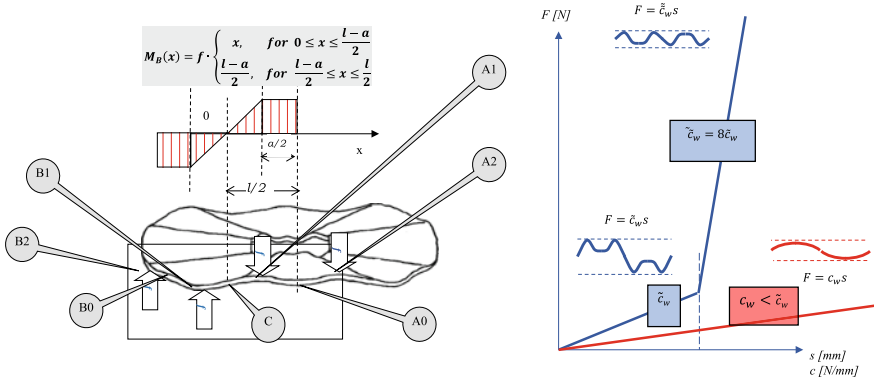


Fig. 4.11 Non-linear wave spring. The spring forces of linear and non-linear wave spring

4.4.3 Design Formulas for Non-Linear Disk Wave Springs

Progressive wave springs have a gradually progressive characteristic. The wave line has a plurality of maxima of different sizes and a plurality of minima of different sizes. A whole wave section consists of four sub-sections, two sub-sections are connected in parallel and two are connected in series (Fig. 4.11). The spring rate of two partial sections that connected in parallel is exactly twice as high as the spring rate of a single partial section. The spring rate in the series connected partial sections is half the spring rate of a section. Consequently, the spring rate of a full shaft is equal to the spring rate of a subsection.

There are two types of the non-linear disk wave springs. For the symmetric spring the directrix line of the middle surface in cylindrical coordinates $\{r, \theta, z\}$ reads (Fig. 4.9b):

$$r = \frac{D}{2}, \theta = 0..2\pi, z = \frac{h}{2} \cos(n_w \theta) \cdot \left(1 - \frac{\varepsilon}{2} \cos(2n_w \theta)\right).$$

Parameter $\varepsilon > 0$ determines the depth of the secondary wave. For the asymmetric spring the directrix line of the middle surface in cylindrical coordinates $\{r, \theta, z\}$ delivers (Fig. 4.9c):

$$r = \frac{D}{2}, \theta = 0..2\pi, z = \frac{h}{2} \cos(n_w \theta) - \frac{\varepsilon h}{2} \cos(2n_w \theta).$$

The mechanical function of both springs is similar. We explain for briefness only the function of the symmetric disk wave spring. If corrugated springs of this type are installed between two uniform contact surfaces and become increasingly loaded, the tips of waves initially come into contact with the contact surfaces with their absolute maxima and absolute minimums (Fig. 4.12a). The forces act in the vertical direction from the top downwards in the points A1 and A2 and from the bottom upwards in the

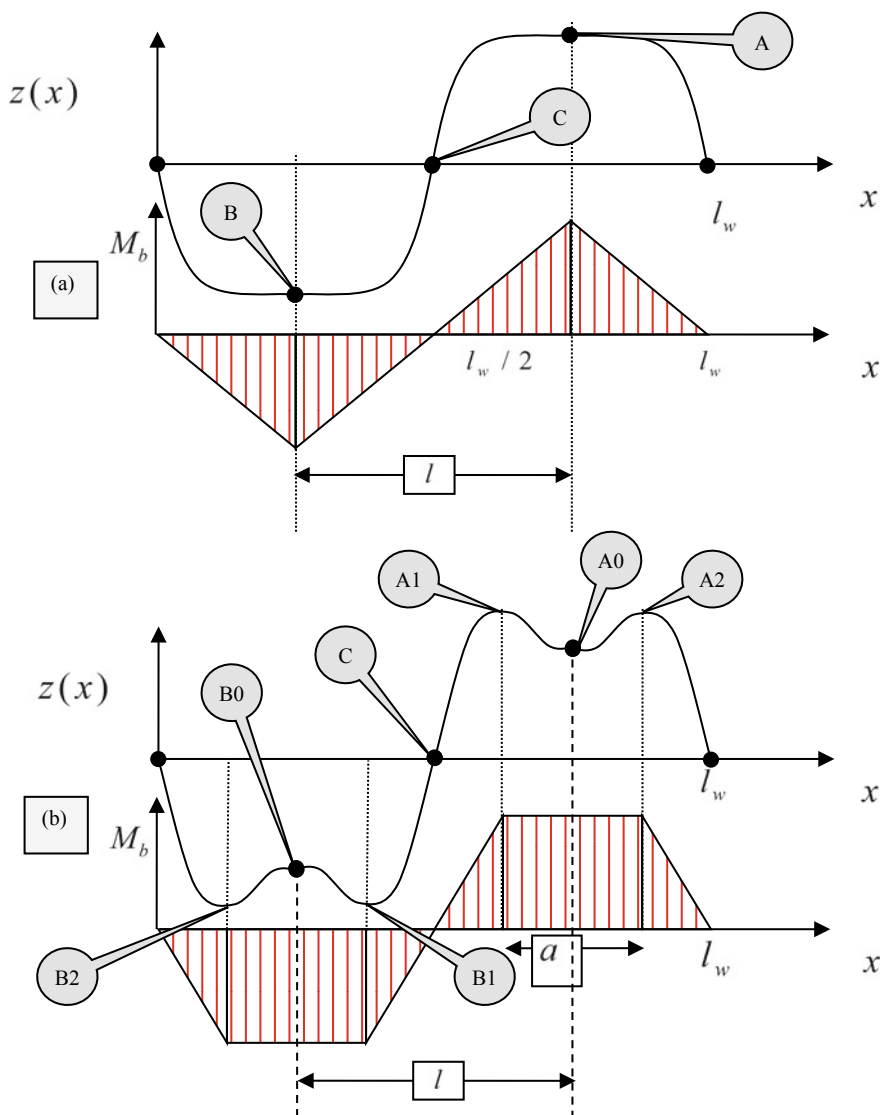


Fig. 4.12 Comparison of bending moments in the linear and progressive wave disk springs

points B1 and B2. Distance in the circumferential direction between points A1 and A2 is referred to as $2a$. The distance between the points B1 and B2 is the same. We now set the point A0 centrally between the points A1 and A2 and correspondingly the point B0 centrally between the points B1 and B4. To calculate the spring rate, we consider the section between two adjacent abutment points A0 and B0 (Fig. 4.12b). The bending moment in the cross-section of the strip vanishes at the point C, because

the point C is positioned centrally between points A0 and B0. At the point C there is a local coordinate system with the axis x in the circumferential direction. The bending moment as a function of the distance to the point C is:

$$M_B(x) = \begin{cases} fx, & \text{if } 0 \leq x \leq \frac{l-a}{2}, \\ f\frac{l-a}{2}, & \text{if } \frac{l-a}{2} \leq x \leq \frac{l}{2}. \end{cases} \quad (4.83)$$

The stored elastic energy of the beam portion between points C and A is:

$$\begin{aligned} U_e &= \int_0^{\frac{l}{2}} \frac{M_B^2}{2EI} dx = \frac{f^2}{2EI} \left[\int_0^{\frac{l-a}{2}} x^2 dx + \int_{\frac{l-a}{2}}^{\frac{l}{2}} \left(\frac{l-a}{2} \right)^2 dx \right] \\ &= \frac{f^2}{16EI} \left[\frac{(l-a)^3}{3} + (l-a)^2 a \right]. \end{aligned} \quad (4.84)$$

The second derivative of the stored elastic energy (4.84) yields the spring rate of a single wave section:

$$\frac{1}{c_a} = \frac{\partial^2 U_e}{\partial f^2} = \frac{1}{8EI} \left[\frac{(l-a)^3}{3} + (l-a)^2 a \right] = \frac{1}{192EI \cdot k_w(\lambda_w)} \left(\frac{\pi D}{n_w} \right)^3. \quad (4.85)$$

Here $\lambda_w = a/l$ is the length ratio and

$$k_w(\lambda_w) = \frac{1}{(1 - \lambda_w)^2 (1 + 2\lambda_w)} \quad (4.86)$$

is the correction factor. This correction factor accounts the bending moment (4.83) for the double-curved form of half-wave.

The disk wave spring consists exactly of four identical sections. The initial (primary) spring rate of the progressive wave spring that comprises n_w waves is:

$$\tilde{c}_w = n_w c_a = \frac{192 n_w^4 EI}{\pi^3 D^3} k_w(\alpha). \quad (4.87)$$

The primary spring rate for small deflections and a small annular width, with the simplest assumption of an elongated rectangular bending beam using (4.80), is therefore:

$$\tilde{c}_w = 16EBn_w \left(\frac{Tn_w}{\pi D} \right)^3 k_w(\lambda_w) \equiv \frac{64}{\pi^3} \frac{D_e - D_i}{(D_e + D_i)^3} En_w^4 T^3 k_w(\lambda_w). \quad (4.88)$$

The initial (primary) spring rate of the double wave form is greater than the spring rate of the conventional wave spring. For example, for $a = l/2$ and

$$\lambda_w = a/l = 1/2,$$

if follows from (4.86) that

$$k_w = 2.$$

The primary spring rate of the considered non-linear wave spring is exactly twice the spring rate of the linear wave spring with the same number of full waves. The spring rate increases because the length of elastic bending section of the disk wave spring with the double wave is shorter than the length of elastic bending section of the linear disk wave spring. For the linear spring the bending happens between the tips of sinusoidal oscillating curve. Contrarily, the elastic bending of the disk wave spring with the double wave occurs between the pairs of maxima and minima. That is the conventional, linear disk wave spring with the simple wave form is two times more flexible than the disk wave spring with the double wave.

The disk wave spring with the double wave initially have a linear characteristic with a first lower spring rate, namely primary spring rate $\tilde{c}_w$. With the increasing of force they come into contact with the next larger relative maxima and relative minima at the contact surfaces. At this moment, the spring characteristic of spring changes to a substantially linear spring force. The disk wave spring with the double wave possesses a markedly higher spring rate after the contact event. At moment of the contact, the spring rate jumps to the secondary spring rate $\tilde{\tilde{c}}_w$. The calculation of the secondary spring rate calculation is as follows. At new contact points A0 and B0 suddenly emerge two reaction forces. The number of waves doubles. From initially existing n_w double waves in contact moment develop $2n_w$ simple waves (Fig. 4.11). The new (secondary) spring rate reads:

$$\tilde{\tilde{c}}_w = 16Eb \left(\frac{2n_w T}{\pi D} \right)^3 2n_w \equiv \frac{64}{\pi^3} \cdot \frac{D_e - D_i}{(D_e + D_i)^3} \cdot 16En_w^4 T^3. \quad (4.89)$$

The ratio of primary spring rate (4.88) and secondary spring rate (4.89) at the moment of contact is:

$$\frac{\tilde{\tilde{c}}_w}{\tilde{c}_w} = \frac{16}{k_w(\alpha)}.$$

For a practically relevant corrugated spring the length ratio is:

$$\lambda_w = 1/2.$$

The most important case is the disk wave spring with the double wave. For this type spring, the ratio of the spring rate after the contact to the spring rate before the contact is exactly eight (Fig. 4.11). The material utilization of the linear disk spring corresponds approximately to the material utilization of a simple leaf spring with

a constant cross-section. The non-linear wave spring possesses a somewhat lower material utilization.

4.5 Conclusions

The developed method could be directly extended to slotted disk springs. A slotted disk spring consists of two segments: a disk segment and a number of lever arm segments. The displacements of a disk spring subjected to an axial load uniformly distributed on inner and outer edges. Currently, the calculation of slotted disk spring is based on the SAE formula (SAE 1996). This formula is limited to a straight slotted disk spring. The dimensions of a disk spring simply refer to the edges of the disk spring cross-section. The same dimensions were also used by Schremmer (1973) as an attempt to propose a new formula for a limited straight slotted disk spring.

In the study (Fawazi et al. 2011), a load–displacement formula for the slotted disk spring is newly developed in the form of energy method by considering both rigid and bending deflections of the two segments.

The slotted disk springs are the essential parts of automatic gear boxes and clutches of passenger cars. The design of a novel friction clutch of exploits the Belleville spring to increase the friction area during operation (Shen and Fang 2007). The load on spring is reduced at a given transmitted torque. Due to the increasing of friction area, the Belleville spring can also act as a friction plate, and the components required for the clutch can be reduced. The maximum transmittable torque of the clutch is easily adjusted by varying the preload on the Belleville spring.

As mentioned above, the results of the current study could be immediately extended to account the slotted disk springs. Namely, the analytical formulas, developed in the actual manuscript, improve the known formulas for the disk segment of the slotted disk springs for the considerably large cone angles and spring travels, because the actual formulas are not based on the usual power series truncation.

Engineering applications of the current theory potentially include Bellville springs and slotted disk springs for automotive and industrial applications with moderate flatness and variable meridional thickness.

4.6 Summary of Principal Results

- The nonlinear governing equations for the disk spring were derived.
- The equations describe the deformation and stresses of thin and moderately thick disk springs with the variable material thickness.
- The variation method is applicable for the springs with free and elastically supported edges.
- The spring rate and stresses for the corrugated spring were derived.

- The behavior of the nonlinear corrugated springs is explained and the expressions for spring forces and spring rate are derived.

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Chapter 5

Thin-Walled Rods with Semi-opened Profiles



Abstract The semi-opened cross tubular sections are manufactured from the standard tubes by their flattening. For the reason that the significant variation of mechanical parameters along the length is possible, the flattening could be easily adjusted for optimal design. From the design viewpoint, the semi-opened cross tubular sections fill the gap between classical closed and open one-wall sections. Due to the high variability of the section's geometry, the simple analytical model is essential for primary design purposes and estimation of numerous opposing static effects. The effective torsion stiffness r_t , effective bending stiffness r_z and effective bending spring rate r_c of the twist-beam in terms of section properties of the twist-beam with the semi-opened cross-section was expressed with the analytical formulas. This chapter is the final section of first part, which studies the elastic elements from viewpoint of modeling and design.

5.1 Thin-Walled Rods with Semi-opened Profiles

5.1.1 Open, Closed and Semi-opened Wall Sections

A thin-walled beam is an extraordinary type of the torsion spring. This type of torsion spring serves as the cross-connection member in several industrial and automotive structures. The thin-walled beam is an elongated elastic structural element whose distinctive geometric dimensions are all different orders of magnitude. Thin-walled beams can be classified by their geometric features. Two classes of thin-walled beam cross-sections are notable, namely thin-walled beams with the open cross-section and thin-walled beams with the closed cross-section (Vlasov 1961; Timoshenko 1945; Flügge and Marguerre 1950). Firstly, an intermediate class of thin-walled beam cross-sections is studied. The cross-section of the beam is closed, but the shape of cross-section is elongated and curved. The walls, which form the cross-section, are nearly equidistant. The St Venant's free torsion behavior of the beam is same as the behavior of closed cross-section beams. However, the total warping function of the semi-opened cross-section is analogous to the warping function of open cross-section beam.

The difference between thin-walled rods is the presence of warping of cross-section and the corresponding force factor, the bi-moment. It is shown, that the thin-walled beams with closed profiles behave differently, depending on the form of the cross-section. There exist two types of the thin-walled beams with closed profiles. The first type of thin-walled beams possesses closed profiles with nearly equal diameters in all directions, so that there is no distinguished direction. The torsion of such closed beams is predominantly of StVenant's type. The effect of bi-moment on the twist behavior of the thin-walled beams of the first type is in most cases negligible in comparison to the StVenant's torsion. The second type of thin-walled beams embraces closed profiles formed by equidistant walls, elongated profiles, and star-like profiles. The contribution of StVenant's torsion stiffness of beams with this type of profile is of the same order of magnitude, as the contribution of sectorial stiffness and the effect of bi-moment should be accounted.

Further, the analytical methods design and analysis of a rear twist-beam axle in the concept design phase are reported. The twist-beam axle is a specific type of semi-solid suspensions. The twist-beam axles are commonly designed with the purpose of improving the vehicle dynamics performance, improving load carrying capacity, reducing weight and cost. The principal element of common twist-beam axle is the thin-walled twist beam. This beam behaves elastically, delivering certain roll rate for the vehicle, but must be sufficiently stiff for bending and guarantee the chamber and lateral stiffness of the vehicle. Insufficient camber and lateral stiffness cause oversteering of the vehicle, unacceptable from viewpoint of vehicle dynamics.

Two classes of thin-walled beam structures are usually distinguished: open and closed (Timoshenko and Gere 1961; Chilver 1967; Librescu and Song 2006). When the locus of points defining the centerline of the wall forms a closed or an open contour (Fig. 5.1), then a thin wall section is said to be closed or open. The thin-walled beam, shown on Fig. 5.1a possesses an open cross-section. Figure 5.1b demonstrates the beam with the closed cross-section. The cross-section of the beam, shown on Fig. 5.1c, will be referred to as semi-opened.

Consider a slender thin-walled structure of cylindrical or prismatic uniform cross-section formed by nearly equidistant walls (Figs. 5.1 and 5.2). Let T be a wall thickness assumed for the beginning to be constant along the beam span but variable along the contour of the cross-section $T = T(s)$, δ is the distance between the walls and L is length of the beam. The middle surface of the wall is defined as the locus of the points equidistant from the upper and the lower bottom surface of the wall. The middle surface is a noncircular cylindrical surface. The straight lines on the middle surface parallel to the beam longitudinal axis are the generators of the middle surface. The intersection of the middle surface with a plane normal to the generators determines the midline of the cross-section contour. The midline of the open cross-section L_m (Fig. 5.1a) is a plane limited line segment bounded by two end points A_0, A_1 . The midline of the closed cross-section is a closed curve (Fig. 5.1b, c). One defines the width of the curve in each direction to be the perpendicular distance between the parallels perpendicular to that direction. The maximal width of the curve is usually defined as its diameter. We denote the most distant points on the curve as B_0, B_1 and refer to them as pole points of the closed mid-line of the cross-section.

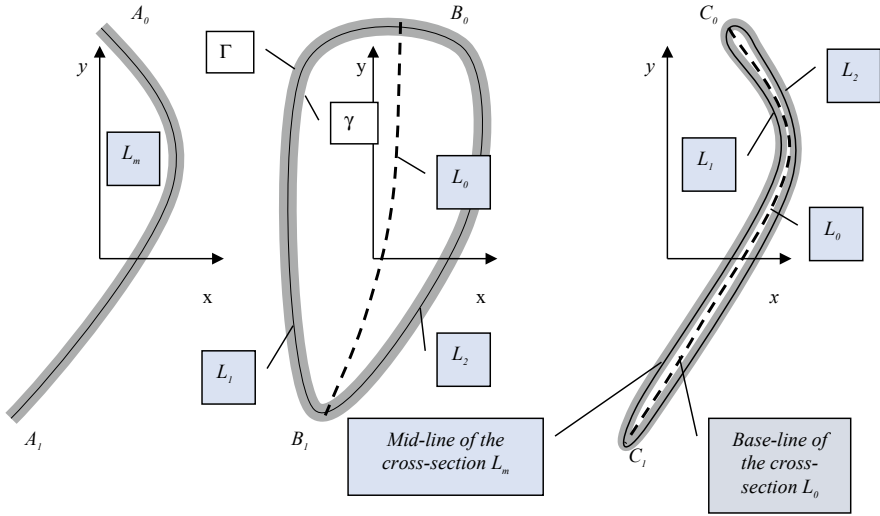


Fig. 5.1 Open, closed and semi-opened profiles

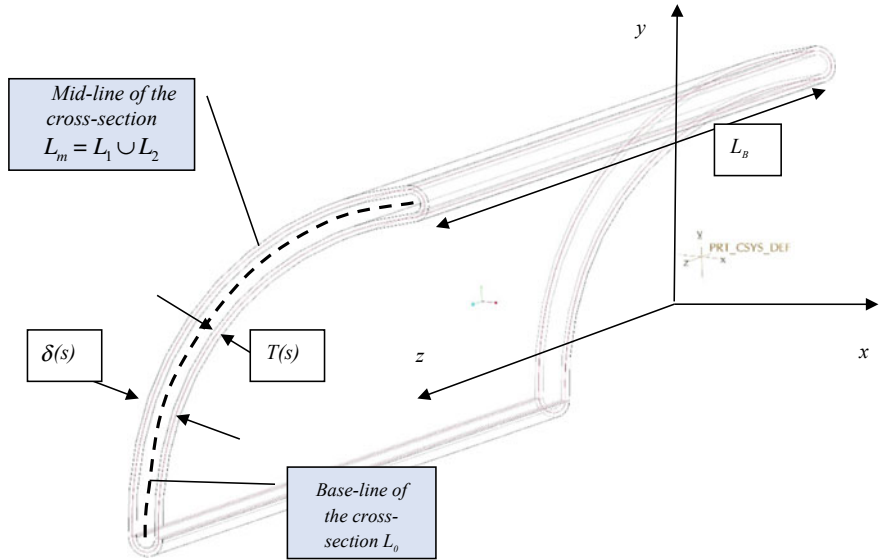


Fig. 5.2 Geometry of semi-open profile and local coordinate system

The points B_0, B_1 divide the closed curve L_m into two open segments L_1 and L_2 . We parameterize both segments with the line parameter θ , such that the point B_0 corresponds the value $\theta = 0$, and the point B_1 corresponds the value $\theta = 1$. For each given value of θ there exist two points $B_1(\theta)$ and $B_2(\theta)$ of two segments L_1 and L_2

correspondingly. The position vector of point $B_1(\theta)$ is $\mathbf{x}_1(\theta)$ and the position vector of point $B_2(\theta)$ is $\mathbf{x}_2(\theta)$. We assume for the beginning the equal thickness of walls at points $B_1(\theta)$ and $B_2(\theta)$. For each $0 \leq \theta \leq 1$ the position vector

$$2\mathbf{x}_0(\theta) = \mathbf{x}_1(\theta) + \mathbf{x}_2(\theta)$$

defines the point on the line segment L_0 with the end points B_0, B_1 . This line segment L_0 is referred to as baseline of the closed cross-section. Consequently, for each closed section there exists a line segment L_0 with both endpoints, which correspond to the pole points B_0, B_1 of the mid-line. Position vectors for two corresponding points of the walls L_1 and L_2 , are identified by arc coordinates

$$s_1 = s_1(\theta), s_2 = s_2(\theta)$$

Generally, for an arbitrary cross-section the introduction of baseline simplifies the modeling only occasionally. For certain practically important case, the mechanical model could be considerably improved with the concept of baseline. Namely, the elongated cross-sections with two nearly equidistant walls, shown on Fig. 5.1b, represent an intermediate case between open and closed cross-sections. The distance δ between points $B_1(\theta)$ and $B_2(\theta)$ is assumed to be much smaller than the length of the curve L_0 . The distance between points δ is in the considered case comparative to the material thickness T .

The pure torsion behavior of such profiles is in some aspects similar to mechanical behavior of thin-walled beams with closed cross-sections, while the warping function and restricted torsion behavior of these beams are analogous to those of open cross-sections (Heilig 1961; Graße 1965).

5.1.2 Baseline of Semi-opened Cross-Section

Each cross-section $z = \text{const}$ is a planar figure with the form of a double-walled curved strip. The thickness of each wall is t and the distance between the midlines of walls is δ . The outer boundary of the cross-section is Γ and the inner boundary is γ . Each value of an arc parameter s matches a dimensionless parameter

$$\theta = \theta(s).$$

A point on the baseline L_0 is identified by the arc coordinate s along the baseline

$$\mathbf{r}^{(0)}(s) = \mathbf{r}^{(0)}(\theta(s)), \quad s = 0..l,$$

where l is the length of the line segment L_0 .

5.1.3 Main Hypotheses of Thin-Walled Open-Profile Bars

The proposed technical theory of thin-walled rods with semi-opened profile is based on the following set of hypotheses of the kinematic and static nature, analogous to the hypotheses of thin-walled open-profile bars (Slivker 2007):

- the unchanged-contour hypothesis, according to which the cross-section of the bar does not change its shape in its plane; this means, that the bending of tubular cross-section vanishes. Particularly, the originally circular cross-section remains circular, so that the ovalization could be neglected.
- a no-shear hypothesis: there is no shear in the median surface of the thin-walled bar;
- a no-pressure hypothesis, according to which the longitudinal fibers of the thin-walled bar do not interact in their normal directions;
- a membrane-shell hypothesis: there are no moments in the longitudinal direction, that is, the distribution of normal stresses σ_x over the thickness of the shell is assumed uniform, and tangential stresses are believed negligibly small and therefore approximately equal to zero;
- a uniform-tangential-stress hypothesis, which assumes tangential stress to be uniformly distributed over the thickness of the shell.

The principal hypotheses match the common hypotheses of thin-walled open-profile bars.

The theory of pure torsion of thin-walled bars with closed profiles is based on the following hypothesis: *for the closed cross-sections, tangential stresses are constant over the thickness of the shell.*

On the contrary, the common theory of thin-walled beams with open profiles uses an alternative hypothesis: *tangential stress is supposed to be linear over the shell thickness for thin-walled beams with open profiles* (Laudiero and Savoia 1990; Bazant and Cedolin 1991; Wang 1999).

The present theory could be considered as the application of the theory developed in Prokic (2003) for the practically important case of elongated thin-wall profile.

5.2 Deformation Behavior of Cross-Sections

5.2.1 Deformation of Rods with Opened and Closed Profiles

The discussion of the torsion phenomena with comparative behavior of open and closed section is given by Tamberg and Mikluchin (1973). Several practical considerations relating to closed sections in torsion are given in Siev (1966). This subject is treated in a few textbooks (Boresi and Schmidt 2003, Chaps. 6–8; Salmon and Johnson 1996), so only a brief treatment follows.

The principal difference of the current model from the common models of open and closed cross-section consists in the assumptions for calculation of the torsion stiffness (Kobelev 2013). The proposed model could be considered as a special case of the open-profile bar with the double bending stiffness of the beam, but with the state of shear stress typical to the closed cross-sections. Namely, the stress of pure torsion is determined by the Bredt's theorem about shear stress circulation.

The motivation for this conjecture could be explained by the thin, curved, and elongated form of the cross-section. Consider such a cross-section where the distance between the walls vanishes and the walls touch each other along their inner surfaces.

Firstly, due to torsion a relative shift or free gliding between the inner surfaces occurs. In this case, the torque stiffness could be described by the Bredt's theorem about shear stress circulation.

Secondly, the relative shift due to torsion disappears and the gliding of walls is fully constrained. The relative shift vanishes if the walls are welded together over the inner surface. The disappearance of relative shift could be the result of friction between inner surfaces of walls. The effective wall thickness doubles. In this case, the torque stiffness considerably increases.

The present theory could be considered as intermediate model between the closed and opened cross-section theories. The warping of cross-section is essentially the same as the warping of open cross-section profiles, but the torsion stress flow is of Bredt type.

In other words, two walls that outline the cross-section of semi-opened profile bend roughly identically in the direction of the beam axis. This behavior is typical for open cross-sections. Contrarily, the shear stress flow due to torsion is constant over contour L_1 and L_2 . With respect to the baseline L_0 the shear stresses are opposite. This stress flow is characteristic for closed cross-sections.

Important for the application of the proposed theory is the lack of bending of the cross-section in its plane xy . The distance between the walls does not alter throughout the deformation. Otherwise, if the distance between walls changes, the bending of the cross-section in its plane occurs. This kind of bending and the ovalization leads to the considerable reduction of bending stiffness. If the ovalization or the bending in plane of cross-section xy occurs, the fundamentally different theories, which based on the thin shell theory, must be developed (Ventsel and Krauthammer 2001).

Another remark concerns the rods with elongated semi-opened cross-section, but with curved axis z . The hypotheses allow the modelling of the curved rods with semi-opened cross-section also. The rigidity of a curved pipe subjected to bending decreases as compared to that of a straight pipe with the same cross-section. It causes significant meridional bending stresses. The classical theory for bending of curved pipes was developed by Karman (1911), where the reasons for significant decrease of their bending rigidity was analyzed. In the cited article and in most subsequent studies it is assumed that actual curvature of longitudinal fibers of a pipe should not be considered. This curvature was considered equal to the centerline curvature. This assumption leads to an error in results if applied to pipes with a small radius of curvature. Clark and Reissner (1951) demonstrated an example in which an actual curvature of longitudinal fibers is given in problem formulation but ignored

in problem solution. Without discussing this issue in detail, we refer only to some basic papers on this subject (Whatham 1981a, b; Cherniy 2001).

5.2.2 Deformation of Rods with Semi-opened Profiles

According to the postulation of perfect rigidity of the cross-section in its own plane, the deformation of the cross-section can be described by only three displacement components, namely two translations U_x , U_y and the angle of twist Θ_z of an arbitrary taken pole P (Fig. 5.3). The position vector of the point S_0 of baseline L_0 with coordinates x , y with respect to the point P is $\mathbf{R}$. In natural coordinate system $(\mathbf{n}, \mathbf{t})$ of the curve L_0 , the components of vector $\mathbf{R}$ are

$$\begin{aligned} R_{\xi} &= (x - x_P)\sin\alpha - (y - y_P)\cos\alpha, \\ R_{\eta} &= (x - x_P)\cos\alpha + (y - y_P)\sin\alpha. \end{aligned} \quad (5.1)$$

The Cartesian coordinates of the points S_1, S_2 of midlines L_1, L_2 are

$$x^{(1)} = x - (\delta/2)\cos \alpha, y^{(1)} = y - (\delta/2)\sin \alpha$$

and

$$x^{(2)} = x + (\delta/2)\cos\alpha, y^{(2)} = y + (\delta/2)\sin\alpha.$$

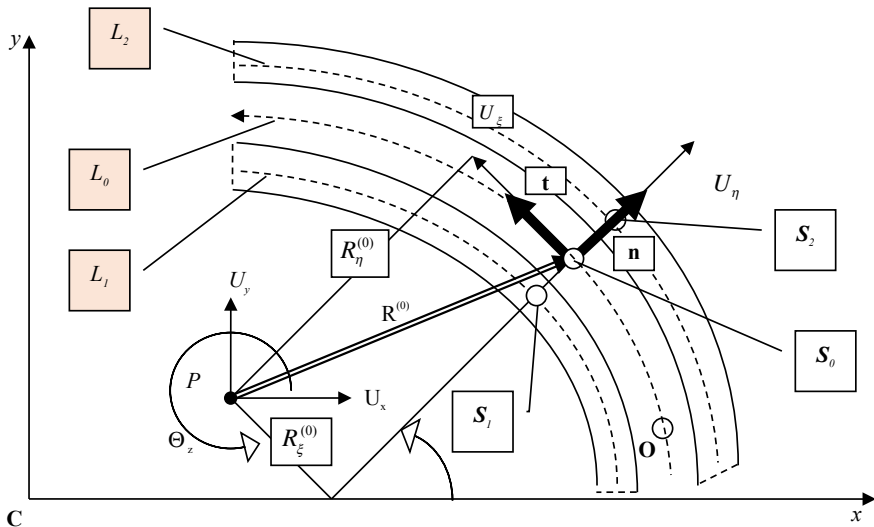


Fig. 5.3 Coordinate system, associated with the base line of the semi-opened profile

The position vectors of these points respectively to the point P are

$$\mathbf{R}^{(1)} = \mathbf{R}_\eta^{(1)} n + \mathbf{R}_\xi^{(1)} t, \quad \mathbf{R}^{(2)} = \mathbf{R}_\eta^{(2)} n + \mathbf{R}_\xi^{(2)} t,$$

where:

$$\mathbf{R}_\eta^{(1)} = \mathbf{R}_\eta - \frac{\delta}{2}, \quad \mathbf{R}_\eta^{(2)} = \mathbf{R}_\eta + \frac{\delta}{2}, \quad \mathbf{R}_\xi^{(1)} = \mathbf{R}_\xi^{(2)} = \mathbf{R}_\xi.$$

Assuming the planar displacement of the whole cross-section as the solid body, the normal $U_\eta^{(i)}$ and tangential displacements $U_\xi^{(i)}$ of the points S_i ($i = 1, 2$) read as

$$\begin{aligned} U_\xi^{(i)} &= U_x \cos \alpha + U_y \sin \alpha + \Theta_z R_\xi^{(i)}, \\ U_\eta^{(i)} &= -U_x \sin \alpha + U_y \cos \alpha + \Theta_z R_\eta^{(i)}. \end{aligned} \quad (5.6.2)$$

The counterclockwise rotation, observed from positive z direction, assumed to be positive. In the absence of shear strain, the longitudinal strains at points S_1, S_2 are

$$\begin{aligned} \varepsilon_z^{(i)} &= \partial_z U_z - x^{(i)} \partial_z^2 U_x - y^{(i)} \partial_z^2 U_y - \omega^{(i)} \partial_z^2 \Theta_z, \\ \partial_z &= \partial / \partial z, \quad i = 1, 2. \end{aligned} \quad (5.6.3)$$

The sectorial areas of lines L_i ($i = 1, 2$) are

$$\omega^{(i)} = \int_0^s R_\eta^{(i)} ds.$$

Substitution of the normal distance $\delta(s)$ between the lines L_1 and L_2 delivers

$$\omega^{(1)} = \omega - \frac{1}{2} \int_0^s \delta ds, \quad \omega^{(2)} = \omega + \frac{1}{2} \int_0^s \delta ds,$$

where

$$\omega = \int_0^s R_\eta ds$$

is the sectorial area of the baseline L_0 .

5.3 Statics of Semi-opened Profile Bars

5.3.1 Normal Stresses in Semi-opened Profile Bars

The no-pressure hypothesis states, that normal stresses in tangential and normal directions could be neglected in Hooke's law. The expression (5.3) gives the normal stresses in longitudinal direction in both walls

$$\sigma_z^{(i)} = E \cdot (\partial_z U_z - x^{(i)} \partial_z^2 U_x - y^{(i)} \partial_z^2 U_y - \omega^{(i)} \partial_z^2 \Theta_z), i = 1, 2. \quad (5.4)$$

The longitudinal force and overall bending moments are

$$\begin{aligned} N &= \int_0^l (\sigma_z^{(1)} + \sigma_z^{(2)}) T ds, \\ M_x &= \int_0^l (\sigma_z^{(1)} + \sigma_z^{(2)}) y T ds, \\ M_y &= \int_0^l (\sigma_z^{(1)} + \sigma_z^{(2)}) x T ds, \end{aligned}$$

where $T = T(s)$.

After substitution of (5.4) and integration we will have the following expressions

$$\begin{aligned} N &= EA \partial_z U_z - S_{\omega_p} \partial_z^2 \Theta_z \\ M_x &= -EI_x \partial_z^2 U_z - I_{x\omega_p} \partial_z^2 \Theta_z, \\ M_y &= -EI_y \partial_z^2 U_z - I_{x\omega_p} \partial_z^2 \Theta_z, \end{aligned} \quad (5.5)$$

where

$$\begin{aligned} S_{\omega_p} &= \int_0^l 2T \omega_p ds, A = \int_0^l 2T ds, \\ I_y &= \int_0^l 2T x^2 ds, I_x = \int_0^l 2T y^2 ds, \\ I_{x\omega_p} &= \int_0^l 2T y \omega_p ds, I_{y\omega_p} = \int_0^l 2T x \omega_p ds. \end{aligned}$$

The principal pole P and the principal origin point O could be chosen in such way that the following conditions should hold

$$S_{\omega_p} = 0, I_{x\omega_p} = 0, I_{y\omega_p} = 0.$$

The normal stresses in the walls of bar cross-section will be

$$\sigma_z^{(i)} = \frac{N}{A} + \frac{M_x}{I_x} y^{(i)} + \frac{M_y}{I_y} x^{(i)} + \frac{B_m}{I_\omega} \omega^{(i)}, \quad (5.6)$$

where the following expressions for sectorial area and bi-moment are used

$$I_\omega = 2 \int_0^l T \omega^2 ds, \quad B = \int_0^l (\sigma_z^{(1)} + \sigma_z^{(2)}) T \omega ds.$$

5.3.2 Torque and Bi-moment

According to the theory of torsion for thin-walled closed profiles, the pure-torsion torque M_H is calculated as

$$M_H = G I_T \partial_z \Theta, \quad (5.7)$$

where $G I_T$ is the section's torsion stiffness

$$I_T = \frac{4A_m^2}{\oint_{L_m} T^{-1}(s) ds}$$

where area enclosed by the curve L_m is A_m (Fig. 5.1c).

Bi-moment in cross-section in terms of displacement is

$$B_m = -E I_\omega \partial_z^2 \Theta_z.$$

5.3.3 Tangential Stresses in Bar Cross-Sections

The tangential stresses in the bar cross-section along the tangent to the profile line is the sum of the average tangential stress τ_S and the stress of pure torsion τ_H (Fig. 5.4). The average tangential stress τ_S is constant over the thickness of the wall. The stress of pure torsion is determined by the Bredt's theorem about shear stress circulation. The magnitude stress of pure torsion is τ_H .

In the closed profiles the tangential stress τ_S and shear stresses of pure torsion τ_H are constant over the thickness of the wall.

Analogously, in semi-opened profiles the tangential stress τ_S and shear stress of pure torsion τ_H are also constant over the thickness of the wall.

The tangential stresses in the walls are

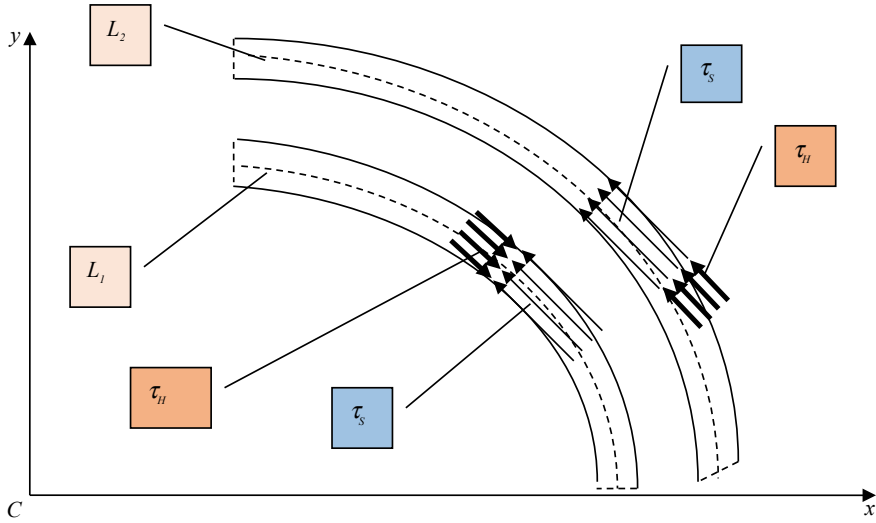


Fig. 5.4 Tangential and shear stresses in the semi-opened profile

$$\tau_{sz}^{(1)} = \tau_S - \tau_H, \quad \tau_{sz}^{(2)} = \tau_S + \tau_H. \quad (5.8)$$

5.3.4 Average Tangential Stress and Equilibrium Conditions

The average tangential stress τ_S could be determined from equilibrium equations. Consider the equilibrium of the two-wall shell-bar element (Fig. 5.5). The element is extracted from the thin-walled bar, so that the element is limited by two longitudinal sections parallel to the generator and located at the distance ds from each other and by two cross-sections at the distance dz from each other.

The equilibrium equation in the absence of external loads for this element for the projection forces on the z-axis reads

$$\partial_s F_\tau + \partial_z F_\sigma = 0, \quad \partial_s = \partial/\partial s, \quad (5.9)$$

where

$$F_\tau = F_\tau(z, s) = (\tau^{(1)} + \tau^{(2)})T \quad (5.10)$$

is the total tangential force per unit length (flow of tangential stresses),

$$F_\sigma = F_\sigma(z, s) = (\sigma_z^{(1)} + \sigma_z^{(2)})T \quad (5.11)$$

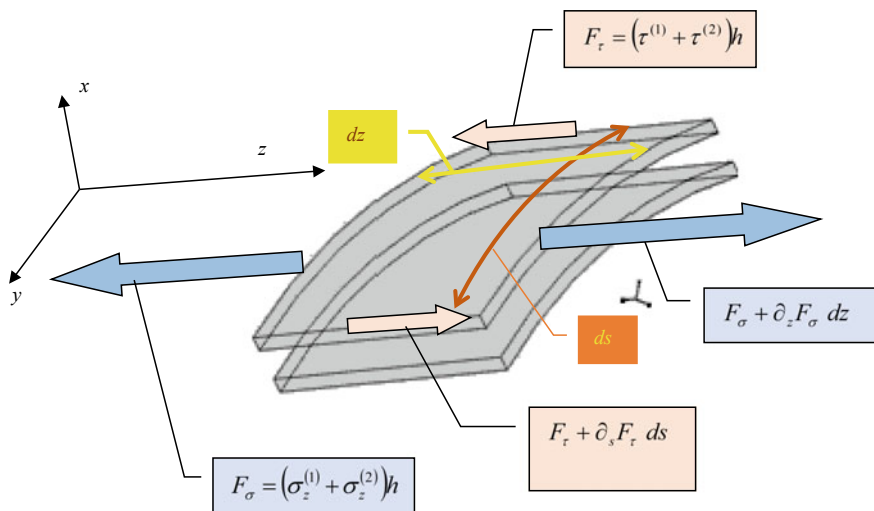


Fig. 5.5 Equilibrium of stresses in the element of semi-opened profile

is the total normal force per unit length at the point of profile.

Integration of the Eq. (5.9) delivers the expression of the tangential force per unit length

$$F_\tau = F_0 - \int_0^s \partial_z F_\sigma ds \equiv F_0 - \int_0^s \partial_z [(\sigma_z^{(1)} + \sigma_z^{(2)})T] ds.$$

Here $F_0 = F_\tau(z, 0)$ is the total normal force per unit length at zero point of profile. In the case of small distance between walls, the normal stresses in both walls are assumed to be equal

$$\sigma_z^{(1)} = \sigma_z^{(2)}.$$

The substitution of the expression (5.5) for normal stress delivers the expression for the flow of tangential stresses

$$F_\tau = F_0 - \frac{A}{A} \partial_z N - \frac{S_x}{I_x} \partial_z M_x - \frac{S_y}{I_y} \partial_z M_y - \frac{S_\omega}{I_\omega} \partial_z B_m. \quad (5.12)$$

The following notations were applied

$$\begin{aligned} A &= 2 \int_0^s T ds, & S_x &= 2 \int_0^s T y ds, \\ S_y &= 2 \int_0^s T x ds, & S_\omega &= 2 \int_0^s T \omega ds. \end{aligned}$$

Table. 5.1 Geometric and stiffness properties of semi-opened twist-beam with V1-shaped and Y-shaped cross-sections

	Static and geometric properties of cross-section	V1-profile, Fig. 5.11 $h = 0, b = 0, \beta = B/H$	Y-profile, Fig. 5.12
I_{xc}	Moment of inertia with respect to x-axis	$\frac{TH^3\beta^2}{12} \frac{1+4\beta^2+4\sqrt{1+4\beta^2}}{1+\sqrt{1+4\beta^2}}$	$\frac{TR^3}{12} (3 - 4\cos\alpha + 4\cos^2\alpha)$
I_{yc}	Moment of inertia with respect to y-axis	$\frac{TH^3}{16} \frac{1+2\beta^2+\sqrt{1+4\beta^2}}{1+\sqrt{1+4\beta^2}}$	$\frac{2TR^3}{3} \sin^2\alpha$
$S_{\omega x}$	Static moment with respect to x-axis	0	0
$S_{\omega y}$	Static moment with respect to y-axis	0	0
x_c	x-coordinate of mass center	0	0
y_c	y-coordinate of mass center	$\frac{H}{2} \frac{\beta\sqrt{1+4\beta^2}}{1+\sqrt{1+4\beta^2}}$	$R(1 + 2\cos\alpha)/6$
I_ω	Sectorial moment of inertia	0	0
α_x	x-coordinate of twist center	0	0
α_y	y-coordinate of twist center	0	0
I_T	Torsion moment of inertia	$(1 + \sqrt{1 + 4\beta^2})TH\delta^2$	$3TR\delta^2$
A	Area of material part	$(1 + \sqrt{1 + 4\beta^2})TH$	$3TR$
A_m	Area enclosed by curve L_m	$(1 + \sqrt{1 + 4\beta^2})\delta H/2$	$3\delta R/2$

The expressions for geometrical integrals for practically important cross-sections are given in Table 5.1.

5.3.5 Strain Energy of Semi-opened Rod

The expression of the strain energy as the functional of displacements vector reads (Reissner 1946):

$$U_e = \frac{1}{2} \int_0^{L_B} \left[EA(\partial_z U_z)^2 + EI_x(\partial_z^2 U_x)^2 + EI_y(\partial_z^2 U_y)^2 + GI_T(\partial_z \Theta)^2 + EI_\omega(\partial_z^2 \Theta)^2 \right] dz. \quad (5.13)$$

The strain energy can be represented as a quadratic function of stresses:

$$U_e = \frac{1}{2} \int_0^{L_B} \left[\frac{N^2}{EA} + \frac{M_x^2}{EI_x} + \frac{M_y^2}{EI_y} + \frac{M_H^2}{GI_T} + \frac{B^2}{EI_\omega} \right] dz. \quad (5.14)$$

Anisotropic generalizations of the developed theory could be developed using the variation methods (Kobelev and Larichev 1988; Kollar and Pluzsik 2002).

The differential equations of equilibrium can be obtained from (5.13) and (5.14) applying the Euler equations of the variation calculus (Kobelev 2013):

$$\partial_z[EA \cdot \partial_z U_z] = -p_z, \quad (5.15)$$

$$\partial_z^2[EI_x \cdot \partial_z^2 U_x] = p_x - \partial_z m_y, \quad (5.16)$$

$$\partial_z^2[EI_y \cdot \partial_z^2 U_y] = p_y + \partial_z m_x, \quad (5.17)$$

$$\partial_z^2[EI_\omega \cdot \partial_z^2 \Theta] - \partial_z[GI_T \cdot \partial_z \Theta] = m_D - \partial_z m_\omega. \quad (5.18)$$

The unknowns in Eqs. (5.15)–(5.18) are four components of the displacement vector and twist of the beam:

$$U_z(z), U_x(z), U_y(z), \Theta(z).$$

The initially known functions of the coordinate z represent respectively external loads per unit length in the x , y , z directions and the externally applied moments per unit length about x , y , z and external distributed bi-moments:

$$p_x, p_y, p_z, m_x, m_y, m_D, m_\omega. \quad (5.19)$$

The stiffness factors

$$EA, EI_x, EI_y, GI_T, EI_\omega$$

in the Eqs. (5.12)–(5.15) are also given functions of the coordinate z .

5.4 Applications of Thin-Walled Rods with Semi-opened Cross-Sections

5.4.1 Semi-Solid Axis with Twist Beam

The principal application of the semi-opened profile is the twist-beam for semi-solid automotive suspensions. This axle, also known as torsion-beam axle is frequently used as rear suspension on a wide variety of cars with the front wheel drive (Fig. 5.6). In this suspension, the lateral carrier for a compound steering axle has two longitudinal steering arms and a lateral carrier bonded to them, of a sheet metal in a closed

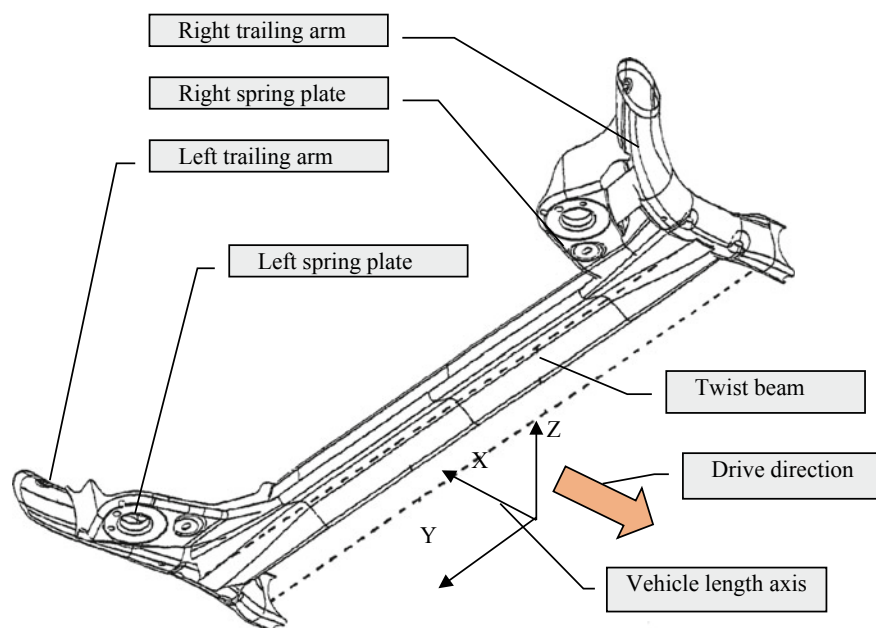


Fig. 5.6 Semi solid axle

profile (Kobeleev et al. 2009). The center section of the carrier has a smaller cross-section surface than the end sections bonded to the steering arms. The center section of cross-section carrier has at least three double-walled struts closed by outer bulges, separated from each other by at least partial drawing. The wall thickness of the center section is thinner than at the outer sections.

This suspension is typically described as semi-independent, meaning that the two wheels can move relatively to each other, but their motion is still somewhat inter-linked, to a greater degree than in a truly independent rear suspension. The latter can gently compromise the handling of the vehicle. The great advantage of the semi solid axle is its simplicity and maintenance. The known difficulties of the semi-solid axis are the induced effect of suspension stiffness on wheel orientation and oversteer due to deformation of suspension under the action of lateral forces at wheel.

A trail arm of the twist-beam axle is intended to semi-isolate one wheel of a vehicle such as an automobile from the opposite wheel. The elements of the semi-solid axle with twist-beam are shown on Fig. 5.7. The elasticity of twist-beam suspension is based on a large “H” or “C” shaped member, which consists of twist-beam and two trail arms. The front of the “H” is attached to the body via rubber bushings, and the rear of the “H” carries each stub-axle assembly, on each side of the car. The cross beam of the “H” holds the two trailing arms together, and provides the roll stiffness of the suspension by twisting as the two trailing arms move vertically, relatively to each other. Connection points are made to the body mounts, wheel spindles, shocks, and springs, as well as to track bars for lateral stability. Individual component pieces

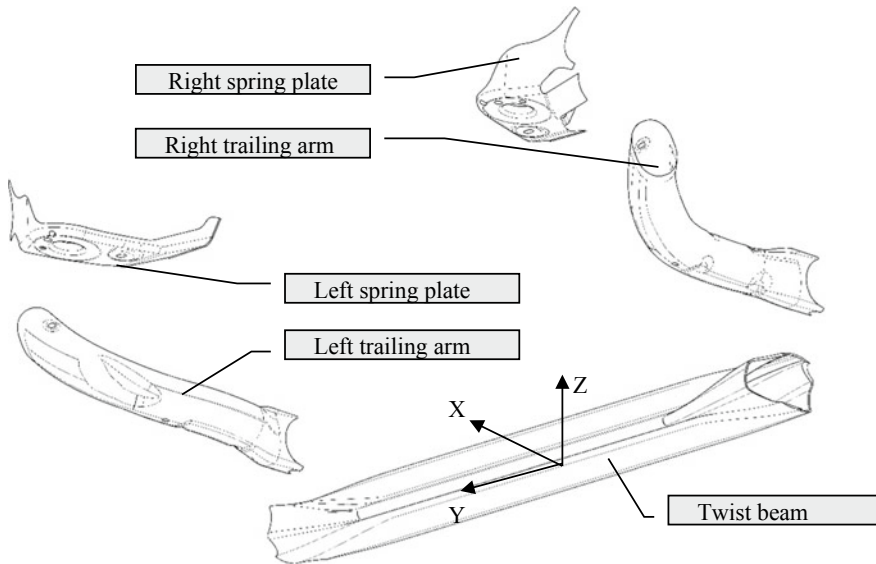


Fig. 5.7 Principal elements of the semi solid axle

typically making up an axle assembly include the twist beam, trail arms, flanges, spring supports, jounce bumpers, shock mounts, bushing mounts, and torsion bar.

The coil springs usually stand on a spring pad alongside the stub-axle. The shock damper is regularly coaxial with the spring. This location gives them a very high motion ratio compared with most suspensions, which improves their performance and reduces their weight.

5.4.2 Mechanical Models of Twist-Beam Axle

A twist-beam typically comprises a twist-beam for bending loads, coupled with a separate solid torsion rod of different material and characteristics from those of the beam, and extending through the open central portion of the beam for accommodating torsion moment (Reimpell et al. 2001; Linnig et al. 2009). The twist-beam is normally a drawn or stamped member, which is often not sufficiently stiff in torsion. Both ends are attached to stubs extending in from the spring seats. The separate rod to control torsion loads is specially fabricated of high strength steel. Its ends are attached separately from, and in a special relationship relative to, the ends of the beam.

The suspension-related kinematic characteristics are of primary importance for vehicle dynamics (Gillespie 1992; Choi et al. 2009). In the present chapter, we consider only the characteristics of the suspension, relating upon the elastic properties of twist beam. The important characteristics of the axle are the roll stiffness, lateral stiffness, and lateral force camber compliance. The contribution of the stiffness of

both trailing arms to the integral stiffness of the axle is usually much higher than the contribution of twist-beam stiffness. For our analysis, we consider the trailing arms stiff and neglect the compliances of the mount bushings. Our aim is to calculate the partial roll stiffness of the axle caused by beam twist. Thus, the contribution of suspension springs on the roll stiffness of the axes is not considered. The total roll stiffness of the axle is just the sum of stiffness due to twist-beam and caused by suspension springs and stabilizer bar.

5.5 Elastic Behavior of Twist-Beam Axles Under Load

5.5.1 Loads and Displacements of Twist-Beam Axles

The loads and displacements of the axle relate to the vehicle coordinate system:

- X positive forward; origin at the front axle position (longitudinal direction);
- Z positive up; origin at ground plane (vertical direction);
- Y positive right; centered on symmetry plane (transversal direction).

Consider the twist-beam axle as shown on Fig. 5.8. The twist-beam local axis z is parallel to Y-axis of vehicle coordinate system. The hard points control the static settings and the kinematics of the suspension.

The geometry of the axle is defined by the associated hard points:

The hard points P_0 and Q_0 are the midpoints of rubber bushing mounts and define the mount axis C_0 ; the points P_1 and Q_1 are located on the shear center axis of the twist beam C_1 ; the points P_2 and Q_2 are located on the center of the left and right wheels and define the axis C_2 . The distance between points P_2 and Q_2 is L_B , the

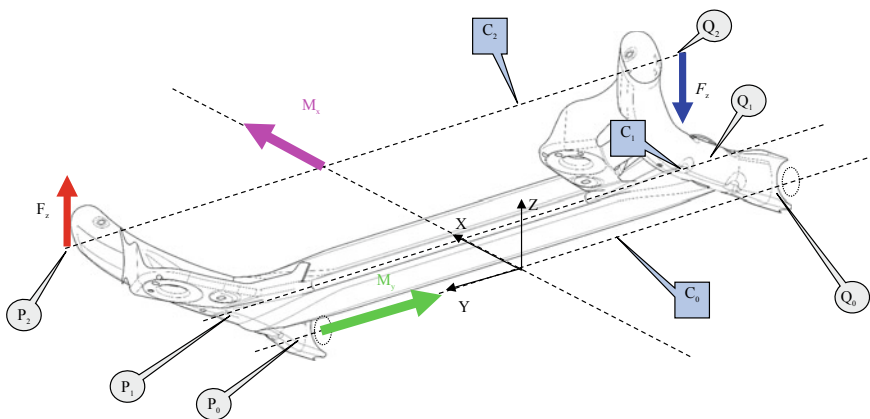


Fig. 5.8 Twist moment of the semi-opened profile M_y and loads on the wheels F_z due to vehicle roll moment M_x

distance between points P_0 and P_2 in the direction of axis x is the effective length of trailing arm L_T .

5.5.2 Roll Stiffness of Twist-Beam Axle

Firstly, we express the roll stiffness of the axle in terms of torsion stiffness of the twist-beam (Fig. 5.8). Let the wheel travel at the points P_2 and Q_2 are u_z and $-u_z$ correspondingly. The associated reaction forces at the points P_2 and Q_2 are F_z and $-F_z$.

The roll moment due to reaction forces is $M_x = 2 \cdot F_z \cdot L_B/2$. The resulting roll angle is $\vartheta_x = u_z/(L/2)$. Consequently, the axle roll stiffness is equal to:

$$r_a = M_x/\vartheta_x = F_z L^2/(2u_z).$$

From the other side, for the twist-beam the torsion moment is $M_y \cong F_z L_T$. The twist angle $\vartheta_y = u_z/L_T$. Consequently, the effective torsion stiffness of the twist-beam due to moment and bi-moment is

$$r_t = M_y/\vartheta_y = F_z L_T^2/(2u_z). \quad (5.20)$$

The roll stiffness of the axle depends on the effective torsion stiffness of the twist-beam r_t

$$r_a = i_t^2 r_t, \quad (5.21)$$

where

$$i_t = L_B/L_T$$

is the geometrical transmission ratio.

The effective torsion stiffness of the twist-beam r_t of the variable section stiffness could be also easily obtained.

5.5.3 Lateral Stiffness of Twist-Beam Axle

Secondly, we express the lateral stiffness of the axle in terms of bending stiffness of the twist-beam (Fig. 5.9). Let the lateral shift of both wheel spindles under the actions of two equal lateral forces of the magnitude F_y be

$$u_y \cong \theta_z L_T.$$

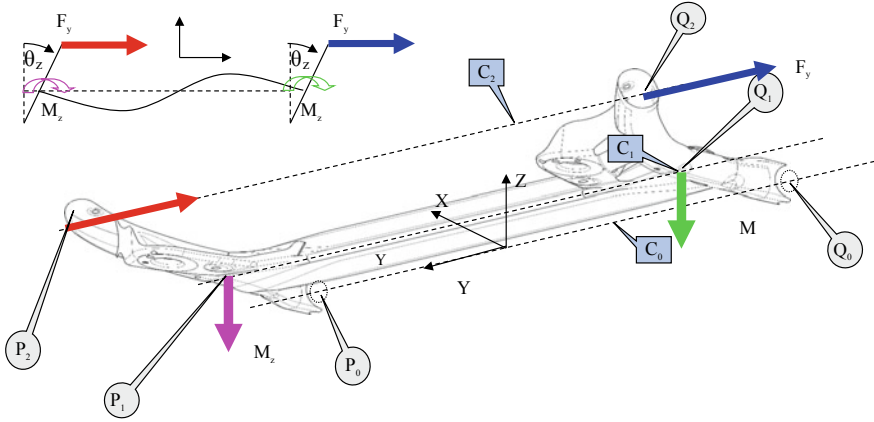


Fig. 5.9 Terminal bending moments of the semi-opened profile M_z due to lateral load F_y in vehicle side direction

The corresponding bending moments on the mount points P_0 and Q_0 are

$$M_z = F_y L_T.$$

The bending moment of the twist-beam can be expressed in terms of its effective bending stiffness r_z :

$$M_z = r_z \theta_z.$$

With this expression the lateral stiffness of the axle is

$$r_l \equiv F_y / u_y = r_z / L_T^2. \quad (5.22)$$

5.5.4 Camber Stiffness of Twist-Beam Axle

Thirdly, the camber stiffness is equal to the effective bending spring rate r_c of twist-beam due to moment in x direction (Fig. 5.10)

$$r_c = m_x / \theta_x. \quad (5.23)$$

The stiffness of the rod depends on the sectional shape of the rod and the boundary conditions at the ends.

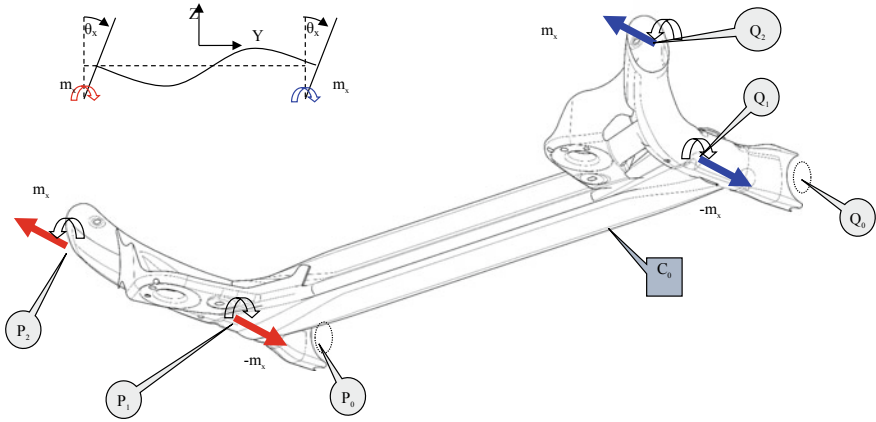


Fig. 5.10 Terminal bending moments of the semi-opened profile m_x due to twist moment on the wheel in vehicle travel direction

5.6 Deformation of Semi-opened Beam Under Terminal Load

5.6.1 Bending of Semi-opened Profile Beam Due to Terminal Moments

Our next aim is to express the effective torsion stiffness r_t , effective bending stiffness r_z and effective bending spring rate r_c of the twist-beam in terms of section properties of the twist-beam with semi-opened cross-section.

Consider the semi-opened profile beam with the constant material thickness and the identical shape of the cross-section profile along the length of the beam $-L/2 \leq z \leq L/2$. Bending of the semi-opened profile beam of the length L with simply supported ends $z = -L_B/2$ and $z = L_B/2$, loaded on both ends by the equal terminal moments M_y is given by the formula

$$U_x = \frac{M_y}{12LEI_y} (2z + L_B)(2z - L_B)z.$$

The terminal angles on the supported ends $z = -L_B/2$ and $z = L_B/2$ are

$$\theta_y = \frac{M_y L_B}{6EI_y}.$$

The bending stiffness of the beam, which is defined as angle due to unit terminal moment, reads

$$r_b = 6EI_y / LL_B. \quad (5.24)$$

The normal stresses in the direction of z axis in the point with coordinates (x, y) are

$$\sigma_z = M_y x / I_y. \quad (5.25)$$

5.6.2 Torsion Stiffness of Beam with Constant Section Due to Terminal Torques

The equilibrium equation of the rod twisted solely by the torque

$$M_z = M_S + M_H$$

is:

$$\partial_z [EI_\omega \cdot \partial_z^2 \Theta] - GI_T \cdot \partial_z \Theta = -M_z, \quad M_z = - \int_0^z m_D dz. \quad (5.26)$$

For the rod with constant stiffness along the span its solution reads:

$$\begin{aligned} \Theta(z) = & -\frac{1}{EI_\omega \lambda} \int_0^z M(z_1) \sinh(\lambda(z - z_1)) dz_1 \\ & + a_1 \sinh(\lambda z) + a_2 \cosh(\lambda z) + a_3. \end{aligned} \quad (5.27)$$

In Eq. (5.27) a_1, a_2, a_3 are the integration constants. The parameter

$$\lambda_c^2 = \frac{GI_T}{EI_\omega}$$

defines the characteristic length λ_c^{-1} of the bi-moment influence. If the length λ_c^{-1} much less than the length of the twist beam L_B , such that $(\lambda_c^{-1} \ll L_B)$, the influence of the bi-moment on the twist stiffness could be neglected. Otherwise, if the length λ_c^{-1} is comparable to the length of the twist beam L_B , $(\lambda_c^{-1} \approx L_B)$, the bi-moment stiffens the twist and the influence of bi-moment must be considered for technical purposes.

The unknown constants A_1, A_2, A_3 depend on the boundary conditions on the ends of the rod. For practically important case of the calculation of roll stiffness of the axle, the boundary conditions are:

$$\partial_z \Theta|_{z=-L_B/2} = -\theta, \quad \partial_z \Theta|_{z=L_B/2} = \theta, \quad \Theta|_{z=0} = 0. \quad (5.28)$$

The solution of (5.7) with the conditions (5.8) is

$$\Theta(z) = M_z \frac{-\lambda_c^{-1} \sin h(\lambda_c z) + z \cos h(\lambda_c L_B/2)}{GI_d \cos h(\lambda_c L_B/2)}. \quad (5.29)$$

The twist angles on the ends of the rod are:

$$\Theta\left(\frac{L_B}{2}\right) = -\Theta\left(-\frac{L_B}{2}\right) = \left(1 - \frac{\tanh(\lambda_c L_B/2)}{\lambda_c L_B/2}\right) \frac{M_z}{GI_T} \frac{L_B}{2} = \frac{M_z}{GI_T} \frac{L_B}{2K(\lambda_c L_B)}, \quad (5.30)$$

where

$$K(\lambda_c L_B) = \left(1 - \frac{\tanh(\lambda_c L_B/2)}{\lambda_c L_B/2}\right)^{-1} > 1$$

is the stiffening factor due to the bi-moment. The stiffness of the twisted rod due to pure torsion stiffness is Kobelev (2012):

$$r_t = \frac{M_z}{\Theta(L_B/2) + \Theta(-L_B/2)} = \frac{GI_T}{L_B} K(\lambda_c L_B). \quad (5.31)$$

The torsion stiffness of the twisted rod without the influence of bi-moment is Kobelev (2012):

$$\bar{r}_t = GI_T/L_B. \quad (5.32)$$

The stiffness of the twisted rod is therefore higher than the stiffness without the influence of bi-moment

$$r_t = K(\lambda_c L_B) \bar{r}_t. \quad (5.33)$$

5.6.3 Stresses in the Beam with Constant Section Due to Terminal Torques

The moments due to pure torsion

$$M_H = GI_T \partial_z \Theta,$$

due to constrained torsion

$$M_S = -\partial_z [E I_\omega \partial_z^2 \Theta]$$

and the bi-moment

$$B_m = E I_\omega \partial_z^2 \Theta$$

correspondingly are

$$M_H(z) = M_z \frac{\cos h(\lambda_c z) - \cos h(\lambda_c L_B/2)}{\cos h(\lambda_c L_B/2)}, \quad (5.34)$$

$$M_S(z) = M_z \frac{\sin h(\lambda_c z)}{\cos h(\lambda_c L_B/2)}, \quad (5.35)$$

$$B_m(z) = M_z \frac{\sin h(\lambda_c z)}{\cos h(\lambda_c L_B/2)}. \quad (5.36)$$

Using the expressions for normal stress, one can obtain the normal stress due to torsion:

$$\sigma_z^{(i)} = \frac{\omega^{(i)}}{I_\omega} B_m = E \omega^{(i)} \partial_z^2 \Theta. \quad (5.37)$$

With (5.36) we get from (5.37):

$$\sigma_z^{(i)} = M_z \frac{\omega^{(i)}}{I_\omega} \frac{\sin h(\lambda_c z)}{\cos h(\lambda_c L_B/2)}. \quad (5.38)$$

Using the equilibrium Eq. (5.11), we get the flow of shear stress

$$F_\tau(z, s) = M_S S_\omega(s) / I_\omega = -\partial_z [E S_\omega(s) \partial_z^2 \Theta].$$

For the wall thickness $T = T(s)$, we calculate from this equation the shear stress τ_S due to bi-moment as

$$\tau_S = \frac{F_\tau}{T} = \frac{S_\omega}{I_\omega T} M_S. \quad (5.39)$$

With (5.39), the shear stress τ_S is

$$\tau_S = M_z \frac{S_\omega}{I_\omega T} \frac{\cos h(\lambda_c z)}{\cos h(\lambda_c L_B/2)}. \quad (5.40)$$

The shear stress due to pure torsion could be calculated using the BREDT theorem

$$\tau_H = \frac{M_H}{2T A_m} = \frac{G I_d}{2T A_m} \partial_z \Theta. \quad (5.41)$$

With (5.41) the shear stress due to pure torsion of the twist beam reads

$$\tau_H(z) = \frac{M_z}{2T A_m} \frac{\cosh(\lambda_c L_B/2) - \cosh(\lambda_c z)}{\cosh(\lambda_c L_B/2)}. \quad (5.42)$$

The total shear stress is Kobelev (2012):

$$\tau = \tau_H + \tau_S = \frac{M_z}{2T A_m} + \frac{M_z}{T} \left(\frac{S_\omega}{I_\omega} - \frac{1}{2A_m} \right) \frac{\cosh(\lambda_c z)}{\cosh(\lambda_c L_B/2)}. \quad (5.43)$$

The area A_m enclosed by the mid-line L_m of the cross-section is

$$A_m = \frac{1}{2} \oint_{L_m} \left(-y \frac{dx}{ds} + x \frac{dy}{ds} \right) ds$$

5.6.4 Equivalent Tensile Stress Due to Simultaneous Bending and Torsion

Normal stress due to bending of the semi-opened profile beam, loaded on both ends by the equal terminal moments $\mathbf{M} = \{M_x, M_y, M_z\}$ is given by the formula

$$\sigma_z = M_x \frac{y}{I_x} + M_y \frac{x}{I_y} + M_z \frac{\omega^{(i)}}{I_\omega} \frac{\sin h(\lambda_c z)}{\cos h(\lambda_c L_B/2)}. \quad (5.44)$$

Finally, the equivalent tensile stress or von Mises stress for both walls of contour is Kobelev (2012):

$$\begin{aligned} \sigma_{eq}^2 = & \left[M_x \frac{y}{I_x} + M_y \frac{x}{I_y} + M_z \frac{\omega^{(i)}}{I_\omega} \frac{\sin h(\lambda_c z)}{\cos h(\frac{\lambda_c L_B}{2})} \right]^2 \\ & + 3 \left[\frac{1}{2T A_m} + \frac{1}{T} \left(\frac{S_\omega}{I_\omega} - \frac{1}{2A_m} \right) \frac{\cos h(\lambda_c z)}{\cos h(\frac{\lambda_c L_B}{2})} \right]^2 M_z^2. \end{aligned} \quad (5.45)$$

This expression demonstrates that the stress along the span of the beam depends on the span coordinate z .

5.6.5 Stiffness Properties of Semi-opened Profiles for Automotive Applications

Various cross-sectional configurations have been suggested for twist-beam and torsion bar elements. The calculation of torsion and bending stiffness of the semi-opened requires the calculation of sectorial areas and the corresponding integrals. Several types of semi-opened profiles are important for automotive applications (Figs. 5.11 and 5.12). The corresponding formulas are given in Table 5.1.

If the constant material thickness T and constant distance between walls δ over the length of the midline the following relations

$$I_d = 4A_m^2 T L_m^{-1}, A_m = L_0 \delta, A = L_m T, L_m = 2L_0$$

are valid for all types of cross-section.

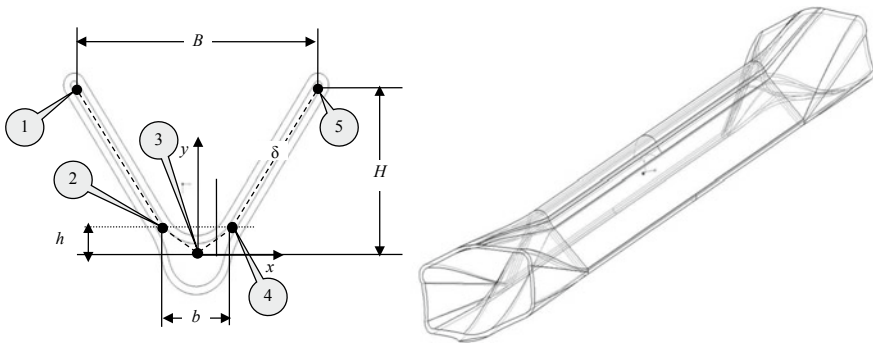


Fig. 5.11 Cross-section of semi-opened twist beam with V1-profile

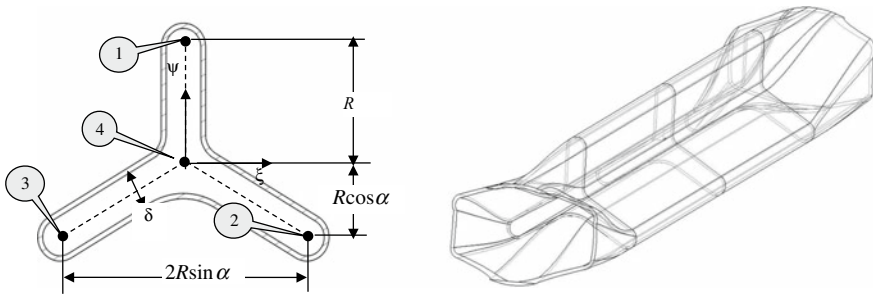


Fig. 5.12 Cross-section of semi-opened twist beam with Y-profile

5.6.6 *Semi-opened Beams with Variable Cross-Sections*

In this chapter was assumed that the cross-section of the beam remains constant along the span of the beam. Overwhelmingly, however, the cross-sections of the beams for the semi-solid axles possess the variable cross-section over the beam length. The cross-sections at the ends of the beam have higher local torsion and bending stiffness comparative to the cross-sections in the middle regions of the beam. The purpose for the increasing of torsion and bending stiffness in the end regions consists in the localization of deformation in the middle regions of the beam. Due to the higher torsion and bending stiffness in the end regions, the bending deformation and rate of twist reduce. The reduction of bending deformation and rate of twist leads to the decrease of tensile and shear stresses in the end regions. Due to this measure, the weld groove between the twist-beam and trailing undergoes lower stresses and its fatigue life considerably increases. To account the variable cross-section the Eqs. (5.15)–(5.18) and the corresponding formulas for the stresses must be applied. These equations are valid for an arbitrary variable cross-section. For conceptual design purposes and for estimation of spring rates, the averaging of reciprocal stiffness over the span length leads to acceptable results. The precise stress and deformation analysis in the final design phase should to be performed using commercial modeling software and finite element codes.

5.7 Conclusions

The new type of thin-walled rods with semi-opened cross-section is suggested and the optimization performed. Descriptive for this class of thin-walled beam-like structures is the closed but flattened profile. The unusual shape of semi-opened thin-walled beams allows the efficient optimization due to wide variability of shapes. An intermediate class of thin walled beam cross-sections is studied. The cross-section of the beam is closed, but the shape of cross-section is elongated and curved. The walls, which form the cross-section, are nearly equidistant. The StVenant's free torsion behavior of the beam is like the behavior of closed cross-section beams. A technical theory of thin walled rods with semi-opened profile can be based on the following set of hypotheses of the kinematic and static nature. The pure torsion behavior of such profiles is in some aspects similar to mechanical behavior of thin-walled beams with closed cross-sections, while the warping function and restricted torsion behavior of these beams is analogous to those of open cross-sections.

The application of the semi-opened profile is the twist-beam for semi-solid automotive suspensions. Twist-beam axles are being used not only in vehicles of the compact cars, but recently in mid-class-segment, in Sport Utility Vehicles and four-wheel drive cars. Due to the low costs, the compact packaging, the reduced weight, and the acceptable axle kinematics it frequently represents a competitive solution. The advantage of the semi solid axle also is its simplicity and maintenance. The

disadvantages of the semi-solid axis are the induced effects of suspension stiffness on wheel orientation and oversteer due to deformation of suspension. The longitudinal location of the twist-beam controls important parameters of the suspension's behavior, such as the roll steer curve and toe and camber compliance. The closer the cross beam to the axle stubs the more the camber and toe changes under deflection. A key difference between the camber and toe changes of a twist-beam versus independent suspension is the change in camber and toe is dependent on the position of the other wheel, not the car's chassis. In a traditional independent suspension, the camber and toe are based on the position of the wheel relative to the body. If both wheels compress together their camber and toe will not change. Thus, if both wheels started perpendicular to the road and car compressed together, they will stay perpendicular to the road. The camber and toe changes are the result of one wheel being compressed relative to the other. Further, we express analytically the effective torsion stiffness and effective bending stiffness of the twist-beam in terms of section properties of the twist-beam with semi-opened cross-section.

In this chapter, an intermediate class of thin-walled beam cross-sections is studied. The cross-section of the beam is closed, but the shape of cross-section is elongated in its plane and curved. The walls, which form the cross-section, are nearly equidistant. The St Venant's free torsion behavior of the beam is same to the behavior of closed cross-section beams. A technical theory of thin-walled rods with semi-opened profile can be based on the following set of hypotheses of the kinematic and static nature. The pure torsion behavior of such profiles is in some aspects analogous to mechanical behavior of thin-walled beams with closed cross-sections, while the warping function and restricted torsion behavior of these beams is analogous to those of open cross-sections.

The application of the semi-opened profile is the twist-beam for semi-solid automotive suspensions. The advantage of the semi solid axle is its simplicity and maintenance. The disadvantages of the semi-solid axis are the induced effects of suspension stiffness on wheel orientation and oversteer due to deformation of suspension. The longitudinal location of the twist-beam controls important parameters of the suspension's behavior, such as the roll steer curve and toe and camber compliance. The closer the cross beam to the axle stubs the more the camber and toe change under deflection.

The twist beam, on the one hand, and conventional independent suspension, on the other hand, behave differently under the load. Namely, the changes of camber and toe under the side load are different for these types of suspensions. The changes in camber and toe are dependent on the relative position of the wheels.

In a conventional independent suspension, the camber and toe are based on the position of the wheel with respect to the body. If both wheels compress together, their camber and toe will not change. Thus, if both wheels start perpendicular to the road and car compressed them together the wheels will stay perpendicular to the road. The camber and toe changes are the result of one wheel being compressed relatively to the other.

The twist-beam suspension behaves more complicated. The suspension changes the steering characteristics of the vehicle. Standard terminology that used to describe

understeer and oversteer is defined by the Society of Automotive Engineers (SAE 2008) and by the International Organization for Standardization (ISO 2010). By these terms, understeer and oversteer are based on differences in steady-state conditions where the vehicle is following a constant-radius path at a constant speed with a constant steering wheel angle, on a flat and level surface. The side load on the twist-beam suspension leads to oversteer effect due to its low stiffness. Oversteer effect could be significant, if the bending stiffness r_z of twist-beam not high enough. The increasing of the bending stiffness, however, leads to the high torsion stiffness r_t and the roll stiffness of the vehicle. The growing roll stiffness alters the dynamics of car and is generally disadvantageous.

5.8 Summary of Principal Results

- The semi-opened cross tubular sections are flattened tubes.
- The flattening could be easily adjusted for optimal design.
- The analytical model estimates principal static effects of semi-opened tubes,
- The principal application of semi-opened tubes is the twist-beam axle.
- For the twist-beam axle the effective torsion stiffness r_t , effective bending stiffness r_z and effective bending spring rate r_c are derived in terms of section properties of tube.

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Chapter 6

Coiling of Helical Springs



Abstract In this chapter the method for calculation of residual stress and plastic bending and torsion moments for combined bending-torsion load is developed. The Bernoulli's hypothesis is assumed for the deformation of the bar. The analysis was provided using deformational theory of plasticity with a nonlinear stress strain law describing active plastic deformation (SAE AE-22, Sect. 8.3.33). The curvature and twist of the bar during the plastic loading increase proportionally, such that the ratio curvature to twist remains constant. The complete solutions based on this approximate material law provide closed analytical solution. The unloading is linear elastic. This chapter is the primary section of second part, which studies mechanical problems arising during the manufacturing of helical springs.

6.1 Elastic–Plastic Bending and Torsion of Wire

The elastic–plastic problem of combined bending and torsion moments of a straight prismatic bar, made of Levy-Mises material and loaded by a terminal bending couple about the axis of symmetry of the cross-section and a twisting couple, was originally considered in Handelman (1944).

Moment–angle relations are reported in Hill and Siebel (1953) for steel bars of circular section plastically strained by combined bending and couples in constant ratio. The bending and torque approach the theoretical values calculated for the fully plastic state of a plastic-rigid material. Good estimates of the latter values are obtained by bracketing between upper and lower approximate values. A general relation is proposed between the fully plastic values of bending moment, torque and axial force when all three are applied together. This relation applies for a wide variety of sections and is suitable for plastic limit design.

A long prismatic member is acted on by combinations of bending moments and torques of such a magnitude as to render the member just fully plastic was discussed in Steele (1954). The citing paper takes Handelman's (Handelman 1944) equation and solves it numerically for a square section member. The moments and torques computed from the numerical solution are compared with a bounded solution due to

Hill and Siebel (1953). Finally, the stress distributions (shear and bending) are given in a member for two critical combinations of moment and torque.

Combinations of twist and extension of a solid circular cylinder are considered in Gaydon (1952). The Reuss equations are used throughout and these are integrated, for different cases, to give the shear stress and tension in the plastic material. It is shown that the stresses rapidly approach their asymptotic values. A more general case, in which the torsion and extension are such as to make the ratio of axial load to torque constant, is solved numerically. Finally, the residual stresses are evaluated, after partial unloading, for a bar, which has been twisted and extended in constant ratio.

Upper and lower approximations are obtained in Gaydon and Nuttal (1957) to the interaction curve of the bending and twisting couples at yield for the combined bending and twisting of cylinders of ideally plastic-rigid material. Rectangular, I-sections and box-sections are dealt with in detail. For the box section, a comparison is made with the thin tube theory (Hill and Siebel 1953).

In the paper (Sankaranarayanan and Hodge 1958), yield criteria expressed in terms of stress resultants are obtained for typical engineering structures, using the first fundamental theorem of limit analysis. These yield criteria, which are often non-linear, are then replaced by inscribed piecewise linear approximations. The complete solutions based on these approximate yield criteria provide lower bounds.

The paper (Imegwu 1960) deals with the plastic flexure and torsion of prismatic beams loaded by terminal bending and torsion moments. Both moments, acting together, cause full plastic flow. The material is assumed to behave according to the Tresca-Levy-Mises Hypotheses, and in non-hardening and rigid plastic. The results were obtained by numerical solution of the second order non-linear differential equation for a Levy-Mises material. The relationships obtained were found to give points lying virtually on a single interaction curve plotted with non-dimensional coordinates.

The elastic-plastic problem of combined bending and torsion is treated analytically in Ishikawa (1973) for an incompressible isotropic work-hardening material obeying a nonlinear stress strain law. Evolving a theory to satisfy the equilibrium and compatibility condition, the basic nonlinear differential equation in the ordinal Cartesian coordinate system can be linearized, adopting the new parameter in the stress space. The Ramberg-Osgood's law is employed as a nonlinear stress strain relation. The stress components, the bending moment and torque can be evaluated by the numerical calculation.

The paper (Ali 2005) deals with two aspects of work to examine the elastic-plastic behavior of pre-loaded circular rods subjected to subsequently applied torque within the plastic region. The uniform diameter and reduced section rods of mild steel, fitted with strain gauges, were subjected to initial axial yield loads using a custom-built torque-tension machine. The specimens were gradually twisted, holding the initial axial displacements constant. Then the measurements of the resulting torque and load were recorded using appropriate load cells as well as by the fitted strain gauges. When the axial displacements of the preloaded rods were held constant, the strain gauges readings increase rapidly with the decrease in the initially applied axial

loads. The experimental results have been compared with reported in Brooks (1969) theoretical predictions.

An analytical model was developed in Baragetti (2006) in order to provide the designer with expressions for estimation the final shape of a wire. The model allows the evaluation with a higher level of accuracy the end shape of wires having different cross-sections after nonlinear bending. The Bernoulli's hypothesis was assumed in the cited paper, such that the model can be used in all the applications where the material behavior of the wire guarantees that plane cross-sections of the wire will remain plane after rotation due to bending.

The residual stresses due to plastic pre-setting the surface of a solid bar were studied in Močilnik et al. (2015).

The results of the elastic–plastic analysis for combined loading were summarized in reference works Życzkowski (1967, 1981).

6.2 Modified Ramberg–Osgood's Law

We analyze the plastic loading using deformational theory of plasticity. The deformational theory of plasticity represents active plastic deformation using a nonlinear stress strain law (SAE AE-22, Sect. 8.3.33; Jones 2009). The curvature and angle of twist pro length unit of the bar during the plastic loading increase proportionally, such that the ratio curvature to twist remains constant.

In this chapter, we use the modified Ramberg–Osgood's law, that is adopted for the analytical calculation of the elastic–plastic problem. For a compressible isotropic work-hardening material, without the distinct yielding point, as observed in the behavior of deformation, the following explicit stress strain expression can be used.:

$$\mathbf{s} = 2G_p(\gamma_{oct})\mathbf{e}, \quad (6.1)$$

$$3K\varepsilon_{oct} = \sigma_{oct} \quad (6.2)$$

In (6.1) the tensors $\mathbf{s}$ and $\mathbf{e}$ are the deviators of stress $\boldsymbol{\sigma}$ and strain tensors $\boldsymbol{\varepsilon}$ respectively. The octahedral normal strain ε_{oct} and octahedral normal stress σ_h are denoted as:

$$3\varepsilon_{oct} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}, 3\sigma_{oct} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}. \quad (6.3)$$

The octahedral shear strain in terms of the second invariant of the deviatoric strain $\mathbf{\Pi}(\mathbf{e})$ reads (Appendix A):

$$\gamma_{oct}/2 \stackrel{\text{def}}{=} \sqrt{2\mathbf{\Pi}(\mathbf{e})/3}. \quad (6.4)$$

The octahedral shear stress is the double square root of the second invariant of the deviatoric stress $\mathbf{II}(s)$:

$$\tau_{oct} \stackrel{\text{def}}{=} \sqrt{2\mathbf{II}(s)/3}. \quad (6.5)$$

From the Eq. (6.1) follows the relation between the octahedral shear stress and octahedral shear strain:

$$\tau_{oct} = G_p(\gamma_{oct})\gamma_{oct}. \quad (6.6)$$

The secant modulus G_p is the function of octahedral shear strain γ_{oct} . For linear elastic medium the Eq. (6.6) reduces to

$$\tau_{oct} = G\gamma_{oct},$$

where G is the elastic shear modulus. The components of deformation tensor increase in the case under consideration proportionally to a single parameter, such that the relation between them remains constant during the deformation history. Under this condition, the deformation theory of plasticity proved to be applicable. In the deformation theory of plasticity, the active plastic deformation the empirical stress–strain relation (6.6) describes fully the plastic deformation law (SAE AE-22, Sect. 8.3.33).

In the present chapter, the following dependence of secant modulus upon the octahedral strain γ_{oct} is used (Kobeleev 2011):

$$G_p(\gamma_{oct}) = \frac{G}{(1 + (\gamma_{oct}/\gamma_w)^2)^k} \quad (6.7)$$

Particularly, in the case $k = 1/2$ the expression (6.7) for secant modulus reads:

$$G_p(\gamma_{oct}) = \frac{G}{\sqrt{1 + (\gamma_{oct}/\gamma_w)^2}} \quad (6.8)$$

The limit value of measure of stress for $\gamma_{oct} \rightarrow \infty$ and for $k = 1/2$ is $\tau_w = G\gamma_w$. For negative values of k , the secant modulus increases with the increasing strain. For positive values of k the secant modulus decreases, as it usually does for common metals and alloys. For $k = 1$, at the limit $\gamma_{oct} \rightarrow \infty$ the stress vanishes.

If the stress is reduced, elastic unloading occurs. The elastic unloading of material is characterized by the elastic shear modulus G and Young modulus E .

Essentially the same material model was used in Neuber (1961a, b) for the solution of notch problems.

6.3 Plastic Deformation of Wire During Coiling

A straight cylindrical solid bar with circular cross-section of length L is loaded from a stress-free state by terminal bending moment M_B and the torque M_T . The end sections are $z = 0$ and $z = L$ (Fig. 6.1). The origin of coordinates is chosen at the centroid of area of one cross-section. The polar coordinates in the cross-section of the wire (ρ, ϕ) relate to Cartesian coordinates:

$$x = \rho \sin \phi, y = \rho \cos \phi, 0 < \rho < r, 0 < \phi \leq 2\pi, \rho = \sqrt{x^2 + y^2}.$$

The distribution of stress due to above combined loading is independent of the variable z . Let the curvature of the axis of cylinder in pure bending is κ . In this article the Bernoulli's hypothesis is assumed, such that plane cross-sections of the wire remain plane after rotation due to bending. This means that tensile strain is linearly linked to the distance from the neutral axis:

$$\varepsilon_{zz} = \kappa x,$$

where $R = 1/\kappa$ is radius of curvature of the bar during bending. The shear strain in pure torsion of cylindrical bar with circular cross-section of radius r is:

$$\gamma_{xz} = -\theta y, \quad \gamma_{yz} = \theta x,$$

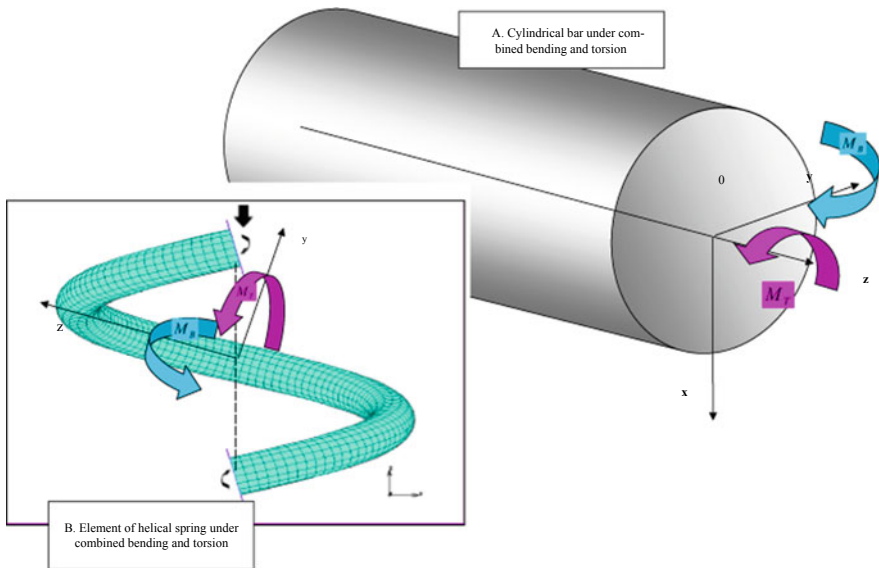


Fig. 6.1 Cylindrical bar under combined bending and torsion

where θ is the angle of twist per unit length of the bar.

In polar coordinates the axial and shear deformations in the points of cross-section for the circular rod are respectively:

$$\varepsilon_{zz}(\rho, \phi) = \kappa \rho \sin \phi, \quad (6.9)$$

$$\gamma_{z\varphi}(\rho, \phi) = 2\varepsilon_{z\varphi} = \theta \rho. \quad (6.10)$$

Maximal axial strain

$$\varepsilon_P = \varepsilon_{zz}(r, \pi/2) \equiv r\kappa$$

and the maximal shear strain

$$\gamma_P = \gamma_{z\varphi}(r, \phi) \equiv \theta r$$

attain on the contour of the circular cross-section. Apparently, the axial strain on the contour ε of the cross-section with radius r depends linearly on curvature of the bar during bending κ . The shear on the contour of the cross-section γ is the linear function of the angle of twist per unit length of the bar θ .

From the viewpoint of dislocation character, the coiling process creates during plastic deformation edge and screw dislocations. Their axes are parallel to the axis of helix. The bending of the initially straight wire into a flat torus induces the edge dislocations. The pitch formation induces the screw dislocations. Both dislocation systems overlaps. Finally, the dislocations in the free helical spring have a mixed edge and screw character, but their axes coincide with the axis of the helix.

6.4 Behavior of Wire in Manufacturing Process

The stress distribution in most structural members loaded into the elastic–plastic range is difficult to determine, because the shape of the elastic–plastic interface is itself related to the stress distribution and is, therefore, unknown until the complete solution is found. However, for a solid circular rod subjected to simultaneous bending and torsion, this restriction is removed since the shape of the interface must be annular to preserve axial symmetry.

The non-vanishing stress components for a solid circular rod subjected to combined tension and torsion are $\tau_{z\varphi}$ and σ_{zz} . Substitution of these stress components in the Eqs. (6.1)–(6.3) delivers the following expressions for strains

$$\varepsilon_{rr} = \varepsilon_{\phi\phi} = -\nu_p \varepsilon_{zz}, \quad \varepsilon_{zz} = \sigma_{zz}/E_p.$$

Both Poisson ratio and Young modulus depend on G_p :

$$\nu_p = \frac{3K - 2G_p}{2(3K + G_p)}, \quad E_p = \frac{3KG_p}{3K + G_p}.$$

The strain and stress tensors for the rod subjected to combined tension and torsion in polar coordinates are

$$\boldsymbol{\varepsilon} = \begin{bmatrix} -\nu_p \varepsilon_{zz} & 0 & 0 \\ 0 & -\nu_p \varepsilon_{zz} & \gamma_{z\varphi}/2 \\ 0 & \gamma_{z\varphi}/2 & \varepsilon_{zz} \end{bmatrix}, \quad \boldsymbol{\sigma} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & G_p \gamma_{z\varphi} \\ 0 & G_p \gamma_{z\varphi} & E_p \varepsilon_{zz} \end{bmatrix}.$$

The considerable simplification of the solution could be attained with the assumption of incompressibility of the material during the plastic deformation $E_p = 3G_p$, $\nu_p = 1/2$.

The stress tensor reads

$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & G_p(\gamma_{oct})\gamma_{z\varphi} \\ 0 & G_p(\gamma_{oct})\gamma_{z\varphi} & 3G_p(\gamma_{oct})\varepsilon_{zz} \end{bmatrix}. \quad (6.11)$$

The measure of shear strain Γ reduces to

$$\gamma_{oct}^2 = 3\varepsilon_{zz}^2 + \gamma_{z\varphi}^2.$$

Substitution of (6.7) in (6.11) delivers the explicit expressions for stress components in terms of strains:

$$\sigma_{zz} = \frac{E\varepsilon_{zz}}{\left(1 + \frac{3\varepsilon_{zz}^2 + \gamma_{z\varphi}^2}{\gamma_w^2}\right)^k}, \quad \tau_{z\varphi} = \frac{G\gamma_{z\varphi}}{\left(1 + \frac{3\varepsilon_{zz}^2 + \gamma_{z\varphi}^2}{\gamma_w^2}\right)^k}. \quad (6.12)$$

The stresses components in each point (r, ϕ) of the cross-section could be calculated using Eqs. (6.9), (6.10) and (6.12).

