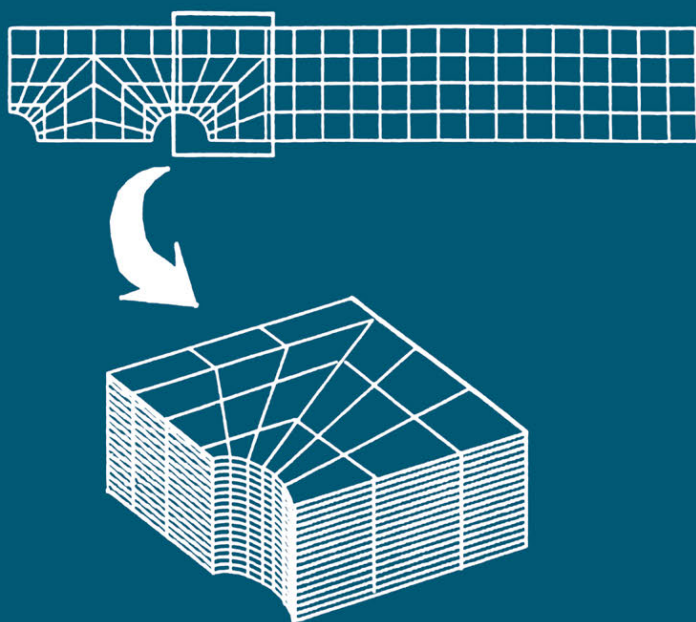


SOLID MECHANICS AND ITS APPLICATIONS

O. O. Ochoa and J. N. Reddy

Finite Element Analysis of Composite Laminates



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FINITE ELEMENT ANALYSIS OF COMPOSITE LAMINATES

SOLID MECHANICS AND ITS APPLICATIONS

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Finite Element Analysis of Composite Laminates

by

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Preface

Composite materials are increasingly used in aerospace, under water, and automotive structures. The application of composite materials to engineering components has spurred a major effort to analyze structural components made from them. Composite materials provide unique advantages over their metallic counterparts, but they also present complex and challenging problems to analysts and designers. To take advantage of the full potential of composite materials, structural analysts and designers must have accurate mathematical models and design methods at their disposal. The most common structural elements are plates and shells. An accurate modelling of stress fields and failures is of paramount importance in the design of such components.

The present monograph has the objective of introducing the mechanics concepts, structural theories, and finite element models of composite laminates. Detailed coverage of the basic mechanics of composite materials, theories of composite plates and shells, and the finite element method are avoided in the interest of providing a general background necessary for engineers to analyze composite structures.

The authors are very grateful to Professor G. M. L. Gladwell of the University of Waterloo for his technical as well as editorial comments and for suggesting improvements. It is a pleasure to acknowledge the help of Mr. Y. S. N. Reddy (Virginia Polytechnic Institute and State University) in reading the manuscript and offering useful suggestions during the preparation of the manuscript. The authors also wish to thank Dr. Nigel Hollingworth (Kluwer) for the encouragement and support in publishing the monograph. The authors are very thankful to Mrs. Vanessa McCoy of the Department of Engineering Science and Mechanics, Virginia Polytechnic Institute and State University, Blacksburg, for typing of the manuscript. Without the patience and cooperation of Mrs. McCoy, it would not have been possible to publish this monograph.

O. O. Ochoa and J. N. Reddy

Chapter One

Overview

1.1 Introduction

The phrase ‘composite material’ refers to a material that is formed by combining two or more materials on a macroscopic scale. Structures made of such materials are called composite structures. Composite materials are fabricated to have better engineering properties than the conventional materials, for example, metals. Some of the properties that can be improved by forming a composite material are: stiffness, strength, weight, corrosion resistance, thermal properties, fatigue life and wear resistance. Most man-made composite materials are made from two materials: a reinforcement material and a parent or matrix material. Composite materials are finding applications in a variety of systems; including aircraft and submarine structures, space structures, automobiles, sports equipment, medical prosthetic devices, and electronic circuit boards. They are most suitable in applications that require high strength-to-weight and stiffness-to-weight ratios. With the increased use of fiber-reinforced composites in structural components, studies involving the behavior of components made of composites are receiving considerable attention. Functional requirements and economic considerations of design are forcing engineers to seek reliable and accurate yet economical methods of determining static and dynamic characteristics of the structural components.

The analytical study and design of composite materials requires knowledge of anisotropic elasticity, structural theories and failure/damage criteria. Unlike isotropic materials, anisotropic materials exhibit complicated mechanical behavior. For example, consider rectangular blocks of isotropic and anisotropic (monoclinic) materials. When an isotropic block is subjected to pure shear stress, it develops only shear strain and no normal strains. Similarly, if an isotropic block is subjected to normal stress, it develops only normal strain and no shear strains. Under

identical loads, a block made of an anisotropic material deforms differently from an isotropic block, as shown in Fig. 1.1-1. When a shear stress is applied to an anisotropic body, it develops normal strains in addition to shear strain; a normal stress similarly produces shear strain as well as normal strain.

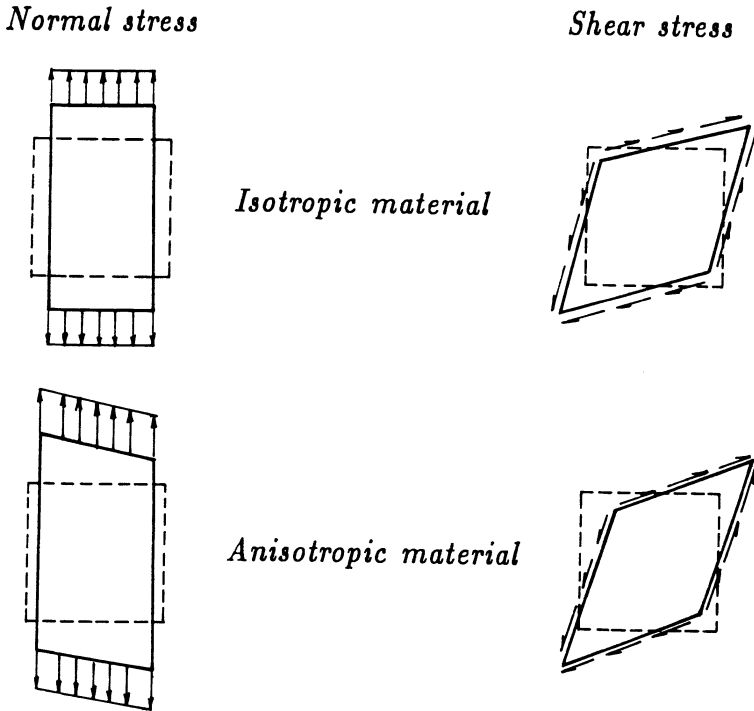


Figure 1.1-1 Deformation of isotropic and anisotropic material elements subjected to normal and shear stresses (broken lines denote undeformed geometry).

Most real world problems involving composite structures do not admit exact solutions, requiring one to find approximate, but representative solutions. The finite element method is an effective approximate method of obtaining numerical solutions to boundary-value, initial-value and eigen-value problems. The method is the most powerful numerical tool available today for predicting the response of composite structures. In the formulation and analysis of any mathematical model

of a physical process, we should include all appropriate details consistent with the objective of the study. For example, representation of true loads and boundary conditions can be achieved to any desired closeness in a numerical model. However, the representation of material properties [physical constants] is inevitably an approximation which demands careful physical measurements. Gross approximation of some parameters and accurate representation of others does not lead to realistic modeling of the overall problem. One must accurately represent the kinematics of deformation as well as the material behavior.

1.2 Present study

The objective of this monograph is to present the laminated plate theories and their finite element models to study the deformation, strength and failure of composite structures. Emphasis is placed on engineering aspects, such as the analytical descriptions, effective analysis tools, modeling of physical features, and evaluation of approaches used to formulate and predict the response of composite structures.

We share our experiences in terms of guidelines and recommendations throughout the book, while focussing on fundamental aspects of finite element modeling of composite laminates. One may consult the list of references provided at the end of each chapter to gain further insight into mechanics and materials aspects and for advanced modeling of composites structures (see [1-8] and references therein).

Chapter 2 is devoted to the introduction of the definitions and terminology used in composite materials and structures. Anisotropic constitutive relations and laminate plate theories are also reviewed. Finite element models of laminated composite plates are presented in Chapter 3. Numerical evaluation of element coefficient matrices, postcomputation of strains and stresses, and sample examples of laminated plates in bending and vibration are discussed. Chapter 4 introduces damage and failure criteria in composite laminates. Finally, Chapter 5 is dedicated to case studies involving various aspects and types of composite structures. Joints, cutouts, woven composites, environmental effects, postbuckling response and failure of composite laminates are discussed by considering specific examples.

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Chapter Two

Mechanics of Composite Laminates

2.1 Introduction

Analysis of structures made of composite materials requires a knowledge of anisotropic elasticity, an appropriate structural theory that accounts for desired kinematics, failure criteria to determine if the structure has failed, and a numerical method to solve the boundary-value problem associated with the structure. The study of anisotropic elasticity and structural theories used to analyze composite laminates constitute the topics for this chapter.

The traditional undergraduate engineering education that emphasizes isotropic materials, e.g., metals, is neither adequate nor appropriate for analyzing this new class of engineered material systems. If the component is a heterogeneous anisotropic medium, we must have a proper structural theory to model the stiffness correctly, predict the stress fields, and determine the initiation and growth mechanisms of different failures. Drawing analogies between isotropic and anisotropic media, with the hope of using existing knowledge of metals and making deductions from them, can lead to incorrect results and conclusions.

In the present study, we shall focus on the fundamentals of anisotropic elasticity and a study of the mechanics of composite laminates. We will review basic assumptions, study constitutive relations of composite materials, and derive structural theories of composite laminates.

2.2 Anisotropic elasticity

2.2–1 Definitions

Since we seek a fundamental understanding of the mechanics of composites materials, we will start with the study of their constitutive behavior. First, we identify the differences between homogeneous and heterogeneous material systems.

A material is said to be *homogeneous* if the material properties remain unchanged throughout. In a heterogeneous system, the material properties are a function of position. Next, we distinguish between isotropic and anisotropic materials. A material is said to be *isotropic* if all its material properties at a point are independent of the direction. That is, if one refers to point A in an isotropic medium of Figure 2.2-1 with a coordinate system with its origin at A, the physical properties at A will remain invariant for any arbitrary rotation of axes. An *anisotropic* material is one which exhibits material properties that are directionally dependent, i.e., a given material property can have different values in different directions. We shall discuss various special cases of anisotropic materials in the next section. Note that a material can be isotropic and homogeneous, isotropic and heterogeneous (or inhomogeneous), anisotropic and homogeneous, or anisotropic and heterogeneous.

2.2-2 Constitutive relations

In a three-dimensional Cartesian coordinate system (see Figure 2.2-2), it is customary to describe the state of deformation by six components of strain and stress, namely, three normal and three shear components. A linear relation between the six stresses and six strains is known as the generalized Hooke's law, and it can be expressed as,

$$\sigma_k = C_{kj} \epsilon_j \quad (k = 1, 2, \dots, 6) \quad (2.2-1)$$

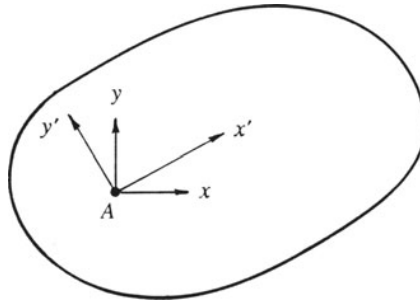
where C_{kj} are known as the *elastic coefficients*. Note that (2.2-1) is an abbreviation of the proper tensor form of Hooke's law, namely (see [1-5]),

$$\sigma_{ij} = c_{ijkl} \epsilon_{kl}$$

When C_{kj} are functions of position the material is heterogeneous, and when they are constant throughout the material it is homogeneous. We note that C_{kj} are entries in the k -th row and j -th column of a 6×6 square matrix; however, C_{kj} are *not* components of a second-order tensor. Also note that the single subscript notation for stress and strain components is based on the convention,

$$\begin{aligned} \sigma_1 &= \sigma_{11}, \sigma_2 = \sigma_{22}, \sigma_3 = \sigma_{33}, \sigma_4 = \sigma_{23}, \sigma_5 = \sigma_{13}, \sigma_6 = \sigma_{12} \\ \epsilon_1 &= \epsilon_{11}, \epsilon_2 = \epsilon_{22}, \epsilon_3 = \epsilon_{33}, \epsilon_4 = 2\epsilon_{23}, \epsilon_5 = 2\epsilon_{13}, \epsilon_6 = 2\epsilon_{12} \end{aligned} \quad (2.2-2)$$

Here $(\sigma_1, \sigma_2, \sigma_3)$ denote the normal stresses and $(\sigma_4, \sigma_5, \sigma_6)$ denote shear stresses; $(\sigma_1, \sigma_2, \sigma_3)$ are the inplane stresses and $(\sigma_3, \sigma_4, \sigma_5)$ are the out-of-plane (i.e., transverse) stresses. Similar terminology is used for the strain components.



m = A material property measured with respect to the (x,y) system
 m' = A material property measured with respect to the (x',y') system

Isotropic Materials: $m=m'$; Anisotropic Materials: $m'\neq m$

Figure 2.2–1 Definitions of isotropic and anisotropic materials

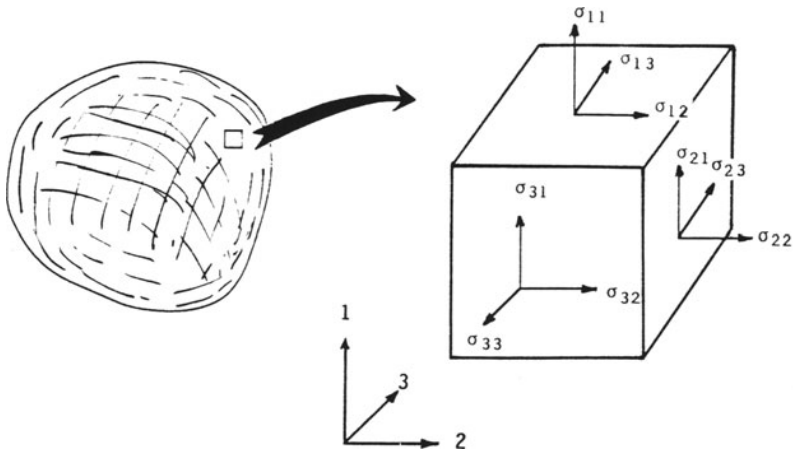


Figure 2.2–2 Components of stress and strain in a rectangular cartesian coordinate system.

The thirty six coefficients C_{ij} are not all independent of each other. The number of independent constants depends on the material constitution. First we show that $C_{kj} = C_{jk}$; that is, they are symmetric for materials for which the strain energy density function U_o is such that

$$\frac{\partial U_o}{\partial \epsilon_k} = \sigma_k \quad (2.2-3)$$

To illustrate this, we consider the strain energy density of the material which may be expressed as

$$U_o = \int_0^{\epsilon_i} \sigma_i d\epsilon_i \quad (2.2-4)$$

Substituting equation (2.2-1) into equation (2.2-4) and integrating, we obtain

$$U_o = \frac{1}{2} C_{ij} \epsilon_i \epsilon_j \quad (2.2-5)$$

Substituting for U_o from Eq. (2.2-5) into Eq. (2.2-3), we arrive at the expression

$$\sigma_k = \frac{1}{2} (C_{kj} + C_{jk}) \epsilon_j \quad (2.2-6)$$

By comparing expressions (2.2-6) and (2.2-1), we conclude that $C_{kj} = C_{jk}$. Because of this symmetry, there are only 21 independent elastic constants for anisotropic materials. In matrix form Eq. (2.2-1) can be expressed as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix} \quad (2.2-7)$$

sym.

It is understood from Eq. (2.2-7) that, in general, the elastic coefficients C_{ij} relating the Cartesian components of stress and strain depend on the coordinate system (x_1, x_2, x_3) used. Referred to another Cartesian coordinate system $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$, the elastic coefficients are $\bar{C}_{ij}$, and in general $\bar{C}_{ij} \neq C_{ij}$. If $\bar{C}_{ij} = C_{ij}$, then they are independent of the coordinate system and the material is said to be isotropic.

Some anisotropic materials may possess material symmetries and their constitutive behavior can be described with fewer than 21 constants. When the elastic coefficients at a point have the same values for every pair of coordinate systems which are mirror images of each other in a certain plane, that plane is called a *plane of elastic symmetry* for the material at that point. Materials with one plane of symmetry are called *monoclinic materials*, and the number of elastic coefficients for such materials reduces to 13. If the plane of symmetry is $x_3 = 0$, the constitutive relations become:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ & C_{22} & C_{23} & 0 & 0 & C_{26} \\ & & C_{33} & 0 & 0 & C_{36} \\ & & & C_{44} & C_{45} & 0 \\ & & & & C_{55} & 0 \\ & sym. & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix} \quad (2.2-8)$$

Note that the out-of-plane shear stresses, namely σ_4 and σ_5 , are independent of normal strains and the inplane shear strain.

If a material system has three mutually perpendicular planes of elastic symmetry, then the number of independent elastic coefficients can be reduced to nine. Such materials are referred to as *orthotropic*. The stress-strain relations for an orthotropic material are given by

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & sym. & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix} \quad (2.2-9)$$

Note that there are no interactions between extensional and shear components for orthotropic materials when loaded along the material coordinates.