The curvature κ and twist θ of the rod increase proportionally to a single parameter, such that their ratio keeps constant during the plastic deformation. The bending and torque moments applied to the end sections of the rod are

$$M_B = \int_0^{2\pi} \left[\int_0^r \sigma_{zz}(\rho, \phi) \rho^2 \sin \phi d\rho \right] d\phi, \quad M_T = \int_0^{2\pi} \left[\int_0^r \tau_{z\varphi}(\rho, \phi) \rho^2 d\rho \right] d\phi \quad (6.13)$$

Using the expression (6.12) for stresses in the cross-section, we can calculate the terminal moments in terms of strains ε_{zz} , $\gamma_{z\varphi}$. Further, with formulas for strains (6.9) and (6.10), the bending and torque moments are derived as the explicit functions of curvature κ and angle of twist per unit length θ . The integrals (6.13) for bending moment M_B and for torque M_T could be explicitly evaluated. The dimensionless

functions $P(\lambda, \mu)$ and $Q(\lambda, \mu)$ are listed below for the most interesting case $k = 1/2$:

$$M_B(\kappa, \theta) = \frac{4}{3}r^4 G\eta P(\lambda, \mu), \quad M_T(\kappa, \theta) = \frac{4}{3}r^4 G\eta Q(\lambda, \mu). \quad (6.14)$$

The dimensionless parameters λ, μ depend on curvature κ and angle of twist per unit length θ :

$$\lambda^2 = \frac{3\kappa^2}{3\kappa^2 + \theta^2 + \eta^2}, \quad \mu^2 = \frac{3\kappa^2}{3\kappa^2 + \theta^2}.$$

Parameter $\eta = \gamma_w/r$ has the same dimension as θ and κ , namely of an inverse length.

The dimensionless function $P(\lambda, \mu)$ for this case reads

$$\begin{aligned} P(\lambda, \mu) &= p_K \mathbf{K}(\lambda) + p_E \mathbf{E}(\lambda) + p_\Pi \mathbf{\Pi}(\mu^2, \lambda) + p_0, \\ p_\Pi &= -\frac{\sqrt{3}(\mu^2 - \lambda^2)^2}{\lambda^3}, \quad p_E = \frac{\mu^2 \sqrt{3}}{\lambda^3}, \\ p_K &= -\frac{\sqrt{3}(\mu^2 - \lambda^4)}{\lambda^3}, \quad p_0 = \frac{\Pi \mu \sqrt{3}(\mu^2 - \lambda^2)^{3/2}}{2\lambda^3 \sqrt{1 - \mu^2}}. \end{aligned}$$

Here

$$\mathbf{K}(k) = \int_0^1 \frac{1}{\sqrt{(1 - k^2 t^2)(1 - t^2)}} dt$$

is the complete elliptic integrals of the first kind, (Appendix D)

$$\mathbf{E}(k) = \int_0^1 \sqrt{\frac{1 - k^2 t^2}{1 - t^2}} dt$$

is complete elliptic integrals of the second kind,

$$\mathbf{\Pi}(\kappa, k) = \int_0^1 \frac{1}{(1 - \kappa t^2) \sqrt{(1 - k^2 t^2)(1 - t^2)}} dt$$

is the complete elliptic integrals of the third kind.

The expression for the dimensionless function $Q(\lambda, \mu)$ is

$$\begin{aligned} Q(\lambda, \mu) &= q_K \mathbf{K}(\lambda) + q_E \mathbf{E}(\lambda) + q_\Pi \mathbf{\Pi}(\mu^2, \lambda) + q_0, \\ q_K &= \frac{2\lambda^4 - \lambda^4 \mu^2 - 2\lambda^2 \mu^2 + \mu^4}{\mu \lambda^3 \sqrt{1 - \mu^2}}, \quad q_E = -\frac{\mu(\mu^2 - \lambda^2)}{\lambda^3 \sqrt{1 - \mu^2}}, \end{aligned}$$

$$q_{\pi} = \frac{(\mu^2 - 2)(\mu^2 - \lambda^2)^2}{\lambda^3 \mu \sqrt{1 - \mu^2}}, \quad q_0 = \pi \frac{2\lambda^2 - 2\mu^2 - \lambda^2 \mu^2 + \mu^4}{2\lambda^3 (\mu^2 - 1)} \sqrt{\mu^2 - \lambda^2}.$$

For the practically significant cases $1/2$, $-1/2$, -1 and -2 of the parameter k the integrals (6.13) are expressed in analytical form Kobelev (2011). The analytical expressions could be analogously found also for integer cases k .

The major advantage of the analytical solutions is that the final formulas could be used for industrial applications and programming of manufacturing machines. These expressions play the fundamental role for the subsequent analysis. With this method we express analytically the bending moment $M_B(\kappa, \vartheta)$ and torque $M_T(\kappa, \vartheta)$ as the functions of curvature of the bar during bending κ and the angle of twist per unit length ϑ . Otherwise, if the moments m_B, m_T are predefined, the κ and ϑ could be determined solving nonlinear algebraic equations

$$M_B(\kappa, \vartheta) = m_B, \quad M_T(\kappa, \vartheta) = m_T.$$

6.5 Elastic Spring-Back and Appearance of Residual Stresses

The simplest definition of residual stresses is as follows: stresses that remain within a part after it has been deformed and all external forces have been removed. More specifically, the deformation must be non-uniform across the material cross-section in order to give rise to residual stresses. The deformation can result not only from forming operations but also from thermal processes. Phase transformations during heat-treating are known to induce sufficient strain to result in plastic deformation, thereby giving rise to residual stresses. In this chapter, we determine analytically the residual stress, which appear in the cylindrical rod after simultaneous plastic torsion and bending.

One of the principal foundations of mathematical theory of conventional plasticity for rate-independent metals is that there exists a well-defined yield surface in stress space for any material point under deformation. A material point can undergo further plastic deformation if the applied stresses are beyond current yield surface, which is generally referred as plastic loading. If the applied stress state falls within or on the yield surface, the metal will deform elastically only. The unloading of the metal is in this case elastic.

Although it has been always recognized throughout the history of development of plasticity theory that there is indeed inelastic deformation accompanying elastic unloading, which leads to metal's hysteretic behavior, its effects are usually negligible and are ignored in the mathematical treatment.

For a bar, which has been twisted and bended in constant ratio, the residual stresses after unloading are evaluated below. In the state of complete elastic unloading

$$\bar{R} = 1/\bar{\kappa}$$

is the unloaded bending radius and $\bar{\vartheta}$ the angle of twist per unit length of the bar respectively. During the elastic unloading, the decrements of maximal axial strain and shear are

$$\Delta\varepsilon = (\kappa - \bar{\kappa}) \cdot r \quad \Delta\gamma = (\vartheta - \bar{\vartheta}) \cdot r$$

In the final state of complete elastic unloading the bending moment and torque disappear. The condition that bending moment and torque vanish are expressed as follows

$$M_B(\kappa, \theta) - (\kappa - \bar{\kappa}) \cdot EI = 0, \quad M_T(\kappa, \theta) - (\vartheta - \bar{\vartheta}) \cdot GI_T = 0.$$

The constants I and I_T signify the second and polar moments of inertia of circular wire respectively.

Thus, the curvature and twist per unit length in the final, unloaded state are respectively

$$\bar{\kappa} = \kappa - \frac{M(\kappa, \vartheta)}{EI}, \quad \bar{\vartheta} = \vartheta - \frac{M_T(\kappa, \vartheta)}{GI_T}. \quad (6.15)$$

The residual stresses in the state of complete elastic unloading $\bar{\sigma}_{zz}$, $\bar{\tau}_{\varphi z}$ could be obtain immediately subtracting the elastic stresses from previously calculated stresses σ_{zz} , $\tau_{\varphi z}$ in the state of maximal plasticization:

$$\bar{\sigma}_{zz} = \sigma_{zz} - (\kappa - \bar{\kappa}) \cdot Ex, \quad \bar{\tau}_{\varphi z} = \tau_{\varphi z} - (\vartheta - \bar{\vartheta}) \cdot G\rho. \quad (6.16)$$

6.6 Post-Coiling Shape of Helical Spring

The overwhelming majority of front-wheel-drive automotive suspension systems use helical springs. The process chosen to produce these is determined by quality, performance, price, environmental issues, etc. The industry develops a potentially cost saving cold-coiling process in which less time is spent treating spring metal at elevated temperatures. Generally, there are two ways to coil a spring: hot coiling and cold coiling. Hot coiling implies that the spring is wound from stock at or above the recrystallization temperature. The strength and fatigue resistance are controlled afterwards by an appropriate heat treatment. Cold-coiling means that the helical winding takes place at a low temperature after the spring has been hardened and tempered. Cold coiling allows the high temperature heat treatments to take place on the bar stock, which is easier to handle than the final coiled spring. The resulting residual

stresses can be essentially eliminated by a relatively low temperature tempering treatment following the cold coiling.

The pronounced residual stress pattern within the cold-coiled spring is undesirable for its unpredictable effect on fatigue and corrosion behavior. These stresses are usually by X-ray measurements of the surface stress field along with modeling of the internal stresses. The success of these procedures requires an independent verification of the actual residual stress field over the cross-section of the original wire stock. The only available well-established method for this is neutron diffraction (Allen et al. 1985).

The coiling process itself introduces residual stresses that detrimental to fatigue, creep and corrosion properties. We apply the obtained formulas for determination of plastic stresses during the manufacturing and residual stresses in helical springs (Kobelev 2011).

In the moment of coiling, the spring wire undergoes the simultaneous bending and torsion. Curvature and twist of spatially deformed wire could be determined using methods of differential geometry (Eisenhart 1940). The shape of the wire in the moment of coiling is given by the spatial curve C , which connects the centers cross-sections of wire. The instantaneous curvature κ and torsion χ of the spatial curve C in the moment of coiling are

$$\kappa = \frac{R}{R^2 + p^2}, \quad \chi = \frac{p}{R^2 + p^2}.$$

where R is the coiling radius, $p = 2\pi R \tan(\alpha)$ is the pitch of one coil in the moment of coiling. The angle α designates the inclination of the active loaded element of helix with any plane perpendicular to the axis of the coil in moment of coiling.

Assume the flat coil of the spring of radius R , such that there is no pitch of this coil. The length of wire for one coil is $2\pi R$. To produce the pitch of H , the coil must be stretched in the direction normal to the initial plane of the coil. The angle of twist during stretching is H/R , such that angle of twist per unit length is ϑ . Thus, the torsion χ of the spatial curve C relates to angle of twist per unit length as

$$\bar{\chi} = 2\pi \bar{\vartheta}.$$

In the moment, which follows coiling, the wire unloads elastically and forms the helical spring. The unloaded shape of the spring is given by the spatial curve $\bar{C}$, which connects the centers cross-sections of final spring. The unloaded curvature $\bar{\kappa}_h$ and residual torsion $\bar{\chi}$ of the spatial curve $\bar{C}$ are calculated with Eq. (6.16). The torsion $\bar{\chi}$ of the spatial curve $\bar{C}$ relates to the residual angle of twist per unit length as

$$\bar{\chi} = 2\pi \bar{\vartheta}.$$

With these values, we calculate finally the unloaded radius and unloaded pitch

$$\bar{R} = \frac{\bar{\kappa}}{\bar{\kappa}^2 + \bar{\chi}^2}, \quad \bar{p} = \frac{\bar{\chi}}{\bar{\kappa}^2 + \bar{\chi}^2}.$$

of the manufactured spring, $\bar{p} = 2\pi \bar{R} \tan(\bar{\alpha})$. The angle $\bar{\alpha}$ is the inclination of helix with any plane perpendicular to the axis of the coil after the elastic spring-back of wire.

For comparison, we use the experimental measurements of residual stresses in cold-coiled helical compression springs (Matejicek et al. 2004). The comparison is performed for as-coiled springs in central coils of reported diameter $2\bar{R} = 160$ mm and pitch $\bar{H} = 100$ mm. The diameter of Cr-Si wire, used for the spring manufacturing is of $2r = 14$ mm. For stress calculation the reported elastic constants are

$$E = 216 \text{ GPa}, \quad \nu = 0.28.$$

The tensile strength of 1968 MPa, elongation 9.3% and area reduction 32% were applied.

The calculation results for plastic state in the spring material during manufacturing process are shown on Figs. 6.2 and 6.3. On Fig. 6.2 the bending moment and torque moment as the function of load parameter λ are plotted. The bending moment

$$M_B = \lambda M_{B.\max}$$

and the torsion moment

$$M_T = \lambda M_{T.\max}$$

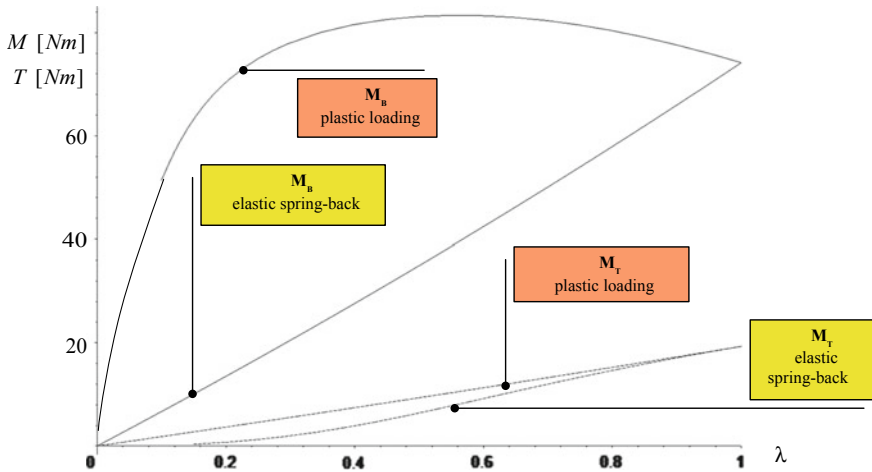


Fig. 6.2 Bending moment and torque as function of loading parameter λ

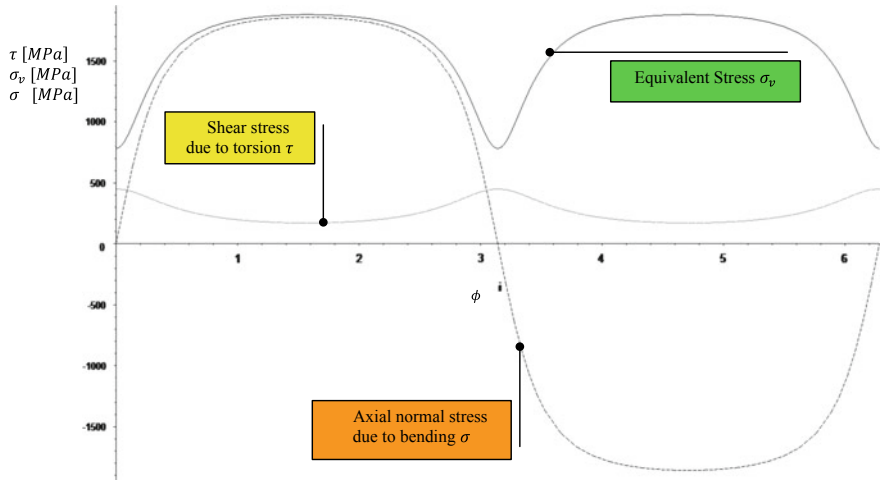


Fig. 6.3 Plastic stresses on the contour of the cross-section in maximal plasticization state

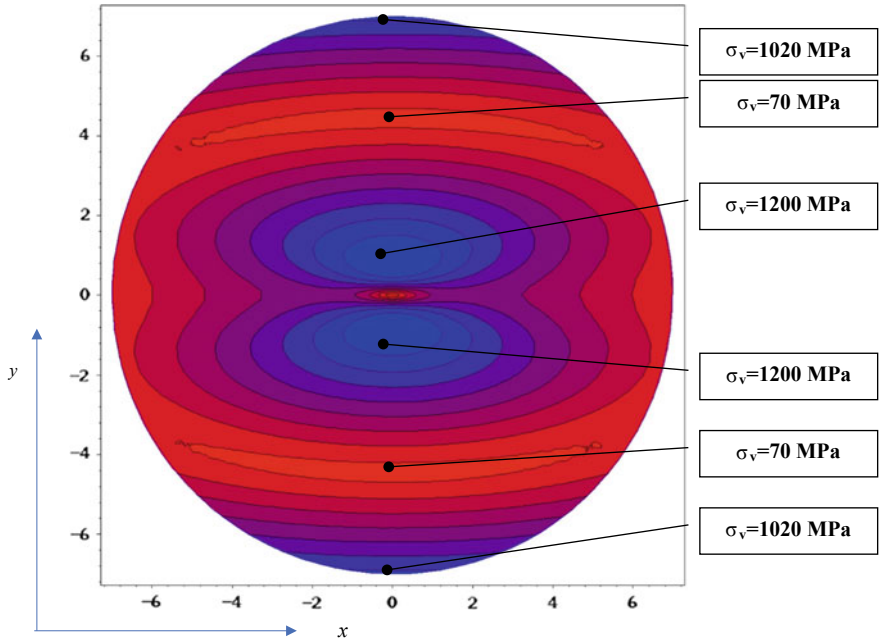


Fig. 6.4 Residual equivalent stress in the unloaded state

increase proportionally to load parameter α . During the plastic loading the load parameter increases from 0 to 1, such that for $\alpha = 1$ the maximal plasticization of the material is achieved. The maximal plastic bending moment is 741 Nm. The plastic torque moment is 193 Nm. Then, during the elastic unloading, the load parameter decreases from 1 to 0, such that for $\lambda = 0$ the free state of the spring with maximal residual stressed of the material is attained. The calculated pitch (Eq. 6.15) at the end of the unload curve ($\lambda = 0$) is $\bar{H} = 106$ mm and the unloaded radius is $2\bar{R} = 136$ mm. For simulation we use the modified Ramberg-Osgood's law.

The stress components and equivalent stress (Eq. 6.16) in the state of maximal plasticization ($\lambda = 1$) are shown on the Fig. 6.3. The maximum of the shear stress is 170 MPa. In the center of the cross-section the stress vanishes. With the increasing radius the stress increases first linearly with radius of the observation point. On the outer surface of the rod the bending dominates, such that the shear stress stagnates. The normal axial stress at the maximal plasticization point $\lambda = 1$ increases from the value $\sigma_{zz} = -980$ MPa on the outmost outer point of the spring body ($x = -r$) to the value $\sigma_{zz} = 980$ MPa on the inner point ($x = r$). At $x = 0$ the bending stress disappears. The distribution of bending stress is mirror-symmetric due to the neutral axis $x = 0$. The Mises equivalent stress (Fig. 6.4) in the spring cross-section is symmetric due to the neutral axis $x = 0$. The maximal value of equivalent stress $\sigma_v = 1020$ MPa is attained on the outmost outer and inner points of the cross-section. In the center of the cross-section both shear and axial stress disappear, such that the equivalent stress vanishes. The plots of equivalent, shear and bending stresses on the outer contour of the circular cross-section are given on Fig. 6.3.

Figures 6.6 and 6.8 show components and equivalent stress in the final unloaded state of spring ($\lambda = 0$, spring as-coiled). The equivalent residual stress is plotted on Fig. 6.6. The equivalent stress in the unloaded state over the spring cross-section is also symmetric due to the neutral axis $x = 0$, but its maximal value is in the inner regions of the cross-section. The graph of residual shear stress and the axial stress are on Figs. 6.5 and 6.6 correspondingly. The shear stress is symmetric over $x = 0$ axis, but the axial stress due to bending is mirror-symmetric. The plots of equivalent, shear and bending stresses on the outer contour of the circular cross-section are given on Fig. 6.6. The profile of equivalent, shear and bending stresses along the line $y = 0$, also $-r < x < r$ is plotted on Fig. 6.8. On Fig. 6.8 the rectangular points show the measured values of stress reported in the cited paper of Matejicek et al. (2004): The simulated values of axial stress demonstrate an excellent correlation to the measured values. It is evidently exposed, that the equivalent, shear and bending stresses attain their maximal values in the inner region of the cross-section. The maximal values of the residual stress in the inner region are around 10–15% higher, as the corresponding maximal stress on the surface of the wire (Fig. 6.9).

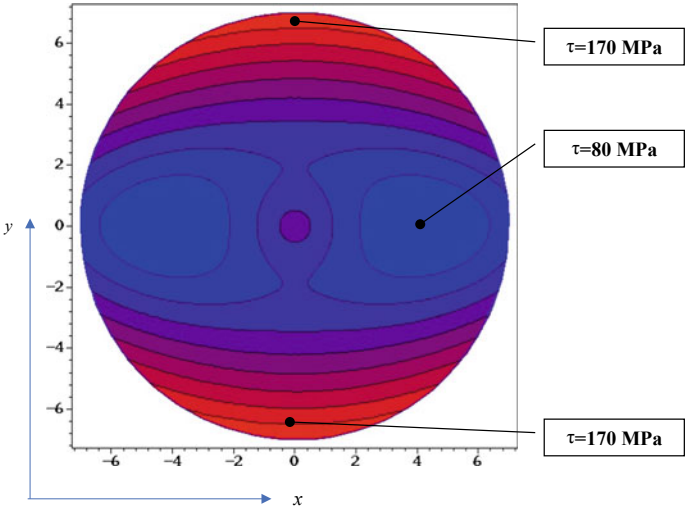


Fig. 6.5 Residual shear stress in the unloaded state

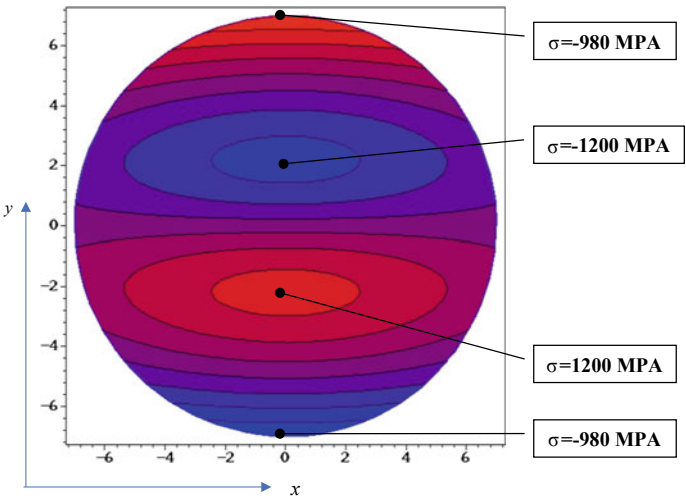


Fig. 6.6 Residual axial normal stress in the unloaded state

6.7 Conclusions

A significant for manufacturing praxis example describes the coiling process of helical spring. The results demonstrate the plasticization process and the origin of residual stresses. The analytical results correspond to the experimentally acquired

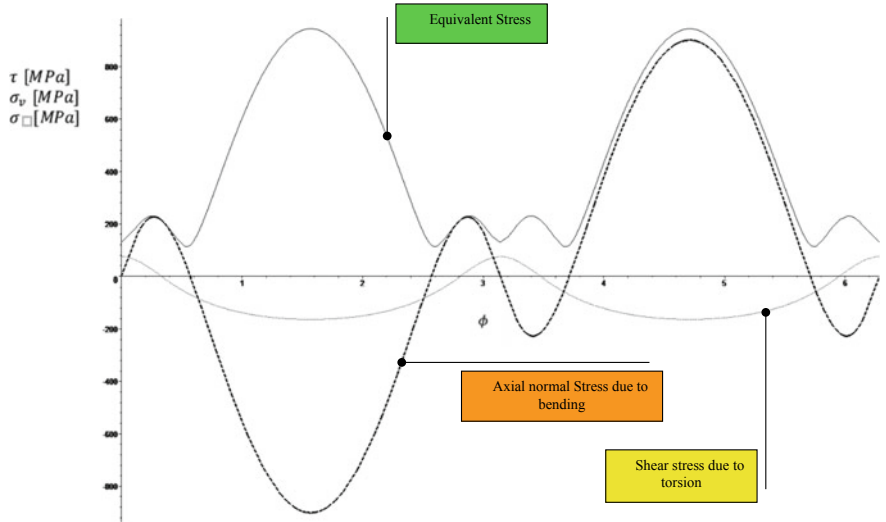


Fig. 6.7 Residual stresses on the contour of the cross-section in the unloaded state

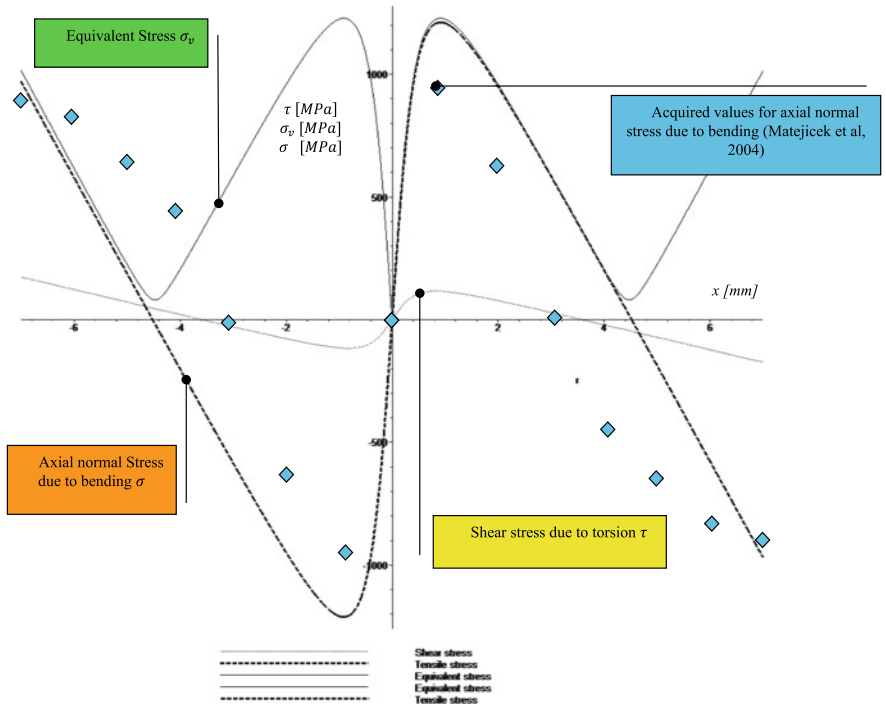


Fig. 6.8 Residual stresses along the positive axis $0 < x < r$ in the unloaded state

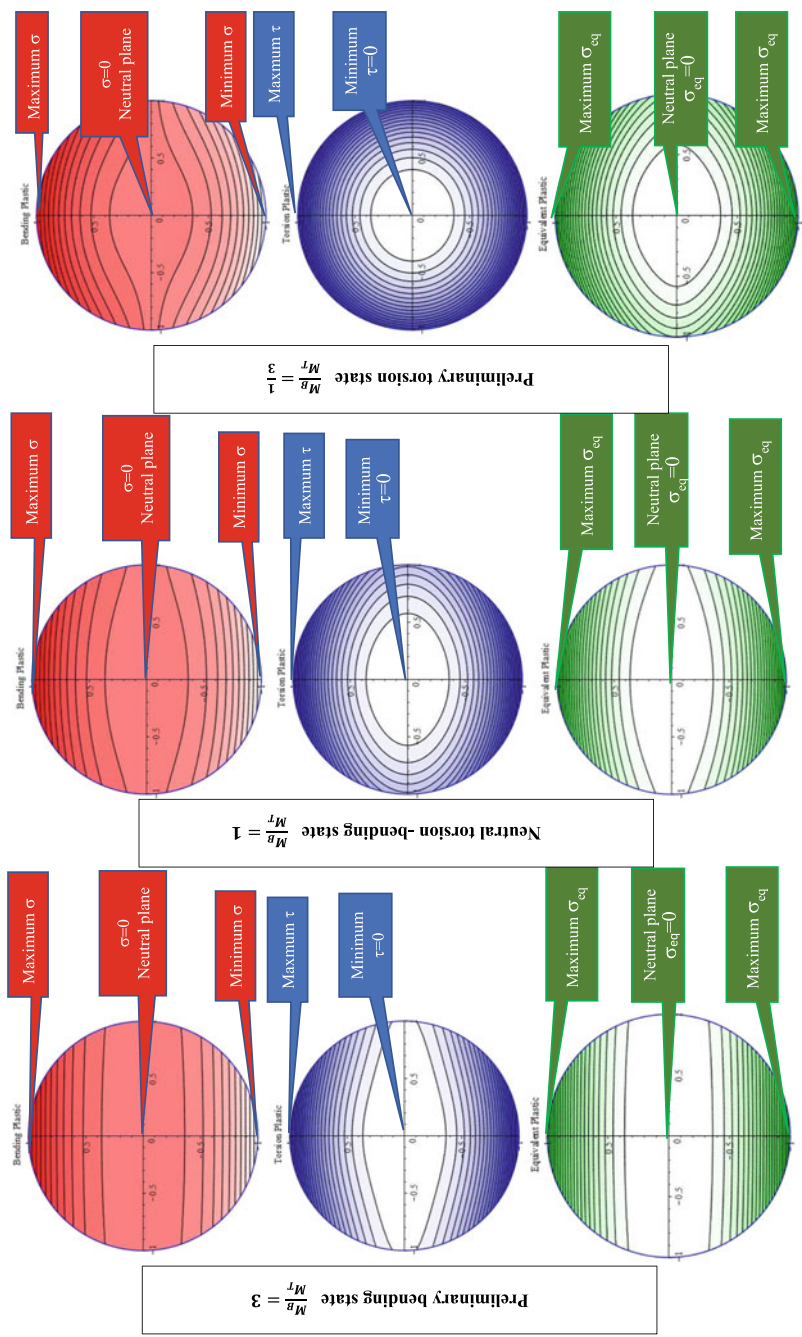


Fig. 6.9 Active stresses along for three different levels of loading. Left—bending moment dominates, middle equal torsion and bending moments, right—bending moment dominates

values. The proposed method does not require numerical simulation and is well-matched for programming of coiling machines. The analytical results display good correlation with the measured residual stresses. The estimation of loads during manufacturing of cold-wound helical springs and for dimensioning and wear calculation of coiling tools could be performed with the closed form solutions.

6.8 Summary of Principal Results

- The coiling process from initially straight wire to the final helical spring are analyzed using deformational theory of plasticity.
- The effects of curvature and pitch during the plastic coiling are accounted.
- The normal, shearing, and octahedral stresses during coiling process and resulting residual stresses are presented with formulas.
- The closed form solutions for the moments and forces in the instruments of the wire coiling machine are derived.

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Chapter 7

Presetting and Residual Stresses in Springs



Abstract In this chapter, the method for calculation of residual stress and enduring deformation of helical springs is developed. The method is based on the deformational formulation of plasticity theory and common kinematic hypotheses (SAE AE-22, Sect. 8.3.33). Two principal types of the helical springs—the compression springs and the torsion springs are studied. For the first type (axial compression or tension springs) the spring wire is twisted. The basic approach neglects the pitch and curvature of the coil, substituting the helical wire by the straight cylindrical bar. The helical spring is in the state of screw dislocation, accordingly, the wire is twisted. The elastic–plastic torsion of the straight bar with the circular cross-section is examined. For the second type (torsion helical springs) the helical wire is in the state edge dislocation, so the wire is in state of flexure. The elastic–plastic bending of the straight bar with the rectangular and circular cross-section is studied using the Bernoulli’s hypothesis. The material for both types of springs is nonlinearly-hardening elastic–plastic with the elastic unloading. The hyperbolic, Ramberg–Osgood, Ludwik–Hollomon, Swift–Voce, Johnson–Cook laws for the material are surveyed. For both problems the elastic–plastic active deformation and elastic spring-back allow the closed-form solutions. Further, this Chapter examines the calculation of remaining deformation and residual stress for helical springs after long-lasting presetting process. The article extends the model for the immediate presetting process accounting the creep deformation of the spring. The method is based on plasticity theory for the instant flow overexposed by the relaxation over the long-term presetting. In this article the following method is used. The plastic deformation of the helical spring with the circular cross-section occurs instantly. If the shortening of the spring in the tool holder persists, the relaxation of stresses occurs and the force of the spring reduces. As the consequence, after the elastic unloading of the long-time presetting, the residual stresses spring reduce gradually with the squeezing time as well. The final length of the springs considerably shortens with the increasing preset duration. The advantage of the discovered closed form solutions is the calculation without the necessity of complex finite-element simulation of spring length loss and residual stresses after presetting process. The analytical expressions are proposed and the exact calibration applied for evaluation of factors for presetting processes. This Chapter is the closing section of second part, which studies manufacturing processes of the helical springs.

7.1 Elastic-plastic Deformation During the Presetting Process of Helical Springs

As already mentioned above, the helical spring is a spiral wound solid bar, usually with a constant coil diameter and uniform pitch (SAE HS 795 1989; DIN EN 13,906:1 2002; DIN EN 13,906:3 2002). Cold coiling process of high tensile steel rod into helical coil springs for the automotive industry is an advanced technique currently implemented by leading spring manufacturers. The most important fabrication stages in the cold-coil forming process are: cold coiling; tempering; presetting; shot-peening. Residual stress plays an important role with respect to the operating performance of helical springs. Its effect on the different properties of a material (fatigue, fracture, corrosion, friction, wear, etc.) can be considerable.

In the modern design of springs, residual stress has therefore to be taken into account. The residual stresses of different origin occur after each manufacturing step. The residual stress fields are normally axisymmetric, that the principal variation in all samples occurred along the hoop direction (helical circumference), at the same time as the radial and axial stresses are considerably lower. The simulations and experimental investigations of two-dimensional stress maps of the rod's cross section have been collected enlightening basic features related with the principal cold coiling steps.

The engineering theory of the presetting process was reported by Graves and O'Malley (1983). In this article the influence of the chemical composition of the spring material and the achieved shear stress was studied experimentally. The influence of the temperature during the presetting on the spring length reduction was evaluated from the spring tests.

The presetting not only reduced stress concentrations at the internal bore, but contributed to the establishment of advantageous surface residual stress conditions that enhance the fatigue life and resistance to creep and relaxation of the manufactured helical spring (Del Llano-Vizcaya et al. 2006; Wakita et al. 2007). The precise evaluation of presetting parameters is obligatory for the optimal adjustment of the presetting process.

Compression and tension springs are made of an elastic wire material formed into the shape of a helix, which returns after the application of a moderate operating load to its natural length after the release of the load. The helical spring with the constant mean helix diameter, wire diameter $d = 2R$ and active coils number n_a is considered in this article. The principal stresses in helical springs are shear stresses due to twisting of the spring wire. If the spring index $D/2R$ significantly greater than one, the influence of wire curvature on the stress distribution in the cross-section of the helical spring could be neglected. For example, if the spring index is equal to 5, the variation of the shear stress over the circumference of the cross-section is about 29%. In this case the equations of torsion of the straight wire are applicable for the stress distribution over the cross-section of the helical spring (Chap. 2).

Presetting of the springs is an operation of loading the spring beyond the plastic limits (Atterbury and Diboll 1960). Before presetting, the initial length of the helical

compression spring is equal to L_0 (Fig. 7.1). The spring is rapidly pressed to a fixed preset length L_1 and is hold in this length in the tool for a definite period of time T_P . Usually, the presetting length L_1 is slightly greater the block length to prevent the clash of coils and their slide shift. Afterward, the spring is released, such that the external loads vanish. In this moment the spring unloads elastically and the residual stresses occur. The final length of the spring is L_2 , such that $L_1 < L_2 < L_0$. Presetting of the springs induces a favorable residual stress field in the wire. This residual stress field improves the operating load capacity.

The stress profiles during the presetting of the helical spring and the straight torsion bar are similar. In the torsion bar, a relatively large presetting torque is applied to cause an outer region of material under the surface to become plastic while the core of the wire remains elastic. Since part of the wire cross-section has yielded, in the moment the presetting torque is removed, the wire recovers elastically but does not return to its original position. Therefore the torsion remains enduringly deformed. The corresponding residual shear stresses are “opposite” at the surface. In this context the word “opposite” means, that the sign of shear stress induced by the applied load, differs from the sign of the residual stress. When the torsion bar is subsequently stressed by the operating load, the residual stresses are subtractive from the stresses attributable to the operating load. Consequently, the surface stresses are less than those which would have occurred in the lack of preset.

Analogously, presetting of the helical compression springs consists of coiling the spring to a longer length than the ultimate desired free length and then compressing it beyond its elastic limit. Presetting allows the use of higher design stresses with the added cost of the secondary operation (Shimoseki et al. 2003; Geinitz et al. 2011). Presetting improves considerably the admissible static loads, decreasing creep and

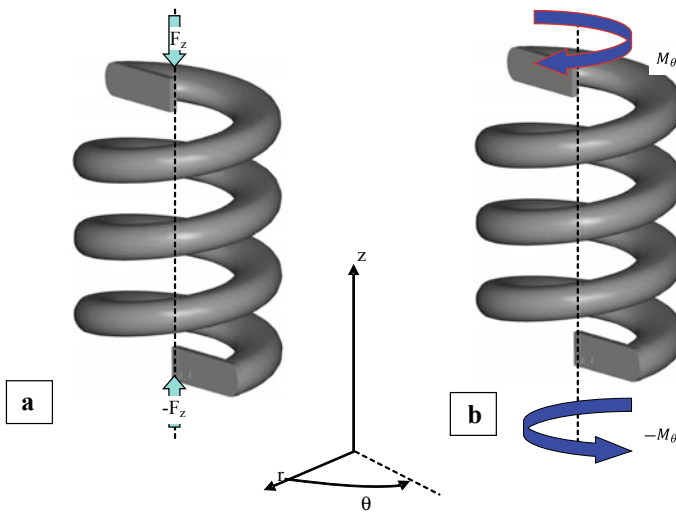


Fig. 7.1 Presetting of the helical compression spring (a) and helical torsion spring (b)

relaxation of springs under operation conditions. Fatigue life is also improved to a certain degree.

The article (Kreuzer 1984) reports the results of the experimental evaluation of presetting of helical compression springs under the elevated temperatures. The reduction of spring force was evaluated as the function of presetting temperature and test shear stress on the surface of wire. Different spring alloys were studied: Ti318, BS 5216 HD3, BS 5216 M4, Ti314A and CrSi/CrV steel wires.

Yang and Wang (1996) studied experimentally the cyclic creep and fracture behavior of SAE spring steel 5160 for two presetting conditions. In this study was demonstrated, that the cyclic creep rate, cyclic plastic strain range, total plastic strain range and cyclic creep life of the material depend strongly on the preset condition.

The residual stresses due to plastic presetting the surface of a solid bar were studied in (Močilnik et al. 2015). The cited article deals with the residual stresses created at the surface through the plastic presetting. Residual stresses were acquired on the surface of a specimen by X-ray diffraction for different angles of following plastic presetting. The residual stresses were also evaluated using analytical and numerical modelling by finite element methods. The approximate analytical method was based on the torsional characteristic, τ - γ , of the material and tension test results. The cited article reveals that the direction of cold rolling on the surface has a significant influence on residual stresses.

The residual stresses are usually analyzed by X-ray measurements of the surface stress field. The internal stresses could not be directly evaluated because the probing depth of X-rays in metal is low. Another reliable method for direct experimental acquisition of the internal residual stresses over the depth of steel probe is neutron diffraction (Allen et al. 1985; Hutchings et al. 1992). The non-destructive measurement of residual stress fields is possible. The neutron diffraction uses the probing power of the neutron into steel, combined with the sensitivity of diffraction (Karpov et al. 2018). The measurement evaluates the separation of lattice planes within grains of polycrystalline steel, thus providing an internal strain gauge. The experimental measurements of residual stresses in cold-coiled helical compression springs were performed in (Matejicek et al. 2004).

In the article (Kobelev 2019) the model for simulation of the residual stresses due to presetting of the helical spring was developed. The analytical formulas permit the assessments of residual stresses and changes of helical spring shape during the manufacturing step. The developed procedures are intended for engineering applications and user interface design of manufacturing equipment and for design of presetting tools. Compression/tension or torsion helical springs are made of an elastic wire material formed into the shape of a helix, which returns to its natural length when unloaded (DIN EN 13,906:1 2002).

The article (Kobelev 2020) introduced the calculation of remaining deformation and residual stress for helical springs after long-lasting presetting process. The simplified ideal-elastic-plastic model for the immediate presetting took into account the creep deformation of the spring. The method was based on the rate-independent material model without hardening, overexposed by the relaxation over the long-term presetting.

The following method is used in this Chapter. The analysis of active plastic deformation requires a certain nonlinear stress strain law. In this article, the irreversible deformation during the presetting process is analyzed using deformational theory of plasticity. One of the principal foundations of mathematical theory of conventional plasticity for rate-independent metals is that there exists a well-defined yield surface in stress space for any material point under deformation. A material point can undergo further plastic deformation if the applied stresses are beyond current yield surface which is generally referred as plastic loading. If the applied stress state falls within or on the yield surface, the metal will deform elastically only and is undergoing elastic unloading. The time-delayed presetting causes the relaxation of the stresses and also the reduction of the residual stresses. This process requires the analysis methods for creeping solids.

The structure of this chapter is the following.

In Sect. 7.2 the deformation formulation of plasticity theory with the typical nonlinear flow functions (“hyperbolic” and “Ramberg, Osgood” laws) is briefly introduced. For analysis were used the material properties of the oil hardened and tempered and patented cold drawn spring steel wire of diameter. For experimental assessment, the spring wires have been heat-treated depending on material specifications. The shear stress shear characteristics of common CrSi spring wire were determined using the procedure described in (Reich 2017).

In Sect. 7.3 the analysis of active plastic torsion and spring-back of the twisted circular wire is performed. This auxiliary solution is used for analysis of presetting for the helical compression springs. The application of the deformation theory of plasticity permits the closed form analytical analysis of the elastic–plastic problem. The analytical derivation makes available the principal formulas for the presetting force over spring travel, active and residual stress on the surface of spring wire and spring length reduction after the completion of the presetting. The major advantage of the analytical solutions is that the final formulas could be used for industrial applications and programming of manufacturing machines. These expressions play the fundamental role for the subsequent analysis. With this method we express analytically the torque as the function the spring travel. The residual stresses in the state of complete elastic unloading on the outer surface of the wire are expressed in closed form.

Section 7.4 studies the helical springs loaded by the axial torque. For analysis the auxiliary problem for the elastic–plastic bending of the wire is solved. The closed-form solutions provide the formulas for the preset moment and elastic spring-back. The analytical expressions for the residual stresses in the state of complete elastic unloading on the outer surface of the wire are provided.

In Sect. 7.5 the alternative method for the description of the material properties. For the simulation of deep plastic deformation, the explicit formulations (Ludwik, Hollomon, Swift and Voce, Johnson and Cook) of the stress–strain curves and strain-hardening of metals (Swift 1952), (Kleemola and Nieminen 1974; Birkert et al. 2013), were used. The nonlinear stress–strain expression due to Ludwik, Hollomon, Swift and Voce express the stress as the explicit function of strain. Section summarizes the coefficients of the stress–strain equations and the relations between the

explicit and implicit formulations. For the closed-form solutions of the plastic deformation problems the relations between Ludwik–Hollomon equation on one side and Ramberg–Osgood equation on the other side are applied. The coefficients of Ludwik–Hollomon and Johnson–Cook equations are expressed through the coefficients of the Ramberg–Osgood relations. The closed form solutions for the problems of torsion and bending of rods are presented. The constitutive laws combine the deformational theory of plasticity with the traditional nonlinear stress–strain laws describing active plastic deformation. The plastic flow during the active load and the origination of the residual stresses are studied.

Section 7.6 studies the relaxation of stresses in the state of time-delayed presetting. After the instantaneous plastic deformation completes, the stage of creep flow occurs the principal hypothesis is the following. The metallic material is in core region elastic and is in the crust region undergoes plastic flow. During the time-delayed presetting, the strain persists, and in both regions of the elastic and plastic deformation occurs the relaxation of stresses.

The closed form solutions allow the assessments of stresses and geometry changes during manufacturing of helical springs. The formulas could be used for industrial applications and programming of presetting tools.

7.2 Implicit Formulations for Plastic Stress–Strain Curves

For the analytical analysis of the elastic–plastic problem, the deformation theory of plasticity with the certain nonlinear flow functions is implemented (SAE AE-22, Sect. 8.3.33; Jones 2009) Commonly accepted, that the spring steels are isotropic, work-hardening materials with the not distinctly observable yielding points. For such behavior of material, the following nonlinear stress–strain expression can be used:

$$\mathbf{e} = f_{\tau}(\tau_{oct}) \frac{\mathbf{s}}{\tau_{oct}}. \quad (7.1)$$

The components of deviatoric stress $\mathbf{s}$ and strain $\mathbf{e}$ tensors respectively are:

$$s_{ij} = \sigma_{ij} - \sigma_{oct} \delta_{ij} \quad (7.2)$$

$$e_{ij} = \varepsilon_{ij} - \varepsilon_{oct} \delta_{ij}. \quad (7.3)$$

The octahedral normal strain and octahedral normal stress are:

$$\varepsilon_{oct} \equiv \mathbf{I}(\boldsymbol{\varepsilon})/3, \sigma_{oct} \equiv \mathbf{I}(\boldsymbol{\sigma})/3. \quad (7.4)$$

The octahedral shear stress and strain in Eq. (7.1) relate to the second invariants of the corresponding deviatoric tensors $\mathbf{II}(\mathbf{e})$, $\mathbf{II}(\mathbf{s})$ (Eqs. (A.5), (A.6) (Appendix A)):

$$\tau_{oct} \stackrel{\text{def}}{=} \sqrt{2\mathbf{\Pi}(\mathbf{s})/3}, \quad (7.5)$$

$$\gamma_{oct}/2 \stackrel{\text{def}}{=} \sqrt{2\mathbf{\Pi}(\mathbf{e})/3}. \quad (7.6)$$

The Eq. (7.1) expresses the dependence of the strain deviator upon the stress deviator. Because the stress could not be resolved in terms of strain, the stress–strain Eq. (7.1) will be referred to as “implicit stress–strain curve”.

From the Eq. (7.1) follows the relation between shear stress and shear strain measures for the “implicit stress–strain function”:

$$\gamma_{oct} = f_{\tau}(\tau_{oct}). \quad (7.7)$$

As it is shown later, the accepted relation (7.1) between the deviators leads to the incompressibility of materials during the plastic flow. For this purpose, consider the state of a pure shear and the state of a uniaxial stress. If the sole components are shear stress $\tau_{xz} = \sqrt{3/2}\tau$ and strain $\gamma_{xz} = \sqrt{3/2}\gamma$, $\gamma = \tau/G$ the intensities of shear stress and strain Eqs. (7.3) and (7.7) read as $\tau_{oct} = \tau$, $\gamma_{oct} = \gamma$. In this case the Eq. (7.7) reduces to $\gamma = f_{\tau}(\tau)$.

Analogously, if the non-zero components are uniaxial stress is $\sigma_{xx} = \sigma$ if follows from (7.1) that the strain $\varepsilon_{xx} = \varepsilon$, $\varepsilon = \sigma/E$. For the incompressible material $\varepsilon_{yy} = \varepsilon_{zz} = -\varepsilon_{xx}/2$. The octahedral shear stress and strain for the uniaxial state are according to Eq. (7.5) correspondingly

$$\tau_{oct} = \frac{\sqrt{2}\sigma}{3}, \gamma_{oct} = \sqrt{2}\varepsilon.$$

The substitution of these formulas in Eq. (7.7) shows, that the following relation must be fulfilled $\sqrt{3}\varepsilon = f_{\tau}(\sigma/\sqrt{3})$. Consequently, the formula for the uniaxial stress follows from the equation for the shear stress:

$$\varepsilon = \frac{1}{\sqrt{3}} f_{\tau} \left(\frac{\sigma}{\sqrt{3}} \right) \stackrel{\text{def}}{=} f_{\sigma}(\sigma). \quad (7.8)$$

For the subsequent derivations we use the simplest assumptions of Eq. (7.1) and (7.8). For more precise applications both stress–strain curves should be acquired experimentally. The corresponding constants for each material law could be properly evaluated numerically using the methods of curve fitting. For example, the stress–strain curves were determined experimentally for steel 54CrSi6 (DIN 10,089, Material number 1.7102) in (Kobeleev 2011). The determination was based on backwards calculation the shear stress vector from the vector of torsion moments. For the determination of stress–strain curve the inverse problem has to be solved. For this purpose, the method of piece-wise linear approximation was implemented. The implemented method is suitable for the finite-element simulation of the plastic deformation of steel wire, but is not easily analyzed for the practical application in manufacture.

The consequent derivation of the closed-form expressions is based on the common plastic deformation laws (Kausel 2017). The first set of expressions for material law will be referenced to as “hyperbolic law”:

$$\gamma = \frac{1}{[1 - (\tau/K_\tau)^n]^l} \cdot \frac{\tau}{G}, \quad \varepsilon = \frac{1}{[1 - (\sigma/K_\sigma)^n]^l} \cdot \frac{\sigma}{E}. \quad (7.9)$$

Here G , E are the elasticity moduli, K_τ , K_σ are the solidifying coefficients and n , l are the solidifying exponents. The material constants in formulas (7.9) are habitually assumed to be unrelated to each other. The common setting for the second solidifying exponent is $l = 1/2$. This material model was used in (Neuber 1961a, b) for the solution of notch problems. For the Neuber material, the formulas (7.9) reduce to the expressions:

$$\gamma = \frac{\tau}{G\sqrt{1 - (\tau/K_\tau)^n}}, \quad \varepsilon = \frac{\sigma}{E\sqrt{1 - (\sigma/K_\sigma)^n}}.$$

The second regularly implemented set of expressions reads according to (Ramberg and Osgood 1943):

$$\gamma = \frac{\tau}{G} + \left(\frac{\tau}{K_\tau}\right)^n, \quad \varepsilon = \frac{\sigma}{E} + \left(\frac{\sigma}{K_\sigma}\right)^n. \quad (7.10)$$

The material constants in formulas (7.10) are also not linked in general case.

It should be noted that the independence of plastic deformation Eq. (7.1) upon the hydrostatic stress impose the certain relations on the material constants. From Eq. (7.8) follow the following relationships between the constants of both laws (7.9) and (7.10):

$$E = 3G, \quad K_\sigma = \sqrt{3}K_\tau.$$

Thus, the hypothesis (7.8) manifests the material incompressibility of the material in course of the plastic flow. For the incompressible material the bulk modulus is infinite, $K \rightarrow \infty$.

Figures 7.2, 7.3, 7.4 relate to the “hyperbolic law” for shear $\gamma = f_\tau(\tau)$ and for uniaxial state $\varepsilon = f_\sigma(\sigma)$ according to Eq. (7.9). Figures 7.5, 7.6, 7.7 use the material laws due to Ramberg and Osgood $\gamma = f_\tau(\tau)$ and $\varepsilon = f_\sigma(\sigma)$ as given in Eq. (7.10). We use the same symbols for the constants in material laws (7.9) and (7.10), remembering in mind, that the numerical values of these constants are different for both laws.

For analysis the material properties of the oil hardened and tempered and patented cold drawn spring steel wire of diameter $d = 3\text{mm}$ were used. Wires have been heat-treated depending on material specifications. Actual shear stress shear characteristic was determined using the procedure described in (Reich 2017). The constants for

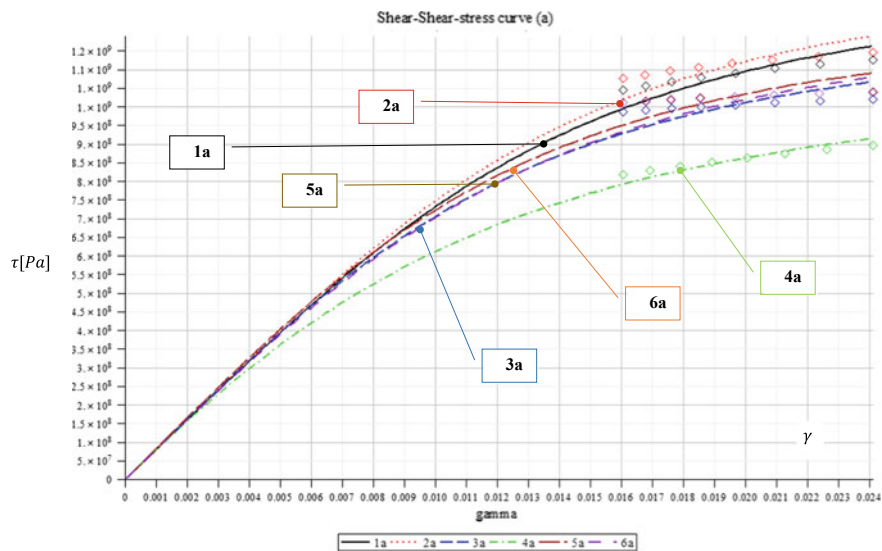


Fig. 7.2 Stress versus shear-strain curve of the oil hardened and tempered spring steel wire (1a,2a,3a) and patented cold drawn spring steel wire (4a,5a,6a) for the “hyperbolic law”

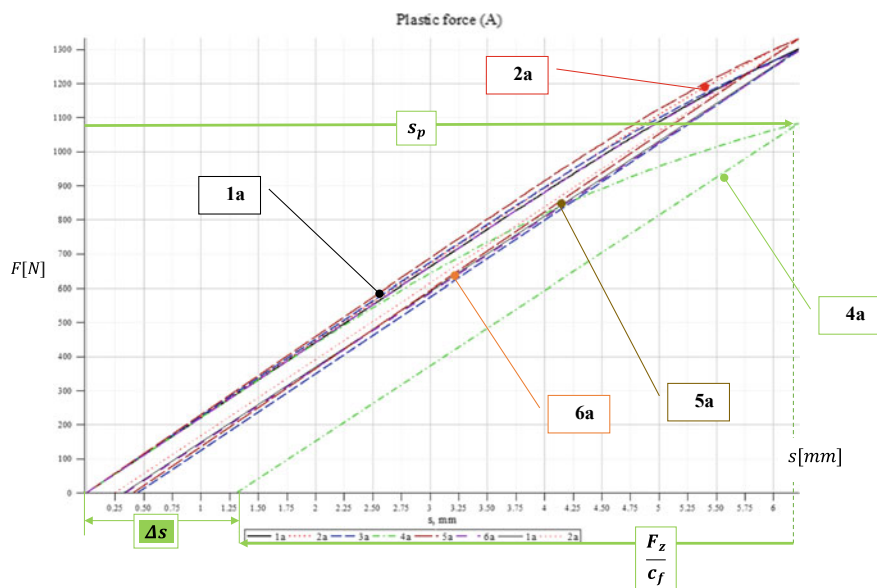


Fig. 7.3 Spring force as functions of the spring stroke for oil hardened and tempered wires (1a,2a,3a) and of patented cold drawn spring steel wires (4a,5a,6a). For calculation the the “hyperbolic law” is used

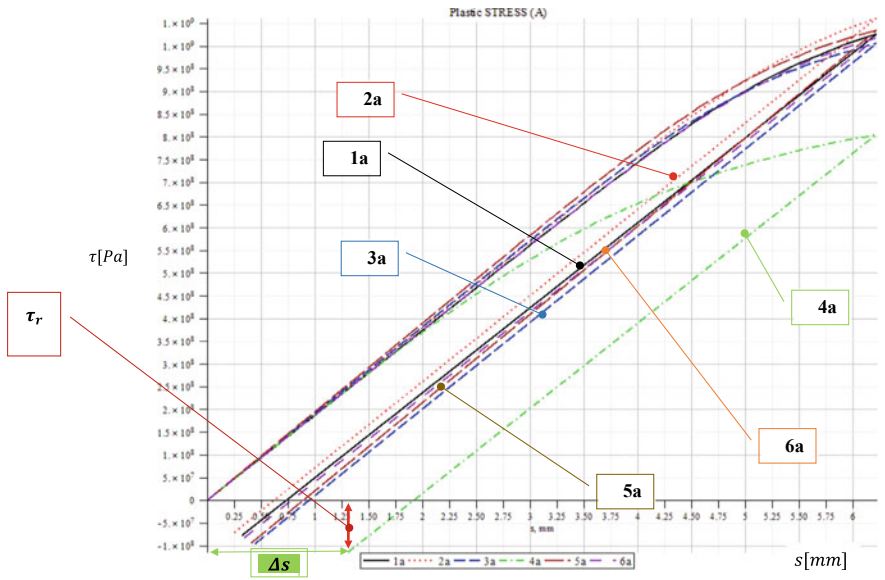


Fig. 7.4 Shear-stress on the surface of spring wire as the function of preset travel for oil hardened and tempered (1a,2a,3a) and of patented cold drawn spring steel wires (4a,5a,6a). The application of the “hyperbolic law”

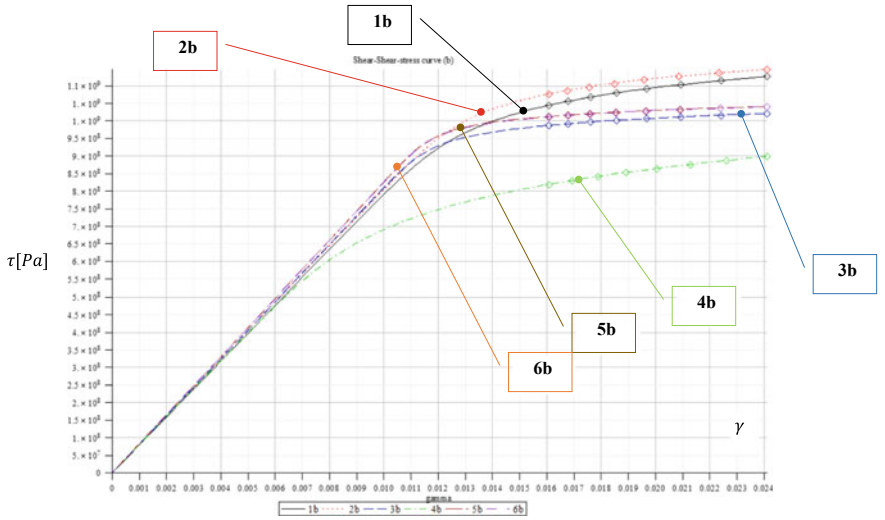


Fig. 7.5 Shear stress as the function of shear for the oil hardened and tempered spring steel wires (1b,2b,3b) and patented cold drawn spring steel wires (4b,5b,6b). Application of the “Ramberg–Osgood-law”

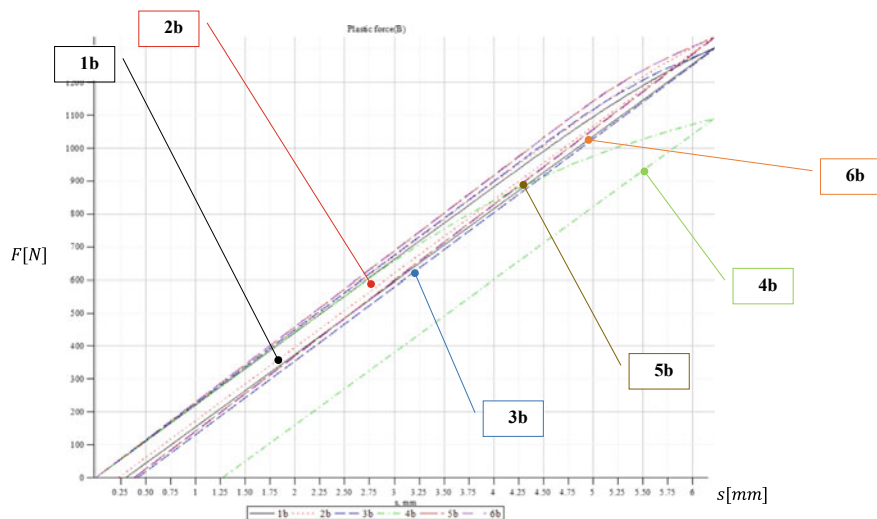


Fig. 7.6 Spring forces as functions of spring travel for oil hardened and tempered (1b,2b,3b) and of patented cold drawn spring steel wires (4b,5b,6b). For calculations the “Ramberg–Osgood-law” is applied

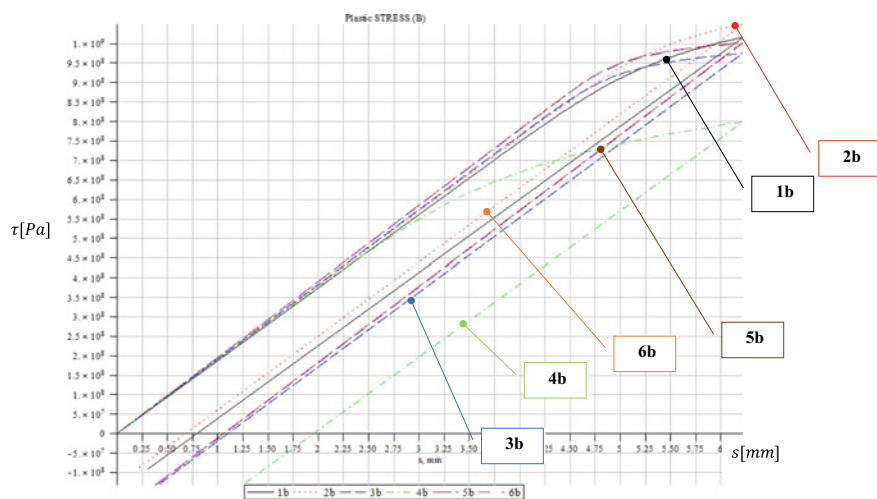


Fig. 7.7 Functional dependence of the shear-stress on the wire surface upon spring travel for oil hardened and tempered (1b,2b,3b) and of patented cold drawn spring steel wires (4b,5b,6b). The “Ramberg–Osgood-law” is used for simulation

Table. 7.1 Material properties for oil hardened and tempered and patented cold drawn spring steel for Ramberg–Osgood law Eq. (7.10)

Material (Reich 2017)	N	Heat treatment	$\gamma = f_{\tau}(\tau)$		
			G	n	K_{τ}
			[GPa]		[MPa]
Oil hardened and tempered spring steel wire	1	Non-heat-treated	79.396	17.26	1496
	2	350 °C/30'	80.405	20.44	1437
	3	420 °C/30'	80.823	32.25	1172
Patented cold drawn spring steel wire	4	non-heat-treated	79.422	8.605	1490
	5	200 °C/30'	82.414	39.84	1164
	6	250 °C/30'	82.434	38.91	1166

Table. 7.2 Material properties for oil hardened and tempered and patented cold drawn spring steel for hyperbolic law, Eq. (7.9), $l = 1/2$

Material	N	Heat treatment	$\gamma = f_{\tau}(\tau)$		
			G	n	K_{τ}
			[GPa]		[MPa]
Oil hardened and tempered spring steel wire	1	Non-heat-treated	79.396	8.650	1219
	2	350 °C/30'	80.405	9.639	1259
	3	420 °C/30'	80.823	8.3672	1169
Patented cold drawn spring steel wire	4	Non-heat-treated	79.422	3.7419	957
	5	200 °C/30'	82.414	8.8377	1201
	6	250 °C/30'	82.434	8.7685	1199

the Ramberg–Osgood law (7.10) are shown in Table 7.1. The methods of curve fitting deliver the material constants for the hyperbolic law, which are listed in Table 7.2. The plots on Fig. 7.2 display the shear stress-shear strain curves of the implemented materials of the hyperbolic law (7.9). Figure 7.5 demonstrates the shear stress-shear strain curves of the implemented materials for the Ramberg–Osgood law (7.10), which was used for derivation of constants for the hyperbolic law. For the consequent comparison of results, the values in Table 7.2 are derived by the curve interpolation of the values of Ramberg–Osgood law. As stated above, for the practical applications the independent evaluation shear and uniaxial stress–strain curves and their interpolations is greatly advantageous.

The traditional methods for calculation of torque during the plastic twist of wire with the circular cross-section, for example reported in (Kachanov 1971), use the resolution of shear stress τ in terms of shear strain γ . The “explicit” formulations of stress–strain laws, which are studied later in Sect. 7.5, express the shear stress over the shear strain.

It must be stressed that the “implicit” material laws, like (7.9) and (7.10), could not be resolved. Specifically, the “implicit” material laws that cannot be explicitly

solved for shear stress as the function of shear strain. The offered method allows the closed-form solutions also for “implicit” formulations.

7.3 Analysis of Active Plastic Torsion and Spring-Back of Circular Wire for Presetting Assessment of Helical Compression Springs

7.3.1 Plastic Deformation of Wire During Active Plastic Torsion of a Solid Round bar

Consider the helical spring with the mean helix diameter D , wire diameter d and active coils number n_a (Fig. 7.1). The compression and extension springs are subjected to the torsional stresses of their wire. This type of springs is described in (SAE HS 795 1989, Sections A and B). For brevity, the diameter of wire, the mean diameter of the helix and its pitch are assumed to be constant, such that the helical spring fits into the cylinder with the diameter $D + d$.

The deflection of the spring is studied on the assumption that it is fixed at the lower end and loaded by the axial load at the upper end. It is assumed that the pitch angle between coils and a plane normal to the axis of the helix is small enough. During the plastic loading, increases the angle of twist per length unit of the bar proportionally to the vertical component of the deflection. For simplification, the effect of pitch angle and curvature of wire are neglected and the deformation consists only of the twisting of the straight wire. T

Presetting of the helical compression springs is explained in (SAE HS 795 1989, Chap. 1, p. 2.10). In this section the mathematical theory of presetting of the helical compression springs is developed. Residual stresses could be induced by the plastic torsion loading and elastic spring-back of a solid bar. A straight cylindrical solid bar with the circular cross-section of the length L is loaded from a stress-free state by the torque M_T . The origin of coordinates is chosen at the centroid of area of one cross-section. The end sections are $z = 0$ and $z = L$.

The shear strain in pure torsion of cylindrical bar with the circular cross-section of the radius $r \equiv d/2$ reads

$$\gamma_{xz} = -\theta y, \gamma_{yz} = \theta x.$$

Here θ is the angle of twist per unit length of the bar. We use the polar coordinates in cross-section for the circular rod:

$$x = \rho \sin \phi, y = \rho \cos \phi, 0 < \rho < r \equiv \frac{d}{2}, \rho = \sqrt{x^2 + y^2}.$$

According to the StVenant's hypothesis, the shear deformation is the linear function of radius (Rubinstein 1977):

$$\gamma_{z\varphi}(\rho, \phi) = 2\varepsilon_{z\varphi} = \theta\rho. \quad (7.11)$$

The shear on the contour of the cross-section γ_p is the linear function of the angle of twist per unit length of the bar θ . The plastic shear strain for the period of the presetting attains its maximum on the contour of the circular cross-section:

$$\gamma_p = \gamma_{z\varphi}(r, \phi) \equiv \theta r. \quad (7.12)$$

In accordance with Eqs. (7.11) and (7.12), the shear strain is the linear function of radius ρ :

$$\gamma(\rho) = \gamma_p \frac{\rho(\tau)}{r}.$$

7.3.2 Torque Moment in Wire Cross-section of Helical Spring During Active Plastic Presetting for Hyperbolic Law

The sole non-vanishing stress component for a solid circular rod subjected to torsion is τ . The “implicit stress–strain functions” express the strain components in terms of stress. The torque moments applied to the end sections of the rod is:

$$M_T = 2\pi \int_0^r \tau(\rho) \rho^2 d\rho. \quad (7.13)$$

For application of Eq. (7.13) requires the expressions for the stress components in each point of the circular cross-section. In the state of torsion, the shear stress $\tau(\rho)$ depend on shear strain along the radius of the circular cross-section. The shear stress on the contour of the cross-section τ_p results from the implicit relation: $\gamma_p = f_\tau(\tau_p)$. If the plastic law $\gamma = f_\tau(\tau)$ could be resolved for the shear stress, the solution greatly simplifies, and evaluation of the integral Eq. (7.13) delivers the torque (Kachanov 1971). Inappropriately, the “implicit stress–strain” laws (7.9) and (7.10) could not be resolved for the shear stress. In this case the change of variables $\rho = \rho(\tau)$ under the integral (7.13) is allows its evaluation:

$$\frac{M_T}{2\pi} = \int_0^r \tau(\rho) \rho^2 d\rho = \int_0^{\tau_p} \tau \rho^2(\tau) \frac{d\rho}{d\tau} d\tau = \int_0^{\tau_p} \tau \rho^2(\tau) \left(\frac{d\tau}{d\rho} \right)^{-1} d\tau. \quad (7.14)$$

For the “hyperbolic law” (7.9) the function $\rho(\tau)$ in Eq. (7.14) reads:

$$\rho(\tau) = r \frac{\gamma}{\gamma_p} \equiv r \frac{f_p(\tau)}{f_p(\tau_p)} \equiv r \frac{\tau}{\tau_p} \left[\frac{1 - \left(\frac{\tau_p}{K_\tau}\right)^n}{1 - \left(\frac{\tau}{K_\tau}\right)^n} \right]^l. \quad (7.15)$$

The new dimensionless variable is introduced for shortness of the formulas:

$$\varphi = \frac{\tau_p}{K_\tau}.$$

The integral (7.14) calculates with the variable φ and with the hypergeometric function ${}_2F_2[]$ as:

$$\begin{aligned} M_T = & \frac{1}{2} \pi \tau_p \left(\frac{d}{2}\right)^3 (1 - \varphi^n)^{3l} \cdot \left\{ {}_2F_2\left(\left[1 + 3l, \frac{4}{n}\right], \left[\frac{n+4}{n}\right], \varphi^n\right) \right. \\ & \left. + {}_2F_2\left(\left[1 + 3l, \frac{n+4}{n}\right], \left[\frac{2(2+n)}{n}\right], \varphi^n\right) \cdot \frac{4(nl-1)}{n+4} \varphi^n \right\}. \end{aligned} \quad (7.16)$$

The common setting for the third solidifying exponent is $l = 1/2$. The straightforward way to derive the expression for torque is to substitute the value of the third solidifying exponent into Eq. (7.15) and perform the integration of torque using Eq. (7.14). The expression for torque in the case $l = 1/2$ reads:

$$\begin{aligned} M_T = & \frac{\pi \tau_p}{2} \left(\frac{d}{2}\right)^3 (1 - \varphi^n)^{3/2} \cdot \left\{ {}_2F_2\left(\left[\frac{5}{2}, \frac{4}{n}\right], \left[\frac{2(2+n)}{n}\right], \varphi^n\right) \right. \\ & \left. + {}_2F_2\left(\left[\frac{5}{2}, \frac{n+4}{n}\right], \left[\frac{2(2+n)}{n}\right], \varphi^n\right) \cdot \frac{2(n-2)}{n+4} \varphi^n \right\} \end{aligned} \quad (7.17)$$

For some values of parameters, the hypergeometric function expresses in terms of the elementary functions. Particularly, the following representations are valid for $n = 2, l = 1/2$:

$$\begin{aligned} M_T &= \frac{2\pi \tau_p}{3\varphi^4} \left(\frac{d}{2}\right)^3 \left(3\varphi^2 - 2 + 2\sqrt{1-\varphi^2} - 2\varphi^2\sqrt{1-\varphi^2}\right), \\ M_T &\underset{\varphi \rightarrow 0}{=} \frac{\pi \tau_p}{2} \left(\frac{d}{2}\right)^3 \left(1 + \frac{1}{6} \left(\frac{\tau_p}{K_\tau}\right)^2 + \frac{1}{16} \left(\frac{\tau_p}{K_\tau}\right)^4 + \dots\right); \end{aligned} \quad (7.18a)$$

for $n = 4, l = 1/2$:

$$\begin{aligned} M_T &= \frac{\pi \tau_p}{3\varphi^4} \left(\frac{d}{2}\right)^3 \left(3\varphi^4 - 1 + \sqrt{1-\varphi^2} - \varphi^4\sqrt{1-\varphi^2}\right), \\ M_T &\underset{\varphi \rightarrow 0}{=} \frac{\pi \tau_p}{2} \left(\frac{d}{2}\right)^3 \left(1 + \frac{1}{4} \left(\frac{\tau_p}{K_\tau}\right)^4 + \frac{1}{24} \left(\frac{\tau_p}{K_\tau}\right)^8 + \dots\right); \end{aligned} \quad (7.18b)$$

for $n = 8, l = 1/2$:

$$M_T = \frac{\pi \tau_p}{6} \left(\frac{d}{2}\right)^3 (\varphi^8 + 3) \equiv \frac{\pi \tau_p}{2} \left(\frac{d}{2}\right)^3 \left(1 + \frac{1}{3} \left(\frac{\tau_p}{K_\tau}\right)^8\right). \quad (7.18c)$$

The limits of the hypergeometric function for the small values of its argument are the following:

$$\lim_{\varphi \rightarrow 0} {}_2F_2 \left(\left[1 + 3l, \frac{4}{n} \right], \left[\frac{n+4}{n} \right], \varphi^n \right) = 1$$

$$\lim_{\varphi \rightarrow 0} {}_2F_2 \left(\left[1 + 3l, \frac{n+4}{n} \right], \left[\frac{2(2+n)}{n} \right], \varphi^n \right) = 1$$

Consequently, for small φ the first term of Taylor series for the twist moment reads:

$$M_T \underset{\varphi \rightarrow 0}{=} \frac{1}{2} \pi \tau_p \left(\frac{d}{2}\right)^3 \underset{\theta \rightarrow 0}{=} \frac{1}{2} \pi G r^4 \theta, \text{ as } \tau_p \underset{\theta \rightarrow 0}{=} G \theta r.$$

This limit case corresponds to the ideal-elastic material with the negligible hardening. The first term of the Taylor series for the moment reveals the moment of the circular rod made of the ideal elastic material. In the case of ideal-elastic deformation, the residual stresses disappear after the elastic spring-back.

7.3.3 Torque Moment in Wire Cross-section of Helical Spring During Active Plastic Presetting for Ramberg–Osgood Law

For the Ramberg–Osgood law, Eq. (7.10):

$$\rho(\tau) = r \frac{\gamma}{\gamma_p} \equiv r \frac{\tau/G + (\tau/K_\tau)^n}{\tau_p/G + (\tau_p/K_\tau)^n}. \quad (7.19)$$

The torque moment is derived as the explicit function of preset stress:

$$M_T = \pi \tau_0 \left(\frac{d}{2}\right)^3 \cdot \frac{\Phi(\varphi)}{\Psi(\varphi)}, \quad (7.20)$$

$$\Phi(\varphi) = c_3 \varphi^3 K_\tau^3 + c_2 \varphi^{2+n} K_\tau^2 G + c_1 \varphi^{1+2n} K_\tau G^2$$

$$+ c_0 \varphi^{3n} G^3, \Psi(\varphi) = (\varphi K_\tau + \varphi^n G)^3 \cdot c_4. \quad (7.21)$$

In Eq. (7.21) the coefficients are the polynomials of the solidifying exponent n :

$$\begin{aligned} c_0 &= 4n(n+3)(n+1), \\ c_1 &= 2(2n+1)(3n+1)(n+3), \\ c_2 &= 4(3n+1)(n+2)(n+1), \\ 2c_3 &= c_4 = 2(n+3)(3n+1)(n+1). \end{aligned} \quad (7.22)$$

In the case $G \rightarrow \infty$ from (7.19) follows:

$$\rho(\tau) = r \frac{\gamma}{\gamma_p} \equiv r \frac{\tau^n}{\tau_p^n}, \quad (7.23)$$

and the Eq. (7.20) reduces to:

$$M_T \underset{G \rightarrow \infty}{=} \pi \tau_p \left(\frac{d}{2} \right)^3 \cdot \frac{2n}{3n+1}. \quad (7.24)$$

The limit expression for small φ the twist moment (7.20) is:

$$M_T \underset{\varphi \rightarrow 0}{=} \frac{\pi \tau_p}{2} \cdot \left(\frac{d}{2} \right)^3 \underset{\theta \rightarrow 0}{=} \frac{\pi G \theta}{2} \cdot \left(\frac{d}{2} \right)^4.$$

The major advantage of the analytical solutions is that the final formulas could be used for industrial applications and programming of manufacturing machines. These expressions play the fundamental role for the subsequent analysis. With this method we express analytically the torque M_T as the function the spring travel.

7.3.4 Elastic Spring-Back and Occurrence of Residual Stresses

The major stresses produced in the compression and tension helical springs are shear stresses due to twisting of the spring wire. For the helical springs made of a wire with the circular cross-section, the shear stress on the outer surface of the wire varies. The inner surface of the wire is highly stressed than the surface of the round straight wire under the action of the equal torque. When the spring index is low, the actual stresses in the wire are excessive due to curvature effect. For the springs with considerably high indices, the stresses equalize over the outer surface of the wire. The finite-element simulations demonstrate, that the influence of the stress variation on the plastic deformation could be neglected for the spring index higher than six. In this case the error in torque evaluation is less than 5% for the common spring steels. In the current study, the spring index is assumed to be high enough, such that the

curvature of wire will be neglected. The shear stress in the helical wire is assumed to coincide to the shear stress in the straight wire under the action of the equal torque.

In terms of the spring wire torque the presetting force reads (Fig. 7.1a):

$$F_z = \frac{2M_T}{D}.$$

With this equation the presetting force of the spring during the plastic deformation results from the expressions (7.16) and (7.20).

The spring travel from the undeformed state to the plastically deformed state is equal to a definite constant value s_p . In the course of the presetting of helical springs the spring force attains the value F_z . When the force disappears, the unloading occurs. The simplest hypothesis for the unloading is the linear elastic spring-back that leads to the increase of the spring length:

$$s_u = \frac{F_z}{c_f}, c_f = \frac{Gd^4}{8D^3n_a}.$$

Consequently, the reduction of the length of the spring after plastic presetting and elastic unloading reads:

$$\Delta s = s_p - s_u \equiv s_p - \frac{2M_T}{c_f D}. \quad (7.25)$$

The unloading could be evaluated analogously for a more realistic unloading model due to (Masing 1926), but the Eqs. (7.24) and (7.25) are sufficient for a practical application.

The calculations were performed for the spring DF219 (Reich 2017) with the mean helix diameter $D = 9\text{mm}$, wire diameter $d = 3\text{mm}$ and active coils number $n_a = 7$. Figs. 7.2 and 7.5 demonstrate the dependences of active plastic and elastic spring-back forces for the hyperbolic and Ramberg–Osgood laws correspondingly. The final points on the force graphs with the zero force show the spring-back length of the spring, i.e. the reduction of spring length in cause of preset. For the hyperbolic law (7.9) preset length loss Δs is shown for six investigated materials on Fig. 7.4. This figure displays also the shear stress during the plastic deformation and elastic spring-back on the surface of the wire. Figure 7.7 reveals the corresponding results for the Ramberg–Osgood law (7.10) for the same materials.

The simplest definition of residual stresses is as follows: stresses that remain within a part after it has been deformed and all external forces have been removed. More specifically, the deformation must be non-uniform across the material cross-section in order to give rise to residual stresses. The deformation can result not only from forming operations but also from thermal processes. Phase transformations during heat treating are known to induce sufficient strain to result in plastic deformation, thereby giving rise to residual stresses. In this paper we determine analytically

the residual stress, which appear in the cylindrical rod after plastic torsion and in the helical spring after presetting.

The residual stresses in the state of complete elastic spring-back τ_r is equal to stresses τ_p in the state of maximal plasticization minus the stresses τ_u due to elastic unloading:

$$\tau_r = \tau_p - \tau_u, \tau_u = \frac{M_T}{W_T}, W_T = \frac{\pi d^3}{16}. \quad (7.30)$$

The residual stresses in the state of complete elastic unloading τ_r on the outer surface of the wire for the hyperbolic law (7.9) is shown for all six investigated materials and the preset stroke of $s_p=7.22$ mm on Fig. 7.4. Analogously, Fig. 7.7 presents residual stresses in the state of complete elastic unloading τ_r on the outer surface for the Ramberg–Osgood law (7.10) for the corresponding material specifications and an equal stroke $s_p=7.22$ mm. The values of preset loss and the residual stress on the surface of the wire are displayed in Table 7.3. The corresponding results for the stroke $s_p=8.22$ mm are shown in Table 7.4.

Figure 7.8 demonstrates the influence of preset stroke on the length loss of the spring and on the induced residual stress on the surface of the wire after preset step. There is the minimal preset travel that causes the reduction of spring length after preset. The higher preset travels lead to the higher length reductions and higher negative residual shear stresses on the surface of the wire. The negative residual stress means that the direction of principal stresses is opposite to the direction of shear stress during the plastic deformation of the wire. There is progressive dependency of length loss upon preset travel, as shown on Fig. 7.8b. Oppositely, dependency of residual stresses upon preset travel, as shown on Fig. 7.8a, is digressive. The published experimental values (Reich 2019) allow the estimations of the preset reductions

Table. 7.3 Length loss and residual stress on the surface for oil hardened and tempered and patented cold drawn spring steel for hyperbolic and for Ramberg–Osgood law (7.10). Preset travel $s_p = 7.22$ mm

Material	N	Hyperbolic law (7.9)		Ramberg–Osgood law, (7.10)		Experiment
		Length loss “a”, Δs	Residual stress “a”, τ_r	Length loss “b”, Δs	Residual stress “b”, τ_r	Length loss (Reich 2019)
		[mm]	[MPa]	[mm]	[MPa]	[mm]
Oil hardened and tempered spring steel wire	1	0.321	−78	0.302	−91	
	2	0.250	−71	0.228	−86	
	3	0.446	−95	0.426	−129	
Patented cold drawn spring steel wire	4	1.312	−114	1.280	−127	1.07
	5	0.407	−94	0.388	−131	0.5
	6	0.344	−81	0.395	−132	0.68

Table. 7.4 Length loss and residual stress on the surface for oil hardened and tempered and patented cold drawn spring steel for hyperbolic and for Ramberg–Osgood law (7.10). Preset travel $s_p=8.22$ mm

Material	N	Hyperbolic law (7.9)		Ramberg–Osgood law, (7.10)	
		Length loss, “a”, Δs	Residual stress “a”, τ_r	Length loss, “b”, Δs	Residual stress “b”, τ_r
		[mm]	[MPa]	[mm]	[MPa]
Oil hardened and tempered spring steel wire	1	1.4216	−154	1.3737	−192
	2	1.3167	−156	1.2797	−201
	3	1.8240	−166	1.7719	−224
Patented cold drawn spring steel wire	4	2.7954	−145	2.7208	−172
	5	1.7777	−171	1.7363	−235
	6	1.6567	−160	1.7458	−232

Table. 7.5 Length loss and residual stress on the surface for oil hardened and tempered and patented cold drawn spring steel for hyperbolic and for Ramberg–Osgood criterion (7.10) for preset travel $= s_p$ 13.22 mm

Material	N	Hyperbolic law (7.9)		Ramberg–Osgood law, (7.10)	
		Tangential stroke loss “a”, Δs	Residual stress “a”, σ_r	Tangential stroke “b”, Δs	Residual stress “b”, σ_r
		[mm]	[MPa]	[mm]	[MPa]
Oil hardened and tempered spring steel wire	1	7.06	−757	4.92	−976
	2	4.99	−783	4.86	−1018
	3	7.83	−746	7.68	−1000
Patented cold drawn spring steel wire	4	7.71	−618	7.53	−775
	5	7.81	−768	7.67	−1032
	6	7.64	−739	7.68	−1030

(Table 7.3). The correlations to the analytically evaluated values for these parameters is satisfactory. The closed solution has to be completely validated in the next step with experiments on helical compression springs with different spring indexes and manufacturing regimes.

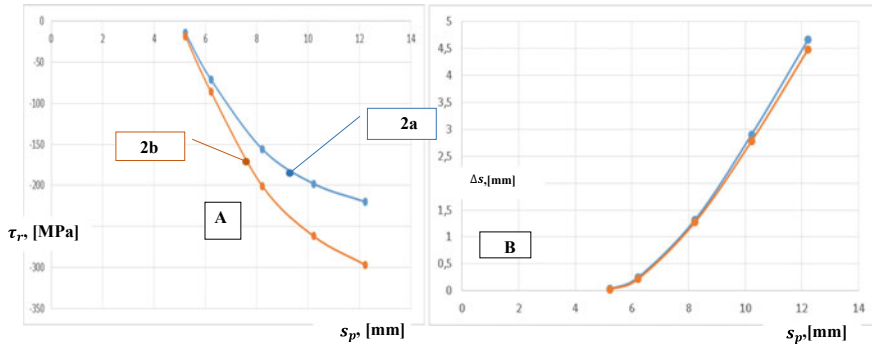


Fig. 7.8 Shear stress on the surface of wire and loss of the spring length as the functions of preset stroke s_p for oil hardened and tempered wires (2a,2b). **a** Surface shear-stress τ_r as function of preset stroke s_p **b** Preset loss Δs as function of preset stroke s_p for two plastic laws for oil hardened and tempered wire

7.4 Analysis of Active Plastic Bending of Rectangular Wire for Evaluation of Presetting for Helical Torsion Springs

7.4.1 Plastic Deformation of Wire During Active Plastic Bending

Torsion springs accumulate elastic energy when exposed to angular deflection. This type of springs is described in (SAE HS 795 1989, Section D). For the torsion spring, the torque M_θ is applied on the upper end of the spring as shown of Fig. 7.1b. The angular deflection leads to an increase in the number coils a relative reduction of the spring diameter. The compression and extension springs, considered in the previous section are subjected to the torsional stresses of wires, while torsion springs are subjected to the bending stresses in the spring wire.

The effect of direction of coiling and beneficial residual stress in the helical torsional springs is explained in (SAE HS 795 1989, Section D2, p. 2.45). In this section the mathematical theory of presetting for helical is developed. The spring wire with rectangular cross-section is loaded from a stress-free state by the bending moment M_B , which is equal to the applied axial torque M_θ . The effect of pitch and curvature of wire for stress calculation will be again neglected for simplicity. This assumption leads to the error in stress estimation less than 5%, if the ratio the mean diameter of spring to the section width is more than 7. The origin of coordinates is chosen at the centroid of area of one cross-section. The uniaxial strain in pure bending of the prismatic bar with the rectangular cross-section of the height T and width B reads:

$$\varepsilon = \kappa x \quad \text{for } -T/2 \leq x \leq T/2. \quad (7.31)$$

In Eq. (7.31), the parameter κ is the curvature of the deformed axis of the bar. The maximal plastic elongation for the duration of presetting:

$$\varepsilon_p = \kappa T/2 \quad (7.32)$$

is attained on the contour of the cross-section. The uniaxial stress on the contour of the cross-section σ_p could be evaluated from the solution of the implicit relation: $\varepsilon_p = f_\sigma(\sigma_p)$.

7.4.2 Bending Moment in Wire Cross-section of Helical Spring During Active Plastic Presetting for Hyperbolic Law

The sole non-vanishing stress component for a solid prismatic rod subjected to bending is σ . Equation (7.1) expresses the strain components in terms of stress. The bending moment of the wire, applied to the end sections of the rod, is:

$$M_B = B \int_{-T/2}^{T/2} \sigma(x) x dx. \quad (7.33)$$

For application of Eq. (7.33) uses the stresses components in each point of the rectangular cross-section. In case of bending, the strain depends linearly on coordinate and implicitly leads to the uniaxial strain $\sigma(x)$:

$$\varepsilon(x) = \frac{2\varepsilon_p x(\sigma)}{T} = f_\sigma(\sigma).$$

As already mentioned above, for the “implicit” laws (7.9) and (7.10) could not be resolved for the shear stress. The change of variables $x = x(\sigma)$ under the integral is allows the evaluation of the integral (7.13):

$$\frac{M_B}{B} = \int_{-T/2}^{T/2} \sigma(x) x dx = 2 \int_0^{\sigma_p} \sigma x(\sigma) \frac{dx}{d\sigma} d\sigma = 2 \int_0^{\sigma_p} \sigma x(\sigma) \left(\frac{d\sigma}{dx} \right)^{-1} d\sigma. \quad (7.34)$$

For the “hyperbolic law” (7.9) the function $\rho(\tau)$ reads:

$$x(\sigma) = x \frac{\sigma}{\sigma_p} \equiv x \frac{f_\sigma(\sigma)}{f_\sigma(\sigma_p)} \equiv x \frac{\sigma}{\sigma_p} \left[\frac{1 - \left(\frac{\sigma_p}{K_\sigma} \right)^n}{1 - \left(\frac{\sigma}{K_\sigma} \right)^n} \right]^l. \quad (7.35)$$

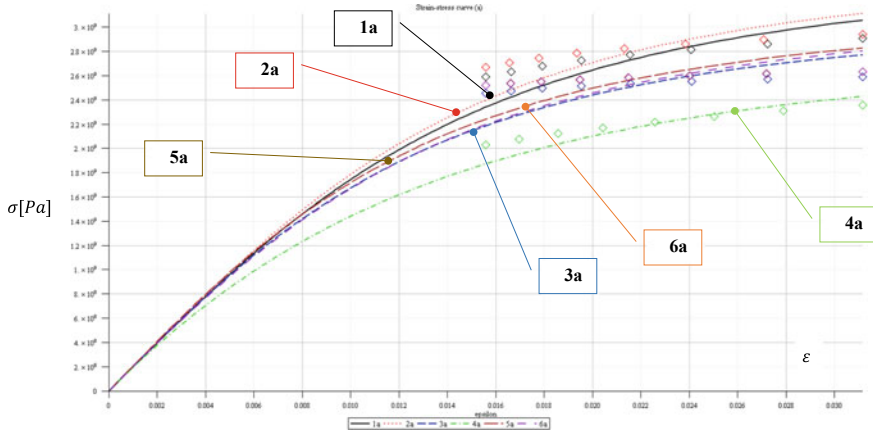


Fig. 7.9 Normal stress as function of tensile strain for the oil hardened and tempered spring steel wires (1a,2a,3a) and patented cold drawn spring steel wires (4a,5a,6a). The “hyperbolic law” is applied

The uniaxial hyperbolic law for the examined materials is shown on Fig. 7.9. We introduce for brevity the new dimensionless variable (dimensionless preset stress):

$$\psi = \frac{\sigma_p}{K_\sigma}.$$

The integral (7.34) evaluates with Eq. (7.35) to:

$$M_B = \frac{\sigma_p B T^2}{6} (1 - \psi^n)^{2l} \cdot \left\{ 2F_2 \left(\left[\frac{3}{n}, 1 + 2l \right], \left[\frac{n+3}{n} \right], \psi^n \right) - 2F_2 \left(\left[\frac{3+n}{n}, 1 + 2l \right], \left[\frac{2n+3}{n} \right], \psi^n \right) \frac{3(l \cdot n - 1)}{n+3} \psi^n \right\} \quad (7.36)$$

For the common setting for the third solidifying exponent $l = 1/2$, the bending moment simplifies to:

$$M_B = \frac{\sigma_p B T^2}{12} \left\{ 3 - (1 - \psi^n)_2 F_2 \left(\left[1, \frac{3}{n} \right], \left[\frac{n+3}{n} \right], \psi^n \right) \right\}. \quad (7.37)$$

For $l = 1/2, n = 2$, the hypergeometric function in (7.37) reduces to the elementary functions:

$$M_B = \frac{B T^2 \sigma_p}{4} \left(\frac{\psi^2 - 1}{\psi^3} \operatorname{arctanh} \psi + \frac{1}{\psi^2} \right). \quad (7.38a)$$

For $l = 1/2, n = 3$, the Eq. (7.37) yields to:

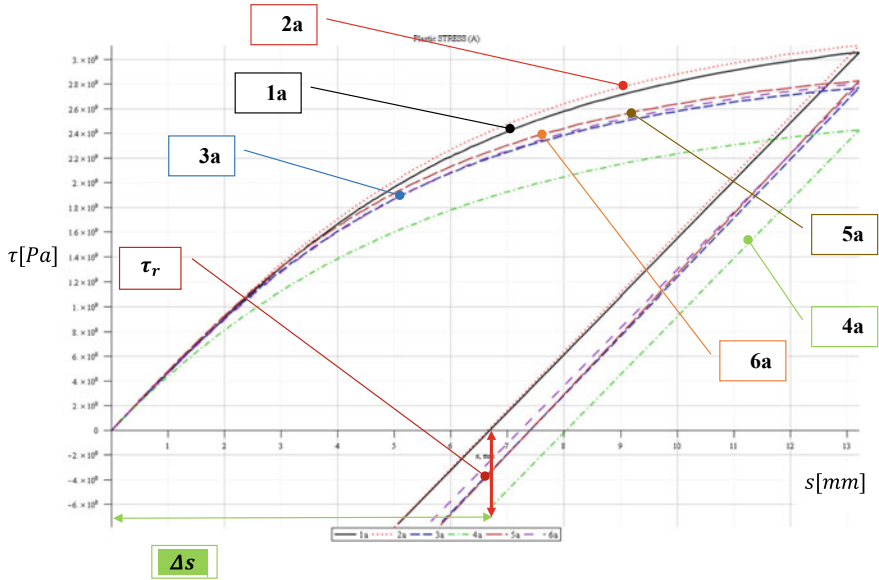


Fig. 7.11 Shear-stress versus as the function of preset of springs meda of oil hardened and tempered (1a,2a,3a) and of patented cold drawn spring steel wires (4a,5a,6a). The “hyperbolic law” is used for calculation

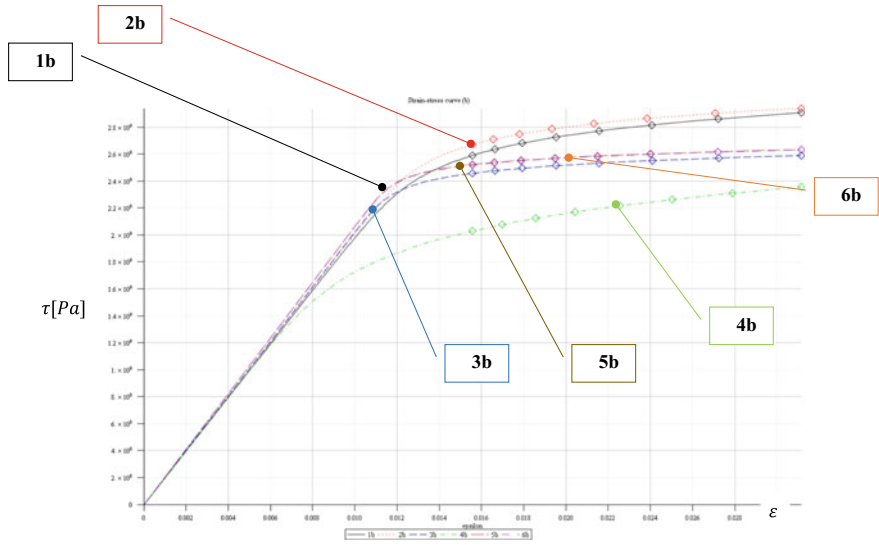


Fig. 7.12 Normal stress as the function of tensile strain curve for the oil hardened and tempered spring steel wires (1b,2b,3b) and patented cold drawn spring steel wires (4b,5b,6b). Assumption of the “Ramberg–Osgood-law”

the moment:

$$M_B = \frac{BT^2\sigma_p}{6} \cdot \frac{\Phi(\psi)}{\Psi(\psi)}, \quad \psi = \frac{\sigma_p}{K_\sigma},$$

$$\Phi(\psi) = c_2\psi^2 K_\sigma^2 + c_1\psi^{1+n} K_\sigma E + c_0\psi^{2n} E^2, \quad (7.40)$$

$$\Psi(\psi) = (\psi K_\sigma + \psi^n E)^2 \cdot c_3. \quad (7.41)$$

In Eq. (7.41) the coefficients are the polynomials of the solidifying exponent m :

$$c_0 = 3n(n+2), \quad c_1 = 3(n+1)(2n+1),$$

$$c_2 = c_3 = (n+2)(2n+1). \quad (7.42)$$

In the case $E \rightarrow \infty$ from (7.39) follows:

$$x(\sigma) = x \frac{\varepsilon}{\varepsilon_p} \equiv x \frac{\sigma^n}{\sigma_p^n} \quad (7.43)$$

and the Eq. (7.40) reduces to:

$$M_B \underset{E \rightarrow \infty}{=} \frac{BT^2\sigma_p}{2} \frac{n}{2n+1}. \quad (7.44)$$

In the elastic case ($\psi \rightarrow 0$) the formula for bending moment (7.40) reduces again to

$$M_B = \sigma_p BT^2/6.$$

7.4.4 Elastic Spring-Back and Evaluation of Residual Stresses

Consider the helical spring with mean helix diameter D and active coils number n_a (DIN EN 13,906:3 2002). The axially directed torsion moment is applied to the end section the helical spring. The torsion moment is identical to the bending moment in all cross-sections of the spring wire M_B . The expressions (7.36) and (7.40) deliver the presetting moment during the plastic deformation. The torsion moment is statically equivalent to the tangential force F_θ (Fig. 7.1b). In the course of the presetting, the spring force attains the value:

$$F_\theta = \frac{2M_B}{D}. \quad (7.45)$$

Torsion moment M_B leads to the travel of the end section of the spring in tangential direction. The twist angle in radians of the torsion spring during plastic presetting is θ_p . The tangential travel from the undeformed state to the plastically deformed state is equal to $s_p = D\theta_p/2$.

Residual stresses appear during the elastic spring-back of the wire. The simplest hypothesis for the unloading is the linear elastic spring-back. The increase of the spring twist angle θ_u during the spring-back reads:

$$s_u = \frac{F_\theta}{c_\theta} = \frac{D}{2}\theta_u, \quad c_\theta = \frac{EBT^3}{3\pi D^3 n_a}. \quad (7.46)$$

Consequently, the final reduction of the opening angle of the spring after plastic presetting and elastic unloading is:

$$\begin{aligned} \Delta s &= s_p - s_u \equiv s_p - \frac{2M_B}{c_\theta D}, \\ \Delta\theta &\equiv \frac{2\Delta s}{D} = \theta_p - \theta_u = \theta_p - \frac{4M_B}{c_\theta D^2} \end{aligned} \quad (7.47)$$

Alternatively, the unloading could be evaluated with the model due (Masing 1926), but the Eqs. (7.44) and (7.45) are satisfactory for everyday practice.

The calculations were performed for the spring with the mean helix diameter = 9mm, wire dimensions $T = 3\text{mm}$, $B = 3\text{mm}$ and active coils number $n_a = 7$. Figures 7.10 and 7.13 demonstrate the dependences of active plastic and elastic spring-back forces for the hyperbolic and Ramberg–Osgood laws correspondingly. The final points on the force graphs with the zero force show the spring-back length of the spring, i.e. the reduction of spring length in cause of preset. For the hyperbolic law (7.9) preset length loss Δs is shown for six investigated materials on Fig. 7.11. This figure displays also the shear stress during the plastic deformation and elastic spring-back on the surface of the wire. Figure 7.14 reveals the corresponding results for the Ramberg–Osgood law (7.10) for the same materials.

The residual stresses in the state of complete elastic unloading σ_r is equal to stresses σ_p in the state of maximal plasticization minus the stresses τ_u due to elastic unloading:

$$\sigma_r = \sigma_p - \sigma_u, \quad \sigma_u = \frac{M_B}{W_B}, \quad W_B = \frac{BT^2}{6}. \quad (7.48)$$

The residual stresses in the state of complete elastic unloading σ_r on the outer surface of the wire for the hyperbolic law (7.9) is shown for all six investigated materials and the preset stroke of $s_p = 13.22\text{ mm}$ on Fig. 7.11. Analogously, Fig. 7.14 presents residual stresses in the state of complete elastic unloading σ_r on the outer

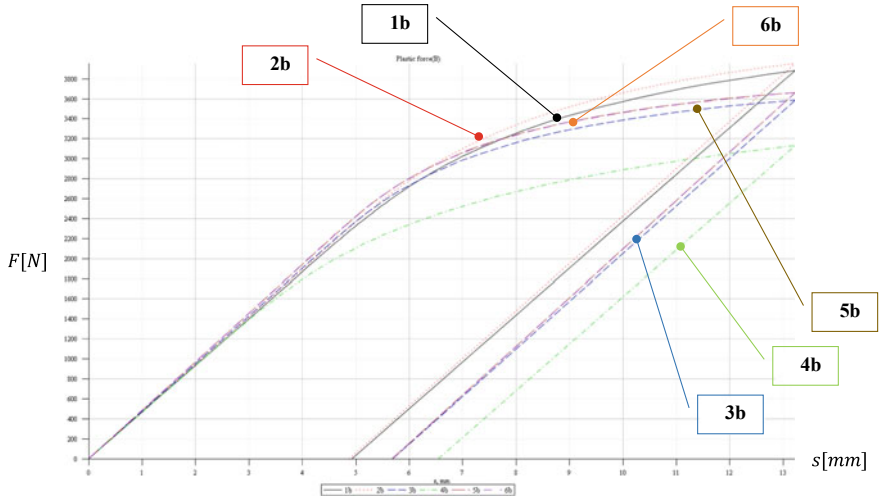


Fig. 7.13 Force-depending on the spring travel for oil hardened and tempered wires (1b,2b,3b) and of patented cold drawn spring steel wires (4b,5b,6b): application of the “Ramberg–Osgood-law”

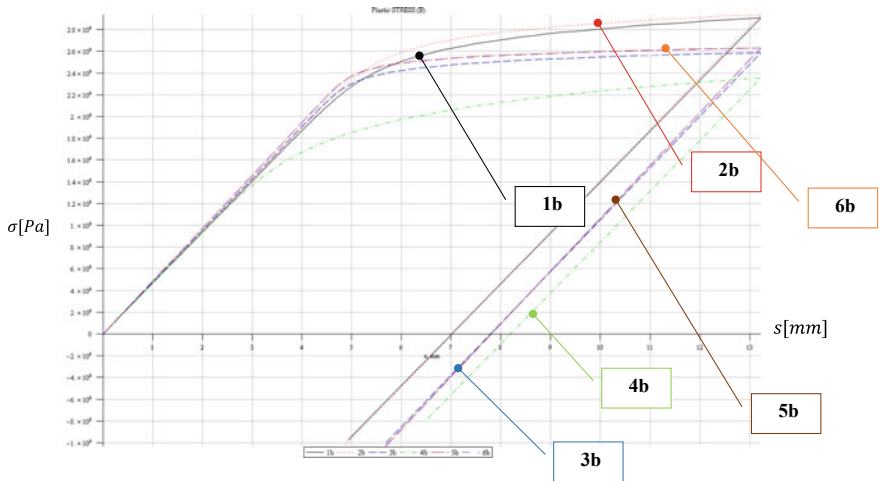


Fig. 7.14 Normal stress versus on the surface as the function of the tangential spring travel for oil hardened and tempered (1b,2b,3b) and of patented cold drawn spring steel wires (4b,5b,6b). The “Ramberg–Osgood-law” is presumed

surface for the Ramberg–Osgood law (7.10) for the corresponding material specifications and an equal stroke $s_p = 13.22$ mm. The values of preset loss and the residual stress on the surface of the wire are displayed in Table 7.7.

7.5 Explicit Formulations for Plastic Stress–Strain Curves

7.5.1 Relations Between Ramberg–Osgood and Johnson–Cook Formulations

For the simulation of deep plastic deformation, the alternative formulations (Ludwik, Hollomon, Swift and Voce, Johnson and Cook) of the stress–strain curves and strain-hardening of metals (Swift 1952; Kleemola and Nieminen 1974; Birkert et al. 2013), are commonly used. The nonlinear stress–strain expression due to Ludwik, Hollomon, Swift and Voce express the stress as the explicit function of strain. The stress–strain equations in tensor form read:

$$\mathbf{s} = \frac{f_{\tau}^{-1}(\gamma_{oct})}{\gamma_{oct}} \mathbf{e}. \quad (7.49)$$

The Eq. (7.49) expresses the dependence of the stress deviator upon the strain deviator. Because the stress components are the functions of strain components, the stress–strain Eq. (7.49) will be referred to as “explicit stress–strain function”. From Eq. (7.49) follows the relation between shear stress and shear strain intensities for the “explicit stress–strain function” in App. A:

$$\frac{\tau_{oct}}{f_{\tau}(\tau_{oct})} = \frac{f_{\tau}^{-1}(\gamma_{oct})}{\gamma_{oct}}.$$

In metallic materials, the relationships metallic material between stress and strain are frequently based on the Johnson–Cook model (Johnson and Cook 1983). The Johnson–Cook model properly exposes the large deformation, high strain rate and elevated temperatures. For the uniaxial deformation the Johnson–Cook stress–strain equation is written in the form (A.8):

$$\sigma = F_1(\varepsilon) F_2(T^{\circ}) F_3(\dot{\varepsilon}). \quad (7.50)$$

The coefficients of the functions in Eq. (7.50) are:

A_{σ} and B_{σ} are the strain coefficients,

T_m° is the melting temperature,

T_{ref}° is a reference (ambient) temperature,

m is the temperature exponent,

$\dot{\varepsilon}$ is the (plastic) strain rate,

$\dot{\varepsilon}_{ref}$ is a reference (quasi-static) strain rate and,

C is the strain rate sensitivity parameter.

The term $F_1(\varepsilon)$ is called the Ludwik–Hollomon equation. This term reveals the quasi-static yielding and work hardening behavior:

$$F_1(\varepsilon) = A_\sigma + B_\sigma \varepsilon^{1/n}. \quad (7.51)$$

The term $F_2(T^\circ)$ determines the temperature dependence :

$$F_2(T^\circ) = 1 - \left(\frac{T^\circ - T_{ref}^\circ}{T_m^\circ - T_{ref}^\circ} \right)^m.$$

The third item demonstrates the strain rate sensitivity. The term $F_3(\dot{\varepsilon})$ includes commonly the logarithm of the strain rate, normalized by a reference value. More general approximations are known (Mauerauch and Vöhringer 1978; El-Magd E. et al. 1985):

$$F_3(\dot{\varepsilon}) = 1 + C \ln \frac{\dot{\varepsilon}}{\dot{\varepsilon}_{ref}}.$$

For example, the strain rate sensitivity is characterized with two dimensionless exponents:

$$F_3(\dot{\varepsilon}) = \left(1 + C \ln^p \frac{\dot{\varepsilon}}{\dot{\varepsilon}_{ref}} \right)^q.$$

For uniaxial stresses and strains, the Ludwik-Hollomon Eq. (7.51) reads:

$$\sigma = A_\sigma + B_\sigma \varepsilon^{1/n}. \quad (7.52)$$

For pure shear stress, the constants of the Ludwik-Hollomon equation are:

$$\tau = A_\tau + B_\tau \gamma^{1/n}, \quad (7.53)$$

$$A_\tau = \frac{A_\sigma}{\sqrt{3}}, B_\tau = \frac{B_\sigma}{\sqrt{3}} \cdot \left(\frac{1}{\sqrt{3}} \right)^{\frac{1}{n}}. \quad (7.54)$$

Table A.1 summarizes the coefficients of the stress–strain equations. The formulas in Table A.1 do not depend on temperature and strain rate. The division of stress in the Ludwik-Hollomon and Ramberg–Osgood equations by the corresponding terms allows to involve these effects into consideration. The term $F_2(T^\circ)$ determines the temperature dependence and $F_3(\dot{\varepsilon})$ describes the strain rate sensitivity. Taking both terms into account, we obtain both explicit and implicit stress–strain relations that depend on temperature and strain rate:

$$\frac{\tau_{oct}}{F_2(T^\circ)F_3(\dot{\varepsilon})} = f_\tau^{-1}(\gamma_{oct}), \quad (7.55)$$

$$\gamma_{oct} = f_{\tau} \left(\frac{\tau_{oct}}{F_2(T^{\circ}) F_3(\dot{\varepsilon})} \right). \quad (7.56)$$

From the Ludwik–Hollomon Eq. (7.51) one gets once again the Johnson–Cook formula, which expresses the octahedral stress τ_{oct} over the octahedral strain τ_{oct} . From Ramberg–Osgood (7.10) follows the inverse formulation, which expresses the strain τ_{oct} through stress τ_{oct} .

Now study the relations between Ludwik–Hollomon equation on one side and Ramberg–Osgood equation on the other side. It is possible to express the coefficients of Ludwik–Hollomon (7.51) and Johnson–Cook Eqs. (7.50) through the coefficients of the Ramberg–Osgood relations. The simplification of the relations could be achieved using the following setting for the constants:

$$B_{\sigma} = K_{\sigma}, A_{\sigma} = \frac{-K_{\sigma}^2}{K_{\sigma} + nE} \approx -\frac{K_{\sigma}^2}{nE}. \quad (7.57)$$

In this case the Ludwik–Hollomon (7.51), Johnson–Cook (7.50) and Ramberg–Osgood (7.10) curves match at $\varepsilon = 1$. Table A.2 recapitulates the simplified stress–strain relations for pure shear and uniaxial elongation states.

7.5.2 Torsion of the Rod with Circular Cross-section

Because the Ludwik–Hollomon Eq. (7.51) expresses stress through the strain, the integrals, which express moments, could be calculated immediately. Using the expressions (7.11) and (7.12) for the radial dependency of shear, the torque for the rod with the circular cross-section results to:

$$M_T = 2\pi \int_0^r \tau(\gamma(\rho)) \rho^2 d\rho \equiv \frac{2\pi}{3} \left(\frac{d}{2} \right)^3 \left(A_{\tau} + \frac{3n}{3n+1} \gamma_p^{\frac{1}{n}} B_{\tau} \right). \quad (7.58)$$

7.5.3 Bending of the Rod with Circular Cross-section

Analogously, using the expressions (7.31) and (7.32) for axial strain over normal coordinate, we get the formula for bending moment to the rod with the circular cross-section:

$$M_B = 2 \int_0^r \sigma(\varepsilon(x)) \sqrt{\left(\frac{d}{2} \right)^2 - x^2} x dx \equiv \frac{2}{3} \left(\frac{d}{2} \right)^3 \left(A_{\sigma} + \frac{3\sqrt{\pi} \Gamma\left(\frac{1+2n}{2n}\right)}{4\Gamma\left(\frac{1+5n}{2n}\right)} \varepsilon_p^{\frac{1}{n}} B_{\sigma} \right). \quad (7.59)$$

The function $\Gamma(x)$ in Eq. (7.59) is the Euler gamma function.

7.5.4 Bending of the Rod with Rectangular Cross-section

The closed-form expression of the bending moment for the prismatic bar with the rectangular cross-section of the height t and width b is:

$$M_B = B \int_{-T/2}^{T/2} \sigma(\varepsilon(x)) x dx \equiv \frac{BT^2}{4} \cdot \left(A_\sigma + \frac{2^{\frac{n-1}{2}} n}{1+2n} \varepsilon_p^{\frac{1}{n}} B_\sigma \right). \quad (7.60)$$

The nonlinear torsion problem for the prismatic rectangular bar is solvable numerically. The method to evaluate the residual stresses and length reduction is developed in Sects. 3.4 and 4.4. The developed method is applicable without difficulties as well for the Eqs. (7.58)–(7.60).

The actual industry standard of the presentation of acquired mechanical data for the spring wire is the Ramberg–Osgood or similar implicit formulas (Sect. 7.2). The above formulas (7.58)–(7.60) require the recalculation to the explicit data presentation (as Johnson–Cook or Ludwik–Hollomon). The modification of the existing material data to an altered representation is necessary for spring wires of different chemical compositions and heat-treating regimes. However, the amount of the currently acquired and stockpiled data in the spring manufacturing industry is immense. Thus, the usage of Ramberg–Osgood and hyperbolic models of the plastic flow looks to be appropriate for praxis.

7.6 Time-Delayed Presetting

7.6.1 Instantaneous Ideal Elastic-Ideal Plastic Flow

Consider the rod of circular cross-section with radius of wire $r = d/2$. The hypothesis of the ideal elastic-ideal plastic material is applied (Rees 2006, Sect. 7.3). The ideal, elastic-perfectly plastic behavior of the material without hardening is assumed. In the initial moment the instant deformation happens. If the stresses are moderate, the pure elastic, reversible deformation occurs. Otherwise, if the stresses are high enough to cause the irreversible deformation, the material flows ideally plastically. For the ideal elastic-ideal plastic material there is the clear division between zones of elasticity and plasticity. The radial distribution of the shear stresses after the initial plastic flow reads:

$$\tau_p(\rho) = \begin{cases} G\theta\rho, & \text{for } 0 < \rho < r_p, \\ \tau_Y, & \text{for } r_p \leq \rho \leq r. \end{cases} \quad (7.61)$$

In Eq. (7.61) θ is the twist angle per unit length, G is the shear modulus, τ_Y is the yield shear stress.

The radius of boundary between the plastic and elastic zones is:

$$r_p = \begin{cases} \tau_Y / \theta G, & \text{if } \theta \geq \theta_Y = \tau_Y / r G, \\ r, & \text{if } \theta < \theta_Y. \end{cases} \quad (7.62)$$

From Eq. (7.62) follows, that if the shear stress on the outer boundary of the wire is less than yield stress, $\tau_p(r) \leq \tau_Y$, the material behaves elastically over the complete cross-section, including the skin of the wire. In this case, the twist angle of wire is not greater than θ_Y . Increasing of the twist angle beyond θ_Y leads to the propagation of a plastic zone inwardly from the outer region.

The torque over the circular cross-section M_T^0 is the sum of the partial torque over the elastic region M_E^0 and the partial torque over the plastic region M_P^0 :

$$M_T^0 = M_E^0 + M_P^0, \quad (7.63)$$

$$M_E^0 = 2\pi \int_0^{r_p} \rho^2 \tau_p(\rho) d\rho, \quad M_P^0 = 2\pi \int_{r_p}^{r=d/2} \rho^2 \tau_p(\rho) d\rho.$$

The expressions for the partial torque at the moment of plastic flow $t = 0$ are:

$$M_E^0 = \begin{cases} \frac{\pi \tau_Y^4}{2\theta^3 G^3}, & \text{for } \theta \geq \theta_Y, \\ \frac{\pi \theta G r^4}{2}, & \text{for } \theta < \theta_Y, \end{cases} \quad (7.64a)$$

$$M_P^0 = \begin{cases} \frac{2\pi \tau_Y (r^3 \theta^3 G^3 - \tau_Y^3)}{3\theta^3 G^3}, & \text{for } \theta \geq \theta_Y, \\ 0, & \text{for } \theta < \theta_Y. \end{cases} \quad (7.64b)$$

The twist angle pro wire length is

$$\theta = \frac{2s_P}{\pi n_a D^2},$$

where $s_P = L_0 - L_1$.

The twist angle θ depends upon the preset spring travel (height reduction of the spring) s_P during the squeezing the spring in the tool holder.

7.6.2 Equations of Creep During Time-Delayed Presetting

After the instantaneous plastic deformation completes, the stage of creep flow occurs (Betten 2008; Krajcinovic 1983). In this Chapter, only the basic equations are

presented. More detailed study of creep is presented in Chap. 8. The principal hypothesis is the following. As shown above, the material in inner region is in elastic state and in the outer region is in plastic state. During the time-delayed presetting, the material is subjected to a definite strain for a certain period. If the strain persists, in both regions of the elastic and plastic deformation occurs the relaxation of stresses. The strain rate is defined by material specific creep law. Typically for metals and alloys, there are the primary and secondary creep stages. Because the presetting time is too short for the secondary creep flow, there occurs only primary creep.

For analysis, we accept for strain rate an isotropic, time-dependent stress function, Eq. (A.9):

$$\dot{\epsilon}_{ij} = \frac{s_{ij}}{\tau_{oct}} f_c(\tau_{oct}, t). \quad (7.65)$$

For the common creep laws the isotropic stress function in (7.65) could be represented as the product of two functions. For details, see Eq. (7.9), Chap. 8. In the case of the shear stress, these functions depend correspondingly on time and on octahedral shear stress:

$$f_c(\tau_{oct}, t) = h_c(t) \cdot g_c(\tau_{oct}). \quad (7.66)$$

The dimensionless time-dependent function $h_c(t)$ in Eq. (7.66) describes the evolution of strain rate:

$$h_c(t) = (t/\bar{t})^{\zeta-1}, \quad (7.67)$$

or

$$h_c(t) = \exp[-(t/\bar{t})^\zeta]. \quad (7.68)$$

The characteristic time constants $\bar{t}, \zeta$ are introduced in Eqs. (7.67) and (7.68). Usual setting is $\bar{t} = 1[s]$ and $0 < \zeta < 1$.

The function $g_c(\tau)$ in Eq. (7.66) portrays the stress dependency of the strain rate. The common interpretations for the stress dependency are the Norton-Bailey law (Odquist and Hult 1962; Betten 2008):

$$g_c(\tau_{oct}) = \bar{\epsilon} \cdot \left(\frac{\tau_{oct}}{\bar{\tau}} \right)^{\xi+1}, \quad (7.69)$$

or Garofalo law (Garofalo 1963):

$$g_c(\tau_{oct}) = \bar{\epsilon} \cdot \sinh^{\xi+1} \left(\frac{\tau_{oct}}{\bar{\tau}} \right), \xi \geq 0. \quad (7.70)$$

The dimension of $\dot{\epsilon}_{ij}$, of $g_c(\tau)$ and of $\bar{\epsilon}$ is $[s^{-1}]$.

7.6.3 Creep Deformation After Instant Plastic Flow

Simultaneously to the plastic flow or instantly after it finishing, happens the relaxation of stresses due to creep. This process starts, if the spring is hold compressed for a definite length s_p . The strain in the wire during relaxation does not alter, but the stress gradually reduces for the certain period of presetting: $0 \leq t \leq T_p$. To investigate this process analytically, we consider the relaxation of stresses in a wire of the circular cross-section under the persistent twist. For briefness, we neglect the curvature and pitch of the helical spring.

Let $\tau_c(\rho, t)$ is the shear stress in the cross-section of the wire in the state of creep. The total shear strain in any moment of time is $\gamma(\rho, t)$, is the sum of the elastic and the creep components of shear strain:

$$\gamma = \gamma_e + \gamma_c. \quad (7.71)$$

The creep component of shear strain is $\gamma_c(\rho, t)$. The elastic component of shear strain is

$$\gamma_e = \frac{\tau_c}{G}. \quad (7.72)$$

The total deformation remains constant in time during the presetting process. Thus, we consider the total strain $\gamma_0(\rho)$ as function of radius only. However, the elastic and the creep components of strain are the functions as well of radius and of time, such that:

$$\gamma(\rho, t) = \gamma_e(\rho, t) + \gamma_c(\rho, t) \equiv \gamma_0(\rho). \quad (7.73)$$

The shear during presetting process $\gamma_0(\rho)$ remains constant. The time differentiation of (7.73) leads to the differential equation for elastic and creep strain rates:

$$\frac{d\gamma(\rho, t)}{dt} = \frac{d\gamma_e(\rho, t)}{dt} + \frac{d\gamma_c(\rho, t)}{dt} \equiv 0. \quad (7.74)$$

The Eq. (7.72) delivers the expression for the shear strain γ_e . However, in the Eq. (7.74) the strain rate is required. Obviously, the differentiation of the Eq. (7.72) over time delivers the elastic component of strain rate:

$$\frac{d\gamma_e}{dt} = \frac{1}{G} \frac{d\tau_c}{dt}. \quad (7.75)$$

The substitution of material laws (7.66) and (7.75) in Eq. (7.74) results in the ordinary, nonlinear differential equation of the first order for total shear stress $\tau_c(\rho, t)$:

$$\frac{d\tau_c}{dt} + G \cdot f_c(\tau_c, t) = 0, \tau_c(\rho, t = 0) = \tau_p(\rho). \quad (7.76)$$

For the Norton-Bailey creep law (7.69), the variables in the Eq. (7.75) could be separated after the substitution:

$$\frac{d\tau_c}{dt} + G \cdot \bar{\varepsilon} \cdot \left(\frac{\tau_c}{\bar{\tau}}\right)^{\xi+1} \frac{dH}{dt} = 0, h_c(t) = \frac{dH}{dt}, \quad H(0) = 0. \quad (7.77)$$

For the compactness of the following expressions an auxiliary function is introduced in Eq. (7.77):

$$H(t) = \int_0^t h_c(t) dt.$$

The initial condition for the Eq. (7.77) is:

$$\tau_c(\rho, t = 0) = \tau_p(\rho). \quad (7.78)$$

With Eq. (7.78) expresses the creep and plastic stresses match, such that the creep stress originates from the plastic stress.

The solution of the ordinary differential Eq. (7.77) yields the shear stress over the cross-section of the twisted rod as the function of time and radius:

$$\frac{1}{[\tau_c(\rho, t)]^\xi} = \frac{\bar{\varepsilon} G \xi H(t)}{\bar{\tau}^{\xi+1}} + \frac{1}{\tau_p^\xi(\rho)} \equiv \begin{cases} \frac{\bar{\varepsilon} G \xi H(t)}{\bar{\tau}^{\xi+1}} + \frac{1}{(G\theta\rho)^\xi}, & \text{for } \rho < r_p, \\ \frac{\bar{\varepsilon} G \xi H(t)}{\bar{\tau}^{\xi+1}} + \frac{1}{\tau_p^\xi}, & \text{for } \rho \geq r_p. \end{cases} \quad (7.79)$$

The total torque over the complete cross-section $M_T(t)$ is the function of time. This function is the sum of the partial torque over the elastic region $M_E(t)$ and the partial torque over the plastic region $M_P(t)$:

$$M_T(t) = M_E(t) + M_P(t), \quad (7.80)$$

$$M_E(t) = 2\pi \int_0^{r_p} \rho^2 \tau_c(\rho, t) d\rho, \quad M_P(t) = 2\pi \int_{r_p}^{d/2} \rho^2 \tau_c(\rho, t) d\rho. \quad (7.81)$$

With the expression for total shear stress (7.79) we can evaluate the integrals (7.81) for torque over the elastic and plastic regions correspondingly in terms of the Meijer G -function (Bateman 1953):

$$M_E(t) = \begin{cases} \mathbf{G}\left(\left[\left[\frac{\xi-4}{\xi}, \frac{\xi-1}{\xi}\right], \square\right], \left[0, \left[-\frac{4}{\xi}\right]\right], \frac{\xi G r_Y^\xi}{\xi \Gamma(1/\xi) \theta^3 G^3}, \frac{2\pi \tau_Y^4}{\xi \Gamma(1/\xi) \theta^3 G^3}, \bar{\varepsilon} H(t)\right), & \text{for } \theta \geq \theta_Y, \\ \mathbf{G}\left(\left[\left[\frac{\xi-4}{\xi}, \frac{\xi-1}{\xi}\right], \square\right], \left[0, \left[-\frac{4}{\xi}\right]\right], \frac{\xi G \cdot (\theta G d/2)^\xi}{\xi \Gamma(1/\xi)}, \frac{2\pi \theta G r^4}{\xi \Gamma(1/\xi)}, \bar{\varepsilon} H(t)\right), & \text{for } \theta < \theta_Y, \end{cases} \quad (7.82)$$

$$M_P(t) = \begin{cases} \frac{2\pi (r^3 \theta^3 G^3 - \tau_Y^3)}{3\theta^3 G^3} \left[\frac{G \bar{\varepsilon} H(t) \xi}{\xi \Gamma(1/\xi)} + \frac{1}{\tau_Y^\xi} \right]^{-1/\xi}, & \text{for } \theta \geq \theta_Y, \\ 0, & \text{for } \theta < \theta_Y. \end{cases} \quad (7.83)$$

For $\xi = 1, 2, 4$ the Meijer $\mathbf{G}$ -function in (7.82) reduces to the elementary functions (Askey 2010):

$$\frac{\mathbf{G}\left(\left[\left[\frac{\xi-4}{\xi}, \frac{\xi-1}{\xi}\right], \square\right], \left[0, \left[-\frac{4}{\xi}\right]\right], Z\right)}{4n\Gamma(1/\xi)} = \begin{cases} -\frac{2}{3} \cdot \frac{6\ln(1-Z) + 6Z + 3Z^2 + 2Z^3}{Z^4} & \text{for } \xi = 1, \\ -\frac{4}{3Z} \cdot \frac{\sqrt{1-Z} - 1 - Z}{\sqrt{1-Z} + 1} & \text{for } \xi = 2, \\ -\frac{4}{3} \cdot \frac{(1-Z)^{3/4} - 1}{Z} & \text{for } \xi = 4. \end{cases}$$

The following series expansions are valid for positive ξ :

$$\begin{aligned} \frac{1}{4\xi\Gamma\left(\frac{1}{\xi}\right)} \mathbf{G}\left(\left[\left[\frac{\xi-4}{\xi}, \frac{\xi-1}{\xi}\right], \square\right], \left[0, \left[-\frac{4}{\xi}\right]\right], Z\right) &= 1 + \frac{4}{\xi(\xi+4)} Z + O(Z^2), \\ \frac{1}{4n\Gamma(1/\xi)} \mathbf{G}\left(\left[\left[\frac{\xi-4}{\xi}, \frac{\xi-1}{\xi}\right], \square\right], \left[0, \left[-\frac{4}{\xi}\right]\right], 0\right) &= 1. \end{aligned}$$

There is the equivalent representation of the Meijer $\mathbf{G}$ -function in terms of hypergeometric function (Bateman 1953):

$$\mathbf{G}\left(\left[\left[\frac{\xi-4}{\xi}, \frac{\xi-1}{\xi}\right], \square\right], \left[0, \left[-\frac{4}{\xi}\right]\right], Z\right) = 4n\Gamma\left(\frac{1}{\xi}\right) \cdot {}_2F_2\left(\left[\frac{1}{\xi}, \frac{4}{\xi}\right], \left[\frac{\xi+4}{\xi}\right], Z\right).$$

According to Eq. (7.77), in the initial moment of creep $H(t=0) = 0$. The former formula verifies that the expressions (7.82), (7.83) and (7.64) match at $t = 0$:

$$M_P(0) = M_P^0, M_E(0) = M_E^0.$$

The plots of the total torque $M_T^0 = M_E^0 + M_P^0$ and partial torques over the elastic and plastic regions in the initial moment $t = 0$ are shown on Fig. 7.2. The abscissa depicts the twist angle per unit length θ in radians pro meter.

With the expressions for moments (7.80)...(7.83) evaluates also the axial force, which is applied on the helical spring. The presetting force reads in terms of the torque and mean diameter of the helical spring:

$$F_z(t) = \frac{2}{D} \begin{cases} M_E^0 + M_P^0, & \text{for } t = 0, \\ M_E(t) + M_P(t), & \text{for } t > 0. \end{cases} \quad (7.84)$$

7.6.4 Elastic Spring-Back and Occurrence of Residual Stresses

When the elastic–plastic bar or helical spring is completely unloaded, elastic stresses recover to leave a residual shear stress field. The residual stresses appear in the moment of removal of the torque $M_T(t = t_P)$ in the moment t_P . If the unloading after initial plastic deformation and unloading occurs not immediately after the accomplishing of the plastic process, the relaxation of the stresses appears. The residual stress field is the difference between the shear stress in elastic and plastic zones under the time-dependent torque $M_T(t_P)$, Eq. (7.80) and the elastic shear stress τ_u that recovers in the moment of removal of the moment $M_T(t_P)$.

The spring travel from the undeformed state to the plastically deformed state is equal to a definite constant values s_P . In the course of the presetting of helical springs the spring force attains the value F_z . When the force disappears, the unloading occurs. The simplest hypothesis for the unloading is the linear elastic spring-back that leads to the increase of the spring length:

$$s_u^0 = \frac{F_z^0}{c_f}, s_u(T_P) = \frac{F_z(T_P)}{c_f}, c_f = \frac{Gd^4}{8D^3n_a}, s_u = L_2 - L_1. \quad (7.85)$$

Consequently, the reduction of the length of the spring after plastic presetting and elastic unloading reads:

$$\Delta s^0 = s_P - s_u^0 \equiv s_P - \frac{2M_T^0}{c_f D}, \Delta s = s_P - s_u \equiv s_P - \frac{2M_T(t_P)}{c_f D}. \quad (7.86)$$

Because the torque reduces with time of presetting, the reduction of length increases with time:

$$\Delta s^0 = L_0 - L_2(t_P = 0) \text{ or } \Delta s = L_0 - L_2 \text{ or for } t_P > 0,$$

such that

$$\Delta s^0 < \Delta s.$$

For an infinite creep time the torque vanishes,

$$M_T(t \rightarrow \infty) \rightarrow 0$$

and the elastic unloading disappears $\Delta s = s_p$. The length of the spring remains equal to its compressed value. In vector form, the corresponding twist angles for the length differences are:

$$[\theta_u^0, \theta_u, \Delta\theta^0, \Delta\theta] = \frac{2}{\pi n_a D^2} [s_u^0, s_u, \Delta s^0, \Delta s].$$

The simplest definition of residual stresses is as follows: stresses that remain within a part after it has been deformed and all external forces have been removed. More specifically, the deformation must be non-uniform across the material cross-section in order to give rise to residual stresses. The deformation can result not only from forming operations but also from thermal processes. Phase transformations during heat treating are known to induce sufficient strain to result in plastic deformation, thereby giving rise to residual stresses. In this paper we determine analytically the residual stress, which appear in the cylindrical rod after plastic torsion and in the helical spring after presetting. The calculation results of the plastic-creep deformation and the final instantaneous spring-back of the material are displayed on Figs. 7.15, 7.16, 7.17, 7.18 and 7.19.

If the unloading happens immediately after the plastic deformation and creep does not happen, the residual stress in the state of complete elastic unloading τ_r^0 is equal to stresses τ_p^0 in the state of maximal plasticization minus the stresses τ_u^0 due to elastic unloading:

$$\tau_r^0 = \tau_p^0 - \tau_u^0, \tau_u^0 = \frac{M_T^0}{W_T}, W_T = \frac{\pi d^3}{16} \quad (7.87)$$

If spring is hold in the tool for a certain period of time t , the unloading takes place after the completing of the creep process. The elastic, plastic and total torque during the plastic deformation as the function of twist angle are shown on Fig. 7.15. The residual stress in the state of complete unloading after creep $\tau_r(t_p)$ is equal to stresses $\tau_c(t_p)$ in the state after creep process minus the stresses $\tau_u(t)$ due to unloading:

$$\tau_r(t_p) = \tau_c(t_p) - \tau_u(t_p), \tau_u(t_p) = \frac{M_T(t_p)}{W_T}. \quad (7.88)$$

The residual stresses in the state of complete elastic unloading $\tau_r(t)$ on the outer surface of the wire depend on the presetting time t_p .

Figure 7.16 demonstrates the effect of creep on the torque for different durations of preset t_p . The ordinate is the value of torques in Nm. For the calculations the following values were used:

$$G = 80\text{GPa}, h = 2.5 \cdot 10^{-9} \cdot t^{0.9}, \tau_Y = 1.3\text{GPa},$$

$$D = 36\text{mm}, d = 3\text{mm}, n = 4, n_a = 7.$$

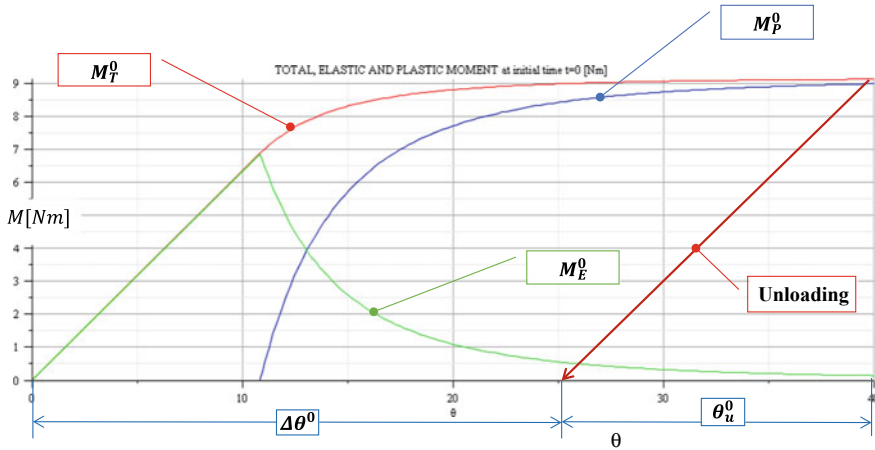


Fig. 7.15 Elastic, plastic and total torque during the plastic deformation as the function of twist angle

For example, the creep durations were assumed to be

$$t_1 \equiv 86400s, t_2 \equiv 172800s, t_3 \equiv 259200s, t_4 \equiv 345600s.$$

In the time moment of the spring-back of the wire, the torque vanishes. According to Eq. (7.88), the higher values of the spring-back lead to the higher values of residual stresses. The maximal length reduction and the uppermost residual stresses appear, if spring-back happens immediately after the instant plastic deformation without relaxation period. The longer relaxation presetting time is, the lower are the residual stresses and the smaller is the elastic spring-back of the spring. The spring in its final length after presetting will be also shorter with the higher grade of relaxation.

Figure 7.18 displays the stresses after the instantaneous plastic flow $\tau_p(r = d/2)$ and the residual stress

$$\tau_r(r = d/2, t = 0) = \tau_r^0(r = d/2)$$

on the surface of the wire. For these values the dotted lines of blue and red colors respectively are used. The stresses after creep period t_P on the surface of the wire are shown with the solid lines: creep stress $\tau_c(r = d/2, t = t_P)$ and the residual stress $\tau_r(r = d/2, t = t_P)$. The calculations were performed with the above material parameters and for creep duration:

$$t_P = t_4 \equiv 345600s.$$

The corresponding distributions of the shear stresses after the mentioned above presetting times, but before the elastic spring-back over the radius of wire are shown on Fig. 7.17. The resulting radial distributions of the residual stresses are shown on Fig. 7.19.

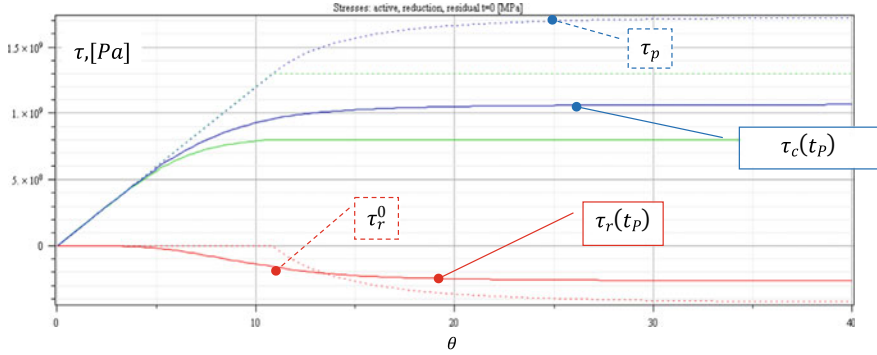


Fig. 7.18 Stress distribution after creep deformation prior to elastic unload for different creep times t

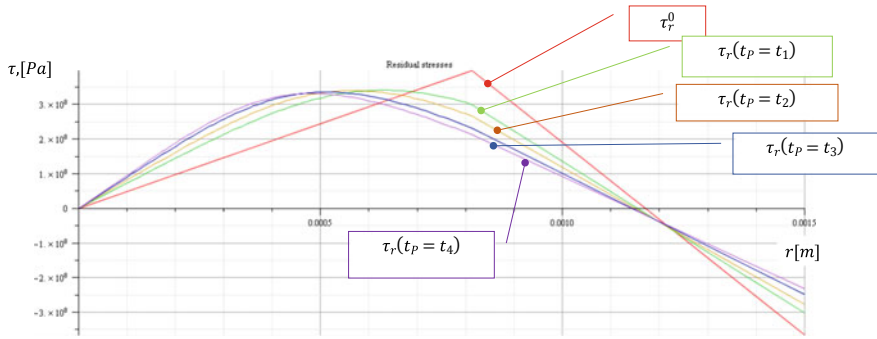


Fig. 7.19 Residual stress distribution over radius after creep deformation for different creep times t

$$\frac{d\tau_c}{dt} + G \cdot c \cdot \sinh^{\xi+1}\left(\frac{\tau_c}{\bar{\tau}}\right) \cdot \frac{dH}{dt} = 0, \quad \tau_c(r, t = 0) = \tau_p(r). \quad (7.89)$$

The initial condition for the Eq. (7.89) presumes the shear stress in the moment $t = 0$ after the instant plastic deformation. For the ideal elastic-ideal plastic material, the shear stresses after the instant plastic flow for the rod with circular cross-section result from Eq. (7.77).

The solution of the ordinary differential Eq. (7.77) for $\xi = 0$ with initial condition (7.78) delivers the shear stress over the cross-section of the twisted rod as the function of time and radius:

$$\tau_c(\rho, t) = \bar{\tau} \cdot \ln \tanh \left[\frac{cGH(t)}{2\bar{\tau}} + \operatorname{arctanh} \left(\exp \left(\frac{\tau_p(\rho)}{\bar{\tau}} \right) \right) \right]. \quad (7.90)$$

According to Eq. (7.80), the couple $M_T(t)$ is the function of time is the sum of the couple over the elastic region $M_E(t)$ and the couple over the plastic region $M_P(t)$.

With the expression for total shear stress (7.85) the partial couples over the elastic and plastic regions express in terms of polylogarithmic function. The final formulas are too bulky and display the analogous functions of the mechanical values, as result after the assumption of the Norton-Bailey law.

7.7 Conclusions

The presetting of the springs is the definite over-elastic loading of the springs at the end of the production process at room temperature or a hot-setting at elevated temperature. After correct application, the presetting has a favourable effect on both the relaxation behaviour and the fatigue behaviour of the springs. The fatigue strength properties of vehicle springs that are only stressed in one working direction (swelling) (coil springs, leaf springs or torsion bars) can be positively influenced by intentional plasticising (over-elastic deformation) in this direction at the end of the production process. This presetting causes a partial flow in the surface layer, which leads to work hardening there and to residual compressive stresses in the direction of operational loading during subsequent relief. In leaf springs or parabolic springs, longitudinal compressive residual stresses occur in the edge layer on the bending tension side, in coil springs corresponding compressive residual stresses under 45° , which is related to the tensile stresses from the torsional load stress state of overlay the built-in spring.

The change in shape (setting loss) resulting from presetting can be used to set the same overall height for series parts and must be kept at the original shape of the springs. While the presetting of leaf springs can be carried out relatively freely, it is limited in helical compression springs by limiting conditions such as buckling or blocking. To increase the effect of the presetting in these cases, it is carried out at sometimes elevated temperatures.

The presetting process of helical springs is substantial for manufacturing praxis. The analysis was performed using deformational theory of plasticity with several nonlinear stress strain laws (SAE AE-22, Sect. 8.3.33). These laws describe the active plastic deformation of the wire together with the linear elastic spring-back. Both twist and curvature of the wire during the plastic loading continuously increase and in course of elastic spring-back reduce. The plasticization process and the appearance of residual stresses are studied. The analytical results correspond roughly to the experimentally acquired values. The closed form solutions allow the estimations of stresses and geometry alternations during manufacturing of helical springs. The final formulas could be used for industrial applications and programming of manufacturing machines and for dimensioning and wear calculation of presetting tools.

A significant for manufacturing methodology designates the time-depending presetting process of helical springs. The twist of the spring wire during the plastic loading grows and in course of elastic unloading reduces. The combined plasticization-relaxation process causes the origin of residual stresses. With the higher temperatures the relaxation process accelerates, such that the length reduction under the higher temperatures will be faster and the residual stresses will be lower.

The proposed method leads to the closed solution. The formulas must to be validated in the next step with experiments on helical compression springs with different spring indexes and manufacturing regimes. The closed form solutions allow the estimations of relaxation and residual stresses in course of fabrication of helical springs. The derived formulas facilitate the industrial applications.

7.8 Summary of Principal Results

- The instant plastic flow in the initial stage of presetting is investigated in closed form.
- The fixation of the compressed spring length in a period of time leads to the relaxation of stress and the lessening of the resulting residual stress.
- The analytical formulas for the length loss and stress reduction as the functions of the presetting time are derived.

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Chapter 8

Creep and Relaxation of Springs



Abstract In this Chapter, we investigate the time-depending behavior of spring elements under constant and oscillating load. The common creep laws are implemented for the description of material. The models are established for the relaxation of stresses under constant stroke and for creep under persistent load. For basic spring elements, closed-form solutions for the models of creep and relaxation are found. The introduction of the “scaled” time variable simplifies thÝe presentation. The explanation of the experimental procedure for the experimental acquisition of creep models is presented. This Chapter forms the first section of the third part, which examines the lifecycle of springs.

8.1 Operational Damage of Spring Elements

During the exploitation period, springs and other machinery elements subjected to various loads. These loads cause different types of stress in the elements. The safe dimensions for springs are determined by methods of failure evaluation. Well known, that any load bearing component is failed if the induced stress exceeds the elastic limit and permanent deformation of the component takes place. For example, the ultimate strength is the limit value of stress at which failure and immediate irreversible deformation occurs. Depending on material behaviour, a distinction must be made between brittle fracture strength (shear strength) and ductile flow strength (ductile deformation). Furthermore, in the case of ultimate strength, a distinction is made between compressive strength and tensile strength. Compressive strength symbolizes the limit stress at which a material suffers complete loss of cohesion along a fracture surface. The compressive strength of spring steel alloys is in the order of 3 GPa. Tensile strength refers to the strength of a material under tensile stress and for the steel alloys approximately equal to the compressive strength. If a certain limit value of is tension strength in normal direction exceeded, failure sometimes occurs through extension fracture, the material tears completely along a fracture surface. The common standard tests for determining strength are uniaxial compression and tensile tests, triaxle compression tests and shear tests.

Depending of the specific application and the character of loading, there are two origins of operational damage of springs.

For the first group, the static loads cause the irreversible deformation as the result of creep. For automotive and industrial springs, the static loads do not lead normally to an immediate failure of material, but cause the remaining distortion of spring. For example, the clearance of the vehicle gradually reduces with time as the consequence of creep. On the other hand, the shape does not alter if the compression of spring persists, but the spring load decreases with time and causes the malfunction of the unit. Character of the creep process is the high ductility of material. The ductility stabilizes the breakage and levels the effects of imperfections. The scattering of failures, which are caused by creep, is much less in comparison to the scattering of fatigue failures. Keeping this behavior in mind, we study the creep phenomena pure deterministically and renounce the statistical analysis of creep. Excessive creep is a frequent source of spring malfunctions and is discussed in this Chapter.

For the second group of springs, the cyclic degradation and breakage of material is by far more relevant. Accepted, that metals break under oscillating stresses whose maximum value is smaller than that required to cause rupture in a static test. Even though, many ferrous alloys show a “fatigue limit,” or stress below which such fracture never occurs even for very high number of load cycles. The operation life under cyclic load is sensitive to the imperfections and production inaccuracies. The scattering of fatigue life in one large batch of tested springs could be very high. In extreme cases, the magnitude of scattering one can cover about one decimal power. The phenomena of springs fatigue will be discussed in Chaps. 9 to 12.

8.2 Common Creep Constitutive Equations

8.2.1 *Constitutive Equations for Creep of Spring Elements*

When a load is applied to a piece of metal, the metal will initially bend elastically; if the load is high enough, it will also bend plastically. Elastic deflection occurs when the material returns to its previous shape/geometry after relief. Plastic deformation is when the metal undergoes permanent deformation and does not return to its previous shape/geometry. It is the elastic part of the deflection which normally experiences the deflection in durable springs.

The stress/deflection function is represented by Hook’s law, which describes the modulus of elasticity E for normal and G for shear stress. A metal of high strength can normally be loaded to a higher stress level without exceeding the limit at which the deflection becomes plastic. The stress value at which the metal begins to deflect plastically is the yield strength σ_Y of the material. The ultimate tensile strength σ_w is the stress at which the metal breaks. In addition to the knowledge of stress–strain relations at low temperatures, the processes that take place at elevated temperatures are

particularly interesting. Metals acquire a hardening effect through plastic deformation. If the temperature is increased, recovery and recrystallization processes begin which continuously counteract the hardening. At certain temperatures, the dislocations of recovery processes caused by deformation are continuously compensated. The permanent deformation begins to grow. This process is called “creep”, a physical phenomenon that designates a slow plastic deformation, which takes place at stress below the yield strength σ_Y of the material. The basic physical processes are dislocation creep, diffusion creep and diffusion-supported grain boundary sliding. The constitutive models and the solution methods for creep problems were discussed in Poirier (1985), Kassner (2008). Comprehensive reviews of creep laws for common engineering materials were provided in Betten (2008), Naumenko and Altenbach (2007) and Yao et al. (2007).

The common material models adequately describe definite creep stages from constant stress uniaxial tests Naumenko (2006). These phenomenological models represent an attempt to approximate in the simplest formulas the experimentally acquired data. From the mechanical viewpoint, the phenomenological models must characterize the stress, time and temperature dependences. However, the behavior of the material dramatically changes with temperature and phase transformation Ashby (2011). Consequently, the unified approximation with the experimentally accessible number of parameters is possible in a narrow temperature interval. If the character temperature of the structural is fixed, it makes sense to evaluate the material constants for each relevant temperature and for each creep stage.

The parametric methods were developed to compute creep and creep fracture in high temperature applications. The distinctive mathematical properties of the power law allowed the development of analytical methods (Boyle and Spence 1983, Chap. 3). The corresponding creep equations can be found commercial numerical simulation codes.

Several other models were able to predict the phenomenon of creep focusing on the rupture time and minimum strain rate of a material: “Theta Projection” (Evans et al. 1982), “Wilshire Equations” (Wilshire and Battenbough 2007), (Wilshire and Scharning 2007) and “Uniaxial Creep Lifting approach” (Wu 2010). In the “Theta Projection” approach creep, the creep curves under uniaxial constant stress measured over a range of stresses and temperatures. The creep curves can be “projected” to other stress/temperature conditions re-constructing full creep curves.

The “Wilshire Equations” provide the facility to describe the stress-rupture behavior, this together with the proper representation of full creep curves.

The “Uniaxial Creep Lifting approach” postulates that the deformation occurs by grain boundary sliding, dislocation glide, dislocation climb and diffusion mechanisms. The total strain rate for a polycrystalline material is the sum of the contributions from each mechanism.

The newly constructed curves define the necessary creep characteristics. The approach for creep data assessment (ECCC) was elaborated European creep collaborative committee (Holdsworth 2008).

The established result are applied to practically important problem of engineering, namely for simulation of creep and relaxation of helical and disk springs. A spring is

considered to be durable if it does not lose any force, or if it does not shorten or break under a given force when a certain spring travel is performed, so that the function is disturbed or eliminated. If a spring is to be durable, the stress in the material must not be higher than the strength of the material allows. In the spring industry this phenomenon is called “creep” when a spring loses length under constant load, and it is called “relaxation” when a spring loses load under constant compression. Over time, every spring suffers a loss of tension, which can be manifested as relaxation or creep, depending on the type of load on the spring.

One speaks of relaxation when the spring is compressed to a constant length and a drop in force becomes noticeable over time. One speaks of creep when the spring loaded with a constant force suffers an additional loss of length over time Δl so that its overall height decreases.

How significant are creep or relaxation effects, depend on the temperature, the stress in the metal, the yield strength of the metal and time. Elevated temperature, stress and time also increase creep/relaxation. Temperature and stress in particular have a high influence.

In diverse types of springs several kinds of stresses cause creep in material. The helical springs are made up of a wire coiled in the form of a helix and are primarily intended for compressive or tensile loads. The springs store the elastic energy by means of either bending or torsion. Respectively, in material dominates either uniaxial or pure shear stress state. These stresses initiate the creep and loss of length or of spring force. The stress-state in the wire of helical spring is highly inhomogeneous. The stress in the circular wire varies from zero on the axis of the wire to its maximum value on the surface. As was pointed out in Boardman (1965), the assumption of the homogeneity of stress is incorrect. The resulting error leads to an erroneous determination of creep law parameters, when with the stress exponent is other than one. However, the frequent values for stress exponents are about four (Gittus 1971). The cross-section of the wire from which the spring is made may be circular, square, or rectangular. A formal technique was developed in Chang (1995) to predict the stress relaxation for compression and torsion springs. The technique uses uniaxial tensile-generated stress-relaxation data for spring wires. Based on the tension-induced stress-relaxation data, the technique was applied to compression springs, where shear stress dominates in predicting the stress relaxation.

The shear stress–strain curve is first constructed based on the octahedral stress–strain curve. The stress relaxation is a phenomenon in which part of the elastic strain responsible for the initial stress is replaced by creep strain. The same argument is valid for the spring elements in predominant state of bending, like disk springs or torsion helical springs. The brief examination of the creep process during the presetting was already discussed in Chap. 7. In this Chapter, we continue to examine systematically the creep effects during the spring operation. One essential task of creep analysis is the derivation of the exact closed form expressions for torsion and bending creep for isotropic materials, which obey the accustomed Norton-Bailey, Garofalo and exponential constitutive laws.

8.2.2 Time-Dependent Constitutive Equations

The creep component of strain rate is defined by material specific creep law. The models are based on the time- and strain-hardening constitutive equations for stress that depends on operation time¹ t . Following the common procedure (Betten 2008), we adopt an isotropic, time-dependent constitutive equation

$$\frac{de_{ij}}{dt} = \frac{s_{ij}}{\tau_{oct}} f_c(\tau_{oct}, t). \quad (8.1)$$

The constants of the constitutive Eq. (8.1) depend on the type, thermal treatment and grain size spring material, the particular creep mechanism, and temperature. For the spring element, typical trial and operation temperatures are all fixed. Therefore, the parameters of the material law correspond to the definite operation temperature.

Hereafter the incompressible behavior of material in the process of creep deformation ($\dot{\epsilon}_{kk} = 0$) is presumed. Thus, the octahedral normal strain remains constant during the creep process: $\epsilon_{oct} = const$.

The constitutive equations Eq. (8.1) comprise the explicit dependence of strain rate upon time. The time dependence expresses the tendency to the strong reduction of creep strain rate in the region of primary creep. Immediately after the application of an external load, the creep rate is maximal, but reduces gradually with time. After a certain period of time, the creep rate stabilizes and remains mainly constant until the final, tertiary creep region achieved and breakage occurs. According to Andrade (1910, 1914), a power or an exponential function of time depicts this reduction of creep strain rate. The reduction does not depend on the stress level. Thus, for all applied stresses the relative strain rate reduction is the same. This feature allows to factorize the function as the product of time function and stress function.

When a plastic material is subjected to a constant load, it deforms with a variable strain rate. The initial strain is roughly predicted by its stress–strain modulus. The experimental results display, that the higher stress in tension accelerates the reduction of strain rate (Poirier 1985). The interval of time, which leads to the stabilization of creep rate and transmission from the primary to the secondary creep stage, reduces with the higher stress level.

The most important region for the technical springs is the section of the primary creep. The primary region is the early stage of loading when the creep rate decreases rapidly with time. Further, the material will continue to deform gradually with time, until yielding or rupture causes failure. The total strain rate is the sum of primary, secondary and tertiary partial strain rates:

$$f_c(\tau_{oct}, t) = f_{Ic}(\tau_{oct}, t) + f_{IIc}(\tau_{oct}, t) + f_{IIIc}(\tau_{oct}, t). \quad (8.2)$$

The creep strain rate in these regions is commonly defined the Norton-Bailey law (Betten 2008). At constant temperature, the creep deformation of metals grows in

¹For material thickness, the symbol T is used. The symbols t and t^* are reserved for time variables.

time following to a power law of stress. According to the Andrade law (Andrade 1910, 1914), the creep strain rate in the primary creep stage could be described by the relation.

$$f_{IC}(\tau_{oct}, t) = \bar{\varepsilon}_I \cdot \left(\frac{t}{\bar{t}}\right)^{S-1} \left(\frac{\tau_{oct}}{\bar{\tau}}\right)^{\xi+1}. \quad (8.3a)$$

We use the constants for proper scaling of stress and time. For example, the suitable setting for scaling constants: $\bar{\tau} = 1[MPa]$, $\bar{\sigma} = 3/\sqrt{2}[MPa]$, $\bar{t} = 1[s]$. We use this setting.

The creep rate in Eq. (8.3) at time moment $t = 0$ is infinite. This context confuses the applicability of the Eq. (8.3). An alternative expression for time function that is free from an initial singularity, uses the exponential time function in primary region:

$$f_{IC}(\tau_{oct}, t) = \bar{\varepsilon}_I \cdot \exp\left[-\left(\frac{t}{\bar{t}}\right)^S\right] \left(\frac{\tau_{oct}}{\bar{\tau}}\right)^{\xi+1}. \quad (8.3b)$$

The creep strain rate reaches later a steady states which are called the secondary and tertiary creep stages.. The recovery and recrystallization processes in this section completely compensate for the strain hardening by dislocations. The creep deformation is proportional to time and the creep rate simultaneously assumes its lowest value. This region therefore frequently covers most of the component's service life. The strain rate functions are:

$$f_{IIC}(\tau_{oct}, t) = \bar{\varepsilon}_{II} \cdot \left(\frac{\tau_{oct}}{\bar{\tau}}\right)^{\xi+1}, \quad (8.4a)$$

$$f_{IIIC}(\tau_{oct}, t) = \bar{\varepsilon}_{III} \cdot \exp\left[\left(\frac{t}{\bar{t}}\right)^\xi - 1\right] \left(\frac{\tau_{oct}}{\bar{\tau}}\right)^{\xi+1} \quad (8.4b)$$

Here $\bar{\varepsilon}_I$, $\bar{\varepsilon}_{II}$, $\bar{\varepsilon}_{III}$ are the experimental constants for the primary, secondary and tertiary stages of creep. The units of these constants is $[1/s]$.

Function (8.5) could be represented as the product of two functions, which depend correspondingly on time and on stress (Kobeleev 2016):

$$f_c(\tau_{oct}, t) = h_c(t) \cdot g_c(\tau_{oct}). \quad (8.5)$$

The creep law (8.5) pronounces the continuous transition between the primary and secondary creep regimes (Cadek 1988) Primary creep occurs at the beginning of the creep test and is characterized first by a high strain rate. The creep rate decelerates gradually to a constant value denoting the beginning of secondary creep. This behavior corresponds to the experimentally acquired creep data (Nabarro and de Villers 1995) and (Es-Souni 2000). However, the continuous transition between creep regimes leads to some mathematical difficulties. The regions of primary

and secondary creep are frequently being separated for reasons of mathematical simplification (Nezhad and O'Dowd 2012, 2015).

The derivation of closed form solution is applied to the generalized creep law (8.5). The most general form of creep law with continuous transition between the primary and secondary creep regimes permits the closed form solutions because of simple geometry of considered stress states.

Particularly, the functions in (8.5) are.

$$h_c(t) = \bar{\varepsilon}_I \cdot (t/\bar{t})^{s-1} + \bar{\varepsilon}_{II}, \text{ or } h_c(t) = \bar{\varepsilon}_I \cdot \exp[-(t/\bar{t})^\xi] + \bar{\varepsilon}_{II}. \quad (8.6a)$$

The time function (8.6) represents the contribution of primary, secondary and tertiary creep regimes. There are several possibilities to proper exemplifications of the creep strain as the function of time. For the “ θ -Projection” (Evans et al. 1982), (Harrison and Evans 2007) and for the “uniaxial creep lifting” (Wu 2010) approaches, the secondary strain rate vanishes ($\bar{\varepsilon}_{II} = 0$):

$$h_c(t) = \bar{\varepsilon}_I \cdot \exp[-(t/\bar{t})^\xi] + \bar{\varepsilon}_{III} \cdot \exp[(t/\bar{t})^\xi - 1]. \quad (8.6b)$$

8.2.3 Experimental Acquisition of Creep Laws

The measurement of the twist of wire under the constant torque provides the straightforward way of the evaluation of creep constant for helical springs (Gooch 2012). The advantage the torsion creep test over tension test is the relative insensitivity to misalignments and the direct account of the possible radial variations of material properties. The state of twist in a straight wire in the test equipment matches the state in a helical spring. The samples of the real spring wire with the same heat treatment and the same surface handling are used in the experiment. During the torsion creep test the function of the twist angle over time is gained. The determination of the stress exponent requires at least three values of the torque. During the torsion creep test the function of the twist angle over time is gained for each value of applied torque. These at least three functions provide together with corresponding values of torque moments, length and diameter of the wire the basis for determination of creep constants.

For example, assume the Norton-Bailey creep law in a form:

$$\gamma = h_c(t) \tau^{\xi+1}. \quad (8.7)$$

As the shear strain is the linear function of radius: $\gamma = \theta \rho$, $0 < \rho < r = d/2$ the solution of Eq. (8.7) leads to the expression for shear stress:

$$\tau(\rho) = (\theta \rho / h_c(t))^{1/(\xi+1)}. \quad (8.8)$$

With Eq. (8.8), the torque reads.

$$M_T = 2\pi \int_0^r \tau(\rho) \rho^2 d\rho = 2\pi \frac{\xi + 1}{4 + 3\xi} \rho^{\frac{4+3\xi}{\xi+1}} \theta^{\frac{1}{\xi+1}} h_c(t)^{-\frac{1}{\xi+1}}.$$

For a constant value of M_T the twist angle is the function of time:

$$\theta(t) = \frac{M_T^{\xi+1} h_c(t)}{r^3 \xi + 4} \left(\frac{1}{2\pi} \frac{4 + 3\xi}{\xi + 1} \right)^{\xi+1}.$$

If two functions θ_1, θ_2 , are gained from experiments for two different moments M_{T1}, M_{T2} , then the stress exponent is:

$$\xi = \frac{\ln(\theta_1/\theta_2)}{\ln(M_{T1}/M_{T2})} - 1.$$

For numerical determination the common curve fitting procedures must be applied. The experimentally determined functions of creep angle over time are shown on Fig. 8.1 (points, right plot, for wire diameter $d = 2.7 \text{ mm}$). The lines of different color correspond to different applied constant torque. The experimentally obtained twist angles for three different moments are plotted with solid lines.

The second step is to resolve the time dependence $h_c(t)$. As the stress dependence is already settled, the function that responsible for time dependence reads:

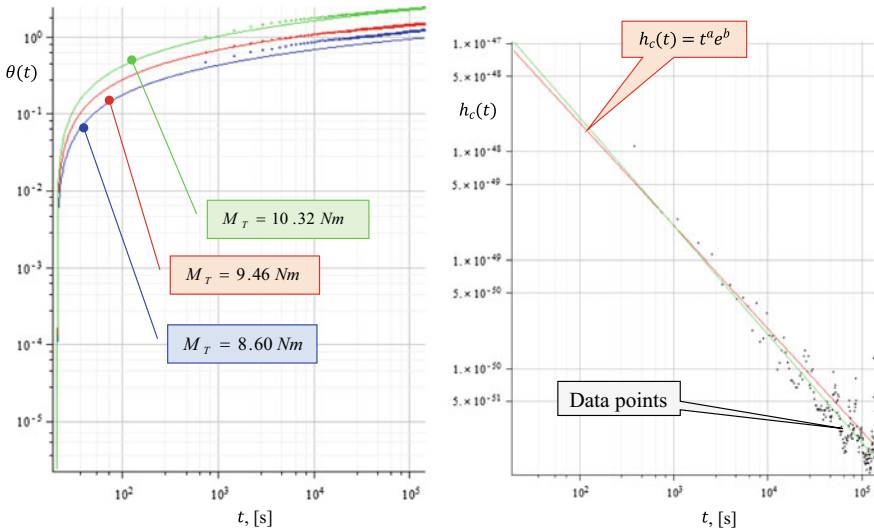


Fig. 8.1 Twist angle as functions of time for different torques (left) and the fitted linear function for time dependency (right)

$$h_c(t) = \frac{r^{3m+4}\theta(t)}{M_T^{\xi+1}} \left(2\pi \frac{\xi+1}{4+3\xi} \right)^{\xi+1}.$$

For commercially available spring materials the time dependence the logarithm of this function is an almost straight line in double logarithmical scale (Fig. 8.1, left):

$$h_c(t) = t^a e^b, \ln(h_c(t)) = at + b.$$

The values a and b are curve fitting parameters. For example, the coefficients of the Norton creep law for JIS G3561 or VDSiCr at 150 °C are:

$$a = -0.9492, b = -105.5357, e^b = 1.4668 \cdot 10^{-46}, \xi = 3.750.$$

According to Eq. (8.7), the Norton creep law for this material is consequently:

$$\dot{\gamma} = 1.4668 \cdot 10^{-46} t^{-0.9492} \tau^{4.75}.$$

The calculation of the twist angles the acquired material law demonstrates an excellent correspondence to the experimental and calculated values (Fig. 8.1, solid lines, right plot). Typical values for ξ are 4 to 5. If the fitted value of ξ is much lower or much higher, it could indicate, that the Norton creep law could not satisfactory fit the stress dependence. The alternative formulations will be proper, as we discuss below.

8.2.4 Time-Invariant Constitutive Equations

The substitution of (8.5) in (8.1) leads to the ordinary differential equation with the separable variables:

$$\frac{de_{ij}}{dt} = \frac{s_{ij}}{\tau_{oct}} h_c(t) \cdot g_c(\tau_{oct}). \quad (8.9)$$

We replace the variables in the Eq. (8.9), using the new variable, “scaled time” :

$$t^* = \int_0^t h_c(p) dp, \frac{de_{ij}}{dt} = \frac{dt^*}{dt} \frac{de_{ij}}{dt^*} = \frac{1}{h_c(t)} \frac{de_{ij}}{dt^*}. \quad (8.10)$$

With Eq. (8.10) the creep constitutive Eq. (8.9) reduces to:

$$\frac{de_{ij}}{dt^*} = \frac{s_{ij}}{\tau_{oct}} g_c(\tau_{oct}). \quad (8.11)$$

Introduction of “scaled time” substantially shortens all formulas. The variable t is not used any more. Hereafter, only “scaled time” t^* is assumed as the time variable. For brevity, we omit the subscript “star”: $t \equiv t^*$ in the following formulas.

The isotropic, time-invariant constitutive equation for creep reads:

$$\dot{\epsilon}_{ij} = \frac{s_{ij}}{\tau_{oct}} g_c(\tau_{oct}). \quad (8.12)$$

We use the symbol “dot” to denote the time derivative with respect to “scaled time” t^* ($\dot{\epsilon}_{ij} = d\epsilon_{ij}/dt^*$) in Eq. (8.12) and the subsequent equations. Applying Eq. (A.4) and (A.6) to (8.12), we get the scalar differential equation for time dependence of octahedral shear strain:

$$\frac{\dot{\gamma}_{oct}}{2} = g_c(\tau_{oct}). \quad (8.13)$$

The octahedral shear stress τ_{oct} is the function of “scaled time”.

For the beginning, consider the function, which covers the creep process from the beginning to the unlimited flow:

$$g_c(\tau_{oct}) = \bar{\epsilon} \cdot \left(\frac{\tau_{oct}}{\bar{t}} \right)^{\xi+1} \cdot \frac{1}{\left(\frac{\tau_{oct}}{\tau_{th}} \right)^{m_1} - 1} \cdot \frac{1}{1 - \left(\frac{\tau_{oct}}{\tau_Y} \right)^{m_2}} \quad (8.14)$$

The value $\tau_{th}[MPa]$ is the lowest stress, for which creep begins. This stress is referred to as the “creep threshold”. Below this threshold, creep cannot be measured. The yield shear stress is for the material $\tau_Y[MPa]$.

We evaluate the total creep life from the earliest occurrence of creep until the unlimited flow of the uniformly tensioned specimen. For the uniaxial stress state we have:

$$\sigma = E\epsilon, \dot{\epsilon}_{11} = \dot{\epsilon}, \tau_{oct} = \sqrt{2}\sigma/3.$$

There is one differential equation for the axial strain component $\epsilon(t)$:

$$\dot{\epsilon} = g_c(E\epsilon)g_c(E\epsilon) = \bar{\epsilon} \cdot \left(\frac{E\epsilon}{\bar{\sigma}} \right)^{\xi+1} \cdot \frac{1}{\left(\frac{E\epsilon}{\sigma_{th}} \right)^{m_1} - 1} \cdot \frac{1}{1 - \left(\frac{E\epsilon}{\sigma_Y} \right)^{m_2}}. \quad (8.15)$$

In the Eq. (8.15), the “creep threshold” and yield stress relate to each other:

$$\sigma_{th} = \frac{3\tau_{th}}{\sqrt{2}}, \sigma_Y = \frac{3\tau_Y}{\sqrt{2}}.$$

The creep process starts, when the stress achieves the threshold tension value σ_{th} , which is minimum necessary stress for creep:

$$\varepsilon(0) = \varepsilon_{th} \equiv \sigma_{th}/E.$$

The initial condition for the differential Eq. (8.15) expresses this requirement. The solution of Eq. (8.15) with reads:

$$t = t(\varepsilon) = c_1 \varepsilon^{m_1+m_2-\xi} + c_2 \varepsilon^{m_2-\xi} + c_3 \varepsilon^{m_1-\xi} + c_4 \varepsilon^{-\xi}. \quad (8.16)$$

The coefficients of Eq. (8.16) are:

$$c_1 = \frac{1}{\bar{\varepsilon} \cdot (\xi - m_1 - m_2)} \left(\frac{\bar{\sigma}}{E} \right)^{\xi+1} \left(\frac{\sigma_Y}{E} \right)^{-m_2} \left(\frac{\sigma_{th}}{E} \right)^{-m_1}, \quad (8.17)$$

$$c_2 = -\frac{1}{\bar{\varepsilon} \cdot (\xi - m_2)(\sigma_C/E)^{m_2}} \left(\frac{\bar{\sigma}}{E} \right)^{\xi+1} \left(\frac{\sigma_Y}{E} \right)^{-m_2}, \quad (8.18)$$

$$c_3 = -\frac{1}{\bar{\varepsilon} \cdot (\xi - m_1)} \left(\frac{\bar{\sigma}}{E} \right)^{\xi+1} \left(\frac{\sigma_{th}}{E} \right)^{-m_1}, \quad (8.19)$$

$$c_4 = \frac{1}{\bar{\varepsilon} \cdot \xi} \left(\frac{\bar{\sigma}}{E} \right)^{\xi+1}. \quad (8.20)$$

The next task is to calculate the total duration of creep process from the initiation of creep to the final rupture. As pointed above, the creep process starts, when the strain arrives the threshold value $\varepsilon_{th} \equiv \sigma_{th}/E$, i.e. $t(\varepsilon_{th}) = 0$. Creep process finishes with the complete rupture of the specimen, when the continuously increasing strain reaches the value of $\varepsilon_Y \equiv \sigma_Y/E$. Consequently, the total time from the beginning of creep process from the first occurrence of creep to the unabridged flow is:

$$t_{total} = t(\varepsilon_Y) = c_1 \varepsilon_Y^{m_1+m_2-\xi} + c_2 \varepsilon_Y^{m_2-\xi} + c_3 \varepsilon_Y^{m_1-\xi} + c_4 \varepsilon_Y^{-\xi}. \quad (8.21)$$

The strain rate as the function of ε reads:

$$\begin{aligned} \dot{\varepsilon} &= \left(\frac{dt}{d\varepsilon} \right)^{-1} \\ &= \frac{\varepsilon^{\xi+1}}{c_1 \cdot (m_1 + m_2 - \xi) \cdot \varepsilon^{m_1+m_2} + c_2 \cdot (m_2 - \xi) \cdot \varepsilon^{m_2} + c_3 \cdot (m_1 - \xi) \cdot \varepsilon^{m_1} - c_4 \xi} \end{aligned} \quad (8.22)$$

For simulation, we used the following material properties:

$$\xi = 3.75, m_1 = m_2 = 1, \sigma_Y = 2GPa, \bar{\varepsilon} = 10^{-26} s^{-1}, E = 200GPa, \bar{\sigma} = 1MPa.$$

For calculation we apply different threshold values: $\sigma_{th} = 300, 350, 400GPa$. The results for calculation are shown in the Table 8.1 and on Figs. 8.2, 8.3 and 8.4.

Table 8.1 Time-independent creep law

	σ_{th} [MPa]	t_{total} [s]	c_1 [s]	c_2 [s]	c_3 [s]	c_4 [s]	$\dot{\epsilon}$ [1/s]
1	300	$1.71 \cdot 10^9 \text{ s} =$ 54.45 years	$2.51 \cdot 10^4$	-24	-160	0.176	$\dot{\epsilon} = -\frac{1.5131\epsilon^{19/4}}{6.66 \cdot 10^4 \epsilon^2 - 766\epsilon + 1}$
2	350	$0.896 \cdot 10^9 \text{ s} =$ 28.42 years	$2.15 \cdot 10^4$	-24	-137	0.176	$\dot{\epsilon} = -\frac{1.5131\epsilon^{19/4}}{5.71 \cdot 10^5 \epsilon^2 - 671\epsilon + 1}$
3	400	$0.504 \cdot 10^9 \text{ s} =$ 15.97 years	$1.89 \cdot 10^4$	-24	-120	0.176	$\dot{\epsilon} = -\frac{1.5131\epsilon^{19/4}}{5.00 \cdot 10^5 \epsilon^2 - 600\epsilon + 1}$

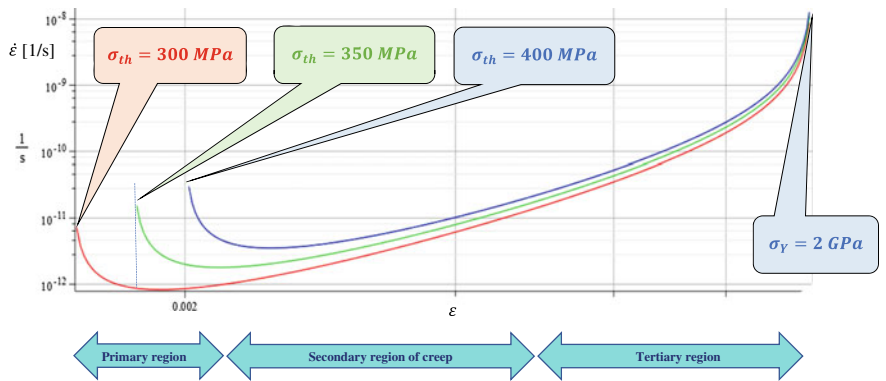


Fig. 8.2 Creep strain rate as function of creep strain. The regions of creep correspond nearly to $\sigma_{th} = 300 \text{ MPa}$

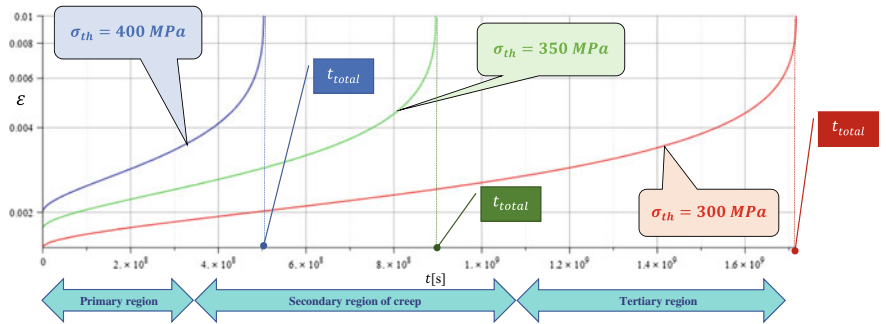


Fig. 8.3 Creep strain as function of time.. The regions of creep correspond nearly to $\sigma_{\gamma} = 2 \text{ GPa}$

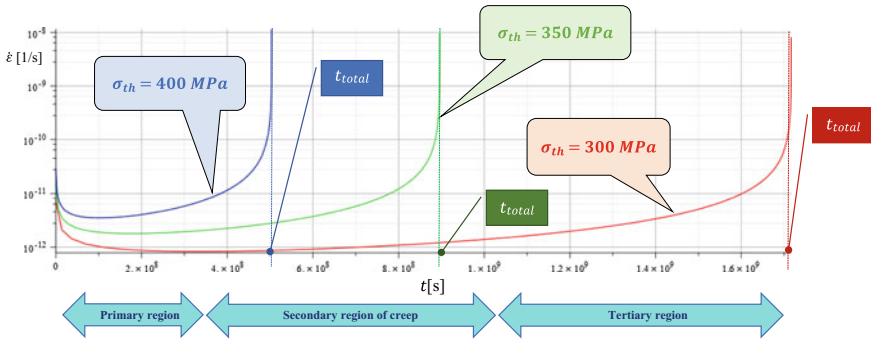


Fig. 8.4 Creep strain rate as function of time.. The regions of creep correspond nearly to $\sigma_Y = 2GPa$

Figure 8.2 shows the creep strain rate as function of creep strain. Three regions of primary, secondary and tertiary creep are approximately marked for: $\sigma_{th} = 300$ MPa. The regions shorten for the higher stress levels. Figure 8.2 displays the creep strain as the function of time over the whole creep region. Figure 8.4 exhibits the creep strain rate as the function of time.

At high stresses, the structural mechanical process determining the exponential growth of temperature is the stress and diffusion-controlled climbing of step dislocations. This is controlled at lower temperatures by diffusion of the lattice atoms along the dislocation lines and at higher temperatures by volume diffusion. The higher

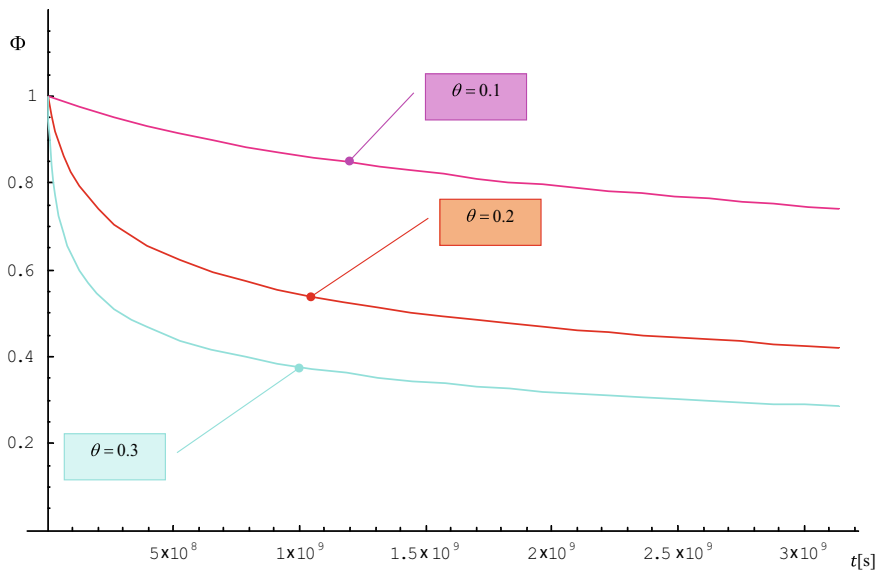


Fig. 8.5 Relaxation function $\Phi(t)$ for different twist rates

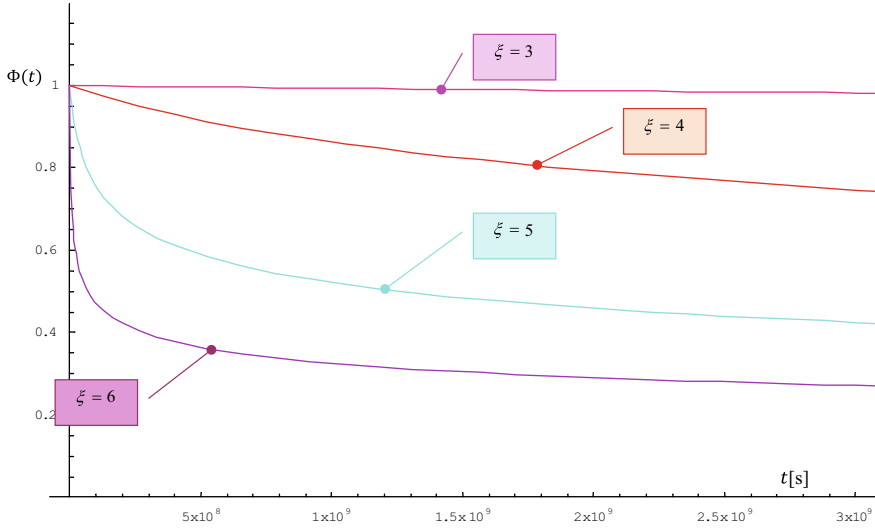


Fig. 8.6 Relaxation function $\Phi(t)$ for twist rate $\theta = 0.1$ and different creep exponents

the stress, the more pronounced the climbing process is at a given temperature and the more sliding dislocations are available. This is called dislocation or power-law creep. Firstly, consider the isotropic stress function for Norton-Bailey law (Odquist and Hult 1962):

$$g_c = \bar{\varepsilon} \cdot \left(\frac{\tau_{oct}}{\bar{\tau}} \right)^{\xi+1}. \quad (8.23)$$

Here ξ is the dimensionless experimental constant for the operational temperature of the spring element. The creep constants for several spring materials were acquired in (IGF 18992 BR 2020). We use the constants $\bar{\sigma}[MPa]$, $\bar{t}[s]$ for proper scaling of stress and time. For example, the common setting for scaling constants $\bar{\tau} = 1[MPa]$, $\bar{\sigma} = 1[MPa]$, $\bar{t} = 1[s]$. The octahedral stress τ_{oct} requires the unit $[MPa]$. With these settings, the units are $[1/s]$ for both $\bar{\varepsilon}$ and $\dot{\varepsilon}_{ij}$. All these constants depend on the particular creep mechanism, which changes with the temperature.

Secondly, the isotropic stress function for Garofalo creep law (Garofalo 1965) reads.

$$g_c = \bar{\varepsilon} \cdot \sinh^{\xi+1} \left(\frac{\tau_{oct}}{\bar{\tau}} \right). \quad (8.24)$$

Thirdly, the exponential creep law (Kassner 2008) is:

$$g_c = \bar{\varepsilon} \cdot \left[\exp \left(\frac{\tau_{oct}}{\bar{\tau}} \right) - 1 \right]. \quad (8.25)$$

Finally, Naumenko-Altenbach-Gorash creep law reads (Naumenko et al. 2009):

$$g_c = \bar{\varepsilon} \cdot \left(\frac{\tau_{oct}}{\bar{\tau}} \right) + \bar{\varepsilon} \cdot \left(\frac{\tau_{oct}}{\bar{\tau}} \right)^{\xi+1}.$$

8.3 Scalar Constitutive Equations for Uniaxial Stresses

8.3.1 Norton-Bailey Law

For the uniaxial stress state, there is only non-vanishing component of stress tensor:

$$\sigma_{11} = \sigma, \tau_{oct} = \sqrt{2}\sigma/3.$$

Correspondingly, the non-vanishing components of strain rate are

$$\dot{\varepsilon} \equiv \dot{\varepsilon}_{11} = -2\dot{\varepsilon}_{22} = -2\dot{\varepsilon}_{33}. \quad (8.26)$$

With Eq. (8.23) the uniaxial strain is:

$$\dot{\varepsilon} = \bar{\varepsilon} \cdot \left(\frac{\sigma}{\bar{\sigma}} \right)^{\xi+1}, \bar{\sigma} = \frac{3\bar{\tau}}{\sqrt{2}}. \quad (8.27)$$

For brevity of equations, we introduce the material constant

$$c_\sigma = \frac{\bar{\varepsilon}}{\bar{\sigma}^{\xi+1}}.$$

With this constant the dependence of uniaxial strain rate upon stress reads

$$\dot{\varepsilon} = c_\sigma \sigma^{\xi+1}. \quad (8.28)$$

For pure shear stress state

$$\sigma_{12} = \sigma_{21} = \tau, \tau_{oct} = \sqrt{6}\tau/3.$$

the non-vanishing components of deformation rate are.

$$\dot{\varepsilon}_{12} = \dot{\varepsilon}_{21} \equiv \dot{\gamma}/2 = \tilde{\gamma} \cdot \left(\tau/\tilde{\tau} \right)^{\xi+1}/2,$$

where $\tilde{\gamma} = \sqrt{3}\bar{\varepsilon}$ and $\tilde{\tau} = \bar{\sigma}/\sqrt{3}$.

The creep constant for shear strain reads:

$$c_\tau = \frac{\tilde{\gamma}}{\tau^{\xi+1}}.$$

The Norton-Bailey creep law for pure shear deformation reduces to.

$$\dot{\gamma} = c_\tau \tau^{\xi+1}. \quad (8.29)$$

The creep law for spring material Nimonic 80A was experimentally acquired in Gittus (1964). In the cited article the similar value of time exponent was reported to the Inconel and EN50 steel.

8.3.2 Garofalo Creep Law

Secondly, the isotropic stress function for Garofalo creep law (8.25) Garofalo (1963) for $\xi = 1$ could be represented as

$$g_c(\tau_{oct}) = \bar{\varepsilon} \cdot \sinh \frac{\tau_{oct}}{\tau}. \quad (8.30)$$

For the uniaxial stress state, the deformation rate reads

$$\dot{\varepsilon} = \bar{\varepsilon} \cdot \sinh(\sigma/\bar{\sigma})$$

or

$$\dot{\varepsilon} \equiv \dot{\varepsilon}_{11} = c_\sigma \sinh(\sigma/\bar{\sigma}) \quad (8.31)$$

Here $c_\sigma = \bar{\varepsilon}$ is the creep constant for uniaxial strain and possess the dimension $[1/s]$.

For pure shear stress state the deformation rate reads

$$\dot{\varepsilon}_{12} = \dot{\varepsilon}_{21} = \frac{1}{2} \tilde{\gamma} \cdot \sinh(\sqrt{3}\tau/\bar{\sigma}) \quad (8.32)$$

Finally, the shear strain rate according to Garofalo creep law is

$$\dot{\gamma} \equiv 2\dot{\varepsilon}_{12} = c_\tau \sinh\left(\tau/\tilde{\tau}\right). \quad (8.33)$$

with the constant of dimension $[1/s]$: $c_\tau = \sqrt{3}\bar{\varepsilon} = \tilde{\gamma} \equiv \sqrt{3}c_\sigma$.

8.3.3 Exponential Law

Equation (8.25) pronounces the isotropic stress function for exponential creep law. For the uniaxial stress state, the strain rate reads:

$$\dot{\varepsilon} \equiv \dot{\varepsilon}_{11} = \bar{\varepsilon} \cdot [\exp(\sigma/\bar{\sigma}) - 1] = -2\dot{\varepsilon}_{22} = -2\dot{\varepsilon}_{33}. \quad (8.34)$$

For pure shear stress, the shear deformation rate reduces to:

$$\dot{\gamma} \equiv 2\dot{\varepsilon}_{12} = \tilde{\gamma} \cdot [\exp(\tau/\tilde{\tau}) - 1]. \quad (8.35)$$

For the above creep laws the closed form solutions of basic creep problems are derived. Numerical values for the creep constants $\bar{\varepsilon}[s^{-1}]$, $\tilde{\gamma} = \sqrt{3}\bar{\varepsilon}[s^{-1}]$ are surely different for diverse creep laws. The constant $\tilde{\tau}[MPa]$ is used for scaling purposes.

8.4 Creep and Relaxation of Twisted Rods

8.4.1 Constitutive Equations for Relaxation in Torsion

In contrast to the creep test (constant load), the relaxation test keeps the strain constant. This process causes components that have been prestressed to a certain elastic strain at the beginning of their use to lose their prestress over time. The stress decreases (relaxes) as the elastic strain is converted into plastic strain over time. The deformation of body during relaxation does not alter, but the stress gradually reduces. Consider the relaxation problem for a rod with circular cross-section under the constant twist. Let $\tau(r, t)$ is shear stress in the cross-section of rod. The total shear strain in any instant of the time is $\gamma(r, t)$, is the sum of the elastic and the creep components of shear strain:

$$\gamma = \gamma_e + \gamma_c.$$

The creep component of shear strain is $\gamma_c(\rho, t)$. The elastic component of shear strain is

$$\gamma_e = \tau/G. \quad (8.36)$$

Firstly, we investigate the creep for the total deformation that remains constant in time. Thus, we consider the total strain $\gamma_0(\rho)$ as function of radius only. However, the elastic and the creep components of strain are the functions as well of radius and of time, such that:

$$\gamma(\rho, t) = \gamma_e(\rho, t) + \gamma_c(\rho, t) \equiv \gamma_0(\rho). \quad (8.37)$$

The time differentiation of (8.37) leads to the differential equation for elastic and creep strain rates:

$$\dot{\gamma}(\rho, t) = \dot{\gamma}_e(\rho, t) + \dot{\gamma}_c(\rho, t) \equiv 0, \quad (8.38)$$

where dot denotes the time derivative.

The differentiation of the Eq. (8.36) over time delivers the elastic component of strain rate.

$$\dot{\gamma}_e = \frac{\dot{\tau}}{G}. \quad (8.39)$$

8.4.2 Torque Relaxation for Norton-Bailey Law

At first, we assume the Norton-Bailey law for the state of shear stress (Boyle 2012):

$$\dot{\gamma}_c(\rho, t) = c_\tau \tau^{\xi+1}. \quad (8.40)$$

The substitution of material law (8.40) in Eq. (8.38) results in the ordinary nonlinear differential equation of the first order for total shear stress $\tau(r, t)$:

$$\dot{\tau}/G + c_\tau \tau^{\xi+1} = 0. \quad (8.41)$$

The shear stresses in the moment $t = 0$ for the rod with circular cross-section of the diameter $d = 2r$ are

$$\tau_0(\rho) = G\theta\rho = \frac{T\rho}{r}, \text{ for } 0 < \rho < r = \frac{d}{2},$$

where θ is the twist angle per unit length, $T = G\theta r$ is the shear stress on the outer surface of the wire in the initial moment of creep. The torque at the initial moment $t = 0$ is pure elastic and is equal to:

$$M_T^0 = \pi T r^3 / 2.$$

The initial condition for the Eq. (8.41) presumes the pure elastic shear stress in the initial moment $t = 0$:

$$\tau(\rho, t = 0) = \tau_0(\rho). \quad (8.42)$$

The average creep time for Norton-Bailey creep is

$$t_c = \frac{1}{T^\xi \cdot G \cdot c_\tau \cdot \xi} \equiv \frac{1}{\tilde{\gamma} \cdot \xi} \cdot \frac{\tilde{\tau}^{\xi+1}}{T^\xi \cdot G}.$$

The solution of the ordinary differential Eq. (8.17) with initial condition (8.18) delivers the shear stress over the cross-section of the twisted rod as the function of time and radius:

$$\tau(\rho, t) = \left[\tau_0^{-\xi}(\rho) + c_\tau G \xi t \right]^{-1/\xi}. \quad (8.43)$$

The function (8.43) pronounced the relaxation of the shear stress from the initial state (8.42). To calculate the relaxation of the couple, we integrate the shear stress over the cross-section of the bar. The couple as the function of time is

$$M_T(t) = 2\pi \int_0^r \rho^2 \tau(\rho, t) d\rho.$$

With the expression for total shear stress (9.43) we can calculate the couple.

$$M_T(t) = 2\pi \int_0^r \rho^2 \left[(T_Q/r)^{-\xi} + c_\tau G \xi t \right]^{-1/\xi} d\rho. \quad (8.44)$$

For evaluation of the integral (9.44) the formula for $J_p(a, m; X)$ from Appendix B is applied for the case $p = 2$. The integral could be expressed in terms of hypergeometric function (Kobelev 2014b):

$$\frac{M_T(t)}{M_T^0} = {}_2F_1\left(\frac{4}{\xi}, \frac{1}{\xi}; \frac{4+\xi}{\xi}; -\frac{t}{t_c}\right). \quad (8.45)$$

8.4.3 Torque Relaxation for Garofalo Law

At second, we presume the Garofalo law for pure share state of stress:

$$\dot{\gamma}_c(\rho, t) = c_\tau \sinh\left(\tau / \tilde{\tau}\right). \quad (8.46)$$

The solution of the differential Eq. (8.46) with initial condition (8.42) for the Garofalo creep law reads (Kobelev 2014b):

$$\tau(\rho, t) = \tilde{\tau} \ln \left\{ \tanh \left[\frac{G c_\tau}{2 \tilde{\tau}} t + \operatorname{arctanh} \left(\exp \left(\frac{G \rho \theta}{\tilde{\tau}} \right) \right) \right] \right\}. \quad (8.47)$$

Using formula for $I_2(a, b; X)$ from the Appendix, the time dependent torque could be expressed in terms of polylogarithms (Lewin 1981). The relaxation function for the Garofalo law reads:

$$\begin{aligned} \frac{M_T(t)}{M_T^0} &= \sum_{i=1}^4 p_i \left(\frac{\tilde{\tau}}{\tilde{T}} \right)^i, \\ p_1 &= \frac{4}{3} \ln(U), U = \frac{e^{t/t_c} - 1}{e^{t/t_c} + 1}, V = e^{T/\tilde{\tau}}. \\ p_2 &= -4 \left[Li_2 \left(-\frac{V}{U} \right) - Li_2(-V \cdot U) \right], \\ p_3 &= 8[Li_3(-V/U) - Li_3(-V \cdot U)], \\ p_4 &= 8[Li_4(-1/U) - Li_4(-U)] - 8 \left[Li_4 \left(-\frac{V}{U} \right) - Li_4(-V \cdot U) \right]. \end{aligned} \quad (8.48)$$

In Eq. (8.48), the average time of creep for the Garofalo and exponential law is:

$$t_c = \frac{\tilde{\tau}}{c_\tau G}. \quad (8.49)$$

8.4.4 Torque Relaxation for Exponential Law

At third, we apply the exponential law for the state of shear stress

$$\dot{\gamma}_r(\rho, t) = \tilde{\gamma} \cdot \left[\exp\left(\tau / \tilde{\tau}\right) - 1 \right]. \quad (8.50)$$

The substitution of material law (9.49) in Eq. (8.38) leads to the ordinary differential equation for total shear stress.

$$\dot{\tau} / G + \tilde{\gamma} \cdot \left[\exp\left(\tau / \tilde{\tau}\right) - 1 \right] = 0. \quad (8.51)$$

The solution of the differential Eq. (8.51) with initial condition (8.42) delivers the shear stress over the cross-section of the twisted rod. Relaxation of torque could be expressed again in terms of polylogarithm (Lewin 1981):

$$\frac{M_T(t)}{M_T^0} = \sum_{i=0}^4 p_i \left(\frac{\tilde{\tau}}{\tilde{T}} \right)^i, U = e^{t/t_c}, p_0 = 1, p_1 = \frac{4t}{3t_c}, \quad (8.52)$$

$$p_2 = 4Li_2(V \cdot (1 - U)), p_3 = -8Li_3(V \cdot (1 - U)),$$

$$p_4 = 8|Li_4(V \cdot (1 - U)) - Li_4(1 - U)|.$$

8.5 Creep and Relaxation of Helical Coiled Springs

8.5.1 Phenomena of Relaxation and Creep

Compression and tension springs are made of an elastic wire material formed into the shape of a helix, which returns to its natural length when unloaded. Compression springs can be commonly referred to as a coil spring or a helical spring. Coil springs are a mechanical device, which is typically used to store energy and subsequently release it to absorb shock, or to maintain a force between contacting surfaces. The major stresses produced in conical and volute springs are shear stresses due to twisting. The subsequent consideration is applicable for both compression or tension helical springs with the minor alternations. For definiteness, only compression springs will be discussed.