The stiffness coefficients C_{ij} for an orthotropic material may be expressed in terms of the engineering constants by (see Reddy [4])

$$\begin{aligned} C_{11} &= \frac{1 - \nu_{23}\nu_{32}}{\Delta E_2 E_3}, \quad C_{12} = \frac{\nu_{21} + \nu_{31}\nu_{23}}{\Delta E_2 E_3} = \frac{\nu_{12} + \nu_{32}\nu_{13}}{\Delta E_1 E_2} \\ C_{13} &= \frac{\nu_{31} + \nu_{21}\nu_{32}}{\Delta E_2 E_3} = \frac{\nu_{13} + \nu_{12}\nu_{23}}{\Delta E_1 E_2}, \quad C_{22} = \frac{1 - \nu_{13}\nu_{31}}{\Delta E_1 E_3} \\ C_{23} &= \frac{\nu_{32} + \nu_{12}\nu_{31}}{\Delta E_1 E_3} = \frac{\nu_{23} + \nu_{21}\nu_{13}}{\Delta E_1 E_2}, \quad C_{33} = \frac{1 - \nu_{12}\nu_{21}}{\Delta E_1 E_2} \end{aligned}$$

$$C_{44} = G_{23}, \quad C_{55} = G_{13}, \quad C_{66} = G_{12}$$

$$\Delta = \frac{1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{21}\nu_{32}\nu_{13}}{E_1 E_2 E_3} \quad (2.2 - 10)$$

where E_i denotes Young's moduli in the i th-material direction, ν_{ij} is the Poisson's ratio for transverse strain in the j -th direction when stressed in the i -th direction

$$\nu_{ij} = -\frac{\epsilon_j}{\epsilon_i},$$

and (G_{23}, G_{31}, G_{12}) are shear moduli in the 2-3, 3-1, and 1-2 planes, respectively. The three material directions (x_1, x_2, x_3) are taken perpendicular to the three planes of elastic symmetry.

If the material is isotropic in one of the planes of elastic symmetry of an orthotropic material system, it is referred to as a *transversely isotropic* material system. For this class, there are only five independent coefficients. If x_1 denotes the coordinate normal to the plane of isotropy, then the stress-strain relationship takes the form

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{23} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{22} & 0 & 0 & 0 \\ & & & \hat{C} & 0 & 0 \\ & & & & C_{66} & 0 \\ & sym. & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix} \quad (2.2 - 11)$$

where $\hat{C} = 0.5(C_{11} - C_{12})$.

Finally, for an isotropic material there are infinite number of planes of elastic symmetry (or no preferred direction), and it can be described in terms of two elastic coefficients. The stress-strain relations for an isotropic material have the form,

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & \hat{C} & 0 & 0 \\ & & & & \hat{C} & 0 \\ & sym. & & & & \hat{C} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix} \quad (2.2 - 12)$$

where $\hat{C} = 0.5(C_{11} - C_{12})$. These two coefficients are related to Young's Modulus E and Poisson's ratio ν by

$$C_{11} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}, \quad C_{12} = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad (2.2-13)$$

A fiber-reinforced composite material is an anisotropic material because it possesses different moduli along the fiber direction and transverse to it. If we assume that all fibers in a lamina are almost parallel, we can characterize it as a homogeneous orthotropic material, with the material coordinate axis $\bar{x}_1$ being parallel to the fiber direction. In the next section, we present constitutive equations for an orthotropic layer. We note that a fiber-reinforced composite is strong along the fibers and weak in the directions transverse to the fibers.

2.3 Orthotropic lamina

The inplane stress-strain relations in material coordinates $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ of a single orthotropic lamina (i.e., a thin layer) are

$$\begin{Bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \\ \bar{\sigma}_6 \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & 0 \\ 0 & 0 & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \bar{\epsilon}_1 \\ \bar{\epsilon}_2 \\ \bar{\epsilon}_6 \end{Bmatrix} \quad (2.3-1)$$

The bars over the σ 's, ϵ 's and Q 's are introduced here to distinguish them from the corresponding variables referred to the coordinate system (x_1, x_2, x_3) , which is used to describe the structure made of composite materials. Thus, $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ denotes the material coordinate system and $(x_1, x_2, x_3) = (x, y, z)$ denotes the coordinate system used to formulate the governing equations of the structural problem (see Figure 2.3-1).

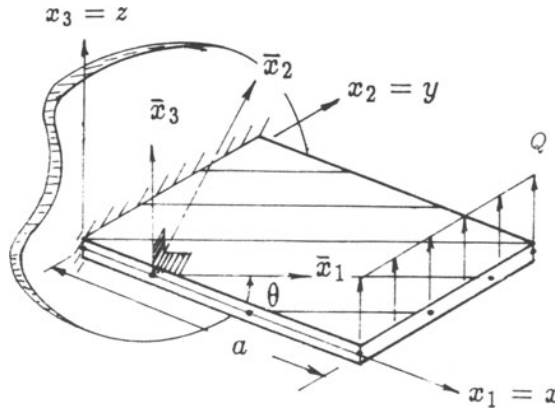


Figure 2.3-1 Coordinate system used for the description of the structure and the fiber direction.

The $\bar{Q}_{ij}$'s in Eq. (2.3-1) are referred to as the *plane stress-reduced stiffnesses* because they are obtained from C_{ij} by setting $\sigma_{33} = 0$ in Eq. (2.2-9). They are related to the engineering constants in the following manner

$$\begin{aligned}\bar{Q}_{11} &= \frac{E_1}{1 - \nu_{12}\nu_{21}}, \quad \bar{Q}_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}} \\ \bar{Q}_{12} &= \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}}, \quad \bar{Q}_{66} = G_{12}\end{aligned}\quad (2.3-2)$$

Note that the fibers in the lamina are aligned with the principal material direction 1, denoted $\bar{x}_1$. In this study, the material directions are always aligned with the fiber directions. When the material directions are not aligned with the coordinate system (x, y, z) used to describe the boundary-value problem associated with a structure, coordinate transformation equations are employed to obtain the stresses in material coordinates, $(\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_6)$, in terms of stresses in the global coordinates, $(\sigma_1, \sigma_2, \sigma_3) = (\sigma_x, \sigma_y, \sigma_{xy})$:

$$\begin{Bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \\ \bar{\sigma}_6 \end{Bmatrix} = \begin{bmatrix} \cos^2 \theta & \sin^2 \theta & 2 \cos \theta \sin \theta \\ \sin^2 \theta & \cos^2 \theta & -2 \cos \theta \sin \theta \\ -\cos \theta \sin \theta & \cos \theta \sin \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} \quad (2.3-3a)$$

$$\begin{Bmatrix} \bar{\sigma}_4 \\ \bar{\sigma}_5 \end{Bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{Bmatrix} \sigma_4 \\ \sigma_5 \end{Bmatrix} \quad (2.3-3b)$$

where θ denotes the angle between the positive x -axis and the positive $\bar{x}_1$ -axis, measured counter-clockwise direction. Hereafter, the angle θ will be referred to as the *lamination angle*.

2.4 The classical laminated plate theory

2.4-1 Introduction

Fiber-reinforced composites are manufactured in the form of thin sheets, called laminae or layers. The composite laminae are bonded together to form a laminate with desired thickness and stiffness. In most applications, the thickness of a laminate is small compared to the planar dimensions. Therefore, two-dimensional theories are used to analyze composite laminates for stresses. The two-dimensional theories are obtained from the three-dimensional elasticity theory by making assumptions concerning the variation of displacements *and/or* stresses through the thickness of the laminate. The *classical laminated plate theory* (CLPT) is an extension of the classical plate theory to laminated plates. In this theory, the inplane

displacements are assumed to vary linearly through the thickness and the transverse displacement is assumed to be constant through the thickness (i.e., transverse normal strain is zero). The classical laminated plate theory is found to be adequate for most applications where the thickness of the laminate is small, by two orders of magnitude compared to the inplane dimensions. When the classical laminated plate theory is not applicable, a refined theory is used. In the present section we develop the classical laminated plate theory. A refinement to the classical laminate theory is provided by the *first-order shear deformation plate theory* (FSDT), which will be discussed in Section 2.5.

2.4–2 Assumptions of the theory

The Kirchhoff-Love hypothesis is used in the derivation of the classical laminated plate theory. The Kirchhoff-Love hypothesis involves the following assumptions: straight lines perpendicular to the midplane before deformation remain (1) straight, (2) inextensible, and (3) normal to the midsurface after deformation. These three assumptions lead to the neglect of the transverse strains, ϵ_{xz} , ϵ_{yz} and ϵ_{zz} . The neglect of ϵ_{zz} leads to the omission of σ_{zz} because the product of ϵ_{zz} and σ_{zz} is zero in the total potential energy of the plate. For orthotropic layers and laminates constructed from orthotropic laminae, irrespective of the lamination angles, the vanishing of ϵ_{xz} and ϵ_{yz} lead to the neglect of σ_{xz} and σ_{yz} . Thus, the classical laminate theory does not account for transverse deformation and stress state. This amounts to assuming that plates are “infinitely” rigid in the transverse direction, whereas in reality they are weaker in the transverse direction.

The classical plate theory is adequate for the analysis of thin plates, especially when the transverse deformation is negligible. However, plates made of advanced fiber-reinforced composite materials, whose elastic to shear modulus ratios (E_1/G_{13} and E_1/G_{23}) are very large, are susceptible to thickness failures because their effective transverse shear moduli (G_{13} and G_{23}) are significantly smaller than the Young’s moduli (E_1) along the fiber direction. Although the transverse shear and normal stresses (σ_{xz} , σ_{yz} , σ_z) are an order of magnitude smaller than in-plane stresses (σ_x , σ_y , σ_{xy}), the material strength *allowables* for the transverse stresses are also an order of magnitude smaller than the allowables for the inplane stresses. Thus, the classical laminate theory developed here is not to be used for composites that are likely to fail in transverse shear or delamination.

2.4–3 Displacements and strains

Consider a plate of total thickness h composed of N orthotropic layers with their principal material coordinates oriented at angles $\theta_1, \theta_2, \dots, \theta_N$ to the laminate coordinate, x . We assume that

- the layers are perfectly bonded together,
- the material of each layer is linearly elastic and has three planes of material

symmetry (i.e., orthotropic),

- each layer is of uniform thickness,
- the strains are small.

(2.4 – 1)

The laminate coordinate system is chosen such that the xy -plane coincides with the midplane of the laminate. A point P , located at the point (x, y, z) in the undeformed plate, moves after deformation to the position $(x + u_1, y + u_2, z + u_3)$ where (u_1, u_2, u_3) are the components of the displacement vector

$$\mathbf{u} = u_1 \hat{\mathbf{e}}_x + u_2 \hat{\mathbf{e}}_y + u_3 \hat{\mathbf{e}}_z \quad (2.4 – 2)$$

and $(\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z)$ are the unit vectors along the (x, y, z) coordinates. The first two assumptions of the Kirchhoff hypothesis require that the displacements (u_1, u_2, u_3) to be such that (see Figure 2.4-1),

$$\begin{aligned} u_1(x, y, z, t) &= u(x, y, t) + z\phi_1(x, y, t) \\ u_2(x, y, z, t) &= v(x, y, t) + z\phi_2(x, y, t) \\ u_3(x, y, z, t) &= w(x, y, t) \end{aligned} \quad (2.4 – 3)$$

where (u, v, w) are the displacements of a point on the xy -plane (i.e., the midplane of the laminate), ϕ_1 is the rotation of a transverse normal about the y -axis, and ϕ_2 is the rotation of the transverse normal about the x -axis and t denotes time. The third assumption of the Kirchhoff hypothesis implies that the rotations ϕ_1 and ϕ_2 are equal to $-\partial w/\partial x$ and $-\partial w/\partial y$, respectively.

$$\phi_1 = -\frac{\partial w}{\partial x}, \quad \phi_2 = -\frac{\partial w}{\partial y} \quad (2.4 – 4)$$

Thus the displacement field of the classical plate theory becomes

$$\begin{aligned} u_1(x, y, z, t) &= u(x, y, t) - z \frac{\partial w}{\partial x} \\ u_2(x, y, z, t) &= v(x, y, t) - z \frac{\partial w}{\partial y} \\ u_3(x, y, z, t) &= w(x, y, t) \end{aligned} \quad (2.4 – 5)$$

It is important to note that in using the kinematic assumptions of a single-layer theory to model a laminated plate composed of multiple layers of possibly dissimilar-material layers, we tacitly assumed that the strains are continuous through the thickness, including the interfaces of dissimilar-material layers. This assumption plays a significant role in developing laminate theories; it allows us to replace a laminate with an equivalent single layer whose material coefficients are

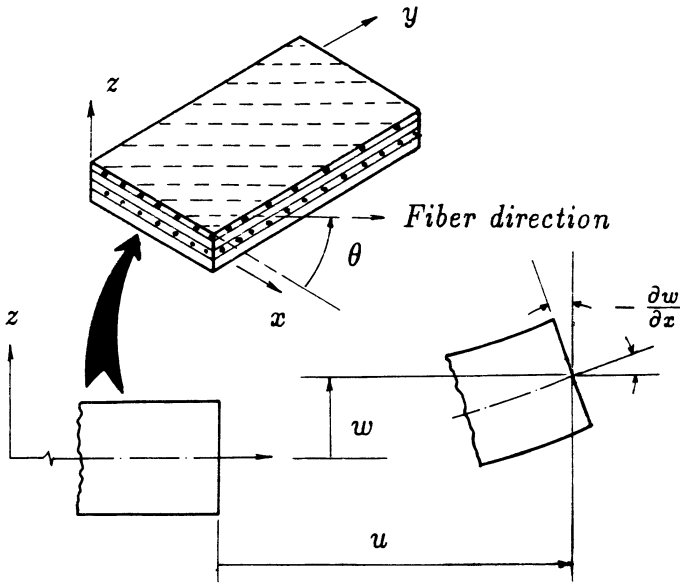


Figure 2.4-1 The Kirchhoff-Love hypothesis for bending of thin laminates.

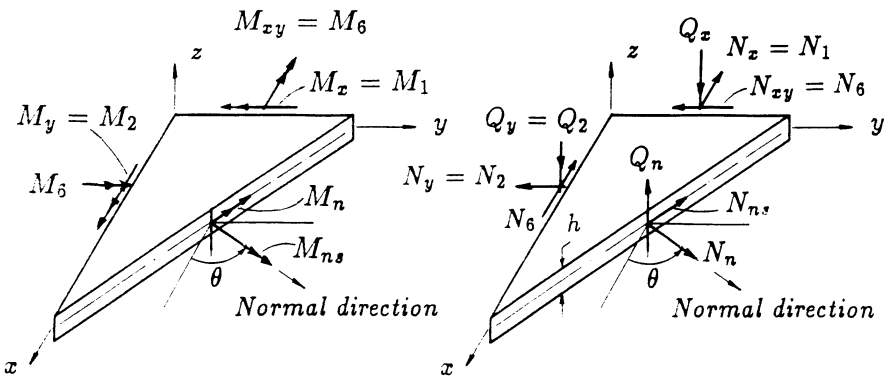


Figure 2.4-2 The stress resultants on an arbitrary edge of a laminate.

averaged over the laminate thickness, as will be shown in the following sections. However, the assumption restricts the use of the theory to model global response characteristics, and it does not represent the *interlaminar stresses* (i.e., stresses at the lamina interfaces) accurately.

Once the displacement field of a continuous body is known, the strains in the body can be computed using the strain-displacement relations (see Reddy [4]),

$$\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i} \frac{\partial u_m}{\partial x_j} \right) \quad (i, j = 1, 2, 3) \quad (2.4-6)$$

Here summation on repeated subscript (m) is implied, and $x_1 = x$, $x_2 = y$, $x_3 = z$ are the coordinates used to describe the deformation of the body. Strains in Eq. (2.4-6) are nonlinear functions of the displacement gradients. The nonlinear part of the strains should be examined closely, and terms that are indeed negligible for the class of problems one wishes to study should be discarded. We restrict our study to problems with small strains, i.e.,

$$\begin{aligned} & \left(\frac{\partial u_1}{\partial x} \right)^2, \left(\frac{\partial u_1}{\partial y} \right)^2, \frac{\partial u_1}{\partial x} \frac{\partial u_1}{\partial y} \ll 1 \\ & \left(\frac{\partial u_2}{\partial x} \right)^2, \left(\frac{\partial u_2}{\partial y} \right)^2, \frac{\partial u_2}{\partial x} \frac{\partial u_2}{\partial y} \ll 1, \left(\frac{\partial u_3}{\partial z} \right)^2 \ll 1 \end{aligned} \quad (2.4-7)$$

Note that (u_1, u_2, u_3) are the *total* displacement components along (x_1, x_2, x_3) coordinates, respectively. Here we consider plate problems that involve moderate rotations (say 10° – 15°). Hence, the following terms associated with rotations of transverse normals,

$$\left(\frac{\partial u_3}{\partial x} \right)^2, \left(\frac{\partial u_3}{\partial y} \right)^2, \frac{\partial u_3}{\partial x} \frac{\partial u_3}{\partial y} \quad (2.4-8)$$

are small but not negligible. Under the restrictions imposed by Eqs. (2.4-7) and (2.4-8), we can simplify Eq. (2.4-6) as follows:

$$\begin{aligned} \epsilon_{xx} &= \frac{\partial u_1}{\partial x} + \frac{1}{2} \left(\frac{\partial u_3}{\partial x} \right)^2, \quad \epsilon_{yy} = \frac{\partial u_2}{\partial y} + \frac{1}{2} \left(\frac{\partial u_3}{\partial y} \right)^2 \\ \epsilon_{zz} &= \frac{\partial u_3}{\partial z}, \quad \epsilon_{xy} = \frac{1}{2} \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial x} \frac{\partial u_3}{\partial y} \right), \\ \epsilon_{xz} &= \frac{1}{2} \left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} + \frac{\partial u_3}{\partial z} \frac{\partial u_3}{\partial x} \right), \quad \epsilon_{yz} = \frac{1}{2} \left(\frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y} + \frac{\partial u_3}{\partial z} \frac{\partial u_3}{\partial y} \right) \end{aligned} \quad (2.4-9)$$

The strains in Eq. (2.4-9) are called the *von Kármán strains*.