Consider the helical spring with mean helix diameter D , wire diameter d and active coils number n_a . The force $F_z(t)$ applied to a spring that causes a deflection s is

$$F_z(t) = \frac{2M_T(t)}{D}.$$

The relaxation of helical springs occurs, when the compressed length of the spring remains constant. During the relaxation process the twist angle per unit length remains constant as well:

$$\theta = \frac{2s}{\pi n_a D^2}.$$

The spring force $F_z(t)$ reduces over time. At initial moment $t = 0$ the spring force due to pure elastic deformation F_z^0 and the shear stress on the outer surface of wire $\tilde{\tau}$ are correspondingly:

$$F_z^0 = \frac{2M_T^0}{D} = \frac{Gd^4s}{8n_aD^3}, \tilde{\tau} = \frac{G\theta d}{2}. \quad (8.53)$$

2°. Creep of spring. The creep of helical springs occurs, when the spring force remains constant. In this situation the compressed length continuously reduces over time and the deflection increases (Geinitz et al. 2011).

8.5.2 Relaxation of Helical Springs

For the solution of relaxation problem we apply the results of relaxation problem for twisted rod with the circular cross-section for helical springs. If the material of helical springs obeys 'Norton-Bailey law, the spring load as the function of time is Kobelev (2014a):

$$F_z(t) = {}_2F_1\left(\frac{4}{\xi}, \frac{1}{\xi}; \frac{4+\xi}{\xi}; -\frac{t}{t_c}\right) F_z^0. \quad (8.54)$$

For certain exponents in Norton-Bailey law, the Eq. (8.54) expresses in terms of elementary functions:

$$F_z(t) = \frac{4}{3} \frac{(1+x)^{3/4} - 1}{x} F_z^0, \quad x = \frac{t}{t_c}, \text{ for } \xi = 4, \quad (8.55)$$

and

$$F_z(t) = \frac{4}{3} \frac{(x-2)\sqrt{1+x} + 2}{x^2} F_z^0, \quad x = \frac{t}{t_c}, \text{ for } \xi = 2. \quad (8.56)$$

The relaxation of spring load over time demonstrates the dimensionless relaxation function:

$$\Phi(t) = \frac{F_z(t)}{F_z^0} = {}_2F_1\left(\frac{4}{\xi}, \frac{1}{\xi}; \frac{4+\xi}{\xi}; -\frac{t}{t_c}\right). \quad (8.57)$$

On Fig. 8.5 the function Φ is plotted for different twist angles per unit length. For illustration of the procedure, we perform all calculations with Norton-Bailey law and the following set of material constants

$$G = 79.74 \text{ GPa}, \quad E = 200 \text{ GPa}, \quad \xi = 4, \quad \bar{\varepsilon} = 10^{-24} \text{ sec}^{-1}, \quad \tilde{\gamma} = 1.73 \cdot 10^{-24} \text{ sec}^{-1}, \\ \bar{\sigma} = 2000 \text{ MPa}, \quad \tilde{\tau} = 1154 \text{ MPa}.$$

The three graphs of the function $\Phi(t)$ are shown on the figure for twist angles per unit length ($\theta = 0.1; 0.2; 0.3$). It means that the higher twist angle per unit length, the quickly occurs the relaxation of torque. This apparently happens because the strain rate is higher for higher stresses.

On Fig. 8.6 the function $\Phi(t)$ is plotted for the same twist angle per unit length ($\theta = 0.1$), but for the different exponents.

$$\xi = 3.0; \xi = 4.0; \xi = 5.0; \xi = 11.0.$$

The rest of material parameters remain the same as in the previous example. Figure 8.6 evidently demonstrates, that the higher the exponent, the higher the relaxation rate.

8.5.3 Creep of Helical Compression Springs

During the creep deformation of springs the spring force remains constant over time. Correspondingly, the spring length reduces with time. The length decrease rate over time is $\dot{s}$. The shear strain rate varies with time:

$$\dot{\gamma}(t, \rho) = \dot{\theta} \rho = 2\dot{s} \rho / \pi n_a D_{im}^2 \text{ for } 0 < \rho < r = d/2.$$

According to Norton-Bailey law (9.4) the shear stress due to creep is.

$$\tau = \left(\frac{\dot{\gamma}}{c_\tau} \right)^{\frac{1}{\xi+1}} \equiv \tilde{\tau} \left(\frac{2\dot{s}}{\tilde{\gamma} \pi n D_{im}^2} \right)^{\frac{1}{\xi+1}} \rho^{\frac{1}{\xi+1}}.$$

Performing the integration over the area of wire we get the moment due to creep (Kobelev 2014a):

$$M_T^0 = 2\pi \int_0^r \rho^2 \tau(\rho, t) dr = \tilde{\tau} \left(\frac{2\dot{s}}{\tilde{\gamma} \pi n D^2} \right)^{\frac{1}{\xi+1}} 2\pi \int_0^r \rho^{2+\frac{1}{\xi+1}} dr. \quad (8.58a)$$

For the solution of Eq. (8.58), we use the following relations:

$$M_T^0 = \frac{D \cdot F_z^0}{2}, \quad \tilde{\gamma} = \tilde{\tau}^{\xi+1} c_\tau, \quad \int_0^{r=d/2} \rho^{2+\frac{1}{\xi+1}} dr = \frac{\xi+1}{4+3\xi} \left(\frac{d}{2} \right)^{\frac{4+3\xi}{1+\xi}}.$$

The solution of Eq. (8.58) provides the reduction rate of spring length. The rate of length reduction for constant spring force F_z^0 is the function of time:

$$\dot{s} = \frac{\pi D^2 n_a c_\tau}{2d^{4+3\xi}} \left(\frac{4DF_z^0}{2\pi} \frac{4+3\xi}{\xi+1} \right)^{\xi+1}. \quad (8.58b)$$

8.6 Creep and Relaxation of Beams in State of Pure Bending

8.6.1 Constitutive Equations for Relaxation in Bending

Consider the problem of stress relaxation in the pure bending of a rectangular cross-section ($B \times T$) beam. In the applied Euler–Bernoulli theory of slender beams subjected to a bending moment M_B , a major assumption is that ‘plane sections remain plane’. In other words, any deformation due to shear across the section is not accounted for (no shear deformation). During the relaxation, experiment the curvature of the neutral axis of beam κ remains constant over time, such that the bending moment M_B continuously decreases. This case describes the relaxation of bending stress, if the flexure deformation of beam does not alter in time. This problem was considered in Boyle and Spence (1983) with the aid of numerical methods.

Let $\sigma(z, t)$ is uniaxial stress in the beam in the direction of beam axis. The total strain in any instant of the time is $\varepsilon(z, t)$; is the sum of the elastic and the creep components of the strain:

$$\varepsilon = \varepsilon_e + \varepsilon_c. \quad (8.59)$$

The elastic component of normal strain is.

$$\varepsilon_e = \sigma/E, \quad (8.60)$$

and $\varepsilon_c(z, t)$ is the creep component of normal strain.

Consider creep under constant in time total strain.

$$\varepsilon(z, t) = \varepsilon_e(z, t) + \varepsilon_c(z, t) \equiv \varepsilon_0(z). \quad (8.61)$$

The normal strain $\varepsilon_0(z) = \varepsilon(z, t = 0)$ is the function of normal coordinate, but remains constant over time during the relaxation process. The time differentiation of (8.61) leads to.

$$\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_c \equiv 0. \quad (8.62)$$

We exercise once again the common constitutive models of creep in bending state.

8.6.2 Relaxation of Bending Moment for Norton-Bailey Law

The Norton-Bailey law for a uniaxial state of stress reads.

$$\dot{\varepsilon}_c(z, t) = c_\sigma \sigma^{\xi+1}, \quad (8.63)$$

The substitution of material laws results in the ordinary differential equation for uniaxial stress.

$$\dot{\sigma}/E + c_\sigma \sigma^{\xi+1} = 0. \quad (8.64)$$

The initial condition for the Eq. (8.64) delivers the pure elastic shear stress in the initial moment.

$$\sigma(z, t = 0) = \sigma_0(z). \quad (8.65)$$

For pure elastic bending the following initial distribution of stresses over the cross-section of the beam is valid:

$$\sigma_0(z) = E\kappa z = \frac{2z}{T} \sigma_p, \quad (8.66)$$

where κ is the bending curvature, which presumed to be constant over time and $-T/2 \leq z \leq T/2$ is the perpendicular distance to the neutral axis. σ_p is the normal stress on the outer surface of the wire in the initial moment of creep.

The solution of the ordinary differential Eq. (8.64) with initial condition (8.65) is.

$$\sigma(z, t) = \left[\sigma_0^{-\xi}(z) + c_\sigma E \xi t \right]^{-1/\xi}. \quad (8.67)$$

The function (9.67) pronounced the relaxation of the normal stress from the initial state of pure tension (9.64). To calculate the relaxation of the bending moment, we integrate the normal stress over the cross-section of the beam. The bending moment for the rectangular cross-section of width B and height H is the function of time.

$$M_B(t) = B \int_{-T/2}^{T/2} z \sigma(z, t) dz. \quad (8.68)$$

With the expression (9.68) we can calculate.

$$M_B(t) = 2EB\kappa \int_0^{T/2} z \cdot \left[\frac{1}{z^\xi} + c_\sigma \kappa^\xi E^{\xi+1} \xi t \right]^{-1/\xi} dz. \quad (8.69)$$

Using the results of Appendix ($J_p(a, m; X)$, case $p = 1$), the integral in (8.69) could be expressed in terms of hypergeometric function (Kobelev 2014a):

$$\frac{M_B(t)}{M_B^0} = {}_2F_1\left(\frac{3}{\xi}, \frac{1}{\xi}; \frac{3+\xi}{\xi}; -\frac{t}{t_{creep}}\right), \quad (8.70)$$

where

$$t_c = \frac{1}{c_\sigma \cdot E \cdot \xi \cdot \Sigma^\xi}.$$

is the characteristic creep time in bending and

$$M_B^0 = \frac{\sigma_p B T^2}{6} \equiv \frac{\kappa E B T^3}{12} \quad (8.71)$$

is the elastic bending moment at time $t = 0$. For example, for $\xi = 3$ we have:

$$\frac{M_B(t)}{M_B^0} = \frac{3}{2} \frac{(1+x)^{2/3} - 1}{x}, \quad x = \frac{t}{t_c}.$$

8.6.3 Relaxation of Bending Moment for Garofalo Law

We consider the now Garofalo law for uniaxial state of stress:

$$\dot{\epsilon}_c = c_\sigma \sinh(\sigma/\bar{\sigma}). \quad (8.72)$$

The solution of the ordinary differential Eq. (8.37) with initial condition (8.38) for the Garofalo creep law leads to the expression of normal stress as function of coordinate z and time t (Kobeleev 2014b):

$$\sigma(z, t) = \bar{\sigma} \ln \left\{ \tanh \left[\frac{E c_\sigma}{2 \bar{\sigma}} t + \operatorname{arctanh} \left(\exp \left(\frac{2 \Sigma}{T \bar{\sigma}} z \right) \right) \right] \right\}. \quad (8.73)$$

The function (9.73) pronounced the relaxation of the normal stress from the initial state of pure tension (9.64). For numerical evaluation, the formula for $I_1(a, b; X)$ from Appendix is to be applied. With this formula, the integral in (8.73) could be expressed in terms of polylogarithm. Relaxation of bending moment could be expressed again in terms of polylogarithm (Lewin 1981):

$$\begin{aligned} \frac{M_B(t)}{M_B^0} &= \sum_{i=1}^3 p_i \left(\frac{\bar{\sigma}}{\Sigma} \right)^i, \quad U = \coth \left(\frac{t}{2 t_c} \right), \quad V = e^{\Sigma/\bar{\sigma}}. \\ p_1 &= -\frac{3}{2} \ln(U), \quad p_2 = -3 \left[Li_2(-V \cdot U) - Li_2 \left(-\frac{V}{U} \right) \right], \\ p_3 &= 3 [Li_3(-V \cdot U) - Li_3(-V/U)] - 3 [Li_3(-U) - Li_3(-1/U)]. \end{aligned} \quad (8.74)$$

8.6.4 Relaxation of Bending Moment for Exponential Law

In this section, we investigate the problem of the pure bending of a rectangular cross-section beam with a modified power law (stress range-dependent constitutive model) subjected to a bending moment. We assume exponential law for the state of uniaxial stress.

The substitution of modified power material law (8.9) results in the ordinary differential equation for uniaxial stress:

$$\dot{\sigma}/E + \bar{\varepsilon} \cdot [\exp(\sigma/\bar{\sigma}) - 1] = 0. \quad (8.75)$$

When loaded by a bending moment, the beam bends so that the inner surface is in compression and the outer surface is in tension. The neutral plane is the surface within the beam between these zones, where the material of the beam is not under stress, either compression or tension. The solution of the ordinary differential Eq. (8.75) with initial condition (8.38) delivers the stress over the cross-section of the beam as the function of time and distance z to neutral plane.

For calculation of the bending moment the formula for $J_p(a, m; X)$ from Appendix is applied for the case $p = 1$. With this formula the bending moment in the cross-section could be expressed in terms of hypergeometric function (Kobelev 2014b):

$$\frac{M_B(t)}{M_B^0} = \sum_{i=0}^3 p_i \left(\frac{\bar{\sigma}}{\Sigma} \right)^i, \quad U = 1 - \exp\left(-\frac{t}{t_c}\right), \quad V = e^{\Sigma/\bar{\sigma}}. \quad (8.76)$$

$$p_0 = 1, \quad p_1 = 3 \frac{t}{2t_c}, \quad p_2 = 3Li_2(V \cdot U), \quad p_3 = -3[Li_3(V \cdot U) - Li_3(U)].$$

The curvature κ of the beam remains constant in time. In the expressions (8.71), (8.74) and (8.76) the bending moment $M_B(t)$ will be the continuously reducing function of time.

8.6.5 Creep in State of Bending

In contrast, during the bending creep deformation the moment M_B^0 remains constant over time. The curvature $\kappa = \kappa(t)$ will slowly increase with the constant rate of growth. The elongation rate of the strip, which locates on the perpendicular distance z to the neutral axis, is:

$$\dot{\varepsilon}(t, z) = \dot{\kappa} z \quad \text{for} \quad -T/2 \leq z \leq T/2.$$

According to Norton-Bailey law (8.3) the shear stress due to creep is:

$$\sigma_c = \left(\frac{\dot{\varepsilon}}{c_\sigma} \right)^{\frac{1}{\xi+1}} = \left(\frac{\dot{\kappa} z}{c_\sigma} \right)^{\frac{1}{\xi+1}}.$$

Performing the integration over the area of wire we get the bending moment due to creep (Kobeleev 2014b):

$$M_B^0 = B \int_{-T/2}^{T/2} z \sigma_c(z, t) dz = \frac{BT^2}{2} \frac{\xi + 1}{3 + 2\xi} \left(\frac{T \dot{\kappa}}{2c_\sigma} \right)^{\frac{1}{1+\xi}}. \quad (8.77)$$

Assuming that bending moment M_B^0 remains constant over time, from this equation the curvature time rate resolves as.

$$\dot{\kappa} = \frac{2c_\sigma}{T} \left(\frac{2M_B^0}{BT^2} \frac{3 + 2\xi}{\xi + 1} \right)^{\xi+1}. \quad (8.78)$$

The rate of flexure $\dot{\kappa}$ remains constant, such that the bending radius linearly increases over time. The similar expressions follow for the other studied above creep laws.

The results of this Section are applicable for springs, which stressed typically by bending loads. For example, the leaf springs and torsion springs are in the bending state. Torsion springs may be of helix or spiral type. Accordingly, the results of this Section could be instantly applied to estimate the effects of creep and relaxation of such springs.

8.7 Creep and Relaxation of Disk Springs

8.7.1 Creep of Disk Springs

One important example of the springs with the uniaxial normal stress is the disk springs. In these springs dominates the state of bending of conical part.

For the calculation of disk springs, the theory of the punched flat disks is usually applied. In this theory, the springs are flat circular disks, resting on their external rim and uniformly loaded along its internal rim (Timoshenko 1948). The shape of the disks is not considered; nevertheless, it has an influence on the shape of the suiting spring diagrams and on stress distribution. The load–deflection curve obtained by calculation is a straight line in this case. The maximum disks flattening load is identical to that obtained by the other calculation methods described hereinafter. The details of the elastic modelling are displayed in Chap. 4. The deformation hypothesis presupposes that the radial stresses are negligible and the cross-section of spring does not distort, but rather that is merely rotates about the inversion point (Sects. 4.1.2 and 4.1.3).

Consider a sector of the disk with the constant thickness T . We use in this chapter the notation T for the thickness of the disk spring, as the notation t is reserved for time. The disk is loaded by a force F_z , which assumed to be constant in time. Consider a sector and in it a strip dx at location x taking the inversion point as the origin. Under the action of force F_z the disk creeps and is deflected through a time dependent angle:

$$\phi = \alpha - \psi,$$

where.

- $\alpha = h/(r_e - r_i)$ the initial cone angle of disk,
 $\psi = H/(r_e - r_i)$ the deformed cone angle of disk,
 h the free height of disk, measured as the elevation of the free truncated cone formed by either the upper or lower surface.
 H the deformed height of disk, measured as the elevation of the deformed truncated cone formed by either the upper or lower surface,
 $r_e = D_e/2$ and are respectively the outer and inner radii of disk.

For small axial deflection of disk s the angle reads.

$$\phi = \frac{s}{(r_e - r_i)},$$

This strip moves slowly into its new deformed position. The resultant tangential strain may be analyzed as the resultant of the radial displacement Δr and the rotation ϕ . The first of these causes a uniform strain throughout the thickness of the disk, if one neglects the small variation in distance to the center of the disk at various points of the section. The second results in a tangential bending strain which is zero in the neutral surface and maximum at the upper and lower surfaces. The tangential stresses produced by these two components of the strain cause a radial moment about point O which resists the moment created by the external and that of the deflected one forces.

For the studied springs, the thickness of spring material T is thin enough. This assumption allows neglecting the tangential stress due to bending. In other words, the influence of tangential bending strain is abandoned.

Calculating the tangential stress due to the radial displacement solely, we can use the expression for tangential strain in terms of rotation ϕ :

$$\varepsilon_\theta = \frac{x}{x - C_i} \left(\alpha - \frac{\phi}{2} \right) \phi. \quad (8.79)$$

Here c_i is the distance of inversion point to the disk symmetry axis in the creep state,

Under the creep conditions the rotation $\phi = \phi(t)$ and, consequently, tangential strain $\varepsilon_\theta = \varepsilon_\theta(t)$ are the functions of time. The tangential strain rate results from (8.79):

$$\dot{\varepsilon}_\theta \equiv \frac{d\varepsilon_\theta}{dt} = \frac{x}{x - c_i} \left[\left(\alpha - \frac{\phi}{2} \right) \dot{\phi} + \left(-\frac{\dot{\phi}}{2} \right) \phi \right] \equiv \frac{x}{x - c_i} (\alpha - \phi) \dot{\phi}. \quad (8.80)$$

Because it is assumed that the radial stresses are negligible, the tangential stress relates to tangential strain by means the uniaxial Norton- Bailey law:

$$\dot{\varepsilon}_\theta(z, t) = c_\sigma \sigma_\theta^{\xi+1}. \quad (8.81)$$

The tangential stress is the function of rotation angle ϕ and its time derivative $\dot{\phi}$:

$$\sigma_\theta = \left[\frac{(\alpha - \phi) \dot{\phi}}{c_\sigma} \frac{x}{x - C_i} \right]^{\frac{1}{\xi+1}}. \quad (8.82)$$

The radial moment of the tangential forces in the section about point O is:

$$dM_\theta = \sigma_\theta T \cdot x \sin(\alpha - \phi) dx d\theta. \quad (8.83)$$

Substituting in Eq. (8.83) the Eq. (8.82) and assuming the deflection as small,

$$\sin(\alpha - \phi) \cong \alpha - \phi,$$

we obtain the radial moment of the tangential forces in the section:

$$dM_\theta = (\alpha - \phi) T \left[\frac{(\alpha - \phi) \dot{\phi}}{c_\sigma} \right]^{\frac{1}{\xi+1}} \left(\frac{x}{x - C_i} \right)^{\frac{1}{\xi+1}} x dx d\theta$$

Integrating from $x = C_i - r_e$ to $x = C_i - r_i$, we get the internal moment of the sector about point O:

$$\begin{aligned} M_\theta(t) &= (\alpha - \phi) 2\pi T \left[\frac{(\alpha - \phi) \dot{\phi}}{c_\sigma} \right]^{\frac{1}{\xi+1}} \int_{C_i - r_e}^{C_i - r_i} \left(\frac{x}{x - C_i} \right)^{\frac{1}{\xi+1}} \\ x dx &= (\alpha - \phi) 2\pi T \left[\frac{(\alpha - \phi) \dot{\phi}}{c_\sigma} \right]^{\frac{1}{\xi+1}} L_{\xi+1}(r_e, r_i, C_i). \end{aligned} \quad (8.84)$$

In Eq. (8.84) we make use of formulae from the Appendix C. The integral could be expressed analytically in terms of incomplete beta function.

The axial force of the disk spring F_z is equal to the radial moment of the tangential forces divided by force arm $r_e - r_i$, such that:

$$\begin{aligned}
 F_z(t) &= \frac{M_\theta}{r_e - r_i} = 2\pi T \left[\frac{(\alpha - \phi)\dot{\phi}}{c_\sigma} \right]^{\frac{1}{\xi+1}} \frac{\alpha - \phi}{r_e - r_i} L_{\xi+1}(r_e, r_i, C_i) \\
 &= 2\pi T \frac{L_{\xi+1}(r_e, r_i, C_i)}{r_e - r_i} \left[\frac{(\alpha - \phi)^{\xi+2}\dot{\phi}}{c_\sigma} \right]^{\frac{1}{\xi+1}}.
 \end{aligned} \tag{8.85}$$

The value of the C_i in the last equation yet remains to be determined.

The sum of all forces action normal to the cross-section F_θ must be equal to zero. Only stresses due to radial displacement need to be considered. To calculate the sum of all forces action normal to the cross-section, we make use of the expression for tangential stress as the function of rotation angle ϕ and its time derivative $\dot{\phi}$:

$$F_\theta = T \int_{C_i - r_e}^{C_i - r_i} \sigma_\theta dx = T \left[\frac{(\alpha - \phi)\dot{\phi}}{c_\sigma} \right]^{\frac{1}{\xi+1}} \int_{C_i - r_e}^{C_i - r_i} \left(\frac{x}{x - C_i} \right)^{\frac{1}{\xi+1}} dx = 0. \tag{8.86}$$

As the factor before integral is positive, the vanishing of the integral in (9.86) determines the position of inversion point C_i :

$$\int_{C_i - r_e}^{C_i - r_i} \left(\frac{x}{x - C_i} \right)^{\frac{1}{\xi+1}} dx \equiv K_{\xi+1}(r_e, r_i, C_i) = 0. \tag{8.87}$$

The integral in the Eq. (8.87) expresses in terms of incomplete beta functions using the formulae from the Appendix C. Further simplification and closed form solution of the Eq. (8.87) for an arbitrary m seems to be impossible in closed analytical form. Avoiding the numerical solution, consider exactly solvable limit cases.

For $\xi = 0$ one get the known value for inversion center (Chap. 5, Eq. 2.10) :

$$c_i = \frac{r_i - r_e}{\ln(r_i/r_e)}. \tag{8.88}$$

The appropriate approximate solution for sufficiently large ξ delivers the expression:

$$c_i = \frac{r_i + r_e}{2}. \tag{8.89}$$

For constant values of r_i, r_e, c_i the expression $K_{\xi+1}(r_e, r_i, c_i)$ is the function of creep exponent ξ .

For higher values of exponent ξ the first function

$$K_{\xi+1}(r_e, r_i, (r_e + r_i)/2) \tag{8.90}$$

asymptotically tends to zero axis.

For higher values of exponent ξ the second function

$$K_{\xi+1}(r_e, r_i, (r_e - r_i)/\ln(r_i/r_e)) \tag{8.91}$$

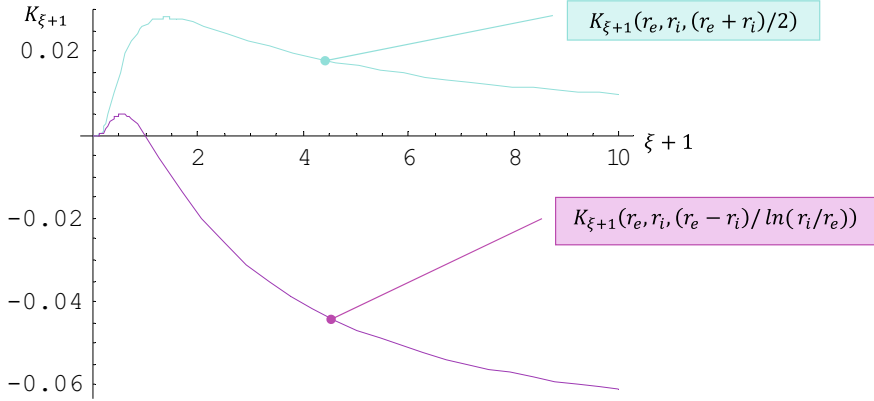


Fig. 8.7 Two functions $K_{\xi+1}(r_e, r_i, (r_e - r_i)/\ln(r_i/r_e))$ and $K_{\xi+1}(r_e, r_i, (r_e + r_i)/2)$ for different exponents ξ

gradually deviates from zero.

For illustration two functions $K_{\xi+1}(r_e, r_i, (r_e - r_i)/\ln(r_e/r_i))$, Eq. (8.91) and $K_{\xi+1}(r_e, r_i, (r_e + r_i)/2)$, Eq. (8.90) are drawn on Fig. 8.7. For lower values of ξ the solution (8.88) is delivers better approximation for the equation

$$K_{\xi+1}(r_e, r_i, c_i) = 0. \quad (8.92)$$

Two functions

$$L_{\xi+1}(r_e, r_i, (r_e - r_i)/\ln(r_e/r_i)) \quad (8.93)$$

and

$$L_{\xi+1}(r_e, r_i, (r_e + r_i)/2). \quad (8.94)$$

are drawn on Fig. 8.8. The approximate solution (8.88) leads to somewhat higher values of function $L_{\xi+1}$, that the solution (8.89).

With the approximate solution (8.89) for the distance of neutral axis to center, we get the final expression of the spring force as the function of rotation angle $\phi(t)$ (Kobelev 2014a):

$$F_z(t) = \left[\frac{(\alpha - \phi)^{\xi+2} \dot{\phi}}{c_\sigma} \right]^{\frac{1}{\xi+1}} \frac{2\pi T}{r_e - r_t} L_{\xi+1} \left(r_e, r_i, \frac{r_e + r_i}{2} \right). \quad (8.95)$$

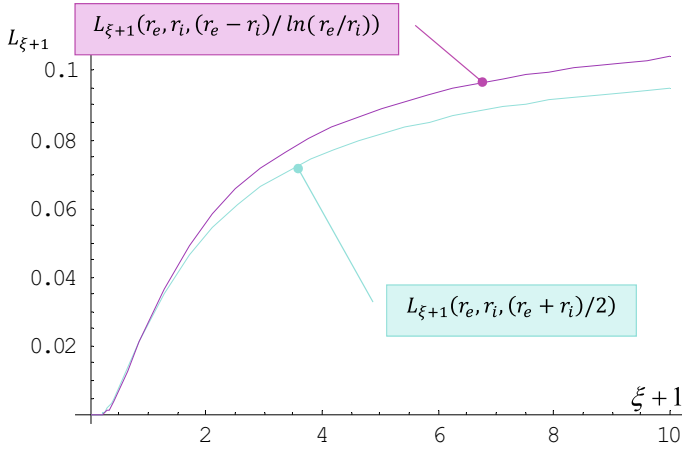


Fig. 8.8 Two functions $L_{\xi+1}(r_e, r_i, (r_e - r_i)/\ln(r_e/r_i))$ and $L_{\xi+1}(r_e, r_i, (r_e + r_i)/2)$ as functions of exponent ξ

If the spring force remains constant over time, the rotation angle $\phi(t)$ is the solution of an ordinary differential equation

$$t^{1-\xi}(\alpha - \phi)^{\xi+2}\dot{\phi} = \tilde{P}, \quad \phi(0) = \phi_0, \quad (8.96)$$

where

$$\tilde{P} = c_\sigma \left[\frac{(r_e - r_i)}{L_{\xi+1}(r_e, r_i, \frac{r_e + r_i}{2})} \frac{F_z}{2\pi T} \right]^{\xi+1},$$

ϕ_0 the initial rotation angle of the cone due to elastic deformation of the spring at time moment $t = 0$.

The solution of the Eq. (8.96) leads to the rotation angle as function of time:

$$\phi(t) = \alpha - \left[(\alpha - \phi_0)^{\xi+3} - \frac{\xi+3}{\xi} \tilde{P} t^\xi \right]^{\frac{1}{\xi+3}}. \quad (8.97)$$

At the critical moment of time

$$t_{crit} = \frac{(\alpha - \phi_0)^{\xi+3}}{(\xi+3) \tilde{P}}, \quad (8.98)$$

the cone angle of disk vanishes and spring turns to be a flat disk, such that.

$$\phi(t_{crit}) = \alpha. \quad (8.99)$$

The axial travel of disk is the function of time:

$$s(t) = \phi(t)(r_e - r_i). \quad (8.100)$$

For example, we calculate the creep of the disk spring with the following parameters:

$$r_e = 25mm, r_i = 12.25mm, T = 3mm, h = 1.1mm.$$

Figures 8.9 and 8.10 demonstrate the influence of parameter $\bar{\sigma}$ on the creep behavior of disk spring. The initial rotation of spring cone is.

$$\phi_0 = 3\alpha/4.$$

In other words, this means that the initial spring travel is.

$$s_0 = \frac{3h}{4} = 0.825mm.$$

The following material parameters were used for calculation:

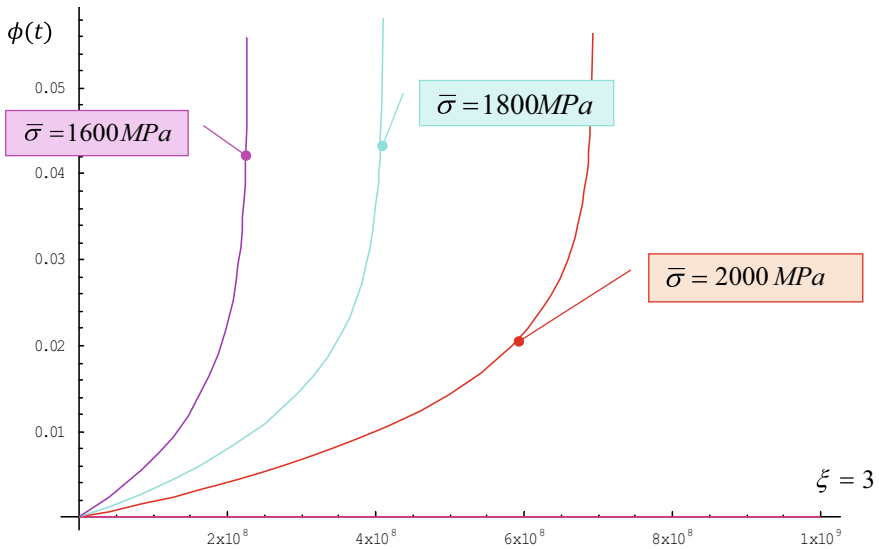


Fig. 8.9 Cone angles of disk springs as functions of creep time for three different material parameters

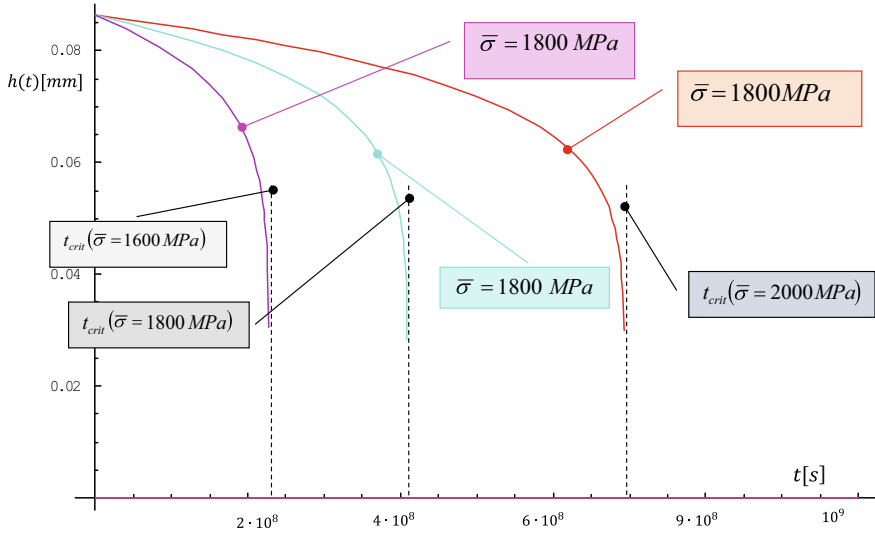


Fig. 8.10 Disk heights $h(t)$ as functions of creep time for three different material parameters

$$E = 200 \text{ GPa}, \xi = 4, \nu = 0.254, \bar{\varepsilon} = 10^{-24} \text{ sec}^{-1}.$$

On Fig. 8.9 the creep curves for three different material parameters are shown:

$$\bar{\sigma} = 1600 \text{ MPa}, \bar{\sigma} = 1800 \text{ MPa} \text{ and } \bar{\sigma} = 2000 \text{ MPa}.$$

The creep curves represent the cone angles as functions of time. The relaxation happens slower for the materials with higher values of parameter $\bar{\sigma}$.

The disk heights $h(t)$ are shown under constant load as function of time for three different material parameters on Fig. 8.10. The cone angles of disk vanish for three different critical moments. Finally, depending on material parameter, the spring turns to be a flat disk. The disk flattening happens evidently at higher time moments for the materials with elevated parameters $\bar{\sigma}$.

8.7.2 Relaxation of Disk Springs

For calculation of the disk spring relaxation the Eq. (8.71) for Norton-Bailey law is applicable. The Eq. (8.71) represents the relaxation function for the beam in flexure. The dominantly flexure stress-state exists in disk springs. Namely, the stress-state in disk springs is the superposition of two states as follow from Chap. 4. Depending upon the geometry of the disk spring, either the flattening of the disk with circumferential mid-surface strain (1.16, Chap. 5) or bending due to circumferential curvature change (1.17, Chap. 5) dominates.

We consider now only the case of complete flattening of moderately thick spring with predominant circumferential curvature change. The disk spring is instantly compressed from its free state to the plane flat state. In the first moment the spring force will be $F_z^0 = F_z(t = 0)$. The force in the first moment arises due to the pure elastic stresses. Correspondingly, for the evaluation of the instant spring force the formulas from Chap. 4 are applicable.

During the relaxation time the spring travel s remains constant, while the stresses relax. The relaxation of stresses leads to the reduction of spring force. The calculation of the relaxation function cannot be performed straightforward, as it was possible for the pure torsion and for the pure bending. The reason is the following. The stresses in the volume of the disk spring are not constant over its surface. The simple consideration allows the fairly accurate estimation of the relaxation function. Keeping in mind, that stresses are caused by bending, we employ the stress relaxation function from the state of flexure that was investigated above. The characteristic of the relaxation function is a “comparative” maximal stress σ_R on the surface. By means of Eq. (8.71) one gets the relaxation function $\Psi(t)$ of the disk spring:

$$F_z(t) = \Psi(t)F_z^0, \Psi(t) = {}_2F_1\left(\frac{3}{\xi}, \frac{1}{\xi}; \frac{3+\xi}{\xi}; -\xi c_\sigma \sigma_R^\xi E t\right). \quad (8.101)$$

As the solution with the general stress distribution cannot be found closed form, the finite element simulations were performed. The finite-element simulation performs the numerical calculation of the relaxation function with the sane geometry and equal material parameters. For comparison the relaxation function $\Psi(t)$ was tabulated for the following six values of “comparative” maximal stress:

$$\sigma_R = |\sigma_{Be} + \sigma_{Bi}|/2; \sigma_R = \frac{\sigma_{Te} - \sigma_{Ti}}{2}, \sigma_R = |\sigma_I|, |\sigma_{II}|, |\sigma_{III}|, |\sigma_{IV}|. \quad (8.102)$$

The first option in Eq. (8.102) fixes the average bending stress $|\sigma_{Be} + \sigma_{Bi}|/2$. The second option designates the average tensile stress $(\sigma_{Te} - \sigma_{Ti})/2$. The subsequent options use the stress on the corners. The finite-element simulation demonstrates, that the best comparison for the relaxation function $\Psi(t)$ and the relaxation is achieved, if “comparative” maximal stress is assumed to be.

$$\sigma_R = \sigma_{III}. \quad (8.103)$$

In this case the numerically calculated the relaxation function and the relaxation function $\Psi(t)$ match (Fig. 8.11). Consequently, it could be advised to use the Eq. (8.101) with the substitution (8.103) for relaxation calculations of disk springs. The relaxation function (8.101) provides the reasonable evaluation of the relaxation effects.

Other common creep laws could be considered the analogously.

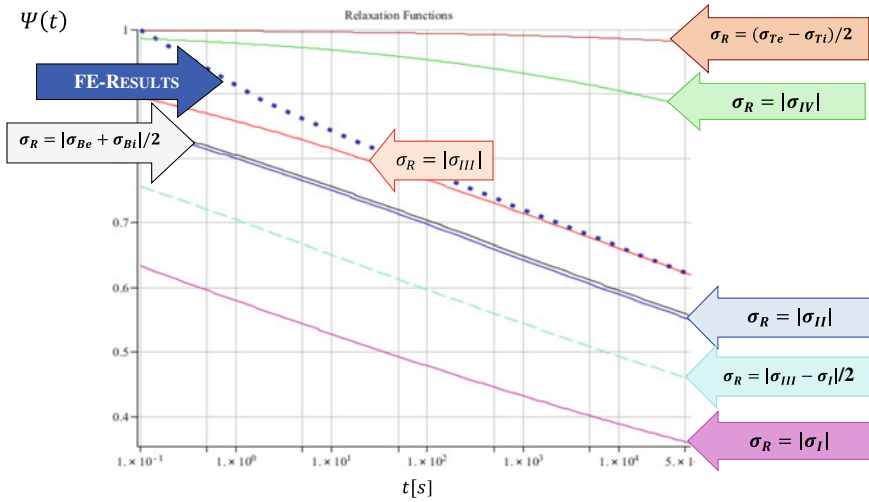


Fig. 8.11 Relaxation function $\Psi(t)$ for several formulations of “comparative” maximal stress

8.8 Cyclic Creep and Fatigue-Creep Interaction

We estimate the influence of the cyclic stress on the creep effect. For this purpose, the load in the spring element will sinusoidal with a constant amplitude and non-zero mean value:

$$\tau_{oct} = \sigma_m + \sigma_a \cos t. \quad (8.104)$$

We investigate the cyclic accumulation creep strain for Norton-Bailey law. Substitution of (8.104) into the creep Eq. (8.23), the increment of the octahedral shear strain after one cycle is:

$$\Delta \gamma_{oct}(\sigma_m, \sigma_a) = \frac{2\bar{\epsilon}}{\tau^{\bar{\epsilon}+1}} \frac{1}{2\pi} \int_0^{2\pi} (\sigma_m + \sigma_a \cos t)^{\bar{\epsilon}+1} dt. \quad (8.105)$$

For evaluation of the integral (8.105), we use the Eqs. (B.13) or (3.661.3) from (Gradshteyn and Ryzhik 2015):

$$\frac{1}{2\pi} \int_0^{2\pi} (a + b \cos t)^q dt = (a^2 - b^2)^{\frac{q}{2}} P_q \left(\frac{a}{\sqrt{a^2 - b^2}} \right). \quad (8.106)$$

Substitution (8.115) into (8.114) gives the increment of the creep strain after one cycle:

$$\Delta\gamma_{oct}(\sigma_m, \sigma_a) = \frac{2\bar{\varepsilon}}{\bar{\tau}^{\xi+1}} (\sigma_m^2 - \sigma_a^2)^{\frac{\xi+1}{2}} P_{\xi+1} \left(\frac{\sigma_m}{\sqrt{\sigma_m^2 - \sigma_a^2}} \right). \quad (8.107)$$

If the amplitude vanishes, the increment of creep during the same time interval will be $\Delta\varepsilon_{cr}(\sigma_m, 0)$. The ratio of $\Delta\varepsilon_{cr}(\sigma_m, \sigma_a)$ to $\Delta\varepsilon_{cr}(\sigma_m, 0)$ gives the expression for the influence of the cyclic stress on the total creep strain:

$$r_{cycl} = \frac{\Delta\gamma_{oct}(\sigma_m, \sigma_a)}{\Delta\gamma_{oct}(\sigma_m, 0)}, \sigma_a = \frac{1 - R_\sigma}{1 + R_\sigma} \sigma_m. \quad (8.108)$$

Substitution of Eq. (8.107) in (8.108) results in the final expression for ratio of cyclic and static creep strains :

$$r_{cycl}(\xi, R_\sigma) = \left(\frac{2\sqrt{R_\sigma}}{1 + R_\sigma} \right)^{\xi+1} \cdot P_{\xi+1} \left(\frac{1 + R_\sigma}{2\sqrt{R_\sigma}} \right) \quad (8.109)$$

Figure 8.12 displays the plot of the function $r_{cycl}(\xi, R_\sigma)$ Eq. (8.109) for $0 < \xi < 4, 0 < R_\sigma < 1$ for equal values of mean stress in cycles σ_m .

The sense of the influence of the stress ratio is the following. If the mean stress in cycle vanishes, the creep elongation according to Norton-Bailey law is positive during the first half of the cycle and negative during the second half of the cycle. Both partial elongations compensate each other and the accumulation of creep over time disappears. If the amplitude in cycle vanishes, the steady creep flow occurs and increases continuously with time.

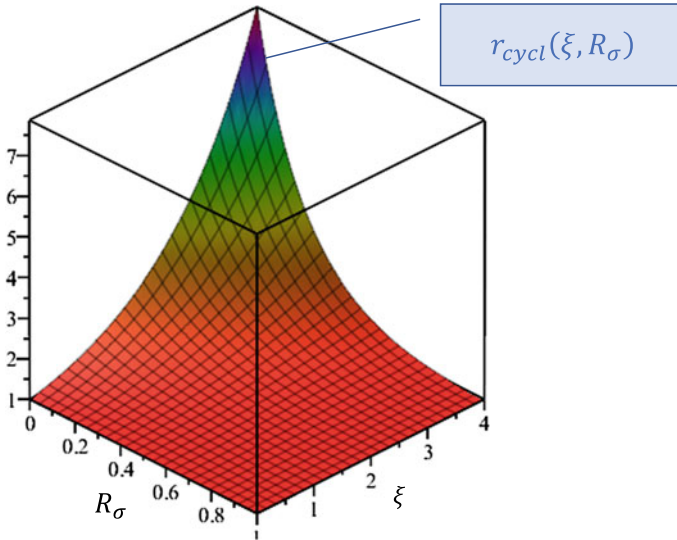


Fig. 8.12 Ratio of cyclic and static creep strains $r_{cycl}(\xi, R_\sigma)$

Interesting to mention, that the fatigue failure follow completely different pattern (Chap. 9). If the mean value vanish but the stress varies symmetrically, the crack elongates and failure occurs after a definite number of cycles. If the amplitude of stress disappears, fatigue damage is absent and occurs no failure.

Both microscopic damage mechanisms of fatigue and creep are different and requires the different mathematical characterization. In details the creep-fatigue interaction was discussed in Chandler and Kwofie (2005), Kwofie and Chandler (2007), Polák et al. (2011), Pineau (1988), Assefpour-Dezfuly and Brownrigg (1989), Yang and Wang (2001).

8.9 Temperature Influence on Creep

1°. The temperature dependence could be incorporated directly in the creep parameters:

$$\dot{\gamma} = g_c(\tau) \cdot h_c(t), h_c(t) = \left(\frac{t}{\bar{t}}\right)^{s-1}. \quad (8.110)$$

The creep parameter $c_\tau(T^\circ)$ is the function of temperature:

$$g_c(\tau) = c_\tau(T^\circ) \tau^{\xi+1} \equiv \tilde{\gamma}(T^\circ) \cdot \left(\frac{\tau}{\bar{\tau}}\right)^{\xi+1}. \quad (8.111)$$

with the characteristic creep rate at the stress level $\bar{\tau}$:

$$\tilde{\gamma}(T^\circ) = c_\tau(T^\circ) \bar{\tau}^{\xi+1}.$$

In this work we assume the scaling parameters in (9.110) and (9.111) as follows:

$$\bar{\tau} = \frac{1}{\sqrt{3}}[MPa], \quad \bar{t} = 1[s], \quad \bar{\tau} = 1[MPa].$$

The representation (8.110) (8.111) is suitable for the direct measurements of creep parameters at certain fixed temperatures.

In the report (IGF 16992, 2018), constitutive equations were evaluated with the usual choice for the stress exponent $\xi = 4$ and using the fitting functions for temperature dependence:

$$\tilde{\gamma}(T^\circ) = \left[\gamma_1 + \gamma_2 \ln\left(\frac{\tau}{\bar{\tau}}\right) \right] \cdot \exp\left(\frac{T^\circ}{273^\circ\text{K}}\right) + \gamma_3 + \gamma_4 \ln\left(\frac{\tau}{\bar{\tau}}\right), \quad (8.112)$$

$$c_{\tau}(T^{\circ}) = \left[c_{\tau 1} + c_{\tau 2} \ln \left(\frac{\tau}{\tau} \right) \right] \cdot \exp \left(\frac{T^{\circ}}{273^{\circ}\text{K}} \right) + c_{\tau 3} + c_{\tau 4} \ln \left(\frac{\tau}{\tau} \right). \quad (8.113)$$

The interrelation between the coefficients of (9.112) and (9.113) is:

$$c_{\tau i} = \frac{\gamma_i}{\tau^{\xi+1}}, i = 1..4$$

The advantage of the coefficients γ_i is their simpler dimension:

$$[\gamma_i] = s^{-1}, [c_{\tau i}] = MPa^{-5} s^{-1}.$$

The region of primary creep was evaluated experimentally. The steady creep rate was not observed for the stress and temperature levels. The constants for the typical spring materials CrSi, CrSiV and 1.4310 are presented in Tables 8.2, 8.3 and 8.4.

Table 8.2 Relative temperature dependence of yield stress, Young modulus and creep constant for VDSiCrV wire

Relative temperature dependence	T °C
$\sigma_w(T^{\circ}\text{C})/\sigma_w(20^{\circ}\text{C})$	$1.0348 - 0.001842 \cdot T^{\circ} + 5.218 \cdot 10^{-6} \cdot T^{\circ 2} - 4.5259 \cdot 10^{-9} \cdot T^{\circ 3} - 2.879 \cdot 10^{-12} T^{\circ 4}$
$E(T^{\circ}\text{C})/E(20^{\circ}\text{C})$	$1.003 - 0.0001273 \cdot T^{\circ} - 1.2673 \cdot 10^{-6} \cdot T^{\circ 2} + 3.2961 \cdot 10^{-9} \cdot T^{\circ 3} - 3.9233 \cdot 10^{-12} T^{\circ 4}$
$\bar{\varepsilon}(T^{\circ}\text{C})/\bar{\varepsilon}(20^{\circ}\text{C})$	$0.58198 * \exp(0.003412 \cdot T^{\circ} + 0.9317) - 0.58198$

Table 8.3 Creep parameters for torsion bars made of oil tempered wires SiCr (IGF 18,992, 2018)

Material SiCr	d , mm	ζ	$\gamma_1 \cdot 10^{50}/s$	$\gamma_2 \cdot 10^{50}/s$	$\gamma_3 \cdot 10^{50}/s$	$\gamma_4 \cdot 10^{50}/s$
350°C/30 min	≤4	0.1217	13.51	−1.754	3.434	−27.32
	>4	0.1151	15.62	−2.027	4.480	−35.25
420°C/30 min	≤4	0.1141	8.175	−0.9631	1.468	6.656
	>4	0.1141	8.175	−0.9631	1.468	6.656

Table 8.4 Creep parameters for torsion bars made of oil tempered wires SiCrV (IGF 18,992, 2018)

Material SiCrV	d , mm	ζ	$\gamma_1 \cdot 10^{50}/s$	$\gamma_2 \cdot 10^{50}/s$	$\gamma_3 \cdot 10^{50}/s$	$\gamma_4 \cdot 10^{50}/s$
350°C/30 min	≤4	0.1050	9.265	−1.174	1.732	−14.63
	>4	0.1050	12.07	−1.546	3.574	−28.16
420°C/30 min	≤4	0.1127	15.44	−2.022	4.779	−36.87
	>4	0.1127	15.44	−2.022	4.779	−36.87

These tables show the constants of Eq. (8.117) for the circular wires after the certain heat treatments. The manufactured springs contain residual stresses, which influence the creep rates. The application of the acquired constants for the fabricated springs requires the correction factors, which regard the manufacturing effects on creep processes. The mentioned correction factors are exposed in the cited report.

2°. Skrotzki et al. (2019) performed the comprehensive review of the testing methods of metals at elevated temperatures. Commonly, the temperature dependence of the stationary creep rate is described with the Arrhenius function:

$$g_c(\tau_{oct}) = \bar{\varepsilon} \cdot \left(\frac{\tau_{oct}}{\bar{\tau}} \right)^{\xi+1} \exp\left(-\frac{Q_a}{R \cdot T^v}\right), \quad R = 8.31 \left[\frac{J}{mol^v K} \right]. \quad (8.114)$$

where $Q_a[kJ/mol]$ is the activation energy for self- or impurity diffusion, R is the gas constant and T° is the absolute temperature in $[^\circ K]$.

The activation energy could be evaluated, if at least two experiments for creep are performed for different temperatures. The first experiment is conducted at absolute temperature T_1° and measures the creep deformation between the start time t_s and the final time t_e :

$$\Delta\gamma_{c1} = \gamma_c(t = t_s) - \gamma_c(t = t_e). \quad (8.115)$$

In the second experiment, which is conducted at absolute temperature T_2° , the creep deformation between the start time t_s and the final time t_e :

$$\Delta\gamma_{c2} = \gamma_c(t = t_s) - \gamma_c(t = t_e). \quad (8.116)$$

The activation energy with the acquired values (8.115) and (8.116) reads:

$$Q_a = R \ln\left(\frac{\Delta\gamma_{c1}}{\Delta\gamma_{c2}}\right) \frac{T_1^\circ T_2^\circ}{T_1^\circ - T_2^\circ}. \quad (8.117)$$

The estimation (9.117) is valid in the region between the absolute temperatures T_1° and T_2° and for order of magnitude the stress, which were applied during the measurements.

For example, the evaluation of creep constants is bases on the results of experimental evaluation of creep curves of VDSiCrV wire ($d = 3$ mm), $350^\circ C/30$ min heat treated and pre-torched with 1450 MPa (IFG 18992, 2018, Abb. 5.54):

$$\frac{\Delta\gamma_{c2}}{\Delta\gamma_{c1}} = \frac{33}{15}, \quad T_1^v = 353^v K, \quad T_2^v = 433^v K.$$

With these constants, the activation energy follows from Eq. (8.117):

$$Q_a = 12.52 \left[\frac{kJ}{mol} \right].$$

Similarly, for each fixed absolute temperature T° , the stress exponent requires two creep tests with different stress on the surface of wire $\tau_{oct.1}, \tau_{oct.2}$. The creep deformation between the start time t_s and the final time t_e for stress $\tau_{oct.1}$ is $\Delta\gamma_{c1}(\tau_{oct.1})$. The creep deformation between the start time t_s and the final time t_e for stress $\tau_{oct.2}$ is $\Delta\gamma_{c1}(\tau_{oct.2})$. Then the stress exponent is:

$$\xi(T) = \frac{\ln[\Delta\gamma_{c2}(\tau_{oct.2})/\Delta\gamma_{c1}(\tau_{oct.1})]}{\ln[\tau_{oct.2}/\tau_{oct.1}]} - 1. \quad (8.118)$$

3°. The experimental results (IFG 18992, 2018) are used for the evaluation of stress exponent ξ and activation energy Q_a for different spring steels. The evaluation with Eqs. (8.117) and (8.118) was performed for the VDSiCrV, VDSiCr and 1.4310 spring steels (Tables 8.5 and 8.6).

As mentioned above, the customary choice for the stress exponent is $\xi = 4$. With this value, the relaxation behavior expresses via the elementary function (8.55) and requires no transcendental function (9.54). This feature greatly simplifies technical calculations.

Well known, that the ultimate tensile stress depends on test temperature as well. The typical relative dependences of mechanical properties of steel are shown on Fig. 8.13. This figure demonstrates the creep rate constant $\bar{\epsilon}(T^\circ C)/\bar{\epsilon}(20^\circ C)$, relative ultimate tensile stress $\sigma_w(T^\circ C)/\sigma_w(20^\circ C)$ and relative Young modulus $E(T^\circ C)/E(20^\circ C)$ as functions of temperature $T^\circ C$. The corresponding functions are presented in Table 8.5. For reference, the plots of the ultimate tensile stress for AISI1095, AISI420, UHB SS 716, AISI 301 and 17–7 PH steel sorts as functions of temperature $T^\circ C$ are shown on Fig. 8.14 (Johansson et al. 1984). Mechanical behavior of metallic materials over wide ranges of elongation was studied in (Emde 2008). The cited results exhibit the greater temperature sensitivity of creep strain rate in comparison to the ultimate tensile stress for common steels. At higher temperatures, the ultimate tensile stress reduces stronger than the Young modulus.

4°. Special applications require the application of high temperature resistant steels for spring manufacturing. One effect at high temperatures is the corrosion of metals caused by oxygen, which is known as scaling. A spring steel is heat-resistant, if it does not scale or only slightly scales. The combination of heat resistance and temperature resistance is referenced as the high temperature resistance. The high temperature resistant spring steels are steels that are optimized for good service life at elevated temperatures and exhibit also the low temperature sensitivity of creep. The high temperature resistant materials are characterized by (Milne et al. 2003):

- High melting point;
- Phase mixture with finely distributed phase (precipitates), temperature resistant microstructure, obstacles with high temperature resistance and strength (obstruction of dislocation movement);

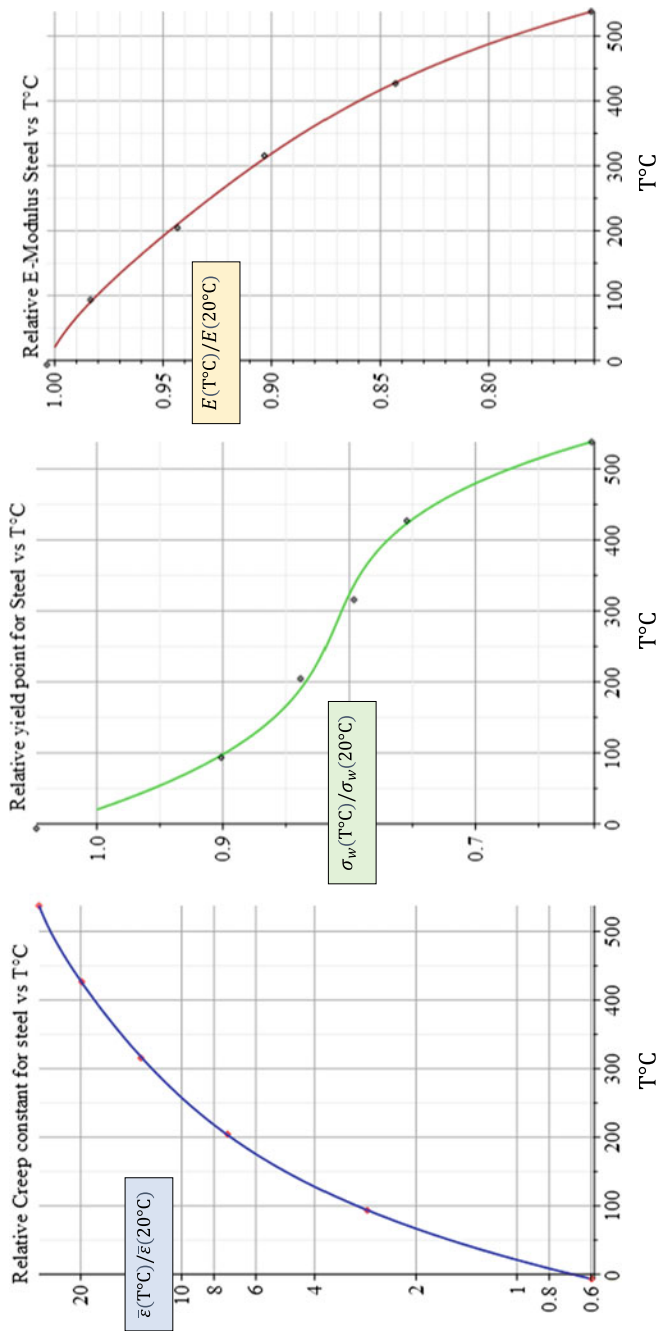


Fig. 8.13 Relative creep rate constant $\bar{\epsilon}(T^{\circ}\text{C})/\bar{\epsilon}(20^{\circ}\text{C})$, relative ultimate tensile stress $\sigma_w(T^{\circ}\text{C})/\sigma_w(20^{\circ}\text{C})$ and relative Young modulus $E(T^{\circ}\text{C})/E(20^{\circ}\text{C})$ as functions of temperature $T^{\circ}\text{C}$

Table 8.5 Creep parameters for torsion bars made of austenitic wires 1.4310 (IGF 18992, 2018)

Material	ζ	$\gamma_1 \cdot 10^{50}/s$	$\gamma_2 \cdot 10^{50}/s$	$\gamma_3 \cdot 10^{50}/s$	$\gamma_4 \cdot 10^{50}/s$
1.4310					
290°C/30 min	0.1318	106.6	−15.17	40.72	−288.9
300°C/30 min	0.1243	26.51	−3.702	3.845	−30.06
400°C/30 min	0.1213	29.98	−4.337	9.689	66.81

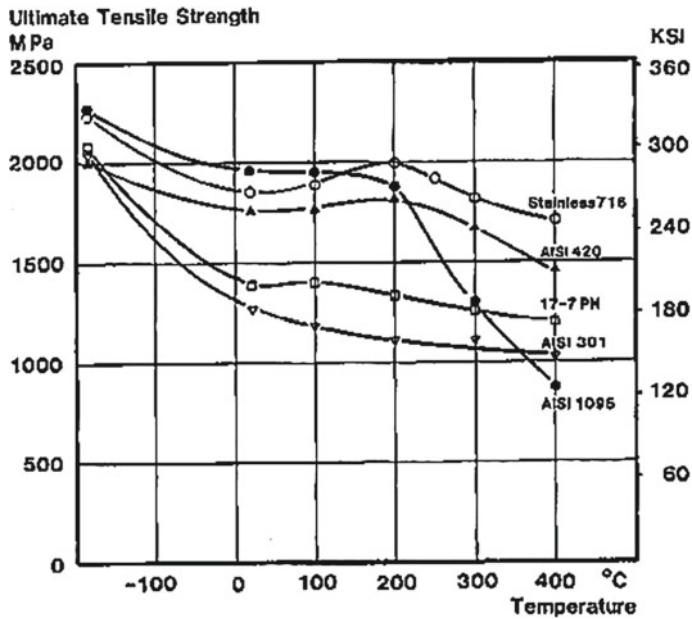


Fig. 8.14 Ultimate tensile stress $\sigma_w(T^\circ\text{C})$ as the function of temperature $T^\circ\text{C}$ for AISI1095, AISI420, UHB SS 716, AISI 301 and 17-7 PH steel sorts (Johansson et al. 1984)

Table 8.6 Temperature dependence of Creep parameters for torsion bars (IGF 18992, 2018)

Wire	Heat treatment		pre-torched	Q_a	ξ	T°
	°C	min	MPa	$\frac{kJ}{mol}$		°C
VDSiCrV	350	30	–	16.35	4.82	160
	350	30	1450	16.07	5.24	160
					5.30	120
					4.06	80
	350	30	1240	13.66	3.66	160
VDSiCr	350	30	–	13.66	3.58	160
	430	30	–	10.61	2.58	160
1.4310	200	30		9.37		
	400	30		9.37		

- Small stacking fault energy (little energy required to produce faults; this leads to more stacking faults that impede dislocation movement);
- Coarse grain (few grain boundaries reduce grain boundary slippage).

8.10 Conclusions

The structures examined comprise a beam in bending, a rod in torsion and helical and disk springs. The closed form solutions demonstrate the basic characteristics of nonlinear creep. As the bending and torsion dominate in spring members, the results are immediately applicable for the majority of springs. Following the procedure we establish closed form solutions for creep and relaxation in helical, leaf and disk springs. Further, the generalized expression for creep law is studied. The knowledge about material creep laws is essential for the creep and relaxation design of the spring elements that withstand high static and cyclic loads. The expression is based on observations of springs creep and unifies the primary, secondary and tertiary regions of creep curve by means of introduction of the new time variable.

8.11 Summary of Principal Results

- Customary constitutive equations for creep (Norton-Bailey, Garofalo, exponential, Naumenko-Altenbach-Gorash) are compared.
- The solutions for creep and stress relaxation for bars in twist and bending states are exhibited in closed form.
- The derived solutions enable the evaluation of creep under constant load and relaxation from the fixed strain for helical, leaf and disk springs.
- The closed form temperature dependence of creep parameters is presented.

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Chapter 9

Fatigue of Spring Materials



Abstract The present chapter explains the customary ways for accounting of the stress amplitude for fatigue life of springs. The fatigue of spring materials for fully reversed, uniaxial loading is reviewed. Two different estimation methods for fatigue life are briefly discussed. The first method implements the stress-life and strain-life procedures. The second method describes the fatigue crack growths (FCG) per cycle. Two analogous unifications of FCG functions are proposed. The expressions for spring length over the number of cycles are derived. This chapter is the second section of final part, which investigates the lifecycle of the elastic elements.

9.1 Phenomenon of Fatigue

9.1.1 Fatigue Influence Factors

The fatigue phenomenon is associated with the nucleation of sub-microscopic surface cracks in the fatigued component early in its life, which initially grow very slowly. Gradually, a crack grows until the effective cross section of the piece is reduced to such a value that the applied stress cannot be supported. Finally, the rapid failure occurs. There are distinct zones are apparent in a typical fatigue fracture surface. These correspond respectively to the period of slow growth and final failure.

Repetitive loading and unloading cycles of significant amplitude cause fatigue of spring materials as well. The level of stresses, which causes fatigue failure, is usually below values that considered as safe for a single static load application. The critical fatigue initiation is usually at a position limited to a small area. The failure may be a result of additional factors such as stress concentration due to component shape, surface finish or corrosion pitting (Juvinal and Marshek 2017, Chap. 8). In the cyclic fatigue test according to ASTM E 739-80 (1986), ASTM E 606-80 (1986), DIN 50100 (2016:12), a material or component is subjected to a periodically changing (cyclic) load. The fatigue test is used to determine the fatigue strength and fatigue limit for tensile, compressive, bending, and torsional stress. Especially for springs, material processing can be evaluated by the fatigue test and improved by design or

material variations. The short-term strength is not considered in the fatigue test. It is investigated in the Low Cycle Fatigue Test.

The influence factors greatly influence the fatigue life of springs. These factors must be carefully accounted. The valuable influence factors are:

- corrosion of spring surface;
- wear factors due to friction on the contact surfaces (Barrois 1979; Sangid 2013; Lamacq et al. 1996; Stachowiak and Batchelor 2014, Chap. 14)
- different residual stresses due to manufacturing, which could affect the durability positively or negatively;
- modification of mechanical material properties due to manufacturing;
- shot peening, which induces residual stresses and influences the roughness of surface;
- hydrogen induced ductility (Akaki et al. 2017; Yoshimoto and Matsuo 2017);
- inclusions, flaws, and other inherited defects of the spring material;
- high temperature, which cause the reversion of crystal structure;
- high level of radiation, especially the neutron radiation, which leads to hydrogen induced ductility.

In the following chapters, we review the fatigue effects of springs. Each chapter examines some specific manifestations of the fatigue phenomena. In the current chapter, only the case for a pure sinusoidal cyclic load of constant amplitude (fully reversed stress amplitude) and zero mean stress is studied. The fatigue laws relate the number of cycles to fully reversed stress amplitude. Excessive majority of tests of spring materials are provided for only for a fixed mean stress with the varying amplitude. The mean stress is usually set to zero during the basic test. The test runs with different constant stress or strain amplitudes to failure. The results of this basic test are organized in form of the fully reversed fatigue law. For the evaluation of all other mentioned above effects, the significantly more fatigue tests must be conducted. The experimental acquisition of stresses ratio requires considerable growth of the duration and expenses of tests. Industry attempts to reduce the experimental effort. To account stress ratio, the industry commonly uses the plausible hypotheses about the influence of mean stress on fatigue life. These hypotheses use correction factors for the fully reversed fatigue laws, as discussed in the next chapter.

In the fatigue test, the load amplitude and the mean load are constant during the single-stage vibration test. Depending on the magnitude of the load amplitude, it can be applied with different frequency before the sample fails. The fatigue strength and fatigue limit of materials or components are determined in fatigue tests. For this purpose, several specimens are loaded cyclically. The cyclic fatigue test is carried out until a defined failure of the specimen (break, crack) occurs. In the cyclic fatigue test, the number of oscillations (number of limit oscillation cycles) is determined. If a specimen reaches the number of cycles without any detectable failure, it is evaluated as fatigue strength or as a runner. Medium stress, high stress and low stress of the cyclic loading are constant for each fatigue test. In tests of the same S–N curve, either only the medium stress or only the ratio between high and low stress is changed.

The typical load case for the uniaxial fatigue analysis is the harmonically oscillating stress:

$$\sigma(t) = \sigma_m + \sigma_a \sin(\omega t). \quad (9.1)$$

This type of stress is typical for torsion springs, disk springs and leaf springs. Usually, the circular frequency ω is not high enough to cause the heating of the spring. In this case, the influence of frequency on fatigue could be ignored at first glance. The parameters of the harmonically oscillating stress are:

$$\sigma_a = (\sigma_{max} - \sigma_{min})/2 \quad (9.2)$$

is the stress amplitude;

$$\Delta\sigma = \sigma_{max} - \sigma_{min} = 2\sigma_a, \quad (9.3)$$

is the stress range;

$$R_\sigma = \frac{\sigma_{min}}{\sigma_{max}} < 1, \quad (9.4)$$

is the stress ratio of cyclic load;

$$\sigma_m = \frac{\sigma_{min} + \sigma_{max}}{2} = \frac{1 + R_\sigma}{1 - R_\sigma} \frac{\Delta\sigma}{2}, \quad (9.5)$$

is the mean value of stress.

The extreme values of the oscillating stress:

$$\sigma_{max} = \frac{1}{1 - R_\sigma} \frac{\Delta\sigma}{2}, \quad (9.6)$$

is the maximum stress per cycle,

$$\sigma_{min} = \frac{R_\sigma}{1 - R_\sigma} \frac{\Delta\sigma}{2}, \quad (9.7)$$

is the minimum stress per cycle.

Torsion and shear stresses dominate in helical compression and tension springs and in twist beams. The harmonically oscillating cyclic shear stress is the function of time:

$$\tau(t) = \tau_m + \tau_a \sin(\omega t). \quad (9.8)$$

The range (or stroke) of shear stress $\Delta\tau$ reads:

$$\Delta\tau = \tau_{max} - \tau_{min} = 2\tau_a, \quad \tau_{min} = \tau_m - \tau_a, \quad \tau_{max} = \tau_m + \tau_a. \quad (9.9)$$

In terms of stress ratio

$$R_\sigma = \frac{\tau_m - \tau_a}{\tau_m + \tau_a} = \frac{\tau_{min}}{\tau_{max}},$$

the parameters of stress cycle are:

$$\begin{aligned}\tau_a &= \frac{1-R_\sigma}{2} \tau_{max}, & \text{shear stress amplitude,} \\ \tau_m &= \frac{1+R_\sigma}{2} \tau_{max}, & \text{mean value of the shear stress.}\end{aligned}$$

In the current chapter, only the case for a pure sinusoidal cyclic load of constant amplitude (fully reversed stress amplitude) and zero mean stress is studied ($R_\sigma = -1$). The influence of stress ratio is discussed in the next chapter.

9.1.2 Stages of Fatigue Fracture

There are three stages of fatigue fracture commonly distinguished: initiation, propagation, and final rupture (SAE AE-22 1997; Totten 2008; Fleck et al. 1994). On a microscopic scale, failure occurs along slip planes in the crystalline structure of the materials. Most metals with a body centred cubic crystal structure have a characteristic response to cyclic stresses. These materials have a threshold stress limit below which fatigue cracks will not initiate. This threshold stress value is referred to as the

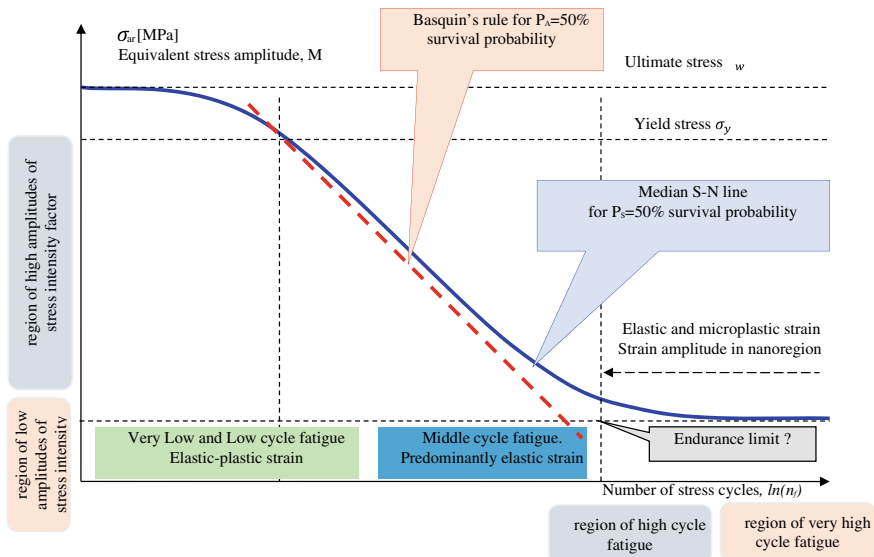


Fig. 9.1 An S-N diagram plotted from the results of completely reversed axial fatigue tests

endurance limit. In spring steels, the life associated with this behaviour is about two million cycles (Fig. 9.1).

The first stage (stage I) of fatigue is referred to as initiation (Polak 2003). Initiation is probably the most complex stage of fatigue fracture. The most significant factor about the initiation stage of fatigue fracture is that the irreversible alterations in the metal are caused by cyclic shear stresses. The microscopic faults accumulate over each load application. The growing with each cycle cumulative damage leads finally to final breakage. At the location of a severe stress concentration, the number of cycles to breakage depends on the geometry of the part as well as on environmental, stress, metallurgical, and strength conditions, as will become apparent. A brief discussion of the role of defects in the initiation phase is given in Sect. 9.3. During the stage I, the spread of fatigue crack per unit cycle decelerates smoothly with number of cycles and the process approaches its second stage.

The second stage of fatigue is known as crack propagation (stage II). The direction of growth can change after the transition from stage I to stage II. The crack grows customary perpendicular to the direction of the maximal tensile stress. For example, the cracks in the cyclically loaded helical compression springs expand in the direction of 45° to the spring axis. The second, or propagation, stage of fatigue is typically the most readily recognizable area of a fatigue fracture. The Paris law describes the stage II fatigue crack growths function (Paris and Erdogan 1963). The traditional form of Paris law pronounces the spread of fatigue crack per unit cycle as a power function of the range of stress intensity factor. Towards the end of the stage II, the spread of fatigue crack per unit cycle begins to accelerate smoothly with number of cycles and the process turns to the third stage.

The final, third stage of fatigue is the final rupture (stage III). As the propagation of the fatigue crack endures, progressively sinking the cross-sectional area of the test specimen, it eventually deteriorates the part. The deterioration occurs so quickly, that final, broad fracture occurs within a pair of load cycles.

The fracture mode may be either ductile (with a dimpled fracture surface) or brittle (with a cleavage, or intergranular, fracture surface). The combinations of both modes occasionally occur, depending upon the metal concerned, the stress level and the environment. In the course of the stage III, the spread of fatigue crack per unit cycle progressively accelerates with each cycle. The in situ observations of the final crack growth are seldom.

The functional relation between the loading properties, stress gradient and the time-depending mechanical characteristics of materials are essential for the fatigue analysis and evaluation number of the cycles to failure (Carpinteri 1994; Suresh 1998; Fatemi and Yang 1998; Christ 1991, 2018). The traditional evaluations concepts based on the Palmgren-Miner's rule of damage accumulation, rain-flow counting of time-dependent loads, Wöhler curve and Basquin equation, Paris-Erdogan law and diverse generalizations of the fracture and damage mechanics approaches (Richard and Sander 2012). The book (Krupp 2007) reviews the numerical treatment of fatigue microscopic crack propagation together with their implementation in fatigue-life prediction models.

9.2 Crack Initiation Approach for Uniaxial Stress State

9.2.1 Stress-Life Approach for Symmetric Cycle

The stress-life approach is applicable to the cases of long fatigue life and moderate cyclic stresses. The fatigue data are acquired by cyclic test specimens at constant amplitude stress until failure occurs. Fatigue data commonly have significant scatter which can be described by probability density functions (Appendix C). The cyclic stresses lead to dominant elastic strain amplitudes and only tiny microscopic plastic strains occurs over one reversal cycle. The fully reversed stress the amplitude (σ_{ar} or τ_{ar}) keeps constant during the basic cyclic test. For the symmetric cycle, the stress is a harmonic function of time with the vanished mean stress. These conditions are suitable for the estimations of fatigue life of common spring materials. Occasionally, the mean stress during the basic is fixed to a certain nonzero constant. For example, for helical compression springs the mean stress is positive, because the minimum force of the spring must be positive to avoid the detachment of the spring from the plates.

It is surely impossible to test the fatigue life of the springs for all imaginable conditions. In the next chapter, we present the usual hypotheses that allows the recalculation of the fatigue life at one-test conditions to the fatigue laws with different test conditions.

For characterization of fatigue life as function of stress amplitude, the fatigue laws traditionally applied (SAE AE-22 1997, Sect. 10.3.4.3). The stress-life approach is based on the Basquin's law. This law is known also as Wöhler's law in German speaking countries. The plot of Basquin's law possess the fatigue life n_f the x-axis and the effective fully-reversed stress amplitude $\sigma_{ar} = \Delta\sigma/2$ on the y-axis. Thus, the number of cycles to failure is n_f . The number of reversals to failure is $2n_f$. The dependence of fatigue life upon the effective fully reversed stress amplitude σ_{ar} derives from the Basquin's law (Basquin 1910):

$$n_f = f_{\sigma}^{-1}(\sigma_{ar}) \equiv \frac{1}{2} \left(\frac{\sigma_{ar}}{\sigma'_{f0}} \right)^{1/b_0}, \quad n_f = f_{\tau}^{-1}(\tau_{ar}) \equiv \frac{1}{2} \left(\frac{\tau_{ar}}{\tau'_{f0}} \right)^{1/b_0}, \quad (9.10)$$

where

$$\sigma_{ar} = f_{\sigma}(n_f), \quad f_{\sigma}(n_f) = \sigma'_{f0} \cdot (2n_f)^{b_0}, \quad (9.11)$$

$$\tau_{ar} = f_{\tau}(n_f), \quad f_{\tau}(n_f) = \tau'_{f0} \cdot (2n_f)^{b_0}. \quad (9.12)$$

The parameters in Eqs. (9.10)–(9.12) are:

σ'_{f0} the fatigue strength coefficient for normal stress,
 τ'_{f0} the fatigue strength coefficient for shear stress,

b_0 the fatigue strength exponent,
 $p_\sigma = -1/b_\sigma$ the reciprocal fatigue exponent.

The index “0” symbolizes that the mean stress is equal to zero. All coefficients are acquired for symmetric stress cycle (S). In this case, stress varies harmonically with the average of zero:

$$\sigma(t) = \sigma_{ar} \sin(\omega t), \quad \sigma_{ar} = \Delta\sigma/2. \quad (9.13)$$

There are alternative proposals, based on stress-life approach, for the description of the fatigue failure line 2–7. Among them, several rarely used criteria were discussed by (Meyer 2014).

The plots of functions (9.10)–(9.12) are referred as S–N curves. Symbol ‘S’ signifies the cyclic stress range and ‘N’ stands for the number of cycles to failure n_f . The scale of ‘N’ axis is usually logarithmical. The scale of ‘S’ axis is logarithmical or linear. In this scaling the S–N curves for laws (9.10)–(9.12) demonstrate the linear dependence.

Hereafter we discuss the median S–N curves. The median S–N curves correspond the cumulative failure probability of 50%. The scattering of fatigue data and the design S–N curves are discussed in Chap. 10.

9.2.2 Strain-Life Approach for Symmetric Cycle

The plastic fully reversed strain amplitude is commonly assumed constant during the cyclic test and is much higher than the elastic strain amplitude (SAE AE-22 1997, Sect. 10.3.4.4). The fatigue data for strain-life approach are obtained by cyclic test specimens at constant amplitude strain until failure occurs (Graham et al. 1968, Chap. 6). This condition fulfils in low cycle fatigue regime, where the Coffin-Manson law is appropriate.

For strain-life approach, the coefficients are acquired for symmetric or for non-zero mean strain cycle. In the first case, strain oscillates harmonically with the vanished mean value:

$$\varepsilon(t) = \varepsilon_{ar} \sin(\omega t), \quad \varepsilon_{ar} = \Delta\varepsilon/2.$$

For the conditions of symmetric cycle, the mean stress is zero. The Coffin-Manson law relates the number of cycles to failure upon the plastic fully-reversed strain amplitude (Manson 1953; Coffin 1954):

$$\varepsilon_{ar} = \frac{\sigma'_{f0}}{E} \cdot (2n_f)^{b_0} + \varepsilon'_{f0} \cdot (2n_f)^{c_0}. \quad (9.14)$$

The values in Eq. (9.14) are:

σ'_{f0} fatigue strength coefficient,
 b_0 fatigue strength exponent,
 ε'_{f0} fatigue ductility coefficient,
 c_0 fatigue ductility exponent

The quantities σ'_{f0} and b_0 are the same as in Eq. (9.14) and ε'_{f0} and c_0 are additional fitting constants for the plastic strain term.

If the plastic fully-reversed strain amplitude is determined with (9.14), the stress amplitude σ_a results from the solution of the Ramberg-Osgood equations:

$$\varepsilon_{ar} = \frac{\sigma_a}{E} + \left(\frac{\sigma_a}{K'_\sigma} \right)^{1/n'} \quad (9.15)$$

The Eq. (9.15) represents the cyclic stress-strain curve with two fitting constants n' and K'_σ . The constant K'_σ the cyclic hardening coefficient and n' is the cyclic hardening exponent. The analogous Ramberg-Osgood representation, but for the static hardening curve, was used in previously.

The solution of the Eq. (9.15) gives the cyclic stress σ_a in terms of the cyclic strain, which is constant in this type of test. The control of the strain-life cyclic test is easier, than the control of stress-life test. It is easier to fix the amplitude of testing, because the stress-life test requires the acquisition of the applied forces during the cyclic test. In the majority of tests of springs and spring materials, the amplitude of deformation keeps during the test time, for example, in R. R. Moore rotating-bending fatigue test (Juvinal and Marshek 2017; Bussoloti et al. 2015).

The corresponding formulas for the shear stress are analogues to Eq. (9.14) with the evident replacements:

$$\gamma_{ar} = \frac{\tau'_{f0}}{G} \cdot (2n_f)^{b_0} + \gamma'_{f0} \cdot (2n_f)^{c_0}, \quad (9.16)$$

$$\gamma_{ar} = \frac{\tau_a}{G} + \left(\frac{\tau_a}{K'_\tau} \right)^{1/n'} \quad (9.17)$$

9.3 Crack Propagation Approach

9.3.1 Crack Growths Functions of Paris-Erdogan Type

The different method for determination of S–N curves starts from a commonly accepted form of law for fatigue crack growth and demonstrates the noticeable features of the fatigue phenomena (SAE AE-22 1997, Sect. 10.4). Using this method, the S–N curve will be a logical consequence from the introduced micromechanical

law, rather than a directly observable phenomenon. Analogously, the mean stress sensitivity will be an immediate consequence of the acknowledged law for the crack propagation. In other words, the cyclic fatigue diagrams will be synthesised based on the micromechanical models of stepwise crack propagation per load cycle.

Numerous load damage models, which extend the celebrated Paris and Erdogan fatigue crack growths function, were proposed (Tanaka 2003). The article (Beden et al. 2009) reviews the articles focused on the prediction of fatigue properties of structures under variable amplitude loading. The reliability and accuracy of prediction models and the physical concept of fatigue damage was discussed. The reviewed fatigue life models were based on the scientific and engineering knowledge about fatigue of material and structures under constant and variable amplitude loading.

The near-threshold deviation of the common fatigue crack growths function was suggested by Donahue et al. (1972). The observed rapidly increasing growth towards ductile tearing was accounted. The brittle fracture correction of the power law was proposed in the cited article in the analytical form.

In the work (Kanninen and Popelar 1985) several combinations of high and low amplitude of stress intensity factors values were studied. A similar approach for transition region between high and low amplitude of stress intensity factors was proposed in McEvily and Groeger (1977).

A more general “unified law” accounts certain deviations from the power-law regime (Pugno et al 2006). The implemented extension of the Paris law for crack propagation results in the generalized Wöhler fatigue curves.

A different expression is proposed for the correction extremely low cycle fatigue behavior of in the paper (Xue 2008). The proposed formulas describe the cyclic life by introducing an exponential function, making possible to cover the entire span of extremely low cycle fatigue to low cycle fatigue. The curves are fitted by means of an additional material parameter.

The methodology to the modeling of material fatigue in cyclic loading has been suggested based on the kinetic theory of strength in Mishnaevsky and Brøndsted (2007). According to this concept, the acting stresses instead of stress changes yield the damage growth in materials under fatigue conditions. In order to model the material degradation, the kinetic theory of strength was applied. The model allows the determination of the stiffness reduction in composite with a definite load history. Moreover, the model resolves the effect of the loading frequency on the lifetime and the damage growths per cycle. It was shown that the number of cycles to failure increases almost linearly. With increasing the loading frequency, the damage growth per cycle decreases.

By applying the quantized fracture mechanics, the unification and extension of the traditional Paris’ and Basquin’s or Wöhler’s laws fatigue crack propagation were derived in Pugno et al. (2007). In the cited article the generalized Paris’, Wöhler’s or unified laws were suggested. The performed analysis demonstrated the applicability of proposed laws for predicting the lifetime of structures containing fatigue cracks. The sizes of fatigue cracks were varied from small (the Basquin’s regime) to large (the Paris’ regime).

The revision of the fatigue crack growth model was proposed by Castillo et al. (2014), CCS model. The crack growth model for derivation of the S–N curves from the curves of crack growth rate was presented. The CCS model complements the common stress based and fracture mechanics methodologies in fatigue lifetime through the enhanced fatigue crack growths function. Firstly, the curve of crack growth rate is defined over all its range as a cumulative distribution function based on a normalized dimensionless range factor of stress intensity. The CCS model assumes that fatigue crack growth takes the form of a Gumbel cumulative distribution function (Gumbel 1954). For this prediction of crack growth rate, the Buckingham theorem was incorporated. Secondly, the proprietary theorem was derived. The derived theorem provides an alternative to self-similarity and assures the significant reduction of experimental planning. Correspondingly, the different crack growth curves for different stress ranges and initial crack lengths are obtained from a specific crack growth curve. The S–N field results from the crack growth curves. The close relation between the fracture mechanics and stress approaches was demonstrated. Thirdly, the model was applied to a certain set of experimental data. The curve of crack growth rate and the S–N curves of a certain material were derived.

The substantial modification of CSS model was proposed in paper (Blasón et al. 2016). This modification accounts the crack opening and closing effects as well as the influence of the stress ratio. The theoretical model incorporates both effects of the average stress as well crack closure and opening action. The model provided an analytical expression of the curve of crack propagation rate by matching experimental data by means of the least squares method. The identification of new variables was introduced for the model of fatigue cracking. The auxiliary variables include the cracking and opening effects using the influence of the stress ratio. The normalized variables allow the values in the range 0–1. Thus, the use of “S-shaped” cumulative distribution functions reproduces the relation between the variation of stress intensity factor and number of cycles. The integration of several models that come from the statistical domain to solve the problem of crack growth is possible. The a–N and S–N curves describe the crack closure effects for the materials on different initial crack size values and loads.

The common laws of fatigue crack propagation must also evaluate opening and closure effects for fatigue crack. For this purpose, a corresponding modification of the CCS law of fatigue crack growth was proposed in the paper (Correia et al. 2016a). The modification incorporated the crack opening and closure effects together with the stress ratio effect. The fatigue crack opening and closure effects were explained with the plasticity-induced crack-closure model. With the introduction of the phenomenological auxiliary parameter, an alternative fatigue crack closure models were implemented. The corresponding modified CCS crack propagation model was based on the effective range factor of stress intensity. The version model was supported by mathematical and physical considerations.

The model (de Castro et al. 2009) predicts crack growth by means of crack initiation properties and critical damage concepts. The crack was modeled as a sharp notch with a very small but finite tip radius. The finite radius removes the singularity of elastic fields at the tip vicinity. Accordingly, the damage caused by each load

cycle and the effects of residual stresses near the crack tip evaluates immediately through the hysteresis loops caused by the loading. The calculation of cycle-bicycle crack growth was based on the introduced method. An agreement between the crack growth forecasts and experiments was achieved in the citing paper for constant and for variable amplitude loadings.

Several prediction models are known for fatigue crack growth under variable amplitude loading histories. However, no of these models predicts adequately considering loading sequence effects. An improved model of crack growth rate has been proposed in Chen et al. (2011). The cited model deals the constant amplitude loading demonstrate its validity for experimental data. The applicability of the improved model of crack growth rate was extended to variable amplitude loading. The extension accounts crack closure level based on the concept of partial crack closure. The concept was based on the crack-tip plasticity. To denote the variation in the affected zone of under- or overload, a modified coefficient was introduced. The identification of the coefficient was based on Wheeler model (Wheeler 1972).

The model of fatigue crack growth (Noroozi et al. 2005) was based on the elastic–plastic crack tip stress–strain history. In the FCG model the fatigue crack growth simulates the stress–strain response in the vicinity of the crack tip and estimating the accumulated fatigue damage. The fatigue crack growth was treated as sequential crack re-initiation near the crack tip. The model predicted the effect of the mean stress and the influence of the compressive stress. A fatigue crack growth was analyzed for the plane strain and for plane stress states. It was demonstrated, that the fatigue crack growth is controlled by a two-parameter driving force. The driving force itself evaluates based on the local stresses and strains at the crack tip using the Smith–Watson–Topper fatigue damage parameter (Smith et al. 1970). The residual (internal) stresses are induced by the reversed cyclic plasticity. The effect of the residual stress could be also accounted with the proposed model. The applied stress intensity factors were modified to the total stress intensity factors to be accounted for the effect of the local crack tip stresses and strains on fatigue crack growth. The fatigue crack growth was forecasted by the stress–strain response near the crack tip on one side and estimating the accumulated fatigue damage on the other side. The fatigue crack growth is the process of successive crack re-initiations in the crack tip region. The model estimates the effect of the mean and residual stresses induced by the cyclic loading. The proposed model evaluates the effect of variable amplitude loadings on fatigue crack growth. The influence of the internal stress induced by the reversed cyclic plasticity alternates the resultant stress intensity factors controlling the fatigue crack growth.

Two singularity fields near the crack tip are known. Namely, the first type of tip field is the Hutchinson-Rice-Rosengren field (Hutchinson 1968; Rice and Rosengren 1968). The second type is the Kujawski-Ellyin field (Kujawski and Ellyin 1984; Ellyin 1986). The effect of these singularity fields was examined in the paper (Shi et al. 2014). The study (Shi et al. 2014) evaluates the common fatigue crack growth model, which is based on energy balance during growth of the crack. The new parameter was introduced. With this parameter, the effect of different types of singularity fields

was incorporated in the crack propagation model. The model predicted fatigue crack growth on the stage-II from the basic low cycle fatigue properties.

In the paper (Correia 2016b) the extended procedure for fatigue life prediction of structural details was presented. The procedure is based on fracture mechanics approach. The model generalizes the normalized the CSS model for fatigue crack growth. Instead of the range factor of stress intensity as reference parameter, the generalization of the CCS model implements the cyclic J-integral range. This allows the comprehensive elastic-plastic conditions. The proposed approach was applied to a notched plate, using the equivalent initial flaw size concept.

9.3.2 *Fatigue Crack Growths Functions for Crack Under Cyclic Loading*

Commonly accepted, that the fatigue crack growth is influenced by the stress intensity factor range and the maximum stress intensity factor. These parameters are usually combined into one formula, which is known as the driving force. Alternative expressions for driving forces were suggested in Noroozi et al. (2008). The driving force concepts could be successfully introduced in the following case. The stress intensity factors are properly linked to the actual elastic-plastic crack tip stress-strain field. However, the correlation between the stress intensity factors and the crack tip stress-strain fields are prejudiced by internal or residual stresses.

The crack propagation is quantified as a function of the range of the stress intensity factor:

$$\Delta K = K_{max} - K_{min}. \quad (9.18)$$

The factors in Eq. (9.18) are:

K_{max} the maximum stress intensity factor;

K_{min} the minimum stress intensity factor per cycle.

The common form of Paris' law quantifies the fatigue life of a specimen for a given particular crack size a . The range of stress intensity factor reads (SAE AE-22 1997, Appendix A):

$$\Delta K = Y \cdot \sqrt{\pi a} \cdot \Delta \sigma. \quad (9.19)$$

The range of stress intensity factor depends on the dimensionless parameter Y that reflects the geometry. The parameter Y possesses the value 1 for a center crack in an infinite sheet. Hereafter, this value is assumed for brevity: $Y = 1$.

It is assumed that the cracks with the initial length δ exist in the material. The cycle count starts with the beginning of crack propagation from its initial length:

$$n_f \big|_{a=\delta} = 0. \quad (9.20)$$

The common form for fatigue crack growths function (propagation law) is used (Tanaka 2003):

$$\frac{da}{dn} = c_f \cdot F(\Delta K), \quad n_f(\delta, a) = \int_{\delta}^a \frac{da}{c_f \cdot F(\sigma \sqrt{\pi a})}, \quad (9.21)$$

The number of cycles for the crack length elongation from the initial length δ to the final length a is the number of cycles to total failure $n_f(\delta, a)$

The value $c_f F(\Delta K)$ determines the expansion of the fatigue crack pro load cycle. The coefficients in Eq. (9.21) are:

$c_f = c_f(R_\sigma)$ the material constant for a given stress ratio R_σ ;

$R_\sigma = \frac{K_{min}}{K_{max}} < 1$, the stress ratio of cyclic load;

$K_m = \frac{K_{min} + K_{max}}{2} = \frac{1 + R_\sigma}{1 - R_\sigma} \frac{\Delta K}{2}$, the mean value of stress intensity factor,

$$K_{min} = \frac{R_\sigma}{1 - R_\sigma} \frac{\Delta K}{2}, \quad K_{max} = \frac{1}{1 - R_\sigma} \frac{\Delta K}{2}.$$

The fatigue crack growths function (FCG) $F(\Delta K)$ describes the infinitesimal crack length growths per load cycle n . For the beginning we suppose that the load ratio R_σ remains constant over the load history. The range ΔK and the mean value K_m of stress intensity factor are thus constant.

All material properties are valid for the definite value of the stress ratio. For the rough estimation, for helical compression springs could be accepted $R_\sigma = 0$. The influence of stress ratio is the subject of the next chapter.

9.4 Fatigue Crack Growths

9.4.1 Unification of Paris Law

As experimentally recognized for some alloys, there is a theoretical value for stress amplitude, below which the material will not fail for any number of cycles. The fatigue cracks will not propagate, of the amplitudes of stress intensity factor are less than this threshold (Ritchie et al. 1999). At the other extreme, approaching the fracture toughness, the material fails almost immediately. The unified functions that reveal the mentioned behavior are known from the literature. A unified relation that covers all three regions was proposed by Freudenthal (1973) :

$$\frac{\ln(K_{IC}/\Delta K)}{\ln(K_{IC}/K_{th})} = \exp \left[- \left(\frac{da/dn}{da/dn|_{\Delta K=\widetilde{K}_m}} \right)^{1/p} \right],$$

$p > 1$ is the fatigue exponent,

K_{IC} is the short-term threshold limit,

K_{th} is the endurance threshold limit $0 < K_{th} < K_{IC}$, $\widetilde{K}_m \cong \frac{K_{th}+K_{IC}}{2}$

Alternative form of unified relation was proposed by Schwalbe (1980) :

$$\frac{da}{dn} = c_f \left(\arctan \frac{\Delta K}{K_{IC}} - \arctan \frac{K_{th}}{K_{IC}} \right).$$

Both previously mentioned unified relations (Freudenthal 1973; Schwalbe 1980) do not lead to closed forms of solutions for spring propagation. The use of these laws to resolve the crack length necessitates the numerical solutions of nonlinear differential equations in terms of cycle numbers. Consequently, the known unified relations are inappropriate for the numerical fitting of unknown parameters. The paper (Miller and Gallagher 1981) presents eight different methods to prediction the fatigue life. Each presented method is applicable to describe the three regions of the crack growth rate curve. The second fatigue crack growth rate (FCGR) description reads:

$$\frac{da}{dn} = c_f \cdot \frac{1 - \left(\frac{K_{th}}{\Delta K} \right)^{m_1}}{1 - \left(\frac{\Delta K}{K_{IC}} \right)^{m_2}}, \quad (9.22)$$

$m_2 > 1$ is the exponent at the short-term limit,

$m_1 > 1$ is the endurance limit exponent

The unified propagation function (9.22) due to Miller and Gallagher (1981) could be studied in closed form, as shown later.