The strains associated with the displacement field (2.4-5) can now be computed using the von Kármán strain-displacement relations (2.4-9):

$$\begin{aligned}
 \epsilon_1 \equiv \epsilon_{xx} &= \frac{\partial u_1}{\partial x} + \frac{1}{2} \left(\frac{\partial u_3}{\partial x} \right)^2 = \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 - z \frac{\partial^2 w}{\partial x^2} \\
 \epsilon_2 \equiv \epsilon_{yy} &= \frac{\partial u_2}{\partial y} + \frac{1}{2} \left(\frac{\partial u_3}{\partial y} \right)^2 = \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 - z \frac{\partial^2 w}{\partial y^2} \\
 \epsilon_6 \equiv 2\epsilon_{xy} &= \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial x} \frac{\partial u_3}{\partial y} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} - 2z \frac{\partial^2 w}{\partial x \partial y} \quad (2.4-10) \\
 \epsilon_4 \equiv 2\epsilon_{yz} &= \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y} = -\frac{\partial w}{\partial y} + \frac{\partial w}{\partial y} = 0 \\
 \epsilon_5 \equiv 2\epsilon_{xz} &= \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} = -\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x} = 0 \\
 \epsilon_3 \equiv \epsilon_{zz} &= \frac{\partial u_3}{\partial z} = \frac{\partial w}{\partial z} = 0 \quad (2.4-11)
 \end{aligned}$$

The single subscript notation used in Eqs. (2.4-10) and (2.4-11) for strains still refers to the laminate coordinates $(x, y, z) \equiv (x_1, x_2, x_3)$.

The strains in Eq. (2.4-10) are of the general form,

$$\epsilon_i = \epsilon_i^{(0)} + z\epsilon_i^{(1)} \quad (i = 1, 2, 6) \quad (2.4-12)$$

Here $\epsilon_i^{(0)} = (\epsilon_1^{(0)}, \epsilon_2^{(0)}, \epsilon_6^{(0)})$ denote the strains associated with the stretching and inplane shearing of the midplane and are called the *membrane strains*. The quantities $\epsilon_i^{(1)} = (\epsilon_1^{(1)}, \epsilon_2^{(1)}, \epsilon_6^{(1)})$ are the *curvatures*. Both $\epsilon_i^{(0)}$ and $\epsilon_i^{(1)}$ are functions of (x, y) only. Equation (2.4-10) shows that the membrane strains $\epsilon_i^{(0)}$ and curvatures $\epsilon_i^{(1)}$ for the classical laminate theory have the following explicit form:

$$\begin{aligned}
 \epsilon_1^{(0)} &= \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2, \quad \epsilon_1^{(1)} = -\frac{\partial^2 w}{\partial x^2} \\
 \epsilon_2^{(0)} &= \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2, \quad \epsilon_2^{(1)} = -\frac{\partial^2 w}{\partial y^2} \\
 \epsilon_6^{(0)} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}, \quad \epsilon_6^{(1)} = -2 \frac{\partial^2 w}{\partial x \partial y} \quad (2.4-13)
 \end{aligned}$$

where the underlined terms are the von Kármán nonlinear strains. In matrix notation, Eq. (2.4-12) takes the form

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{Bmatrix} = \begin{Bmatrix} \epsilon_1^{(0)} \\ \epsilon_2^{(0)} \\ \epsilon_6^{(0)} \end{Bmatrix} + z \begin{Bmatrix} \epsilon_1^{(1)} \\ \epsilon_2^{(1)} \\ \epsilon_6^{(1)} \end{Bmatrix} \quad (2.4 - 14)$$

We note from Eq. (2.4-14) that the strains vary linearly through the laminate thickness, and they are independent of the lamination scheme, as noted earlier. For a fixed value of z , the strains vary only with respect to the x and y coordinates.

2.4-4 Equations of motion

The equations of motion of a solid body can be derived using either energy principles or vector mechanics (i.e., Newton's laws). The energy principles require the minimization of the total potential energy (or maximization of the total complementary energy) to derive the equations of motion and associated boundary conditions. Thus, in energy methods, one is required to write expressions for the appropriate energy of the body. In vector mechanics, a representative volume element of the body is isolated with all its applied and reactive forces (i.e., a free-body diagram is used), and the governing equations of motion are obtained by summing forces and moments. The advantage of the energy principles is that they yield not only the equations of motion, but also the form of the boundary conditions consistent with the equations of motion. The vector mechanics approach is closer to physical intuition. In certain cases, the two methods may give different, but closely related, sets of equations of motion.

Here we use an energy principle, namely, the principle of virtual displacements. The principle of virtual displacements states that if a body is in (static or dynamic) equilibrium, the total virtual work done by all externally applied and internally generated forces in moving through their respective virtual displacements must be zero (see Reddy [4]). The total virtual work is equal to the sum of the internal virtual work and the virtual work done by external forces. In the interest of brevity and consistent with the objectives of the book (i.e., not to dwell too much on fine theoretical points), we omit the details, and summarize the equations of motion resulting from the principle of virtual displacements (see [4]):

$$\frac{\partial N_1}{\partial x} + \frac{\partial N_6}{\partial y} = I_o \ddot{u} - I_1 \frac{\partial \ddot{w}}{\partial x} \quad (2.4 - 15a)$$

$$\frac{\partial N_6}{\partial x} + \frac{\partial N_2}{\partial y} = I_o \ddot{v} - I_1 \frac{\partial \ddot{w}}{\partial y} \quad (2.4 - 15b)$$

$$\frac{\partial^2 M_1}{\partial x^2} + 2 \frac{\partial^2 M_6}{\partial y \partial x} + \frac{\partial^2 M_2}{\partial y^2} + N(w) + q = I_o \ddot{w} + I_2 \left(\frac{\partial^2 \ddot{w}}{\partial x^2} + \frac{\partial^2 \ddot{w}}{\partial y^2} \right) + I_1 \left(\frac{\partial \ddot{u}}{\partial x} + \frac{\partial \ddot{v}}{\partial y} \right) \quad (2.4 - 15c)$$

where, for example, $\ddot{w} \equiv \partial^2 w / \partial t^2$,

$$N(w) = \frac{\partial}{\partial x} \left(N_1 \frac{\partial w}{\partial x} + N_6 \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial y} \left(N_6 \frac{\partial w}{\partial x} + N_2 \frac{\partial w}{\partial y} \right) \quad (2.4 - 16)$$

q is the distributed transverse load, (N_i, M_i) , are the force and moment resultants (see Figure 2.4-2),

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_6 \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} dz, \quad \begin{Bmatrix} M_1 \\ M_2 \\ M_6 \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} z dz \quad (2.4 - 17)$$

$(\sigma_1 = \sigma_{xx}, \sigma_2 = \sigma_{yy}, \sigma_6 = \sigma_{xy})$ and I_i are the mass inertias

$$\begin{Bmatrix} I_o \\ I_1 \\ I_2 \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} 1 \\ z \\ z^2 \end{Bmatrix} \rho dz \quad (2.4 - 18)$$

The second term on the right side of the equality in the first two equations of Eq. (2.4-15) is called the *rotatory inertia* term, and it is often omitted because it is small in comparison to the $I_o \ddot{u}$ and $I_o \ddot{v}$ terms. The same comment applies to the second term on the right hand side of Eqs. (2.4-15).

The essential (i.e., geometric) and natural boundary (i.e., force) conditions of the theory are given by (see Reddy [4]):

$$u_n, \quad u_{ns}, \quad w, \quad \frac{\partial w}{\partial n} \text{ (essential)} \quad (2.4 - 19a)$$

$$N_n, \quad N_{ns}, \quad \hat{Q}_n, \quad M_n \text{ (natural)} \quad (2.4 - 19b)$$

where (see Figure 2.4-2)

$$u_n = u n_x + v n_y, \quad u_{ns} = -u n_y + v n_x$$

$$\frac{\partial w}{\partial n} = \frac{\partial w}{\partial x} n_x + \frac{\partial w}{\partial y} n_y \quad (2.4 - 20a)$$

$$\begin{aligned}
N_n &= N_1 n_x^2 + N_2 n_y^2 + 2N_6 n_x n_y \\
N_{ns} &= (N_2 - N_1) n_x n_y + N_6 (n_x^2 - n_y^2) \\
M_n &= M_1 n_x^2 + M_2 n_y^2 + 2M_6 n_x n_y \\
M_{ns} &= (M_2 - M_1) n_x n_y + M_6 (n_x^2 - n_y^2) \\
\hat{Q}_n &= Q_1 n_x + Q_2 n_y - \frac{\partial M_{ns}}{\partial s}
\end{aligned} \tag{2.4-20b}$$

and n_x and n_y are the direction cosines of the unit normal on the boundary of the plate.

It should be noted that the vector mechanics approach would have given the following equations, in the linear case, by summing forces along the x , y and z directions and summing moments about the x and y axes:

$$\begin{aligned}
\frac{\partial N_1}{\partial x} + \frac{\partial N_6}{\partial y} &= I_o \ddot{u} - I_1 \frac{\partial \ddot{w}}{\partial x} \\
\frac{\partial N_6}{\partial y} + \frac{\partial N_2}{\partial x} &= I_o \ddot{v} - I_1 \frac{\partial \ddot{w}}{\partial y} \\
\frac{\partial Q_1}{\partial x} + \frac{\partial Q_2}{\partial y} + q &= I_o \ddot{w} \\
\frac{\partial M_1}{\partial x} + \frac{\partial M_6}{\partial y} - Q_1 &= I_2 \frac{\partial \ddot{w}}{\partial x} - I_1 \ddot{u} \\
\frac{\partial M_6}{\partial x} + \frac{\partial M_2}{\partial y} - Q_2 &= I_2 \frac{\partial \ddot{w}}{\partial y} - I_1 \ddot{v}
\end{aligned} \tag{2.4-21}$$

where Q_1 and Q_2 are the transverse shear forces. We can solve the fourth and fifth equations in (2.4-21) for Q_1 and Q_2 , substitute the expressions into the third equation of (2.4-21) and obtain the third equation in (2.4-15) without the nonlinear term.

Equations (2.4-21) can also be derived from the equations of motion of 3-D elasticity (in the absence of body forces),

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = \rho \frac{\partial^2 u_1}{\partial t^2} \tag{2.4-22a}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = \rho \frac{\partial^2 u_2}{\partial t^2} \tag{2.4-22b}$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho \frac{\partial^2 u_3}{\partial t^2} \quad (2.4-22c)$$

Integrating Eq. (2.4-22) with respect to z over the thickness of the plate $(-\frac{h}{2}, \frac{h}{2})$, we obtain the first three equations in (2.4-21) without the rotatory inertia terms. The boundary conditions on σ_{xz} , σ_{yz} and σ_{zz} are

$$\sigma_{xz} = \sigma_{yz} = 0 \text{ at } z = \pm \frac{h}{2}, \sigma_{zz} = q \text{ at } z = \frac{h}{2}, \sigma_{zz} = 0 \text{ at } z = -\frac{h}{2}$$

Similarly, by multiplying the equations (2.4-22) by z and integrating over the plate thickness, we obtain the remaining two equations. The details are left as an exercise to the interested reader.

2.4-5 Laminate constitutive equations

The laminate constitutive equations relate the force and moment resultants (N_i, M_i) to the strains $(\epsilon_i^{(0)}, \epsilon_i^{(1)})$. By substituting the stresses σ_i in terms of strains ϵ_i , both referred to the laminate coordinates, into Eq. (2.4-17) and carrying out the indicated integration with respect to z , we obtain

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_6 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1^{(0)} \\ \epsilon_2^{(0)} \\ \epsilon_6^{(0)} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1^{(1)} \\ \epsilon_2^{(1)} \\ \epsilon_6^{(1)} \end{Bmatrix} \quad (2.4-23a)$$

$$\begin{Bmatrix} M_1 \\ M_2 \\ M_6 \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1^{(0)} \\ \epsilon_2^{(0)} \\ \epsilon_6^{(0)} \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1^{(1)} \\ \epsilon_2^{(1)} \\ \epsilon_6^{(1)} \end{Bmatrix} \quad (2.4-23b)$$

where A_{ij} denote the extensional stiffnesses, D_{ij} the bending stiffnesses, and B_{ij} the bending-extensional coupling stiffnesses of a laminate:

$$(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^N \int_{z_k}^{z_{k+1}} Q_{ij}^{(k)}(1, z, z^2) dz \quad (2.4-24a)$$

where $Q_{ij}^{(k)}$ are the material stiffnesses of the k -th lamina, as referred to the laminate coordinates. Since $Q_{ij}^{(k)}$ are constant within each layer, the integration over the thickness can be carried out easily. We obtain,

$$A_{ij} = \sum_{k=1}^N Q_{ij}^{(k)}(z_{k+1} - z_k) \quad (2.4-24b)$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^N Q_{ij}^{(k)} (z_{k+1}^2 - z_k^2) \quad (2.4-24c)$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^N Q_{ij}^{(k)} (z_{k+1}^3 - z_k^3) \quad (2.4-24d)$$

where N is the number of layers in the laminate, and (z_k, z_{k+1}) are the thickness coordinates of the bottom and top of the k -th layer.

The material stiffnesses $Q_{ij}^{(k)}$ can be expressed in terms of the $\bar{Q}_{ij}^{(k)}$ [see Eq. (2.3-1)] by the following transformation equations (the superscript k is omitted for brevity),

$$\begin{aligned} Q_{11} &= \bar{Q}_{11} \cos^4 \theta + 2(\bar{Q}_{12} + 2\bar{Q}_{66}) \sin^2 \theta \cos^2 \theta + \bar{Q}_{22} \sin^4 \theta \\ Q_{12} &= (\bar{Q}_{11} + \bar{Q}_{22} - 4\bar{Q}_{66}) \sin^2 \theta \cos^2 \theta + \bar{Q}_{12} (\sin^4 \theta + \cos^4 \theta) \\ Q_{22} &= \bar{Q}_{11} \sin^4 \theta + 2(\bar{Q}_{12} + 2\bar{Q}_{66}) \sin^2 \theta \cos^2 \theta + \bar{Q}_{22} \cos^4 \theta \\ Q_{16} &= (\bar{Q}_{11} - \bar{Q}_{12} - 2\bar{Q}_{66}) \sin \theta \cos^3 \theta + (\bar{Q}_{12} - \bar{Q}_{22} + 2\bar{Q}_{66}) \sin^3 \theta \cos \theta \\ Q_{26} &= (\bar{Q}_{11} - \bar{Q}_{12} - 2\bar{Q}_{66}) \sin^3 \theta \cos \theta + (\bar{Q}_{12} - \bar{Q}_{22} + 2\bar{Q}_{66}) \sin \theta \cos^3 \theta \\ Q_{66} &= (\bar{Q}_{11} + \bar{Q}_{22} - 2\bar{Q}_{12} - 2\bar{Q}_{66}) \sin^2 \theta \cos^2 \theta + \bar{Q}_{66} (\sin^4 \theta + \cos^4 \theta) \end{aligned} \quad (2.4-25)$$

Note that $\bar{Q}_{ij}^{(k)}$ are the material stiffnesses of the k -th layer referred to the material coordinates $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$, whereas $Q_{ij}^{(k)}$ are the transformed material stiffnesses referred to the laminate coordinates (x_1, x_2, x_3) . The lamination angle θ is measured counter-clockwise from the positive x_1 -axis to the positive $\bar{x}_1$ -axis, which is aligned with the fiber direction (see Figure 2.3-1).

The equations of motion in Eqs. (2.4-15) can be expressed in terms of the displacements (u, v, w) for the purpose of developing either analytical or displacement finite element models. The first step is to introduce the strain-displacement relations (2.4-10) for the strains into the laminate constitutive equations (2.4-23). In the second step, the force and moment resultants from Eq. (2.4-23) are introduced into the equations of motion (2.4-15). The result is a system of three differential equations in three displacements (u, v, w) . We obtain:

$$\begin{aligned} \frac{\partial}{\partial x} \left[A_{11} \frac{\partial u}{\partial x} + A_{12} \frac{\partial u}{\partial y} + A_{16} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + B_{11} \left(-\frac{\partial^2 w}{\partial x^2} \right) + B_{12} \left(-\frac{\partial^2 w}{\partial y^2} \right) \right. \\ \left. + B_{16} \left(-2 \frac{\partial^2 w}{\partial x \partial y} \right) \right] + \frac{\partial}{\partial y} \left[A_{16} \frac{\partial u}{\partial x} + A_{26} \frac{\partial u}{\partial y} + A_{66} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] \end{aligned}$$

$$\begin{aligned}
& + B_{16} \left(-\frac{\partial^2 w}{\partial x^2} \right) + B_{26} \left(-\frac{\partial^2 w}{\partial y^2} \right) + B_{66} \left(-2 \frac{\partial^2 w}{\partial x \partial y} \right) \Big] \\
& = I_o \frac{\partial^2 u}{\partial t^2} + I_1 \frac{\partial^3 w}{\partial x \partial t^2}
\end{aligned} \tag{2.4-26}$$

$$\begin{aligned}
& \frac{\partial}{\partial x} \left[A_{16} \frac{\partial u}{\partial x} + A_{26} \frac{\partial v}{\partial y} + A_{66} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + B_{16} \left(-\frac{\partial^2 w}{\partial x^2} \right) + B_{26} \left(-\frac{\partial^2 w}{\partial y^2} \right) \right. \\
& \quad \left. + B_{66} \left(-2 \frac{\partial^2 w}{\partial x \partial y} \right) \right] + \frac{\partial}{\partial y} \left[A_{12} \frac{\partial u}{\partial x} + A_{22} \frac{\partial v}{\partial y} + A_{26} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right. \\
& \quad \left. + B_{12} \left(-\frac{\partial^2 w}{\partial x^2} \right) + B_{22} \left(-\frac{\partial^2 w}{\partial y^2} \right) + B_{26} \left(-2 \frac{\partial^2 w}{\partial x \partial y} \right) \right] \\
& = I_0 \frac{\partial^2 v}{\partial t^2} + I_1 \frac{\partial^3 w}{\partial y \partial t^2}
\end{aligned} \tag{2.4-27}$$

$$\begin{aligned}
& \frac{\partial^2}{\partial x^2} \left[B_{11} \frac{\partial u}{\partial x} + B_{12} \frac{\partial v}{\partial y} + B_{16} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + D_{11} \left(-\frac{\partial^2 w}{\partial x^2} \right) + D_{12} \left(-\frac{\partial^2 w}{\partial y^2} \right) \right. \\
& \quad \left. + D_{16} \left(-2 \frac{\partial^2 w}{\partial x \partial y} \right) \right] + 2 \frac{\partial^2}{\partial x \partial y} \left[B_{16} \frac{\partial u}{\partial x} + B_{26} \frac{\partial v}{\partial y} \right. \\
& \quad \left. + B_{66} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + D_{16} \left(-\frac{\partial^2 w}{\partial x^2} \right) + D_{26} \left(-\frac{\partial^2 w}{\partial y^2} \right) + D_{66} \left(-2 \frac{\partial^2 w}{\partial x \partial y} \right) \right] \\
& \quad + \frac{\partial^2}{\partial y^2} \left[B_{12} \frac{\partial u}{\partial x} + B_{22} \frac{\partial v}{\partial y} + B_{26} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right. \\
& \quad \left. + D_{12} \left(-\frac{\partial^2 w}{\partial x^2} \right) + D_{22} \left(-\frac{\partial^2 w}{\partial y^2} \right) + D_{26} \left(-2 \frac{\partial^2 w}{\partial x \partial y} \right) \right] + N(w) + q \\
& = I_o \frac{\partial^2 w}{\partial t^2} + I_1 \left(\frac{\partial^3 u}{\partial x \partial t^2} + \frac{\partial^3 v}{\partial y \partial t^2} \right)
\end{aligned} \tag{2.4-28}$$

Equations (2.4-26) and (2.4-27) may be uncoupled from Eq. (2.4-28) when the extensional-bending coupling stiffnesses B_{ij} are zero. In that case Eqs. (2.4-26) and (2.4-27) can be solved independently for the inplane displacements u and v , and Eq. (2.4-28) can be solved for the transverse deflection, w . If $B_{ij} = 0$ and the applied inplane displacements and forces are zero, then u and v will be identically zero. For the uncoupled case, the transverse deflection w under applied transverse force q is governed by

$$\begin{aligned}
& - \left[D_{11} \frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} \right] \\
& - 4 \left(D_{16} \frac{\partial^4 w}{\partial x^3 \partial y} + D_{26} \frac{\partial^4 w}{\partial x \partial y^3} \right) + N(w) + q = I_0 \frac{\partial^2 w}{\partial t^2}
\end{aligned} \tag{2.4-29}$$

For the orthotropic case $B_{ij} = 0$, and $A_{16} = A_{26} = 0$, and $D_{16} = D_{26} = 0$, and for the isotropic case $D_{11} = D_{22} = D$, $D_{12} + 2D_{66} = 2D$, where $D = Eh^3/12(1 - \nu^2)$.

2.5 First-order shear deformation theory

2.5-1 A review of refined plate theories

Numerous plate theories that include transverse shear deformations are documented in the literature (see the review articles in [6-10]). In these theories the displacement or stress components are expanded as linear combinations of the thickness coordinate and undetermined functions of position in the reference surface; this reduces the 3-D elasticity problem to a two-dimensional (2-D) one. The governing equations are derived either using virtual work principles or by weighted-integrals of the 3-D stress-equilibrium equations through the thickness of the laminate – yielding an equivalent single-layer plate theory. In this study we consider plate theories based on assumed displacement expansions.

The theories based on displacement expansion can be found in the works of Basset [11], Hildebrand et al. [12], Hencky [13], Mindlin [14], and others [15,16]. In the displacement-based theories, the three components of the displacement vector are expanded in power series of the thickness coordinate and unknown functions. The principle of virtual displacements or the method of moments is used to derive the equations of equilibrium. In all single-layer plate theories, the displacements and strains are continuous through the laminate thickness (i.e., single-valued at interfaces of layers). This leads to a discontinuous interlaminar stress field because of different elastic coefficients at layer interfaces when the constitutive equations are used to compute the stresses.

The most widely used displacement based theory is the *first-order shear deformation theory*. The theory is based on the displacement field in Eq. (2.4-3):

$$\begin{aligned} u_1(x, y, z) &= u(x, y) + z\phi_1(x, y), \\ u_2(x, y, z) &= u(x, y) + z\phi_2(x, y), \\ u_3(x, y, z) &= w(x, y) \end{aligned} \quad (2.5 - 1)$$

where (u_1, u_2, u_3) denote the displacements of a point (x, y, z) in the laminate, (u, v, w) are the displacements of a point on the midplane $(x, y, 0)$, and (ϕ_1, ϕ_2) are the rotations of the transverse normal about the (y, x) -axes, respectively. The first-order shear deformation theory is known more commonly as “the Mindlin plate theory”; the original idea of such an expansion can be found in earlier works by Basset [11], Hildebrand, Reissner, and Thomas [12], and Hencky [13]. The first-order shear deformation theory yields a constant value of transverse shear strain through the thickness of the plate, and thus requires shear correction factors. The

shear correction factors are dimensionless quantities introduced to account for the discrepancy between the constant state of shear strains in the first-order theory and the quadratic or higher-order distribution of shear strains in the elasticity theory. For composite laminates, the shear correction factors, in general, depend on the constituent ply properties, lamination scheme, and type of structure (i.e., geometry and boundary conditions) [17-19]. Stavsky [20] extended the first-order shear deformation theory of isotropic plates to plates laminated of isotropic layers of dissimilar materials. The work of Stavsky [20] was extended to anisotropic laminates by Yang, Norris and Stavsky [21] and Whitney and Pagano [22,23], who studied the effect of shear deformation on bending and free vibration response of simply supported rectangular laminates according to the first-order shear deformation theory.

Second and higher-order plate theories involve higher-order expansions of the displacement field. For example, the *third-order theory of Reddy* [24,25] is based on the displacement field

$$\begin{aligned} u_1(x, y, z) &= u(x, y) + z\phi_1 + z^3 \left(-\frac{4}{3h^2} \right) \left(\phi_1 + \frac{\partial w}{\partial x} \right) \\ u_2(x, y, z) &= v(x, y) + z\phi_2 + z^3 \left(-\frac{4}{3h^2} \right) \left(\phi_2 + \frac{\partial w}{\partial y} \right) \\ u_3(x, y, z) &= w(x, y) \end{aligned} \quad (2.5-2)$$

The displacement field accommodates the vanishing of transverse shear strains (and hence stresses) on the top and bottom of a laminate. The theory was extended in [25] to include the von Kármán strains.

The displacement field of the type in Equation (2.5-2) was also considered by others [26-35]. The works of Vlasov [26], Jemilita [27], Schmidt [28], Krishna Murty [29], and Levinson [30] were restricted to isotropic plates, and Schmidt [28] and Levinson [30] used the equilibrium equations based on the method of moments. Murthy [32] used the same approach to obtain the equilibrium equations, but cast the approach in terms of the thickness-averaged displacement field. The technical note of Krishna Murty [29] suggested a general, higher-order displacement field of the type in Equation (2.5-2). The work of Schmidt [28] also accounts for the von Kármán strains, but is restricted to isotropic plates. The displacement field used by Bhimaraddi and Stevens [33] is the same as that used by Levinson and Reddy but recast in terms of different variables which are convenient in a finite element model development.

All displacement-based theories up to and including third-order theories can be obtained from a single displacement field, as shown recently by Reddy [10]. *Reddy's*

strain-consistent third-order theory [10] is by far the most general technical theory of plates. It is based on the following displacement field [36],

$$\begin{aligned} u_1(x, y, z) &= u + \alpha z \frac{\partial w}{\partial x} + \beta z \phi_1 + \lambda z^2 \psi_1 + \gamma z^3 \theta_1 \\ u_2(x, y, z) &= v + \alpha z \frac{\partial w}{\partial y} + \beta z \phi_2 + \lambda z^2 \psi_2 + \gamma z^3 \theta_2 \\ u_3(x, y, z) &= w + \lambda_1 z \psi_3 + \gamma_1 z^2 \theta_3. \end{aligned} \quad (2.5-3)$$

Here (u, v, w) denote the midplane displacement components in the (x, y, z) directions, ϕ_1 and ϕ_2 are the rotations of a transverse normal about the y - and x -axes, respectively, and ψ_i and θ_i are undetermined functions. All of the generalized displacements are functions of only x and y . The displacement fields of various single-layer theories can be obtained from Eq. (2.5-3) by assigning proper values to the constants, called *tracers*: $\alpha, \beta, \lambda, \lambda_1, \gamma$, and γ_1 . For example, we have

Classical theory of Eq. (2.4-5):

$$\alpha = -1, \beta = \lambda = \gamma = \lambda_1 = \gamma_1 = 0; \quad (2.5-4a)$$

First-order theory of Eq. (2.5-1):

$$\alpha = 0, \beta = 1, \lambda = \gamma = \lambda_1 = \gamma_1 = 0; \quad (2.5-4b)$$

Second-order theory [15,16]:

$$\alpha = 0, \beta = 1, \lambda = 1, \gamma = \lambda_1 = \gamma_1 = 0 \quad (2.5-4c)$$

Third-order theory of Reddy [24] in (2.5-2):

$$\alpha = 0, \beta = 1, \lambda = 0, \gamma = 1, \lambda_1 = \gamma_1 = 0;$$

$$\begin{aligned} \theta_1 &= -\frac{4}{3h^2} \left(\phi_1 + \frac{\partial w}{\partial x} \right) \\ \theta_2 &= -\frac{4}{3h^2} \left(\phi_2 + \frac{\partial w}{\partial y} \right) \end{aligned} \quad (2.5-4d)$$

The displacement fields of many other single-layer third-order theories can be deduced from the displacement field (2.5-3) (see Table 1 of Reddy [10]).

Displacement-based, single-layer theories more refined than third-order are not considered in the literature because they are cumbersome and the increase in accuracy is outweighed by the increase in the number of unknowns. The third-order theory (i.e., theory with cubic terms of the thickness coordinate) yields quadratic variation of transverse shear strains within each layer and represents the true distribution of the strains through thickness. Thus, it does not require shear correction factors. The first-order theory is by far the most efficient (i.e., increased accuracy without an increase in computational effort) theory. Here we develop the first-order shear deformation theory for laminated plates.

2.5-2 Kinematics

In the first-order shear deformation, one of the three assumptions of the classical laminate theory is removed: a straight line normal to the midplane of the laminate remains straight and inextensible after deformation, but it does not necessarily remain perpendicular to the midsurface. Thus, the rotations ϕ_1 and ϕ_2 [see Eq. (2.5-1)] of a transverse normal are independent of $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$.

The displacement field of the first-order shear deformation theory is given by Eq. (2.5-1), where ϕ_1 and ϕ_2 are the rotations of a transverse normal. The von Kármán strains associated with the displacement field (2.5-1) are

$$\epsilon_i = \epsilon_i^{(0)} + z\epsilon_i^{(1)} \quad (i = 1, 2, 6, 4, 5) \quad (2.5-5a)$$

where

$$\begin{aligned} \epsilon_1^{(0)} &= \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2, \quad \epsilon_1^{(1)} = \frac{\partial \phi_1}{\partial x} \\ \epsilon_2^{(0)} &= \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2, \quad \epsilon_2^{(1)} = \frac{\partial \phi_2}{\partial y} \\ \epsilon_6^{(0)} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}, \quad \epsilon_6^{(1)} = \frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial x} \\ \epsilon_3^{(0)} &= 0, \quad \epsilon_3^{(1)} = 0, \quad \epsilon_4^{(0)} = \frac{\partial w}{\partial y} + \phi_2, \quad \epsilon_4^{(1)} = 0 \\ \epsilon_5^{(0)} &= \frac{\partial w}{\partial x} + \phi_1, \quad \epsilon_5^{(1)} = 0. \end{aligned} \quad (2.5-5b)$$

2.5-3 Equations of motion

The equations of motion of the first-order shear deformation theory are

$$\frac{\partial N_1}{\partial x} + \frac{\partial N_6}{\partial y} = I_o \frac{\partial^2 u}{\partial t^2} + I_1 \frac{\partial^2 \phi_1}{\partial t^2} \quad (2.5-6a)$$

$$\frac{\partial N_6}{\partial x} + \frac{\partial N_2}{\partial y} = I_o \frac{\partial^2 v}{\partial t^2} + I_1 \frac{\partial^2 \phi_2}{\partial t^2} \quad (2.5-6b)$$

$$\frac{\partial Q_1}{\partial x} + \frac{\partial Q_2}{\partial y} + N(w) + q = I_o \frac{\partial^2 w}{\partial t^2} \quad (2.5-6c)$$

$$\frac{\partial M_1}{\partial x} + \frac{\partial M_6}{\partial y} - Q_1 = I_1 \frac{\partial^2 u}{\partial t^2} + I_o \frac{\partial^2 \phi_1}{\partial t^2} \quad (2.5-6d)$$

$$\frac{\partial M_6}{\partial x} + \frac{\partial M_2}{\partial y} - Q_2 = I_1 \frac{\partial^2 v}{\partial t^2} + I_o \frac{\partial^2 \phi_2}{\partial t^2} \quad (2.5-6e)$$

where

$$(Q_1, Q_2) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_5, \sigma_4) dz \quad (\sigma_5 = \sigma_{xz}, \sigma_4 = \sigma_{yz}) \quad (2.5-7)$$

and (N_i, M_i) , $N(w)$, and I_i are the same as those defined by Eqs. (2.4-17), (2.4-16) and (2.4-18), respectively.

The boundary conditions for the first order theory are of the form [see Eq. (2.4-20)]

$$u_n, u_{ns}, w, \phi_n, \phi_{ns} \quad (\text{essential}) \quad (2.5-8a)$$

$$N_n, N_{ns}, Q_n, M_n, M_{ns} \quad (\text{natural}) \quad (2.5-8b)$$

The transverse shear forces Q_i can be expressed in terms of the displacements as,

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_5 \\ \sigma_4 \end{Bmatrix} dz = \begin{bmatrix} A_{55} & A_{45} \\ A_{45} & A_{44} \end{bmatrix} \begin{Bmatrix} \epsilon_5^{(0)} \\ \epsilon_4^{(0)} \end{Bmatrix} \quad (2.5-9)$$

where

$$\begin{aligned} A_{ij} &= K_{ij} \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{ij}^{(k)} dz \quad (i, j = 4, 5) \\ &= K_{ij} \sum_{k=1}^N Q_{ij}^{(k)} (z_{k+1} - z_k) \end{aligned} \quad (2.5-10)$$

and K_{ij} are the shear correction coefficients. The material stiffnesses $Q_{ij}^{(k)}$ in the laminate coordinates are related to the $\bar{Q}_{ij}^{(k)}$ in the material coordinates by

$$Q_{44} = \bar{Q}_{44} \cos^2 \theta + \bar{Q}_{55} \sin^2 \theta$$

$$\begin{aligned}
Q_{45} &= (\bar{Q}_{55} - \bar{Q}_{44}) \cos \theta \sin \theta \\
Q_{55} &= \bar{Q}_{55} \cos^2 \theta + \bar{Q}_{44} \sin^2 \theta
\end{aligned} \tag{2.5 - 11}$$

The equations of motion (2.5-6) can be expressed in terms of the displacements (u, v, w, ϕ_1 and ϕ_2) as:

$$\begin{aligned}
&\frac{\partial}{\partial x} \left[A_{11} \frac{\partial u}{\partial x} + A_{12} \frac{\partial v}{\partial y} + A_{16} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + B_{11} \frac{\partial \phi_1}{\partial x} + B_{12} \frac{\partial \phi_2}{\partial y} \right. \\
&\quad \left. + B_{16} \left(\frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial x} \right) \right] + \frac{\partial}{\partial y} \left[A_{16} \frac{\partial u}{\partial x} + A_{26} \frac{\partial v}{\partial y} + A_{66} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right. \\
&\quad \left. + B_{16} \frac{\partial \phi_1}{\partial x} + B_{26} \frac{\partial \phi_2}{\partial y} + B_{66} \left(\frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial x} \right) \right] \\
&= I_o \frac{\partial^2 u}{\partial t^2} + I_1 \frac{\partial^2 \phi_1}{\partial t^2}
\end{aligned} \tag{2.5 - 12}$$

$$\begin{aligned}
&\frac{\partial}{\partial x} \left[A_{16} \frac{\partial u}{\partial x} + A_{26} \frac{\partial v}{\partial y} + A_{66} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + B_{16} \frac{\partial \phi_1}{\partial x} + B_{26} \frac{\partial \phi_2}{\partial y} \right. \\
&\quad \left. + B_{66} \left(\frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial x} \right) \right] + \frac{\partial}{\partial y} \left[A_{12} \frac{\partial u}{\partial x} + A_{22} \frac{\partial v}{\partial y} + A_{26} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right. \\
&\quad \left. + B_{12} \frac{\partial \phi_1}{\partial x} + B_{22} \frac{\partial \phi_2}{\partial y} + B_{26} \left(\frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial x} \right) \right] \\
&= I_o \frac{\partial^2 v}{\partial t^2} + I_1 \frac{\partial^2 \phi_2}{\partial t^2}
\end{aligned} \tag{2.5 - 13}$$

$$\begin{aligned}
&\frac{\partial}{\partial x} \left[A_{45} \left(\frac{\partial w}{\partial y} + \phi_2 \right) + A_{55} \left(\frac{\partial w}{\partial x} + \phi_1 \right) \right] + \frac{\partial}{\partial y} \left[A_{44} \left(\frac{\partial w}{\partial y} + \phi_2 \right) \right. \\
&\quad \left. + A_{45} \left(\frac{\partial w}{\partial x} + \phi_1 \right) \right] + N(w) + q = I_o \frac{\partial^2 w}{\partial t^2}
\end{aligned} \tag{2.5 - 14}$$

$$\begin{aligned}
&\frac{\partial}{\partial x} \left[B_{11} \frac{\partial u}{\partial x} + B_{12} \frac{\partial v}{\partial y} + B_{16} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + D_{11} \frac{\partial \phi_1}{\partial x} + D_{12} \frac{\partial \phi_2}{\partial y} \right. \\
&\quad \left. + D_{16} \left(\frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial x} \right) \right] + \frac{\partial}{\partial y} \left[B_{16} \frac{\partial u}{\partial x} + B_{26} \frac{\partial v}{\partial y} + B_{66} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right. \\
&\quad \left. + D_{16} \frac{\partial \phi_1}{\partial x} + D_{26} \frac{\partial \phi_2}{\partial y} + D_{66} \left(\frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial x} \right) \right] - \left[A_{45} \left(\frac{\partial w}{\partial y} + \phi_2 \right) \right. \\
&\quad \left. + A_{55} \left(\frac{\partial w}{\partial x} + \phi_1 \right) \right] = I_o \frac{\partial^2 \phi_1}{\partial t^2} + I_1 \frac{\partial^2 u}{\partial t^2}
\end{aligned} \tag{2.5 - 15}$$

$$\begin{aligned}
& \frac{\partial}{\partial x} \left[B_{16} \frac{\partial u}{\partial x} + B_{26} \frac{\partial v}{\partial y} + B_{66} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + D_{16} \frac{\partial \phi_1}{\partial x} + D_{26} \frac{\partial \phi_2}{\partial y} \right. \\
& \quad \left. + D_{66} \left(\frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial x} \right) \right] + \frac{\partial}{\partial y} \left[B_{12} \frac{\partial u}{\partial x} + B_{22} \frac{\partial v}{\partial y} + B_{26} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right. \\
& \quad \left. + D_{12} \frac{\partial \phi_1}{\partial y} + D_{22} \frac{\partial \phi_2}{\partial y} + D_{26} \left(\frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial x} \right) \right] - \left[A_{44} \left(\frac{\partial w}{\partial y} + \phi_2 \right) \right. \\
& \quad \left. + A_{45} \left(\frac{\partial w}{\partial x} + \phi_1 \right) \right] = I_o \frac{\partial^2 \phi_2}{\partial t^2} + I_1 \frac{\partial^2 v}{\partial t^2} \quad (2.5-16)
\end{aligned}$$