9.4.2 Unification of Paris Law Type I

9.4.2.1 Closed Form Solution for Unified Paris Law Type I

In this chapter we employ the method of representation of crack propagation functions through appropriate elementary functions. The proper choice of the elementary functions is motivated by the phenomenological data and covers a broad region of possible parameters. With the introduced crack propagation functions differential equations describing the crack propagation are solved rigorously. The resulting closed form solutions allow the evaluation of crack propagation histories on one side, and the effects of stress ratio on crack propagation, on the other side.

For closed form solutions, two unified functions (type I and type II) are proposed. The functions are suggested in the form that incorporates the three commonly

accepted stages of fatigue. The advantage of the newly proposed functions is the closed form solution of crack propagation. The solution immediately delivers dependency of crack length over load history and to the number of stress cycles that a specimen sustains before failure occurs. The proposed functions allow fitting of acquired experimental data. Most known generalized laws follow from the newly introduced functions as the special cases.

The type I unified propagation function, F_I for the damage growths per cycle accounts the Paris fatigue crack growths function together with transition regions at high and low amplitudes of stress intensity factors (Kobeleev 2017a):

$$F_I(\Delta K) = \Delta K^p \cdot \frac{1 - \left(\frac{K_{th}}{\Delta K}\right)^{m_1}}{1 - \left(\frac{\Delta K}{K_{IC}}\right)^{m_2}} \quad (9.23)$$

The mentioned above fatigue laws demonstrate that if the applied stress is below the threshold limit, destruction of spring material occurs for technically significant number of cycles. Initiation of the fatigue happens when the stress amplitude increases and the stress intensity factor arrives the threshold value. The deceive role for this process play the defects in spring material. The behavior of spring steels near the threshold limit is the subject of fatigue analysis at very high number of cycles.

The main advantage of the function (9.23) is the closed form analytical solution. On one side, the closed form analytical solution facilitates the universal fitting of the constants of the fatigue law over all stages of fatigue. On the other side, the closed-form solution eases the application of the fatigue law, because the solution of nonlinear differential equation turns to be dispensable. Moreover, the mean stress dependence could be derived from Eq. (9.23).

The proposed function delivers the closed form solution of the ordinary differential Eq. (9.21) with the initial condition $n_f|_{a=\delta} = 0$, such that:

$$n_f = n_f(a, \delta, \Delta\sigma).$$

Precisely, the function $n_f(a, \delta, \Delta\sigma)$ delivers the number of cycles for growth of the crack length from the initial value δ to the given value a , assuming that both stress range σ and mean stress σ_m remains constant. As discussed later in Sect. 9.4, the size of existing defect is taken as the initial value δ of crack or of inhomogeneity.

From the Eq. (9.21) two ultimate crack sizes could be determined immediately.

Firstly, the value:

$\pi a_C = K_{IC}^2 / \sigma_r^2$ is the critical crack length at which instantaneous fracture will occur

Secondly, the value:

$\pi a_{th} = K_{th}^2 / \sigma_r^2$ is the initial crack length at which fatigue crack growth starts for the given stress range

The number of cycles to failure if $\delta \leq a_{th}$ is numerically infinite. For a finite number of cycles before fracture the initial crack length δ must satisfy the condition:

$$a_{th} < \delta < a_C.$$

The relations between the crack growth rate $da/dn = c_f F_I(\Delta K)$ and the range of stress intensity factor ΔK for simulated materials are shown on the Fig. 9.2. The endurance and short time threshold exponents were experimentally acquired oil-tempered Si–Cr steel for valve springs (Akiniwa et al. 2008) (the alloy type JIS G3561, SWOSC-V, equivalent to alloys VDSiCr (EN 10270-2:2011), ASTM A877/877M Grade A).

For a nickel chromium molybdenum steel (JIS SNCM439, equivalent to AISI 4340, DIN 34NiCrMo6) the fatigue date were obtained by Akiniwa and Tanaka (2004). For these materials the numerically fitted parameters of fatigue law are recapitulated in the Table 9.1.

The solution of the ordinary differential Eq. (9.21) with the initial condition $n|_{a=\delta} = 0$ delivers closed form analytical expression for the remaining number of cycles to total failure:

$$n_f(a, \delta, \Delta\sigma) = n_I(a, \Delta\sigma) - n_I(\delta, \Delta\sigma), \quad (9.24)$$

where the auxiliary functions are:

$$n_I(a, \Delta\sigma) = \frac{2aK_1^{2-p}}{c_f m_1 \Delta K^2} \cdot N_I\left(\frac{\Delta K}{K_{th}}\right)^{m_1}, \quad (9.25)$$

$$N_I(q) = (K_{th}/K_{IC})^{m_2} \beta_2(q) - \beta_1(q), \quad (9.26)$$

$$\beta_1(q) = \mathbf{B}\left(q; \frac{2+m_1-p}{m_1}, 0\right), \beta_2(q) = \mathbf{B}\left(q; \frac{2-p+m_1+m_2}{m_1}, 0\right). \quad (9.27)$$

The incomplete beta-function $\mathbf{B}(q; x, y)$ appear in the expressions (9.27):

$$\mathbf{B}(q; x, y) = \int_0^q t^{x-1} (1-t)^{y-1} dt, \quad (9.28)$$

The functions β_1 and β_2 in (9.27) relate to each other as:

$$\beta_1(a, \Delta\sigma) = \lim_{m_2 \rightarrow 0} \beta_2(a, \Delta\sigma). \quad (9.29)$$

The following example illustrates the behavior of the metallic material with the unified propagation function (9.23). The diagrams that express the dependence of cycles to failure, S–N curves (9.24) are presented on Fig. 9.3. The initial length of the crack was assumed for definiteness in all cases $\delta = 10^{-4}$ m. The stress intervals for calculation depend on threshold stress intensity factors:

$$\sigma_{th} < \Delta\sigma < \sigma_C, \quad \sigma_{th} = K_{th}\sqrt{\pi\delta}, \quad \sigma_C = K_{IC}\sqrt{\pi\delta}.$$

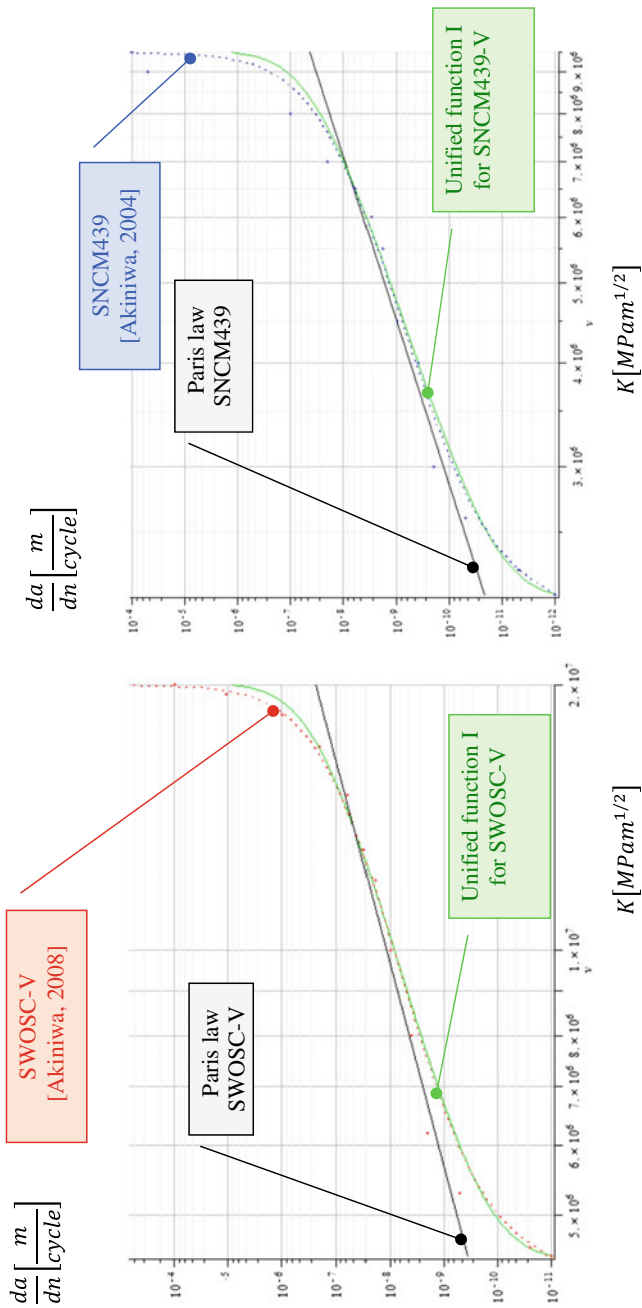


Fig. 9.2 The plot of the relations between the crack growth rate da/dn and the range of stress intensity factor K for materials SWOSC-V (Akiniwa et al. 2008) and SNCM439 (Akiniwa and Tanaka 2004)

Table 9.1 Mechanical properties of simulated materials SWOSC-V (Akiniwa et al. 2008) and SNCM439 (Akiniwa and Tanaka 2004)

Alloys	Oil tempered silicon/chromium alloyed valve spring wire	Nickel chromium molybdenum steel	Units
Alloys type	JIS G3561, SWOSC-V VDSiCr (EN 10270-2:2011) ASTM A877/877M Grade A	JIS SNCM439, AISI 4340 DIN 34NiCrMo6	
Exponents at short-term limit and endurance limit, F_I	$m_1 = 3/2; m_2 = 3/2;$		
Fatigue exponent, p, F_I	2.17	2.51	
Material constant, c_f, F_I	$0.1514 \cdot 10^{-23}$	$0.4738 \cdot 10^{-26}$	$(\text{Pa}\sqrt{\text{m}})^{p_1} \text{m}$
Endurance threshold limit K_{Ih}	$0.45 \cdot 10^7$	$0.21 \cdot 10^7$	$\text{Pa}\sqrt{\text{m}}$
Short-term threshold limit K_{IC}	$0.2 \cdot 10^8$	$0.95 \cdot 10^7$	$\text{Pa}\sqrt{\text{m}}$
exponents at short-term limit and endurance limit F_{II}	$k = 1, m_1 = 3/2; m_2 = 3/2;$		
Fatigue exponent, p, F_{II}	2.17	2.51	
Material constant, c_f, F_{II}	$0.4621 \cdot 10^{-23}$	$0.1537 \cdot 10^{-25}$	$(\text{Pa}\sqrt{\text{m}})^{p_2} \text{m}$

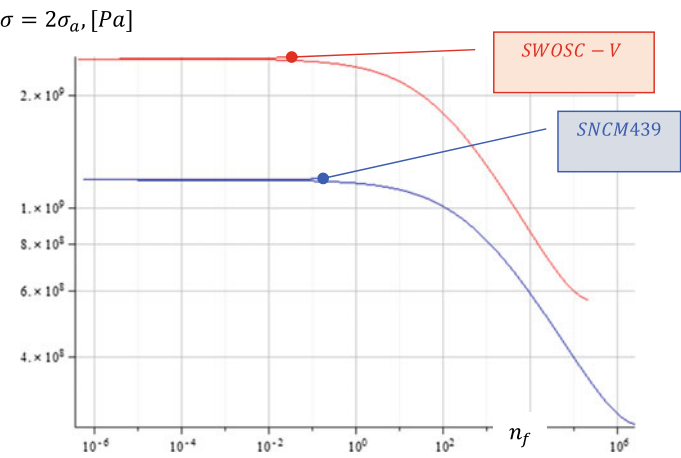


Fig. 9.3 The calculated dependencies between cycles to failure n_f for the given initial crack lengths upon the stress range $\Delta\sigma$ (s–N-curves) for SWOSC-V (Akiniwa et al. 2008) and SNCM439 (Akiniwa and Tanaka 2004)

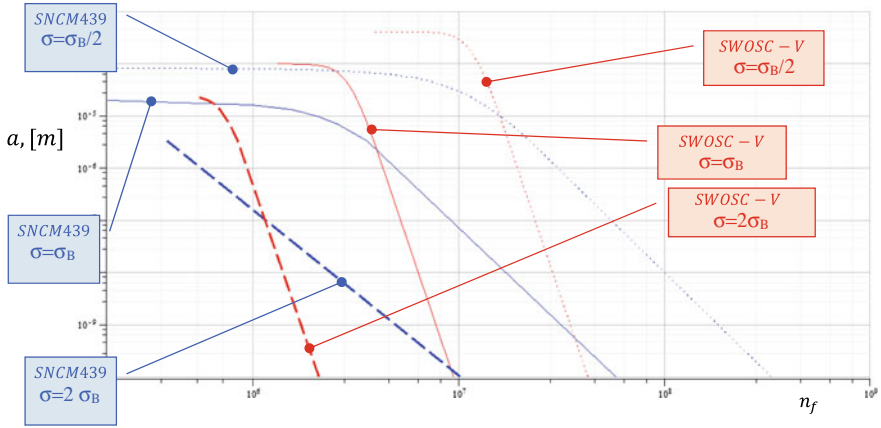


Fig. 9.4 The calculated dependencies between cycles to failure n_f for the given stress upon the length of crack (a-N-curves) for SWOSC-V (Akiniwa et al. 2008) and SNCM439 (Akiniwa and Tanaka 2004), $\sigma_B = 250$ MPa

The dependencies of crack lengths on cycles, a-N curves are drawn on the right pictures. The number of cycles was calculated as the function of length of the crack.

The S-N curves are shown on the Fig. 9.3 for Si-Cr and for Cr-Mo alloys. The a-N curves are shown on the Fig. 9.4 for these alloys.

9.4.2.2 Limit Cases of Type I Propagation Function

There are three corresponding limit cases for the function (9.23):

$$F_{I,1}(\Delta K) = \lim_{K_1 \rightarrow 0} F_I = \frac{\Delta K^p}{1 - (\Delta K / K_{IC})^{m_2}}, \quad (9.30)$$

$$F_{I,2}(\Delta K) = \lim_{K_2 \rightarrow \infty} F_I = \Delta K^p \cdot [1 - (K_{th} / \Delta K)^{m_1}], \quad (9.31)$$

$$F_{I,3}(K) = \lim_{K_1 \rightarrow 0} \lim_{K_2 \rightarrow \infty} F_I = \Delta K^p. \quad (9.32)$$

The crack growth rate $F_{I,1}$ in (9.30) express the fatigue with zero the short-term threshold limit:

$$K_{th} \rightarrow 0.$$

The crack growth rate $F_{I,2}$ in (9.31) describes the materials with the infinite endurance threshold

$$K_{IC} \rightarrow \infty.$$

The crack growth rate $F_{I,3}$ in (9.32) corresponds to Paris-Erdogan propagation function with both $K_{th} \rightarrow 0$ and $K_{IC} \rightarrow \infty$.

The Eq. (9.24) could be simplified for the limit cases of vanishing short-term threshold $K_{th} \rightarrow 0$ and of infinite endurance threshold $K_{IC} \rightarrow \infty$.

The expression for the remaining number of cycles to fracture for three limit cases for the materials with the short-term threshold limit $K_{th} \rightarrow 0$ in (9.30) are:

$$\begin{aligned} n_{I,1}(a, \Delta\sigma) &= \lim_{K_{th} \rightarrow 0} n_I(a, \Delta\sigma) \\ &= \frac{2aK^{-p}}{(p-2)(p-2-m_2)c_f} \left[m_2 + (p-2) \left(\left(\frac{\Delta K}{K_{IC}} \right)^{m_2} - 1 \right) \right] \end{aligned} \quad (9.33)$$

For the materials with the endurance threshold limit $K_{IC} \rightarrow \infty$ in (9.31) we have:

$$n_{I,2}(a, \Delta\sigma) = \lim_{K_{IC} \rightarrow \infty} n_I(a) = -\frac{2aK_{th}^{2-p}}{c_f m_1 \Delta K^2} \cdot \beta_1 \left(\frac{\Delta K^{m_1}}{K_{th}^{m_1}} \right). \quad (9.34)$$

The function $n_{I,1}$ represents the fatigue curve in the region of Low Cycle Fatigue (LCF), but possesses no endurance limit:

$$\begin{aligned} n_{I,1}(a, \Delta\sigma) &= \lim_{K_{th} \rightarrow 0} n_I(a, \Delta\sigma) \\ &= \frac{2a\Delta K^{-p}}{(p-2)(p-2-m_2)c_f} \left[m_2 + (p-2) \left(\left(\frac{\Delta K}{K_{IC}} \right)^{m_2} - 1 \right) \right]. \end{aligned}$$

The function $n_{I,2}$ possesses the endurance limit, but does not represent the fatigue curve in the region of Low Cycle Fatigue (LCF).

For the regime with both $K_{th} \rightarrow 0$ the number of loads to failure is equal:

$$n_{I,3}(a, \Delta\sigma) = \lim_{K_{th} \rightarrow 0} \lim_{K_{IC} \rightarrow \infty} n_I(a, \Delta\sigma) = \frac{2a}{(2-p)c_f \Delta K^p}. \quad (9.35)$$

The function $n_{I,3}$ possesses neither endurance limit nor the short-term threshold and represents Basquin's or Wöhler's laws. In the logarithmical coordinates the function

$$n_{I,3}(a, \delta, \Delta\sigma) = \frac{2}{\pi^{p/2}(2-p)c_f \Delta\sigma^p} \cdot \left(a^{1-\frac{p}{2}} - \delta^{1-\frac{p}{2}} \right)$$

is straight line:

$$\ln n_{I,3}(a, \delta, \Delta\sigma) = -p \ln \Delta\sigma + \ln[n_{I,3}(a, \delta, 1)]$$

Table 9.2 Propagation function $F_I(K)$ and the corresponding number of cycles to failure as function of stress intensity range

	Propagation function, $\Delta K = \Delta \sigma \cdot \sqrt{\pi a}$	Number of cycles to failure as function of stress intensity range ΔK
	$F_I(\Delta K) = K^p \frac{1-(K_{th}/\Delta K)^{m_1}}{1-(\Delta K/K_{IC})^{m_2}}$	$n_I(a, \Delta \sigma) = \frac{2aK_1^{2-p}}{cm_1 \cdot \Delta K^2} N_I \left(\frac{\Delta K^{m_1}}{K_{th}^{m_1}} \right),$ $N_I(q) = \left(\frac{K_{th}}{K_{IC}} \right)^{m_2} \cdot \beta_2(q) - \beta_1(q),$ $\beta_1(q) = \mathbf{B} \left(q; \frac{2+m_1-p}{m_1}, 0 \right),$ $\beta_2(q) = \mathbf{B} \left(q; \frac{2-p+m_1+m_2}{m_1}, 0 \right).$
$K_{th} \rightarrow 0$	$F_{I,1}(\Delta K) = \Delta K^p / [1 - (\Delta K/K_{IC})^{m_2}]$	$n_{I,1}(a, \Delta \sigma) = \frac{2a \cdot \Delta K^{-p}}{(p-2)(p-2-m_2)c} \cdot \left[m_2 + (p-2) \left(\left(\frac{\Delta K}{K_{IC}} \right)^{m_2} - 1 \right) \right]$
$K_{IC} \rightarrow \infty$	$F_{I,2}(\Delta K) = \Delta K^p \cdot [1 - (K_{th}/\Delta K)^{m_1}]$	$n_{I,2}(a, \Delta \sigma) = -\frac{2aK_1^{2-p}}{cm_1 \cdot \Delta K^2} \cdot \beta_1 \left(\frac{\Delta K}{K_{th}} \right)^{m_1}$
$K_{th} \rightarrow 0$ $K_{IC} \rightarrow \infty$	$F_{I,3}(K) = \Delta K^p$	$n_{I,3}(a, \Delta \sigma) = \frac{2a}{(2-p)c \cdot \Delta K^p}$

with

$$n_{I,3}(a, \delta, 1) = \frac{2}{\pi^{p/2}(2-p)c_f} \cdot \left(a^{1-\frac{p}{2}} - \delta^{1-\frac{p}{2}} \right).$$

The outline of the expressions for the unified fatigue crack growths function I displays the Table 9.2.

9.4.3 Unification of Fatigue Law Type II

9.4.3.1 Closed Form Solution for Fatigue Law Type II

An alternative generalization for the fatigue law is achieved by the introduction of the unified propagation function type II (Kobelev 2017b):

$$\frac{da}{dn} = c_f \cdot F_{II}(\Delta K), \text{ where } F_{II}(\Delta K) = \Delta K^p \frac{\left[1 - \left(\frac{K_{th}}{\Delta K} \right)^k \right]^{m_1}}{\left[1 - \left(\frac{\Delta K}{K_{IC}} \right)^k \right]^{m_2}}, \quad (9.36)$$

k is the dimensionless material constant. The additional the dimensionless material constant permits more precise fitting of the experimental data

The solution of the ordinary differential Eq. (9.36) with the initial condition $n|_{a=\delta} = 0$ delivers closed form analytical expression for the remaining number of cycles to total failure:

$$n_f(a, \delta, \Delta\sigma) = n_{II}(a, \Delta\sigma) - n_{II}(\delta, \Delta\sigma). \quad (9.37)$$

The auxiliary function $n_{II}(a, \sigma)$ in the Eq. (9.37) with $\Delta K = \Delta\sigma\sqrt{\pi a}$ is:

$$n_{II}(a, \Delta\sigma) = \frac{2a}{c_f \Delta K^p} \cdot \left(\frac{\Delta K}{K_{th}} \right)^{km_1} \cdot N_{II}(\Delta K), \quad (9.38)$$

$$N_{II}(\Delta K) = \frac{e^{i\pi m_1}}{2 + km_1 - p} \cdot \mathbf{F}_1 \left(\left[\frac{2 + km_1 - p}{k}, -m_2, m_1, \frac{2 + k + km_1 - p}{k} \right]; \left(\frac{\Delta K}{K_{IC}} \right)^k, \left(\frac{\Delta K}{K_{th}} \right)^k \right).$$

The transcendental function in the Eq. (9.38) is known as the Appell hypergeometric function of two variables:

$$\mathbf{F}_1([A, B_1, B_2, C]; x, y) = \frac{\Gamma(C)}{\Gamma(A)\Gamma(C-A)} \cdot \int_0^1 t^{A-1} (1-t)^{C-A-1} (1-tx)^{-B_1} (1-ty)^{-B_2} dt.$$

The fatigue crack growth rate (FCGR) description (9.2) due to Miller and Gallagher (1981) is the special case of the Eq. (9.36) for the following values of parameters $k = 1$, $p = m_1$:

$$F_{II}^*(\Delta K) = K_{th}^{m_1} \cdot \frac{\left(\frac{\Delta K}{K_{th}} - 1 \right)^{m_1}}{\left(1 - \frac{\Delta K}{K_{IC}} \right)^{m_2}}. \quad (9.39)$$

The solution of the differential Eq. (9.21) provides in this special case the expression for the remaining number of cycles to fracture:

$$n_f(a, \delta, \Delta\sigma) = n_{II}^*(a, \Delta\sigma) - n_{II}^*(\delta, \Delta\sigma). \quad (9.40)$$

The auxiliary function $n_{II}^*(a, \Delta\sigma)$ reads in this case as:

$$n_{II}^*(a, \Delta\sigma) = \frac{ae^{i\pi m_1}}{c_f K_{th}^{m_1}} \cdot \mathbf{F}_1 \left([2, -m_2, m_1, 3]; \frac{\Delta K}{K_{IC}}, \frac{\Delta K}{K_{th}} \right). \quad (9.41)$$

9.4.3.2 Limit Cases of Type II Propagation Function

The expressions could be simplified for the limit cases of vanishing short-term threshold $K_{th} \rightarrow 0$ and an infinite endurance threshold $K_{IC} \rightarrow \infty$ as well. There are three corresponding limit cases for the function (9.38):

$$F_{II,1}(\Delta K) = \lim_{K_{th} \rightarrow 0} F_{II} = \Delta K^p \cdot \left[1 - \left(\frac{\Delta K}{K_{IC}} \right)^k \right]^{-m_2}, \quad (9.42)$$

$$F_{II,2}(\Delta K) = \lim_{K_{IC} \rightarrow \infty} F_{II} = \Delta K^p \cdot \left[1 - \left(\frac{K_{th}}{\Delta K} \right)^k \right]^{m_1}, \quad (9.43)$$

$$F_{II,3}(\Delta K) = \lim_{K_{th} \rightarrow 0} \lim_{K_{IC} \rightarrow \infty} F_{II} = \Delta K^p. \quad (9.44)$$

The matching expressions for the remaining number of cycles to total fracture for three limit cases of Eq. (9.40) respectively are:

$$n_{II,1}(a, \Delta\sigma) = \lim_{K_{th} \rightarrow 0} n_{II}(a, \Delta\sigma) = \frac{2aK_2^{2-p}}{c_f k \Delta K^2} \mathbf{B} \left(\left(\frac{\Delta K}{K_{IC}} \right)^k; \frac{2-p}{k}, 1+m_2 \right) \quad (9.45)$$

for the materials with the short-term threshold limit $K_{th} \rightarrow 0$;

$$\begin{aligned} n_{II,2}(a, \Delta\sigma) &= \lim_{K_{IC} \rightarrow \infty} n_{II}(a, \Delta\sigma) \\ &= \frac{2ae^{i\pi m_1} K_1^{2-p}}{c_f k \Delta K^2} \cdot \mathbf{B} \left(\left(\frac{\Delta K}{K_{th}} \right)^k; \frac{2-p}{k} + m_1, 1-m_1 \right). \end{aligned} \quad (9.46)$$

for the materials with the endurance threshold limit $K_{IC} \rightarrow \infty$;

$$n_{II,3}(a, \Delta\sigma) = \lim_{K_{th} \rightarrow 0} \lim_{K_{IC} \rightarrow \infty} n_{II}(a, \Delta\sigma) \equiv n_{I,3}(a, \sigma_r) = \frac{2a}{(2-p)c_f \Delta K^p}, \quad (9.47)$$

for regime with $K_{th} \rightarrow 0$ and $K_{IC} \rightarrow \infty$.

The later expression turns again into the Basquin's' or Wöhler's law, which is given by Eq. (9.35).

The three limit terms for the function (9.39) are:

$$F_{II,1}^*(\Delta K) = \lim_{K_{th} \rightarrow 0} F = K_{th}^{m_1} \cdot \left(1 - \frac{\Delta K}{K_{IC}} \right)^{-m_2}, \quad (9.48)$$

$$F_{II,2}^*(\Delta K) = \lim_{K_{IC} \rightarrow \infty} F_{II}^* = K_{th}^{m_1} \cdot \left(\frac{\Delta K}{K_{th}} - 1 \right)^{m_1}, \quad (9.49)$$

$$F_{II,3}(\Delta K) = \lim_{K_{th} \rightarrow 0} \lim_{K_{IC} \rightarrow \infty} F_{II}^* = \Delta K^{m_1}. \quad (9.50)$$

The matching expressions for the second fatigue crack growth rate (FCGR) description (9.2) due to Miller and Gallagher (1981) respectively are:

$$n_{II,1}^*(a, \Delta\sigma) = \lim_{K_{th} \rightarrow 0} n_{II}^*(a, \sigma_r) = \frac{2aK_2^{2-m_1}}{c_f \Delta K^2} \cdot \mathbf{B} \left(\frac{\Delta K}{K_{IC}}; 2 - m_1, 1 + m_2 \right). \quad (9.51)$$

for the materials with the short-term threshold limit and vanishing endurance threshold limit $K_{th} \rightarrow 0$;

$$n_{II,2}^*(a, \Delta\sigma) = \lim_{K_{IC} \rightarrow \infty} n_{II}^*(a, \Delta\sigma) = \frac{2aK_1^{2-m_1}}{c_f \Delta K^2} \cdot \mathbf{B} \left(\frac{\Delta K}{K_{th}}; 2, 1 - m_1 \right) e^{i\pi m_1}. \quad (9.52)$$

for the materials with the endurance threshold limit and infinite short-term threshold limit $K_{IC} \rightarrow \infty$.

The second fatigue crack growth rate (FCGR) description can also represent the regime with $K_{th} \rightarrow 0$ and $K_{IC} \rightarrow \infty$:

$$n_{II,3}^*(a, \Delta\sigma) = \lim_{K_{th} \rightarrow 0} \lim_{K_{IC} \rightarrow \infty} n_{II}^*(a, \Delta\sigma) = \frac{2a}{(2 - m_1)c_f \Delta K^{m_1}}. \quad (9.53)$$

The summary of the solutions for the unified fatigue crack growths function II and its limit cases represents the Tables 9.3 and 9.4.

The curve fitting of coefficients for fatigue crack growths functions of the oil-tempered Si-Cr steel for valve springs (JIS G3561, SWOSC-V) results in:

$$\begin{aligned} \frac{da}{dn} &= 0.4399 \cdot 10^{-38} \cdot \Delta K^{4.3457} & \text{Paris Law} \\ \frac{da}{dn} &= 0.1514 \cdot 10^{-38} \Delta K^{2.17} \cdot \frac{0.1047 \cdot 10^{-9} \Delta K^{3/2} - 1}{1 - 0.1118 \cdot 10^{-10} \Delta K^{3/2}} & \text{Law } F_I \\ \frac{da}{dn} &= 0.4621 \cdot 10^{-23} \Delta K^{2.17} \cdot \frac{(1 - 0.45 \cdot 10^7 / \Delta K)^{3/2}}{(1 - 0.5 \cdot 10^{-7} \Delta K)^{3/2}} & \text{Law } F_{II} \end{aligned} \quad (9.54)$$

The acquired coefficients for nickel chromium molybdenum steel (SNCM439) deliver the expressions for fatigue crack growths functions:

$$\begin{aligned} \frac{da}{dn} &= 0.3867 \cdot 10^{-42} \cdot \Delta K^{5.0228} & \text{Paris Law} \\ \frac{da}{dn} &= 0.4738 \cdot 10^{-26} \Delta K^{2.51} \cdot \frac{0.3286 \cdot 10^{-9} \Delta K^{3/2} - 1}{1 - 0.3415 \cdot 10^{-10} \Delta K^{3/2}} & \text{Law } F_I \\ \frac{da}{dn} &= 0.1537 \cdot 10^{-25} \Delta K^{2.51} \cdot \frac{(1 - 0.21 \cdot 10^7 / \Delta K)^{3/2}}{(1 - 0.1052 \cdot 10^{-6} \Delta K)^{3/2}} & \text{Law } F_{II} \end{aligned} \quad (9.55)$$

Table 9.3 Propagation function $F_{II}(K)$ and the corresponding number of cycles to failure as function of stress intensity range K

	Propagation function, $\Delta K = \Delta \sigma \cdot \sqrt{\pi a}$	Number of cycles to failure as function of stress intensity range K $q_1 = \left(\frac{\Delta K}{K_{th}}\right)^k, q_2 = \left(\frac{\Delta K}{K_{IC}}\right)^k$
	$F_{II}(\Delta K) = \Delta K^p \frac{[1 - (K_{th}/\Delta K)^k]^{m_1}}{[1 - (\Delta K/K_{IC})^k]^{m_2}}$	$n_{II}(a, \Delta \sigma) = \frac{2a}{cK^p} \cdot \left(\frac{\Delta K}{K_{th}}\right)^{km_1} \cdot N_{II}(K),$ $N_{II}(\Delta K) = \frac{e^{i\pi m_1}}{2 + km_1 - p} \cdot \mathbf{F}_1\left[\frac{2 + km_1 - p}{k}, -m_2, m_1, \frac{2 + k + km_1 - p}{k}\right]; q_2, q_1$
$K_{th} \rightarrow 0$	$F_{II,1}(\Delta K) = \Delta K^p \cdot [1 - (\Delta K/K_{IC})^k]^{-m_2}$	$n_{II,1}(a, \Delta \sigma) = \frac{2aK_2^{2-p}}{ck \cdot \Delta K^2} \cdot \mathbf{B}\left(q_2; \frac{2-p}{k}, 1 + m_2\right),$
$K_{IC} \rightarrow \infty$	$F_{II,2}(\Delta K) = \Delta K^p \cdot [1 - (K_{th}/\Delta K)^k]^{m_1}$	$n_{II,2}(a, \Delta \sigma) = \frac{2aK_1^{2-p}}{ck \cdot \Delta K^2} \cdot \mathbf{B}\left(q_1; \frac{2-p}{k} + m_1, 1 - m_1\right)e^{i\pi m_1},$
$K_{th} \rightarrow 0$ $K_{IC} \rightarrow \infty$	$F_{II,3}(\Delta K) = \Delta K^p$	$n_{II,3}(a, \Delta \sigma) = \frac{2a}{(2-p)c \cdot \Delta K^p}$

Table 9.4 Propagation function F_{II}^* and the corresponding number of cycles to failure as function of stress intensity range

	Propagation function, $\Delta K = \Delta \sigma \cdot \sqrt{\pi a}$	Number of cycles to failure as function of stress intensity range ΔK , $q_1^* = \frac{\Delta K}{K_{th}}, q_2^* = \frac{\Delta K}{K_{IC}}$
	$F_{II}^*(\Delta K) = K_{th}^{m_1} \left[\frac{\frac{\Delta K}{K_{th}} - 1}{1 - \frac{\Delta K}{K_{IC}}} \right]^{m_2}$	$n_{II}^*(a, \Delta \sigma) = \frac{a}{c K_{th}^{m_1}} \cdot \mathbf{F}_1([2, -m_2, m_1, 3]; q_2^*, q_1^*) e^{i\pi m_1}$
$K_{th} \rightarrow 0$	$F_{II,1}^*(\Delta K) = K_{th}^{m_1} \cdot \left[1 - \frac{\Delta K}{K_{IC}} \right]^{-m_2}$	$n_{II,1}^*(a, \Delta \sigma) = \frac{2a K_{th}^{2-m_1}}{c \cdot \Delta K^2} \cdot \mathbf{B}(q_2^*, 2 - m_1, 1 + m_2)$
$K_{IC} \rightarrow \infty$	$F_{II,2}^*(\Delta K) = K_{th}^{m_1} \cdot \left[\frac{\Delta K}{K_{th}} - 1 \right]^{m_1}$	$n_{II,2}^*(a, \Delta \sigma) = \frac{2a K_{th}^{2-m_1}}{c \cdot \Delta K^2} \cdot \mathbf{B}(q_1^*, 2, 1 - m_1) e^{i\pi m_1}$
$K_{th} \rightarrow 0$ $K_{IC} \rightarrow \infty$	$F_{II,3}^*(\Delta K) = \Delta K^{m_1}$	$n_{II,3}^*(a, \Delta \sigma) = \frac{2a}{(2-m_1)c \cdot \Delta K^{m_1}}$

The plots of propagation functions together with experimentally acquired points are shown on the Fig. 9.5. The curves for spring steel JIS G3561, SWOSC-V are shown with the red color. The data for nickel chromium molybdenum steel SNCM439 are plotted in the blue color. The points represent the experimental data, which acquired for the nickel chromium molybdenum steel JIS SNCM439 (Akiniwa and Tanaka 2004). The fatigue strength of an oil-tempered SiCr steel for valve springs JIS G3561, SWOSC-V was investigated in Akiniwa et al. (2008). In the cited research, no

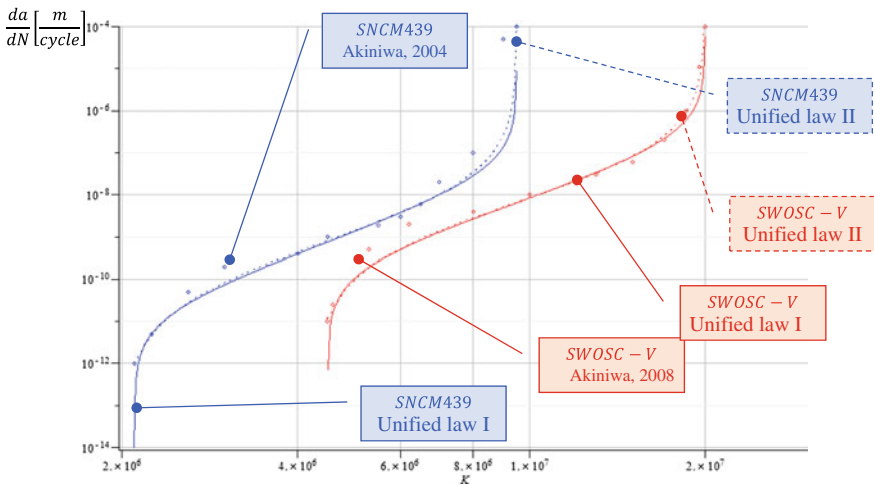


Fig. 9.5 Comparison of different propagation laws for materials SWOSC-V (Akiniwa et al. 2008) and SNCM439 (Akiniwa and Tanaka 2004)

valuable residual surface stresses were found in the rotational-symmetric specimens. The specimens were cyclically tested with ultrasonic fatigue testing machines.

In this chapter, we discuss the loads, which cause the harmonically varying scalar stress in the spring element with the zero mean value (“symmetric cycle”). In this case, the stress parameter could be either amplitude or stroke of the scalar varying stress. The scalar varying stress, is the dominant component of stress tensor. This dominant component of stress tensor, depending on spring type, could be shear or normal stress. The dominant component cause fatigue failure of spring. The mean stress in this cycle could vanish. This kind of stress appears rarely in practical application cases. More frequently, the dominant component of stress oscillates sinusoidal with a constant amplitude and non-zero mean value. Moreover, several significant components of the stress tensor oscillate simultaneously in numerous applications. The simultaneous oscillations of several significant stress components could be synchronous, shifted or even arbitrary. In the last case, the multiaxial fatigue effects must be considered. Several appropriate definitions of scalar “stress parameters” for more general cases are discussed in later chapters. Furthermore, the stress variations could be arbitrary, non-harmonic functions of time. For example, one or more components of stress could be stochastic functions of time. These general cases of fatigue loading were reviewed in Milne et al. (2003a, b), Juvinall and Marshek (2017) and SAE AE-22 (1997) and are not addressed in this book.

9.5 Conclusions

The closed form analytical expression for crack length over number of cycles is studied. Two functions that generalize the damage growth per cycle are introduced. These functions allow the unification of different fatigue laws in a single expression. The unified fatigue law provides closed form analytical solutions for crack length upon the mean value and range of cyclic variation of stress intensity factor. The solution expresses the number of cycles to failure as the function of the initial size of the crack and eliminates the solution of nonlinear ordinary differential equation of the first order. The explicit formulas for stress against the number of cycles to failure are delivered for both proposed unified fatigue laws.

9.6 Summary of Principal Results

- The fatigue of spring materials for fully reversed, uniaxial loading was studied.
- Two traditional ways of fatigue description are referred: the stress-life or strain-life techniques versus the cyclic growth of fatigue crack.
- The closed form expressions for spring length over the number of cycles are derived.

- The method originates the S–N curves from the micromechanical model of the crack propagation.
- The alteration of the micromechanical constants of the spring material results into the certain variation of the S–N curve.

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Chapter 10

Factors Affecting the Fatigue Life of Springs



Abstract In this chapter, the effects of mean stress, multiaxial loading, and residual stresses on fatigue life of springs are reviewed. We study the fatigue life of the homogeneously stressed material subjected to the cyclic load with a non-zero mean stress. Traditional methods for the estimation of fatigue life are based on Goodman and Haigh diagrams. The formal analytical descriptions, namely stress-life and strain-life approaches, turn out to be more appropriate for the numerical methods. The reported above method of fatigue analysis, which describes the crack growths per cycle, is extended. The extensions enable the accounting of the mean stress of load cycle. The closed form expressions for the crack length over the number of cycles are derived. The complement effects on the fatigue life of springs, which eventually significantly influence the fatigue life of springs, are concisely pronounced. This chapter is the second section of the third part, which investigates the lifecycle of the elastic elements.

10.1 Fatigue Life Estimation Based on Empirical Damage Models

10.1.1 Influence of Stress Ratio, Amplitude and Mean Stress

The presentation of the method is the following. Such important factors as the surface roughness, residual stresses, environmental influence, temperature, and several metallurgical aspects are fixed. The other main factors, which influence spring fatigue, are the stress amplitude and the mean stress. Namely, we study the influence of the stress amplitude and mean stress in cycle on the fatigue life. The common methods for the fatigue life estimations of spring materials are reviewed, based on Juvinal and Marshek (2017) and SAE AE-22 (1997).

The traditional approach for the fatigue calculation of springs exploits the concept the Goodman diagrams. The fatigue analysis of leaf springs presents (SAE HS 788 1980, Chap. 8, Operating Stress and Fatigue Life). The consideration of helical springs is given in (SAE HS 795 1997, Chap. 5, Design of Helical Springs). Similar procedures for fatigue life estimation present European norms (EN 13906 2013a, b,

2014). The design stresses for disk springs specify the norms (EN 16984 2017; SAE HS 1582 1988). The Section 24 “Test Results for Component Parts” of ASM (1986) provides results for spring elements fatigue testing for coil springs, leaf springs and torsion bars.

The thesis (Hattingh 1998) deals with a mathematical evaluation of actual theories of fatigue strength of automotive springs and a case study for spring made of 55Cr3 material. The experimental verification follows the mathematical modelling. For verification, the strains were measured with the aid of strain gauges located at different positions in the coil spring. The proposed evaluation gave a better understanding of the operational stress distribution for input into the fatigue methods. The thesis studies the relationship between fatigue life, process effects and residual stresses. The relationship between fatigue failures and process effects exhibits the mechanism responsible for component fatigue.

In this chapter, the load in the spring element is sinusoidal with a constant amplitude and non-zero mean value:

$$\sigma(t) = \sigma_m + \sigma_a \sin(\omega t), \tau(t) = \tau_m + \tau_a \sin(\omega t).$$

The two most common approaches for modeling fatigue damage under cyclic loading are a stress-life approach and a strain-life approach. In these approaches, the number of cycles to failure is plotted as a function of alternating stress and mean stress, alternating stress and R_σ -ratio (defined as a ratio of the minimum stress amplitude to the maximum stress amplitude), or alternating strain and mean strain. When the loading is low enough that the deformation is linear elastic, the number of cycles to failure will be plotted against the stress variable and the approach is referred to a stress-life approach. The fatigue data for the stress-life approach (Sect. 9.2.1) are attained by cyclic test specimens at constant amplitude stress until failure occurs. The traditional Palmgren-Miner cumulative damage rule ignores load sequence effects, and the damage accumulation rate is assumed to be independent of stress amplitude.

The fatigue data for strain-life approach (Sect. 9.2.2) are obtained by cyclic test specimens at constant amplitude strain until failure occurs. The damage data are acquired via strain recorders. The test and service data and cumulative damage analyses are integrated into a computer codes that includes an algorithm for cycle counting. Usually applied ‘Rainflow’ counting was described in Bannantine et al. (1990).

10.1.2 Evaluation of Fatigue Life with Goodman Diagrams

In spring engineering, the graphical representation of fatigue properties of springs and spring materials is common. The Goodman diagrams are based on the condensed representation of experimentally acquired fatigue data. The charts used to predict the fatigue life of a metal component are based on empirical data from a large numbers of tests and provide us with the most consistent method for fatigue life prediction. The

stress life method is one of the favoured methods. The S-N curve is either plotted as a log-linear or a log-log plot, and it is obtained from data of constant-amplitude cyclic tests for different alternating stresses with a same stress ratio. Different S-N curves are therefore generated for different R_σ -ratios. Alternately, the constant life diagrams are used to the S-N curves. In constant life diagrams, the alternating stress is plotted against the mean stress for a given fixed number of cycles to failure. As mentioned earlier, S-N curve and constant life diagrams are only applicable to a constant amplitude loading. However, springs are normally subjected to variable amplitude and spectrum types of operating loads. In order to evaluate the fatigue failure for structures under the latter type of loading, a Palmgren-Miner rule is normally used together with the S-N curves or constant.

Fatigue diagrams in the spring industry possess in the minimum operation stress on the x-axis and the maximum operation stress on the y-axis. The isolines of failure probability are plotted in coordinates $(\sigma_{\max}, \sigma_{\min})$. Each particular diagram is specified by the certain number of full cycles (peak-to peak) to total failure n_f . In coordinates $(\sigma_{\max}, \sigma_{\min})$, the isoline of survival probability of the element is $P_S(n_f)$. For this isoline, the probability the fatigue of failure after n_f cycles is $P_F(n_f) = 1 - P_S(n_f)$. Usually the survival failure probability for the construction of Goodman diagrams is $P_S(n_f) = 50\%$. The probability evaluations for fatigue life of helical springs is presented in this chapter and Chap. 11.

The normal stresses $(\sigma_{\max}, \sigma_{\min})$ are used in Goodman diagrams for leaf springs, disk springs and twisted helical springs. Goodman diagrams in shear-stress coordinates $(\tau_{\max}, \tau_{\min})$ are suitable for compression and extension helical springs and for torsion bars. Goodman diagrams are plotted in basic stresses (Chap. 2, Eq. (2.4)) or in corrected stresses (Chap. 2, Eq. (2)).

The modified Goodman diagrams (SAE HS 788 1980) are plotted in dimensionless coordinates. The minimum and maximum stresses are normalized by dividing them by the ultimate tensile strength of the material σ_w . The normalization allows the diagram to be used for various similar materials with different ultimate tensile strengths. Thus, coordinates of the modified Goodman diagrams are the normalized minimum $\sigma_{w\cdot\min}$ and the normalized maximum stress $\sigma_{w\cdot\max}$.

A particular Goodman diagram depicts the immediate failure under static load. For this diagram number of cycles to failure is exact one, $n_f = 1$. The interior single Goodman diagram corresponds to the endurance limit of material. The construction of the interior Goodman diagram is shown on the Fig. 10.1. The straight line that connects the points 2 and 3 is referred to as “relaxation limit”. The relaxation limit is as a rule identified to the yield stress σ_y . The load is static and creep or relaxation of spring occurs. Usually no breakage of spring occurs, but creep or relaxation lead to the malfunction of the spring. Therefore, the relaxation limit is included into Goodman diagram. Another mechanism of spring malfunction takes place when the stress approaches the line between points 2 and 7. In this case, the complete breakage of the spring material due to material fatigue occurs.

The “complete” Goodman diagram is a sequence of nested particular diagrams Δ_n , each of them corresponds a certain number of load cycles to failure (Fig. 10.2).

Each complete Goodman diagram corresponds to a definite specimen geometry, surface condition, and material characteristics. The major advantage of Goodman diagrams is their clear exposition of fatigue life at any stress ratio and stress level. The major disadvantage of Goodman diagrams is an enormous experimental effort for their characterization. Each complete Goodman diagram requires the acquisition of S-N curves for every relevant stress ratio. The cyclic tests to failure of the samples accomplished for every relevant stress ratio and for different stress amplitudes. The experimentally acquired data is ordered in a specific for Goodman diagram manner. The properly evaluated Goodman diagrams demonstrate the mean stress sensitivity, relaxation and endurance limits, different behaviour for tension and compression, sensitivity to shear, normal and effective stresses over the whole range of admissible deformations.

The detailed presentation of diagrams provides (FKM 2018, Sect. 1.4).

There are also several secondary effects, which influence the fatigue life of springs. The influence of creep crack growth interaction, temperature, and environmental effects on fatigue life of structural elements is reviewed in Coffin (1983), SAE AE-22 (1997), Juvinall and Marshek (2017).

10.1.3 Evaluation of Fatigue Life with Haigh Diagrams

Another type of a constant life diagram is known as a Haigh diagram (Haigh 1915). As shown on Fig. 10.3, the x-axis on this diagram is the mean stress σ_m and the y-axis is the stress amplitude σ_a . The diagram demonstrates the fatigue limit as a function of the mean stress plotted in the (σ_a, σ_m) -plane. Thus, the Haigh diagram is a 45°

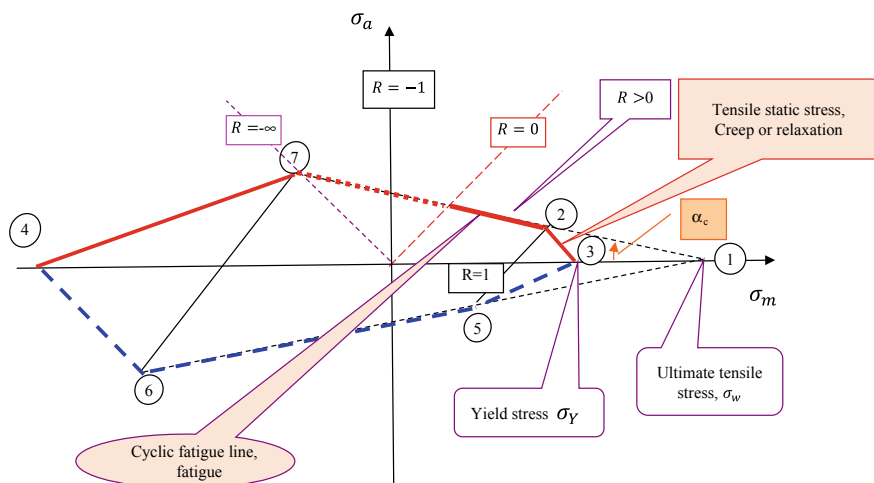


Fig. 10.3 Haigh diagram. The inclination angle α_c represents the mean value sensitivity

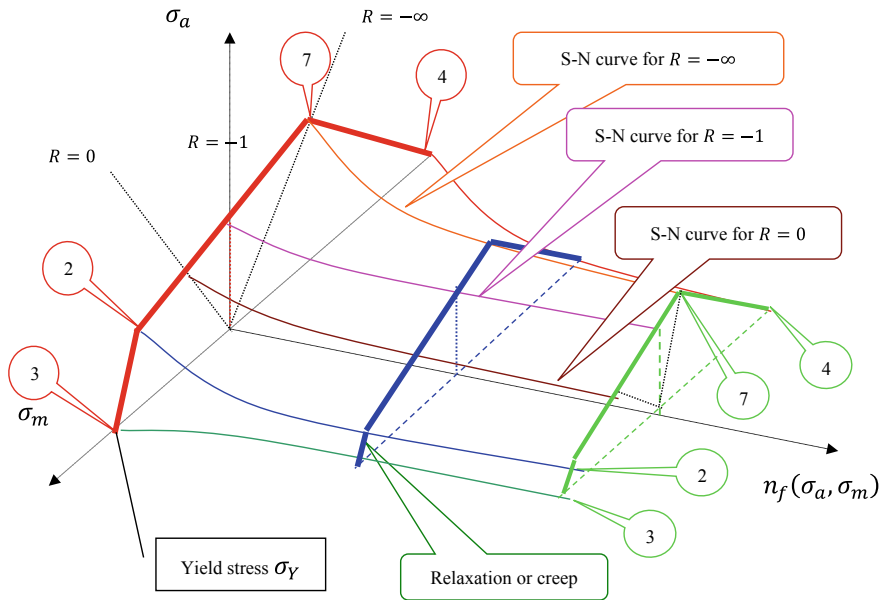


Fig. 10.4 Schematic representation of the M-A-N surface $N_f(\sigma_a, \sigma_m)$

clockwise-rotated Goodman diagram. The negative inclination of the fatigue line 2–7 on the Haigh diagram increases with the growing mean stress sensitivity of the material μ_σ . Tangent of the inclination angle of the line 2–7 depicts the mean stress sensitivity of the material. The three-dimensional M-A-N surface (Mean-Amplitude-Number of cycles) represents the number of cycles to failure n_f as the function of mean σ_m and amplitude stresses σ_a (Fig. 10.4). Each M-A-N surface correspond to a definite failure probability, for example for $P_F(n_f) = 50\%$. Each isoline of the M-A-N surface for the constant n_f depicts the single Haigh diagram.

The Goodman and Haigh diagrams portray graphically the experimentally acquired fatigue data with apparent amplitude and mean value of stresses. For general oscillating loads with time-variable amplitudes and mean values, the diagram technique is inappropriate. For the numerical applications, the combination of data in form of a particular damage parameter is advantageous. Damage parameter presents the level of microscopic breakage accrued in a component from the single stress reversals. The damage accumulates over load history. When the total accumulated damage will be equal to an experimentally acquired ultimate value, the final breakage occurs. The possibility of numerical evaluation of accumulated damage is the main advantage of damage parameters over the graphic diagram techniques. Among damage parameters most often assessed is the Smith, Watson, Topper parameter $\mathbf{P}_{\text{SWT}}$ (Smith et al. 1970). The value of $\mathbf{P}_{\text{SWT}}$ accounts both effects of the range and the maximum of stress. $\mathbf{P}_{\text{SWT}}$ traditionally used in spring manufacturing industry for the estimation of fatigue effects, as discussed in Chap. 1. However, another parameters could predict better the fatigue life, as discussed later in this and the next chapters.

10.1.4 Consistence of Fatigue Life Diagrams and Creep Diagrams

There is a similarity between stress-life test for cyclic loading and creep test for the static loading. In both tests the load is fixed and spring length alters with the number of cycles or time of load application. This kind of loading is typical for the springs, which carry the weight of a vehicle or exposed to some fluctuating external forces, for example wind or water pressure. The suspension springs are the typical examples of this application. Another example are the springs for the resonance machinery (resonance crushers for concrete recycling or rubbleization). Because the stresses sustain during the operation time, the fatigue life of these springs culminates with their destruction.

The alternative analogy exists between strain-life-test for cyclic and relaxation test for the static loading. In both tests the spring travel is fixed and the reaction spring force reduces with the number of reciprocations or with time of load application. This kind of loading is characteristic for the springs, which oscillate between the fixed bounds. The valve and fuel-pump springs, the springs for fuel injectors to fuel rail of combustion engines are the typical examples for the dynamically loaded strain-loaded spring. The Belleville springs and spring washers are the corresponding static applications of the strain-life exploited springs,. Because the spring load and stresses reduce with time, the operational life terminates with the functional degradation but not the breakage of the spring element.

The complete constant life diagram depicts the region of the oscillating stress (Fig. 10.1, Line 2–7) and the region of the constant stress (Fig. 10.1, Line 2–3). Both lines must be acquired in the consistent tests, both in stress-life and creep tests or, alternatively, both in strain-life and relaxation tests.

10.2 Stress Ratio Influence on Uniaxial Fatigue

10.2.1 Stress-Life Approach with Variable Stress Ratio

For majority of spring types, the stress state is uniaxial. The helical compression springs are in the state of pure torsion. The wire of the helical twist springs is the state of pure bending. The circumferential stresses form the stress state in the disk springs. For the beginning, the uniaxial fatigue is discussed.

Consider two harmonically oscillating stresses:

$$\sigma(t) = \sigma_a \sin(\omega t), \quad (\text{S})$$

$$\sigma(t) = \sigma_m + \sigma_a \sin(\omega t). \quad (\text{N})$$

The first, fully reversed, symmetric stress cycle (S) has the amplitude σ_a and zero mean stress. The mean stress of the second cycle (N) is non-zero σ_m and the amplitude σ_a . Maximal stress over cycle for the process (N) with non-zero mean stress is $\sigma_{\max} = \sigma_m + \sigma_a$. The fatigue life of the cycle (S) is longer, than the fatigue life of the cycle (N), if $\sigma_m > 0$ and the amplitudes σ_a are equal.

The cycle with zero mean value and the amplitude σ_{ar} is referred as an effective fully-reversed cycle (E). The third cycle is the harmonic, fully reversed cycle with the effective amplitude σ_{ar} :

$$\sigma(t) = \sigma_{ar} \sin(\omega t), \quad (E)$$

For the certain amplitude σ_{ar} , the fatigue life of the cycle (E) is equal to the fatigue life of the cycle (N). For the positive mean stress $\sigma_m > 0$, we have $\sigma_{ar} > \sigma_a$. The issue is, how to determine the effective amplitude σ_{ar} as the function of the amplitude σ_a and mean stress σ_m . There are several acknowledged approaches to solve this task.

The stress-life approach is applicable to the cases of long fatigue life and moderate cyclic stresses. The cyclic stresses lead to dominant elastic strain amplitudes and only tiny microscopic plastic strains occurs over one reversal cycle.

The stress-life approach is based on the Basquin's law. The plot of Basquin's law possess the fatigue life n_f the x-axis and the effective stress amplitude σ_{ar} on the y-axis. The dependence of fatigue life (number of cycles to total failure) upon the effective stress amplitude σ_{ar} derives from the Basquin's law (or Wöhler's law) (Basquin 1910) :

$$n_f = f_{\sigma}^{-1}(\sigma_{ar}) \equiv \frac{1}{2} \left(\frac{\sigma_{ar}}{\sigma'_f(R_{\sigma})} \right)^{1/b(R_{\sigma})} \equiv \lambda_{\sigma} \sigma_{ar}^{-p_{\sigma}}, \quad (10.1a)$$

$$n_f = f_{\tau}^{-1}(\tau_{ar}) \equiv \frac{1}{2} \left(\frac{\tau_{ar}}{\tau'_f(R_{\sigma})} \right)^{1/b(R_{\sigma})} \equiv \lambda_{\tau} \tau_{ar}^{-p_{\sigma}}. \quad (10.1b)$$

The constants of Eqs. (10.1a, 10.1b) are:

$\sigma'_f(R_{\sigma})$	fatigue strength coefficient for normal stress,
$\tau'_f(R_{\sigma})$	fatigue strength coefficient for shear stress,
$b(R_{\sigma})$	fatigue strength exponent,
$p_{\sigma} = -1/b(R_{\sigma}) > 0$	reciprocal strength exponent, and

$$\lambda_{\sigma} \equiv \frac{1}{2} (\sigma'_f(R_{\sigma}))^{p_{\sigma}}, \lambda_{\tau} \equiv \frac{1}{2} (\tau'_f(R_{\sigma}))^{p_{\sigma}}.$$

These coefficients depend upon the stress ratio R_{σ} Eq. (10.3) (Table 10.1). The coefficients of Eqs. (10.1a, 10.1b) are the limit case of the functions in the case of vanishing mean stress:

Table 10.1 Stress ratios for different stresses

Static stress $R_\sigma = 1$	Pulsating stress $R_\sigma = 0$	Alternating stress $R_\sigma = -1$
$\sigma_a = 0$	$\sigma_a = \sigma_m$	$\sigma_m = 0$
$\sigma_{\max} = \sigma_{\min}$	$\sigma_{\min} = 0, \sigma_{\max} = 2\sigma_a$	$\sigma_{\max} = -\sigma_{\min}$

Table 10.2 Empirical curves for fatigue life based on uniaxial stress-life approach for fully reversed cycle $\sigma_{ar} = \sigma'_f (2n_f)^b$

Gerber (1874)	$\sigma_a = \left[1 - \left(\frac{\sigma_m}{\sigma^*}\right)^\gamma\right] \cdot \sigma_{ar}$	$\sigma^* = \sigma_u,$ $\gamma = 2 \rightarrow \sigma_a = \left[1 - \left(\frac{\sigma_m}{\sigma_u}\right)^2\right] \cdot \sigma_{ar}$
Goodman (1899)		$\sigma^* = \sigma_u,$ $\gamma = 1 \rightarrow \sigma_a = \left[1 - \left(\frac{\sigma_m}{\sigma_u}\right)\right] \cdot \sigma_{ar}$
Soderberg (1930)		$\sigma^* = \sigma_y,$ $\gamma = 1 \rightarrow \sigma_a = \left[1 - \left(\frac{\sigma_m}{\sigma_y}\right)\right] \cdot \sigma_{ar}$
Morrow (1968)		$\sigma^* = \sigma_f,$ $\gamma = 1 \rightarrow \sigma_a = \left[1 - \left(\frac{\sigma_m}{\sigma_f}\right)\right] \cdot \sigma_{ar}$
Walker (1970) $\mathbf{P_W}$	$\sigma_{ar} = \sigma_a^{\tilde{\gamma}} \sigma_{\max}^{1-\tilde{\gamma}}$	$\sigma_{ar} = \sigma_a \cdot \left(\frac{2}{1-R_\sigma}\right)^{\tilde{\gamma}}$
Smith et al. (1970) $\mathbf{P_{SWT}}$ $\tilde{\gamma} = 1/2$		
Bergmann (1983) $\mathbf{P_B}$	$\sigma_{ar} = \sqrt{\sigma_a} \sqrt{\sigma_{\max} + \mu_r \sigma_m}$	$\sigma_{ar} = \sqrt{1 + (1 + \mu_r) \frac{1+R_\sigma}{1-R_\sigma}} \cdot \sigma_a$
Haigh (1915) $\mathbf{P_H}$	$\sigma_{ar} = \sigma_{\max} + \mu_r \sigma_m$	$\sigma_{ar} = \left(1 + (1 + \mu_r) \frac{1+R_\sigma}{1-R_\sigma}\right) \cdot \sigma_a$
Bergman-Walker, $\mathbf{P_{BW}}$	$\sigma_{ar} = \sigma_a^{\tilde{\gamma}} (\sigma_{\max} + \mu_r \sigma_m)^{1-\tilde{\gamma}}$	$\sigma_{ar} = \sigma_a \cdot \left(1 + (1 + \mu_r) \frac{1+R_\sigma}{1-R_\sigma}\right)^{\tilde{\gamma}}$

Table 10.3 Empirical curves for fatigue life based on stress-life approach for shear stresses life approach for fully reversed cycle $\tau_{ar} = \tau'_f (2n_f)^b$

Walker (1970) $\mathbf{P_W}$	$\tau_{ar} = \tau_a^{\tilde{\gamma}} \tau_{\max}^{1-\tilde{\gamma}}$	$\tau_{ar} = \tau_a \cdot \left(\frac{2}{1-R_\sigma}\right)^{\tilde{\gamma}}$
Smith et al. (1970) $\mathbf{P_{SWT}}$, $\tilde{\gamma} = 1/2$		
Bergmann (1983) $\mathbf{P_B}$	$\tau_{ar} = \sqrt{\tau_a} \sqrt{\tau_{\max} + \mu_r \tau_m}$	$\tau_{ar} = \sqrt{1 + (1 + \mu_r) \frac{1+R_\sigma}{1-R_\sigma}} \cdot \tau_a$
Haigh (1915) $\mathbf{P_H}$	$\tau_{ar} = \tau_{\max} + \mu_r \tau_m$	$\tau_{ar} = \left(1 + (1 + \mu_r) \frac{1+R_\sigma}{1-R_\sigma}\right) \cdot \tau_a$
Bergmann-Walker, $\mathbf{P_{BW}}$	$\tau_{ar} = \tau_a^{\tilde{\gamma}} (\tau_{\max} + \mu_r \tau_m)^{1-\tilde{\gamma}}$	$\tau_{ar} = \tau_a \cdot \left(1 + (1 + \mu_r) \frac{1+R_\sigma}{1-R_\sigma}\right)^{\tilde{\gamma}}$

Table 10.4 Units for the application of small inclusion analysis

Ultimate nominal stress Cahoon (1972)	$\sigma_{w,n} = \frac{HV}{2.9} \left(\frac{n'}{0.217} \right)^{n'}$	MPa
Hardening exponent	n'	
Vickers pyramid hardness	HV	$\text{kgf mm}^{-2} = 9.81 \text{ N mm}^{-2}$
Endurance threshold limit	K_{th}	$\text{MPa m}^{1/2}$
Inclusion size	$\sqrt{area}$	μm
Fatigue limit stress	σ_w	MPa

$$\sigma'_{f0} = \sigma'_f(R_\sigma = -1), \tau'_{f0} = \tau'_f(R_\sigma = -1), b_0 = b(R_\sigma = -1).$$

The inverse functions for (10.1a) evaluate the effective stress amplitude for a given number of full cycles (peak-to-peak cycles):

$$\sigma_{ar} = f_\sigma(n_f), \quad f_\sigma(n_f) = \sigma'_f(R_\sigma) \cdot (2n_f)^{b(R_\sigma)}, \quad (10.1c)$$

$$\tau_{ar} = f_\tau(n_f), \quad f_\tau(n_f) = \tau'_f(R_\sigma) \cdot (2n_f)^{b(R_\sigma)}. \quad (10.1d)$$

Fatigue strength coefficients $\sigma'_f(R_\sigma)$ or $\tau'_f(R_\sigma)$ are the certain functions of R_σ . The fatigue strength exponent is commonly accepted to be independent upon stress ratio R_σ : $b(R_\sigma) = b_0$.

Both values σ_m, σ_a of the cycle (N) are specified. As already specified, the task is to determine the equivalent stress amplitude σ_{ar} , which will cause the same fatigue life. To solve this task, the functions $\sigma'_f(R_\sigma), \tau'_f(R_\sigma), b(R_\sigma)$ must be determined as functions of stress ratio R_σ .

10.2.2 Equivalent Stress Amplitude

In this Section, we calculate the equivalent amplitude σ_{ar} of the symmetric cycle (S), which leads to the same fatigue life n_f as achieved in the asymmetric cycle (N). There are several hypotheses that reasonable solve this task (Tables 10.2 and 10.3)

- 1°. Smith, Watson, Topper declared, that all harmonically oscillated normal stresses with the same maximal stress $\sigma_{\max}$, but different stress ratios R_σ cause the same fatigue damage, as caused by the pure sinusoidal oscillating stress with the amplitude σ_{ar} and $R_\sigma = -1$. $\mathbf{P}_{\text{SWT}}$ amplitude of the sinusoidal oscillating stress reads ($\sigma_{ar} > \sigma_a$ for $R_\sigma > 0$):

$$\sigma_{ar} = \sqrt{\sigma_a \sigma_{\max}} = \sqrt{\sigma_a} \sqrt{\sigma_a + \sigma_m} = \sqrt{\frac{2}{1 - R_\sigma}}, \quad (10.2a)$$

$$n_f = f_\sigma^{-1}(\sigma_{ar}) \equiv \frac{1}{2} \left(\frac{\sigma_{\max}}{\sigma'_{f0}} \right)^{\frac{1}{b_0}} \left(\frac{1 - R_\sigma}{2} \right)^{\frac{1}{2b_0}}. \quad (10.2b)$$

According to Smith, Watson and Topper, the positive mean stress increase the cycle damage and lead to the reduction of fatigue life. For shear stress, the expressions for $\mathbf{P}_{\text{SWT}}$ in terms of shear stresses are:

$$\tau_{ar} = \sqrt{\tau_a \tau_{\max}}, \quad n_f = f_\tau^{-1}(\tau_{ar}) \equiv \frac{1}{2} \left(\frac{\tau_{\max}}{\tau'_{f0}} \right)^{\frac{1}{b_0}} \left(\frac{1 - R_\sigma}{2} \right)^{\frac{1}{2b_0}}. \quad (10.3)$$

Comparison of Eqs. (10.2a, 10.2b) with (10.1a, 10.1b, 10.1c, 10.1d) demonstrates that the $\mathbf{P}_{\text{SWT}}$ approach is equivalent to Basquin's law with the following functions:

$$\sigma'_f(R_\sigma) = \sigma'_{f0} \sqrt{\frac{2}{1 - R_\sigma}}, \quad \tau'_f(R_\sigma) = \tau'_{f0} \sqrt{\frac{2}{1 - R_\sigma}}.$$

That is, if the coefficients σ'_{f0} and τ'_{f0} are determined in one series of fatigue experiments with $R_\sigma = -1$, the estimation of the fatigue life could be performed for all other stress ratios. This hypothesis greatly simplifies the testing of spring materials. But this approach could lead to significant errors in fatigue analysis, of the real spring material behaves in reality differently, as the hypothesis presumes.

According to the $\mathbf{P}_{\text{SWT}}$ rule, all materials possess the same dependence upon mean stress. This hypothesis overestimates the influence of mean stress. However, the mean stress sensitivity is apparently different for different materials. This drawback limits the application $\mathbf{P}_{\text{SWT}}$ for the evaluation of fatigue life of springs.

- 2°. An alternative hypothesis dependence upon the stress ratio was proposed by Walker (1970) by introduction of the adjustable fitting exponent $0 \leq \tilde{\gamma} \leq 1$. For normal stresses the effective cyclic stress $\mathbf{P}_W$:

$$\sigma_{ar} = \sigma_a^{\tilde{\gamma}} \sigma_{\max}^{1-\tilde{\gamma}}, \quad \sigma_{ar} = \sigma'_a \left(\frac{2}{1 - R_\sigma} \right)^{\tilde{\gamma}}, \quad (10.4a)$$

$$n_f = f_\sigma^{-1}(\sigma_{ar}) \equiv \frac{1}{2} \left(\frac{\sigma_{\max}}{\sigma'_{f0}} \right)^{\frac{1}{b_0}} \left(\frac{1 - R_\sigma}{2} \right)^{\frac{\tilde{\gamma}}{b_0}}, \quad (10.4b)$$

and for shear stress:

$$\tau_{ar} = \tau_a^{\tilde{\gamma}} \tau_{\max}^{1-\tilde{\gamma}}, \quad \tau_{ar} = \tau_a \cdot \left(\frac{2}{1 - R_\sigma} \right)^{\tilde{\gamma}}, \quad (10.5a)$$

$$n_f = f_\tau^{-1}(\tau_{ar}) \equiv \frac{1}{2} \left(\frac{\tau_{\max}}{\tau'_{f0}} \right)^{\frac{1}{b_0}} \left(\frac{1 - R_\sigma}{2} \right)^{\frac{\tilde{\gamma}}{b_0}} \quad (10.5b)$$

The expressions for fatigue coefficients of Basquin's law according to Walker approach are:

$$\sigma'_f(R_\sigma) = \sigma'_{f0} \cdot \left(\frac{2}{1 - R_\sigma} \right)^{\tilde{\gamma}}, \quad \tau'_f(R_\sigma) = \tau'_{f0} \cdot \left(\frac{2}{1 - R_\sigma} \right)^{\tilde{\gamma}}. \quad (10.6)$$

$\mathbf{P}_{\text{SWT}}$ (10.2a, 10.2b) follows for $\tilde{\gamma} = 1/2$ as the special case of (10.3). For steels (Dowling et al. 2009), $\tilde{\gamma}$ was found to correlate with the ultimate tensile strength, In the cited article, the trend of decreasing $\tilde{\gamma}$ with the increasing strength was stated. In accordance to the above reported results, the values $\tilde{\gamma} = 0.15 \dots 0.23$ fit the acquired data for the spring steels (this chapter). This behavior indicates an increasing sensitivity to mean stress with the strength and hardness growth. Geilen et al. (2020) applied recently the Walker approach for fatigue damage of disk springs.

- 3°. Bergmann (1983) suggested extending the original $\mathbf{P}_{\text{SWT}}$ (10.2a, 10.2b) by introduction of an auxiliary constant μ_r , which permits the correct accounting of the mean stress:

$$\sigma_{ar} = \sqrt{\sigma_a} \sqrt{\sigma_{\max} + \mu_r \sigma_m}, \quad \sigma_{ar} = \sqrt{1 + (1 + \mu_r) \frac{1 + R_\sigma}{1 - R_\sigma}} \cdot \sigma_a. \quad (10.7)$$

- 4°. The traditional Haigh approach $\mathbf{P}_H$ assumes the linear dependence of the effective fully reversed stress amplitude upon the mean stress (Haigh 1915). For positive τ_m , $\tau_{ar} > \tau_a$, $\sigma_{ar} > \sigma_a$. That is, the positive mean stress increase the effective fully reversed amplitude τ_{ar} , which leads to the growing damage pro load cycle and consequently reduces the total number of cycles to breakage. In Haigh diagrams, the functions for the effective, fully reversed stress amplitudes as the function of mean stress are linear. The fatigue parameter $\mathbf{P}_H$ uses the linear functions with the mean stress sensitivity μ_r :

$$\sigma_{ar} = \sigma_{\max} + \mu_r \sigma_m, \quad \sigma_{ar} = \left(1 + (1 + \mu_r) \frac{1 + R_\sigma}{1 - R_\sigma} \right) \cdot \sigma_a, \quad (10.8a)$$

$$\tau_{ar} = \tau_{\max} + \mu_r \tau_m, \quad \tau_{ar} = \left(1 + (1 + \mu_r) \frac{1 + R_\sigma}{1 - R_\sigma} \right) \cdot \tau_a. \quad (10.8b)$$

The mean stress sensitivity μ_r is occasionally taken as the function of R_σ . For example, (FKM 2018, Abb. 2.4-1 and Abb. 2.4-2) uses the piecewise-linear function $\mu_r = \mu_r(R_\sigma)$.

The slightly different definition of mean stress sensitivity is traditionally accepted in the industry. According to Schütz (1967), Haibach (2013) the mean stress

sensitivity is defined as:

$$M_r = \frac{\sigma_a(R_\sigma = -1) - \sigma_a(R_\sigma = 0)}{\sigma_m(R_\sigma = 0)} > 0. \quad (10.9)$$

From Eqs. (10.8a, 10.8b) follows, that:

$$\sigma_a(R_\sigma = -1) = \sigma_{ar}, \sigma_a(R_\sigma = 0) = \sigma_{ar}/(2 + \mu_r), \sigma_m(R_\sigma = 0) = \sigma_m(R_\sigma = 0)$$

The substitution of these formulas to Eq. (10.9) demonstrates the relation of mean stress sensitivity according to Schütz (1967) to mean stress sensitivity according to Bergmann (1983):

$$M_r = \frac{\sigma_a(R_\sigma = -1) - \sigma_a(R_\sigma = 0)}{\sigma_m(R_\sigma = 0)} = \frac{\sigma_{ar} - \sigma_{ar}/(2 + \mu_r)}{\sigma_{ar}/(2 + \mu_r)} \equiv 1 + \mu_r. \quad (10.10a)$$

It follows with (10.10a) from (10.8a, 10.8b):

$$\tau_{ar} = \tau_a + M_r \tau_m, \quad \sigma_{ar} = \sigma_m + M_r \sigma_m.$$

With Eq. (10.10a) we get the relation of stress amplitude σ_a to fatigue parameter $\mathbf{P}_L$, which uses mean stress sensitivity M_r . In term of the stress amplitude, $\mathbf{P}_L$ reads:

$$\sigma_{ar} = \left(1 + M_r \frac{1 + R_\sigma}{1 - R_\sigma}\right) \cdot \sigma_a, \quad \tau_{ar} = \left(1 + M_r \frac{1 + R_\sigma}{1 - R_\sigma}\right) \cdot \tau_a. \quad (10.10b)$$

Murakami (2001) reports the relation of mean stress sensitivity to the Vickers hardness HV :

$$M_r(HV) = \exp(0.0693 \cdot 10^{-3} \cdot HV + 0.157) - 1, \\ \mu_r(HV) = M_r(HV) - 1.$$

In its turn, the Vickers hardness HV can be expressed in terms of ultimate stress (Cahoon 1972).

The dependence of mean stress sensitivity upon the ultimate stress σ_w for steel (Hänel et al. 2003, Sect. 2.4.2.4):

$$M_r(\sigma_w) = 0.35 \cdot 10^{-3} \cdot \sigma_w - 0.1, \quad \mu_r(\sigma_u) = M_r(\sigma_u) - 1. \quad (10.10c)$$

According to Haibach (2013), the dependency of mean stress sensitivity upon R_σ is :

$$\widetilde{M}_r(R_\sigma) = \begin{cases} M_r, & R_\sigma < 0 \\ M_r/3, & R_\sigma > 0 \end{cases}. \quad (10.10d)$$

5°. As an extension of the above approximations, the expression with two sensitivity parameters ($\tilde{\gamma}$, μ_r) could be considered:

The generalized, or Bergmann-Walker parameter $\mathbf{P}_{\text{BW}}$ reads:

$$\sigma_{ar} = \sigma_a^{\tilde{\gamma}} (\sigma_{\max} + \mu_r \sigma_m)^{1-\tilde{\gamma}}, \quad \tau_{ar} = \tau_a^{\tilde{\gamma}} (\tau_{\max} + \mu_r \tau_m)^{1-\tilde{\gamma}}. \quad (10.11a)$$

In terms of stress amplitudes and mean stress the Eq. (10.11a) reads:

$$\sigma_{ar} = \sigma_a^{1-\tilde{\gamma}} (\sigma_a + (1 + \mu_r) \sigma_m)^{1-\tilde{\gamma}}, \quad \tau_{ar} = \tau_a^{1-\tilde{\gamma}} (\tau_a + (1 + \mu_r) \tau_m)^{1-\tilde{\gamma}}. \quad (10.11b)$$

For comparison, we study now, how the Bergmann-Walker parameters $\mathbf{P}_{\text{BW}}$ behaves for varying fitting exponent $\tilde{\gamma}$. For definiteness, we assume, that the mean stress sensitivity of symmetric stress cycle (S) in the point $R_\sigma = -1$ does not vary. In this point the mean stress is: $\sigma_m = 0$. In other words, the effective stresses are equal for all different $\tilde{\gamma}$:

$$\sigma_{-1} = \sigma_{ar}(R_\sigma = -1).$$

The second assumption is the following. The slopes of Bergmann-Walker parameters $\mathbf{P}_{\text{BW}}$ in the point $R_\sigma = -1$ must not depend on $\tilde{\gamma}$. From both conditions one can find the expression for the Bergmann-Walker parameter $\mathbf{P}_{\text{BW}}$, which depends only upon fitting exponent $\tilde{\gamma}$ and μ_r . In this case, the parameter $\mathbf{P}_{\text{BW}}$ takes the form:

$$\sigma_{ar} = \sigma_a^{1-\tilde{\gamma}} \tilde{\gamma}^{-\tilde{\gamma}} (\tilde{\gamma} \sigma_a + (\mu_r + 1) \sigma_m)^{1-\tilde{\gamma}}. \quad (10.11c)$$

Dividing (10.11c) by σ_{-1} , we get the dimensionless form for this expression:

$$\frac{\sigma_{ar}}{\sigma_{-1}} = \left(\frac{\sigma_a}{\sigma_{-1}} \right)^{1-\tilde{\gamma}} \tilde{\gamma}^{-\tilde{\gamma}} \left(\tilde{\gamma} \frac{\sigma_a}{\sigma_{-1}} + (\mu_r + 1) \frac{\sigma_m}{\sigma_{-1}} \right)^{1-\tilde{\gamma}}. \quad (10.11d)$$

The plots of Eq. (10.11d) for diverse values μ_r and $\tilde{\gamma}$ are shown on Fig. 10.5. The plots are displayed in the dimensionless coordinates σ_m/σ_{-1} and σ_a/σ_{-1} , taken correspondingly as abscise and ordinate. The curves on this figure correspond to the Haigh modulus for arrangement of fatigue data. The values μ_r and $\tilde{\gamma}$ are displayed to each plot on Fig. 10.5. These two values are used to fit the experimental data. The displayed graphs show, that the Bergmann-Walker parameters $\mathbf{P}_{\text{BW}}$ smoothly fit the reducing mean stress sensitivity at the higher values of the mean stress σ_m . Contrarily, the Eq. (10.10d) due to (Haibach 2013) displays reducing of mean stress sensitivity abruptly in a certain point. This sharp alternation of Eq. (10.10d) is questionable from the physical lookout. The smooth transition of the Bergmann-Walker parameter $\mathbf{P}_{\text{BW}}$ is also favorable for numerical computations and optimization methods. The curves on the actual diagram correspond the lines (2–7) of the Haigh diagram (Fig. 10.3).

The plots of Eq. (10.11d) for varying values μ_r and $\tilde{\gamma}$ are shown on Fig. 10.6.

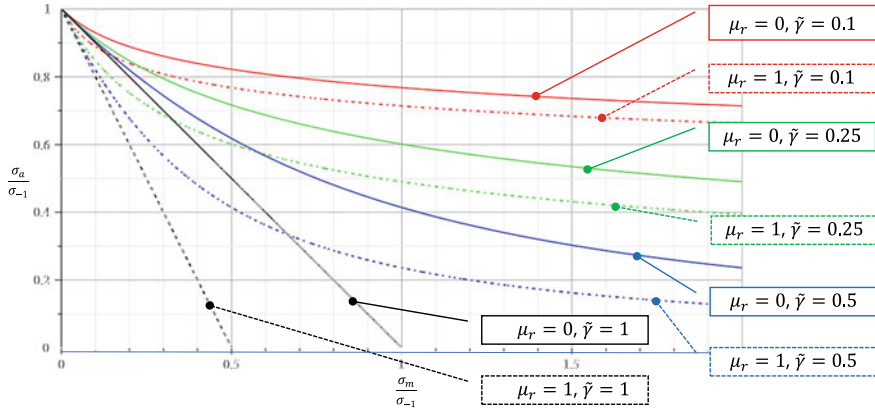


Fig. 10.5 The isolines of Bergmann-Walker parameter $\mathbf{P}_{\text{BW}}$ in coordinates (σ_m, σ_a) , Eq. (10.11d) for different values μ_r and $\tilde{\gamma}$

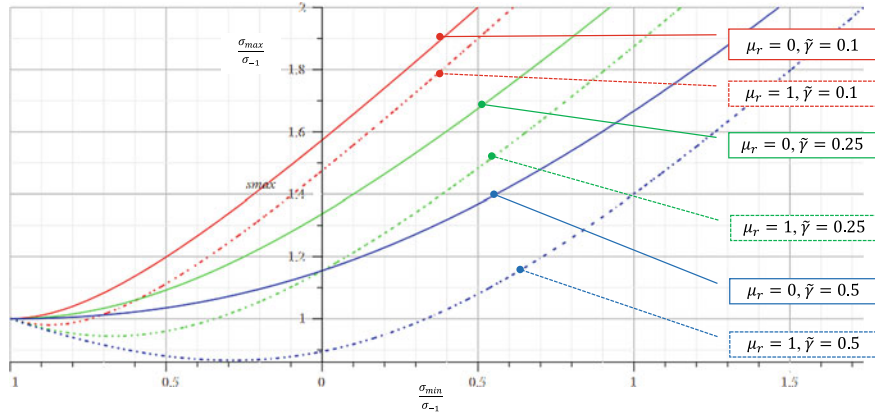


Fig. 10.6 The isolines of Bergmann-Walker parameter $\mathbf{P}_{\text{BW}}$ in coordinates $(\sigma_{\min}, \sigma_{\max})$ for different values μ_r and $\tilde{\gamma}$

The plots are displayed in the dimensionless coordinates $\sigma_{\min}/\sigma_{-1}$ and $\sigma_{\max}/\sigma_{-1}$, as abscise and ordinate. As $\sigma_{\min} = \sigma_m - \sigma_a$, $\sigma_{\max} = \sigma_m + \sigma_a$, the curves on this drawing (Fig. 10.6) are simply counterclockwise rotated curves from the previous picture (Fig. 10.5). The values μ_r and $\tilde{\gamma}$ are displayed for every curve on Fig. 10.6. The curves on this figure correspond to the Goodman way for presentation of fatigue data. The curves display the lines (2–7) of the Goodman diagram (Fig. 10.2).

In terms of stress ratio, the expressions for $\mathbf{P}_{\text{BW}}$ and for the number of cycles to failure are:

$$\sigma_{ar} = \sigma_a \cdot \left(1 + (1 + \mu_r) \frac{1 + R_\sigma}{1 - R_\sigma} \right)^{\tilde{\gamma}} \equiv \sigma_a \cdot \left(1 + M_r \frac{1 + R_\sigma}{1 - R_\sigma} \right)^{\tilde{\gamma}}, \quad (10.12a)$$

$$\tau_{ar} = \tau_a \cdot \left(1 + (1 + \mu_r) \frac{1 + R_\sigma}{1 - R_\sigma}\right)^{\tilde{\gamma}} \equiv \tau_a \cdot \left(1 + M_r \frac{1 + R_\sigma}{1 - R_\sigma}\right)^{\tilde{\gamma}}, \quad (10.12b)$$

$$n_f = f_\sigma^{-1}(\sigma_{ar}) \equiv \frac{1}{2} \left(\frac{\sigma_{\max}}{\sigma'_{f0}} \right)^{\frac{1}{b_0}} \left(1 + M_r \frac{1 + R_\sigma}{1 - R_\sigma} \right)^{-\frac{\tilde{\gamma}}{b_0}}, \quad (10.12c)$$

$$n_f = f_\tau^{-1}(\tau_{ar}) \equiv \frac{1}{2} \left(\frac{\tau_{\max}}{\tau'_{f0}} \right)^{\frac{1}{b_0}} \left(1 + M_r \frac{1 + R_\sigma}{1 - R_\sigma} \right)^{-\frac{\tilde{\gamma}}{b_0}}. \quad (10.12d)$$

All traditional formulations are the distinct cases of the generalized formulas (10.11a):

- Walker parameter $\mathbf{P}_W$, Eqs. (10.4a, 10.4b), follows from (10.11a) with $\mu_r = 0$, $M_r = 1$;
- Smith, Watson and Topper parameter $\mathbf{P}_{SWT}$, Eqs. (10.2a, 10.2b) is the distinct case of (10.12a, 10.12b, 10.12c, 10.12d): $\tilde{\gamma} = 1/2$, $\mu_r = 0$, $M_r = 1$;
- Bergmann parameter $\mathbf{P}_B$, Eq. (10.7) is the case of (10.11a): $\tilde{\gamma} = 1/2$. For spring steels, the mean stress sensitivities are according to standard (Hänel et al. 2003), $\mu_r < 0$, $M_r < 1$;
- Haigh parameter $\mathbf{P}_H$, Eqs. (10.8a, 10.8b) follows from (10.11a) after the substitution $\tilde{\gamma} = 0$.

There are also alternative proposals, based on stress-life approach, for the description of the fatigue failure line 2–7. Among them, several rarely used criteria were discussed by Meyer (2014).

If the range of mean stress sensitivity is considerably narrow, for example $0 < R_\sigma < 0.5$, the Haigh parameter $\mathbf{P}_H$ with the linearly dependence, roughly depicts the experimental data in this region (for example, Mayer et al. 2016). The better approximation provides the Walker parameter $\mathbf{P}_W$. Practically this means, that for a certain type of springs in most cases these parameters could be used for fatigue estimations. If the range for stress ratio spreads, the adequate ordering of the experimental fatigue data requires the nonlinear relation between stress amplitude and mean stress for a given fatigue life. Particularly, the piecewise-linear relations are recommended in Hänel et al. (2003), Haibach (2013) or FKM (2018, Sect. 2.4.2.2).

All discussed above stress parameters represent the approximation methods for the experimental data. The generalized, or Bergmann-Walker parameter $\mathbf{P}_{BW}$ (10.11a) uses two fitting parameters instead of only one of the traditional parameters. Consequently, parameter $\mathbf{P}_{BW}$ covers much wider range of mean stress variation. This parameter has the advantages of the smoothness of stress isolines and easiness for the tabular representation of two fitting coefficients μ_r and $\tilde{\gamma}$.

10.2.3 Strain-Life Approach with Variable Stress Ratio

The constant strain tests are easy to control and the testing equipment is simple. Thus, these tests are applicable for the estimations of fatigue life of common springs in industrial applications.

The strain-life approach presumes the dominant plastic strains over a load cycle. The plastic strain amplitude ε_p is assumed constant during the cyclic test. The plastic strain amplitude is roughly constant in low cycle fatigue regime, where the Coffin-Manson law is suitable.

For zero mean stress, the Coffin-Manson law relates the number of cycles to failure upon the plastic fully-reversed strain amplitude (Manson 1953; Coffin 1954):

$$\varepsilon_{ar} = \frac{\sigma'_f(R_\sigma)}{E} (2n_f)^{b(R_\sigma)} + \varepsilon'_f(R_\sigma) \cdot (2n_f)^{c(R_\sigma)}, \quad \varepsilon_a = \frac{\sigma_a}{E} + \left(\frac{\sigma_a}{K'_\sigma} \right)^{1/n'} \quad (10.13)$$

In Eq. (10.13) we use the following parameters:

- $\sigma'_f(R_\sigma)$ fatigue strength coefficient,
- $b(R_\sigma)$ fatigue strength exponent,
- $\varepsilon'_f(R_\sigma)$ fatigue ductility coefficient,
- $c(R_\sigma)$ fatigue ductility exponent, $c_0 = c(R_\sigma = -1)$.

The quantities $\sigma'_f(R_\sigma)$ and $b(R_\sigma)$ are the same as in Eqs. (10.1a, 10.1b, 10.1c, 10.1d) and $\varepsilon'_f(R_\sigma)$ and $c(R_\sigma)$ are additional fitting constants for the plastic strain term. The fatigue strength exponent is commonly accepted to be independent upon stress ratio R_σ : $c(R_\sigma) = c_0$.

The coefficient ε'_{f0} of Eqs. (10.4a, 10.4b) appears as the limit case:

$$\varepsilon'_f(R_\sigma = -1) = \varepsilon'_{f0}.$$

The second equation (10.13) represents the cyclic stress-strain curve with two fitting constants n' : and K'_σ . The constant K'_σ the cyclic hardening coefficient and n' is the cyclic hardening exponent. Similar Ramberg-Osgood representation, but for the static hardening curve, was used in Chap. 7.

The corresponding formulas are analogues to Eq. (10.13) with the evident replacements:

$$\gamma_{ar} = \frac{\tau'_f(R_\sigma)}{G} (2n_f)^{b(R_\sigma)} + \gamma'_f(R_\sigma) \cdot (2n_f)^{c(R_\sigma)}, \quad \gamma_{ar} = \frac{\tau_a}{G} + \left(\frac{\tau_a}{K'_\tau} \right)^{1/n'}. \quad (10.14)$$

As mentioned above, the fatigue exponents demonstrate no significant sensitivity to variations of stress ratio. The fatigue exponents are accepted to be constant:

$$b(R_\sigma) = b_0, \quad c(R_\sigma) = c_0.$$

The situation with the strain-life approach is similar to those of stress-life approach. Theoretically, the strain-life testing must be performed for different stress amplitudes and for different maximal strain rates. The testing is an expensive matter and the engineer wish to reduce the number of test series. For this purpose, the common hypotheses for dependence upon stress ratio are used. The material fatigue test is performed for a given fixed stress ratio, usually for a fully-reversed sinusoidal test $R_\sigma = -1$. Alternatively, for the helical compression spring one standard spring design will be fixed. For this standard design the stress ratio is some constant. The fatigue experiments are performed for the standard design and extrapolated for different design with some usual hypotheses for stress ratio functions.

There are several hypotheses to account the dependence upon stress ratio.

1°. The simplest approach, according to Morrow (1968) , is:

$$\sigma'_f(R_\sigma) = \left(1 - \frac{\sigma_m}{\sigma'_{f0}}\right) \sigma'_{f0}, \quad \varepsilon'_f(R_\sigma) = \varepsilon'_{f0}, \quad \sigma_a = \sigma_{ar} \left(1 - \frac{\sigma_m}{\sigma'_{f0}}\right). \quad (10.15)$$

Accordingly, for nonzero mean stress, the elastic term of Coffin-Manson law was modified to account the mean stress (Morrow 1968, Eq. (3.16), p. 27):

$$\varepsilon_a = \left(1 - \frac{\sigma_m}{\sigma'_{f0}}\right) \frac{\sigma'_{f0}}{E} (2n_f)^{b_0} + \varepsilon'_f (2n_f)^{c_0}. \quad (10.16)$$

2°. Manson and Halford (1981) accounted the mean stress in both elastic and plastic terms:

$$\varepsilon_a = \left(1 - \frac{\sigma_m}{\sigma'_{f0}}\right) \frac{\sigma'_{f0}}{E} (2n_f)^{b_0} + \varepsilon'_f \left(1 - \frac{\sigma_m}{\sigma'_{f0}}\right)^{c_0/b_0} (2n_f)^{c_0}. \quad (10.17)$$

3°. The similar strain-life approach accounts the comparable values of elastic and plastic strains over a load cycle (Socie and Morrow 1980) . Finally, the expression of the effective fully-reversed strain (Bannantine and Socie 1989) is:

$$\varepsilon_{ar} = \sqrt{\varepsilon_a \sigma_{\max}} = \sqrt{\varepsilon_a \sigma_{\max}} = \sqrt{\frac{(\sigma'_{f0})^2}{E} (2n_f)^{2b_0} + \sigma'_{f0} \varepsilon'_f (2n_f)^{b_0+c_0}}. \quad (10.18)$$

4°. To account for the stress ratio effect on fatigue life, a adjustment of the Morrow and the Smith, Watson, Topper mean stress correction models was proposed in Ince and Glinka (2011). This approach improves the estimation of the fatigue life in low-cycle-fatigue regime. The proposed modification is based on the strain-life approach. The model improves the low-cycle fatigue calculations. The capability and accuracy of the proposed model were compared to those

of the original Morrow rules and $\mathbf{P}_{\text{SWT}}$ using available fatigue test data with different mean stress. The mean stress correction model was found to provide a better correlation, than $\mathbf{P}_{\text{SWT}}$ and the Morrow rules in the case of a superalloy and the ASTM A723 steel.

- 5°. An alternative approach (Kwofie 2001; Kwofie and Chandler 2007) represents the dependence of coefficients (10.1a, 10.1b, 10.1c, 10.1d) and (10.13) upon stress-ratio in form of the exponential function:

$$\begin{aligned}\sigma'_f(R_\sigma) &= \sigma'_{f0} \exp\left(-\lambda_d \frac{\sigma_m}{\sigma_a}\right) = \sigma'_{f0} \exp\left(-\lambda_d \frac{1 + R_\sigma}{1 - R_\sigma}\right), \\ \varepsilon'_f(R_\sigma) &= \varepsilon'_{f0},\end{aligned}\tag{10.19a}$$

$$\begin{aligned}b(R_\sigma) &= b_0 \exp\left(-\lambda_d \frac{\sigma_m}{\sigma_a}\right) = b_0 \exp\left(-\lambda_d \frac{1 + R_\sigma}{1 - R_\sigma}\right), \\ c(R_\sigma) &= c_0.\end{aligned}\tag{10.19b}$$

The value λ_d is the fitting constant, representing the mean stress sensitivity of the material. This constant is for the tested materials (Kwofie and Chandler 2007) of the order $\lambda_d \approx 1$.

For explanation of Coffin-Manson law (Sornette et al. 1992) proposed a theory, which is based on the observation of the “mesoscopic” grain scale, intermediate between the dislocation scale and the macroscopic crack scale. Well known, that under extremely low cycle fatigue conditions the Coffin-Manson law tends to over-predict the cyclic life.

- 5°. The variations of range ΔK and the mean value K_m of stress intensity factor over the load history could be accounted by applications of damage accumulation theories (Sorensen 1969; Wheeler 1972).

10.3 Stress Ratio Influence on Fatigue Crack Growth

10.3.1 Influence of Stress Ratio on Fatigue Threshold

As already mentioned, the stress ratio is defined as the ratio of the algebraically minimum over the maximum stress. The experiments demonstrate that the stress ratio affects the fatigue crack growth and threshold behavior. Namely, the fatigue crack propagation rate and threshold value vary with the applied stress ratio (Walker 1970). If the stress ratio is positive, the experiments reveal that the necessary range of stress intensity factor decreases with increasing positive values. In the region of the negative stress ratio, the required stress intensity factor range for growth (threshold stress of intensity range) decreases as stress ratio decreases. Reaching a definite negative value, known as saturation point, the required range of stress intensity factor stabilizes.

Consider now the variation of stress ratio for the proposed propagation laws. SAE AE-22 (1997, Sect. 9.4.4) uses the Eqs. (10.32) and (10.33) for estimation of stress-ratio effect.

In accordance with the phenomenological approach (Klesnil and Lukas 1972), the damage growths per cycle depends upon the stress ratio:

$$c_f(R_\sigma) = (1 - R_\sigma)^{1-\zeta} \cdot c_f(R_\sigma = -1). \quad (10.20)$$

The exponent ζ accounts the influence of the stress ratio on the crack growth rate, $0 < R_\sigma < 1$ is an arbitrary stress ratio.

According to Forman et al. (1967), the crack closure decrease the fatigue crack growth rate by reducing the short-term threshold:

$$K_{IC}(R_\sigma) = (1 - R_\sigma)^{\gamma_2} K_{IC,0} \quad (10.21)$$

is the function of the stress ratio R_σ , where

$$K_{IC,0} \stackrel{\text{def}}{=} K_{IC}(R_\sigma = 0)$$

is the threshold value at $R_\sigma = 0$ and γ_2 is a material dependent constant. The model (Forman et al. 1967) leads to the extension of the Paris equation, considering the fracture toughness K_{crit} :

$$\frac{da}{dn} = c_f \cdot \frac{[\Delta K]^p}{(1 - R_\sigma) \cdot K_{\text{crit}} - \Delta K}.$$

Hartman and Schijve (1970) suggested that da/dn should be dependent on the amount by which ΔK exceeds the fatigue threshold K_{th} :

$$\frac{da}{dn} = c_f \cdot \frac{[\Delta K - K_{th}]^p}{(1 - R_\sigma) \cdot K_{\text{crit}} - \Delta K}.$$

Branco et al. (1976) proposed an alternative relationship:

$$K_{IC}(R_\sigma) = \sqrt{1 - R_\sigma^2} K_{IC,0}$$

The endurance threshold limit might be also considered as a function of stress ratio:

$$K_{th}(R_\sigma) = (1 - R_\sigma)^{\gamma_1} K_{th,0}. \quad (10.22)$$

Here

$$K_{th,0} \stackrel{\text{def}}{=} K_{th}(R_\sigma = 0)$$

is the endurance threshold value at $R_\sigma = 0$ and γ_1 is a material dependent constant.

The mentioned above Eqs. (10.20)–(10.22) are the empirically determined phenomenological dependences (Kondo et al. 2003).

An equivalent driving force approach was proposed by one of the current authors for correlation of growth data for fatigue crack at different stress ratios (Kwofie and Zhu 2019).

10.3.2 Analytical Models for the Stress Ratio

The closed form solution for the cycles to failure for a given initial crack length upon the stress amplitude could be found for given stress ratio for both type I and type II of unified laws. Instead of an empirical approach, we use the Eqs. (10.12a, 10.12b, 10.12c, 10.12d) to derive the influence of the stress ratio on the crack growth rate explicitly. For this purpose, consider two harmonically varying loads with equal amplitude, but with the different mean values. The mean values and amplitudes are assumed to be positive, such that:

$$K_m > 0, K_m^* > 0, \Delta K > 0.$$

Both loads lead to the harmonically varying stress intensity factors. The harmonically varying stress intensity factors correspondingly are:

$$K(t) = K_m + \frac{\Delta K}{2} \sin t, \quad K^*(t) = K_m^* + \frac{\Delta K}{2} \sin t. \quad (10.23)$$

For both considered cyclic loads (10.23) the ratios are equal correspondingly to:

$$R_\sigma = \frac{2K_m - \Delta K}{2K_m + \Delta K}, \quad R_\sigma^* = \frac{2K_m^* - \Delta K}{2K_m^* + \Delta K}.$$

Consider at first the positive load cycles, according to Eq. (10.23):

$$K(t) > 0, \quad K^*(t) > 0.$$

For the positive load cycles the stress ratios are in the ranges:

$$-1 < R_\sigma < 1, \quad -1 < R_\sigma^* < 1$$

The damage caused by the varying load depends linearly upon the stress intensity factor in the power p along the cycle. Consequently, the relation between fatigue coefficients, which correspond to the different stress ratios $c_f(R_\sigma)$ and $c_f(R_\sigma^*)$, reduces to (Kobeleev 2017c, d):

$$\Lambda[R_\sigma, R_\sigma^*] = \frac{c_f(R_\sigma)}{c_f(R_\sigma^*)}, \quad (10.24a)$$

$$c_f(R_\sigma) = \left[\frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{(1 + R_\sigma \sin t)^p} \right]^{-1} \cdot c_f(R_\sigma = -1). \quad (10.24b)$$

$$c_f(R_\sigma^*) = \left[\frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{(1 + R_\sigma^* \sin t)^p} \right]^{-1} \cdot c_f(R_\sigma = -1). \quad (10.24c)$$

The closed-form expression of integrals in (10.24b)–(10.24c) is:

$$\Upsilon(R_\sigma) = \frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{(1 + R_\sigma \sin t)^p} = {}_2F_1\left(\left[\frac{1+p}{2}, \frac{p}{2}\right][1], R_\sigma^2\right) = \frac{P_{-p}\left(\frac{1}{\sqrt{1-R_\sigma^2}}\right)}{(1 - R_\sigma^2)^{p/2}}, \quad (10.25)$$

$$\Upsilon(0) = 1.$$

Here P_p is the Legendre function and ${}_2F_1$ is the hypergeometric function : (Appendix F).

Consequently, the dependence of the fatigue coefficient upon the stress ratio the for positive load cycles reads:

$$\Lambda[R_\sigma, R_\sigma^*] = \frac{\Upsilon(R_\sigma)}{\Upsilon(R_\sigma^*)} = \left[\frac{1 - R_\sigma^2}{1 - R_\sigma^{*2}} \right]^{p/2} \cdot \frac{P_{-p}\left((1 - R_\sigma^{*2})^{-1/2}\right)}{P_{-p}\left((1 - R_\sigma^2)^{-1/2}\right)}. \quad (10.26)$$

The plot of the function (10.26) for different values of p is shown on the Fig. 10.7.

With the function (10.25) the mean sensitivity follows directly from the first principles. The Eq. (10.26) is based on the Basquin law and summarizes the damage on each cycle. The Eq. (10.26) rescales the material constant from a certain stress ratio R_σ to the material constant for the changed stress ratio R_σ^* .

10.3.3 Stress Ratio Influence in NASGRO Model

Thirdly, we extend the validity region for the fatigue coefficient upon the stress ratio. For this purpose, the effective stress intensity concept is implemented, following the results (Elber 1970, 1971). In these papers, the fatigue crack in a center-cracked-tension panel subjected to zero-to-tension loading was studied. It was demonstrated that a fatigue crack is fully open for only a part of the loading cycle. This effect was attributed to contact of residual plastic deformation around the crack tip. The cause for plasticity induced crack closure is the plastic wake, which is developed

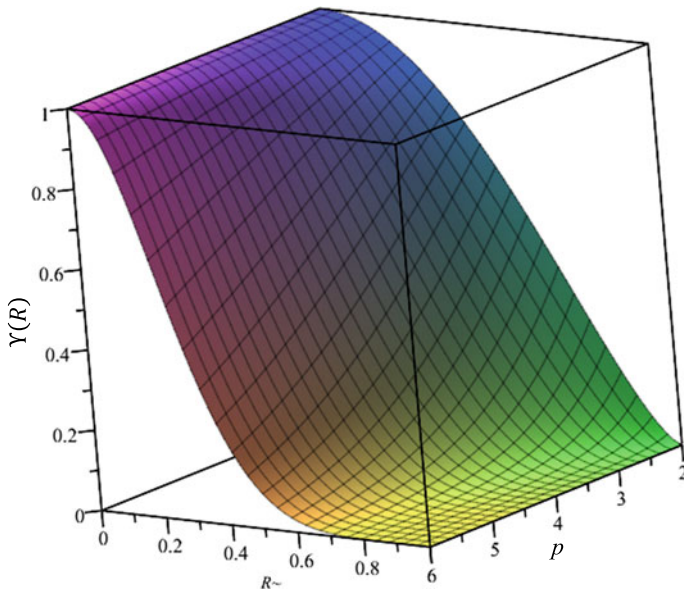


Fig. 10.7 The plot of the function $\Upsilon(R)$ for different values of p

during crack propagation. The faces of crack could be subjected to considerable compressive stresses. In the article (Boyce and Ritchie 2001) was proposed that the crack tip stress intensity factor range, must be modified by an experimental factor. The experimental factor accounts the exposed crack closure effect.

AFGROW code was developed by NASA (Harter 1999). This code is used for simulation of fatigue crack growth. In AFGROW code, strain-life based crack initiation analysis method to predict crack initiation life is incorporated. In fatigue case and at the notch tip, local strains are obtained by using the Neuber's rule (Neuber 1961a, b) or (Glinka 1987) expressed. The NASGRO model is applied, when entirety of curves, which describe fatigue crack growth, is evaluated. The standard NASGRO model reads:

$$\frac{da}{dn} = c_f \cdot \left[\left(\frac{1 - \tilde{f}(R_\sigma)}{1 - R_\sigma} \right) \Delta K \right]^p \cdot \frac{\left(1 - \frac{K_{th}}{\Delta K} \right)^{m_1}}{\left(1 - \frac{K_{max}}{K_{crit}} \right)^{m_2}} \quad (10.27)$$

The function $\tilde{f}$ presents the contribution of crack closure, Newmann's crack opening function (Newman 1984). For constant amplitude loading, the function $\tilde{f}$ can be written as:

$$\tilde{f}(R_\sigma) = \max(R_\sigma, A_0 + A_1 R_\sigma + A_2 R_\sigma^2 + A_3 R_\sigma^3) \quad (10.28)$$

The parameters c_f, p, m_1, m_2 are determined experimentally and K_{th} is the threshold value of the stress–intensity factor range for crack propagation. The basic mechanical properties for technical alloys are given in DOT (2005).

Reworking the Eq. (10.27) in form of the unified propagation law (10.23) leads to the

$$\frac{da}{dn} = c_f \cdot \left[\left(\frac{1 - \tilde{f}(R_\sigma)}{1 - R_\sigma} \right) \Delta K \right]^p \cdot \frac{\left(1 - \frac{K_{th}}{\Delta K} \right)^{m_1}}{\left(1 - \frac{K_{max}}{K_{crit}} \right)^{m_2}} \quad (10.29)$$

The formula (10.29) depends on both K_{max} and ΔK . Substitution of the expression

$$K_{max} = \frac{1}{1 - R_\sigma} \frac{\Delta K}{2}.$$

into (10.29) leads to the equation depending solely on range ΔK :

$$\frac{da}{dn} = c_f \left[\frac{1 - \tilde{f}(R_\sigma)}{1 - R_\sigma} \right]^p (\Delta K)^p \cdot \frac{\left(1 - \frac{K_{th}}{\Delta K} \right)^{m_1}}{\left(1 - \frac{\Delta K}{2(1 - R_\sigma)K_{crit}} \right)^{m_2}}. \quad (10.30)$$

The expression (10.30) could be further rephrased in the form of the type II unified propagation function with the exponent $k = 1$.

In this case, the coefficients of the type II unified propagation function are the functions of stress ratio R_σ :

$$\tilde{c}_f(R_\sigma) = c_f \cdot \left[\frac{1 - f(R_\sigma)}{1 - R_\sigma} \right]^p, \quad \tilde{K}_2(R_\sigma) = 2(1 - R_\sigma)K_{crit}. \quad (10.31)$$

With the above expressions (10.31), the NASGRO assumes the form:

$$\frac{da}{dn} = \tilde{c}_f(R_\sigma) \Delta K^p \cdot \frac{\left(1 - K_{th}/\Delta K \right)^{m_1}}{\left(1 - \Delta K/\tilde{K}_2 \right)^{m_2}}. \quad (10.32)$$

Comparison with (10.26) states, that the following substitution reveals the dependence of the coefficients of the unified propagation function upon stress ratio R_σ :

$$c_f \rightarrow \tilde{c}_f(R_\sigma), \quad K_2 \rightarrow \tilde{K}_2(R_\sigma), \quad k \rightarrow 1.$$

Consequently, the NASGRO model (10.27) and (10.32) transforms to the form of unified propagation function type II (10.26) with the explicit dependence upon stress ratio R_σ :

$$\frac{da}{dn} = \tilde{c}_f \tilde{F}_{II}(\Delta K), \quad \tilde{F}_{II}(\Delta K) = \Delta K^p \cdot \frac{(1 - K_{th}/\Delta K)^{m_1}}{(1 - \Delta K/\tilde{K}_2)^{m_2}}. \quad (10.33)$$

The expressions for the crack length evolution (10.27)–(10.43) were derived in Chap. 9 for fatigue law with the unified propagation function type II, with the constant $R_\sigma = 0$. Finally, all derived expressions are valid for an arbitrary R_σ with the formulas (10.31) and $k = 1$.

10.4 Influence of Defects on Fatigue Resistance

10.4.1 Influence of Area of Defects for Uniaxial Stress

The initiation of the fatigue and fatigue at high number of cycles profit of the observation, that limit of most materials is not sensitive to the critical stress for crack initiation (Murakami 2003).

When inclusions or another material imperfection does not affect fatigue crack initiation, fatigue crack initiates from slip bands, which are formed on the material surface. In this case, the linear relationship exists between material strength or hardness and fatigue limit. Industry uses traditionally the relation (Garwood et al. 1951):

$$\sigma_w \approx 0.45 \dots 0.5 \cdot \sigma_u \approx 1.5 \dots 1.4 \cdot HV \text{ for } HV < 400.$$

The recent research (Pang et al. 2013) demonstrates the quadratic dependence of the fatigue strength σ_w to tensile strength σ_u :

$$\sigma_w \approx (0.70 - 1.85 \cdot 10^{-4} \sigma_u) \cdot \sigma_u.$$

The most important factor is the threshold stress for arrest of crack propagation. This threshold stress originates from a defect or from an inhomogeneity, from a non-metallic inclusion or from an original flaw, but not by the microstructure itself. If thermal and surface treatment are properly performed, they improve the microstructure of materials. In this case, fatigue limit could be observed only in a narrow range of applied stress. The important parameter *area* is the area of the defect, projected on the plane perpendicular to the directions of the maximum principal stress. The fatigue analysis accounts only the endurance threshold limit K_{th} and the original defect size of the value $\sim \sqrt{area}$. The endurance threshold limit K_{th} correlates with the value $\sqrt{area}$. The three-dimensional crack analysis (Murakami 1985), demonstrates a strong correlation between K_{th} along the crack front and $\sqrt{area}$:

$$K_{th} \sim (\sqrt{area})^{1/3}.$$

In terms of Vickers hardness HV , the formula for the endurance threshold limit reads:

$$K_{th} = 3.3 \cdot 10^{-3} \cdot (HV + 120) (\sqrt{area})^{1/3}.$$

The expression the endurance threshold limit K_{th} in terms of fatigue limit stress σ_w is (Murakami and Isida 1985):

$$K_{th} = 0.65 \cdot \sigma_w \sqrt{\pi \sqrt{area}}.$$

Combination of the above equations yields the fatigue limit stress in terms of the inclusion size:

$$\sigma_w = \frac{1.43(HV + 120)}{(\sqrt{area})^{1/6}}. \quad (10.34)$$

The units for the application are shown in the Table 10.4. The above analysis applies for helical springs at very high number of cycles, particularly for valve and injection pump springs in internal combustion engines.

The fatigue analysis accounts only the endurance threshold limit K_{th} and the original defect size $\sim \sqrt{area}$, where $area$ is the area of the defect, projected on the plane perpendicular to the directions of the maximum principal stress. The model (Murakami and Beretta 1999; Takahashi and Murakami 2002) predicts the lower limit of fatigue limit for the case of uniaxial tension load:

$$\sigma_w = C_0 \frac{HV + 120}{(\sqrt{area})^{1/6}} \left(\frac{1 - R_\sigma}{2} \right)^{\zeta_{HV}}, \quad \zeta_{HV} = 0.226 + 10^{-4} \cdot HV. \quad (10.35)$$

C_0 is a dimensionless constant depending on whether the inclusion is located on the surface or inside the probe (Murakami 1989). For surface cracks or inclusions this constant is $C_0 = 1.43$, for subsurface inclusions $C_0 = 1.40$ and for interior inclusions $C_0 = 1.56$. The Eq. (10.34) accounts the stress ratio and generalizes the Eqs. (10.47a, 10.47b).

10.4.2 Influence of Area of Defects for Shear Stress

Gadouini and Nadot (2007) studied the evaluation of defect tolerance design under multi-axial fatigue loading. The method permits the determination of the maximum defect size, which is tolerable for a component for a given fatigue life. The idea is to represent the defect by the mean of the stress gradient at the mesoscopic scale using kinematic hardening. This approach was used to model in multiaxial cyclic stress fields. Gadouini et al. (2008) studied the influence of mean stress on the fatigue

limit of steel containing spherical artificial defects under tension and torsion. For the shear stress fields the new parameter $area_P$ was introduced. This parameter describes the area of the defect projected on the plane perpendicular to the directions of the maximum tensile stress (45° to the direction of shear stress). For torsion loading, the fatigue limit is given by the following equation (Gadouini et al. 2008):

$$\tau_w = \frac{C_0(HV + 120)}{F_{GNR}(b/a)(\sqrt{area})^{1/6}} \left(\frac{1 - R_\sigma}{2} \right)^{\zeta_{HV}}, \quad (10.36)$$

where

$$F_{GNR}(b/a) = 0.095 + 2.11 \cdot \left(\frac{b}{a} \right) - 2.26 \cdot \left(\frac{b}{a} \right)^2 + 1.09 \cdot \left(\frac{b}{a} \right)^3 - 0.196 \cdot \left(\frac{b}{a} \right)^4,$$

- a material parameter in Billaudeau's criterion describing defect influence (μm) (Billaudeau et al. 2004),
- b material parameter in Chaboche's model (Chaboche 1989).

10.4.3 Influence of Corrosion-Induced Defects

Experimental investigations display; that the fatigue strength decreases sharply as corrosion of the component progresses (AIF 5996 1988). Corrosion is responsible for the formation of cracks and the support of crack growth in fatigue-stressed components. The investigations (Schnattinger and Beste 1995; Angelova et al. 2014) confirm; that the vibration resistance under the medium used, 5%-NaCl solution, is greatly reduced compared to the reference Wöhler line in air. The mechanical properties of tested spring steels were reported in Angelova et al. (2014) and Schnattinger and Beste (1995) (Tables 10.5 and 10.6). Figure 10.8 shows the graph of the shear strain amplitudes γ to failure as the function of load cycles n_f (Angelova et al. 2014).

Figure 10.9 displays the dependence of the shear stress amplitudes τ to failure as the function of load cycles n_f (Angelova et al. 2014).

Figure 10.10 displays S-N curves for τ_{ar} according to Smith, Watson and Topper P_{SWT} , Eq. (10.3) (Schnattinger and Beste 1995). The test results determined under corrosion are far below the limit values required by the standards. The influence of a corrosive medium on the tolerable load capacity is currently not considered in the analytical design guidelines. Therefore, investigations into an increase in vibration resistance in corrosive environments are recommended. The thermal processes (for example, induction hardening) could increase the fatigue strength of steels in corrosive ambient media by introducing residual compressive stresses.

The fatigue lifetime was investigated for different spring manufacturing technologies and surface protection systems in Hoffmann et al. (2015). The relevant operating effects (deflection, abrasion, grit impact and corrosion) were considered. The substantial reduction of the fatigue strength can be determined by means of

Table 10.5 Mechanical properties of spring steels (Kong et al. 2019; Boardman 1982; Angelova et al. 2014)

Properties	Units	SAE 5160	SAE 4340	EN 10270-1 SH				BS 250A53	
DIN				17223C				55Si7	
Source/year			Kong et al. (2019)	Boardman (1982)				Angelova et al. (2014)	
				BHN Prediction	Su Prediction	Regression	Air	5% NaCl	5% NaCl
Ultimate tensile strength	MPa	σ_w	1584	826			1522	1610	
Yield strength	MPa		1487	634			1217	1440	
Modulus of elasticity	GPa	E	205	193					
Shear modulus	GPa	G					78		
Fatigue strength coefficient	MPa	σ'_f	2063	1180	1088	1232			
Fatigue strength coefficient	MPa	τ'_f					3.944	19.649	1.8875
Fatigue strength exponent		b	-0.08	-0.075	-0.070	-0.10	-0.1465	-0.368	-0.0449
Fatigue ductility exponent		c	-1.05	-0.6	-0.574	-0.56			
Fatigue ductility coefficient		ϵ'_f	9.56	0.70	0.57	0.53			
Hardening coefficient	MPa	K'	1940	1234	1165	1384			
Hardening exponent		n'	0.05	0.125	0.122	0.17			

Table 10.6 Effective fatigue life equations for different environments and coating (Schnattinger and Beste 1995; Angelova et al. 2014)

Spring coating	Material	Environment	Fatigue life equation
Uncoated spring	Suspension spring material Schnattinger and Beste (1995)	Dry	$\tau_{ar} = 2.214 \cdot 10^3 \cdot n_f^{-0.0854} (P_{SWT})$
		Corrosive NaCl, 5%	$\tau_{ar} = 4.28 \cdot 10^4 \cdot n_f^{-0.38461} (P_{SWT})$
Coated spring			$\tau_{ar} = 3.276 \cdot 10^3 \cdot n_f^{-0.125} (P_{SWT})$
Uncoated spring	EN 10270-1 SH Angelova et al. (2014)	Dry	$\tau_c = 3.563 \cdot 10^3 \cdot n_f^{-0.1465}$
		Corrosive, NaCl, 5%	$\tau_c = 1.522 \cdot 10^4 \cdot n_f^{-0.368}$
	BS 250A53 Angelova et al. (2014)	Dry	$\tau_c = 1.829 \cdot 10^3 \cdot n_f^{-0.0449}$
		Corrosive, NaCl, 5%	$\tau_c = 3.91 \cdot 10^4 \cdot n_f^{-0.3347}$

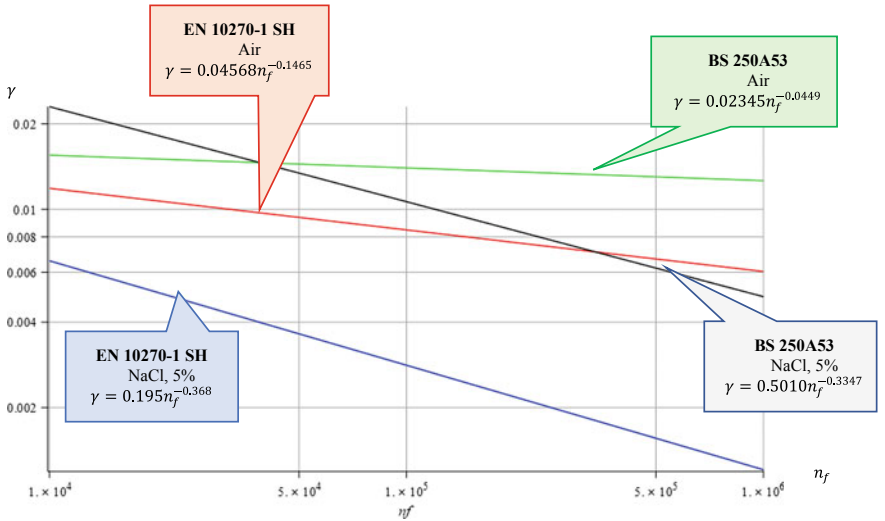


Fig. 10.8 Graph of the shear strain amplitudes γ to failure as the function of load cycles n_f (Angelova et al. 2014)

mechanical damage of the spring surface and subsequent corrosion. The double-coating or a thick film coating prevents the lifetime decrease due to corrosion and mechanical damage.

Li et al. (2006) reports the study of fatigue behavior of 14 disc springs types under corrosive conditions. The report covers the results of investigation for the uncoated

Table 10.7 Experimental data high strength steels (Chapetti 2010), T-C: Tension-Compression, R-B: Rotating-Bending, Tor: torsion, HEL: helical spring

Steel	Source	Range of cycles	Test
SCM435	Murakami et al. (1999)	10^8	T-C
0.46Carbon	Murakami et al. (1998)	$5 \cdot 10^8$	T-C
SUJ2	Shiozawa et al. (2001)	10^{10}	R-B
SCM435 H	Nakamura et al. (1998)	10^8	T-C
SNM439	Furuya et al. (2002)	10^{10}	T-C
Cr-V	Almaraz (2008)	10^{10}	T-C
Cr-Si (54SC6)		$5 \cdot 10^9$	T-C
Cr-Si (55SC7)		10^{10}	T-C
Cr-Si (55SC7)		$3 \cdot 10^9$	T-C
SUP7	Murakami et al. (1988)	10^{10}	R-B
SUP7		10^8	T-C
SUP7		10^8	T-C
SUP10M	Wang et al. (1999)	$0.2 \cdot 10^8$	T-C
SUP12	Abe et al. (2001)	10^8	T-C
SWOSC-V		10^8	T-C
100C6Mart	Murakami et al. (2006)	$7 \cdot 10^9$	T-C
100C6Bain		10^{10}	T-C
SUJ2		10^9	
VDSiCr, TDSiCr, DH, DM, 1.4568, 1.4310	AiF 12287N (2002)	10^7	HEL
VDSiCr	Kaiser et al. (2011)	$1.5 \cdot 10^9$	HEL
VDSiCr, VDSiCrV, X7CrNiAl17-7 (1.4568)	AiF 15064N (2010)	10^9	HEL
VDSiCr, VDSiCr (WBH), DH, 1.4310, 1.4568 (conv, new)	(IGF16873)	$\sim 10^9$	HEL
VDSiCr	Mayer et al. (2015, 2016)	$5 \cdot 10^9 - 1.4 \cdot 10^{10}$	Tor

springs, made of stainless steel as well as coated springs, made of tempered spring steel, both of them under the influence of different corrosive media.

10.5 Influence of Multiaxial Stresses

One of the most important requirements for the properly designed metal springs is the absence of the multiaxial stress. The properly designed spring must withstand one type of stress, for example, pure shear or pure tensile stress. Consequently, in the majority of cases the simplest forms of stress tensor are assumed. However,

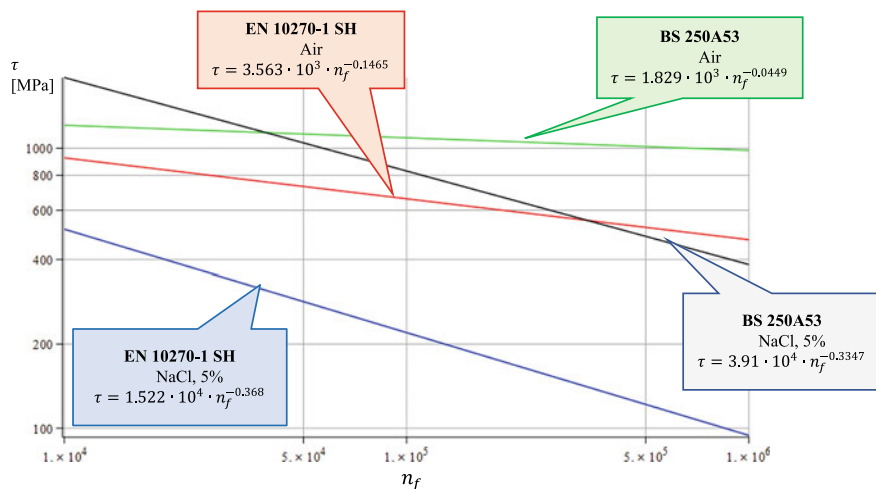


Fig. 10.9 Graph of the shear stress amplitudes τ to failure as the function of load cycles n_f (Angelova et al. 2014)

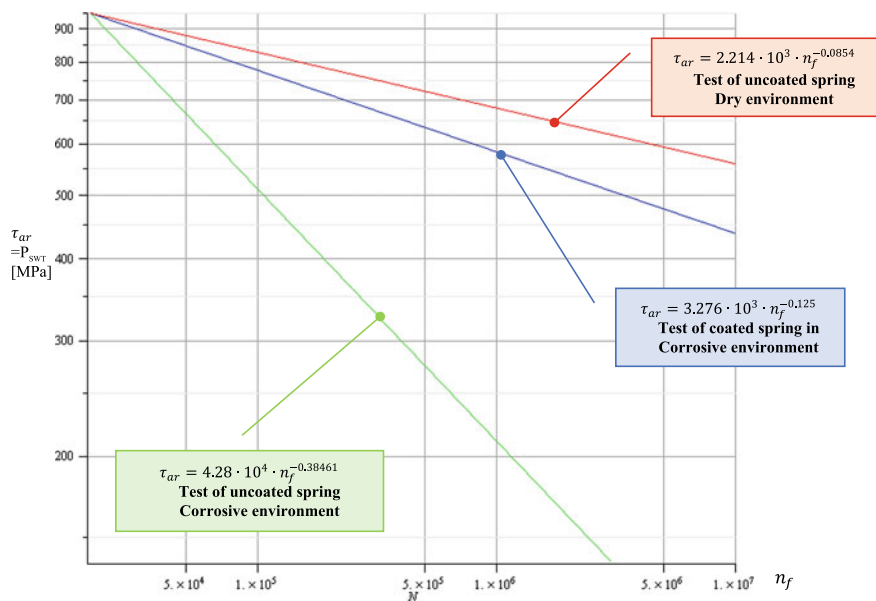


Fig. 10.10 S-N curves for τ_{ar} according to Smith, Watson and Topper P_{SWT} , Eq. (10.3) (Schnattinger and Beste 1995)

fluctuating loads of different amplitudes and frequencies in two or three directions expose the springs under real practical conditions. For example, the automotive springs subjected to axial load. The bending loads occasionally occur during the cornering. The axle spring of the rear suspension is ideally in the state of pure axial load, but the large deformation leads to the bending of spring. The shorter length of the suspension arms favours the excessive bending of springs. Occasionally, the springs are specifically designed for multiple stress cases, for example, the simultaneous torque and axial loads. The axial load causes the shear stresses in the wire. In the same time, the bending stress leads to the normal stresses. Consequently, the stresses in springs are the combination of pure normal stress and pure shear stress. In this case, the stress tensor holds several non-zero components. Such stress fields cause biaxial or multiaxial fatigue (Brawn 1989; Fateimi and Socie 1989). The evaluation of fatigue life under biaxial or multiaxial stresses is an important design requirement for safe and reliable functionality of structural members (ESIS 31 2003).

In ideal case, possible loads systems in properly designed springs correlate. Nevertheless, the uncorrelated or random loading could occur under real exploitation environments. For multiaxial fatigue damage evaluation under random loading, three common steps should be included: multiaxial cycle counting, multiaxial fatigue damage evaluation for each counted cycle, and fatigue damage accumulation for all counted cycles. Table 10.7 from Chapetti (2010) reviews the experimental data for high strength steels.

10.5.1 Stress-Life Approach in Multiaxial Fatigue

10.5.1.1 Proportionally Oscillating Stress Tensor Fields

For the oscillating stress fields, the proper definitions of a scalar, fully reversed equivalent fatigue stress is essential. For the assessment of stress- and strain-based methods, the several definitions for equivalent parameters for multiaxial fatigue were introduced.

Such behavior of stresses is typical for the springs, which are loaded by oscillating forces, for example for the springs in the automotive suspensions. The torque and side loads of the twist beam suspension do not correlate. If properly designed, the suspension springs carry a single axial load. Sometimes, however, the stiffness of the suspension is somewhat low and the side forces act on the spring. The circumferential and radial stresses in disk springs, as discussed in Chap. 4, oscillate synchronously, but their amplitudes and stress ratios differ. Generally, if the spring is subjected to predefined external oscillating forces, the stress-life approach is recommended for fatigue life evaluation. Typically for this kind of load, that the sprig length slowly reduces or elongates over the exploitation period. In other words, the strain will be the sum of an oscillation component and the mean strain, slowly increasing with time. The cyclic creep occurs in the spring material.

For this purpose, consider time-dependent principal stresses

$$\sigma_i(t) = \sigma_{m,i} \cdot (1 + A_i \cdot \sin(\omega t)). \quad (10.37)$$

In Eq. (10.37) the amplitude ratios (ASM 2008, Eq. 14.5) for each principal stress are:

$$A_i = \frac{\sigma_{a,i}}{\sigma_{m,i}}, \quad i = 1 \dots 3$$

In the simplest case of multiaxial loading, all three amplitude ratios are equal:

$$A \equiv A_1 = A_2 = A_3, \quad A = \frac{1 - R_\sigma}{1 + R_\sigma}.$$

In this case, all principal stresses oscillate synchronously and proportionally to a harmonic function $1 + A \cdot \sin(\omega t)$ of time:

$$\sigma_i(t) = \sigma_{m,i} \cdot (1 + A \cdot \sin(\omega t)). \quad (10.38)$$

In this case, the effective stress is also the harmonic function of time:

$$\sigma_{eq,ar}(t) = \sigma_{eq,m} \cdot (1 + A \cdot \sin(\omega t)), \quad (10.39)$$

$$\sigma_{eq,m} = \frac{1}{\sqrt{2}} \sqrt{(\sigma_{m,1} - \sigma_{m,2})^2 + (\sigma_{m,2} - \sigma_{m,3})^2 + (\sigma_{m,3} - \sigma_{m,1})^2}$$

is the mean value of equivalent fully reversed fatigue stress.

For the synchronously harmonically oscillating stress tensor fields, the Goodman diagram uses the following axes:

$$\sigma_{eq,min} = \sigma_{eq,m} - \sigma_{eq,a}, \quad \sigma_{eq,max} = \sigma_{eq,m} + \sigma_{eq,a}, \quad (10.40)$$

here $\sigma_{eq,a} = A \cdot \sigma_{eq,m}$ is the amplitude of the of equivalent fully reversed fatigue stress (Table 10.2). Goodman diagrams are plotted in terms the scalar effective stresses $\{\sigma_{eq,a}, \sigma_{eq,m}\}$.

As far as the fatigue equivalent stress is defined, the number of cycles to total failure for tensor stress fields are the functions of effective stress $\sigma_{eq,ar}$:

$$n_f = f_{eq}^{-1}(\sigma_{eq,ar}) \equiv \frac{1}{2} \left(\frac{\sigma_{eq,ar}}{\sigma'_{eq,f}(R_\sigma)} \right)^{1/b(R_\sigma)}, \quad (10.41a)$$

$$\sigma_{eq,ar} = f_{eq}(n_f), \quad f_{eq}(n_f) = \sigma'_{eq,f}(R_\sigma) \cdot (2n_f)^{b_0}, \quad (10.41b)$$

here $\sigma'_{eq,f}(R_\sigma)$ is the fatigue strength coefficient for effective stress.

10.5.1.2 Synchronously, Harmonically Oscillating Stress Tensor Fields

Commonly, the relations of the amplitudes to the mean values of the principal stresses are not all equal: $\sigma_1 \neq \sigma_2 \neq \sigma_3$. In this case the proportionality of all principal stresses upon one single parameter is lost and simple relation (10.39) is no more valid.

The consideration (Sines 1955) is restricted to the stresses, which harmonically oscillate from a minimum to a maximum with a mean value, while the stresses are in-phase and the principal axes are fixed. Thus, the principal stresses are synchronous functions of time:

$$\sigma_i = \sigma_{m,i} + \sigma_{a,i} \sin(\omega t), i = 1 \dots 3. \quad (10.42)$$

The equivalent stress plays the role of the scalar, uniaxial fully reversed fatigue stress (Table 10.8). This stress is expected to give the same life on smooth specimens as the multiaxial state. If intensive yielding is avoided, the effect of static bending dominates the static torsion on alternating fatigue strength. According to (Sines 1955), equivalent stress for multiaxial fatigue is defined as:

$$\begin{aligned} \sigma_{eq.ar} = & \frac{1}{\sqrt{2}} \sqrt{(\sigma_{a,1} - \sigma_{a,2})^2 + (\sigma_{a,2} - \sigma_{a,3})^2 + (\sigma_{a,3} - \sigma_{a,1})^2} \\ & + \frac{C_1}{\sqrt{2}} (\sigma_{m,1} + \sigma_{m,2} + \sigma_{m,3}), n_f = f_{eq}^{-1}(\sigma_{eq.ar}). \end{aligned} \quad (10.43)$$

The influence factor $C_1 > 0$ for normal stress acting on the maximum shear stress plane was introduced in Findley (1957, 1959). For a given fatigue life, the allowable alternating shear stress decreases, if the maximum normal stress increases.

$$\tau_a + \mu_{\sigma\tau} \sigma_{n,max} = f_\tau(n_f), \quad (10.44)$$

Table 10.8 Common equivalent stresses for multiaxial fatigue analysis of springs

Maximal shear stress for static load	$\sigma_{eq} = \frac{1}{2} \max(\sigma_1 - \sigma_3 , \sigma_3 - \sigma_2 , \sigma_2 - \sigma_1)$
Equivalent MISES stress for static load	$\sigma_{eq} = \frac{1}{\sqrt{2}} \sqrt{(\sigma_1 - \sigma_3)^2 + (\sigma_3 - \sigma_2)^2 + (\sigma_2 - \sigma_1)^2}$
Equivalent stress for multiaxial fatigue (Sines 1955)	$\begin{aligned} \sigma_{eq.ar} = & \frac{1}{\sqrt{2}} \sqrt{(\sigma_{a,1} - \sigma_{a,2})^2 + (\sigma_{a,2} - \sigma_{a,3})^2 + (\sigma_{a,3} - \sigma_{a,1})^2} + \\ & \frac{C_1}{\sqrt{2}} (\sigma_{m,1} + \sigma_{m,2} + \sigma_{m,3}) \end{aligned}$
Equivalent stress for multiaxial fatigue (Findley 1957, 1959)	$\begin{aligned} \sigma_{eq} = & \frac{1}{\sqrt{2}} \sqrt{(\sigma_{a,1} - \sigma_{a,2})^2 + (\sigma_{a,2} - \sigma_{a,3})^2 + (\sigma_{a,3} - \sigma_{a,1})^2} + \\ & \frac{C_1}{\sqrt{2}} (\sigma_{m,1} + \sigma_{m,1} + \sigma_{m,1}) + \frac{C_2}{\sqrt{2}} (\sigma_{a1} + \sigma_{a,1} + \sigma_{a,1}) \end{aligned}$

$\sigma_{n,\max}$ is the maximum normal stress on the plane of the critical alternating shear stress,

τ_a is the allowable alternating shear stress.

The fatigue effective stress yields (Sines 1955; Findley 1959):

$$\sigma_{eq,ar} = \frac{1}{\sqrt{2}} \sqrt{(\sigma_{a.1} - \sigma_{a.2})^2 + (\sigma_{a.2} - \sigma_{a.3})^2 + (\sigma_{a.3} - \sigma_{a.1})^2} + \frac{C_1}{\sqrt{2}} (\sigma_{m.1} + \sigma_{m.2} + \sigma_{m.3}) + \frac{C_2}{\sqrt{2}} (\sigma_{a.1} + \sigma_{a.2} + \sigma_{a.3}), \quad (10.45)$$

The constant C_2 represents the impact of the hydrostatic alternating component. This constant could be positive or negative indicating the damaging or favorable effects of hydrostatic alternating component of stress.

The fatigue failure under hydrostatic alternating stresses was studied experimentally in bending and torsion fatigue testing by Kakuno and Kawada (1979).

The modification in the stress approach was introduced by McDiarmid (1991, 1993):

$$\tau_a \cdot \left(1 + \frac{\sigma_{n,\max}}{2\sigma_w} \right) = f_\tau(n_f).$$

In this equation σ_w is the tensile strength. The function $f_\tau(n_f)$ depends on whether the cracking grows along the surface (mode III) or grows into the surface (mode II). The first case occurs in torsion loading and the second one under biaxial tension loading.

Dang Van (1993) proposes an approach for determining fatigue strength which—like the approaches of Sines (1955), Findley (1959) and McDiarmid (1991)—is based on a combination of shear stress range $\Delta\tau$ and a hydrostatic stress component σ_{oct} . Dang Van's approach is based on an observation of microscopic stresses (on the order of magnitude of a microstructural grain) to map crack initiation within a microstructural grain:

$$\Delta\tau + a\sigma_{oct} = b \quad (10.46)$$

Papadopoulos (1997, 2001) formulated a fatigue criterion both for the problem of fatigue strength, which was based on a consideration of the mesoscopic size scale. Analogous to the micro-macro scale approach (Dang Van 1993), the size scale refers to the sliding processes within a microstructure grain. Papadopoulos (2001) assumed that the plastic displacement along a slip plane accumulated by cyclic loading is proportional to the macroscopic shear stress amplitude. To determine the fatigue strength, the average shear stress $\langle\tau\rangle$ was limited, taking hydrostatic stress σ_{oct} into account. The approach (Papadopoulos 2001) was formulated with the material constants and as follows:

$$\langle \tau \rangle + a\sigma_{oct} = b, \quad (10.47a)$$

$$\langle \tau \rangle = \sqrt{\frac{5}{8\pi} \int_{\varphi=0}^{2\pi} \int_{\psi=0}^{\pi} \int_{\theta=0}^{2\pi} \tau_a(\varphi, \psi, \theta)^2 d\theta \sin \psi d\psi d\varphi}. \quad (10.47b)$$

10.5.2 Strain-Life Approach in Multiaxial Fatigue

If the spring oscillates between two fixed limits, the fatigue estimation of these springs is based on strain-life approach. The strain amplitude and mean value remain equal over the load history, but the stresses in material and external loads continuously decrease with time. The spring is in the process of cyclic relaxation. Such behavior is typical for the valve and diesel injection pump springs.

The models for fatigue life under combined cyclic axial- torsional loading conditions were proposed in Haverd and Topper (1971), Miller and Brown (1984), Fatemi and Kurath (1988), Fatemi and Socie (1988), Socie (1987). The models include parameters based on the von Mises equivalent strain range, the principal stresses or principal strains. The models use also a combination of maximum shear strain and the normal stress and/or normal strain acting on the maximum shear plane. The life prediction models were verified with at room temperature using axial or torsional test rigs. The common approaches for the description of the multiaxial fatigue are based on the following models:

- maximum principal strain amplitude model;
- maximum shear strain amplitude model;
- von Mises effective strain amplitude model;
- Brown Miller model;
- Lohr-Ellison model;
- Energy based model.

10.5.2.1 Model of Maximum Principal Strain Amplitude

In the simplest case of multiaxial loading, all principal strains oscillate synchronously and proportionally to one harmonic function $1 + \tilde{A} \cdot \sin(\omega t)$ of time:

$$\varepsilon_i(t) = \varepsilon_{m,i} \cdot (1 + \tilde{A} \cdot \sin(\omega t)), \quad \varepsilon_{m,1} \geq \varepsilon_{m,2} \geq \varepsilon_{m,3}. \quad (10.48)$$

In Eq. (10.48) the amplitude ratios for each principal strain are:

$$\tilde{A} = \frac{\varepsilon_{a,i}}{\varepsilon_{m,i}}, i = 1 \dots 3 \quad \text{or} \quad \tilde{A} = \tilde{A}_1 = \tilde{A}_2 = \tilde{A}_3.$$

Consider now the cyclic multiaxial strain with different amplitude ratios for some principal strains. The amplitude of principal strain ε_1 is equal to

$$\Delta\varepsilon_1 = \tilde{A}\varepsilon_{m.1}.$$

This amplitude is greater than the amplitude of both other principal strains. The strain amplitude $\tilde{A}\varepsilon_{m.1}$ is considered the dominant parameter to describe damage. The damage is caused by the amplitude of the principal strain $\Delta\varepsilon_1$ and

$$\Delta\varepsilon_1 = \frac{\sigma'_f(R_\sigma)}{E}(2n_f)^{b_0} + \varepsilon'_f(R_\sigma) \cdot (2n_f)^{c_0}, \quad \Delta\varepsilon_1 = \frac{\sigma_a}{E} + \left(\frac{\sigma_a}{K'_\sigma}\right)^{1/n'} \quad (10.49)$$

10.5.2.2 Model of Shear Strain Amplitude

The amplitude of dominant shear strain $\Delta\gamma_{13}$ is equal to

$$\Delta\gamma_{13} = \tilde{A} \cdot (\varepsilon_{m.1} - \varepsilon_{m.3}).$$

This amplitude is greater than the amplitude of both other principal strains. The strain amplitude $\varepsilon_{m.1}$ is considered the dominant parameter to describe damage. The damage is caused by the amplitude of the Tresca shear strain $\Delta\gamma_{13}$ and according to (10.14):

$$\Delta\gamma_{13} = \frac{\tau'_f(R_\sigma)}{G}(2n_f)^{b_0} + \gamma'_f(R_\sigma) \cdot (2n_f)^{c_0}, \quad \Delta\gamma_{13} = \frac{\tau_a}{G} + \left(\frac{\tau_a}{K'_\tau}\right)^{1/n'}. \quad (10.50)$$

10.5.2.3 Equivalent Strain Amplitude

The von Mises equivalent strain amplitude is the octahedral shear strain:

$$\Delta\gamma_{eq} = \frac{\tilde{A}}{\sqrt{6}} \cdot \sqrt{(\varepsilon_{m.1} - \varepsilon_{m.3})^2 + (\varepsilon_{m.3} - \varepsilon_{m.2})^2 + (\varepsilon_{m.2} - \varepsilon_{m.1})^2}. \quad (10.51)$$

The damage is caused by the amplitude of the von Mises equivalent shear strain $\Delta\gamma_{eq}$ and according to (10.14):

$$\Delta\gamma_{eq} = \frac{\tau'_f(R_\sigma)}{G}(2n_f)^{b_0} + \gamma'_f(R_\sigma) \cdot (2n_f)^{c_0}, \quad \Delta\gamma_{eq} = \frac{\tau_a}{G} + \left(\frac{\tau_a}{K'_\tau}\right)^{1/n'}. \quad (10.52)$$

10.5.2.4 Corrections of Amplitude of Shear Strain

There are two outmost strain measures in three-dimensional solid. The first is the maximum amplitude of the shear strain: $\Delta\gamma_{13} = \tilde{A} \cdot (\varepsilon_{m.1} - \varepsilon_{m.3})$. The second is the amplitude of normal strain, acting on the plane where the former occurs: $\Delta\varepsilon_n = \tilde{A} \cdot (\varepsilon_{m.1} + \varepsilon_{m.3})$.

Brown and Miller (1973) observed that failure under multiaxial fatigue is caused if the certain relation between $\Delta\gamma_{13}$ and $\Delta\varepsilon_n$ is fulfilled: $\Delta\gamma_{13} = \Phi(\Delta\varepsilon_n)$. In other words, the amplitude of normal strain influences the damage, which is caused by the amplitude of the shear strain. To account this phenomenon, the maximum amplitude of the shear strain must be corrected:

$$\widetilde{\Delta\gamma_{13}} = \Delta\gamma_{13} - \Phi(\Delta\varepsilon_n).$$

Kandil et al. (1982) used a linear correction

$$\widetilde{\Delta\gamma_{13}} = \Delta\gamma_{13} + k_1 \Delta\varepsilon_n, k_1 = 1.$$

and derived equation for the evaluation of the cyclic fatigue limit:

$$\widetilde{\Delta\gamma_{13}} = A_1 \frac{\sigma'_f(R_\sigma)}{E} (2n_f)^{b_0} + A_2 \varepsilon'_f(R_\sigma) \cdot (2n_f)^{c_0}, \quad (10.53)$$

$$A_1 = 3.3, A_2 = 3.5.$$

Another correction was proposed by Lohr and Ellison (1980). They studied the crack propagation, which is driven by shear. This type of damage is typical for the helical compression and tension springs. If the stress is considerably high and the fatigue life expectation is low, the crack originates from the surface of the material.

10.5.2.5 Critical Plane Criteria

For multiaxial low-cycle fatigue damage models, the critical plane criteria are the modern approaches due to their specific physical meaning and acceptable capability of the multiaxial fatigue life prediction (Ince 2012; Socie and Marquis 1999; Karolczuk and Macha 2005; Fatemi and Shamsaei 2011).

The strain-based approaches are usually used in combination with the concept of the critical plane (Brown-Miller 1973; Kandil et al. 1982; Fatemi-Socie 1988). The energy-based models on the critical plane were also proposed (Socie 1987; Varvani and Farahani 2000; Xue et al. 2019).

The methods, developed in the cited works, could be principally applied for the spring elements. The essential precondition are:

- Evaluation of multiaxial load collectives for questioned applications.
- Experimental acquisition of the fatigue properties for the multiaxial loading of spring materials.

10.6 Influence of Environment and Manufacturing

10.6.1 Influence of Wire Diameter

The next task is the evaluation of the effective stress for the different diameters of wire. For this purpose, the correction factor $\vartheta(d)$ is developed. The reduction of tolerant stress for the material SH or DH is presented in EN 1270-1:2001. The rational interpolation of the data provides the correction of the effective stress as the function of wire diameter:

$$\frac{\tau_{ar}(d)}{\tau_{ar}(d = 1 \text{ mm})} = \vartheta(d), \quad (10.54)$$

$$\vartheta(d) = \frac{61683 \cdot d^2 - 101769 \cdot d + 373390}{3748 \cdot d^3 + 46996 \cdot d^2 + 11960 \cdot d + 269600} \quad (10.55)$$

because $\vartheta(d = 1 \text{ mm}) = 1$. Equation (10.55) is based on rational interpolation of the values from standard EN 13906-1:2002 (D). For example, if the fatigue data is acquired for some diameter of wire, $d = 2 \text{ mm}$, the recalculation for any other diameter d , according to Eq. (10.55), uses the formula:

$$\frac{\tau_{ar}(d)}{\tau_{ar}(d = 2 \text{ mm})} = \frac{\vartheta(d)}{\vartheta(d = 1 \text{ mm})}. \quad (10.56)$$

Matek et al. (1992) in TB 10-2c recommend the different correction factors for different spring materials $\sigma_w(d)/\sigma_w(1)$ (Table 10.9). The comparison of both correction factors presents Fig. 10.11

There are several explanations of the influence of wire diameter on fatigue life (Shelton and Swanger 1935). At first, fatigue limit depends greatly upon the conditions at the surface, such as surface decarburization in steels, tool marks, notches, and some protective metallic coatings. At second, the processing is different for the wires of greater and smaller diameters. At third, the survival probability for the wires of greater diameter is lower due to the multiplication of survival probabilities of the elementary material sub-volumes (weakest link concept). We discuss the application of the weakest link concept to evaluation of survival probability of springs in Chap. 11.

Table 10.9 Dependence of minimal wire strength $\sigma_w = R_m$ upon wire diameter and correction factors

Wire grade	Wire diameter	Ultimate strength	Correction factors
	d (mm)	$\sigma_w(d)$, (MPa)	$\sigma_w(d)/\sigma_w(1)$
A	1 to 10	$1720 - 660 \log_{10} d$	$1 - 0.166 \log_{10} d$
B	0.3 to 20	$1980 - 740 \log_{10} d$	$1 - 0.162 \log_{10} d$
C	2 to 20	$2220 - 820 \log_{10} d$	$1 - 0.160 \log_{10} d$
D	0.2 to 20	$2220 - 820 \log_{10} d$	$1 - 0.160 \log_{10} d$
FD	0.5 to 17	$1846 - 480 \log_{10} d$	$1 - 0.113 \log_{10} d$
VD	0.5 to 10	$1800 - 415 \log_{10} d$	$1 - 0.1 \log_{10} d$

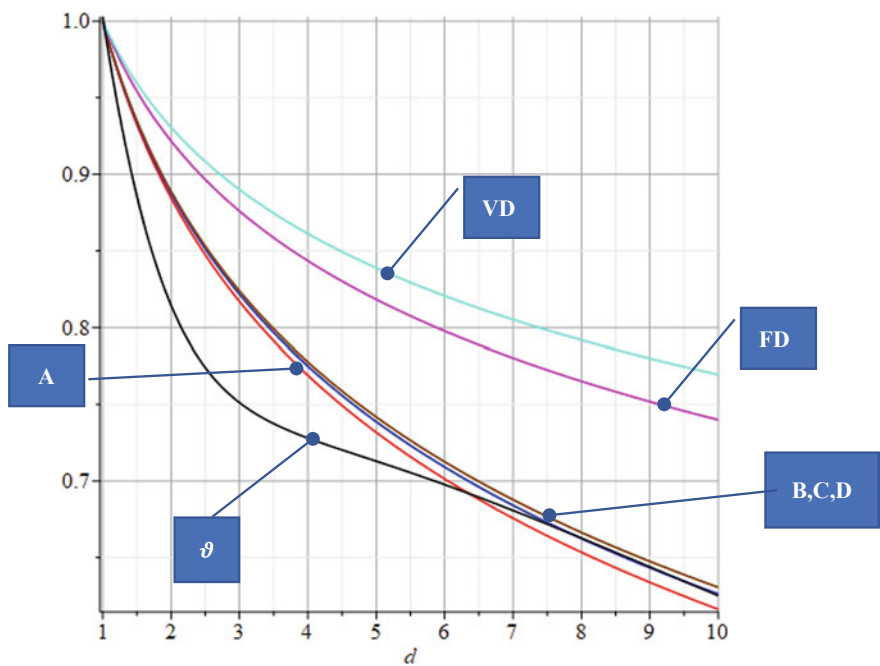


Fig. 10.11 Correction factors for materials A, B, C, D, VD, FD (Matek et al. 1992) and $\vartheta(d)$ for material SH or DH, EN 1270-1:2001

10.6.2 Influence Shot Peening

Shot peening is a special type of surface treatment and is an essential measure to increase the service life and fatigue strength of springs made of high-strength steels (Niku-Lari 1996; Kirk 1999; Wagner 2003). In this process, compressed air or blast wheels shoot the abrasive at high speed onto the springs. Blast wheel blast machines are usually used for the standard blast treatment of vehicle springs. In these plants

the abrasive is accelerated in one or more blast wheel units by centrifugal force to the desired abrasive speed (discharge speed). The abrasive usually reaches the centre of the blast wheel, which is usually equipped with 8 blades, indirectly via a mechanical pre-acceleration. By means of the rotating cage-like distributor, uniform portions of abrasive are taken up and fed through the opening of the fixed inlet piece onto the blades. When leaving the blades the jet fans out and creates an elliptical blasting pattern which can be adjusted to the surface to be blasted by positioning the inlet piece accordingly. The abrasive consists of round or rounded grains, which give the process its name.

The aim of shot peening is to increase the surface volume of the material and to create a residual compressive stress. The result is a compression of the material and thus a hardening of the surface. It introduces residual compressive stresses into the edge layer, the level of which correlates with the level of material strength. In this way, any surface defects such as scoring or scars are largely defused. Therefore, the fracture origin is recurrently shifted to the inside regions of the material. For larger helical compression springs, shot peening under preload (stress peening) is also used. This process significantly improves the fatigue strength. The blasting medium speed (depending on air pressure or blast wheel speed) together with the grain mass influences the kinetic energy of the steel medium and thus the blasting intensity. The amount of blasting medium and blasting time are mainly decisive for the degree of surface coverage (i.e. the percentage of the surface deformed by ball impacts in relation to the initial surface).

There are following common conclusions about the formation of residual stresses in shot-blasted components (Hirsch et al. 1979):

- The magnitude of the beam residual stresses at the blasting material surface is primarily determined by the material strength or hardness. Blasting parameters such as blasting medium speed, coverage ratio and blasting grain diameter influence the residual compressive stress on the surface of the blasting material are also significant for the fatigue of the processed part.
- The maximum value of the induced residual compressive stresses usually occurs below of the blasting material surface. The maximum occurs on the surface of the exposed material in cases of low material strengths or low resulting residual stress.
- The thickness of the surface layer subject to compressive residual stress decreases with the higher material harnesses.
- The thickness of the surface layer subject to compressive residual stress increases with increasing speed of shot-peen particles, abrasive grain size and final coverage.
- With hardened blasting material (>700 HV) the transition from a soft (40–530 HV) to a hard blasting medium (580–655 HV) causes an increase of the coverage and maximum residual stresses as well as of the thickness of the edge layer subjected to residual compressive stresses with otherwise identical blasting parameters.
- If the blasting material is blasted under static tensile (pressure) pretension, the following values are then in the material areas close to the surface larger

(smaller) amounts residual compressive stresses than with non-pre-stressed blasting material (stress blasting).

- Shot peening with vertically incident blasting medium induces rotationally symmetrical residual compressive stress state

Due to the large number of parameters in shot peening there are several attempts to describe the effect of the blasting treatment by easily measurable parameters. Nikulari (1981) considered that a certain source of stress is introduced to the metallic component while the shot peening process. One of the analytical models to calculate the induced residual stress was based on Al-Obeid (2007) where he proposed that the distribution of the residual stress was a result of stretching action followed by an elastic unloading of a metallic plate that was being shot peened. The major assumption in this case was that the plane section of the metallic plate remains constant. This assumption was in dispute with the consideration of the plastic deformation that occurs for the formation of the residual stress. The idea of stress source was introduced to overcome this dispute of local plasticity

Despite frequent criticism and numerous other suggestions, the measurement of shot peening intensity is still usually carried out today using the so-called Almen test (Almen 1943). In this test, standardized steel plates are irradiated on one side in a clamped condition. The deflection (“arc height”) of the Almen strip measured after loosening serves as a measure of the blasting intensity and has proved to be a good process control and criterion for the adjustment of blasting machines. The test strip is to be arranged in such a way that the respective decisive component area is detected. However, there is no direct connection between Almen intensity and the improvement of springs’ durability, for which the following reasons are decisive:

- there is no definite relationship between the stress changes caused by shot peening and the change of the Almen arc height;
- the relationship between the Almen arc height and the achieved degree of coverage must be experimentally evaluated for each material and process;
- the influence of material properties, such as hardening behaviour and notch sensitivity, on the Almen arc height is not clear.

For unambiguous determination of the shot peening effect on the fatigue stroke strength of coil springs vibration tests should therefore be carried out.

The coverage indicates the percentage of the abrasive impact surface. The degree of surface coverage increases with the blasting time and approaches asymptotically close to the 100% limit, in general a value of 98% is regarded as total coverage assumed. Longer blasting times leave are expressed as a multiple of the value 98%. The degree of area coverage is given by planimetry measurement or by comparison with reference series.

The increase in fatigue strength achievable by shot peening and pre-setting is approx. 10 to 20% with an ideal initial condition of the surface layer (Kaiser 2006). Depending on the geometric conditions, the fatigue crack can be shifted below the surface, whereby the weak point effect of the surface is eliminated and in addition a fatigue strength increase corresponding to the stress gradient is achieved. The

improvement in fatigue strength as a result of presetting and in particular as a result of shot peening or stress peening can be substantial. Firstly, the mean stress reduces due to residual compressive stresses. Secondly, the surface topography could be improved by shot peening. Thirdly, the barrier to crack propagation at defects due to residual compressive stresses (crack closure) can also contribute the fatigue life.

The residual stress of shot peened helical compression springs without pre-stressing is changed by a sufficiently high torsion threshold load or, alternatively, due to presetting. The residual stresses near the edges in the longitudinal and transverse directions of the wire and under the 45° direction of the load are redistributed, but in the 45° direction of the load-tension stresses are maintained or even increased Fig. 10.12 (Muhr 1970).

The explanation of the effect of shot-peening on fatigue life provides the method (Findley 1957, 1959) The method of fatigue evaluation for synchronously, harmonically oscillating stress tensor fields was displayed above in Sect. 10.5.1.2. On the surface of the wire and slightly below it there is an isotropic residual compressive stress $-\sigma_{sp}$, due to shot-peening of the material, where $\sigma_{sp} > 0$ is the pressure due to shot-peening. The twist moment of the spring wire causes an oscillating shear stress

$$\tau(t) = \tau_m + \tau_a \cdot \sin(\omega t). \quad (10.57)$$

As shown above, the acting stresses are tension stress and compressive stress. There are two stress fields in the spring, namely the static due to shot-peening and the oscillating due to functioning of the spring. Both stress fields overlap during the spring operation. Consider the overlapping of the stress field on the surface of the wire. The surface of the wire is stress free and thus the normal stress vanishes. Consequently, three components of stress tensor remain on the surface layer of the wire. For simplification, look on these stress fields from the viewpoint of the Cartesian coordinate system, which is rotated on the angle 45° to the axis of the wire. For definiteness, the x axis of this coordinate system makes the angle 45° , and y axis 135° to the wire tangent. The stress tensor is diagonal in this coordinate system. Two components of tensor remain on the surface of wire. We study the stresses in the coordinate system of the principal stresses. The stress tensor in the surface layer in the coordinate system, which form the angles $\pm 45^\circ$ to the wire axis, reads:

$$\sigma = \begin{bmatrix} -\sigma_{sp} + \tau_m + \tau_a \cdot \sin(\omega t) & 0 & 0 \\ 0 & -\sigma_{sp} - \tau_m - \tau_a \cdot \sin(\omega t) & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (10.58)$$

The two non-vanishing components are the principal stresses. These stresses are synchronous, harmonically oscillating functions. From the Eq. (10.58), follows, that the stress ratios of two principal stresses are different:

$$R_{\sigma,1} = \frac{\sigma_{sp} - \tau_m + \tau_a}{\sigma_{sp} - \tau_m - \tau_a}, R_{\sigma,s} = \frac{\sigma_{sp} + \tau_m - \tau_a}{\sigma_{sp} + \tau_m + \tau_a}. \quad (10.59)$$

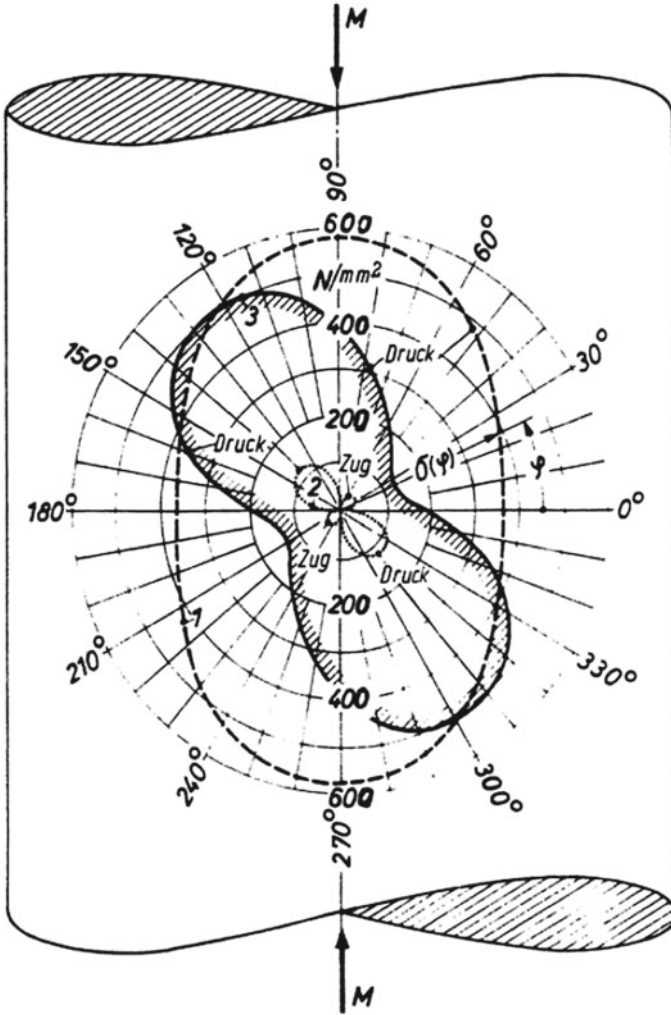


Fig. 10.12 Polar diagram of the residual stresses on the surface of the CrV wire for the valve springs. Curve 1 shows the residual stresses due to shot-peening. Curve 2 displays the residual stresses after the application of the relatively small torque. Curve 3 shows the residual stresses after shot-peening and presetting (Muhr 1970)

If the relations of the amplitudes to the mean values are not all equal, the proportionality of all principal stresses upon one single parameter is missing (Sect. 10.5.1.2). In this case we apply the method (Sines 1955). The equivalent stress for multiaxial fatigue is defined by Eq. (10.43). Using Eq. (10.58), from Eq. (10.43) follows:

$$\sigma_{ar} = \sqrt{3}\tau_a - C_1\sqrt{2}\sigma_{sp}. \quad (10.60)$$

Because the factor C_1 and pressure σ_{sp} are both positive, the equivalent stress reduces with the increasing of the residual stresses due to presetting. The fatigue life increases with the growth of the residual pressure in material. This explains from the mechanical viewpoint the effect of shot peening.

Shot peening of larger leaf springs and coil springs is carried out in a wheel blast machine (longitudinal blast single-train continuous blast machine), whereby leaf springs are usually only treated on the tension side. For uniform blasting of the entire coil spring surface, the springs are individually treated with a defined rotary motion and constant Feed guided through the system. For smaller helical compression springs that are not individually can be shot peened, blast wheel drum systems are used. The inside of helical tension springs can only be removed in the stretched state using conventional methods of the spring can be radiated well. Occasionally, these springs are blasted by inserting a free jet hose with curved nozzle treated on the inside. In order to achieve optimum endurance stroke strength values in shot peening of smaller helical compression springs in barrel machines, the filling quantity must be selected according to Hirsch et al. (1979), in such a way that an approximately constant spring wire-total surface is present.

In case of very small winding ratios and/or distances between windings, the blasting time should be extended by 50% in order to sufficiently cover the most stressed inner side of the windings. Springs with low dimensional stability must be treated with small blasting agent grains and low blasting intensity. Hooking and coiling of coil springs in drum systems can result in almost not-blasted or too weakly blasted surface areas. Blasting grains jammed under the applied end coils should be removed in order to avoid spring fatigue fractures or disturbances during operation. Shot-blasted springs are very susceptible to corrosion and must be protected accordingly.

Shot peening without pre-stressing usually creates a symmetrical residual compressive stress state in the surface layer, which does not have a particular preferred direction has. Such a symmetrical stress state with not too high values is with alternating stresses on components with no or only low medium voltage to otherwise residual stress reduction through superimposition of compressive residual stresses and pressure load stresses are to be expected. The presetting is used for purely alternating stressed components are not used for the same reason. Most springs are subject to a swelling load. For these springs are directional residual stresses due to prestressing and shot peening under prestress recommended

Coil springs are partially hot-set, then (stress) blasted and then again set or released at room temperature. A final shot peening can lead to a stronger relaxation of the springs than a final setting. For some time now, studies have been carried out on the effect of shot peening at elevated temperatures (Müller et al. 2006).

The influence of the defects with different sizes $\sqrt{(4area/\pi)}$ on the S-N-curves is displayed on the Figs. 10.13 and 10.14. These figures show the calculation of the fatigue life for the VDCrSiV material without peening (left) and peening surface treatment (right). The parameter P_{SWT} for normal stresses, Eq. (10.2a) is displayed as the ordinate of Fig. 10.13. Additionally, Fig. 10.14 uses the parameter P_{SWT} for shear stresses Eq. (10.3), as the ordinate.

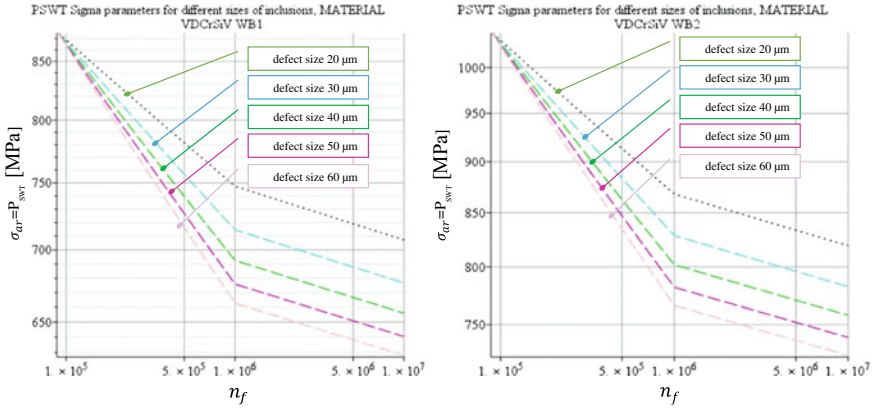


Fig. 10.13 Calculation of the influence of defects with the size $\sqrt{(4area/\pi)}$ on the S-N-curves for the VDCrSiV material without peening (left) and peening surface treatment (right). The parameter P_{swt} Eq. (10.2a) is displayed as the ordinate

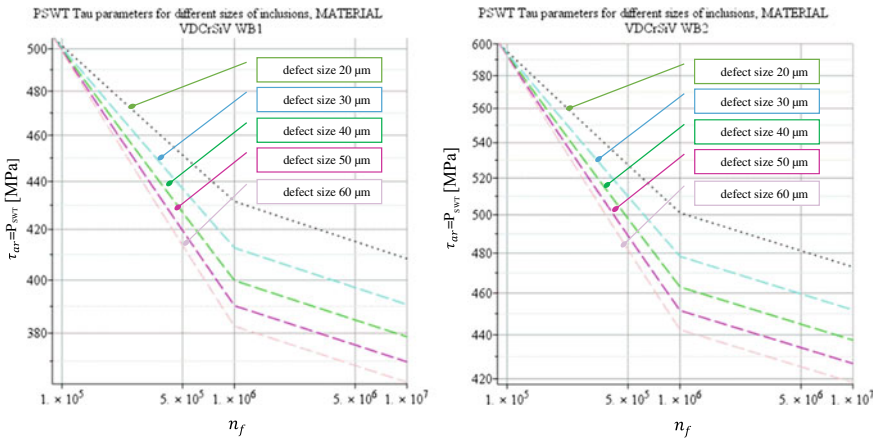


Fig. 10.14 Calculation of the influence of defects with the size $\sqrt{(4area/\pi)}$ on the S-N-curves for the VDCrSiV material without peening (left) and peening surface treatment (right). The parameter P_{swt} Eq. (10.3) is displayed as the ordinate

10.7 Conclusions

The different common expressions, which account the influence of the stress ratio, are immediately applicable. For the proposed unified propagation functions, the ranges for the stress load factor are extended using the introduction of effective stress intensity. The solution leads to the factor range for effective stress intensity, effective mean value of stress intensity factor and effective stress ratio.

The developed in this Chapter method assumes the homogenous stress state in the whole spring element. For the non-homogeneously loaded structural elements, the weak-link concept will be applied to account fatigue. The weak-link model is applied for the evaluation of fatigue life of helical spring later in Chap. 11.

10.8 Summary of Principal Results

- Mean stress, multiaxial loading, and residual stresses significantly affect the fatigue life of springs and must be accounted during their design and calculation.
- For accounting of these effects, the stress-life and strain-life techniques were augmented.
- The traditional damage parameters (Walker, Smith-Watson-Tooper, Haigh, Bergmann) are the special cases of the new generalized, or Bergmann-Walker parameter.
- The new Bergmann-Walker parameter has two fitting constants and allows a broad fitting of experimental data
- The method of the fatigue crack growths per cycle explicitly accounts the mean stress
- The environmental and corrosion influences on fatigue life of springs are exhibited
- The effect of shop-peening on fatigue explained from the viewpoint of Sines model for the multiaxial fatigue.

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Chapter 11

Failure Analysis Based on Weakest Link Concept



Abstract In this Chapter we begin to study the statistical effects in failure analysis of springs. It presents the procedure for dimensional analysis of failures of the helical springs. The dimensional analysis spring element is examined with the weakest-link concept. This concept is applicable to the extremely brittle material, for which the failure of one link causes the complete destruction of the complete structural element. The variation of stress over the surface of the wire is accounted for the analytical calculation of Weibull failure probability. Chapter delivers analytical formulas for failure probability of helical springs. The derived solution clarifies the effects of spring index and wire diameter on the fatigue life. The methodology is valid for different types of springs or basic structural elements.

This Chapter is the third section of the third part, which studies the lifecycle of the elastic elements.

11.1 Evaluation of Failure Probability of Springs

Well known, the stress variation on the surface of the spring depends on the spring index (EN 13906 2013). However, the customary estimations of fatigue life of cyclically loaded helical springs does not account accurately the stress variation on the wire surface. The known experiments report substantial dependence of fatigue life upon the spring index and the diameter of wire (Kaiser 1981), (Reich 2016). On one side, the experimental results indicate the evident reduction of fatigue life of springs with thicker wire in comparison to the equally stressed springs made of the thinner wire. On the other side, the springs with the same stress maximum but different spring indices demonstrate different statistical expectation of fatigue life (Reich and Kletz 2013). The springs with higher indices suffer earlier breakage than the equally loaded springs with lower spring indices. The explanation of this behavior is the following. The springs with higher indices are equally stressed over the surface of the wire. The stress on the wire surface of springs with lower indices varies considerably. The maximal stress of both types of the springs is the same, but the springs with higher

spring indices are higher stressed over the total surface. That is the defects on the outer regions of the spring wire provoke more damage in the springs with higher indices. The existing methodology does not explain accurately these dependencies.

Consequently, one of aims of this Chapter is the quantitative analytical description of the scaling and stress-gradient effects in cyclic failure analysis of helical springs. The second aim is to provide the closed form solution for the estimation of these effects.

The essence of the proposed approach is the following:

- Fatigue crack propagation is based on the linear elastic fracture mechanics;
- Advancement of fatigue crack per unit cycle as a function of the amplitude of stress intensity factor;
- Survival probability is based on the weakest link concept;
- Fatigue life is accounted by means of the Weibull distribution;
- Failure probability as the function of stress and of number of cycles is derived for helical springs with circular wire of the constant mean diameter.

The results of the actual article provide the method for estimation some important parameters (stress ratio, spring index) on the fatigue behavior of the springs. The method can be adapted for the other types of springs and structural elements.

The fatigue damage, which is studied in the Chap. 8, must account the influence of spatial inhomogeneity of stress field. With this enhancement, the fatigue life of the non-homogeneously stressed structural element could be studied. The addressed problem is investigated with the concept of weakest link. The weakest link concept has been used for material failure problems by many researchers (Wormsen et al. 2007). Traditionally, the weakest link concept is adopted for different shape coefficients of the Weibull distribution. This approach permits to calculate the probability distribution of the fatigue life for helical springs with different spring indices and different stress levels. The assumed statistical model of the weakest link theory is based on the Weibull probability distribution of failure. The theory describes the statistical distribution of strength properties of fragile materials and materials with low ductility. This assumption describes exactly the materials for springs, because of the absence of plastic region due to extraordinary manufacturing of helical springs.

The fatigue life of a large diameter wire rope was estimated from test data for small diameter rope in (Chien et al. 1988). Dimensional analysis and the technique of interpolation and extrapolation were employed. The method was applied for analysis of size effects on the strength of wires and for prediction of fatigue strength.

A fatigue crack is initiated in high-stressed helical springs under moderate loads by the randomly distributed material defects. The similar crack initiation occurs in most of brittle materials. In high-cycle, fatigue tests to determine a fatigue life the different theories are used.

11.2 Weakest Link Concepts for Homogeneously Loaded Elements

11.2.1 Failure Probability of Elements with Distributed Defects

The theory of weakest link is based on statistically distributed defects in a material per volume unit. The defects are non-homogeneities, inclusions, cracks, and precipitates. In a specific volume where the most dangerous defect or the weakest link exists, the crack initiates. Fatigue cracks propagate independently from each other in different areas of the structural element. The effect of load and cross-sectional area with the fatigue life is accounted by means of the Weibull distribution. The Weibull distribution presumes, that for identical elements (at the macroscopic scale) loaded by time dependent but equal stresses, the logarithm of the number of cycles n to crack initiation is a random variable with a given probability density distribution. “The most harmful defect” (“weakest link”) exhibits different features in a set of successive specimens. Therefore, crack initiation occurs under a different number of cycles (Wormsen and Härkegård 2004).

The mathematical formulation must be reconsidered for heterogeneous stresses in order to consider the spatial stress gradients. The considered structural element is divided into several subdomains. In each subdomain, the stresses are different. The probability that a failure of the whole element will not occur within the lifetime interval $0..N$ means that any elementary subdomain not failed (weakest link concept). Indicating that $P_S^{(i)}$ is the survival probability means that the sub-domain with the number (i) is not failed within the number of cycles $0..N$. The survival probability for the whole structural element (or component) is the product of all the individual probabilities $P_S^{(i)}$:

$$P_S = \prod_{i=1}^k P_S^{(i)}.$$

The product rule is valid because no interaction between dangerous defects indifferent subdomains is assumed.

The dangerous, or critical defect causes the destruction of the whole spring. Assuming that the defects are uniformly distributed over the volume of the whole volume V of structural element, the survival probability is the integral:

$$P_{S,V} = \exp\left(-\frac{1}{V_0} \int_V p_f(\sigma) dV\right). \quad (11.1)$$

In Eq. (11.1) we use the following values:

$p_f(\sigma)$ the function called by Weibull “risk of rupture”, which represents the fatigue probability density (see Chap. 8),

V_o an average volume that contains one critical defect.

Alternatively, if the dangerous defects are uniformly distributed over the surface A_e of the structural element, the survival probability is the integral:

$$P_{S,A} = \exp\left(-\frac{1}{A_0} \int_{A_e} p_f(\sigma) dA\right) \quad (11.2)$$

where A_0 is an average surface element that contains one critical defect.

Let the number of loads during the operation of spring is N . Our task is to evaluate the failure probability after this given number of cycles in operation. For this purpose we use the cumulative distribution functions. The Weibull form of failure probability reads:

$$P_{F,A}(N) = 1 - P_{S,A}(N), P_{F,V}(N) = 1 - P_{S,V}(N) \quad (11.3)$$

What Eq. (11.1) or Eq. (11.2) should be used for computations depends on the material and application of the spring. For example, the springs for use in aggressive or corrosive media suffer failure due to corrosion on the outer surface. For these springs source of the defects is the surface and correspondingly the Eq. (11.2) should be used for computations. Otherwise, for springs those operate in oil or inert gases the frequent source of defects is the inclusions or flaws inside the material. These defects are distributed evenly over the whole volume of spring and therefore appropriate estimation is (11.1). Two traditional forms of the Weibull function $p_f(\sigma)$ are commonly used:

$$p_f(\sigma) = \left(\frac{\sigma}{S_W}\right)^{m_W} \text{ or } p_f(\sigma) = \left(\frac{\sigma - S_{sh}}{S_W}\right)^{m_W} \quad (11.4)$$

The parameters S_{sh} , S_W , m_W are known as the stress shift, stress scale and shape parameters, respectively. The forms (11.4) are appropriate for the immediate failure. The Weibull function must be generalized for cyclic load and fatigue estimations. This is the subject of the next paragraph.

11.2.2 Application of Weakest Link for Fatigue

The application of the concept of weakest link for fatigue was established in the case of fatigue processes in a uniaxial loaded component with a homogenous stress distribution. The failure probability P_F is a certain function of the stress amplitude σ_{ar} and the number of cycles in operation of the spring N .

When the required number of cycles is given, then the failure probability could be determined as the function of stress amplitude σ_{ar} . Otherwise, if the failure probability and fatigue life are prescribed by design requirements, the allowable stress amplitude σ_{ar} could be determined from this equation.

Consider a spatially inhomogeneous stress field within the structural element Ω . The spatially inhomogeneous stress field results in an inhomogeneous fatigue damage field, such that the probability of failure also depends on the coordinates. However, for design purposes the probability of failure of the structural element is required. The concept of weakest link permits to determine the fatigue calculations of the structural element with a spatially varying stress field.

For this purpose, at each point of the structural element, Ω consider the infinitesimally small element $d\Omega$. The stress in the element $d\Omega$ is constant. Therefore, for the element $d\Omega$ the mean cycles to failure N could be computed using the equations of Chap. 8. The mean cycles to failure is the function of the stress amplitude σ_{ar} in the element $d\Omega$. The dependence of maximal cycles to failure of the element $d\Omega$ is

$$n_f = n_f(\sigma_{ar}).$$

The cycles to failure of the small element is given by the equations of Chaps. 9 and 10. Particularly, the simplest form expresses the intermediate Wöhler-Basquin regime. The number of cycles to failure n_f on the effective stress amplitude σ_{ar} , according to Eq. (10.1), in the homogeneously stressed element reads:

$$n_f = \frac{1}{2} \left(\frac{\sigma_{ar}}{\sigma'_f(R_\sigma)} \right)^{1/b(R_\sigma)} \equiv \lambda_\sigma \sigma_{ar}^{-p_\sigma} \quad (11.5)$$

The “risk of rupture” function of the small element is assumed to be analogous to Weibull expression $p_f(\sigma)$, Eq. (11.4). The essential difference is that for the cyclic loading the “risk of rupture” $p_h(N, \sigma_{ar})$ depends on effective stress amplitude σ_{ar} and also on the mean cycles to failure of the heterogeneously stressed structural element.

Thus, the required number of cycles N and operation stress are given, and our aim is to determine the failure probability of the structural element. The following simplest form of Weibull approximation as the “Weibull:risk of rupture” function is used for subsequent computations:

$$p_h(N, \sigma_{ar}) = \begin{cases} 0 & \text{for } N < N_F \\ \left(\frac{N - (1 - k_f)n_f}{k_f n_f} \right)^m & \text{for } N_L < N < N_L \\ 1 & \text{for } N > N_L. \end{cases} \quad (11.6)$$

The values in Eq. (11.6)

$$N_F = N_F(\sigma_{ar}), N_L = N_L(\sigma_{ar})$$

depend on the maximal cycles to failure $n_f = n_f(\sigma_{ar})$. The value $N_F = N_F(\sigma_{ar})$ is the number of cycles to failure of the first element and $N_L = N_L(\sigma_{ar})$ is the number of cycles to failure of the last element. At first, the function (11.6) corresponds to 100% of “risk of rupture” probability and consequently $N_L = n_f$, according to Eq. (11.5). But some element will fail even at the lower number of cycles. To account these first occurrence of failure, we introduce the extent of the region $N_F..N_L$. The extent linearly depends on the value n_f , such that

$$N_F = (1 - k_f)n_f.$$

The extent constant equals to:

$$k_f = 1 - N_F/N_L.$$

The extent constant k_f characterizes the range of the failure region from the first occurrence of failure to the final failure of the last element. For the given operation stress, there is no failure prior to N_F operational cycles and all elements will guaranteed fail after N_L cycles. In other words, we test a collection of specimens. The first specimen from the collection breaks exactly at N_F cycles. At the cycle count N_L all specimens from the collection are broken. With Eq. (11.5), these numbers are:

$$N_F = (1 - k_f)\lambda_\sigma \sigma_{ar}^{-p_\sigma}, N_L = \lambda_\sigma \sigma_{ar}^{-p_\sigma}.$$

The proposed “risk of rupture” function (11.6) is the simplest form, adopted form of second Weibull function for fatigue computations.

We apply the results of testing of collection of specimens to the whole structure. The structure is assembled from several structural elements. If one element fails, fails the whole structure. Thus, the survival probability of heterogeneously stressed structure depends on load cycles N for the structural element. The survival probability of each element is the decreasing function of cycles. The failure probabilities $P_{F,A}(N)$, $P_{F,V}(N)$ increase with the cycle number. Finally, we multiply all risk of rupture of all single element and get the risk of rupture of the whole structure:

$$P_{S,A}(N) = \exp\left(-\frac{1}{A_0} \int_{A_e} p_h(N, \sigma_{ar}(dA)) dA\right), P_{F,A}(N) = 1 - P_{S,A}(N) \quad (11.7)$$

$$P_{S,V}(N) = \exp\left(-\frac{1}{V_0} \int_V p_h(N, \sigma_{ar}(dV)) dV\right), P_{F,V}(N) = 1 - P_{S,V}(N) \quad (11.8)$$

With the expressions (11.8) and (11.9) we estimate the failure and survival probabilities of the complex system, based on failure probabilities of its elements for a give number of operational cycles. Evidently, that the failure probability increases with the stress level in the structural element as well. As a result, the longer structural element is in service and higher the stress, the significantly the failure probability is. It is evident as well, that the higher number of elements increase the probability

of failure of the whole structure, if each failed element leads to complete disaster. Exactly this behavior is typical for springs. If the large enough crack occurs somewhere in the spring inner volume or on the surface, it spreads over the whole section of the spring and causes the breakage of the spring. If there are no possibility to observe the spring, to control the state of spring and to repair the local breakages, the theory of weakest link must be applied. This theory allows the conservative, pessimistic evaluation of failure probability of a complex element and permits the evaluation of design parameters on the failure probability. We apply this theory to springs.

11.3 Analysis of Springs with Weakest Link Method

11.3.1 Failure Probability of Helical Springs

The failure probability as the function of stress and of number of cycles will be derived below for helical springs in closed analytical form. The springs with the constant mean diameter $D = 2R$ that made of the circular wire with the diameter $d = 2r$ are considered.

The spring index for helical springs is a measure of coil curvature of the wire. For springs of circular section wire the spring index is the ratio of mean coil diameter to wire diameter.

The equations assume that the spring coil is loaded centrally along the spring axis, which remains straight and the ends can rotate about the spring axis relative to one another, such that no bending of wire occurs.

Owing to different material properties on the surface and in volume due to manufacturing process, the failure probability is commonly considered for the surface but not over the volume of the structural element. The following consideration is typically appropriate to the failure of springs. The predominant source of failure is the skin of the wire but not its core regions. This assumption is natural for the helical springs, because the stresses decline rapidly from surface of wire to its center.

For proper application of the weak link concept, the accurate study of stresses on the surface of the wire is essential. The standard reference sources (EN 13906–1:2013–11 2013) and (SAE 1997) provide the stress correction factor that accounts the stress conditions at the inside of the coil by using a “corrected stress” (Chaps. 1 and 2). The stress as function of polar angle is necessary for the subsequent estimation of failure probability based on weak link concept. This auxiliary problem for the stress on the surface as function of the polar angle was treated in Chap. 2. The expressions provided for the stress on the surface of the wire as function of the polar angle φ were provided.

All essential formulas are prepared now and we can precede the estimation of failure probabilities.

11.3.2 Effect of Spring Index on Immediate Failure

The developed formulas allow the estimation of failure probability of the springs. At first consider the case of failure due to surface defects. The survival probability is estimated with the aid of Eq. (11.2). The Weibull function is given by the second Eq. (11.4). The evaluation leads to the following expression of the survival probability of the helical spring with the spring index w (Kobeleev 2017b):

$$P_{S.1} = \exp\left(-\frac{rL_W}{A_0} \int_0^{2\pi} p_f(\tau_s(w, \varphi)) d\varphi\right) = \exp\left(-\frac{rL_W}{A_0} \int_0^{2\pi} \left(\frac{\tau_s(w, \varphi)}{S_W}\right)^{m_W} d\varphi\right). \quad (11.9)$$

The shear stress $\tau_s(w, \varphi)$ is the function of the spring index w and of the polar angle φ , Eq. (2.36). The outer surface of spring wire is

$$A_e = 2\pi r L_W.$$

The expression for survival probability could be easily determined also for different values of m_W :

$$P_{S.1} = \exp\left(-\frac{A_e}{A_0} \frac{\tau_b^2}{S_W^2} k_5(w, m_W)\right),$$

$$k_5(\infty, m_W) = \lim_{w \rightarrow \infty} k_5(w, m_W) = 1.$$

The factor $k_5(w, m_W)$ is the stress correction factor for surface defects. Performing integration in (11.9) with the expression for shear stress on the outer surface of the wire from Eq. (2.9), demonstrates the formula for correction factor for surface defects:

$$k_5(w, m_W) = \frac{4096w^2 + 961m_W^2 + 880m_W}{4096w^2}.$$

This correction factor is applicable for stress evaluation in fatigue problems, as already discussed in Chap. 10. This factor averages the failure probability over the outer surface. The factor k_5 is analogous to the empirical averaging factors $\bar{k}_2(w)$, $\bar{k}_3(w)$, which were used in Chap. 10 for fatigue estimations. Conversely, the traditional correction factors $k_i(w, m_W)$, $i = 1, 4$, were referred Chaps. 1 and 2 defects. The traditional correction factors deliver the local, maximal value of stress in a single point on the inner side of the section. The traditional factors underestimate the risks of fatigue failure for the springs with higher spring ratios with the evenly stressed surfaces.

For comparison, consider the straight rod with circular cross-section. The cross-section of the straight rod is the same as the cross-section of the helical spring. The

torsion moment in the cross-section of the straight rod is again the same as the torsion moment in the cross-section of the helical spring. As already mentioned, the straight rod possesses the constant stress over the surface. The shear stress on the surface of the rod is the same as basic stress σ_b of the helical spring. Obviously, the shear stress for the rod equals to the shear stress on the surface of helical spring with the infinite spring index w . Consequently, the survival probability of the straight rod is equal to the survival probability of the helical spring with infinite spring index and the same length and diameter of the wire. It follows from the Eq. (11.9) that:

$$P_{R,1} = \lim_{w \rightarrow \infty} P_{S,1} = \exp\left(-\frac{A_e}{A_0} \frac{\tau_b^2}{S_W^2}\right) \quad (11.10)$$

The ratio of survival probabilities of the spring (11.9) and the rod (11.10) in case of immediate rupture reads:

$$\alpha_1 = \frac{P_{S,1}}{P_{R,1}} = \exp\left(-\frac{A_e}{A_0} \frac{\tau_b^2}{S_W^2} (k_5(w, m_w) - 1)\right) \quad (11.11)$$

At second consider the volume defects that cause the immediate failure of the spring .. The intensity of shear stress $\tau_q(\rho, \varphi)$ for the circular wire of radius r is given by Eq. (2.30). The following expressions of the survival probability of the helical spring with the spring index w are valid in this case (Kobeleev 2017b):

$$\begin{aligned} P_{S,2} &= \exp\left(-\frac{L_W}{V_0} \int_0^{2\pi} \int_0^r p_f(\tau_q(\rho, \varphi)) \rho d\rho d\varphi\right) \\ &= \exp\left(-\frac{L_W}{V_0} \int_0^{2\pi} \int_0^r \left(\frac{\tau_q(\rho, \varphi)}{S_W}\right)^{m_w} \rho d\rho d\varphi\right). \end{aligned} \quad (11.12)$$

The volume of spring wire is:

$$V = \pi r^2 L_W.$$

For an arbitrary value of the exponent m the integrals are:

$$P_{S,2} = \exp\left(-\frac{V}{V_0} \frac{\tau_b^2}{S_W^2} k_6(w, m_w)\right),$$

The factor $k_6(w, m_w)$ is the stress correction factor for volume defects. Integration in (11.12) with the expression for shear stress on the outer surface of the wire from Eq. (2.9), demonstrates the formula for correction factor for surface defects:

$$\begin{aligned} k_6(w, m_w) &= \frac{961m_w^3 + 1414m_w^2 + 4096m_w w^2 - 2168m_w + 16384w^2}{2048(4 + m_w)(2 + m_w)w^2}, \\ k_6(\infty, m_w) &= \lim_{w \rightarrow \infty} k_6(w, m_w). \end{aligned}$$

The survival probability of the straight rod with the same torque, length and diameter of wire is:

$$P_{R.2} = \lim_{w \rightarrow \infty} P_{S.2} = \exp\left(-k_6(\infty, m_w) \frac{V}{V_0} \frac{\sigma_b^2}{S_w^2}\right). \quad (11.13)$$

The basic stress τ_b of the helical spring and the rod are equal. From Eqs. (11.12), (11.13) it follows the ratio of survival probabilities of the spring and the rod

$$\alpha_2 = \frac{P_{S.2}}{P_{R.2}} = \exp\left(-\frac{V}{V_0} \frac{\tau_b^2}{S_w^2} (k_6(w, m_w) - k_6(\infty, m_w))\right). \quad (11.14)$$

Important, that the dependence of wire diameter is different in both considered cases.

11.3.3 Effect of Spring Index on Fatigue Life

The falling out of helical spring during its operation depends primary upon the failures in the thin layer on the surface of the spring. The distribution of stress over the surface of the spring Chap. 2, Eq. (11.3.6) is essential for the calculation of failure and survival probabilities (11.7) and (11.8).

At third consider again the surface defects. The shear stress $\tau_s(w, \varphi)$ is the function of the spring index w and of the polar angle φ , Eq. (2.36). The survival probability in this case is the function of stress on the surface and load cycle (Kobelev 2017b):

$$\begin{aligned} P_{S.3} &= \exp\left(-\frac{rL_w}{A_0} \int_0^{2\pi} p_h(N, \tau_s(w, \varphi)) d\varphi\right) \\ &= \exp\left(-\frac{rL_w}{A_0} \int_0^{2\pi} \left[\frac{N}{N_0} \left(\frac{\tau_s(w, \varphi)}{\sigma_0}\right)^{p_\sigma}\right]^{m_w} d\varphi\right). \end{aligned} \quad (11.15)$$

Thus, the survival probability reads:

$$P_{S.3} = \exp\left(-\frac{A_e}{A_0} \left(\frac{N \tau_b^p}{N_0 \sigma_0^p}\right)^m k_1(w, p_\sigma m_w)\right).$$

The stress on the surface σ_c depends on basic stress σ_b and spring index w . In the limit case of infinite spring index, the Eq. (11.15) provides the expression for the failure probability of the straight rod:

$$P_{R.3} = \lim_{w \rightarrow \infty} P_{S.3} = \exp\left(-\frac{A_e}{A_0} \frac{\tau_b^2}{\sigma_0^2} \frac{N}{N_0}\right) \quad (11.16)$$

The ratio of both formulas (11.21) and (11.16) consequently is:

$$\alpha_3 = \frac{P_{S.3}}{P_{R.3}} = \exp\left(-\frac{A_e}{A_0} \frac{\tau_b^2}{S_W^2} (k_1(w, p_\sigma m_w) - 1)\right) \quad (11.17)$$

At fourth determine the survival probability of spring that undergoes due to the volume defect (Kobeleev 2017b):

$$\begin{aligned} P_{S.4} &= \exp\left(-\frac{L_W}{V_0} \int_0^{2\pi} \int_0^r p_h(N, \tau_q(\rho, \varphi)) \rho d\rho d\varphi\right) \\ &= \exp\left(-\frac{L_W}{V_0} \int_0^{2\pi} \int_0^r \left[\frac{N}{N_0} \left(\frac{\tau_q(\rho, \varphi)}{\sigma_0}\right)^{p_\sigma}\right]^{m_w} \rho d\rho d\varphi\right) \end{aligned} \quad (11.18)$$

$$P_{S.4} = \exp\left(-\frac{V}{V_0} \left(\frac{N \tau_b^{p_\sigma}}{N_0 \sigma_0^{p_\sigma}}\right)^m k_2(w, p_\sigma m_w)\right) \quad (11.19)$$

The intensity of shear stress $\tau_q(\rho, \varphi)$ for the circular wire of the radius r is given by Eq. (2.30). Its limit case of the straight rod, the survival probability reads:

$$P_{R.4} = \lim_{w \rightarrow \infty} P_{S.4} = \exp\left(-k_6(\infty, m_w) \frac{V}{V_0} \frac{\tau_b^2}{\sigma_0^2} \frac{N}{N_0}\right) \quad (11.20)$$

The ratio of survival probabilities (11.19) and (11.20) follows:

$$\alpha_4 = \frac{P_{S.4}}{P_{R.4}} = \exp\left(-\frac{V}{V_0} \frac{\tau_b^2}{\sigma_0^2} \frac{N}{N_0} (k_6(w, m_w) - k_6(\infty, m_w))\right) \quad (11.21)$$

Once again, the dependence of wire diameter is different in cases of harmful defects on the surface and in the material core. This is an important dimensional effect, which could explain the sensitivity of experimental fatigue results upon the wire diameter.