For uncoupled laminates (i.e., $B_{ij} = 0$, $I_1 = 0$), Eqs. (2.5-14)-(2.5-16), involving (w, ϕ_1, ϕ_2) , can be solved independent of equations (2.5-12) and (2.5-13). In this case, Eqs. (2.5-15)-(2.5-16) take the form

$$\begin{aligned}
& D_{11} \frac{\partial^2 \phi_1}{\partial x^2} + D_{16} \frac{\partial^2 \phi_2}{\partial x^2} + (D_{12} + D_{66}) \frac{\partial^2 \phi_2}{\partial x \partial y} + 2D_{16} \frac{\partial^2 \phi_1}{\partial x \partial y} + D_{66} \frac{\partial^2 \phi_1}{\partial y^2} \\
& \quad + D_{26} \frac{\partial^2 \phi_2}{\partial y^2} - A_{45} \left(\phi_2 + \frac{\partial w}{\partial y} \right) - A_{55} \left(\phi_1 + \frac{\partial w}{\partial x} \right) = I_o \frac{\partial^2 \phi_1}{\partial t^2} \quad (2.5-17)
\end{aligned}$$

$$\begin{aligned}
& D_{16} \frac{\partial^2 \phi_1}{\partial x^2} + D_{66} \frac{\partial^2 \phi_2}{\partial x^2} + (D_{12} + D_{66}) \frac{\partial^2 \phi_1}{\partial x \partial y} + 2D_{26} \frac{\partial^2 \phi_2}{\partial x \partial y} + D_{26} \frac{\partial^2 \phi_1}{\partial y^2} \\
& \quad + D_{22} \frac{\partial^2 \phi_2}{\partial y^2} - A_{44} \left(\phi_2 + \frac{\partial w}{\partial y} \right) - A_{45} \left(\phi_1 + \frac{\partial w}{\partial x} \right) = I_o \frac{\partial^2 \phi_2}{\partial t^2} \quad (2.5-18)
\end{aligned}$$

For orthotropic plates, the laminate stiffnesses A_{16} , A_{26} , D_{16} , D_{26} , and A_{45} are zero, and Eqs. (2.5-14), (2.5-17) and (2.5-18) simplify to

$$A_{55} \left(\frac{\partial \phi_1}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) + A_{44} \left(\frac{\partial \phi_2}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) + N(w) + q = I_o \frac{\partial^2 w}{\partial t^2} \quad (2.5-19)$$

$$D_{11} \frac{\partial^2 \phi_1}{\partial x^2} + (D_{12} + D_{66}) \frac{\partial^2 \phi_2}{\partial x \partial y} + D_{66} \frac{\partial^2 \phi_1}{\partial y^2} - A_{55} \left(\phi_1 + \frac{\partial w}{\partial x} \right) = I_o \frac{\partial^2 \phi_1}{\partial t^2} \quad (2.5-20)$$

$$D_{66} \frac{\partial^2 \phi_2}{\partial x^2} + (D_{12} + D_{66}) \frac{\partial^2 \phi_1}{\partial x \partial y} + D_{22} \frac{\partial^2 \phi_2}{\partial y^2} - A_{44} \left(\phi_2 + \frac{\partial w}{\partial y} \right) = I_o \frac{\partial^2 \phi_2}{\partial t^2} \quad (2.5-21)$$

2.6 The layerwise theory of Reddy

2.6-1 Background

In the analysis of composite laminates with embedded delaminations, free edges, or regions of 3-D stress fields, one must use a theory based on 3-D kinematics and develop a computational model that is more efficient than the conventional 3-D finite element model. These requirements motivated Reddy [37] to propose a novel layerwise theory, in which the displacements are expanded within each layer using the Lagrange family of finite elements. The nodal values through the thickness are functions of the inplane coordinates (x, y) . Each layer of the theory can be treated either as a mathematical or a physical layer, giving the flexibility to the analyst to treat several physical layers as a sublaminates wherever necessary (for computational saving), and model delaminations, ply terminations (i.e., ply-drop-offs), ply splits, and so on. The most significant aspect of the *layerwise theory of Reddy* is that, compared to the conventional 3-D displacement finite element model, it has a data structure that saves computational time while giving exactly the same results for comparable meshes (see Robbins and Reddy [38-40]).

2.6-2 Displacements and strains

The layerwise theory of Reddy is based on the following displacement expansion through the laminate thickness. The i -th displacement component is expressed as,

$$u_i(x, y, z) = \sum_{J=1}^N U_i^J(x, y) \Phi^J(z) = U_i^J(x, y) \Phi^J(z) \quad (2.6-1)$$

where $i = 1, 2, 3$; N is the number of subdivisions (e.g., the number of nodes in the finite element discretization) through the thickness of the laminate and Φ^J are known functions of the thickness coordinate, z . Summation on repeated indices is implied in Eq. (2.6-1). While the same interpolation functions are used in Eq. (2.6-1) for all three displacements for simplicity, independent interpolation of the displacements (especially u_3) can be used. The functions Φ^J are the *global interpolation functions*, defined (see Reddy [4]) in terms of the Lagrange interpolation functions associated with the layers connected to the J -th interface through the laminate thickness, and (U^J, V^J, W^J) denote the nodal values of (u, v, w) at the nodes through the thickness (see Fig. 2.6-1). Because of this local nature of Φ^J , the displacements are continuous through the thickness, but their derivatives with respect to z are discontinuous. This implies that all transverse strains will be discontinuous at layer interfaces. Therefore, the interlaminar transverse stresses computed from the layer constitutive equations (i.e., the products of the strains with the elastic stiffnesses from the two layers at the interface) can be

continuous. The inplane strains ($\epsilon_x, \epsilon_y, \epsilon_{xy}$) will be continuous but the inplane stresses ($\sigma_x, \sigma_y, \sigma_{xy}$) will be discontinuous at layer interfaces because of the difference in material properties of adjacent layers. The resulting theory will have $3N$ variables and $3N$ two-dimensional differential equations. An advantage of the layerwise theory is that it requires only 2-D finite elements. The element aspect ratio in the layerwise models is restricted to only two dimensions, as opposed to three-dimensions in the conventional 3-D finite elements.

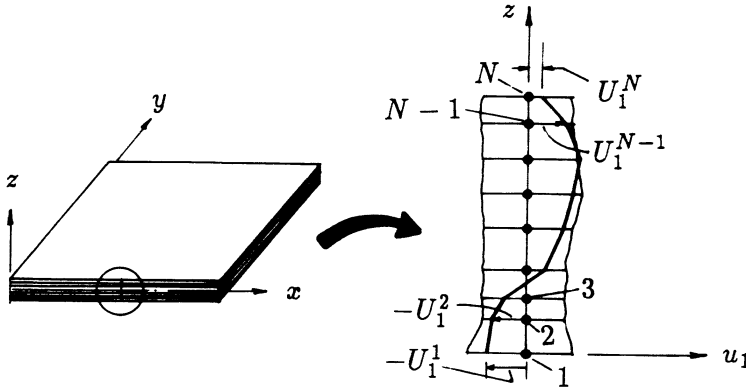


Figure 2.6-1 Kinematics of Reddy's layerwise theory.

The value of N in Eq. (2.6-1) can be appropriately selected. When N is chosen such that at least one element per physical layer is used, the interlaminar stress distributions can be determined accurately. The sublaminates concept can be used to model several layers as one equivalent single layer, i. e., N is less than the number of physical layers in the laminate. The layerwise laminate theory also yields the conventional, single-layer, plate theories as special cases, as shown by Reddy [37].

Here we consider a laminated composite plate with total thickness h . We use the strain-displacement relations of the von Kármán nonlinear theory. We have

$$\begin{aligned}\epsilon_1 &= \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 = \frac{\partial U^J}{\partial x} \Phi^J + \frac{1}{2} \left(\frac{\partial W^J}{\partial x} \Phi^J \right) \left(\frac{\partial W^J}{\partial x} \Phi^J \right), \\ \epsilon_2 &= \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 = \frac{\partial V^J}{\partial y} \Phi^J + \frac{1}{2} \left(\frac{\partial W^J}{\partial y} \Phi^J \right) \left(\frac{\partial W^J}{\partial y} \Phi^J \right), \\ \epsilon_3 &= \frac{\partial w}{\partial z} = W^J \frac{d\Phi^J}{dz},\end{aligned}\tag{2.6-2a}$$

$$\begin{aligned}
\epsilon_4 &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = V^J \frac{d\Phi^J}{dz} + \frac{\partial W^J}{\partial y} \Phi^J \\
\epsilon_5 &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = U^J \frac{d\Phi^J}{dz} + \frac{\partial W^J}{\partial x} \Phi^J \\
\epsilon_6 &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} = \left(\frac{\partial U^J}{\partial y} + \frac{\partial V^J}{\partial x} \right) \Phi^J + \left(\frac{\partial W^J}{\partial x} \Phi^J \right) \left(\frac{\partial W^J}{\partial y} \Phi^J \right) \quad (2.6-2b)
\end{aligned}$$

2.6-3 Governing equations

The governing equations for the nodal variables (U^J, V^J, W^J) can be derived using the principle of virtual displacements. The equations of equilibrium of the layerwise theory are:

$$\frac{\partial M_1^I}{\partial x} + \frac{\partial M_6^I}{\partial y} - Q_1^I = 0 \quad (2.6-3a)$$

$$\frac{\partial M_6^I}{\partial x} + \frac{\partial M_2^I}{\partial y} - Q_2^I = 0 \quad (2.6-3b)$$

$$\frac{\partial K_1^I}{\partial x} + \frac{\partial K_2^I}{\partial y} - Q_3^I + N^I (W^J) = 0 \quad (2.6-3c)$$

for $I = 1, 2, \dots, N$, where

$$N^I (W^J) = \frac{\partial}{\partial x} \left(S_1^{IJ} \frac{\partial W^J}{\partial x} + S_6^{IJ} \frac{\partial W^J}{\partial y} \right) + \frac{\partial}{\partial y} \left(S_6^{IJ} \frac{\partial W^J}{\partial x} + S_2^{IJ} \frac{\partial W^J}{\partial y} \right) \quad (2.6-4)$$

and the resultants are defined by

$$\begin{aligned}
M_i^I &= \int_{-h/2}^{h/2} \sigma_i \Phi^I(z) dz, \quad S_i^{IJ} = \int_{-h/2}^{h/2} \sigma_i \Phi^I(z) \Phi^J(z) dz, \quad (i = 1, 2, 6) \\
Q_1^I &= \int_{-h/2}^{h/2} \sigma_5 \frac{d\Phi^I}{dz} dz, \quad Q_2^I = \int_{-h/2}^{h/2} \sigma_4 \frac{d\Phi^I}{dz} dz, \quad Q_3^I = \int_{-h/2}^{h/2} \sigma_3 \frac{d\Phi^I}{dz} dz, \\
K_1^I &= \int_{-h/2}^{h/2} \sigma_5 \Phi^I dz, \quad K_2^I = \int_{-h/2}^{h/2} \sigma_4 \Phi^I dz. \quad (2.6-5)
\end{aligned}$$

Equations (2.6-3) consist of $3N$ two-dimensional differential equations in $3N$ variables (U^I, V^I, W^I), where N denotes the number of interfaces, including the top

and bottom faces of the laminate. The resultants $M_i^I, S_i^{IJ}, Q_i^I, K_i^I$ can be expressed, using the layer constitutive equations and strain-displacement relations, in terms of the interface displacement functions (U^I, V^I, W^I). The finite element model of the layerwise theory is presented in [38].

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Chapter 3

Finite Element Analysis of Composite Laminates

3.1 Introduction

The partial differential equations governing composite laminates (see Section 2.4) of arbitrary geometries and boundary conditions cannot be solved in closed form. Analytical solutions of plate theories are available (see Reddy [1-5]) mostly for rectangular plates with all edges simply supported (i.e., the Navier solutions) or with two opposite edges simply supported and the remaining edges having arbitrary boundary conditions (i.e., the Lévy solutions). The Rayleigh-Ritz and Galerkin methods can also be used to determine approximate analytical solutions, but they too are limited to simple geometries because of the difficulty in constructing the approximation functions for complicated geometries. The use of numerical methods facilitates the solution of these equations for problems of practical importance. Among the numerical methods available for the solution of differential equations defined over arbitrary domains, the finite element method is the most effective method. A brief introduction to the finite element method is presented in Section 3.2.

There are several types of finite element models developed for plate theories. These can be grouped into three major categories: (i) displacement models, (ii) mixed and hybrid models, and (iii) equilibrium models. The displacement finite element models of plate theories are based on the principles of virtual displacements, where all governing equations are expressed in terms of the displacements [see, for example, Eqs. (2.4-26)–(2.4-29)]. The mixed and hybrid finite element models are based on modified or mixed variational statements of the plate theories, in which both displacements and stresses are independently approximated. The equilibrium models are based on the principle of virtual forces. Among the three types of models, the displacement finite element models are most natural and commonly used in commercial finite element programs.

The objective of this chapter is to review the basic idea of the finite element method and develop the displacement finite element models of the classical laminated plate theory (CLPT) and the first-order shear deformation plate theory (FSDT), which were discussed in Chapter 2. The finite element models are then used to analyze several laminate problems.

We begin with the discussion of the finite element method (see Reddy [2,6]).

3.2 The finite element method

3.2-1 Basic features

The finite element method is the most powerful numerical technique ever devised for solving solid and structural mechanics problems in geometrically complicated regions. The finite element analysis of a problem is so systematic that it can be divided into a set of logical steps that can be implemented on a digital computer and can be used to solve a wide range of problems by merely changing the data input defining the domain, its physical properties and initial and boundary conditions. It is this feature that gave the finite element method such remarkable success in the modeling and simulation of practical engineering problems.

The finite element technique is endowed with two distinct features that no other method shares:

1. The *domain* of the problem is viewed as a collection of nonintersecting simple subdomains, called *finite elements*. The word domain is used to denote the physical system, or material region over which the governing equations are to be solved. The subdivision of a domain into elements is termed *finite element discretization*. The collection of the elements is called the *finite element mesh* of the domain. In general, the finite element mesh is an approximation to the domain.
2. Over each finite element, the solution of the governing equations is approximated by a linear combination of undetermined parameters and preselected approximation functions, almost always polynomials.

Since the solution is represented by polynomials on each element, a continuous approximation of the solution of the whole can be obtained only by imposing the continuity of the finite element solution, and possibly its derivatives, at element interfaces. The procedure of putting the elements together is called the *assembly* of elements.

In the finite element method, a geometrically complicated domain is represented as a collection of subdomains that allow an easy construction of the approximation functions. By representing the solution in terms of its values at the nodes of the element (i.e., by interpolating), we identify the approximation func-

tions for an element as appropriate interpolation polynomials, and we can write

$$u(x) \approx U_e(x) = \sum_{j=1}^n U_j^e \psi_j^e(x) \quad (3.2-1)$$

where U_e represents the finite element interpolant of u on a typical element Ω^e , U_j^e is the value of U_e at the j -th node, and $\psi_j^e(x)$ are the interpolation functions. For example, the linear finite element approximation of a function $u(x, y)$ in two-dimensional problems using triangular finite elements consists of representing the domain with triangular elements and then approximating the function $u(x, y)$ over each element by linear polynomials of the form (see Reddy [2,6])

$$u \approx a + bx + cy = \sum_{j=1}^3 U_j^e \psi_j^e(x, y) \quad (3.2-2)$$

Apart from the two basic features, (i) the discretization of a domain into a set of finite elements, and (ii) the representation of the solution u by equations of the form (3.2-1), the finite element method is a general numerical method in which a given differential equation is recast in an equivalent form, called a *weighted-integral* or *variational* statement, over each element and the parameters U_j^e are determined such that the variational statement is satisfied in each element. The satisfaction of the variational statement leads to a set of algebraic relations among the parameters U_j^e of an element. Since the domain is divided into a set of elements, they are put together into their original positions using continuity of $U_e(x)$ at element interfaces (*i.e.*, *assembly* of elements). The finite element method typically entails the solution of a very large number of equations for the nodal values of the function being sought. The number of equations is equal to the number of unknown nodal values. Thus the method requires so much computation that it is practical only if the calculations are carried on a computer.