The stress variation over the surface of the wire is higher for the springs with the lower spring indices and with correspondingly higher wire curvature. Accordingly, the failure probability for the same number of cycles and the same basic stress is higher for the springs with lower spring indices.

Consider the spring with spring index w to cycles to failure for the straight rod with the constant stress. For this spring, we determine the ratio of cycles to failure over its surface. In case of surface defects, cycles to failure for the straight rod and for the helical spring are to be determining from the equation:

$$\exp\left(-\frac{A_e}{A_0} \frac{\tau_b^2}{\sigma_0^2} k_5(w, p m_w) \frac{N_S}{N_0}\right) = P_S^*, \quad (11.22)$$

$$\exp\left(-\frac{A_e}{A_0} \frac{\tau_b^2}{\sigma_0^2} \frac{N_R}{N_0}\right) = P_S^* \quad (11.23)$$

These equations deliver the fatigue life for springs and for rod respectably::

$$N_S = -N_0 \ln(P_S^*) \left[\frac{A_e}{A_0} \frac{\tau_b^2}{\sigma_0^2} k_5(w, p_\sigma m_W) \right]^{-1}, \quad (11.24)$$

$$N_R = -N_0 \ln(P_S^*) \left[\frac{A_e}{A_0} \frac{\tau_b^2}{\sigma_0^2} k_5(\infty, p_\sigma m_W) \right]^{-1}, \quad (11.25)$$

such that for surface defects

$$N_R/N_S = k_5(w, p_\sigma m_W)/k_5(\infty, p_\sigma m_W) > 1. \quad (11.26)$$

Correspondingly, the ratio of cycles to failure for volume defects reads:

$$N_R/N_S = k_6(w, p_\sigma m_W)/k_6(\infty, p_\sigma m_W) > 1. \quad (11.27)$$

The functions $k_5(w, p_\sigma m_W)$ and $k_6(w, p_\sigma m_W)$ characterize the influence of spring index on the life time depending on Weibull parameter m_W , spring index w and reciprocal fatigue exponent $p_\sigma = -1/b_\sigma$.

Compare for this purpose two helical springs with equal Weibull parameters, basic stresses, desired survival probability and fatigue exponents. These two springs possess different spring indices w . The spring with high spring index possesses approximately the constant stress over it surface and therefore has nearly the same number of cycles to failure as the straight rod (Huhnen 1970), (Kloos and Kaiser 1977). The number of cycles to failure of the spring with low spring index differs significantly from the number of cycles to failure of the spring with high values of spring index. This difference greatly increases for the higher stress exponent of fatigue law. This circumstance must be considered in the design process and during the testing of springs.

The actual design procedure assumes that the springs with the same corrected stress (but not basic stress) possess the same fatigue life independently on the spring index. Strictly speaking, this statement is not accurate. Namely, the springs with low spring index and the same corrected stress have lower stress on the most outer surface in comparison with the springs with high spring index and the same corrected stress. As consequence, the springs with low spring index and the same corrected stress must have longer fatigue life in comparison with the springs with high spring index. In contrast, the springs with low spring index and the same basic stress must have shorter fatigue life compared with the springs with high spring index.

The derived formulas estimate precisely this effect and provide the direct estimation of fatigue life as the function of desired survival probability and spring index. The different thought was proposed by (Reich 2016, Sect. 8.6.2).

11.4 Conclusions

The weakest link concept is adopted using the Weibull distribution approach. The influence of stress variation in course of cyclic loading is accounted together with the effect of stress variation over the surface of the wire due to its curvature. The approach allows one to calculate the global probability distribution of the fatigue life for helical springs with different spring indices and different stress levels. The approach is applied to calculate the number of cycles to crack initiation of helical springs under different probability levels. The probability function of the fatigue limit for helical compression springs is given in closed form as the function of spring index and Weibull shape parameters solely. The springs with low spring indices exhibit in the case of high survival probabilities greater sensitivity to the variation of stress. The number of cycles to failure of the spring with low spring index differs significantly from the number of cycles to failure of the spring with high spring index.

11.5 Summary of Principal Results

- The traditional expressions for failure probability of springs account only the pointwise stress maximum and neglect the scattering of defects over the surface or over the volume of spring.
- The derived formulas correct the traditional expressions.
- The influence of variation of stress over the surface of the wire is exposed.
- The stress variation depends upon the spring index and wire diameter.
- The failure probability is expressed in terms of stress variation over the wire surface or over the wire volume.
- The derivation of failure probability is based on weakest link concept.

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Chapter 12

Statistical Effects on Fatigue of Spring Materials



Abstract In the present Chapter, we continue to study the stochastic influences on fatigue life of springs. At first, we evaluate the probabilistic effects prevailing at low amplitudes of cyclic stresses. The question is the calculation of failure probability as the function of stress amplitude. The answer to this question results from the examination of the experimental fatigue life data. The experimental data demonstrate different behaviors in the regions of low and high amplitudes of stress. To describe this phenomenon, we introduce the randomization to the crack propagation, which accompanies by the accidental deviation and branching of crack. The randomization of crack propagation escalates with the reducing stress amplitude. The high inhomogeneity of the polycrystalline structure on the micro level cause hypothetically the random propagation. This hypothesis leads to the mathematical model for the randomly propagating crack. The differential equation with stochastic coefficients describes the randomly travelling crack. This equation is analogous to the equation of the enforced Brownian motion. The examples of the solutions for the Brownian stochastic differential equation are presented. This Chapter is the final section of the last part of the book, which studies the lifecycle of springs.

12.1 Fatigue Analysis at Very High Number of Cycles

12.1.1 Fatigue Strength and Failure Mechanisms

Several intensively stressed springs (e.g. in valve trains or fuel injection systems) achieve the numbers of oscillation cycles of a billion or even higher magnitude during their service life. The fatigue analysis of structural materials at very high number of cycles was the subject of several studies, as reviewed at (Murakami 2003; Bathias and Paris 2005; Bathias 2014; Sander 2018; Christ 2018).

To make the processes of failure at very high numbers of cycles understandable, the structural materials are subdivided into Type I and Type II in (Mughrabi 2006). Type I materials are pure, single-phase, annealed and ductile metals without significant internal defects. The persistent slip band limit is defined here as the limit for crack initiation and the minimum stress for crack growth as the conventional fatigue limit.

Accumulation of less irreversible strains results in surface roughening. As the number of oscillation cycles increases, the roughness of the surface increases and thus leads to local stress concentration, which reaches the limit value for the formation of persistent slip bands. This leads to the formation of micro-persistent slip bands, in which strain localization takes place. As a result, the persistent slip bands expand in the direction of the inside of the specimen and can thus lead to crack formation on the surface.

12.1.2 Experimental Investigation for Low-Stress Springs' Fatigue

The scientific-technical results of the research project (AiF 12 287 N 2002) were the experimental investigation of the fatigue strength behaviour (Goodman diagrams) and the relaxation behaviour (relaxation diagrams) of coil compression springs made of the following material groups:

- Patented drawn steel wires (spring strength from a combination of preheat treatment and work hardening of the wire),
- Oil tempered steel wires (spring strength from final tempering of the wire),
- Stainless spring steel wires (spring strength from work hardening of the wire).

The aim of the research project (AiF 12 287 N 2002) was to determine and evaluate statistically verified results on the fatigue and relaxation behaviour of helical compression springs made of six types of spring steel wire (VDSiCr, TDSiCr, DH, DM, 1.4568 and 1.4310). The springs with wire diameters of 1, 2, 3, 5 and 8 mm were prepared for testing. The results of the project were intended for spring calculation and dimensioning. The fatigue behaviour of the test springs was investigated with the aid of three testing machines. Statistically evaluated S-N-lines were determined for 27 basic variants at different levels of medium and low stress. From these, fatigue strength diagrams according to Goodman were generated for a number of limit cycles of 10^7 cycles. These fatigue strength diagrams were compared with the previous diagrams in DIN 2089 for the survival probability $P_S = 90\%$. The experimentally determined endurance stroke strength values at 90% survival probability were determined for the materials:

- VDSiCr by about 300 to 400 MPa,
- TDSiCr by about 300 to 400 MPa,
- DH by about 200 to 350 MPa,
- DM by about 200 to 300 MPa,
- 1.4568 by about 80 to 100 MPa and
- 1.4310 by about 50 MPa.

The detailed results of the cited project were published in several articles. Specifically, (Kaiser and Berger 2005) reviewed the mentioned above assessments of fatigue

behaviour of springs. The significance of defects on the surface and in the surface-near layer decrease the fatigue properties of cold formed springs were evaluated. Besides of the internal “inclusion-type” defects, surface defects originate during hot forming or heat treatment, were observed. The influence of other defects, which result from drawing or coiling (“coiling marks”), was debated as well. In conclusion, the article (Berger et al. 2008) presents conclusions of experimental examination of the fatigue behaviour for helical springs at a very high number of load cycles.

The objective of the research project (AiF 15 064 N 2010) was to determine and evaluate the long-term vibration behaviour of coil springs in the oscillation range up to 10^9 vibration cycles. In the cited research project, the long-term fatigue tests were performed with a spring testing machine of the type “Reicherter DV8-S2”. The examined springs were the shot peened cylindrical helical compression springs with wire diameters of 1.6 and 3 mm. The materials for the spring production were the oil tempered valve spring steel wires VDSiCr and VDSiCrV as well as the stainless spring steel wire X7CrNiAl17-7 (material number 1.4568). The fatigue examination was accompanied by stereomicroscopic and scanning electron microscopic fracture surface investigations, metallographic investigations, X-ray residual stress measurements and microstructure determinations as well as torsion tests on wires.

The works (Pyttel et al. 2011; Kaiser et al. 2011) review the results of fatigue strength and failure mechanisms at very high number of cycles. Fatigue testing machinery for very high number of cycles is accounted. The classification of investigated materials is performed with respect to characteristic S–N curves and influencing factors like notches, residual stresses and environment.

The article (Schwerdt 2011) provides details on the discovered failure mechanisms, which occur especially in the VHCF-region. An important role on the emerging of a failure mechanism plays the microstructure of the homogeneities. The cited article proposes a double S–N curve for description of fatigue behaviour. The proposed method considers different failure mechanisms. Investigated materials are different metals with body-centred cubic lattice. Such materials, like low- or high strength steels and quenched and tempered steels, are used frequently for spring production. In (Schwerdt 2011) the aluminium alloys (EN AW-6082-T5, EN AW-6056-T6) and high-strength tempered alloy steel 42CrMo4 were studied experimentally in the VHCF-region.

The aim of the succeeding project (IGF 16873 N 2014) was to determine the influence of different materials and manufacturing parameters on the VHCF stroke strength of coil compression springs. For this purpose, the springs made of oil-quenched and tempered valve spring steel wire VDSiCr with $d = 1.6 \text{ mm}$ were manufactured. The influence of the end coil geometries, shot peening times and heat treatment temperatures after spring coiling was considered for the test springs. Primary tests of torsion influence, tensile and rotating bending as well as metallographic microstructure determinations or chemical analyses were carried out on the examined wires.

Test springs made of the low-cost stainless spring steel wire 1.4310 with wire diameter $d = 1.6 \text{ mm}$ as well as springs made of patented drawn spring steel wire DH with $d = 3.15 \text{ mm}$ were examined. In addition, springs made of high-quality

stainless spring steel 1.4568 were examined in two wire production variants with $d = 3.15$ mm. Springs made of the stainless spring steel 1.4310 were tested up to 10^9 load cycles in fatigue tests. In comparison to springs made of stainless spring steel 1.4568, springs made of 1.4310 have a lower VHCF lifting strength and service life.

With regard to springs made from the low-cost patented drawn spring steel wire DH, it was also possible to demonstrate their suitability for use in the VHCF sector depending on the specific requirements.

For springs with protruding and adjacent end coils, the oscillation tests up to $5 \cdot 10^8$ vibration cycles were performed for different stress amplitudes.

An increase in the amplitude of stress in VHCF region due to the protruding spring end was found. This design of the spring end coils leads to the favourable stress pattern along the length of the helically wound spring wire. Failures of the end coil were observed in both end coils.

The increasing of the duration time of shot peening showed a positive effect on the VHCF lifting strength of springs made of VDSiCr. Particularly, the heat treatment temperature after spring coiling has an improving effect on the strength amplitude in VHCF region.

For springs with a heat treatment temperature of $400^\circ\text{C}/30$ min occur relatively few failures in the regions of 10^9 oscillating cycles. However, the springs with a heat treatment temperature of $360^\circ\text{C}/30$ min and half the test duration ($5 \cdot 10^8$ oscillating cycles) produced more than twice as many fracture events. Thus, the expectancy of fatigue life for the latter type of springs is generally lower, than former ones.

The test springs were made of the alternatively manufactured wires of the material 1.4568 with a wire diameter $d = 3.15$ mm. The fatigue life of the alternatively manufactured springs was roughly equal.

In general, a change of the failure location from on the surface at high stress to under the surface at medium and low stress was observed. This feature could be explained. For the high loading, the damage due to defects on the surface overwhelms the positive effect of internal stresses. The detailed accounts of discussed above results for the project (IGF 16873 N 2014) were published in several works.

Pyttel et al. (2014) reported the results of long-term fatigue tests on shot-peened helical compression springs. The testing performed with the spring fatigue-testing machine at 40 Hz. Test springs were made of three different spring materials. The investigated materials were oil hardened and tempered SiCr- and SiCrV-alloyed valve spring steel and stainless steel. With a special test strategy in a test run, a batch of springs with different wire diameters were tested simultaneously at different stress levels. Based on fatigue investigations of springs up to a number of cycles 10^9 cycles an analysis was done. The test continued to $1.5 \cdot 10^9$ cycles. The results of both tests were compared. The influence of different shot-peening conditions is studied. The fractured test springs inspected for of fracture behaviour and the failure mechanisms. The paper (Pyttel et al. 2014) includes a comparison of the results of the different spring sizes, materials, number of cycles and shot-peening conditions.

Schuller et al. (2013) investigated very high cycle fatigue properties of VDSiCr spring steel were with ultrasonic equipment under fully reversed cyclic torsion loading and under cyclic axial loading at load ratios.

The spring manufacturing regularly uses the austenitic stainless steel. Lukacs (2010) acquired of the basic data for fatigue crack propagation limit or design curves on austenitic stainless steel (type 321) in corrosive environment and at elevated temperatures. This work estimated the design curves based on statistical analysis of measured data. The propagation law of fatigue crack was evaluated.

Mayer et al. (2017) examined VHCF properties of high-strength spring steel SWOSC-V for different load ratios in cyclic torsion test. For comparison, cyclic tension-compression and cyclic tension tests at the same load ratio were also performed. The ultrasonic fatigue testing technique was adopted for both torsional and axial loading fatigue examination. The mechanisms of crack initiation and strength in VHCF region were compared for torsion and axial load.

Nishimura et al. (2017) studied the torsional fatigue behavior in high cycle and very high cycle fatigue regimes. Definite tiny longitudinal notches were introduced on the surface of test specimens. Both uniaxial and torsion fatigue examinations were performed. A significant reduction of the uniaxial fatigue limit was observed. In contrast, the torsional fatigue limit was not considerably reduced. The analysis of the broken specimens with small longitudinal notch in torsion fatigue experiments confirmed, that the propagation of mode I crack leads to the final failure.

Fatigue tests were performed on four compositions of the 18Ni maraging steels in precipitation hardened condition in (Karr et al. 2017). One material contained the addition of titanium, the other increased *Co* content. The third type of material used alloying with *Al* and *Cr*. The *Al* content was increased in the fourth type of maraging alloy. For fatigue tests, the thin sheets with nitrated surfaces were manufactured. The VHCF-fatigue properties of the 18Ni maraging steels have been investigated at load ratio $R_\sigma = 0.1$. The origin of fatigue cracks in three material compositions were internal inclusions. However, one material composition with an increased *Al* content demonstrated the surface and internal crack initiation. Only the initial failures in this material were caused by internal inclusions.

In this study we perform the evaluation of the technically relevant S-N-curves for different material and load parameters, based on the reported experimental results (Kaiser and Berger 2005; AiF 12 287 N 2002; AiF 15 064 N 2010).

The results of the cited projects combine the fatigue tests of helical springs made of steel wires. Especially, *CrSi* and *CrSiV*-alloyed oil hardened and tempered spring steel wires) conformed to (DIN EN 10270, 2011) were selected. The springs were manufactured in so-called super-clean quality from peeled or ground raw material. The fatigue test was performed on the shot peened helical compression springs of *CrSi* -alloyed spring steel wire with diameter of 2 mm, a spring diameter of 12 mm and a free length of 50 mm and test results were presented in (Kaiser and Berger 2007; AiF 12287 N 2002; AiF 15 064 N 2010).

The investigations (Mayer et al. 2015, 2016) provide further information, especially the influence of stress ratios on the fatigue life.

The effect of the ultimate number of cycles of fatigue tests on fatigue time prediction for compression springs produced from *VDSiCr* spring wire was investigated in (Geilen et al. 2020). The state-of-the art of the long-term fatigue examination was reviewed. In this paper, the new kind of experiment, referred to as the “Artificial

Censoring Experiment”, was proposed. The “Artificial Censoring Experiment” was executed for the extrapolation of the results of fatigue tests on compression springs and for the reduction of the needed testing volume.

12.1.3 Modelling Hypothesis for Low-Stress Springs’ Fatigue

The reported above experimental results demonstrate two features of the fatigue process of low-stress fatigue of the spring materials.

At first, the stochastic scattering of fatigue data. The scattering widens with the decreasing amplitude of cyclic stress. To account the scattering, the known statistical methods are briefly reviewed. This task is important for applications, because the testing procedures provide usually only the median S-N curves and the scatter range (statistical moment of the second order). The applications require the guaranteed much lower probability of failures. The recalculation of S-N curves for this lower failure probability is reported.

At second, the clear alternation of the fatigue mechanism with the reduced stress amplitude was stated experimentally. This effect is visible as the flattening of the S-N curve in the region of low stress amplitude. The traditional Paris-Irvin model does not describe the alternation of the propagation mechanism in the region of low cyclic stresses. As reported in Chaps. 9 and 10, the propagation functions of the Paris-Irvin model are based on the constant stress exponent in the central region between the threshold and ultimate rupture stress. In logarithmic scaling, this description for the crack elongation pro load cycle proclaims the linear dependence of fatigue life upon the stress amplitude. To fit the observe experimental data, we divide the propagation function of Paris-Irvin law in the regions of mid- and low-stress level. As expected, the resulted S-N curve will have also two regions with the smooth transition between mid- and low-stress levels. To explain this division from the viewpoint of micromechanics, we associate the mid-stress crack propagation with the grain destruction by the crack. The crack crosses the grain, because it is the shortest way for crack propagation. On the low-stress level, the energy release will be insufficient for grain-destruction, and the crack travels along the weaker grain boundaries, which considerably increases the total area of crack and decreases the projected elongation pro load cycle. This explains the nonproportional reduction of the total crack length with the lower stress amplitudes. The mathematical description of this process is based on the differential equation with the stochastically varying coefficients. The introduced equation is analogues to the equation of Brownian motion. The crack has a definite propagation direction, like the enforced Brownian motion with the directionally enforced drift.

There exist alternative explanations of the above transition process from mid-stress level to low-stress level. Particularly, pure geometical explantion could be also imagined (Kobelev 2017a). The idea was a fractal nature of the crack surface. In the mis-stress region the crack is nearly planar. The reduction of cyclic stress causes the deviation of crack from the grain-destruction to the separation of grain interfaces, as described above. The deviations causes the transition from plane to

fractal surface of the crack. The total area of the fractal surface is a power function of the projected area. The creation of free fractal surfaces requires much more released elastic energy, which depends only upon the projected area of crack. Therefore, the crack propagation slows down in the low-stress level. The mathematical description of fractal surface is based on fractional calculus. The fractional-differential extension of Paris-Irvin equations, which describe the crack propagation were proposed. The unified propagation function describes the infinitesimal crack length growths per increasing number of load cycles, supposing that the load ratio remains constant over the load history. Two unification fractional-differential functions with different number of fitting parameters were discussed. The number of cycles to failure from a given initial crack length upon the stress amplitude follows from the solution of fractional differential equation.

12.2 Statistical Assessment of Springs Fatigue Test

12.2.1 *Fatigue Scattering of Spring Steels*

The technically relevant number of cycles to failure are in region of several millions to hundreds of millions. The practical reasons restrict the number of springs in the test series and the durations of tests. The heavy utilization of test equipment for high and ultrahigh number of cycles confines amount of the examined springs. The direct acquisition of S-N curves for different parameters of material and load is impossible for all springs in production. Consequently, the technical springs could be tested to failure only for the moderate number of cycles. Based on these series, the S-N curves for the average failure probabilities are generated. These S-N curves are not directly applicable for the failure forecasts in technically relevant numbers of cycles. However, these S-N curves are suitable for the extrapolation to the required S-N curves. We describe below the calculation of technically relevant S-N-curves (“Wöhler” lines) for different failure probabilities

$$P_F = 1 - P_S. \quad (12.1)$$

The value P_S in Eq. (12.1) is the survival probability.

In the probability plot, the scattering curve appears as a piecewise-straight function of load cycles to failure, as reported in (Kaiser and Berger 2007). The experimentally acquired data display the median S-N lines for the failure probability P_F of 50%. For the given number of cycles to failure, the values of the corresponding stress amplitudes were plotted. The presented method calculates the fatigue lines for the technically relevant failure probabilities, based on the S-N line of 50% failure probability.

We adopt the experimentally evaluated median S-N line for development of the model. Based on the experimental results, the fatigue crack growths (FCG) per cycle

is estimated. The generalizations of FCG describe crack extension rate, implementing the piecewise linear or continuous function.

Further, we use the FCG estimation for the probabilistic modelling on the microlevel of material structure. With this data, the stochastic differential equation for the travelling crack is established. The stochastic equation is analogous to the equation of the enforced Brownian motion. The generalizations of FCG (gFCG) functions explain the experimentally observed S-N curve in HCF and VHCF regions. From gFCG functions follow the closed form expressions of S-N curves for different failure probabilities, stress ratios and the geometry of microstructure.

12.2.2 Uniformity of Stress on Wire Surface and Averaged Fatigue Stress

There are several standard stress parameters for the helical springs. These factors were reviewed in Chaps. 1 and 2. The first known stress parameter is the “basic stress” τ_b (1.25). This stress parameter represents the average shear stress on the surface of the wire, the “basic”, or “uncorrected stress”, Eq. (2.4) (SAE 1997, Chap. 5, Design of Helical Springs).

The second known parameter is the “corrected stress”. It is equal to the maximum shear stress on the surface of the coil spring. The maximum shear stress is the product of the “basic stress” with the correction factor $k_\tau(w)$ (1.27). For example, the correction factor due to Bergsträsser (1.30) :

$$\tau_c = k_2(w)\tau_b, \quad (12.2)$$

$$k_2(w) = \frac{w + 0.5}{w - 0.75}. \quad (12.3)$$

However, only the maximum stress, but also the uniformity of stress distribution on the surface of the wire influences the failure probability of springs in series production. The more evenly the stress is distributed on the surface, the higher is the probability of defects and inclusions in the stressed area on the inner surface of the spring. This sounds paradoxical on the first thought. The explanation is the following. Consider two springs with different spring indices, but equal value of maximal stress on the surface. The region of increased stress on the inner surface of the spring with higher spring index is smaller, that of the spring with the lower spring index. Obviously, the probability of defects in a smaller area is lower, than probability of defects in a larger area. Thus, the stress, which concentrates in a smaller area, cause less damage, than the same stress, which spreads in a larger area. Both springs must reach the same number of load cycles until breakage. There are special stress correction factors, which assure, that if the corrected stress is equal in two different springs, the damage probability in series for two springs will be equal

(Kloos and Kaiser 1977; Huhnen 1970; FKM 2018, Sect. 2.1). The factors are based on the results of fatigue tests on coil springs.

There are other stress factors, which average the stress over the surface of the wire. These factors reflect the possibility of failure from possible defects in the fewer stressed regions. The simplest factor of this type $\bar{k}_2(w)$ is based on Bergsträsser correction factor (12.3):

$$\bar{k}_2(w) = \frac{k_2(w) + 1}{2} \equiv \frac{w - 0.125}{w - 0.75}. \quad (12.4)$$

Evidently, that the fatigue stress correction factor (12.4) averages the weighted stress over the wire section:

$$\tau_f = \bar{k}_2(w) \tau_b = \frac{\tau_b + \tau_c}{2}. \quad (12.5)$$

Figure 12.1 displays both correction factors and their relation. In (12.4) any other correction factor could be applied instead of Bergsträsser correction factor. However, the tiny difference between the known correction factors is not significant for fatigue calculations. The value τ_f is referred to as “fatigue stress”. Numerous experimental findings indicate, that the fatigue stress τ_f better correlates to the fatigue test results, than the basic stress τ_b or the corrected stress τ_c .

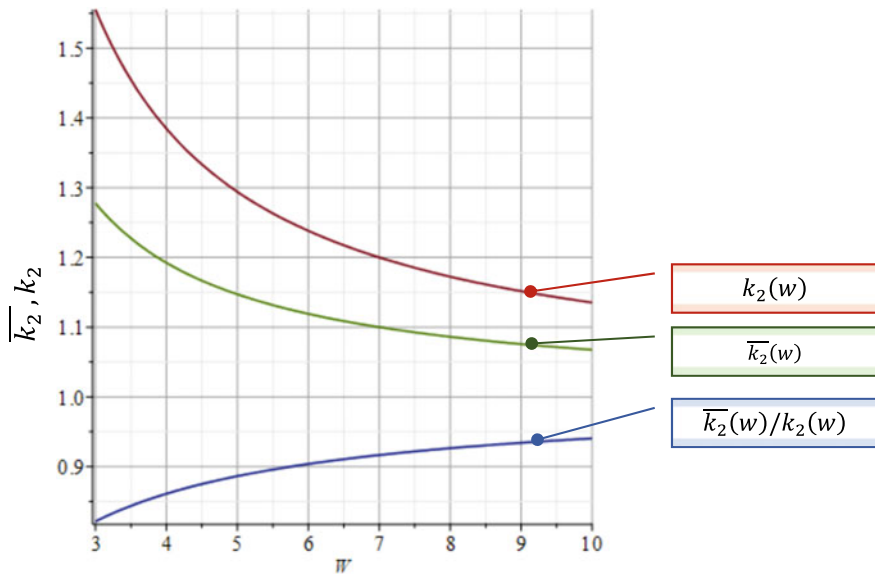


Fig. 12.1 Bergsträsser correction factor k_2 and averaged correction factor $\bar{k}_2$ as functions of spring index w

The correction factors for fatigue stress were discussed in Chaps. 1, 2. Based on the weakest link concept, several expressions for the stress correction factors and stress parameters were determined in Chap. 11.

12.2.3 *Experimental Acquisition for Sensitivity to Stress Ratio and Spring Indices*

The next important issue for fatigue estimation is the influence of the stress ratio, as discussed in Chaps 10. As shown above, there are three possible definitions for a scalar stress. The stresses are the harmonically oscillating functions of time with constant amplitudes and non-zero mean values (N):

$$\tau_b(t) = \tau_{m.b} + \tau_{a.b} \sin(\omega t), \quad \tau_{b.a} = \Delta \tau_b / 2 \quad (12.6)$$

$$\tau_f(t) = \tau_{m.f} + \tau_{a.f} \sin(\omega t), \quad \tau_{f.a} = \Delta \tau_f / 2 \quad (12.7)$$

$$\tau_c(t) = \tau_{m.c} + \tau_{a.c} \sin(\omega t), \quad \tau_{c.a} = \Delta \tau_c / 2. \quad (12.8)$$

The cycles are asymmetric (N) for three scalar, time-dependent stresses $\tau_b(t)$, $\tau_f(t)$, $\tau_c(t)$. The coefficients of the Eq. (12.6) to (12.8):

$$\tau_{a.b}, \quad \tau_{a.f} = \overline{k_2}(w) \tau_{a.b}, \quad \tau_{a.c} = k_2(w) \tau_{a.b},$$

are amplitudes of cycles (N) and

$$\tau_{m.b}, \quad \tau_{m.f} = \overline{k_2}(w) \tau_{m.b}, \quad \tau_{m.c} = k_2(w) \tau_{m.b}.$$

are mean values of cycles (N).

The stress ratios for the functions (12.6) to (12.8) are all equal:

$$R_\sigma = \frac{\tau_{m.b} - \tau_{a.b}}{\tau_{m.b} + \tau_{a.b}} = \frac{\tau_{m.f} - \tau_{a.f}}{\tau_{m.f} + \tau_{a.f}} = \frac{\tau_{m.c} - \tau_{a.c}}{\tau_{m.c} + \tau_{a.c}}. \quad (12.9)$$

For fatigue calculations, we use the experimental data, reported in the cited above reports. The data is ordered in the double logarithmic graphs. The abscise axis presents the number of cycles. The ordinate axis displays the amplitude of the corrected shear stress $\tau_{a.c}$. The immediate application of this data is not convenient, considering the following. As we already know, the amplitude of the corrected shear stress $\tau_{a.c}$ for the certain number of cycles depends on several factors. Firstly, $\tau_{a.c}$ depends upon the mean stress $\tau_{m.c}$, and secondly upon the spring index w . If the mean stress or the spring index changes, the ordinates of the S-N line also change. Accordingly, S-N lines are not invariant and do not display “true” material properties.

The material properties must be independent upon design parameter w . Therefore, we must “clean” the S-N lines from the design-dependent influences. The only difference between proposed S-N lines is the proper scaling of ordinate axis. After some preparations, the invariant S-N lines are calculated below.

To “clean” the S-N lines from the influence of the stress ratio, the $\mathbf{P}_{\text{SWT}}$ (Smith, Watson, Topper) and $\mathbf{P}_W$ (Walker) parameters are habitually applied (Chap. 10). However, these two parameters account the stress ratio (12.9) slightly different. We proceed with the parameter $\mathbf{P}_W$, because $\mathbf{P}_{\text{SWT}}$ parameter appears as the special case ($\tilde{\gamma} = 1/2$) of the first mentioned.

For each relevant harmonically oscillating functions $\tau_b(t)$, $\tau_f(t)$, $\tau_c(t)$, there are three effective parameters $\tau_{ar.b}$, $\tau_{ar.f}$, $\tau_{ar.c}$. The Walker parameters $\mathbf{P}_W$ are defined with Eq. (10.5):

$$\tau_{ar.b} = \tau_{a.b}^{\tilde{\gamma}} \tau_{max.b}^{1-\tilde{\gamma}}, \quad (12.10)$$

$$\tau_{ar.f} = \tau_{a.f}^{\tilde{\gamma}} \tau_{max.f}^{1-\tilde{\gamma}} = \overline{k_2}(w) \tau_{ar.b}, \quad (12.11)$$

$$\tau_{ar.c} = \tau_{a.c}^{\tilde{\gamma}} \tau_{max.c}^{1-\tilde{\gamma}} = k_2(w) \tau_{ar.b}. \quad (12.12)$$

The fitting parameter must satisfy $0 \leq \tilde{\gamma} \leq 1$. The recommended values of the fitting parameter for high strength spring steels are $\tilde{\gamma} = 0.15..0.25$.

The first version of the S-N lines applies the Walker parameter $\mathbf{P}_W$ for corrected stresses (12.12) as the ordinate axis. The dependence of S-N lines upon the mean stress disappears for the Walker exponent of $\tilde{\gamma} = 0.15$. For this setting of fitting parameter, several S-N lines with different stress ratios $0 < R_\sigma < 1$ overlap. Thus, this setting is invariant with respect to the stress ratio. If the range of mean stress sensitivity is considerably narrow, for example $0 < R_\sigma < 0.5$, the Walker parameter $\mathbf{P}_W$ describes the mean stress sensitivity properly. In the extended range, $-1 < R_\sigma < 0.5$ (Mayer et al. 2016), the adequate description requires the nonlinear dependence upon R_σ , as already discussed in Ch.10. For example, the generalized, or Bergmann-Walker parameter $\mathbf{P}_{\text{BW}}$ (10.11) covers the wide range of mean stress variation.

The second version uses the “corrected” $\mathbf{P}_{\text{SWT}}$ parameter (12.12) for ordinate axis. In this case one can put in the Eq. (12.10) to (12.12) the fixed value of fitting parameter $\tilde{\gamma} = 0.5$. We use this version only for reference, because numerous calculations in spring industry traditionally use $\mathbf{P}_{\text{SWT}}$ approach for corrected stresses.

Similarly, the third and fourth versions use either “basic” $\mathbf{P}_W$ or $\mathbf{P}_{\text{SWT}}$ parameters, which are grounded on “basic” stresses $\tau_{ar.b}$ instead of corrected stress, Eq. (12.10).

The springs with different spring indices still display different S-N lines, because the different spring indices would lead to different S-N curves. To clean this influence, the choose the averaged stress (12.5) as the ordinate. The ordinate in fifth and sixth versions is the mean value of “corrected” and “basic” effective stresses:

$$\tau_{ar.f} = (\tau_{ar.c} + \tau_{ar.b})/2. \quad (12.13)$$

This preferable scaling provides the best fitting of the fatigue data. The springs with different spring indices display coincident S-N lines.

The search of the optimal setting for the fitness parameters determines the following results. The best formulation adopts the $\mathbf{P}_w$ effective stress $\tau_{ar.f}$ (12.13) with the Walker exponent $\tilde{\gamma} = 0.15$. For this setting, the displayed S-N lines match independently on different stress ratios $0 < R_\sigma < 1$ and on different spring indices. Thus, these S-N curves in ordinates (12.11) could be considered as the true material characteristics.

12.2.4 Median S-N Curves

The number of cycles to total failure for wide ranges of corrected stress $\Delta\tau_c = 2\tau_{a.c}$ for common spring materials were reported in the cited above documents (AiF 12 287 N 2002) and (AiF 15 064 N 2010), (IGF 16873 N 2014). These results provide the start point for our calculations. The amplitude of corrected stress is

$$\tau_{a.c} = \frac{1}{2} \Delta\tau_c = \frac{(1 - R_\sigma)}{2} \Delta\tau_{max.c}.$$

According to Walker, Eq. (10.5), the relation between the corrected effective shear stress $\tau_{ar.c}$ and the range of corrected shear stress $\Delta\tau_c$ reads:

$$\tau_{ar.c} = C_R \Delta\tau_c, \quad C_R = \frac{1}{2} \Delta \left(\frac{1 - R_\sigma}{2} \right)^{\tilde{\gamma}-1}. \quad (12.14)$$

Using experimental results from the cited above references, the median S-N lines for different conceivable effective stresses could be evaluated:

$$\tau_{ar50} = \tau_{ar}(P_F = 50\%).$$

The median S-N lines are plotted in three different types of scaling for effective stresses, namely corrected, basic, and averaged.

Firstly, the median S-N lines for each tested spring are plotted in coordinates $(\tau_{ar.c}, n_f)$ with corrected effective stress (12.14) for ordinate.

Secondly, the basic effective stress evaluates from (12.14) with the formula:

$$\tau_{ar.b} = \tau_{ar.c} / k_2(w). \quad (12.15)$$

Thirdly, the median S-N lines for each tested spring are portrayed in coordinates $(\tau_{ar.b}, n_f)$, with the basic effective stress as ordinate.

As discussed in above, all three differently scaled S-N lines usually indicate the dependence upon the spring indices of the tested springs. This dependence is undesirable and must be eliminated. For practical calculations, the true material parameters

are necessary. To make the S-N –lines independent upon the spring indices, the averaged stress correction factor $\overline{k_2}(w)$ for fatigue estimations is applied. The averaged effective stress $\tau_{ar.f}$ is invariant to spring index and is also the true material constant:

$$\tau_{ar.f} = \overline{k_2}(w)\tau_{ar.b} \equiv \frac{\overline{k_2}(w)}{k_2(w)} \cdot \tau_{ar.c}. \quad (12.16)$$

Finally, the median S-N lines for each tested material are plotted in coordinates $(\tau_{ar.f}, n_f)$, with the averaged effective stress (12.16) and (12.13) shown on vertical axis. This exposition of the fatigue results is the most insensitive to the variations of spring indices and stress ratios.

One remark concerns the dependence of the strength of the wire over its diameter. This issue was discussed in Ch.10. As the relevant influence factors are accounted, the method is appropriate for design purposes of the mainstream industrial springs.

12.2.5 Valuation of S-N Lines for Prearranged Failure Probabilities

The experimental evaluation of S-N lines delivers the median S-N line, for which the failure probability is 50%. The cumulative failure probability of springs in industrial applications must be much lower. Its value P_F^* is prearranged by customer. For example, in automotive applications the fracture of valve springs in combustion motors repeatedly leads to the severe or even irreparable failure of an engine. The experimental evaluation of the cumulative failure probabilities in the region of their low values is a very consuming matter. At first, such test requires a great number of probes. At second, the stress level is so low, that the number of cycles must be of the order of hundred million. For this fatigue life, direct industrial tests are nearly impossible. In praxis, the S-N lines of high survival probability are derived from the median S-N line with survival probability 50%. Starting from the median S-N line, the S-N lines for different, prearranged failure probabilities P_F^* are evaluated using the experimentally acquired scatter range values. Commonly, the lognormal distribution is assumed (Kalbfleisch, Prentice 2002), (Castillo, Fernández-Canteli 2009), (Forbes et al. 2011). The lognormal distribution is briefly explained in Appendix C.

For example, failure probabilities the standard scatter range to $T_\sigma = 1/1.2 = 0.833..$ It is the ratio of the stresses, which lead after the number of cycles to failure with 90 and 10% probabilities. According to Eq. (C.6), this value corresponds to the scale factor $\mu = 0.071133$.

The effective stress for $P_F = 10\%$ for the same number of cycles to failure is equal to:

$$\tau_{ar10} = \tau_{ar}(P_F = 10\%).$$

Table 12.1 Recalculation of stress levels that correspond to different failure probabilities

Failure probability	Stress level			
P_F	$\tau_{ar}(P_F)/\tau_{ar50}$			
T_σ	1/1.2	1/1.234	1/1.3845	1/1.58
μ	0.071133	0.08203	0.1269	0.1784
50%	100%	100%	100%	100%
10%	91.2%	90%	85%	80%
1%	84.74%	84.3%	76.5%	68.6%
0.1%	80.26%	79.9%	69.9%	59.9%
100 ppm	76.755%	76.3%	64.5%	46.2%
10 ppm	73.83%	73.1%	59.7%	40.5%
1 ppm	71.36%	70.2%	55.4%	35.2%

The τ_{ar} stays for one effective stress $\tau_{ar,b}$, $\tau_{ar,c}$, $\tau_{ar,f}$. We determine another S-N line, which displays any other failure probability P_F in the same coordinates. In other words, we look for the effective stress $\tau_{ar}(P_F)$ for a given survival probability P_F^* and for the same number of cycles to failure. The calculation is based on the inverse error function Eq. (C.3). With the inverse error function, we have the relation:

$$\frac{\tau_{ar}(P_F = 50\%) - \tau_{ar}(P_F)}{\tau_{ar}(P_F = 50\%) - \tau_{ar}(P_F = 10\%)} = \frac{\text{ierf}(1 - P_F)}{\text{ierf}(1 - 10\%)}. \quad (12.17)$$

Resolution of the Eq. (12.17) delivers the effective stress $\tau_{ar}(P_F)$ for the certain P_F :

$$\tau_{ar}(P_F) = \tau_{ar50} - \frac{\text{ierf}(1 - P_F)}{\text{ierf}(1 - 10\%)} (\tau_{ar50} - \tau_{ar10}). \quad (12.18)$$

For automotive applications, the typical requirement for cumulative failure probability of springs is $P_F^* = 1 \text{ ppm}$. This the corresponding S-N line demonstrates the stresses of the level 71.36% of the median-S-N line. The corresponding S-N curves is the “design S-N curve” for the cumulative failure probability of $P_F^* = 1 \text{ ppm}$. Apparently, the correction “shifts” the 10%—S-N-Line towards abscise of the S-N graph. Table 12.1 demonstrates the calculation of the effective stress reduction for different vertical placements of S-N-lines. This calculation finishes the solution of the initially formulated problem. The relative reduction of the corrected effective stress amplitude $\tau_{ar}(P_F)/\tau_{ar50}$ are shown on Figs. 12.2, 12.3. The horizontal axes for failure probability P_F are given in logarithmical scale.

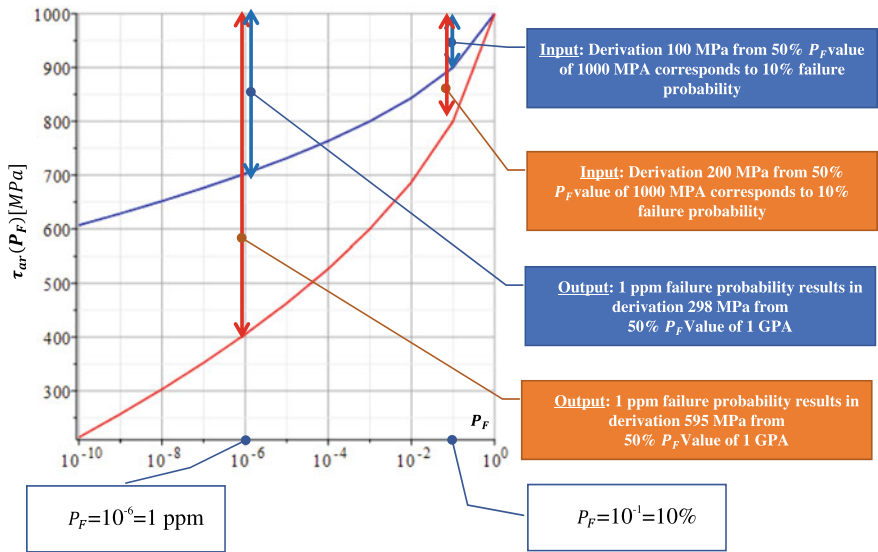


Fig. 12.2 Relative reduction of stresses $\tau_{ar}(FP)/\tau_{ar50}$ for different failure probabilities FP . The horizontal axis for failure probability is given in logarithmical scale

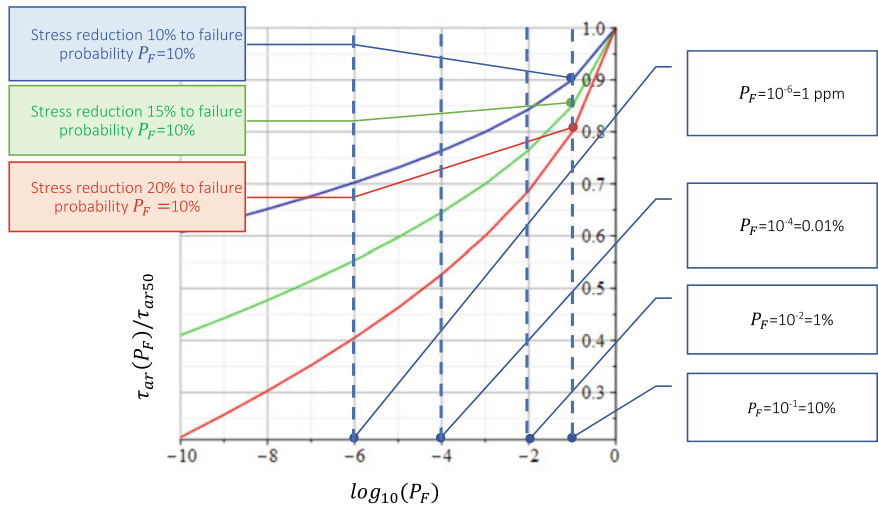


Fig. 12.3 Evaluation of stresses for different failure probabilities. The horizontal axis for failure probability is given in logarithmical scale

12.2.6 S-N Curves of Spring Steels in VHCF Range

The experimental results [IGF16873], (Mayer et al. 2015, 2016) for the cumulative failure probabilities for the tested springs are displayed on Figs. 12.4 and 12.5. The test springs are numbered from 1 to 11. The static mechanical properties, stress ratios R_σ and the amplitude of corrected stress $\tau_{c,a}$ are listed for the tested springs in Table 12.2. For the tested springs the cumulative failure probabilities are displayed on Fig. 12.4 in median probability range $R_\sigma = 0.1; 0.5; 0.9$. In this range the springs are tested and the scattering is evaluated. The values

$$R_\sigma = 10^{-1}; 10^{-2}; 10^{-3}; 10^{-4} 10^{-5}; 10^{-6}$$

are marked on Fig. 12.5 for cumulative probabilities in low range. This range is relevant for applications, as already discussed above.

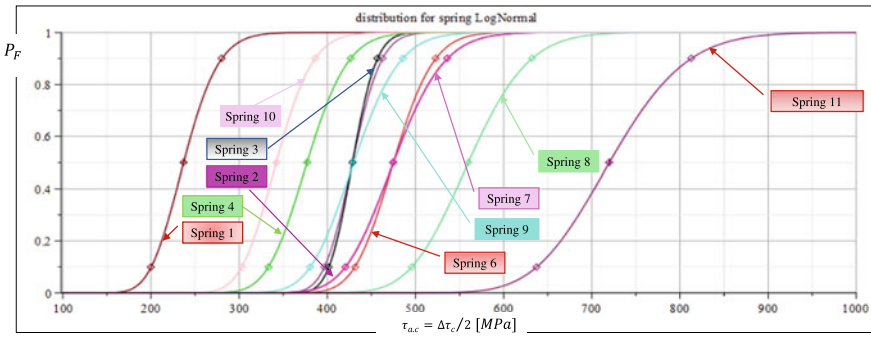


Fig. 12.4 Cumulative failure probabilities in the median range for tested springs. Abscise axis presents the amplitude of corrected stress $\tau_{a,c} = \Delta\tau_c/2$

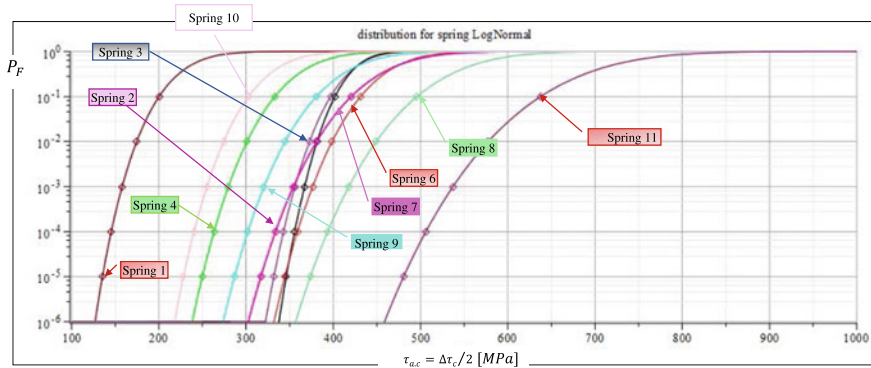


Fig. 12.5 Cumulative failure probabilities in the low range for tested springs. Abscise axis presents the amplitude of corrected stress $\tau_{a,c} = \Delta\tau_c/2$

Table 12.2 Experimentally determined values of scatter range T_σ and scale parameter μ of the lognormal distributions

	Source	Material		Tension Strength, MPa	Torsion Strength MPa	Torsional yield point, $\tau_{0.04}$, MPa	R_σ	Amplitude of corrected stress, MPa $\tau_{c,a}$
1	[IGF16873]	1.4310 X10CrNi18-8		1958	1491	694	0.3763	237
2		1.4568 X7CrNiAl17-7	New composition	1398	1400	677	0.2020	428
3			Conventional composition	1441	1345	640	0.2020	428
4		DH		1942	1389	846	0.286	377
5		VDSiCr 54SiCrV6	400 °C	2103	1645	1118	0.203	475
6			360 °C				0.203	475
7			400 °C ¹				0.203	475
8	(Mayer et al. 2015, 2016)			1984	1558		0.1	560
9							0.35	430
10							0.5	342
11							-1	720

1. flattened coils

Table 12.3 Values scatter range T_σ , scale parameter μ of the lognormal distributions for Walker exponents $\tilde{\gamma}$ (Exp. Data [IGF16873] [Mayer et al. 2015, 2016])

	$1/T_\sigma$	μ	$\tilde{\gamma}$	$p_\sigma = -1/b_0$	b_0	τ'_{f0b}	τ'_{f0f}	τ'_{f0c}
	Scatter range of failure probabilities	scale parameter of lognormal distribution	Walker exponent	Reciprocal strength exponent	Strength exponent	basic, MPa	Averaged fatigue, MPa	Corrected, MPa
1	1.4	0.131275	0.15	10	– 0.1	2065	2299	2534
2	1.166	0.059919	0.15	16	– 0.0625	1392	1550	1708
3	1.137	0.050093	0.15	12	– 0.833..	2143	2387	2631
4	1.28	0.096312	0.15	72	– 0.01388..	482	536	591.1
5	1.275	0.094786	0.15	29	– 0.034482..	864	963	1061
6	1.213	0.075337	0.15	23	– 0.04348..	1041	1160	1278
7	1.275	0.094786	0.15	33	– 0.03030..	792	882	972
8	1.275 ²	0.094786	0.15	33 ³	– 0.03030..	884	985	1086
9			0.15			895	998	1099
10			0.15			890	992	1093
11			0.15			577	642	708.3

2. Estimated value

3. Estimated value

The first method is based on application of Walker parameter ($\mathbf{P}_W$) for the effective stress. Table 12.3 summarizes the results of the fatigue evaluations. This table displays the values of scatter ranges T_σ , scale parameters μ , $b(R_\sigma)$, the reciprocal stress exponent $p_\sigma = -1/b(R_\sigma)$, fitted Walker exponent $\tilde{\gamma}$ and the coefficients τ'_{f0c} , τ'_{f0b} , τ'_{f0f} for evaluations of fatigue life. The stress-life approach is applicable, because of low stress and high number of cycle the plastic deformation is excessively low. Hereafter assumed, that the stress exponent $b(R_\sigma) = b_0$ is independent on stress ratio. For evaluation of the median number of cycles to failure, one of the effective Walker parameters $\mathbf{P}_W$ must be substituted in the corresponding formulas for the Basquin's or Wöhler's law Eq. (10.1b), (10.5b). The substitution of Eq. (12.10) to (12.12) leads to the expressions of number of cycles in terms of three different effective stresses:

$$n_f = \frac{1}{2} \left(\frac{\tau_{ar.c}}{\tau'_{f0c}} \right)^{\frac{1}{b(R_\sigma)}}, n_f = \frac{1}{2} \left(\frac{\tau_{ar.b}}{\tau'_{f0b}} \right)^{\frac{1}{b(R_\sigma)}}, n_f = \frac{1}{2} \left(\frac{\tau_{ar.f}}{\tau'_{f0f}} \right)^{\frac{1}{b(R_\sigma)}}. \quad (12.19)$$

The coefficients τ'_{f0c} , τ'_{f0b} , τ'_{f0f} are listed in the Table 12.3.

Table 12.4 Values scatter range T_σ , scale parameter μ of the lognormal distributions for P_{SWT} (Exp. Data [IGF16873] (Mayer et al. 2015, 2016))

	$1/T_\sigma$	μ	$\tilde{\gamma}$	$p_\sigma = -1/b_0$	b_0	τ'_{f0b}	τ'_{f0f}	τ'_{f0c}
	Scatter range of failure probabilities	scale parameter of lognormal distribution	Walker exponent	Reciprocal strength exponent	Strength exponent	basic, MPa	Averaged fatigue, MPa	Corrected, MPa
1	1.4	0.131275	0.5	10	– 0.1	1373	1529	1685
2	1.166	0.059919	0.5	16	– 0.0625	1009	1123	1238
3	1.137	0.050093	0.5	12	– 0.833..	1554	1730	1907
4	1.28	0.096312	0.5	72	– 0.01388..	339	377	415
5	1.275	0.094786	0.5	29	– 0.034482..	626	697	769
6	1.213	0.075337	0.5	23	– 0.04348..	755	840	926
7	1.275	0.094786	0.5	33	– 0.03030..	574	639	705
8	1.275	0.094786	0.5	33	– 0.03030..	669	745	821
9			0.5			605	673	742
10			0.5			548	610	673
11			0.5			577	642	708.3

Strictly saying, the Eq. (12.19) calculate three slightly different numbers of cycles to failure. If significant differences between three calculated number of cycles arise, we recommend an additional, exhaustive experimental study.

The value $\tilde{\gamma} = 0.15$ fits the acquired data for the spring steels. In this choice, for different $0 < R_\sigma < 1$ the S-N lines coincide undependably on stress ratio. The small value of $\tilde{\gamma}$ indicates an increasing sensitivity to mean stress for the high-strength spring wires. This feature is typical for high the strength and hardness alloys.

The second way is based on the parameter of Smith, Watson, Topper (P_{SWT}). The effective P_{SWT} parameters for corrected, basic and averaged formulations according to (10.3) are:

$$\tau_{ar.c} = \sqrt{\tau_{a.c}\tau_{max.c.}}, \quad \tau_{ar.b} = \sqrt{\tau_{a.b}\tau_{max.b}}, \quad \tau_{ar.f} = \sqrt{\tau_{a.f}\tau_{max.f}}.$$

With these definitions of effective stress, the number of cycles to failure provides once again the Eq. (12.19) with the matching coefficients. The proper coefficients for fatigue evaluations, based on P_{SWT} effective stress are presented in Table 12.4. This table summarizes the corresponding values of scatter ranges T_σ , scale parameters μ , strength exponent $b(R_\sigma)$, the reciprocal stress exponent $p_\sigma = -1/b(R_\sigma)$ and the coefficients $\tau'_{f0.c}$, τ'_{f0b} , τ'_{f0f} for evaluations of fatigue life. Clearly, that solitary coefficients $\tau'_{f0.c}$, τ'_{f0b} , τ'_{f0f} differ from the similar values of Table 12.3. The stress exponents and scatter ranges are in both cases equal.

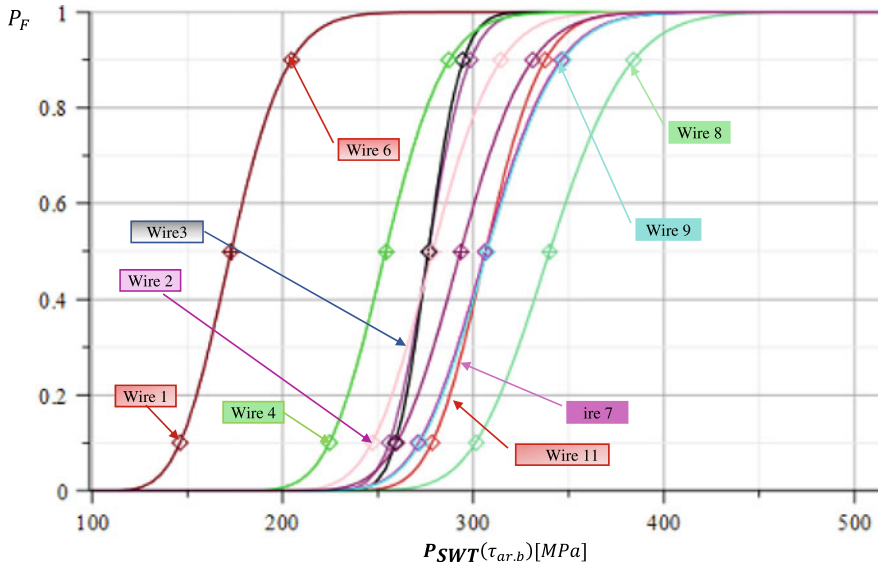


Fig. 12.6 Cumulative failure probabilities in the median range for wires. Abscise axis presents the averaged P_{SWT} parameter $P_{SWT}(\tau_{ar.b})$

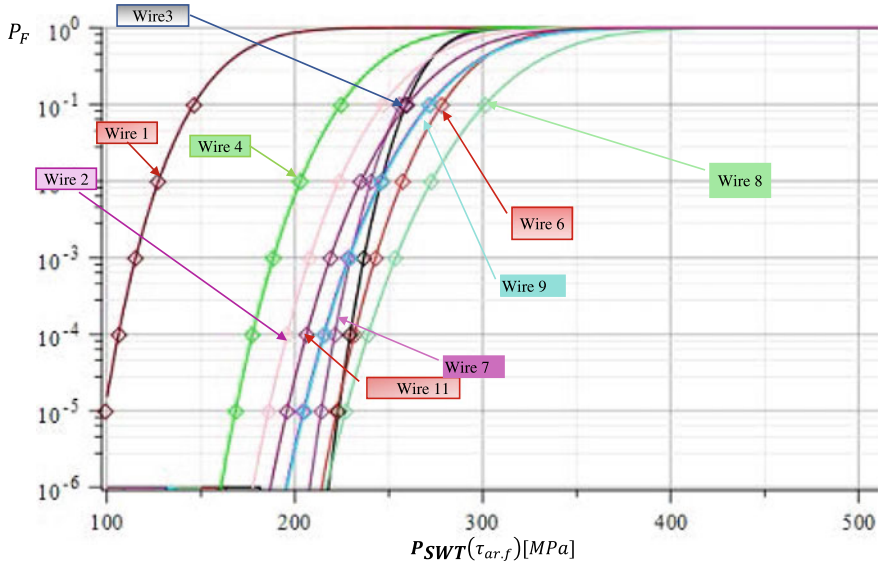


Fig. 12.7 Cumulative failure probabilities in the low range for wires. Abscise axis presents the averaged P_{SWT} parameter $P_{SWT}(\tau_{ar.f})$

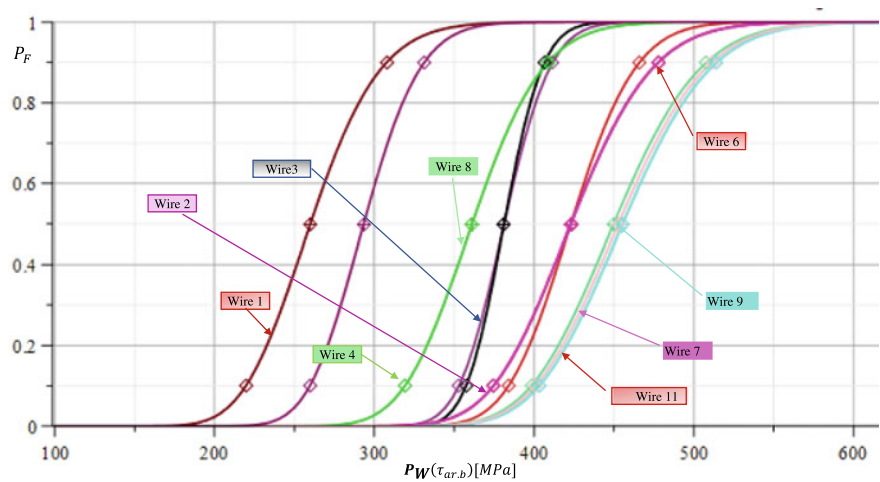


Fig. 12.8 Cumulative failure probabilities in the median range for wires. Abscise axis presents the averaged Walker parameter $PW(\tau_{ar,b})$

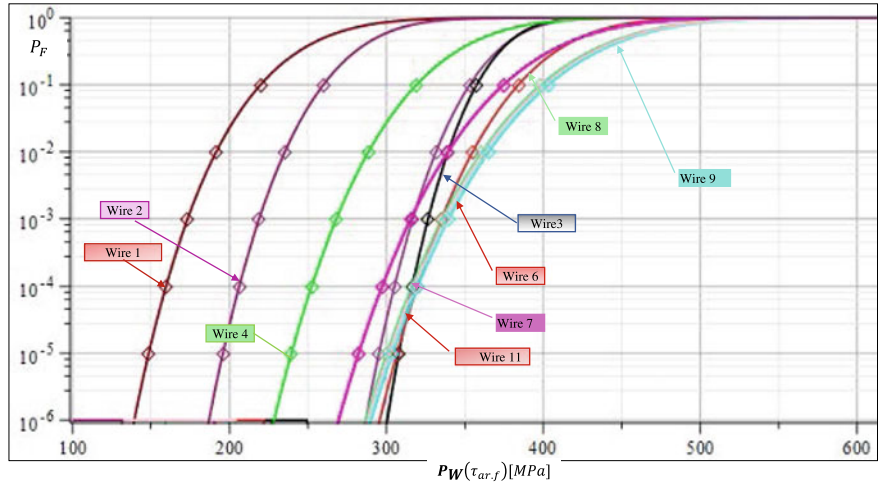


Fig. 12.9 Cumulative failure probabilities in the low range for wires. Abscise axis presents the averaged Walker parameter $PW(\tau_{ar,f})$

Cumulative failure probabilities for the wires are presented on the plots (Figs. 12.6, 12.7, 12.8 to 12.9). These curves display actually the material parameters, which are independent on spring indices and stress ratios. The curves of cumulative failure probabilities for the application of the effective $\mathbf{P}_{SWT}$ parameters on abscise axis are displayed on Figs. 12.6 and 12.7 in two different ranges (medium and low

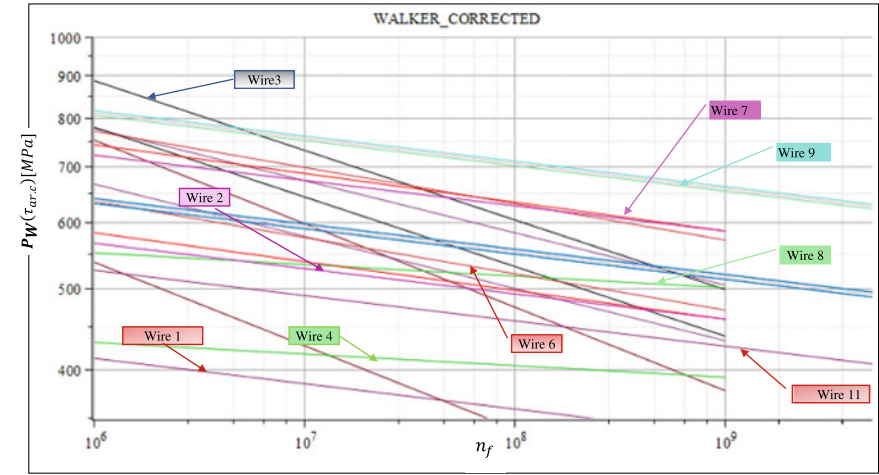


Fig. 12.10 Median S-N lines of wires. Ordinate presents the amplitude of corrected Walker parameter $PW(\tau_{ar,c})$

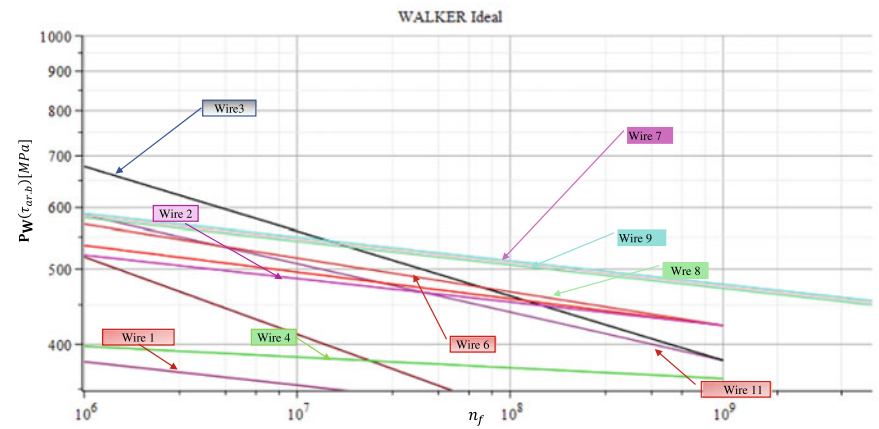


Fig. 12.11 Median S-N lines of wires. Ordinate presents the basic Walker parameter $PW(\tau_{ar,b})$

failure probabilities). Figures 12.8 and 12.9 show cumulative failure probabilities for the effective Walker parameters P_W on the abscise axis. Two graphs display the probability functions in medium and low failure probability ranges. (Figures 12.10, 12.11)

12.3 Scale-Dependent Propagation of Straight Crack

12.3.1 S-N Curves of Spring Steels in Transition Range

Well known (Abi et al. 2001, Sect. 12.3), (FKM 2018, Sect. 2.1.2.2), that around a certain point occurs the transition from high to low amplitudes of the stress. In logarithmic scaling, the fatigue life curves are nearly piecewise-straight lines with a corner point. The slopes of the straight lines left and right from the corner point are different. Because of significant increasing of the numbers of cycles to failure for the stresses below the corner point, the corresponding stress is commonly referred to as the endurance strength. It is frequently accepted, that the stress level below the endurance strength can be applied to a material without causing fatigue failure for an infinite number of loading cycles. For the models of crack propagation, the assumption of endurance strength appears as endurance threshold limit for the stress intensity factor (Sect. 8.3.2). However, the assumptions of endurance limits could be unacceptable for the advanced machinery.

The S-N-curves evaluation for *SiCrV* valve spring wire is based on the experimental results (Kaiser and Berger 2005). The curves represent fatigue life up to $7 \cdot 10^8$ cycles of shot peened helical compression springs of *SiCr*-alloyed valve spring wire, DIN EN 10270, with $d = 2 \text{ mm}$. The S-N curves were obtained for $R_\sigma = 0.05$. The original Fig. 12.12 from (Kaiser and Berger 2007) displays S-N curves for 10, 50 and 90% survival probability. The ordinate is the range of corrected shear stress $\Delta\tau_c$. To present the results in invariant form, the P_{SWT} parameter was built, starting from the corrected shear stress $\Delta\tau_c$. The equations if the S-N lines in terms of effective

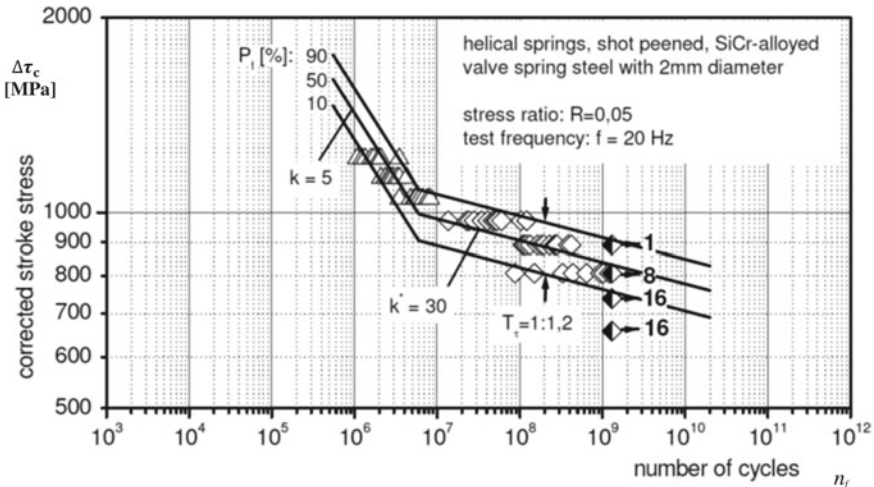


Fig. 12.12 S-N curves for 10, 50 and 90% survival probability for shot-peened helical compression springs of Si-Cr-alloyed valve spring wire with $d = 2 \text{ mm}$ (Kaiser and Berger 2005)

Table 12.5 Formulas for S-N lines in terms of effective stresses τ_{ar} for $T_\sigma = \frac{1}{1.234}$. The influence of mean stress is accounted using P_{SWT} approach (Exp. Data (Kaiser et al. 2011))

Walker exponent	$\tilde{\gamma} = \frac{1}{2} \leftrightarrow P_{SWT}$	
Region	UHCF ⁴	HCF ⁵
	$k_{UHCF} = 30$	$k_{HCF} = 5$
	$2 \cdot 10^6 \leq n_f \leq 2 \cdot 10^{10}$	$7 \cdot 10^5 \leq n_f \leq 2 \cdot 10^6$
$P_F = 90\% \ T_\sigma = 1/1.234$	$\tau_{ar} = 1342 \cdot n_f^{-1/k_{UHCF}}$	$\tau_{ar} = 18098 \cdot n_f^{-1/k_{HCF}}$
$P_F = 50\% \ P_F$	$\tau_{ar} = 1220 \cdot n_f^{-1/k_{UHCF}}$	$\tau_{ar} = 16453 \cdot n_f^{-1/k_{HCF}}$
$P_F = 10\% \ P_F$	$\tau_{ar} = 1098 \cdot n_f^{-1/k_{UHCF}}$	$\tau_{ar} = 14807 \cdot n_f^{-1/k_{HCF}}$
$P_F = 0.1\% = 10^{-3}$	$\tau_{ar} = 976 \cdot n_f^{-1/k_{UHCF}}$	$\tau_{ar} = 13161 \cdot n_f^{-1/k_{HCF}}$
$P_F = 1 \text{ ppm} = 10^{-6}$	$\tau_{ar} = 857 \cdot n_f^{-1/k_{UHCF}}$	$\tau_{ar} = 11560 \cdot n_f^{-1/k_{HCF}}$

4. Ultra-high cycle fatigue, $2 \cdot 10^6 \leq n_f \leq 2 \cdot 10^{10}$

5. High cycle fatigue, $7 \cdot 10^5 \leq n_f \leq 2 \cdot 10^6$

stresses τ_{ar} are shown in Tables 12.5 and 12.6. Thus, the fatigue life curves are nearly piecewise-straight lines with the corner point at $5 \cdot 10^6$ cycles.

To the left from the corner point there is the region of high amplitudes of stress intensity factor and the number of cycles to failure is less than $5 \cdot 10^6$ cycles. To the right of the corner point the amplitudes of stress intensity factor are below the endurance strength and the fatigue life is significantly longer. Using the preferred method (Sect. 12.2.3) for displaying the material properties, the S-N lines for different failure probabilities are displayed. The results of the evaluation of the S-N lines for 0.1%, 1 ppm failure probability are displayed on Figs. 12.13, 12.14, 12.15. For the characterization the Walker parameter P_W with different exponents was applied. Figure 12.13 shows S-N curves for Walker exponent $\tilde{\gamma} = 2/3$, Fig. 12.14 for Walker exponent $\tilde{\gamma} = 1/3$ and Fig. 12.15 for $\tilde{\gamma} = 1/2$ or P_{SWT} effective stress. The lower values of Walker exponents preferably match the S-N lines for different mean stress. Regrettably, the experimental results (Kaiser and Berger 2005) do not cover sufficiently the different mean stresses.

In this Section, we apply the standard methods for evaluation of the effective S-N lines. The explanations of such material behavior are based on the micromechanical models. The possible models to explain the alternation of micromechanical mechanism will be discussed in Sect. 12.4.

12.3.2 Piecewise Linear Paris –Irvin Equation

As shown above, the numerous experimental studies display the S-N curves as nearly piecewise-linear graphs. This phenomenon could be explained from the crack propagation viewpoint. As the first step, we examine the planar crack with two regimes

Table 12.6 Formulas for S-N lines in terms of effective stresses τ_{ar} for $T_\sigma = 1/1.234$. The influence of mean stress is accounted using Pw approach with different Walker exponents $\tilde{\gamma}$ (Exp. Data [Kaiser et al. 2011])

Walker Exponent	$\tilde{\gamma} = \frac{1}{3}$				$\tilde{\gamma} = \frac{1}{2} \leftrightarrow P_{\text{SWT}}$				$\tilde{\gamma} = \frac{2}{3}$	
Region	UHCF $k_{\text{UHCF}} = 30$	HCF $k_{\text{HCF}} = 5$	UHCF $k_{\text{UHCF}} = 30$	HCF $k_{\text{HCF}} = 5$	UHCF $k_{\text{UHCF}} = 30$	HCF $k_{\text{HCF}} = 5$	UHCF $k_{\text{UHCF}} = 30$	HCF $k_{\text{HCF}} = 5$	UHCF $k_{\text{UHCF}} = 30$	HCF $k_{\text{HCF}} = 5$
n_f	$2 \cdot 10^6 \leq n_f \leq 2 \cdot 10^{10}$	$7 \cdot 10^5 \leq n_f \leq 2 \cdot 10^6$	$2 \cdot 10^6 \leq n_f \leq 2 \cdot 10^6$	$7 \cdot 10^5 \leq n_f \leq 2 \cdot 10^{10}$	$2 \cdot 10^6 \leq n_f \leq 2 \cdot 10^6$	$7 \cdot 10^5 \leq n_f \leq 2 \cdot 10^6$	$2 \cdot 10^6 \leq n_f \leq 2 \cdot 10^{10}$	$7 \cdot 10^5 \leq n_f \leq 2 \cdot 10^{10}$	$2 \cdot 10^6 \leq n_f \leq 2 \cdot 10^{10}$	$7 \cdot 10^5 \leq n_f \leq 2 \cdot 10^6$
	$\left(\frac{\tau_{FL}}{\tau_{ar}}\right)^{-k_{\text{UHCF}}}$	$\left(\frac{\tau_{FL}}{\tau_{ar}}\right)^{-k_{\text{HCF}}}$	$\left(\frac{\tau_{FL}}{\tau_{ar}}\right)^{-k_{\text{UHCF}}}$	$\left(\frac{\tau_{FL}}{\tau_{ar}}\right)^{-k_{\text{HCF}}}$	$\left(\frac{\tau_{FL}}{\tau_{ar}}\right)^{-k_{\text{UHCF}}}$	$\left(\frac{\tau_{FL}}{\tau_{ar}}\right)^{-k_{\text{HCF}}}$	$\left(\frac{\tau_{FL}}{\tau_{ar}}\right)^{-k_{\text{UHCF}}}$	$\left(\frac{\tau_{FL}}{\tau_{ar}}\right)^{-k_{\text{HCF}}}$	$\left(\frac{\tau_{FL}}{\tau_{ar}}\right)^{-k_{\text{UHCF}}}$	$\left(\frac{\tau_{FL}}{\tau_{ar}}\right)^{-k_{\text{HCF}}}$
90% P_F	$\tau_{FF} = \tau_{90\%}$									
	1519	20489	1324	18098	1185				15986	
50% P_F	$\tau_{FF} = \tau_{50\%}$									
Median line	1381	18626	1220	16452	1078				14531	
10% P_F	$\tau_{FF} = \tau_{10\%}$									
	1243	16764	1098	14807	970				13080	
0.1% = $\tau_{FF} = \tau_{0.1\%}$										
$10^{-3} P_F$	1105	14900	976	13161	862				11626	
$1 \text{ ppm} = \tau_{FF} = \tau_{1 \text{ ppm}}$										
$10^{-6} P_F$	970	13087	857	11560	757				10211	

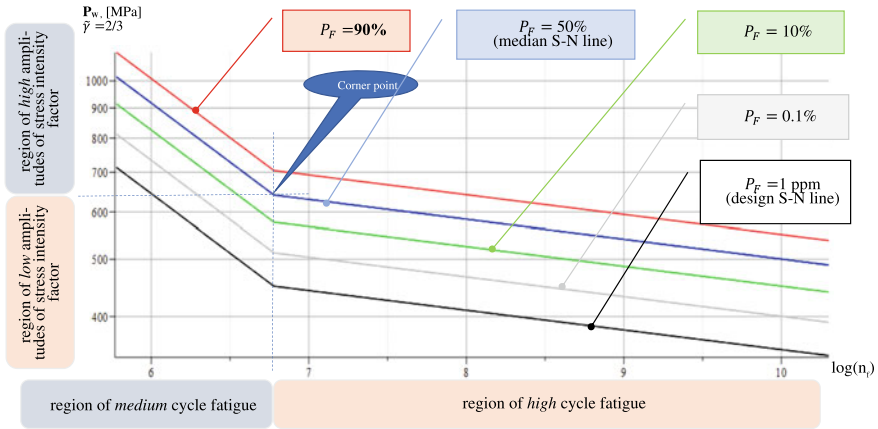


Fig. 12.13 S-N curves for Walker exponent $\tilde{\gamma} = 2/3$ (Kaiser and Berger 2005)

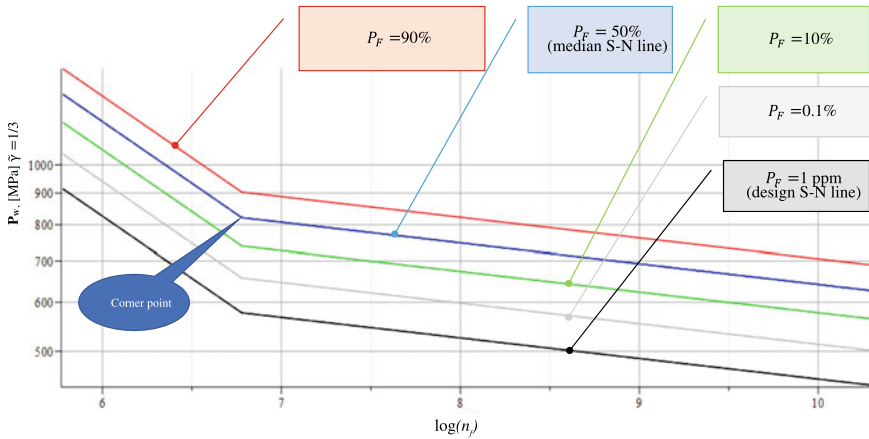


Fig. 12.14 S-N curves for Walker exponent $\tilde{\gamma} = 1/3$ (Kaiser and Berger 2005)

of propagation. The growth of cracks occurs in the direction that attempt to maximize the subsequent energy release rate and minimize the strain energy density. The optimum propagation path maintains an orientation normal to the maximum extension strains. Thus, the shape of the crack should be an ideally straight line. The above formulations FCG were based on endurance limit concepts, such (9.22), (9.23) and (9.36). These formulations FCG cannot correctly describe the piecewise-linear S-N curves. As an alternative, the piecewise-linear propagation function is introduced for the explanation of this phenomenon.

For this purpose, we consider the crack, which propagates in a plane plate. The straight line, which connects both tips of the crack, will be referred to as “director vector” of the crack. In the XY coordinates, the director vector of the ideally straight

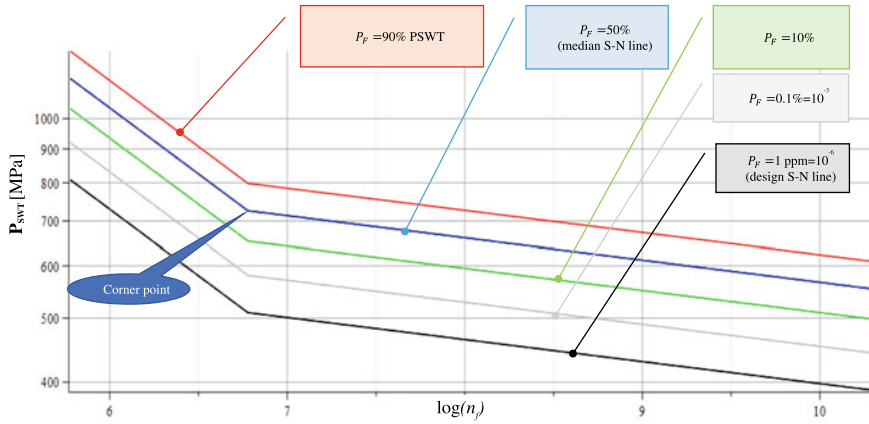


Fig. 12.15 S-N curves for Walker exponent $\tilde{\gamma} = 1/2$, P_{SWT} , (Kaiser and Berger 2005)

crack lengthens *exactly* towards the X -axis. The tensile stress acts in the direction of the Y -axis. The range of the component tension stress $\sigma_{Y\gamma}$ is $\Delta\sigma$. Thus, the range of stress intensity factor of the planar crack is proportional to the square root of its length a :

$$\Delta K = \Delta\sigma \cdot \sqrt{\pi a}.$$

If the length of crack is tiny, the elongation of crack length pro cycle gives δ_l . This region is the low range fatigue region, $\Delta K \leq \Delta K^*$. Crack tries to follow the outer grain boundaries, because of the high defect density of the crystal lattice and weakness of polycrystalline structure.

When the crack grows, the process of crack extension accelerates and the elongation of crack length pro cycle will be higher δ_s . Crack acquire enough strength to deform the polycrystalline on its tip and could spread in a shortest way across the grain regions. Both processes occur simultaneously, but the tendency of grain-destruction dominates for the higher stresses, $\Delta K \geq \Delta K^*$. The piecewise-linear propagation function is shown on Fig. 12.16. The value ΔK^* matches the ordinate of the “corner point” in coordinates $\Delta K - \delta$ (Fig. 12.16).

The increments of the crack length pro cycle are determined by two Paris-Irvin relations. Planar crack propagates along the X -axis. The length increments for both regions pro one load cycle are:

$$\delta_s = c_s \Delta K^{p_s}, \text{ if } \Delta K \geq K^*; \quad \delta_l = c_l \Delta K^{p_l}, \text{ if } \Delta K \leq K^*. \quad (12.20a)$$

The parameters c_s, c_l, p_s, p_l express the crack growth for both ranges of stress intensity factor.

To describe the transition region between both regimes, the total length increment pro one load cycle is assumed as:

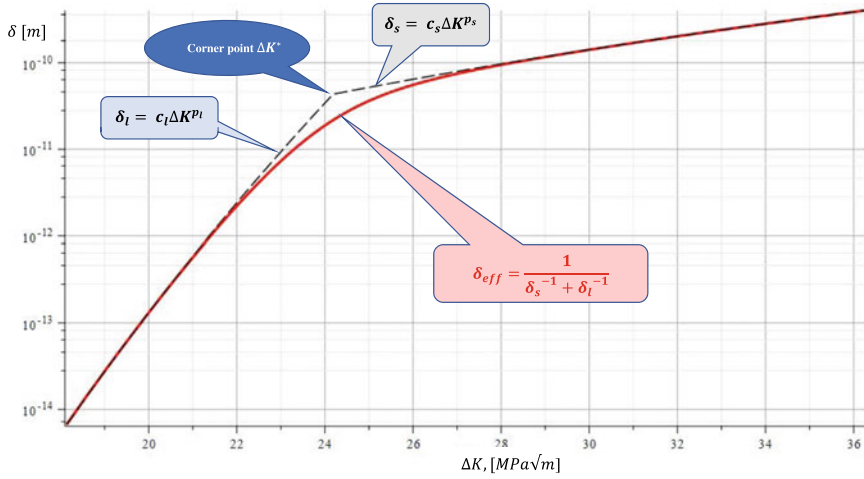


Fig. 12.16 Piecewise-linear propagation function

$$\delta_{eff} = (\delta_s^{-1} + \delta_l^{-1})^{-1}. \quad (12.20b)$$

The hypothesis (12.20a) leads to the piecewise-linear propagation function and abrupt changing of the propagation character. On the other hand, the expression (12.20b) accounts the parallel contributions of the in-grain (grain- destruction) and the grain-boundary growth. For the setting (12.20b), the smooth transition region arises instead of a solitary cross-point between two regimes (Fig. 12.16).

The value of stress intensity factor at the “corner point” ΔK^* follows from the condition $\delta_s = \delta_l$:

$$\Delta K^* = \left(\frac{c_s}{c_l} \right)^{\frac{1}{p_l - p_s}}. \quad (12.20c)$$

The application of Eq. (12.20) modifies the Paris-Irvin equation Eq. (9.21):

$$\frac{da}{dn} = F(\Delta K), \quad F(\Delta K) = \left(\frac{1}{c_s \Delta K^{p_s}} + \frac{1}{c_l \Delta K^{p_l}} \right)^{-1}. \quad (12.21)$$

From (12.21) follow the limit asymptotic S-N lines for both low and high ranges of stress intensity factor:

$$F_s(\Delta K) = c_s \Delta K^{p_s}, \quad \text{if } \Delta K \geq \Delta K^*, \quad (12.22a)$$

$$F_l(\Delta K) = c_l \Delta K^{p_l}, \quad \text{if } \Delta K \leq \Delta K^*. \quad (12.22b)$$

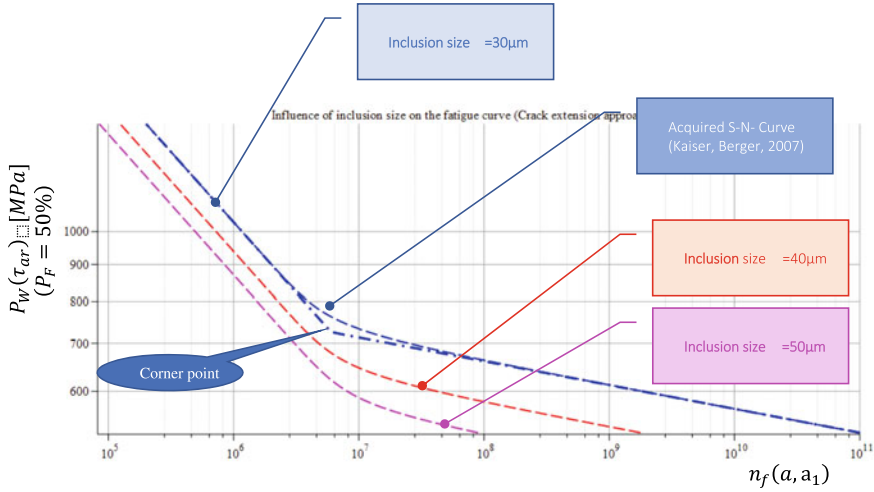


Fig. 12.17 Influence of inclusion size on the fatigue curve (Crack extension approach)

The crack spreads from the initial inclusion size δ to the final size of wire diameter d . The solution of Eq. (12.21) with the initial condition $n(a = \delta) = 0$ delivers the number of cycles to total failure $n_f = n(a = d)$:

$$n_f = n_f(c_s, c_l) = \left[\frac{2}{c_s(p_s - 2)\sigma^{p_s} a^{\frac{p_s}{2} - 1} \pi^{\frac{p_s}{2}}} + \frac{2}{c_l(p_l - 2)\sigma^{p_l} a^{\frac{p_l}{2} - 1} \pi^{\frac{p_l}{2}}} \right]_{a=\delta}^{a=d}. \quad (12.23)$$

The function $n_f(c_s, c_l)$ depends upon physical parameters c_s, c_l and upon geometrical parameters δ and d . Using the reported experimental results (Kaiser and Berger 2007), it is possible to determine the coefficients and exponents of Eq. (12.22), which lead to the experimental S-N line (blue line, Fig. 12.17). We assume, that the failure caused by the defect of the size $\delta = 30 \mu m$. With this assumption, the reported S-N line delivers for valve spring wire DIN EN 10270 the following physical coefficients of the Eq. (12.20):

$$c_s = 1.064 \cdot 10^{-18} m * \left(MPa * m^{\frac{1}{2}} \right)^{-p_s}, \quad p_s = 5.5,$$

$$c_l = 2.7448 \cdot 10^{-53} m * \left(MPa * m^{\frac{1}{2}} \right)^{-p_l}, \quad p_l = 30.5.$$

According to Eq. (12.20c), the “corner point” of stress intensity factor is equal to :

$$\Delta K^* = 24.185.. MPa\sqrt{m}.$$

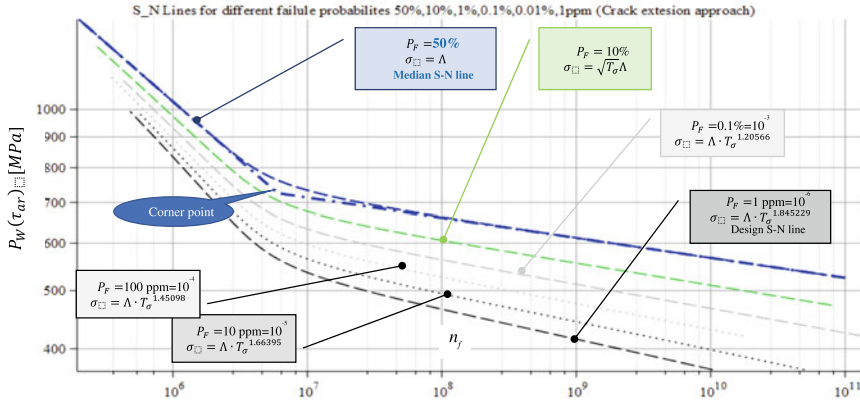


Fig. 12.18 ‘S-N curves for different failure probabilities and for an inclusion size $30 \mu\text{m}$

Equation (12.23) estimate of the influence of inclusion size δ on the fatigue life of springs. For the estimation, we vary the size of the inclusion δ and come back to the different S-N lines with the fixed physical coefficients and wire diameter δ . The results of evaluation of the inclusion influence on the S-N median lines are presented on Fig. 12.17. The calculation was performed for the defects of the sizes $\delta = 40 \mu\text{m}$ (red line, Fig. 12.17) and $\delta = 50 \mu\text{m}$ (magenta line, Fig. 12.17).

Further, we can estimate the influence of probabilistic effects on the S-N lines Eq. (12.23). The Eq. (12.23) delivers the median S-N line for the stress, which corresponds the mean value of the lognormal distribution $\sigma = \Lambda$. To reduce the failure probability, the cycling stress σ must be reduced, as shown in Table C-4, Appendix C. To draw the S-N line, which corresponds the certain cumulative failure probability, one determines the ratio of stress parameter stress to the median stress (Table C-4). The median stress $\sigma = \Lambda$ should be replaced by the stress parameter to the requisite failure probability.

For example, for $P_F = 10^{-6}$ the Table C-4 gives the value

$$\sigma = \Lambda \cdot T_{\sigma}^{1.845229} = 0.6478 \dots \Lambda.$$

Substitution of this value for stress into (12.23) results the S-N line for $P_F = 10^{-6}$.

Based on crack propagation approach, Fig. 12.18 draws S-N lines for different cumulative failure probabilities.

12.4 Scale-Dependent Propagation of Probabilistic Crack

12.4.1 *Simulation of Crack Propagation with Random Deviation*

In Chap. 9, the method to derive the S-N curves from the micromechanical model of the crack propagation was presented. The method opens the way to somewhat adopt the micromechanics of propagation and derive the new S-N curves.

We assumed above the planar crack propagation in the energetically preferred direction. Thus, if the material is homogeneous, the crack spreads in the energetically preferred direction. This direction results the maximal possible elongation for the given direction of the greatest principal stress. In the second step, we propose the mechanism of crack extension, which accounts the deviation of crack from its energetically preferred direction. The consideration is the following. Evidently, that the process of fracture is an anisotropic phenomenon. The growth of micro-cracks occurs in the directions, which attempt to maximize the energy release rate and minimize the strain energy density. The crack spreading along the grain boundaries requires less energy in comparison to the spreading across the perfect grain crystals. The grain boundaries are randomly oriented. Consequently, the optimum propagation path deviates randomly from the ideal straight path, normal to the maximum principal stress. The deviation occurs also due to the disordered orientations of anisotropy in grains and the highly warped grain boundaries. Thus, due to an arbitrary anisotropy of single polycrystalline domains and their irregular shapes, the crack does not follow the straight direction and randomly deviates from the energetically preferred direction. The models of disordered crack propagation were reviewed in (Schwalbe 1980). The damage mechanisms on the VHCF regime are reviewed in (Zimmermann 2019). We use these considerations for the physical background of the proposed mathematical model.

Our aim is to evaluate quantitatively the influence of steel grain sizes on the extension of fatigue crack. For this purpose, we emphasize the various dimensions of single crystals for diverse steel crystalline structures (Verhoeven 2007, Chap. 1, Fig. 1.2). For explanation, we show the crystalline structures, which appear in the duration of the quenching of steel alloys (Fig. 12.19). The schematic representation of the temperature- and time-dependent transformation ranges shows the ranges of martensite, bainite and pearlite formation):

- (1) *Quenching to martensite;*
- (2) *Intermediate grades;*
- (3) *Cooling to bainite;*
- (4) *Pearlite area;*
- (5) *Bainite area.*

The material data of 51CrV4 are given in (DIN EN 10132-4:2003-04) .

Consider a zigzag-shaped crack, which propagates in a plane plate (Fig. 12.20). The straight line, which connects both tips of the crack, will be referred to as “director

vector” of the crack. In the XY coordinates, the director vector of the crack lengthens *roughly* towards the X -axis. The tensile stress acts in the direction of the Y -axis. The range of the component tension stress σ_{YY} is $\Delta\sigma$.

At low stress intensity factor range a highly faceted fracture surface resulted due to crystallographic fracture along slip bands. We apply a simple model for the description of the crack branching. We assume, that the crack propagates on each load cycle in different, stochastic directions. The crack spreading is analogous to the Brownian motion with drift. The applied stress causes the energy release near the tip of the crack and consequently the drift of the crack path.

As already discussed in Sect. 12.3.2, there are at least two diverse mechanisms of crack elongation and correspondingly two length scales.

The first mechanism governs the propagation of the crack, if the range of stress intensity factor is high. The crack spreads across the grains, but its direction must not coincide with the X -axis. The random angle between the macroscopically crack director and direction of the microscopical crack spreading is

$$-\pi < \theta < \pi.$$

This angle is a stochastically function on each propagation event. If the elongation of crack, projected on the macroscopically crack plane of crack, is positive, the crack spreads. In this case

$$-\pi/2 < \theta < \pi/2.$$

Otherwise, if

$$\theta < -\pi/2 \text{ or } \pi/2 < \theta,$$

the crack spreads backwards. However, the backward spreading of crack is energetically intolerable, and the crack suspends one elongation cycle.

The second mechanism is responsible for the crack spreading at low ranges of stress intensity factor. In this case a highly faceted fracture surface resulted due to crystallographic fracture along slip bands. The crack spreads over the grain region. The random angle between the macroscopically crack plane and direction of the slip band is:

$$-\pi < \alpha < \pi.$$

The crack propagates forward, if the angle keeps lying between:

$$-\pi/2 < \alpha < \pi/2.$$

Otherwise, if

$$-\frac{\pi}{2} > \alpha > -\pi, \text{ or } \pi > \alpha > \pi/2$$

the direction of crack extension is opposite to the director direction. The backward extension of crack is energetically unfavorable and will be jammed.

For each cycle, both angles (θ, α) are the statistically independent random values.

The increments of the crack length pro cycle are determined by two Paris-Irvin relations. In a similar manner as in Eq. (12.20), the length increments correspondingly inside the grains and along slip bands are:

$$\delta_s = c_s [\cos(\theta) \cdot \Delta K]^{p_s}, \text{ if } \Delta K \geq \Delta K^*; \quad (12.24a)$$

$$\delta_l = c_l [\cos(\alpha) \cdot \Delta K]^{p_l} \text{ if } \Delta K \leq \Delta K^*. \quad (12.24b)$$

The components of the director vector $X_p(n)$, $Y_p(n)$ are the functions of number of cycles n . The projected length of the crack is X_p and this value replaces the length of the planar crack in the equation for crack intensity factor for the considered uniaxial stress state. Thus, the range of stress intensity factor of the zigzag crack is proportional to the square root of the projected length X_p :

$$\Delta K = \Delta \sigma \cdot \sqrt{\pi X_p}.$$

The critical projected length separates for the given stress range both regions:

$$X^* = \frac{1}{\pi} \left(\frac{\Delta K^*}{\Delta \sigma} \right)^2.$$

One of two potential mechanisms of crack propagation will dominate. The dominating mechanism is responsible for the crack spreading. The travel of the crack tip in the XY-plane according to (12.24) reads:

$$\begin{cases} \frac{dX_p}{dn} = F_X(\theta, \alpha), \\ \frac{dY_p}{dn} = F_Y(\theta, \alpha). \end{cases} \quad (12.25)$$

The Eq. (12.25) are the stochastic differential equations. Obviously, that the nonplanar crack deviates from the abscise axis. Two variants of the propagation functions were tested. The first, stiffer variant for components of propagation function in (12.25) is:

$$\begin{cases} F_X(\theta, \alpha) = \min(\cos(\theta) \Delta \delta_s, \cos(\alpha) \Delta \delta_l), \\ F_Y(\theta, \alpha) = \min(\sin(\theta) \Delta \delta_s, \sin(\alpha) \Delta \delta_l). \end{cases} \quad (12.26a)$$

The second, smoother variant reads:

$$\begin{cases} F_X(\theta, \alpha) = \left(\frac{1}{\cos(\theta)\Delta\delta_s} + \frac{1}{\cos(\alpha)\Delta\delta_l} \right)^{-1}, \\ F_Y(\theta, \alpha) = \left(\frac{1}{\sin(\theta)\Delta\delta_s} + \frac{1}{\sin(\alpha)\Delta\delta_l} \right)^{-1}. \end{cases} \quad (12.26b)$$

12.4.2 Closed-Form Solution of the Stochastic Differential Equation

The closed-form solutions are advantageous for analysis. The closed-form solution of the probabilistic differential equation is based on the averaging of the cyclical crack elongations. For averaging procedure, the angular elongation in Eq. (12.25) must be integrated over the limits:

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad -\frac{\pi}{2} < \alpha < \frac{\pi}{2}.$$

The evaluation of integrals is based on the formula from Appendix C:

$$S(p) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \cos^{2p} \varphi d\varphi = \frac{1}{2\sqrt{\pi}\Gamma(p+1)} \Gamma\left(p + \frac{1}{2}\right). \quad (12.27)$$

Applying the Eq. (12.27), the averaged propagation functions for short or for long range read:

$$\overline{F_s(\Delta K)} = \frac{c_s}{2\pi} \int_{-\pi/2}^{\pi/2} [\cos(\theta) \cdot \Delta K]^{p_s} d\theta = S(p_s) c_s \Delta K^{p_s}, \quad (12.28a)$$

$$\overline{F_l(\Delta K)} = \frac{c_l}{2\pi} \int_{-\pi/2}^{\pi/2} [\cos(\alpha) \cdot \Delta K]^{p_l} d\alpha = S(p_l) c_l \Delta K^{p_l}, \quad (12.28b)$$

Following from (12.26a) or (12.26b), the averaged propagation functions for both short and long ranges correspondingly read:

$$\overline{F_X(\theta, \alpha)} = \min(S(p_s) c_s \Delta K^{p_s}, S(p_l) c_l \Delta K^{p_l}) \quad (12.29a)$$

or

$$\overline{F_X(\theta, \alpha)} = \left(\frac{1}{S(p_s) c_s \Delta K^{p_s}} + \frac{1}{S(p_l) c_l \Delta K^{p_l}} \right)^{-1} \quad (12.29b)$$

There are two variants of the deterministic differential equations for both propagation functions (12.29a) and (12.29b). The differential equations for the mean values of projected lengths $\bar{X}(n)$, $\bar{Y}(n)$ read according to the stiffer variant Eq. (12.29a):

$$\begin{cases} \frac{d\bar{X}}{dn} = \min(S(p_s)c_s \Delta K^{p_s}, S(p_l)c_l \Delta K^{p_l}), \\ \frac{d\bar{Y}}{dn} = 0. \end{cases} \quad (12.30a)$$

Similarly, for smoother variant (12.29b) the equations for the mean values of projected lengths $\bar{X}(n)$, $\bar{Y}(n)$ are:

$$\begin{cases} \frac{d\bar{X}}{dn} = \frac{1}{(S(p_s)c_s \Delta K^{p_s})^{-1} + (S(p_l)c_l \Delta K^{p_l})^{-1}}, \\ \frac{d\bar{Y}}{dn} = 0. \end{cases} \quad (12.30b)$$

In both cases (12.30a) and (12.30b), the mean value of $\bar{X}$ extends randomly with the cycle number and shows the crack elongation. The $\bar{X}$ component is a kind of Brownian motion with the stress-enforced drift. Because the stress $\sigma_{XX} = 0$, the mean value of $\bar{Y}$ is zero. The travelling crack oscillates over abscises axis. The component $\bar{Y}$ of oscillation is the classical Brownian motion without stress-enforcement.

If probabilistic effects are suppressed, both factors $S(p_s)$ and $S(p_l)$ turn to one. This case delivers again the deterministic S-N curves (12.23).

If the crack propagates randomly either in low amplitude or in high amplitude region or in both regions, we derive three S-N curves. Table 12.7 displays the review of possible combinations. The number of cycles for both short and long ranges follows from (12.23) after the correction of coefficients c_s , c_l .

The first S-N curve

$$\widetilde{n}_f = n_f(c_s, S(p_l)c_l), \quad (12.30)$$

represents the case of the reduction of crack length increments in the region of high amplitudes of stress intensity factor. The number of cycles to failure $\widetilde{n}_f$ increases in high stress region, because the crack spreads slowly.

The second S-N curve

$$\widetilde{\widetilde{n}}_f = n_f(S(p_s)c_s, c_l), \quad (12.31)$$

characterizes the case of the reduction of crack length increments in the region of low amplitudes of stress intensity factor. The crack propagates slowly, and the number of cycles in low stress region $\widetilde{\widetilde{n}}_f$ grows. The crack spreads for low amplitudes and for low lengths slowly.

The third S-N curve

$$\widetilde{\widetilde{\widetilde{n}}}_f = n_f(S(p_s)c_s, S(p_l)c_l). \quad (12.32)$$

Table 12.7 Propagation patterns (Pattern D- Deterministic planar crack, S- Stochastic Brownian crack)

Stress intensity factor	Crack length	Pattern	Propagation function	Number of cycles to failure
High $\Delta K \geq \Delta K^*$	Short $X \leq X^*$	D	$F(\Delta K) = \left(\frac{1}{c_s \Delta K^{p_s}} + \frac{1}{c_l \Delta K^{p_l}} \right)^{-1} = \frac{9.3914 \cdot 10^{17}}{\Delta K^{3.5}} + \frac{3.643 \cdot 10^{52}}{\Delta K^{30.5}}$	$n_f = n_f(c_s, c_l) = \frac{5.4153 \cdot 10^{92}}{\Delta \sigma^{3.5}} + \frac{3.8188 \cdot 10^{21}}{\Delta \sigma^{5.5}}$
Low $\Delta K \leq \Delta K^*$	Long $X \geq X^*$	D		
High $\Delta K \geq \Delta K^*$	Short $X \leq X^*$	D	$\tilde{F}(\Delta K) = \left(\frac{1}{c_s \Delta K^{p_s}} + \frac{1}{S(p_l)c_l \Delta K^{p_l}} \right)^{-1} = \tilde{F}(\Delta K) = \frac{9.3914 \cdot 10^{17}}{\Delta K^{3.5}} + \frac{7.1617 \cdot 10^{53}}{\Delta K^{30.5}}$	$\tilde{n}_f = n_f(c_s, S(p_l)c_l) = \frac{5.4153 \cdot 10^{92}}{\Delta \sigma^{3.5}} + \frac{3.2477 \cdot 10^{22}}{\Delta \sigma^{5.5}}$
Low $\Delta K \leq \Delta K^*$	Long $X \geq X^*$	S		
High $\Delta K \geq \Delta K^*$	Short $X \leq X^*$	S	$\tilde{\tilde{F}}(\Delta K) = \left(\frac{1}{S(p_s)c_s \Delta K^{p_s}} + \frac{1}{c_l \Delta K^{p_l}} \right)^{-1} = \frac{7.9688 \cdot 10^{18}}{\Delta K^{3.5}} + \frac{3.643 \cdot 10^{52}}{\Delta K^{30.5}}$	$\tilde{\tilde{n}}_f = n_f(S(p_s)c_s, c_l) = \frac{1.06453 \cdot 10^{94}}{\Delta \sigma^{3.5}} + \frac{3.8188 \cdot 10^{21}}{\Delta \sigma^{5.5}}$
Low $\Delta K \leq \Delta K^*$	Long $X \geq X^*$	D		
High $\Delta K \geq \Delta K^*$	Short $X \leq X^*$	S	$\approx \tilde{F}(\Delta K) = \left(\frac{1}{S(p_s)c_s \Delta K^{p_s}} + \frac{1}{S(p_l)c_l \Delta K^{p_l}} \right)^{-1} = \frac{7.9688 \cdot 10^{18}}{\Delta K^{3.5}} + \frac{7.1617 \cdot 10^{53}}{\Delta K^{30.5}}$	$\approx \tilde{n}_f = n_f(S(p_s)c_s, S(p_l)c_l) = \frac{1.06453 \cdot 10^{94}}{\Delta \sigma^{3.5}} + \frac{3.2477 \cdot 10^{22}}{\Delta \sigma^{5.5}}$
Low, $\Delta K \leq \Delta K^*$	Long $X \geq X^*$	S		

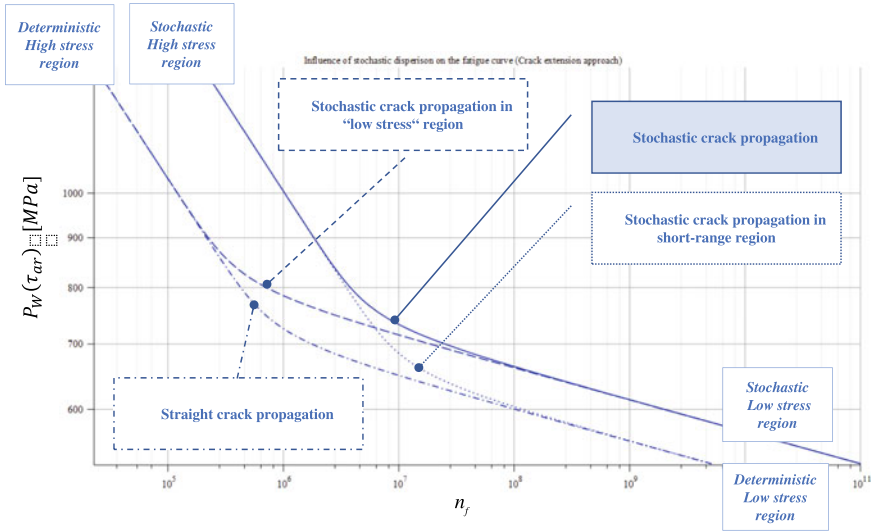


Fig. 12.21 Closed form solutions for ‘S-N curves for different character of crack propagation (straight crack in both regions, straight crack in short and in long cycle regions, stochastic crack propagation in one and an the regions) initial inclusion size 30 μm

takes both weakenings of elongations in regions of high and low stress intensity factors. The cumulation of both effects leads to the maximal fatigue life $\tilde{n}_f$.

The results of analytical evaluation of possible the averaged propagation functions are shown on Fig. 12.21. The influence of size for initial defect displays Fig. 12.22. Three solid curves present the S-N curves for the randomly propagated crack for the different with the different initial lengths 30, 40, and 50 μm . The dashed curves present the S-N curves for planar, deterministic crack for the different with the different initial lengths 30, 40, and 50 μm . The randomly propagated cracks indicate roughly twenty times more cycles to achieve the same crack length in comparison to the planar, deterministically spreading crack.

The plot of fatigue crack growth rate da/dn versus stress intensity factor range for different characters of crack propagation displays Fig. 12.23. At low stress intensity factor range a highly faceted fracture surface resulted due to crystallographic fracture along slip bands and stochastic crack expected (green lines). This effect explains the observed increasing of the fatigue life.

The numerically calculated S-N curves are presented on Fig. 12.24. The numerical results deliver the single points of S-N lines. Green points display the results of numerical simulation for planar elongation of crack. Red points show the results of numerical simulation for Brownian crack spreading. For comparison, the closed form solution with the averaged coefficients and simulation of the stochastic differential equations are shown. The figure displays one simulated S-N curve for straight and

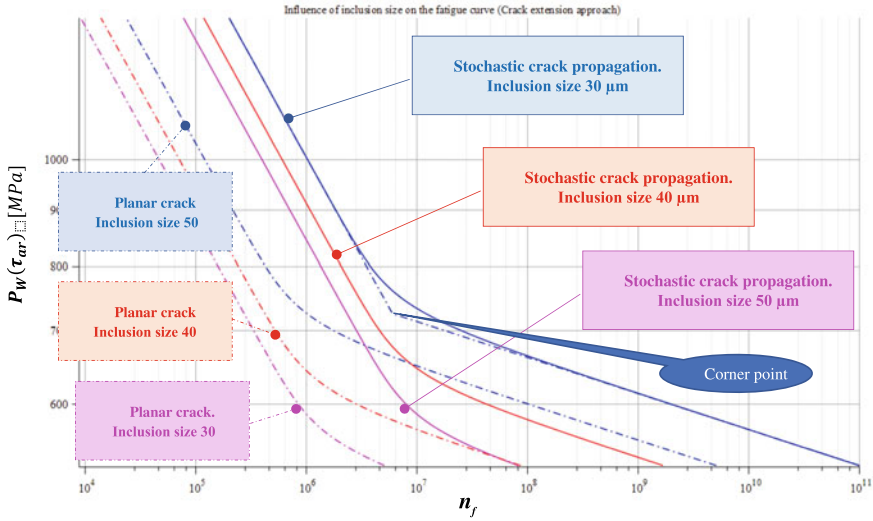


Fig. 12.22 S-N curves for different character of crack propagation (straight crack in both regions and stochastic crack propagation in in both regions) initial inclusion size 30, 40 and 50 μm

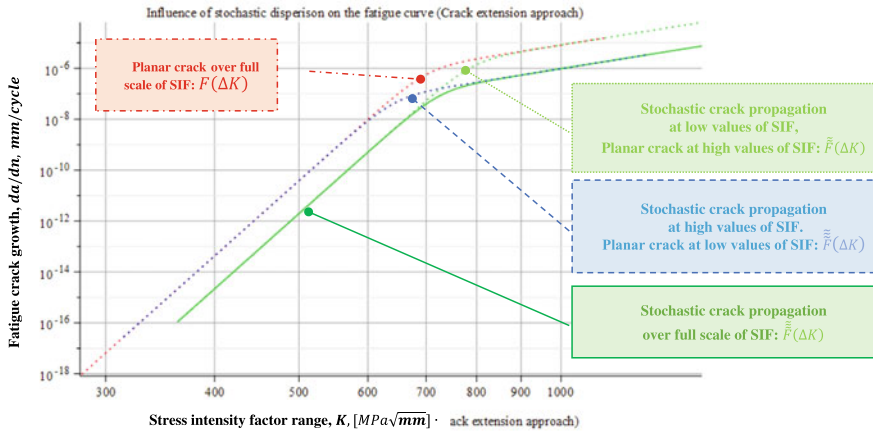


Fig. 12.23 The plot of fatigue crack growth rate da/dn versus stress intensity factor range for different characters of crack propagation. At low stress intensity factor range a highly faceted fracture surface resulted due to crystallographic fracture along slip bands and stochastic crack growth expected (green lines)

one for randomly travelling crack together with closed-form solutions $\tilde{n}_f, n_f$. For comparison, the S-N curves from closed-form solutions and simulation of stochastic crack growth are portrayed.

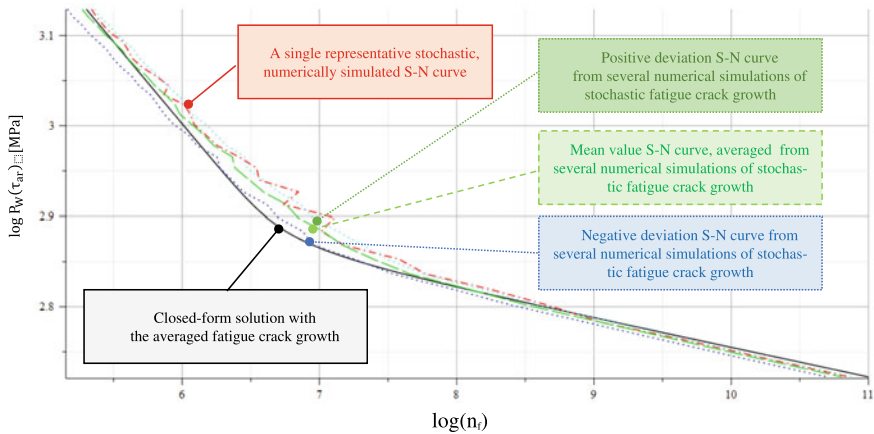


Fig. 12.24 S-N curves for stochastic crack propagation. Comparison of the closed form solution with the averaged coefficients and simulation of the stochastic differential equations

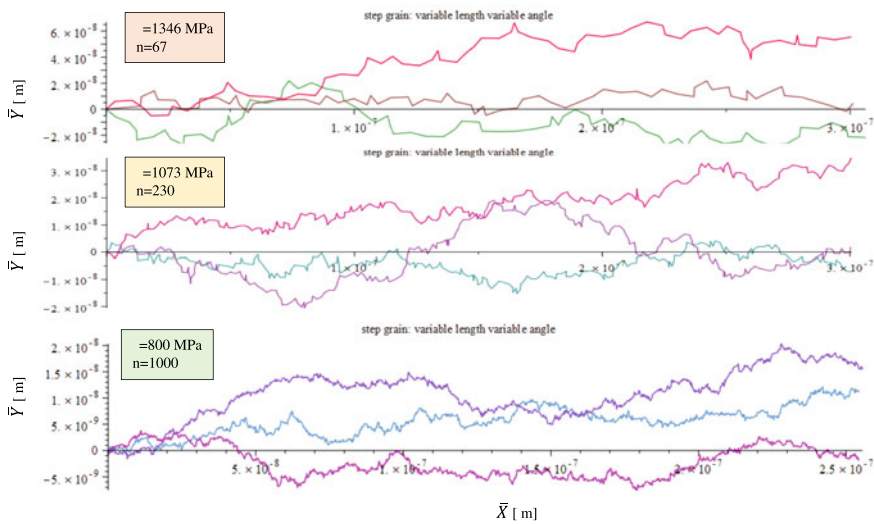


Fig. 12.25 Simulation of stochastic crack growth under different applied stresses and different number of cycles. The initial crack size of 60 μm

The examples of numerical simulation of drifted Brownian travelling of the cracks are displayed on Fig. 12.25. The above three curves show the jumps of the cracks for high value of stress $\sigma_{YY} = 1346 \text{ MPa}$. Initially, the crack tip matches origin of coordinates. With each cycle the crack tip travels from the left to the right. The results

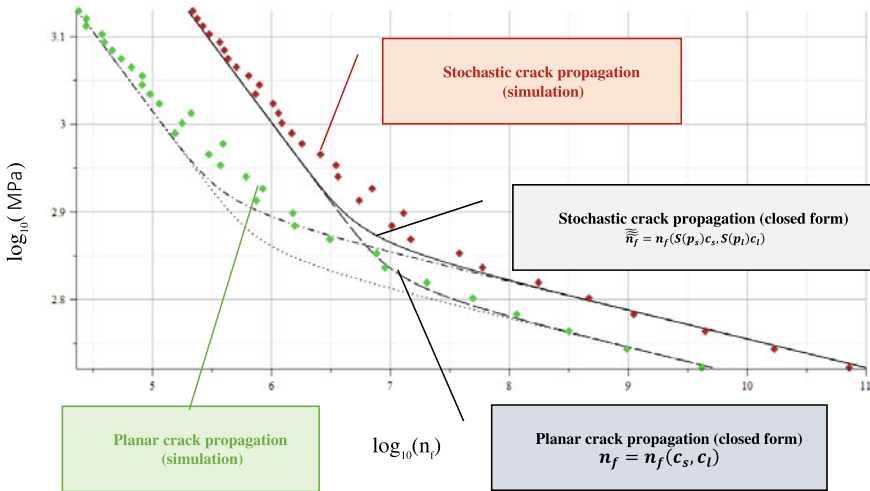


Fig. 12.26 Comparison of S-N curves from closed-form solutions and 'simulation of stochastic crack growth. One simulated S-N curve for straight and one for travelling stochastic crack together with closed-form solutions) $\tilde{n}_f, n_f$

of three simulations are shown, and the cracks travel on each load cycle considerably long. The fatigue life is the shortest, because the considerable spreading of crack.

The central batch of three curves show the jumps of the cracks for mean value of stress $\sigma_{YY} = 1073 \text{ MPa}$. The cracks start also from the origin of coordinates. The results of three crack paths are shown. The cracks travel on each load cycle evidently shorter, than in the first case. The propagation rate of crack is middling.

The bottommost family of three curves show the jumps of the cracks for lowest value of stress $\sigma_{YY} = 800 \text{ MPa}$. The cracks travel on each load cycle is the shortest and the fatigue life is longest of all.

The numerical solution is presented on the Fig. 12.26. Several simultaneous evaluations of randomly-driven S-N lines were performed. The single simulated lines oscillate with respect to the averaged S-N line. The several simultaneous S-N lines allows the numerical evaluation of shape factors and the deviations from the averaged S-N line as well. The simulations were performed for 100 loads in the region $\sigma_{YY} = 800..1346 \text{ MPa}$. Each batch of simulations for a single load delivers a point of the S-N line. The red curve presents an example of stochastic simulation of S-N line. Each point of this curve is based on a single evaluation of crack spreading to failure for each stress level. Each of S-N lines were evaluated eight times. From the generated S-N lines the median fatigue lines and the shape factors of distributions were determined. The green line displays the mean value (median) S-N curve. Two dotted lines show the deviation of the fatigue life based on 8 stochastic simulations.

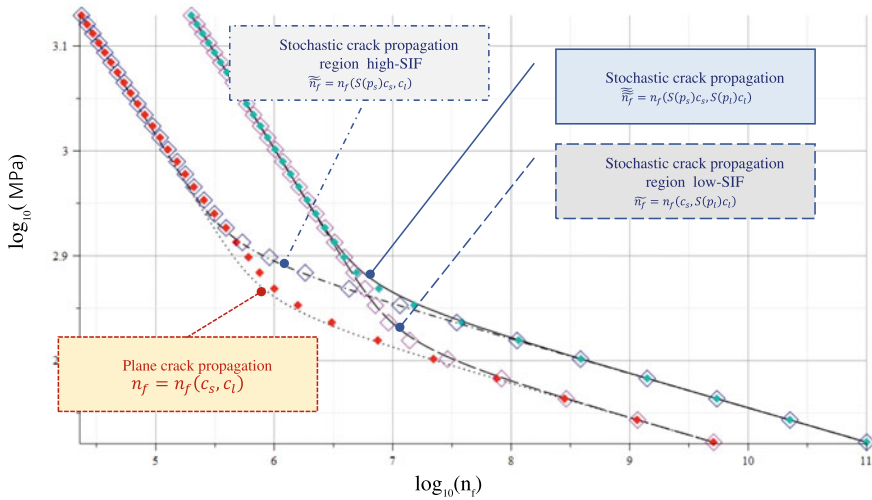


Fig. 12.27 Comparison of S-N curves from closed-form solutions and ‘simulation of stochastic crack growth. Mean-values of simulated S-N curves overlapped with closed-form solutions. propagation (straight crack in both regions, straight crack in short and in long cycle regions, stochastic crack propagation in one and in the regions)

Simulation show, that there are no valuable differences between the numerical results with both variants of propagation functions.

The results of averaging for several simulations are shown on Fig. 12.27. The combinations of stochastic effects, as presented in Table 12.7 and given for closed-form solutions Eqs. (12.23), (12.30) to (12.32) were simulated. Evidently, that the numerical and analytical results display the perfect correlation. The numerical simulation confirms the averaging method for the solution of stochastic differential equation, which leads to the closed-form solutions.

12.5 Conclusions

The experimentally acquired data from (AiF. 12 287 N 2002) and (AiF 15 064 N 2010), (IGF 16873 N 2014) provides the source for the statistical modelling of springs’ fatigue. Different variants for introduction of effective stresses were compared. Calculations with P_{SWT} were performed. Beside this, calculations with Walker stress parameter P_W were presented. The optimal setting for fitting parameter $\tilde{\gamma}$ was evaluated using variations of stress ratio (Schuller et al. 2013; Mayer et al. 2015, 2016). The fitted value $\tilde{\gamma} = 0.15$ leads to the conformity of the experimentally obtained curves for the range of stress ratio $0 < R_\sigma < 1$.

Further, we discussed the mechanism of crack extension, which accounts the deviation of crack from its energetically preferred direction. The deviation occurs due to the chaotic orientations of anisotropy in grains and the stochastic grain boundaries. If material is homogeneous, the crack spreads in the energetically preferred direction which leads to the maximal possible elongation. However, due to arbitrary anisotropy of microscopic crystal grains and their irregular shapes, the crack does not follow the straight direction and deviates from it randomly. The shape of nonplanar crack looks like the path a Brownian particle in the force-drifted motion.

12.6 Summary of Principal Results

- The acknowledged experimental results were analyzed from two different viewpoints.
- At first, the traditional Walker and averaging of stresses over the wire surface techniques were applied.
- The best fitting of the exponential data provides the Walker parameter with $\tilde{y} \approx 0.15$ and the mean value of basic and corrected stress
- At second, the micromechanics of fatigue crack grow at low amplitudes of cyclic stress was studied
- The modified micromechanics of crack grows portrays the specialties of low cyclic amplitudes
- The fatigue data demonstrates the significant prolongation of the fatigue life of springs in the region of the small stress amplitudes.
- This behavior asserts as the reduction of damage and the arrest of the crack propagation on inclusions and obstacles.
- Remarkable is the influence of the microstructure refinement due to the heat treatment of spring steels.
- The effect of structural refinement was displayed from geometric viewpoint.
- The modelling was based on the stochastic branching and the Brownian travelling of the crack, which leads to the piecewise linear terms in Paris-Irvin equations of crack growth and to the reduction of cyclic elongation.
- The closed form solutions of the stochastic equations were presented.

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Appendix A

Models of Spring Materials

A.1 Plastic Deformation

1°. The stress σ and strain tensors ϵ reads (Chen and Han 1988):

$$\sigma = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix}, \epsilon = \begin{bmatrix} \epsilon_{xx} & \gamma_{xy}/2 & \gamma_{xz}/2 \\ \gamma_{xy}/2 & \epsilon_{yy} & \gamma_{yz}/2 \\ \gamma_{xz}/2 & \gamma_{yz}/2 & \epsilon_{zz} \end{bmatrix} \equiv \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{bmatrix}. \quad (\text{A.1})$$

The octahedral normal stress σ_{oct} and the octahedral normal strain ϵ_{oct} are denoted as:

$$3\sigma_{oct} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \equiv \sigma_{kk} = \mathbf{I}(\sigma), \quad 3\epsilon_{oct} = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} \equiv \epsilon_{kk} = \mathbf{I}(\epsilon). \quad (\text{A.2})$$

The octahedral normal stress is equal to hydrostatic pressure or spherical stress. The $\mathbf{I}(\sigma)$ and $\mathbf{I}(\epsilon)$ are the first invariants of stress and strain tensors.

The tensors s and e are the deviators of tensors (A.1) of with the components respectively:

$$s_{ij} = \sigma_{ij} - \sigma_{oct}\delta_{ij}, \quad e_{ij} = \epsilon_{ij} - \epsilon_{oct}\delta_{ij}, \quad (\text{A.3})$$

$$s = \begin{bmatrix} s_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & s_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & s_{zz} \end{bmatrix}, e = \begin{bmatrix} e_{xx} & \gamma_{xy}/2 & \gamma_{xz}/2 \\ \gamma_{xy}/2 & e_{yy} & \gamma_{yz}/2 \\ \gamma_{xz}/2 & \gamma_{yz}/2 & e_{zz} \end{bmatrix} \equiv \begin{bmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{xy} & e_{yy} & e_{yz} \\ e_{xz} & e_{yz} & e_{zz} \end{bmatrix} \quad (\text{A.4})$$

The octahedral shear strain is:

$$\frac{\gamma_{oct}}{2} \stackrel{\text{def}}{=} \sqrt{\frac{2\mathbf{\Pi}(e)}{3}}, \quad \mathbf{\Pi}(e) = \frac{1}{2}e_{ij}e_{ij}. \quad (\text{A.5})$$

$\mathbf{II}(\mathbf{e})$ is the second invariant of deviatoric strain tensor $\mathbf{e}$.

The octahedral shear stress, with reads as:

$$\tau_{oct} \stackrel{\text{def}}{=} \sqrt{2\mathbf{II}(\mathbf{s})/3}, \quad \mathbf{II}(\mathbf{s}) = \frac{1}{2}s_{ij}s_{ij}. \quad (\text{A.6})$$

$\mathbf{II}(\mathbf{s})$ is the second invariant of deviatoric stress $\mathbf{s}$.

The equivalent stress is $\sigma_{eq} = 3\tau_{oct}/\sqrt{2}$.

2° For a linear elastic medium

$$\mathbf{s} = 2G\mathbf{e}, \quad 3K\varepsilon_{oct} = \sigma_{oct}, \quad (\text{A.7})$$

where G is the shear modulus, K is the modulus of cubic compressibility or bulk modulus of elasticity. From this equation follows, that in case of linear elasticity:

$$\tau_{oct} = G\gamma_{oct}.$$

For a compressible isotropic work-hardening material, without the distinct yielding point the stress could be presented as the function of strain:

$$\mathbf{e} = f_{\tau}(\tau_{oct}) \frac{\mathbf{s}}{\tau_{oct}}, \quad 3K\varepsilon_{oct} = \sigma_{oct}. \quad (\text{A.8})$$

The dependence of stress in terms of strain reads:

$$\mathbf{s} = 2G_p(\gamma_{oct})\mathbf{e}, \quad \mathbf{s} = f_{\tau}^{-1}(\gamma_{oct}) \frac{\mathbf{e}}{\gamma_{oct}} \quad (\text{A.9})$$

The relation must be satisfied:

$$\frac{\tau_{oct}}{f_{\tau}(\tau_{oct})} = \frac{f_{\tau}^{-1}(\gamma_{oct})}{\gamma_{oct}} = 2G_p(\gamma_{oct}). \quad (\text{A.10})$$

From the Eq. (A.9) follows the relation of the octahedral shear stress upon the octahedral shear strain :

$$\tau_{oct} = G_p(\gamma_{oct})\gamma_{oct}. \quad (\text{A.11})$$

3° The Johnson–Cook model (Johnson and Cook 1983) expresses the stress upon strain. The Johnson–Cook model is appropriate for the large deformation, high strain rate and elevated temperatures. In the case of an uniaxial deformation, the Johnson–Cook stress–strain equation is written in the form:

$$\sigma = F_1(\varepsilon)F_2(T^{\circ})F_3(\dot{\varepsilon}), \quad (\text{A.12})$$

$$F_1(\varepsilon) = A_{\sigma} + B_{\sigma}\varepsilon^{1/n}, \quad (\text{A.13})$$

$$F_2(T^\circ) = 1 - \left(\frac{T^\circ - T_{ref}^\circ}{T_m^\circ - T_{ref}^\circ} \right)^{m^\circ}, \quad (\text{A.14})$$

$$F_3(\dot{\varepsilon}) = 1 + C \ln \frac{\dot{\varepsilon}}{\dot{\varepsilon}_{ref}}. \quad (\text{A.15})$$

where A_σ and B_σ are the strain coefficients, T_m° is the melting temperature, T_{ref}° is a reference (ambient) temperature, m° is the temperature exponent, $\dot{\varepsilon}$ is the (plastic) strain rate, $\dot{\varepsilon}_{ref}$ is a reference (quasi-static) strain rate and C is the strain rate sensitivity parameter.

The term $F_1(\varepsilon)$ is called the Ludwik–Hollomon equation. This term reveals the quasi-static yielding and work hardening behavior at a certain temperature.

The term $F_2(T^\circ)$ determines the temperature dependence.

The third term demonstrates the strain rate sensitivity. The term $F_3(\dot{\varepsilon})$ includes commonly the logarithm of the strain rate, normalized by a reference value. More general approximations are known (Mauerauch and Vöhringer 1978; El-Magd et al. 1985).

3°. Relations between the coefficients of explicit and implicit stress–strain functions are listed in Table A.1. Simplified relations between the coefficients of Ludwik–Hollomon, Johnson–Cook and Ramberg–Osgood stress–strain functions are shown in Table A.2.

A.2 Creep Deformation

The isotropic, time dependent stress function (Betten 2008) for creep laws (Table A.3):

$$\frac{d\mathbf{e}_{ij}}{dt} = \frac{s_{ij}}{\tau_{oct}} f_c(\tau_{oct}, t), \quad (\text{A.16})$$

$$f_c(\tau_{oct}, t) = h_c(t) g_c(\tau_{oct}). \quad (\text{A.17})$$

Table A.1 Relations between the coefficients of explicit and implicit stress–strain functions

Relations between the coefficients of explicit and implicit stress–strain functions		
	Implicit stress–strain function	Explicit stress–strain function
Relation between octahedral shear stress and strain	$\gamma_{oct} = f_{\tau}(\tau_{oct})$	$\tau_{oct} = f_{\tau}^{-1}(\gamma_{oct})$
Quasi-static shear-stress-shear-strain relation	$\gamma = f_{\tau}(\tau)$	$\tau = f_{\tau}^{-1}(\gamma)$
Quasi-static tensile-stress-tensile-strain relation	$\varepsilon = \frac{1}{\sqrt{3}} f_{\tau} \left(\frac{\sigma}{\sqrt{3}} \right) \stackrel{\text{def}}{=} f_{\sigma}(\sigma)$	$\sigma = \sqrt{3} f_{\tau}^{-1} \left(\sqrt{3} \varepsilon \right) \stackrel{\text{def}}{=} f_{\sigma}^{-1}(\varepsilon)$
Quasi-static stress–strain flow equations	Ramberg–Osgood equation	Ludwik–Hollomon equation
Shear deformation	$\gamma = \frac{\tau}{G} + \left(\frac{\tau}{K_{\tau}} \right)^n$	$\tau = A_{\tau} + B_{\tau} \gamma^{1/n}$
Uniaxial deformation	$\varepsilon = \frac{\sigma}{E} + \left(\frac{\sigma}{K_{\sigma}} \right)^n$	$\sigma = A_{\sigma} + B_{\sigma} \varepsilon^{1/n}$
Stress–strain relation Eq. (6.8), $\tau_w = G \gamma_w$	$\gamma_{oct} = \frac{\tau_{oct}}{G \cdot \sqrt{1 - (\tau_{oct}/\tau_w)^2}}$	$\tau_{oct} = \frac{G \gamma_{oct}}{\sqrt{1 + (\gamma_{oct}/\gamma_w)^2}}$
Secant modulus in stress–strain relation Eq. (6.8)	$\gamma_{oct} = \tau_{oct} / G_p(\tau_{oct}),$ $G_p(\tau_{oct}) =$ $G \cdot \sqrt{1 - (\tau_{oct}/\tau_w)^2}$	$\tau_{oct} = G_p(\gamma_{oct}) \gamma_{oct},$ $G_p(\gamma_{oct}) = \frac{G}{\sqrt{1 + (\gamma_{oct}/\gamma_w)^2}}$
Relations between coefficients for shear and uniaxial plastic flow for incompressible medium	$E = 3G,$ $K_{\sigma} = \sqrt{3} K_{\tau}$	$A_{\tau} = \frac{A_{\sigma}}{\sqrt{3}},$ $B_{\tau} = \frac{B_{\sigma}}{\sqrt{3}} \cdot \left(\frac{1}{\sqrt{3}} \right)^{1/n}$
Johnson–Cook equations	$\gamma_{oct} =$ $f_{\tau} \left(\frac{\tau_{oct}}{F_2(T^{\circ}) F_3(\dot{\gamma}_{oct})} \right)$	$\frac{\tau_{oct}}{F_2(T^{\circ}) F_3(\dot{\gamma}_{oct})} = f_{\tau}^{-1}(\gamma_{oct})$
Dynamic tensile-stress-tensile-strain relation	$\varepsilon = f_{\sigma} \left(\frac{\sigma}{F_2(T^{\circ}) F_3(\dot{\varepsilon})} \right)$	$\frac{\sigma}{F_2(T^{\circ}) F_3(\dot{\varepsilon})} = f_{\sigma}^{-1}(\varepsilon)$
Dynamic static shear-stress-shear-strain relation	$\gamma = f_{\tau} \left(\frac{\tau}{F_2(T^{\circ}) F_3(\dot{\gamma})} \right)$	$\frac{\tau}{F_2(T^{\circ}) F_3(\dot{\gamma})} = f_{\tau}^{-1}(\gamma)$

Table A.2 Ludwik–Hollomon, Johnson–Cook and Ramberg–Osgood stress-strain functions

	Pure shear deformation	Uniaxial deformation
Explicit stress–strain functions (A.8): Ludwik–Hollomon and Johnson–Cook equations	$\tau = -\frac{K_{\sigma}^2}{\sqrt{3}(K_{\sigma} + nE)} + \frac{K_{\sigma}}{\sqrt{3}} \left(\frac{\gamma}{\sqrt{3}} \right)^{1/n}$	$\sigma = -\frac{K_{\sigma}^2}{K_{\sigma} + nE} + K_{\sigma} \varepsilon^{1/n}$
Implicit stress–strain function (A.7): Ramberg–Osgood equation	$\gamma = \frac{3\tau}{E} + \left(\frac{\sqrt{3}\tau}{K_{\sigma}} \right)^n$	$\varepsilon = \frac{\sigma}{E} + \left(\frac{\sigma}{K_{\sigma}} \right)^n$

Table A.3 Common phenomenological laws for creep

Creep laws	$\bar{\sigma} = \frac{3\bar{\tau}}{\sqrt{2}}, \bar{\tau} = \frac{\bar{\sigma}}{\sqrt{3}}$
Norton-Bailey law	$g_c = \bar{\varepsilon} \cdot \left(\frac{\tau_{ocL}}{\bar{\tau}}\right)^{\xi+1}$
Garofalo creep law:	$g_c = \bar{\varepsilon} \cdot \sinh^{\xi+1}\left(\frac{\tau_{ocL}}{\bar{\tau}}\right)$
Exponential creep law:	$g_c = \bar{\varepsilon} \cdot \left[\exp\left(\frac{\tau_{ocL}}{\bar{\tau}}\right) - 1\right]$
Naumenko-Altenbach-Gorash creep law:	$g_c = \bar{\varepsilon} \cdot \left(\frac{\tau_{ocL}}{\bar{\tau}}\right) + \bar{\varepsilon} \cdot \left(\frac{\tau_{ocL}}{\bar{\tau}}\right)^{\xi+1}$
Time independent stress function $\frac{\sigma_{th}}{\bar{\sigma}} = \frac{\tau_{th}}{\bar{\tau}}, \frac{\sigma_Y}{\bar{\sigma}} = \frac{\tau_Y}{\bar{\tau}}$	$g_c = \bar{\varepsilon} \cdot \left(\frac{\tau_{ocL}}{\bar{\tau}}\right)^{\xi+1} \cdot \frac{1}{\left(\frac{\tau_{ocL}}{\tau_{th}}\right)^{m_1} - 1} \cdot \frac{1}{1 - \left(\frac{\tau_{ocL}}{\tau_Y}\right)^{m_2}}$

Appendix B

Some Special Functions

1°. Integrals with polylogarithm.

The weighted integrals of the function

$$f(x) = \ln(\tanh(a + \operatorname{arctanh}(e^{bx}))) \quad (\text{B.1})$$

are:

$$I_0(a, b; X) \equiv \int_0^X f(x) dx = X \ln(\tanh(a)) + \frac{1}{b} (\Lambda_2 - \mathbf{M}_2 + \mu_2 - \lambda_2), \quad (\text{B.2})$$

$$\begin{aligned} I_1(a, b; X) &\equiv \int_0^X f(x) x dx = \\ &= \frac{1}{6b^2} [\pi^2 \ln(\coth(a)) + \ln^3(\coth(a)) + 3b^2 X^2 \ln(\tanh(a))] + \\ &+ \frac{1}{b^2} (\mathbf{M}_3 - \Lambda_3) - \frac{X}{b} (\mathbf{M}_2 - \Lambda_2), \end{aligned} \quad (\text{B.3})$$

$$\begin{aligned} I_2(a, b; X) &\equiv \int_0^X f(x) x^2 dx = \frac{X^3}{3} \ln(\tanh(a)) + \\ &+ \frac{2}{b^3} (\Lambda_4 - \mathbf{M}_4 + \mu_4 - \lambda_4) + \frac{2X}{b^2} (\mathbf{M}_3 - \Lambda_3) - \frac{X^2}{b} (\mathbf{M}_2 - \Lambda_2). \end{aligned} \quad (\text{B.4})$$

The following abbreviations are used:

$$\begin{aligned} \mathbf{M}_k &= Li_k(-\coth(a) \exp(bX)), \quad \Lambda_k = Li_k(-\tanh(a) \exp(bX)), \\ \mu_k &= \mathbf{M}_k|_{X=0} \equiv Li_k(-\coth(a)), \quad \lambda_k = \Lambda_k|_{X=0} \equiv Li_k(-\tanh(a)). \end{aligned}$$

In these expressions is $Li_k(z)$ the polylogarithm (also known as Jonquière's function) of order k and argument z .

2°. Integrals with hypergeometric function.

The weighted integrals of the function $g = (a + x^{-m})^{-1/m}$ are ($p \geq 0$):

$$J_p(a, m; X) \equiv \int_0^X x^p g(x) dx = {}_2F_1\left(\frac{1}{m}, \frac{2+p}{m}; \frac{2+p+m}{m}; -aX^m\right) \frac{X^{2+p}}{2+p}, \quad (\text{B.5})$$

For some cases, the integrals could be expressed in terms of elementary functions:

$$\begin{aligned} J_1(a, 1; X) &= -\frac{X}{a^2} + \frac{X^2}{2a} + \frac{\ln(1+aX)}{a^3}, \quad J_2(a, 1; X) = \frac{X}{a^3} - \frac{X^2}{2a^2} + \frac{X^3}{3a} - \frac{\ln(1+aX)}{a^4} \\ J_1(a, 2; X) &= \frac{X\sqrt{1+aX^2}}{2a} - \frac{\arcsin(X\sqrt{A})}{2a^{\frac{3}{2}}}, \quad J_2(a, 2; X) = \frac{2}{3a^2} - \frac{2\sqrt{1+aX^2}}{3a^2} + \frac{X^2\sqrt{1+aX^2}}{3a} \\ J_1(a, 3; X) &= -\frac{1}{2a} + \frac{(1+aX^3)^{\frac{2}{3}}}{2a}, \quad J_2(a, 4; X) = -\frac{1}{3a} + \frac{(1+aX^4)^{\frac{3}{4}}}{3a}. \end{aligned}$$

3°. Integrals with incomplete beta function.

The weighted integrals of the function $(|c - r|/r)^{1/n}$ are:

$$\begin{aligned} K_n(a, b, c) &\equiv \frac{1}{c} \int_a^b \left(\frac{|c-r|}{r}\right)^{1/n} dx = \\ &= \frac{(-1)^{-1/n} \pi}{n} \left(i + \cot\left(\frac{\pi}{2n}\right)\right) - \mathbf{B}\left(\frac{a}{c}; \frac{n-1}{n}, \frac{n+1}{n}\right) - (-1)^{1/n} \mathbf{B}\left(\frac{b}{c}; \frac{n-1}{n}, \frac{n+1}{n}\right) \end{aligned} \quad (\text{B.6})$$

and

$$\begin{aligned} L_n(a, b, c) &\equiv \frac{1}{c^2} \left[\int_a^b \left(\frac{|c-r|}{r}\right)^{1/n} (c-r) dx \right] = \\ &= \frac{i\pi}{\exp\left(\frac{i\pi}{n}\right) - 1} \frac{1+n}{n^2} + \mathbf{B}\left(\frac{a}{c}; \frac{2n-1}{n}, \frac{n+1}{n}\right) - \mathbf{B}\left(\frac{b}{c}; \frac{n-1}{n}, \frac{n+1}{n}\right) + \\ &+ (-1)^{-1/n} \left[\mathbf{B}\left(\frac{b}{c}; \frac{2n-1}{n}, \frac{n+1}{n}\right) - \mathbf{B}\left(\frac{b}{c}; \frac{n-1}{n}, \frac{n+1}{n}\right) \right]. \end{aligned} \quad (\text{B.7})$$

In these expressions

$$\mathbf{B}(x; p, q) = \int_0^x z^{p-1} (1-z)^{q-1} dz \quad (\text{B.8})$$

is the incomplete beta-function (Pearson 1968). This function expresses in terms of hypergeometric functions (Abramowitz and Stegun 1972) by:

$$\mathbf{B}(x; p, q) = {}_2\mathbf{F}_1(p, 1-q; p+1; x) \cdot \frac{x^p}{p}. \quad (\text{B.9})$$

4°. Complete elliptic integrals

$$\mathbf{K}(k) = \int_0^1 \frac{1}{\sqrt{(1-k^2t^2)(1-t^2)}} dt \quad \text{complete elliptic integrals of the first kind.}$$

$$\mathbf{E}(k) = \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}} dt \quad \text{complete elliptic integrals of the second kind.}$$

$$\mathbf{\Pi}(\kappa, k) = \int_0^1 \frac{1}{(1-\kappa t^2)\sqrt{(1-k^2t^2)(1-t^2)}} dt \quad \text{complete elliptic integrals of the third kind.}$$

5°. Appell hypergeometric function.

Appell hypergeometric function of two variables (Erdélyi 1950; Kampe de Fériet 1957) reads:

$$\begin{aligned} & \mathbf{F}_1([A, B_1, B_2, C]; x, y) \\ &= \frac{\Gamma(C)}{\Gamma(A)\Gamma(C-A)} \int_0^1 t^{A-1} (1-t)^{C-A-1} (1-tx)^{-B_1} (1-ty)^{-B_2} dt. \end{aligned} \quad (\text{B.10})$$

6°. Legendre and hypergeometric function

The expression of the integral for $|R| < 1$ (Gradshteyn and Ryzhik 2015) in terms of the Legendre function P_p and the hypergeometric function :

$$\Upsilon(R) = \frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{(1+R \sin t)^p} = {}_2\mathbf{F}_1\left(\left[\frac{1+p}{2}, \frac{p}{2}\right]; [1], R^2\right) = \frac{P_{-p}\left(\frac{1}{\sqrt{1-R^2}}\right)}{(1-R^2)^{p/2}}. \quad (\text{B.11})$$

7°. Integrals with trigonometric function

The expressions of the integral (Gradshteyn and Ryzhik 2015, 3.621) in terms of Gamma function are

$$\int_0^{\pi/2} \cos^{2p} \varphi d\varphi = \int_0^{\pi/2} \sin^{2p} \varphi d\varphi = \frac{\sqrt{\pi}}{2} \frac{\Gamma(p)}{\Gamma(p+1/2)}, \quad (\text{B.12})$$

$$\int_0^{\pi/2} \cos^{2p+1} \varphi d\varphi = \int_0^{\pi/2} \sin^{2p+1} \varphi d\varphi = \frac{\sqrt{\pi}}{2} \frac{\Gamma(p+1/2)}{\Gamma(p+1)}, \quad (\text{B.13})$$

$$\int_0^{\pi/2} \sin^{p-1} \varphi \cos^{q-1} \varphi d\varphi = \frac{\Gamma(p/2)\Gamma(q/2)}{2\Gamma((p+q)/2)}, \quad \text{Re } p > 0, \text{Re } q > 0. \quad (\text{B.14})$$

The following integral evaluates with Legendre function Eq. (3.661.3) (Gradshteyn and Ryzhik 2015):

$$\frac{1}{2\pi} \int_0^{2\pi} (a + b \cos t)^q dt = (a^2 - b^2)^{\frac{q}{2}} P_q \left(\frac{a}{\sqrt{a^2 - b^2}} \right). \quad (\text{B.15})$$

Appendix C

Statistical Assessment for Scattering of Fatigue Data

1°. Prediction of failure probability under a specific stress level is an important tool for analysis of springs under static and cyclic loading. Several distributions are customary applied for the statistical evaluation of the fatigue data (Forbes et al. 2011).

Scattering analysis of experimental data in fatigue tests was traditionally performed by using:

- Normal (Gaussian) distribution
- Lognormal distribution
- Exponential distribution
- Weibull distribution
- $\arcsin\sqrt{P}$ —Arcsine distribution.

The variability of a material characteristic is described by using lognormal distribution function, which cover a large scale of applications. The lognormal distribution is recommended in ASTM E 739-80 (1986), ASTM E 606-80 (1986), DIN 50100 (2016:12) norms. Schijve (2005) proposed “log(N-N0)-normal distribution” for fatigue data analysis.

The probability density function is $p_f(\sigma_\delta)$. The symbol σ_δ designates the variate or random variable. In case of fatigue analysis, the variate characterizes the intensity of the harmonically variable stress. The stress amplitude, or stress range, or parameter of Smith, Watson and Topper, or Walker parameter is used for characterization the variable stress in fatigue experiments.

The cumulative failure probability of the argument σ is equal to the probability that the variate δ takes a value less than or equal to σ :

$$P_f(\sigma) = Pr[\sigma_\delta \leq \sigma] = \int_{-\infty}^{\sigma} p_f(\sigma_\delta) d\sigma_\delta. \quad (\text{C.1})$$

For a fixed stress parameter, the cumulative failure $P_f(\sigma)$ is equal to the fraction of failures among the large series of samples.

Table C.1 Normal distribution

Normal distribution	
Distribution mean	Λ
Scale parameter	μ
Probability density of failures	$p_f(\sigma_\delta) = \frac{1}{\mu\sqrt{2\pi}} \exp\left(-\frac{(\Lambda-\sigma_\delta)^2}{2\mu^2}\right)$
Maximum of probability density of failures	$\max p_f(\sigma_\delta) = p_f(\sigma_{max}) = 1/\sqrt{2\pi}\mu$
Stress at maximum	$\sigma_\delta = \Lambda$
Cumulative failure probability P_F (for given stress value σ)	$P_f(\sigma) = \frac{1}{2} - \frac{1}{2}\operatorname{erf}\left(\frac{\Lambda-\sigma}{\mu\sqrt{2}}\right)$
Standard deviation	$S = \mu$
Mean value (median)	$\sigma_{mean} = \Lambda, P_f(\sigma_{mean}) = 1/2$

Table C.2 Lognormal distribution

Lognormal distribution	
Mean log parameter	$\ln\Lambda$
Scale parameter	μ
Probability density of failures	$p_f(\sigma_\delta) = \begin{cases} \frac{1}{\sigma_\delta\mu\sqrt{2\pi}} \exp\left(-\frac{1}{2\mu^2}(\ln\frac{\sigma_\delta}{\Lambda})^2\right), & \text{if } \sigma_\delta \geq 0; \\ 0, & \text{if } \sigma_\delta < 0 \end{cases}$
Maximum of probability density	$\max p_f(\sigma_\delta) = p_f(\sigma_\delta) = \exp(\mu^2/2)/\Lambda\sqrt{2\pi}\mu$
Stress at maximum	$\sigma_{max} = \Lambda/\exp(\mu^2/2)$
Cumulative failure probability P_F (for given stress value σ)	$P_f(\sigma) = \frac{1}{2} - \frac{1}{2}\operatorname{erf}\left(\frac{\ln\Lambda-\ln\sigma}{\mu\sqrt{2}}\right)$
Standard deviation	$S = \Lambda\sqrt{\exp(\mu^2) - 1 \exp(\mu^2/2)}$
Mean value (median)	$\sigma_{mean} = \Lambda\exp(\mu^2/2), P_f(\sigma_{mean}) = 1/2$

In order to obtain a relationship between stress level and failure probability we use the lognormal distribution. The parameters of probability density functions for normal and lognormal distributions are shown in Tables C.1 and C.2. The normal and lognormal distribution function have two parameters. The scale parameter of the distributions is μ . The location parameter of the distribution is Λ . Each parameter has a specific significance as function of analyzed application. Figures C.1 and C.2 display the graphs of functions $p_f(\sigma_\delta)$, $P_f(\sigma)$. Apparent is the influence of scale parameter μ on the width of scaling region.

For normal distribution, the location parameter Λ is the level of stress for which most fracture events occur. The scale parameter μ is a measure of experimental data scattering. These parameters have the similar meaning for lognormal distribution as well.

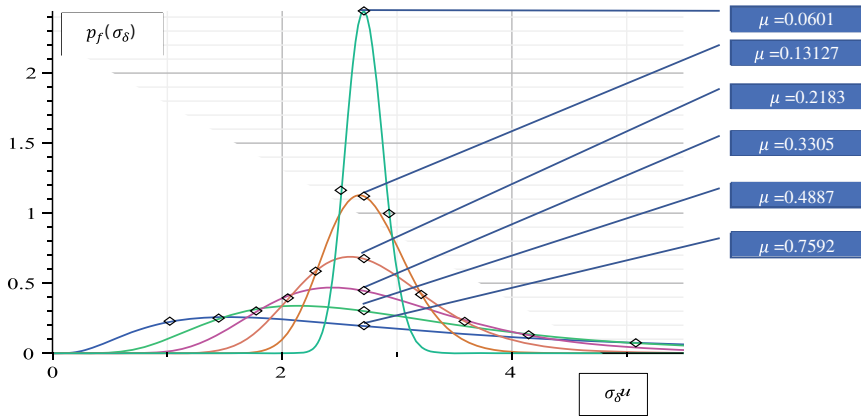


Fig. C.1 Probability density of lognormal distribution for mean value $\ln \Lambda = 1$ and different scaling factors μ

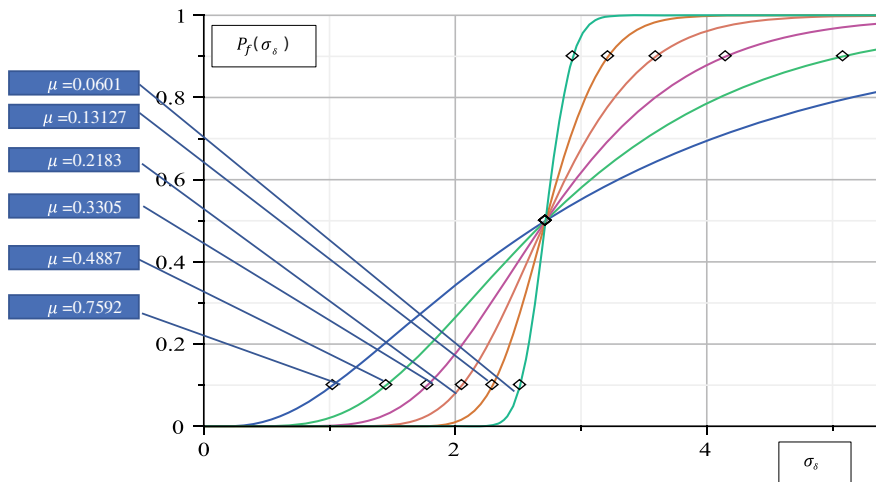


Fig. C.2 Cumulative probability of lognormal distribution for mean value $\ln \Lambda = 1$ and different scaling factors μ

2°. The method of calculation of failure probability is applicable for the statistical failure analysis of samples under static loading. The same method is applicable cyclic loading, if the number of cycles to failure is fixed. For fatigue tests, the number of cycles is fixed. For example, the simplest “stress parameter” is the amplitude of stress for a symmetric stress cycle. Because of practical reasons, the fatigue tests are performed to failure for several values of fixed stress parameters, such that cycles to failure scatter. To account the scattering of cycles to failure for a fixed stress value, the data must be reorganized, in other words, transposed (Wirsching 1983). For design

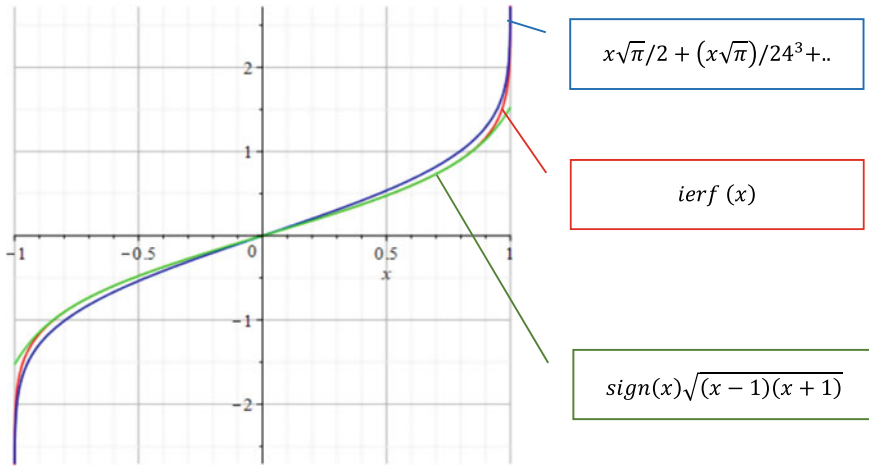


Fig. C.3 Inverse error function

purposes, the designer quests the levels of stress σ for the prescribed the fraction of failures P_F^* among the large series of samples.

The cumulative failure probability for the specified intensity of cyclic load $P_f(\sigma)$ must obey the condition:

$$P_f(\sigma) \leq P_F^*. \quad (C.2)$$

Accordingly, we search the fatigue lines, which express the number of cycles to failure as the functions of stress for each fixed failure probability.

2°. The graph of the inverse error function is shown on Fig. C.3. The inverse error function approximately evaluates as:

$$ierf(x) \approx \begin{cases} sign(x)\sqrt{(x-1)(x+1)}, & |x| < 1 \\ \frac{x\sqrt{\pi}}{2} + \frac{(x\sqrt{\pi})^3}{24} + \frac{7(x\sqrt{\pi})^5}{960} + \frac{127(x\sqrt{\pi})^7}{80640} + \dots, & |x| < 1 \end{cases} \quad (C.3)$$

The inverse error function is the odd function of its argument x (Carlitz 1963):

$$ierf(x) = -ierf(-x). \quad (C.4)$$

Apparently, most samples will fail if the stress parameter is approximately X_0 . Thus, the at least 50% of all samples fail, if the stress parameter is less or equal to X_0 .

3°. For the design of springs, the acceptable level of the stress parameter of certain prearranged cumulative failure probability $50\% - \varepsilon$ after a certain number of cycles must be determined. The value ε is referred to as confidence parameter. For example,

if $\varepsilon = 50\% - 1 \text{ ppm}$, the failure probability for corresponding stress level will be less or equal $1 \text{ ppm} = 10^{-6}$.

To evaluate the acceptable level of the stress parameter of certain prearranged cumulative probability, the following questions must be answered. Which stress σ_m cause the failure of samples with the probability $50\% - \varepsilon$? For which stress parameter σ_p the spring fails with the definite failure probability $50\% + \varepsilon$? The answers for these questions deliver the admissible stress level for the guaranteed failure probability in series applications.

For a given confidence parameter ε , the stress scatter range $T(\varepsilon)$ is the ratio of both stresses σ_p, σ_m for the given confidence parameter:

$$\frac{\sigma_p(\varepsilon)}{\sigma_m(\varepsilon)} = \frac{1}{T(\varepsilon)}. \quad (\text{C.5})$$

Once the stress scatter range $T(\varepsilon)$ is experimentally evaluated, the scale parameter reads:

$$\mu = \frac{\ln(T(\varepsilon))}{2\sqrt{2} \operatorname{ierf}(2\varepsilon)} \quad (\text{C.6})$$

The scatter range $T(\varepsilon)$ is the acquired quantity and is assumed in this book to be known for the traditional setting of confidence parameter $\varepsilon_0 = 4/10$. The ratio of the stresses, which lead after a definite number of cycles to failure with 90 and 10% probabilities, is referred to as the “standard stress scatter range” T_σ (Haibach 2006). For the acquired curves, the usual assessment of the confidence parameter is $\varepsilon = \varepsilon_0$.

4°. The fatigue data arranges data sets in accordance to the number of cycles to failure. The described procedure is applicable directly for these data sets. Baseline fatigue data are obtained by cycling test specimens at constant-amplitude stress (or strain) until failure occurs. For each stress level, the scattering of the number cycles to failure is performed. The “cycle scatter range” T_N for a fixed stress is evaluated. The data could be rearranged, that “standard stress scatter range” T_σ evaluates (Wirsching 1983). For this purpose, two S–N lines are evaluated. These S–N lines display the levels of failure probabilities 90% and 10% correspondingly:

$$\sigma_{p0} = \sigma_p(\varepsilon_0), \quad \sigma_{m0} = \sigma_m(\varepsilon_0), \quad \varepsilon_0 = \frac{4}{10} \quad (\text{C.7})$$

The stress scatter range of the lognormal distribution is:

$$\frac{\sigma_{p0}}{\sigma_{m0}} = \frac{1}{T_\sigma}, \quad T_\sigma = T(\varepsilon_0), \quad 0 < T_\sigma < 1. \quad (\text{C.8})$$

The scale parameter μ of lognormal distribution relates to the standard stress scatter range as follows:

Table C.3 Calculations for lognormal distribution for given failure confidence parameter ε

Failure probability is 50% for stress X_0	$P_f(\sigma_0) = \frac{1}{2}$	$\sigma_0 = \Lambda$
Failure probability is 50% + ε for stress X_p	$P_f(\sigma_p) = \frac{1}{2} + \varepsilon$	$X_p(\varepsilon) = \Lambda \exp(\sqrt{2} \mu \operatorname{ierf}(2\varepsilon))$
Failure probability is 50% - ε for stress X_m	$P_f(\sigma_m) = \frac{1}{2} - \varepsilon$	$X_m(\varepsilon) = \Lambda \exp(-\sqrt{2} \mu \operatorname{ierf}(2\varepsilon))$
Scatter range of failure probabilities, $0 < T(\delta) < 1$	$\frac{\sigma_p(\varepsilon)}{\sigma_m(\varepsilon)} = \frac{1}{T(\delta)}$	$\mu = \frac{\ln(T(\varepsilon))}{2\sqrt{2} \operatorname{ierf}(2\varepsilon)}$

$$\mu = \frac{\ln(T_\sigma)}{2\sqrt{2} \operatorname{ierf}(2\varepsilon_0)} = -0.3901520 \ln(T_\sigma). \quad (\text{C.9})$$

Generally, the stress scatter range and the scale parameters depend on the cyclic stress or on number of cycles. The acquired data habitually assumes the constant scatter range over the total interval of stress parameter. In this case, the σ_{p0} and σ_{m0} are parallel to the baseline S–N line.

5° Once the scale parameter μ is determined, the cumulative failure probability distribution could be evaluated. For this purpose, of the formula for the cumulative failure probability distribution from the Table C.3 is applicable in the fatigue experiments, the value σ represents the measure of cyclic stress, for example, their amplitude or stroke. For practical purposes, the inverse problem must be solved. The engineer must determine the amplitude of the cyclic stress, which leads to the certain cumulative failure probability. Accordingly, the acceptable value σ stress parameter is the function of the required failure probability P_F^* . The design of spring is performed for this acceptable value of stress parameter.

For example, assume for standard stress scatter range the following value $T_\sigma = 1/1.25 = 0.8$. If the obligatory failure probability is equal to $P_F^* = 0.1\% = 10^{-3}$, the reduction of stress parameter relative to its median is:

$$T_\sigma^{1.20566..} = 0.7641 \dots \quad (\text{C.10})$$

This means, that the acceptable value of stress is 76% of the median stress parameter. If the required failure probability is equal to $P_F^* = 1 \text{ ppm} = 10^{-6}$, we have

$$T_\sigma^{1.8455..} = 0.6478 \dots \approx 65\%. \quad (\text{C.11})$$

The applicability of the lognormal distribution is restricted to the range of the cumulative failure probability between 1 and 99%. The estimations of cyclic stress for the lower values of the cumulative failure probability are displayed for recommendation. Figure C.4 displays the relation between between T_σ and scaling factor μ of the lognormal distribution (Table C.4).

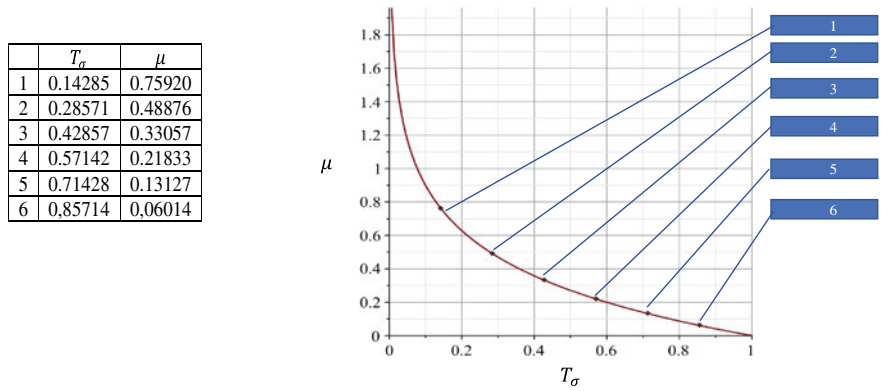


Fig. C.4 Relation between T_σ and scaling factor μ

Table C.4 Acceptable stress parameters for required cumulative failure probability P_F

Cumulative failure probability P_F	Stress parameter to requisite failure probability	Ratio of stress parameter stress to the median stress Λ
$P_f(\sigma)$	σ	σ/Λ
0.9	$\Lambda/\sqrt{T_\sigma}$	$1/\sqrt{T_\sigma}$
0.5 median	Λ	1
0.1	$\Lambda\sqrt{T_\sigma}$	$\sqrt{T_\sigma}$
10^{-2}	$\Lambda \cdot T_\sigma^{0.90762}$	$T_\sigma^{0.90762}$
10^{-3}	$\Lambda \cdot T_\sigma^{1.20566}$	$T_\sigma^{1.20566}$
10^{-4}	$\Lambda \cdot T_\sigma^{1.45098}$	$T_\sigma^{1.45098}$
10^{-5}	$\Lambda \cdot T_\sigma^{1.66395}$	$T_\sigma^{1.66395}$
10^{-6}	$\Lambda \cdot T_\sigma^{1.845229}$	$T_\sigma^{1.845229}$

List of symbols

Description	Symbol ^a	Unit ^b	Chapters
Activation energy of diffusion	Q_a	kJ/mol	12
Angle between meridian and principal material axis	χ	rad	4
Angle of inclination of the bent axis	ϕ_Q	rad	3
Angle of pitch of helical spring	α	rad	1, 2
Angle of twist per unit length	ϑ	rad	6
Angle of twist per unit length after spring-back	$\overline{\vartheta}$	rad	6
Angle, of slope, deformed middle surface of disk spring	$\psi = H/(r_e - r_i)$	rad	4, 12

(continued)

(continued)

Description	Symbol ^a	Unit ^b	Chapters
Angle, of slope, free middle surface of disk spring	$\alpha = h/(r_e - r_i)$	rad	4, 12
Angle, rotation of the middle surface disk spring	ϕ	rad	12
Area of the material part of the cross-section	A	m ²	1, 3, 5
Area, enclosed by the curve L_m	A_m	m ²	5
Auxiliary functions	$\phi_1(r), \phi_2(z)$	1	2
Auxiliary functions for survival probabilities and ratio of cycles to failure of spring to straight rod	k_1, k_2	1	11
Auxiliary scaling constants, $\sigma'_f = \sigma_0(2N_0)^{-b_0}$	σ_0, N_0	1	11
Axes of wire cross-section (thickness and width)	T, B	m	11
Bending radius after unloading (after spring-back)	$\bar{R} = 1/\bar{\kappa}$	m	6
Bending radius in during active coiling	$R = 1/\kappa$	m	6
Circumferential angle along wire length	$\theta_a = 2\pi n_a$	rad	1
Coordinate on the meridian of free conical shell	$x_e \leq x \leq x_i$	m	4
Coordinates of the center of mass of the cross-section	$x_c, y_c,$	m	5
Coordinates of the twist center of the cross-section	α_x, α_y	m	5
Coordinates, Cartesian of the circular cross-section	$x = \rho \cos \phi,$ $y = \rho \sin \phi$	m	2
Coordinates, polar of the circular cross-section	$0 \leq \rho \leq r,$ $0 \leq \phi < 2\pi$	m rad	2
Correction factors for stress, $i = 1, \dots, 4$	$k_i = k_i(w)$	1	1
Creep constant, for shear strain	c_τ	$\frac{s^{-\xi}}{\text{Pa}^{\xi+1}}$	7, 12
Creep constant, for uniaxial strain	c_σ	$\frac{s^{-\xi}}{\text{Pa}^{\xi+1}}$	7, 12
Creep, average time	t_c	s	12
Creep, Norton-Bailey constant	$\bar{t}$	s	7, 12
Creep, strain rate constants	$\bar{\epsilon}, \bar{\gamma}$	1/s	7, 12
Creep, stress exponent	ξ	1	7, 12
Creep, stress scaling constants	$\bar{\sigma}, \bar{\tau}$	Pa	7, 12

(continued)

(continued)

Description	Symbol ^a	Unit ^b	Chapters
Creep, time exponent	ς	1	7, 12
Curvature in moment of plastic deformation	κ	1/m	6
Curvature principal, $i = 1, 2$	κ_i	1/m	4
Cycles number to the failure for a given stress amplitude, highest (failure event of the last homogeneously stressed specimen)	N_L	1	11
Cycles number to the failure for a given stress amplitude, lowest (failure event of the first homogeneously stressed specimen)	N_F	1	11
Deflection at loading and unloading, critical	μ_+^*, μ_-^*	1	3
Density of material	ρ_m	kg/m ³	1, 3
Derivative, fractional of order $\hat{\alpha}$	$D^{\hat{\alpha}}$	1/s ^{$\hat{\alpha}$}	12
Diameter of circular wire or bar	$d = 2r$	m	1, 3, 11
Diameter of middle surface of free spring, external (outside)	D_e	m	4
Diameter of middle surface of free spring, internal (inside)	D_i	m	4
Diameter of wire, optimal	d_{opt}	m	1
Dimensionless length, character	μ^*	1	3
Displacement, axial, measured from upper inside edge to lower outside edge	$\tilde{s}$	m	4
Displacement, caused by bending moment	s_b	m	3
Displacement, caused by shearing force	s_s	m	3
Displacement, lateral	s_Q	m	3
Endurance limit for completely reversed stress	τ_e	Pa	1
Endurance threshold limit	K_{th}	N $\sqrt{m}$	8, 9, 10
Energy, elastic strain	$U_e, U_1 \dots U_5$	J	1, 4
Energy, potential of applied forces	U_f	J	1, 4
Energy, total potential	Π	J	4
Equivalent bending stiffness	$\langle EI_B \rangle$	Pam ⁴	3
Equivalent shear stiffness	$\langle GS \rangle$	Pam ²	3
Euler's critical load for compression	F^*	N	3

(continued)

(continued)

Description	Symbol ^a	Unit ^b	Chapters
Exponent, at short-term limit	$m_2 > 1$	1	8, 9, 10
Exponent, endurance limit	$m_1 > 1$	1	8, 9, 10
Exponent, fatigue	$p > 1$	1	8
Exponent, fatigue ductility	$c(R_\sigma)$	1	9
Exponent, fatigue in high cycle range	p_S	1	10
Exponent, fatigue in moderate cycle range	p_L	1	10
Exponent, of fatigue strength	$b_0(R_\sigma) = -1/p_\sigma$	1	8, 9, 10, 11
Exponent, secant	k	1	6
Extent constant of failure region	k_f		11
Factor for safety	S_f	1	1
Fatigue ductility coefficient, shear	$\gamma'_f(R_\sigma)$	1	9
Fatigue ductility coefficient, uniaxial	$\varepsilon'_f(R_\sigma)$	1	9
Fatigue equation, constant	$\lambda \equiv \left(\sigma'_f\right)^{-\frac{1}{b_\sigma}}/2$	1	11
Fatigue strength coefficient for normal stress	$\sigma'_f(R_\sigma)$	Pa	8, 9, 10
Fatigue strength coefficient for shear stress	$\tau'_f(R_\sigma)$	Pa	8, 9, 10
Force acting on the upper middle surface, radial	F_R	N	4
Force tangential, pro unit length	$F_\tau(z, s)$	N/m	5
Force, axial on the spring	F	N	2, 3
Force, Circumferential in the wire direction	F_θ	N	1
Force, corrected total axial	$\tilde{F}_{1Z}$	N	4
Force, normal, pro unit length	$F_\sigma(z, s)$	N/m	5
Force, of spring as the function of time	$F_z(t)$	N	12
Force, shear	Q	N	3
Force, Spring at the moment $t = 0$	F_z^0	N	12
Force, spring loads at lengths L_{comp}	F_{min}	N	1
Force, spring loads at lengths L_{ref}	F_{max}	N	1
Force, total axial acting on the upper middle surface	F_z	N	4
Force, total axial due to Almen and Laszlo	F_{AL}	N	4

(continued)

(continued)

Description	Symbol ^a	Unit ^b	Chapters
Force, total axial, DIN standard	F_{zDIN}	N	4
Forces, meridional, circumferential and shear direct	N_1, N_2, N_{12}	N	4
Frequency of compressed spring, natural	ω_N	rad/s	3
Frequency of free spring, natural	ω_N^0	rad/s	3
Frequency, fundamental, dimensionless, relative, $\Omega_N(\mu) = \omega_N/\omega_N^0$	Ω_N	1	3
Gas constant	R	$\frac{\text{J}}{\text{molK}^\circ}$	12
Geometry parameter, dimensionless	Y	1	8, 9, 10
Greenhill's critical twist buckling moment	M_T^*	Nm	3
Height of middle surface of disk spring, free state	h	m	4, 12
Height, of middle surface of disk spring in the deformed state	H	m	4
Height, of the inside edge of the middle surface	z_i	m	4
Height, of the outer edge of the middle surface	z_e	m	4
Height, total between utmost edges of the conical spring in its free state	$\tilde{h}$	m	4
Height, total between utmost edges of the deformed conical spring	$\tilde{H}$	m	4
Index of spring	$w = D/d$	1	1, 2
Invariant of deviatory strain tensor, second	$\mathbf{II}(\boldsymbol{\varepsilon})$	Pa^2	6, 7
Invariant of deviatory stress tensor, second	$\mathbf{II}(\mathbf{s})$	Pa^2	6, 7
Invariant of strain tensor, first	$\mathbf{I}(\boldsymbol{\varepsilon})$	Pa	6, 7
Invariant of stress tensor, first	$\mathbf{I}(\mathbf{s})$	Pa	6, 7
Inversion center point for cross-section	c_i	m	4, 12
Length of crack	a	m	8, 9, 10
Length of the spring, actual under action of load	L	m	3
Length of the spring, Free	L_0	m	1, 3
Length parameter, inverse	$\eta = \varepsilon_p/r$	1	6
Length ratio of disk wave spring	$\lambda_w = a/l$	1	4

(continued)

(continued)

Description	Symbol ^a	Unit ^b	Chapters
Length, characteristic	$\lambda_c^2 = \frac{GI_T}{EI_\omega}$	m	5
Length, compression	L_{comp}	m	1
Length, dimensionless relative	$\mu = L/L_0$	1	3
Length, of trailing arm	L_T	m	5
Length, of twist-beam length	L_B	m	5
Length, of wire	L_W	m	11
Length, released	L_{rel}	m	1
Limit cases for number of cycles to fault (type I), $i = 1, 2, 3$	$n_{I,i}$	1	8, 9, 10
Limit cases for number of cycles to fault (type II), $i = 1, 2, 3$	$n_{II,i}$	1	8, 9, 10
Limit cases for propagation function of type I, $i = 1, 2, 3$	$F_{I,i}$	1	8, 9, 10
Limit cases for propagation function of type II, $i = 1, 2, 3$	$F_{II,i}$	1	8, 9, 10
Load, external in the transverse direction	f_Q	N	3
Magnitude of the Burgers vector, screw dislocation	B_s		
Mass, absolute lowest	m_{opt}	kg	1
Mass, of the spring material	m	kg	1
Material constant for a given stress ratio R_σ	$c_f(R_\sigma)$	m	8, 9, 10
Maximum stress intensity factor per cycle	K_{max}	$\text{N}\sqrt{\text{m}}$	8, 9, 10
Maximum stress per cycle	σ_{max}	Pa	8, 9, 10
Mean coil diameter	$D = 2R$	m	1, 3, 11
Mean diameter of ring, disk wave spring	$D_m = \frac{D_o + D_i}{2}$	m	4
Mean value of stress intensity factor,	$K_m = \frac{K_{max} + K_{min}}{2}$	$\text{N}\sqrt{\text{m}}$	8, 9, 10
Minimal slenderness ratio for instability	ξ^*	1	3
Modulus, bulk	$K = \frac{2(1+\nu)}{3(1-2\nu)} G$	Pa	7
Modulus, secant	G_p	Pa	6
Modulus, sectional for twist of wire	W_t	m^3	1
Modulus, shear	G	Pa	1, 3, 4, 6, 7, 11
Modulus, Young	$E = 2(1 + \nu)G$	Pa	1, 3, 4, 7
Moment of inertia of the cross-section with respect to the x-axis	I_x	m^4	5

(continued)

(continued)

Description	Symbol ^a	Unit ^b	Chapters
Moment of inertia of the cross-section with respect to the y-axis	I_y	m ⁴	5
Moment of inertia of the section, Sectorial	I_ω	m ⁵	5
Moment of inertia of wire, area (for helical springs: about the helix axis)	I	m ⁴	1, 6
Moment of inertia of wire, area (for helical springs: about the radius of helix)	I_r	m ⁴	1, 6
Moment of the cross-section, polar	$S_{\omega p}$	m ⁴	5
Moment, bending	M_B	Nm	2, 3, 4, 6, 12
Moment, bending at the moment $t = 0$	M_B^0	Nm	12
Moment, bending the helical spring wire	m_B	Nm	1
Moment, circumferential	M	Nm	4
Moment, circumferential due to Almen and Laszlo	M_{AL}	Nm	4
Moment, due to constrained torsion	M_S	Nm	5
Moment, due to pure torsion	M_H	Nm	5
Moment, of resistance for bending load of wire	W_B	N	7
Moment, of Resistance for twist of wire	W_T	m ³	1, 7
Moment, torque	M_T	Nm	2, 7, 12
Moment, torque at the moment $t = 0$	M_T^0	Nm	12
Moment, twisting the helical spring wire	m_T	Nm	1
Moments, meridional, circumferential and twist	M_1, M_2, M_{12}	Nm	4
Number of active coils, helical spring	n_a	1	1, 3
Number of cycles to failure as function of stress amplitude and mean stress	$n_f(\sigma_a, \sigma_m)$	1	8, 9, 10
Number of cycles to failure as function of stress range	$n_f(a, \delta, \Delta\sigma)$	1	9
Number of cycles to failure of helical spring	n_S	1	11

(continued)

(continued)

Description	Symbol ^a	Unit ^b	Chapters
Number of cycles to failure of straight rod	n_R	1	11
Number of reversals to failure	$2n_f$	1	8, 9, 10
Parameters, dimensionless	λ, μ	1	6
Pitch	p	m	1, 6
Pitch angle	α	1	1, 6
Pivot radius at maximum height of ovate wire	$r_m = \sqrt{\frac{r_o^2 + r_i^2}{2}}$	m	2
Poisson coefficient	ν	1	4
Prandtl stress function	$\varphi(r, z)$	1	2
Probability, failure	P_F	1	11
Probability, prescribed survival of the structural element, helical spring	P_F^*	1	11
Probability, survival	P_S	1	11
Probability, survival, of the spring with defects	$P_{S,i}$	1	11
Probability, survival, of the straight rod with defects	$P_{R,i}$	1	11
Projected length of zigzag crack crack, y- component	Y_p	m	10
Projected length of zigzag crack, x- component	X_p	m	10
Projected length, critical	$X^* = \frac{1}{\pi} \left(\frac{\Delta K^*}{\Delta \sigma} \right)^2$		10
Quality parameter, spring	Q_p		1
Radius during plastic coiling	R	m	6
Radius of free spring, external (outside)	$r_e = D_e/2$	m	2, 4, 12
Radius of free spring, internal (inside)	$r_i = D_i/2$	m	2, 4, 12
Radius of wire or bar	r	m	1–3, 6–12
Radius, after spring-back unloaded	$\bar{R}$	m	6
Radius, polar	$\rho = \sqrt{x^2 + y^2}$	m	6
Ratio of material thickness to inside radius	$\mu = T/r_i$	1	4
Ratio of outside radius to inside radius	$\Delta = r_e/r_i$	1	4
Ratio of survival probabilities of spring and rod	α_i	1	11

(continued)

(continued)

Description	Symbol ^a	Unit ^b	Chapters
Ratio stress amplitude to Walker effective stress	$C_R(\gamma)$		10
Ratio, geometrical transmission	$i_t = L/L_T$	1	5
Ratio, of amplitude for principal strains	$\tilde{A}_i$	1	9
Ratio, of amplitude for principal stresses	A_i	1	9
Reciprocal strength exponent	$p_\sigma = -1/b(R_\sigma) > 0$	1	9, 11
Reduction of the spring angle due to compression force and torque respectively	θ_F, θ_M	rad	1
Relation between fatigue coefficients	$\Lambda[R_\sigma, R_\sigma^*]$	1	8, 9, 10
Relaxation function for coiled and disk springs	$\Phi(t), \Psi(t)$	1	12
Residual stresses after spring-back	τ_r, σ_r	Pa	7
Root of eigenvalue problem	Λ	1	3
Section modulus, for bending of wire (for helical springs: with respect to helix axis)	W_b	m ³	1
Section modulus, for bending of wire (for helical springs: with respect radius of helix)	W_{br}	m ³	1
Sections of inner curve, upper and lower	$z_+(r), z_-(r)$	m	2
Sections of outer curve, upper and lower	$Z_+(r), Z_-(r)$	m	2
Sectorial moment of the cross-section with respect to the x-axis	S_x	m ³	5
Sectorial moment of the cross-section with respect to the y-axis	S_y	m ³	5
Shear stresses, second degree Taylor polynomials of	$\tau_{r\theta}^{(2)}, \tau_{z\theta}^{(2)}$	Pa	2
Short-term threshold limit	K_{IC}	N $\sqrt{\text{m}}$	8, 9, 10
Slenderness ratio	$\xi = L_o/D$	1	3
Smith–Watson–Topper parameter for shear stress	$p_{SWT,\tau}$	Pa	1
Solidifying coefficients for shear and tension states	K_τ, K_σ	Pa	7

(continued)

(continued)

Description	Symbol ^a	Unit ^b	Chapters
Solidifying exponents for shear and tension states	n, l	1	7
Solutions of characteristic equation	β_i	1	3
Spring rate of disk spring, corrected	$\tilde{c}_{1Z}$	N/m	4
Spring rate of the wave spring, total initial	c_w	N/m	4
Spring rate, compression (or extension)	c	N/m	1
Spring rate, compression-twist	$c_{\theta F}$	N	1
Spring rate, design compression	c^*	N/m	1
Spring rate, DIN standard	c_{zDIN}	N/m	4
Spring rate, for axial loading	c_f	N/m	7
Spring rate, for twist loading	c_{θ}	Nm	1, 7
Spring rate, of disk spring	c_Z	N/m	4
Spring rate, of one complete coil	c_w	N/m	2
Stiffening factor due to bi-moment	$K(\lambda_c L)$	1	5
Stiffness camber, axle	r_c	Nm/rad	5
Stiffness of axle, roll, $r_a = i_t^2 r_t$	r_a	Nm/rad	5
Stiffness, bending of the twist beam	r_z	Nm/rad	5
Stiffness, bending of wire (for helical springs: about the helix axis)	EI	Pam ⁴	1
Stiffness, bending of wire (for helical springs: with about radius of helix)	EI_r	Pam ⁴	1
Stiffness, lateral of the axle	r_l	N/m	5
Stiffness, torsion of the twist beam	r_t	Nm/rad	5
Stiffness, torsion of the twisted rod without the influence of bi-moment	$\bar{r}_t$	Nm/rad	5
Strain coefficients of Johnson–Cook and Ludwik–Hollomon equations	A_{σ}, B_{σ}	1	7
Strain rate	$\dot{\epsilon}$	1/s	7
Strain rate, deviatoric components	$\dot{e}_{ij}$	1/s	12
Strain rate, fractional, deviatoric components	$D^{\hat{\alpha}} e_{ij}$	1/s ^{$\hat{\alpha}$}	12
Strain rate, reference (quasi-static)	$\dot{\epsilon}_{ref}$	1	7
Strain rate, sensitivity parameter	C	1	7
Strain tensor	$\boldsymbol{\epsilon}$	1	1, 6, 7, 12
Strain tensor, components	ϵ_{ij}	1	6, 7, 12
Strain tensor, deviatoric	$\boldsymbol{e}$	1	6, 7, 12

(continued)

(continued)

Description	Symbol ^a	Unit ^b	Chapters
Strain tensor, deviatoric components	$e_{ij} = \varepsilon_{ij} - \varepsilon_{oct}\delta_{ij}$	1	1, 6, 7, 12
Strain, amplitude of shear	$\gamma_a = \tau_a/G$	1	1
Strain, decrements of maximal normal	$\Delta\varepsilon$	1	6
Strain, decrements of maximal shear	$\Delta\gamma$	1	6
Strain, mid-surface, principal, $i = 1, 2, 3$	ε_i	1	4
Strain, octahedral normal	ε_{oct}	1	6, 7, 12
Strain, octahedral shear	$\gamma_{oct} = 2\varepsilon_{oct}$	1	6, 7, 12
Strain, on the contour of the cross-section during active load (shear and normal)	γ_P, ε_P	1	6, 7
Strain, principal, $i = 1, 2, 3$	E_i	1	4
Strain, shear component, elastic	γ_e	1	12
Strain, shear, creep component of	γ_c	1	12
Strain, ultimate shear	γ_w	1	6
Strength, ultimate shear	τ_w	Pa	1, 6
Strength, ultimate tensile	$\sigma_w = R_m$	Pa	1, 9
Stress amplitude in operation	τ_a	Pa	1
Stress due to bending, normal	σ_z	Pa	5
Stress due to torsion, normal	$\sigma_z^{(i)}$	Pa	5
Stress function, isotropic creep	$f_c(\tau_{oct}, t) = h_c(t)g_c(\tau_{oct})$	1/s	7, 12
Stress function, isotropic primary creep stage	$f_{Ic}(\tau_{oct}, t)$	1/s	7, 12
Stress function, isotropic secondary creep stage	$f_{IIc}(\tau_{oct}, t)$	1/s	7, 12
Stress function, isotropic tertiary creep stage	$f_{IIIc}(\tau_{oct}, t)$	1/s	7, 12
Stress intensity factor per cycle, maximum	K_{max}	$\text{Pa}\sqrt{\text{m}}$	
Stress intensity factor per cycle, minimum	K_{min}	$\text{Pa}\sqrt{\text{m}}$	8, 9, 10
Stress intensity factor, range	$\Delta K = K_{max} - K_{min}$	$\text{Pa}\sqrt{\text{m}}$	8, 9, 10
Stress intensity factor, transition range	ΔK^*	$\text{Pa}\sqrt{\text{m}}$	10
Stress per cycle, maximum	σ_{max}	Pa	8, 9, 10
Stress per cycle, minimum	σ_{min}	Pa	8, 9, 10
Stress range, $\Delta\sigma = 2\sigma_a$	$\Delta\sigma = \sigma_{max} - \sigma_{min}$	Pa	8, 9, 10
Stress ratio of cyclic load	R_σ	1	1, 8, 9, 10
Stress tensor	σ	Pa	1, 6, 7, 12

(continued)

(continued)

Description	Symbol ^a	Unit ^b	Chapters
Stress tensor, components	σ_{ij}	Pa	1, 6, 7, 12
Stress tensor, deviatory	s	Pa	1, 6, 7, 12
Stress tensor, deviatory components	$s_{ij} = \sigma_{ij} - \sigma_{oct}\delta_{ij}$	Pa	1, 6, 7, 12
Stress, “comparative” maximal	σ_R	Pa	12
Stress, average bending	$ \sigma_{Be} + \sigma_{Bi} /2$	Pa	12
Stress, average tensile	$(\sigma_{Te} - \sigma_{Ti})/2$	Pa	12
Stress, basic	σ_b	Pa	2
Stress, corner	$ \sigma_I , \sigma_{II} , \sigma_{III} , \sigma_{IV} $	Pa	12
Stress, corrected	τ_c	Pa	1, 2, 10, 11
Stress, deviatory tensor	s	Pa	6, 7
Stress, due to bending, inside diameter D_i	σ_{Bi}	Pa	4
Stress, due to bending, outside diameter D_e	σ_{Be}	Pa	4
Stress, due to circumferential strain, inside diameter D_i	σ_{Ti}	Pa	4
Stress, due to circumferential strain, outside diameter D_e ;	σ_{Te}	Pa	4
Stress, equivalent Mises	$\sigma_{eq} = 3\tau_{oct}/\sqrt{2}$	Pa	1, 6, 7, 12
Stress, for the circular wire	$\tau(\rho, \phi)$	Pa	2
Stress, mean in operation	τ_m	Pa	1
Stress, octahedral normal	σ_{oct}	Pa	6, 7, 12
Stress, octahedral shear	τ_{oct}	Pa	6, 7, 12
Stress, on the contour of the cross-section during active load	σ_p, τ_p	Pa	6
Stress, principal, $i = 1, 2, 3$	σ_i	Pa	4
Stress, residual after spring-back	$\bar{\sigma}_{zz}, \bar{\tau}_{\varphi z}$	Pa	6
Stress, shear	τ	Pa	5
Stress, shear average	$\langle \tau \rangle$	Pa	9
Stress, shear due to bi-moment	τ_S	Pa	5
Stress, shear due to pure torsion	τ_H	Pa	5
Stress, shear on the outer surface of the circular wire	$\sigma_c(\phi) = \tau(\rho = r, \phi)$	Pa	2
Stress, shear, components in the cross-section of wire	$\tau_{r\theta}, \tau_{z\theta}$	Pa	11
Stress, shear, first degree Taylor polynomials	$\tau_{r\theta}^{(1)}, \tau_{z\theta}^{(1)}$	Pa	2
Stresses on corner points of disk spring	$\sigma_I, \sigma_{II}, \sigma_{III}, \sigma_{VI}$	Pa	4

(continued)

(continued)

Description	Symbol ^a	Unit ^b	Chapters
Surface element that contains one critical defect, average	A_0	m^2	11
Surface of the whole structural element, spring	A	m^2	11
Surface, middle in the undeformed state	ϖ	m^2	4
Surface, middle of the shell in the deformed state	Ω	m^2	4
Temperature exponent	m^ν	1	7
Temperature of deformation, actual	T^ν	$^\nu\text{K}$	7
Temperature, melting	T_m^ν	$^\nu\text{K}$	7
Temperature, Reference (ambient)	T_{ref}^ν	$^\nu\text{K}$	7
Thickness of material outside diameter	T_e	m	4
Thickness of material, inside diameter	T_i	m	4
Thickness or height or of the cross-section (for helical springs: in the direction of helix axis)	T	m	1, 2, 4
Time in seconds	t	s	12
Time, scaled	$t^* = \int_0^t h_c(p) dp$	s	12
Torque of the helical torsion spring	M_θ	Nm	1
Torque of wire during plastic coiling	M_{TW}	Nm	6
Torque, external per unit length	m_B	Nm	3
Torsion constant for the section of wire	I_T	m^4	1
Torsion of helix after spring-back	$\bar{\chi} = 2\pi\bar{\theta}$	rad/m	6
Torsion of helix during plastic coiling, instantaneous	$\chi = 2\pi\theta$	rad/m	6
Torsional rigidity of wire about wire axis	GI_T	Nm^2	1, 3
Travel or axial displacement of spring	s	m	1, 3, 4
Ultimate nominal stress	$\sigma_{w,n}$	Pa	9
Unified propagation function of type I	F_I	1	8, 9, 10
Unified propagation function of type II	F_{II}	1	8, 9, 10
Variable, dimensionless in flexure problem	$\psi = \sigma_p/K_\sigma$	1	7

(continued)

(continued)

Description	Symbol ^a	Unit ^b	Chapters
Variable, dimensionless in torque problem	$\varphi = \tau_p / K_\tau$	1	7
Volume of active material	$V = \pi r^2 L$	m ³	1, 11
Volume of one coil	V_1	m ³	2
Volume of the whole structural element, spring	Ω	m ³	11
Volume that contains one critical defect, average	V_0	m ³	11
Walker exponent	$\tilde{\gamma}$	1	9, 10
Warping function	$\psi(r, z)$	1	2
Wave length of disk wave spring	l_w	m	4
Wave spring, number of waves of disk	n_w	m	4
Weibull “risk of rupture” of the element	p_h	1	11
Weibull immediate “risk of rupture”	p_f	1	11
Weibull stress shift, stress scale and shape parameters	S_{sh}, S_W, m_W	1	11
Width between utmost edges of the conical spring in its free state	$\tilde{b}$	m	4
Width of disk spring in deformed state of width of disk wave spring	B	m	1, 2, 4
Width of the deformed conical spring, total	$\tilde{B}$	m	4
Width of the middle surface of the unloaded disk spring	b	m	4
Yield stress	σ_Y, τ_Y		

^aFor the sake of continuous consistency in the manuscript, the designations may be different from those used in the standards

^bOnly the units of SI system are referred. In industry, instead of m or Pa the units mm and MPa, instead of rad the degrees are usually used

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