The finite element method can be regarded as a technique of generating approximation functions for variational methods of approximation (*e.g.*, the method of Rayleigh-Ritz, Galerkin, least squares, collocation, subdomain, and so on; see Reddy [2,6,7]). The generation of approximation functions is made convenient by dividing the domain into finite elements and restricting the choice to polynomials. While there is only one finite element method, there can be several different *finite element models* of a given equation, depending on the variational method used to satisfy the governing equation.

3.2-2 Modeling considerations

Finite element analysis of a structural problem is a numerical analysis of the mathematical model used to represent the behavior of the structure. Therefore, finite element modeling involves assumptions concerning the representation of the

geometry of the structure and its behavior. Valid assumptions can be made only if one has a qualitative understanding of how the structure behaves. A good knowledge of the basic equations governing the deformation of the structure and its finite element model enable the development of a good mathematical model of the actual process.

Here we discuss several aspects of finite element model development. Guidelines concerning element geometries, mesh refinements, and load representations are given (from Reddy [6]).

Mesh generation

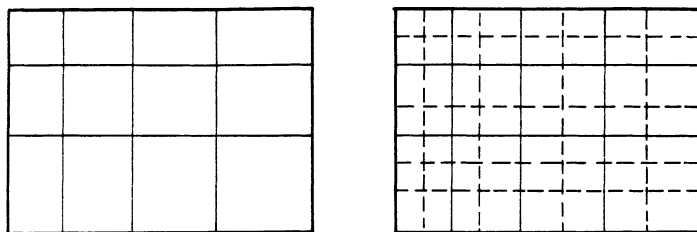
Generation of a finite element mesh of a given structure should follow the guidelines listed below:

1. The mesh should represent the geometry of the computational domain and load representation accurately.
2. The mesh should be such that large gradients in the solution (displacements *and/or* stresses) are adequately represented.
3. The mesh should not contain elements with very large aspect ratios (i.e., ratio of the smallest side to the largest side of an element), especially in regions of large gradients.

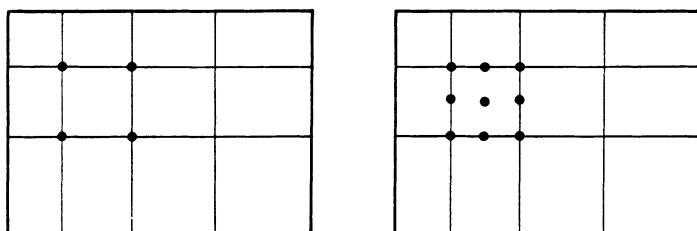
Within the above guidelines, the mesh used can be *coarse* (i.e., few elements) or *refined* (i.e., many elements), and may consist of one or more orders and types of elements (e.g., linear and quadratic, triangular and quadrilateral). A judicious choice of element order and type could save computational cost for a given accuracy in the results. It should be noted that the choice of elements and mesh is problem-dependent. What works well for one problem may not work well for another. An analyst with physical insight into the process being simulated can make a better choice of elements and mesh for the problem at hand than one who does not have the physical insight. One should start with a coarse mesh that meets the three requirements listed above, exploit symmetries available in the problem, and evaluate the results thus obtained in the light of physical understanding and approximate analytical *and/or* experimental information. These results can be used to guide subsequent mesh refinements and analyses.

Generation of meshes of single element type (i.e., linear elements or quadratic elements) is easy because elements of the same degree are compatible with each other. Mesh refinements involve several options. Refine the mesh by subdividing existing elements into two or more elements of the same type (see Figure 3.2-1a). This is called the *h-version mesh refinement*. Refine the mesh by replacing existing elements by elements of higher order (see Figure 3.2-1b). This type of refinement

is called the *p-version mesh refinement*. The *h, p-version mesh refinement* is one in which elements are subdivided into two or more elements in some places and replaced by higher-order elements in other places. Generally, local mesh refinements should be such that very small elements are not placed adjacent to very large elements.



(a) The *h-refinement*



(b) The *p-refinement*

Figure 3.2-1 The two types of mesh refinements in finite element analysis.

(a) *h-refinement* (refinement with the same order of elements).

p-refinement (refinement with higher-order elements).

There are many problems in solid and structural mechanics in which it is necessary to combine elements of different *kinds*. For example, 2-D plate bending elements can be connected to a 1-D beam element. If the plate element is based on the classical plate theory, the beam element should be one based on the Euler-Bernoulli beam theory so that they have the same degrees of freedom at the connecting nodes. When a plane elasticity element is connected to a beam element, and the degrees of freedom of the connected elements are incompatible (because

the rotational degree of freedom of the beam element has no counterpart in the elasticity element), one must construct a special element that makes the transition from the 2-D plane elasticity element to the 1-D beam element. Such elements are called *transition elements*.

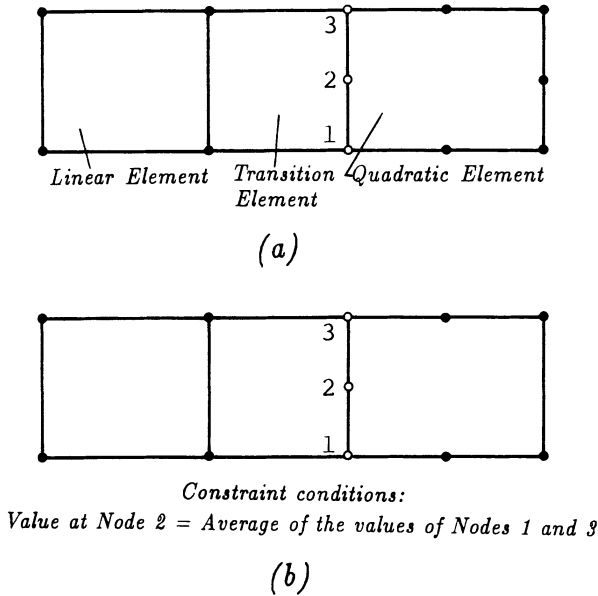


Figure 3.2-2 Connections between different order elements. (a) Use of transition elements. (b) Use of constraint conditions.

To accomplish local mesh refinements it may be necessary to combine elements of different orders, say linear and quadratic elements. There are two ways to combine elements of different orders. One way is to use transition elements which have different number of nodes on different sides of the element (see Figure 3.2-2a). The other way is to impose a condition that constrains the midside node to have the same value as that at the node of the lower-order element (see Figure 3.2-2b). Neither combination enforces interelement continuity of the solution. Figure 3.2-3 shows a few examples of such refinements.

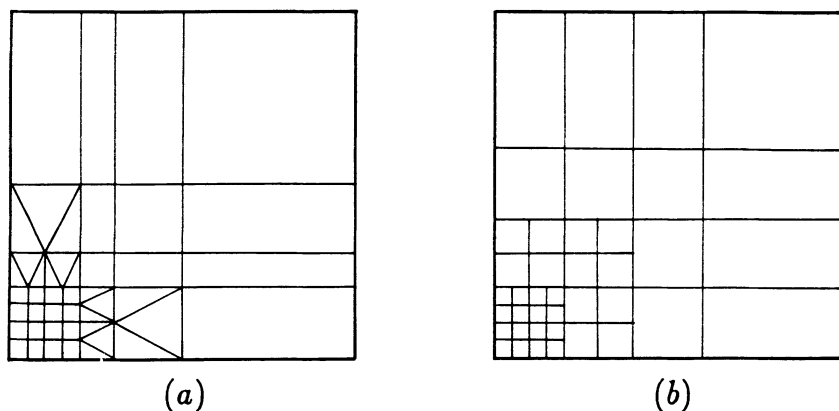


Figure 3.2-3 Examples of local mesh refinements. (a) Mesh refinement with compatible elements, and (b) mesh refinement with transition elements.

Load representation

Suppose that a solid body is subject to a distributive force on its boundary. When the displacement of the body is modeled by a set of finite elements, the force distribution must be modeled also, i.e., replaced by a set of nodal forces. This is done in such a way that the work done by the nodal forces on the finite element nodal displacements is equal to the work done by the distributed force in moving through the displacement. The accuracy of the solution depends on the element type and mesh used to represent the domain and the representation of actual forces. Use of linear elements, for example, to represent a curved boundary will change the actual distribution (see Figure 3.2-4). Of course, mesh refinements by h -version or p -version will improve the representation of the specified boundary force. Also, the load vector due to distributed force should be computed using the interpolation functions of the element used to represent the solution. For example, if the solution is approximated using the nine-node quadratic elements, the nodal contributions of a distributed force should not be evaluated using four-node linear elements.

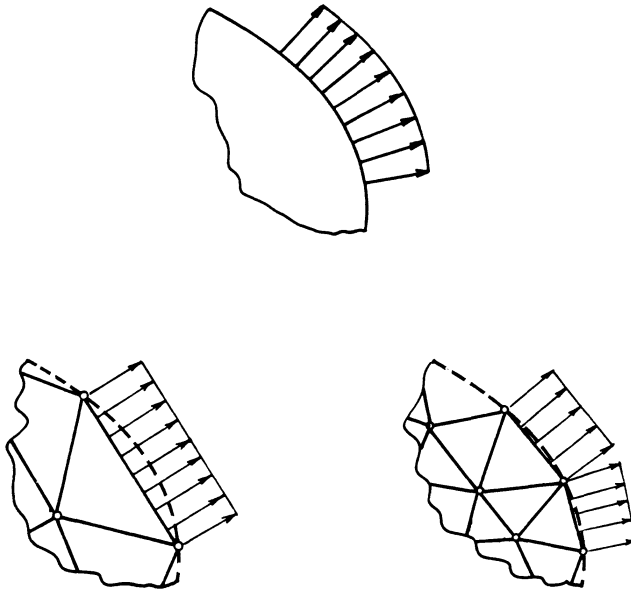


Figure 3.2-4 Representation of specified boundary forces in the finite element analysis.

Another situation where a boundary force is subject to different types of approximations is contact between two bodies. For example, a solid plate in contact with a circular disc generates a reactive force that can be represented either as a point load or as a locally distributed force. Representation of the contact force between deformable bodies as a point load is an approximation of the true distribution. A sine distribution might be more realistic representation of the actual force.

Imposition of boundary conditions

In most problems one encounters situations where the portion of the boundary on which forces are specified has points in common with the portion of the boundary on which the displacements are specified. In other words, at a few nodal points of the mesh, both the force and displacement degrees of freedom may be specified. Such points are called *singular points*. Obviously, one cannot impose both boundary conditions at the same point. As a general rule, one should im-

pose the geometric (i.e., displacement) boundary condition at the singular points and disregard the force boundary condition. Of course, if the true situation in a problem is that the force boundary conditions are imposed and the geometric boundary conditions are a result of it, then consideration must be given to the former one.

Another type of singularity one encounters in the analysis of boundary-value problems is the specification of two different values of a displacement or force variable at the same boundary point. In the finite element analysis, one must make a choice between the two values or take a weighted average of the two. When a choice is made between the two values, the larger value is recommended, as it represents the more conservative choice. In any case, one must note that the true boundary condition is replaced by an approximate condition. The closeness of the approximate boundary condition to the true one depends on the size of the element containing the point. It is often necessary to make a mesh refinement in the vicinity of the singular point to obtain an acceptable solution.

Since most practical problems are approximated in their engineering formulations, one should not be overly concerned with the numerical accuracy of the solution. A feel for the relative proportions and directions of various errors introduced into the analysis helps the finite element practitioner to make a decision on when to stop refining a mesh. In summary, scientific (or engineering) knowledge and practical experience with a given class of problems is an essential part of any approximate analysis.

3.3 Classical laminated plate theory

3.3-1 Governing equations

The classical laminated plate theory is an extension of the Kirchhoff plate theory of thin isotropic plates to laminated plates. The assumptions of the Kirchhoff plate theory were discussed in Section 2.4-2, and the governing equations of motion were derived in Section 2.4-4. For static case, the equilibrium equations of the classical laminated plate theory are [see Eqs. (2.4-15) and (2.4-19)–(2.4-21)]:

$$-\left(\frac{\partial N_1}{\partial x} + \frac{\partial N_6}{\partial y}\right) = 0 \quad (3.3-1a)$$

$$-\left(\frac{\partial N_6}{\partial x} + \frac{\partial N_2}{\partial y}\right) = 0 \quad (3.3-1b)$$

$$-\left(\frac{\partial^2 M_1}{\partial x^2} + 2\frac{\partial^2 M_6}{\partial x \partial y} + \frac{\partial^2 M_2}{\partial y^2} + N(w) + q\right) = 0 \quad (3.3-1c)$$

where (N_1, N_2, N_6) and (M_1, M_2, M_6) are force and moment resultants defined

in Eqs. (2.4-17) and $N(w)$ is nonlinear expression defined in Eq. (2.4-16):

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_6 \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} dz, \quad \begin{Bmatrix} M_1 \\ M_2 \\ M_6 \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} z dz \quad (3.3-2a)$$

$$N(w) = \frac{\partial}{\partial x} \left(N_1 \frac{\partial w}{\partial x} + N_6 \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial y} \left(N_6 \frac{\partial w}{\partial x} + N_2 \frac{\partial w}{\partial y} \right) \quad (3.3-2b)$$

The equations of equilibrium can be expressed in terms of the displacements (u, v, w) [see Eqs. (2.4-26)–(2.4-28)], using the laminate constitutive equations. Here we develop the finite element model of Eqs. (3.2-1), assuming that the force and moment resultants are known in terms of the displacements.

3.3-2 Finite element model

The development of the finite element model of the equations follows two basic steps: (1) the construction of weighted-integral statements of the equations, and (2) the approximation of the displacements, using ideas from the interpolation theory, and derivation of the algebraic equations among the nodal displacements and forces using the weighted-integral statements. In the first step, the differential equations to be solved are recast in a weighted-integral form. This is necessary in order to derive a set of linearly independent algebraic relations among the nodal displacements and forces. The weighted-integral statement can be derived either directly from the governing equations or from the principle of virtual displacements. Both approaches give exactly the same weighted-integral statement, called the *weak form*. In the second step, the dependent variables are approximated by appropriate finite element interpolations of the form (3.2-1), and they are substituted into the weak form of the equations to obtain the algebraic relations among the nodal values of the displacements and forces. These two steps are discussed in detail in the following paragraphs. For simplicity of exposition, we omit the nonlinear term $N(w)$ in the third equation of (3.3-1) and develop the finite element model of the linear equations. Once the linear model is discussed, we return to the nonlinear case [i.e., account for the nonlinear expression $N(w)$ in Eq. (3.3-1)] and derive the nonlinear finite element model. The time-dependent case will also be discussed in the sequel.

Weak Form

The principle of virtual displacements (see Reddy [2]) naturally gives the weak form of the differential equations in (3.3-1). Alternatively, the weak form of Eq. (3.3-1) can be constructed as described here. Each equation is multiplied by an appropriate weight function and integrated over a typical element. Since the statement holds for any arbitrary choice of the weight function, one can choose as many different, linearly independent, functions for the weight function as there are

nodal values in the expansion of the displacement component associated with the equation. The weight function has the meaning of a virtual variation in the displacement component. The three equations in (3.3-1) correspond to equilibrium of forces in the three coordinate directions, (x, y, z) . Hence, the weight functions for the three equations have the meaning of virtual variations $(\delta u, \delta v, \delta w)$, respectively.

Consider the first equation in (3.3-1). Multiply it with a weight function δu and integrate over the element domain Ω^e :

$$0 = \int_{\Omega^e} \delta u \left[-\frac{\partial N_1}{\partial x} - \frac{\partial N_6}{\partial y} \right] dx dy \quad (3.3 - 3)$$

Recall that N_1 and N_6 are functions of the derivatives of the displacements u , v and w . To reduce the differentiability of the interpolation functions used in the finite element approximation of u , v , and w , the differentiation on N_1 and N_6 is traded to the weight function δu by using integration-by-parts (or the gradient theorem):

$$0 = \int_{\Omega^e} \left(\frac{\partial \delta u}{\partial x} N_1 + \frac{\partial \delta u}{\partial y} N_6 \right) dx dy - \int_{\Gamma^e} (N_1 n_x + N_6 n_y) \delta u ds$$

where (n_x, n_y) are the direction cosines of the unit normal on the boundary Γ^e of the element domain Ω^e . We note that the expression in the parenthesis of the boundary integral denotes the inplane normal force

$$N_1 n_x + N_6 n_y = N_n$$

Hence, Eq. (3.3-3) becomes

$$0 = \int_{\Omega^e} \left(\frac{\partial \delta u}{\partial x} N_1 + \frac{\partial \delta u}{\partial y} N_6 \right) dx dy - \int_{\Gamma^e} \delta u N_n ds \quad (3.3 - 4)$$

Equation (3.3-4) is called the *weak form* of Eq. (3.3-1a) because it requires less differentiability (i.e. weaker continuity) of the dependent variables (u, v, w) than the original differential equations (3.3-1a).

Following the same procedure for Eqs. (3.3-1b) and (3.3-1c), we obtain the following weak forms,

$$0 = \int_{\Omega^e} \left(\frac{\partial \delta v}{\partial x} N_6 + \frac{\partial \delta v}{\partial y} N_2 \right) dx dy - \int_{\Gamma^e} N_{ns} \delta v ds \quad (3.3 - 5)$$

$$0 = - \int_{\Omega^e} \left(\frac{\partial^2 \delta w}{\partial x^2} M_1 + 2 \frac{\partial^2 \delta w}{\partial x \partial y} M_6 + \frac{\partial^2 \delta w}{\partial y^2} M_2 + \delta w q \right) dx dy - \int_{\Gamma^e} \left(\delta w V_n + \frac{\partial \delta w}{\partial n} M_n \right) ds \quad (3.3-6)$$

where δv and δw are the virtual variations of the displacements v and w , respectively.

Interpolation and finite element model

The displacements (u, v, w) are approximated over an element by interpolations of the form,

$$u = \sum_{j=1}^n u_j \psi_j(x, y), \quad v = \sum_{j=1}^n v_j \psi_j(x, y) \\ w = \sum_{j=1}^m \Delta_j \phi_j(x, y) \quad (3.3-7)$$

Here (u_j, v_j) denote the nodal values of (u, v) and Δ_j denote the nodal values of w and its derivatives. All nodal displacements and rotations, called *generalized displacements*, are associated with the element Ω^e .

There are two basic types of interpolation functions that are used in finite element analyses. The Lagrange interpolation is one in which only the function is interpolated, whereas the Hermite interpolation is one in which the function and its derivatives are interpolated. The finite elements developed using the Lagrange type interpolation are called C^0 elements, and finite elements developed using Hermite type interpolation are called C^m elements, where $m > 0$ is the order of the derivatives included in the interpolation.

Based on the order of differentiation in Eqs. (3.3-4)–(3.3-6), it can be concluded that ψ_i are the Lagrange interpolation functions and ϕ_i are the Hermite interpolation functions (see [6,7]). Clearly, C^0 -interpolation of (u, v) and C^1 -interpolation of w are required to develop a plate bending element of the classical plate theory when using the weak forms in Eqs. (3.3-4)–(3.3-6).

For linear Lagrange interpolation of (u, v) and Hermite cubic interpolation of w using four-node rectangular elements, we have $n = 4$ and $m = 16$. In this case the four nodal values associated with w are:

$$w, \quad \frac{\partial w}{\partial x}, \quad \frac{\partial w}{\partial y}, \quad \text{and} \quad \frac{\partial^2 w}{\partial x \partial y} \quad (3.3-8)$$

The Lagrange interpolation functions ψ_i and Hermite interpolation functions ϕ_i are presented in Table 3.3-1 (taken from Table 7.9 of Reddy [7]; also see Reddy [6]). The Hermite interpolation functions for the conforming plate element are obtained by taking the tensor product of the Hermite cubic polynomials associated with the Euler-Bernoulli beam element (see [6]).

This choice of interpolation of (u, v, w) guarantees continuity of $(u, v, w, \frac{\partial w}{\partial n})$ at the interface of elements, and therefore such elements are called *conforming elements*. If the mixed derivative, $\frac{\partial^2 w}{\partial x \partial y}$, is not included in the interpolation, the resulting element violates the inter element continuity of the slope of the transverse displacement. Hence, it is termed a *nonconforming element*. The conforming element has a total of six degrees of freedom per node, whereas the nonconforming element has a total of five degrees of freedom per node, as indicated in Figure 3.3-1.

The finite element model of Eqs. (3.3-1) is obtained by substituting expressions (3.3-7) for (u, v, w) , and $\delta u = \psi_i$, $\delta v = \psi_i$ and $\delta w = \phi_i$ into Eqs. (3.3-4)–(3.3-6). This gives the i -th algebraic equation of each weak form. There are a total of $(2n + m)$ nodal degrees of freedom per element. We obtain,

$$0 = \int_{\Omega^e} \left(\frac{\partial \psi_i}{\partial x} N_1 + \frac{\partial \psi_i}{\partial y} N_6 \right) dx dy - \int_{\Gamma^e} N_n \psi_i ds \quad (3.3-9a)$$

$$0 = \int_{\Omega^e} \left(\frac{\partial \psi_i}{\partial x} N_6 + \frac{\partial \psi_i}{\partial y} N_2 \right) dx dy - \int_{\Gamma^e} N_{ns} \psi_i ds \quad (3.3-9b)$$

$$0 = - \int_{\Omega^e} \left(\frac{\partial^2 \phi_i}{\partial x^2} M_1 + 2 \frac{\partial^2 \phi_i}{\partial x \partial y} M_6 + \frac{\partial^2 \phi_i}{\partial y^2} M_2 + \phi_i q \right) dx dy \\ - \int_{\Gamma^e} \left(\phi_i V_n + \frac{\partial \phi_i}{\partial n} M_n \right) ds \quad (3.3-9c)$$

These equations can be expressed in the form,

$$\sum_{\beta=1}^3 \sum_{j=1}^{n(\beta)} K_{ij}^{\alpha\beta} \Delta_j^\beta - F_i^\alpha = 0, \quad i = 1, 2, \dots, n(\alpha) \quad (3.3-10a)$$

or

$$[K^e] \{\Delta^e\} - \{F^e\} = \{0\} \quad (3.3-10b)$$

where $\alpha = 1, 2, 3$; $n(1) = n(2) = 4$ and $n(3) = 16$. The variables Δ_j^β and stiffness coefficients $K_{ij}^{\alpha\beta}$ are defined by

Table 3.3–1 The Lagrange and Hermite interpolation functions for the plate bending elements ((ξ_i, η_i) denote the coordinates of the i -th node of the element, and (x_c, y_c) are the global coordinates of the center of the element).

Element type	Interpolation functions	Remarks
Lagrange elements:		
Linear	$\frac{1}{4}(1 + \xi\xi_i)(1 + \eta\eta_i)$	Node i ($i = 1, \dots, 4$)
Quadratic	$\frac{1}{4}\xi\xi_i(1 + \xi\xi_i)\eta\eta_i(1 + \eta\eta_i)$ $\frac{1}{2}\eta\eta_i(1 + \eta\eta_i)(1 - \xi^2)$ $\frac{1}{2}\xi\xi_i(1 + \xi\xi_i)(1 - \eta^2)$ $(1 - \xi^2)(1 - \eta^2)$	Corner node Side node, $\xi_i = 0$ Side node, $\eta_i = 0$ Interior node
Serendipity element:		
Quadratic	$\frac{1}{4}(1 + \xi\xi_i)(1 + \eta\eta_i)(\xi\xi_i + \eta\eta_i - 1)$ $\frac{1}{2}(1 - \xi^2)(1 + \eta\eta_i)$ $\frac{1}{2}(1 + \xi\xi_i)(1 - \eta^2)$	Corner node Side node, $\xi_i = 0$ Side node, $\eta_i = 0$
Hermite cubic element:		
Interpolation functions for variable u	$\frac{1}{16}(\xi + \xi_i)^2(\xi\xi_i - 2)(\eta + \eta_i)^2(\eta\eta_i - 2)$ $-\frac{1}{16}\xi_i(\xi + \xi_i)^2(\xi\xi_i - 1)(\eta + \eta_i)^2(\eta\eta_i - 2)$ $-\frac{1}{16}(\xi + \xi_i)^2(\xi\xi_i - 2)\eta_i(\eta + \eta_i)^2(\eta\eta_i - 1)$ $\frac{1}{16}\xi_i(\xi + \xi_i)^2(\xi\xi_i - 1)\eta_i(\eta + \eta_i)^2(\eta\eta_i - 1)$	For node j ($i = 1, \dots, 4$)
Interpolation functions for derivative $\partial u / \partial \xi$	$\frac{1}{2}(\xi_0 + 1)(\eta_0 + 1)(2 + \xi_0 + \eta_0 - \xi^2 - \eta^2)$ $\frac{1}{2}\xi_i(\xi_0 + 1)^2(\xi_0 - 1)(\eta_0 + 1)$ $\frac{1}{2}\eta_i(\xi_0 + 1)(\eta_0 + 1)^2(\eta_0 - 1)$	For node i ($i = 1, \dots, 4$)
Interpolation functions for derivative $\partial^2 u / \partial \xi \partial \eta$	$\xi = (x - x_c)/a, \quad \eta = (y - y_c)/b$ $\xi_0 = \xi\xi_i, \quad \eta_0 = \eta\eta_i$	($2a$ and $2b$ are the sides of the rectangular element)

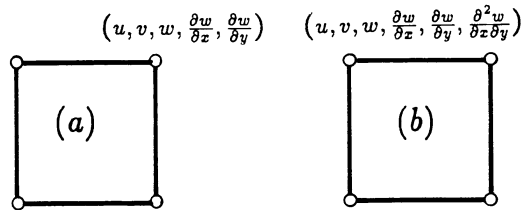


Figure 3.3–1 Plate bending elements of the classical plate theory. (a) Non-conforming element $(u, v, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y})$. (b) Conforming element $(u, v, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x \partial y})$.

$$\Delta_j^1 = u_j, \Delta_j^2 = v_j, \Delta_j^3 = \Delta_j \quad (3.3 - 11)$$

and

$$\begin{aligned}
K_{ij}^{1\alpha} &= \int_{\Omega^e} \left(\frac{\partial \psi_i}{\partial x} N_{1j}^\alpha + \frac{\partial \psi_i}{\partial y} N_{6j}^\alpha \right) dx dy \\
N_{1j}^1 &= A_{11} \frac{\partial \psi_j}{\partial x} + A_{16} \frac{\partial \psi_j}{\partial y} \\
N_{1j}^2 &= A_{16} \frac{\partial \psi_j}{\partial x} + A_{12} \frac{\partial \psi_j}{\partial y} \\
N_{1j}^3 &= - \left(B_{11} \frac{\partial^2 \phi_j}{\partial x^2} + 2B_{16} \frac{\partial^2 \phi_j}{\partial x \partial y} + B_{12} \frac{\partial^2 \phi_j}{\partial y^2} \right) \\
K_{ij}^{2\alpha} &= \int_{\Omega^e} \left(\frac{\partial \psi_i}{\partial x} N_{6j}^\alpha + \frac{\partial \psi_i}{\partial y} N_{2j}^\alpha \right) dx dy \\
N_{6j}^1 &= A_{16} \frac{\partial \psi_j}{\partial x} + A_{66} \frac{\partial \psi_j}{\partial y} \\
N_{6j}^2 &= A_{66} \frac{\partial \psi_j}{\partial x} + A_{26} \frac{\partial \psi_j}{\partial y} \\
N_{6j}^3 &= - \left(B_{16} \frac{\partial^2 \phi_j}{\partial x^2} + 2B_{66} \frac{\partial^2 \phi_j}{\partial x \partial y} + B_{26} \frac{\partial^2 \phi_j}{\partial y^2} \right) \\
N_{2j}^1 &= A_{12} \frac{\partial \psi_j}{\partial x} + A_{26} \frac{\partial \psi_j}{\partial y}, \\
N_{2j}^2 &= A_{26} \frac{\partial \psi_j}{\partial x} + A_{22} \frac{\partial \psi_j}{\partial y} \\
N_{2j}^3 &= - \left(B_{12} \frac{\partial^2 \phi_j}{\partial x^2} + 2B_{26} \frac{\partial^2 \phi_j}{\partial x \partial y} + B_{22} \frac{\partial^2 \phi_j}{\partial y^2} \right) \\
K_{ij}^{3\alpha} &= - \int_{\Omega^e} \left(\frac{\partial^2 \phi_i}{\partial x^2} M_{1j}^\alpha + 2 \frac{\partial^2 \phi_i}{\partial x \partial y} M_{6j}^\alpha + \frac{\partial^2 \phi_i}{\partial y^2} M_{2j}^\alpha \right) dx dy \quad (3.3 - 12) \\
M_{1j}^1 &= B_{11} \frac{\partial \psi_j}{\partial x} + B_{16} \frac{\partial \psi_j}{\partial y}, \quad M_{2j}^2 = B_{16} \frac{\partial \psi_j}{\partial x} + B_{12} \frac{\partial \psi_j}{\partial y} \\
M_{1j}^3 &= - \left(D_{11} \frac{\partial^2 \phi_j}{\partial x^2} + 2D_{16} \frac{\partial^2 \phi_j}{\partial x \partial y} + D_{12} \frac{\partial^2 \phi_j}{\partial x^2} \right)
\end{aligned}$$

$$\begin{aligned}
M_{6j}^1 &= B_{16} \frac{\partial \psi_j}{\partial x} + B_{66} \frac{\partial \psi_j}{\partial y}, \quad M_{6j}^2 = B_{66} \frac{\partial \psi_j}{\partial x} + B_{26} \frac{\partial \psi_j}{\partial y} \\
M_{6j}^3 &= - \left(D_{16} \frac{\partial^2 \phi_j}{\partial x^2} + 2D_{66} \frac{\partial^2 \phi_j}{\partial x \partial y} + D_{26} \frac{\partial^2 \phi_j}{\partial y^2} \right) \\
M_{2j}^1 &= B_{12} \frac{\partial \psi_j}{\partial x} + B_{26} \frac{\partial \psi_j}{\partial y}, \quad M_{2j}^2 = B_{26} \frac{\partial \psi_j}{\partial x} + B_{22} \frac{\partial \psi_j}{\partial y} \\
M_{2j}^3 &= - \left(D_{12} \frac{\partial^2 \phi_j}{\partial x^2} + 2D_{26} \frac{\partial^2 \phi_j}{\partial x \partial y} + D_{22} \frac{\partial^2 \phi_j}{\partial y^2} \right) \quad (3.3-13)
\end{aligned}$$

$$\begin{aligned}
F_i^1 &= \int_{\Gamma^e} N_n \psi_i ds, \quad F_i^2 = \int_{\Gamma^e} N_{ns} \psi_i ds \\
F_i^e &= \int_{\Gamma^e} \left(\phi_i V_n + \frac{\partial \phi_i}{\partial n} M_n \right) ds - \int_{\Omega^e} \phi_i q dx dy \quad (3.3-14)
\end{aligned}$$

All other coefficients are zero.

The computation of the coefficients in Eq. (3.3-12)–(3.3-14) require evaluation of integrals over each element domain. When the element geometry is nonrectangular or the integrands are complicated functions of the coordinates (x, y) , exact evaluation of the integrals by conventional methods is not possible. In finite element computer programs, such integrals are evaluated using numerical integrations methods. We will discuss numerical evaluation of integrals in Section 3.5-1.

This completes the development of the finite element model of the classical laminated plate theory.

Equations (3.3-10) of an element are assembled by using the interelement continuity of the nodal displacements and balance of nodal forces. Then the displacement and force boundary conditions of the problem are imposed on the assembled system of equations and the equations are solved for nodal displacements of the total mesh. Equations (3.3-7) can be used to determine displacements at any arbitrary point $(x, y, 0)$, and the displacement field in Eq. (2.4-5) can be used to determine the total displacements at a point (x, y, z) . The strains (2.4-12) and stresses in the global (i.e., problem) coordinates (x, y, z) can be computed using the displacement field, and they can be transformed to lamina coordinates using the transformation in Eq. (2.3-3). Additional discussion of the strain and stress computation in finite element analyses is given in Section 3.5-2.

3.3-3 Nonlinear model

If the von Kármán nonlinear strains [see Eq. (2.4-13)] are to be included in the finite element model, the procedure discussed above does not change, except that the resulting algebraic equations (3.3-10) will be nonlinear. The additional stiffness coefficients for the nonlinear model are:

$$\begin{aligned}
K_{ij}^{13} &= \frac{1}{2} \int_{\Omega^e} \left(\frac{\partial \psi_i}{\partial x} \hat{N}_{1j} + \frac{\partial \psi_i}{\partial y} \hat{N}_{6j} \right) dx dy, \quad K_{ij}^{3\alpha} = 2K_{ji}^{\alpha 3} \\
K_{ij}^{23} &= \frac{1}{2} \int_{\Omega^e} \left(\frac{\partial \psi_i}{\partial x} \hat{N}_{6j} + \frac{\partial \psi_i}{\partial y} \hat{N}_{2j} \right) dx dy \\
K_{ij}^{33} &= \int_{\Omega^e} \left[\frac{\partial \phi_i}{\partial x} \left(\bar{N}_1 \frac{\partial \phi_j}{\partial x} + \bar{N}_6 \frac{\partial \phi_j}{\partial y} \right) + \frac{\partial \phi_i}{\partial y} \left(\bar{N}_6 \frac{\partial \phi_j}{\partial x} + \bar{N}_2 \frac{\partial \phi_j}{\partial y} \right) \right] dx dy
\end{aligned} \tag{3.3-15a}$$

where

$$\begin{aligned}
\hat{N}_{\alpha j} &= A_{\alpha 1} \frac{\partial w}{\partial x} \frac{\partial \phi_j}{\partial x} + A_{\alpha 2} \frac{\partial w}{\partial y} \frac{\partial \phi_j}{\partial y} + A_{\alpha 6} \left(\frac{\partial w}{\partial x} \frac{\partial \phi_j}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \phi_j}{\partial x} \right) \quad (\alpha = 1, 2, 6) \\
\bar{N}_I &= \frac{1}{2} A_{I1} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{1}{2} A_{I2} \left(\frac{\partial w}{\partial y} \right)^2 + A_{I6} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \quad (I = 1, 2, 6)
\end{aligned} \tag{3.3-15b}$$

The nonlinear algebraic equations of the nonlinear model are solved by iterative methods, which will be discussed in Section 3.5-3. In iterative methods, the expressions involving the dependent variables and their derivatives are computed using their values from the previous iteration, so that the integrals can be evaluated (by numerical integration methods).

3.4 First-order shear deformation theory

3.4-1 Governing equations

In this section, the displacement finite element models of the first-order shear deformation theory are presented. The procedure used is the same as that used for the classical laminated plate theory in Section 3.3. First we consider the static equilibrium equations of the theory.

The equilibrium equations of the first-order theory are obtained by omitting the time-derivative terms (i.e., assuming that all resultants are functions of spatial coordinates only) in Eq. (2.5-6):

$$-\left(\frac{\partial N_1}{\partial x} + \frac{\partial N_6}{\partial y} \right) = 0, \quad -\left(\frac{\partial N_6}{\partial x} + \frac{\partial N_2}{\partial y} \right) = 0 \tag{3.4-1a,b}$$

$$-\left(\frac{\partial Q_1}{\partial x} + \frac{\partial Q_2}{\partial y} + N(w) + q \right) = 0 \tag{3.4-1c}$$

$$-\left(\frac{\partial M_1}{\partial x} + \frac{\partial M_6}{\partial y} - Q_1 \right) = 0, \quad -\left(\frac{\partial M_6}{\partial x} + \frac{\partial M_2}{\partial y} - Q_2 \right) = 0 \tag{3.4-1d,e}$$

The weak forms of Eqs. (3.4-1) can be obtained in the same manner as described for the classical laminated plate theory. The first equation is multiplied by δu , the second by δv , the third by δw , the fourth by $\delta \phi_1$, and the fifth by $\delta \phi_2$. Once again we use integration-by-parts to trade differentiation from the resultants to the weight functions to obtain the weak forms.

3.4-2 Finite element model

We note that the force and moment resultants contain, at the most, only the first derivatives of the dependent variables $(u, v, w, \phi_1, \phi_2)$. Therefore, they can all be approximated using the Lagrange interpolation functions. Thus, the plate bending element of the first-order plate theory is a C^0 element. The displacements $(u, v, w, \phi_1, \phi_2)$ are approximated as,

$$\begin{aligned} u &= \sum_{j=1}^n u_j \psi_j(x, y), \quad v = \sum_{j=1}^n v_j \psi_j(x, y) \\ w &= \sum_{j=1}^n w_j \psi_j(x, y) \\ \phi_1 &= \sum_{j=1}^n S_j^1 \psi_j(x, y), \quad \phi_2 = \sum_{j=1}^n S_j^2 \psi_j(x, y) \end{aligned} \quad (3.4-2)$$

where ψ_j are Lagrange family of interpolation functions. One can use linear, quadratic, or higher-order interpolation functions.

For the linear case, the weak form of Eqs. (3.4-1) over a typical element domain Ω^e is given by,

$$0 = \int_{\Omega^e} \left(\frac{\partial \psi_i}{\partial x} N_1 + \frac{\partial \psi_i}{\partial y} N_6 \right) dx dy - \int_{\Gamma^e} N_n \psi_i ds \quad (3.4-3a)$$

$$0 = \int_{\Omega^e} \left(\frac{\partial \psi_i}{\partial x} N_6 + \frac{\partial \psi_i}{\partial y} N_2 \right) dx dy - \int_{\Gamma^e} N_{ns} \psi_i ds \quad (3.4-3b)$$

$$0 = \int_{\Omega^e} \left(\frac{\partial \psi_i}{\partial x} Q_1 + \frac{\partial \psi_i}{\partial y} Q_2 + \psi_i q \right) dx dy - \int_{\Gamma^e} Q_n \psi_i ds \quad (3.4-3c)$$

$$0 = \int_{\Omega^e} \left(\frac{\partial \psi_i}{\partial x} M_1 + \frac{\partial \psi_i}{\partial y} M_6 + \psi_i Q_1 \right) dx dy - \int_{\Gamma^e} M_n \psi_i ds \quad (3.4-3d)$$

$$0 = \int_{\Omega^e} \left(\frac{\partial \psi_i}{\partial x} M_6 + \frac{\partial \psi_i}{\partial y} M_2 + \psi_i Q_2 \right) dx dy - \int_{\Gamma^e} M_{ns} \psi_i ds \quad (3.4-3e)$$

Substituting Eqs. (3.4-2) for $(u, v, w, \phi_1, \phi_2)$ into Eq. (3.4-3), we obtain the finite element model of the first-order theory,

$$\sum_{\beta=1}^5 \sum_{j=1}^n K_{ij}^{\alpha\beta} \Delta_j^\beta - F_i^\alpha = 0, \quad (\alpha = 1, 2, \dots, 5) \quad (3.4-4a)$$

or

$$[K^e]\{\Delta^e\} - \{F^e\} = \{0\} \quad (3.4-4b)$$

where the stiffness and force coefficients are defined by $(\alpha = 1, 2, \dots, 5)$

$$\begin{aligned} K_{ij}^{1\alpha} &= \int_{\Omega^e} \left(\frac{\partial \psi_i}{\partial x} N_{1j}^\alpha + \frac{\partial \psi_i}{\partial y} N_{6j}^\alpha \right) dx dy \\ K_{ij}^{2\alpha} &= \int_{\Omega^e} \left(\frac{\partial \psi_i}{\partial x} N_{6j}^\alpha + \frac{\partial \psi_i}{\partial y} N_{2j}^\alpha \right) dx dy \\ K_{ij}^{3\alpha} &= \int_{\Omega^e} \left(\frac{\partial \psi_i}{\partial x} Q_{1j}^\alpha + \frac{\partial \psi_i}{\partial y} Q_{2j}^\alpha \right) dx dy \\ K_{ij}^{4\alpha} &= \int_{\Omega^e} \left(\frac{\partial \psi_i}{\partial x} M_{1j}^\alpha + \frac{\partial \psi_i}{\partial y} M_{6j}^\alpha + \psi_i Q_{1j}^\alpha \right) dx dy \\ K_{ij}^{5\alpha} &= \int_{\Omega^e} \left(\frac{\partial \psi_i}{\partial x} M_{6j}^\alpha + \frac{\partial \psi_i}{\partial y} M_{2j}^\alpha + \psi_i Q_{2j}^\alpha \right) dx dy \end{aligned} \quad (3.4-5)$$

The coefficients N_{Ij}^α , M_{Ij}^α , and Q_{Ij}^α for $\alpha = 1, 2, \dots, 5$ and $I = 1, 2, 6$ are given by

$$\begin{aligned} N_{1j}^1 &= A_{11} \frac{\partial \psi_j}{\partial x} + A_{16} \frac{\partial \psi_j}{\partial y}, \quad N_{1j}^2 = A_{12} \frac{\partial \psi_j}{\partial y} + A_{16} \frac{\partial \psi_j}{\partial x} \\ N_{1j}^4 &= B_{11} \frac{\partial \psi_j}{\partial x} + B_{16} \frac{\partial \psi_j}{\partial y}, \quad N_{1j}^5 = B_{12} \frac{\partial \psi_j}{\partial y} + B_{16} \frac{\partial \psi_j}{\partial x} \\ N_{2j}^1 &= A_{12} \frac{\partial \psi_j}{\partial x} + A_{26} \frac{\partial \psi_j}{\partial y}, \quad N_{2j}^2 = A_{22} \frac{\partial \psi_j}{\partial y} + A_{26} \frac{\partial \psi_j}{\partial x} \\ N_{2j}^4 &= B_{12} \frac{\partial \psi_j}{\partial x} + B_{26} \frac{\partial \psi_j}{\partial y}, \quad N_{2j}^5 = B_{22} \frac{\partial \psi_j}{\partial y} + B_{26} \frac{\partial \psi_j}{\partial x} \\ N_{6j}^1 &= A_{16} \frac{\partial \psi_j}{\partial x} + A_{66} \frac{\partial \psi_j}{\partial y}, \quad N_{6j}^2 = A_{26} \frac{\partial \psi_j}{\partial y} + A_{66} \frac{\partial \psi_j}{\partial x} \\ N_{6j}^4 &= B_{16} \frac{\partial \psi_j}{\partial x} + B_{66} \frac{\partial \psi_j}{\partial y}, \quad N_{6j}^5 = B_{26} \frac{\partial \psi_j}{\partial y} + B_{66} \frac{\partial \psi_j}{\partial x} \end{aligned}$$

$$\begin{aligned}
M_{1j}^1 &= B_{11} \frac{\partial \psi_j}{\partial x} + B_{16} \frac{\partial \psi_j}{\partial y}, \quad M_{1j}^2 = B_{12} \frac{\partial \psi_j}{\partial y} + B_{16} \frac{\partial \psi_j}{\partial x} \\
M_{1j}^4 &= D_{11} \frac{\partial \psi_j}{\partial x} + D_{16} \frac{\partial \psi_j}{\partial y}, \quad M_{1j}^5 = D_{12} \frac{\partial \psi_j}{\partial y} + D_{16} \frac{\partial \psi_j}{\partial x} \\
M_{2j}^1 &= B_{12} \frac{\partial \psi_j}{\partial x} + B_{26} \frac{\partial \psi_j}{\partial y}, \quad M_{2j}^2 = B_{22} \frac{\partial \psi_j}{\partial y} + B_{26} \frac{\partial \psi_j}{\partial x} \\
M_{2j}^4 &= D_{12} \frac{\partial \psi_j}{\partial x} + D_{26} \frac{\partial \psi_j}{\partial y}, \quad M_{2j}^5 = D_{22} \frac{\partial \psi_j}{\partial y} + D_{26} \frac{\partial \psi_j}{\partial x} \\
M_{6j}^1 &= B_{16} \frac{\partial \psi_j}{\partial x} + B_{66} \frac{\partial \psi_j}{\partial y}, \quad M_{6j}^2 = B_{26} \frac{\partial \psi_j}{\partial y} + B_{66} \frac{\partial \psi_j}{\partial x} \\
M_{6j}^4 &= D_{16} \frac{\partial \psi_j}{\partial x} + D_{66} \frac{\partial \psi_j}{\partial y}, \quad M_{6j}^5 = D_{26} \frac{\partial \psi_j}{\partial y} + D_{66} \frac{\partial \psi_j}{\partial x} \\
Q_{1j}^4 &= A_{55} \psi_j, \quad Q_{2j}^4 = A_{45} \psi_j, \quad Q_{1j}^5 = A_{45} \psi_j, \quad Q_{2j}^5 = A_{44} \psi_j \\
Q_{1j}^3 &= A_{55} \frac{\partial \psi_j}{\partial x} + A_{45} \frac{\partial \psi_j}{\partial y}, \quad Q_{2j}^3 = A_{45} \frac{\partial \psi_j}{\partial x} + A_{44} \frac{\partial \psi_j}{\partial y}
\end{aligned} \tag{3.4-6}$$

$$\begin{aligned}
F_i^1 &= \int_{\Gamma^e} \psi_i N_n ds, \quad F_i^2 = \int_{\Gamma^e} \psi_i N_{n,s} ds, \quad F_i^3 = \int_{\Omega^e} q \psi_i dx dy + \int_{\Gamma^e} \psi_i Q_n ds \\
F_i^4 &= \int_{\Gamma^e} \psi_i M_n ds, \quad F_i^5 = \int_{\Gamma^e} \psi_i M_{n,s} ds
\end{aligned} \tag{3.4-7}$$

All other coefficients are zero.

When the four-node rectangular element is used, the element stiffness matrices are of the order 20×20 ; and for the nine-node element they are 45×45 (see Figure 3.4-1). If the inplane degrees of freedom (u, v) are omitted (for the uncoupled case), the stiffness matrices will be of orders 12×12 and 27×27 for the linear and quadratic rectangular elements, respectively (see [1,8-13]). The coefficients in Eq. (3.4-5)–(3.4-7) are evaluated, in general, using numerical integration methods (see Section 3.5-1).

Once the finite element equations of all elements are available, they are put together by using the continuity of the nodal displacements and the balance of nodal forces. Then the boundary conditions on displacements *and/or* forces are applied on the assembled system of equations, and the equations are solved for the nodal values of the displacements. Equations (3.4-2) can be used to determine the displacements, strains, and stresses at any point inside the element Ω^e , as discussed in more detail in Section 3.5-2.

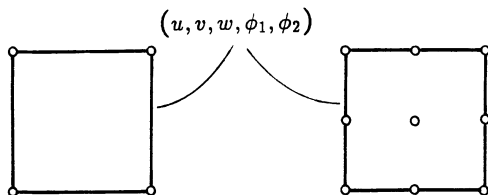


Figure 3.4-1 The C^0 plate bending elements of the first-order shear deformation plate theory. Each node has the following degrees of freedom: $(u, v, w, \phi_1, \phi_2)$.

The geometric nonlinearity (in the von Kármán sense) can be included in the same manner as was done for the classical laminate theory. In fact, the nonlinear coefficients in Eqs. (3.3-15) are also valid for the first-order shear deformation theory.

3.4-3 The dynamic case

In developing the finite element models (3.3-10) and (3.4-4) of the classical and first-order shear theories, we considered only the static case. For the dynamic case, we should consider the equations of motion (2.4-26)–(2.4-28) and (2.5-12)–(2.5-16) to develop the finite element models. For time dependent problems it is often assumed that the variables $[u(x, y, t), v(x, y, t), w(x, y, t), \phi_1(x, y, t), \phi_2(x, y, t)]$ each can be represented as a product of a function of time t and a function of coordinates (x, y) :

$$u = \sum_{j=1}^n u_j(t) \psi_j(x, y), \quad v = \sum_{j=1}^n v_j(t) \psi_j(x, y), \quad w = \sum_{j=1}^n w_j(t) \psi_j(x, y)$$

$$\phi_1 = \sum_{j=1}^n S_j^1(t) \psi_j(x, y), \quad \phi_2 = \sum_{j=1}^n S_j^2(t) \psi_j(x, y) \quad (3.4-8)$$

where the nodal values of the generalized displacements are now treated as functions of time t . In developing the finite element equations, we first approximate the spatial variation of $(u, v, w, \phi_1, \phi_2)$ for an arbitrarily fixed time $t > 0$. This step is called the *semidiscretization*. This leads to a set of ordinary differential equations in time, which are then further approximated to obtain algebraic equations for the nodal values at a fixed time. The approximation of the time derivatives is known

as the *temporal approximation*. These two stages of approximation are discussed below.

Semidiscretization (spatial approximation)

Substitution of Eq. (3.4-8) into the weak form of Eqs. (2.5-12)-(2.5-16) results in a set of $5n$ ordinary differential equations in time for $(u_j, v_j, w_j, S_j^1, S_j^2)$:

$$[K^e]\{\Delta^e\} + [M^e]\{\ddot{\Delta}^e\} = \{F^e\}, \text{ in } \Omega^e \text{ for } t > 0 \quad (3.4-9)$$

where $[K^e]$ is the element stiffness matrix and F^e is the force vector in Eq. (3.4-4b), and $[M^e]$ is the mass matrix whose coefficients are given below:

$$[M^e] = \begin{bmatrix} [M^{11}] & [M^{12}] & [M^{13}] & [M^{14}] & [M^{15}] \\ & [M^{22}] & [M^{23}] & [M^{24}] & [M^{25}] \\ & & [M^{33}] & [M^{34}] & [M^{35}] \\ & & & [M^{44}] & [M^{45}] \\ & \text{sym.} & & & [M^{55}] \end{bmatrix} \quad (3.4-10a)$$

and [see the right-hand sides of Eqs. (2.5-12)-(2.5-16)]

$$\begin{aligned} M_{ij}^{11} &= \int_{\Omega^e} I_0 \psi_i \psi_j dx dy, \quad M_{ij}^{12} = 0, \quad M_{ij}^{13} = 0, \quad M_{ij}^{15} = 0, \\ M_{ij}^{14} &= \int_{\Omega^e} I_1 \psi_i \psi_j dx dy, \quad M_{ij}^{22} = M_{ij}^{11}, \quad M_{ij}^{23} = 0, \quad M_{ij}^{24} = 0, \\ M_{ij}^{25} &= M_{ij}^{14}, \quad M_{ij}^{33} = M_{ij}^{11}, \quad M_{ij}^{34} = 0, \quad M_{ij}^{35} = 0, \quad M_{ij}^{44} = M_{ij}^{11}, \\ M_{ij}^{45} &= M_{ij}^{14}, \quad M_{ij}^{55} = M_{ij}^{11} \end{aligned} \quad (3.4-10b)$$

Equation (3.4-9) can be reduced to appropriate form depending on the type of analysis. For static analysis, we set the acceleration vector, $\{\ddot{\Delta}\}$ to zero and obtain the problem [see Eq. (3.4-4)],

$$[K^e]\{\Delta^e\} = \{F^e\}. \quad (3.4-11)$$

For natural vibration problems, we assume periodic motion and replace the acceleration vector by,

$$\{\ddot{\Delta}^e\} = -\omega^2 \{\Delta_o^e\} \quad (\text{or } \{\Delta^e\} = \{\Delta_o^e\} e^{-i\omega t}) \quad (3.4-12)$$

Then Eq. (3.4-9) takes the form of an eigenvalue problem,

$$([K^e] - \omega^2[M^e])\{\Delta_o^e\} = \{F_o^e\} \quad (3.4-13)$$

For buckling analysis (i.e., to determine the value of inplane compressive force at which the plate buckles), we replace the mass matrix in Eq. (3.4-13) with the stability matrix $[S^e]$ and ω^2 with the buckling load λ . The stability matrix comes from the geometric nonlinearity of Eq. (2.5-14):

$$\frac{\partial}{\partial x} \left(\bar{N}_1 \frac{\partial w}{\partial x} + \bar{N}_6 \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial y} \left(\bar{N}_6 \frac{\partial w}{\partial x} + \bar{N}_2 \frac{\partial w}{\partial y} \right) \quad (3.4-14)$$

where $(\bar{N}_1, \bar{N}_2, \bar{N}_6)$ are applied inplane forces (see Figure 3.4-2). For the first order theory, the stability (or geometric stiffness) matrix $[S^e]$ is given by

$$[S^e] = \begin{bmatrix} [0] & [0] & [0] & [0] & [0] \\ & [0] & [0] & [0] & [0] \\ & & [S^{33}] & [0] & [0] \\ & & & [0] & [0] \\ & & & [0] & [0] \\ & sym. & & & [0] \end{bmatrix} \quad (3.4-15a)$$

where

$$S_{ij}^{33} = \int_{\Omega^e} \left[\bar{N}_1 \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + \bar{N}_2 \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} + \bar{N}_6 \left(\frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial y} + \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial x} \right) \right] dx dy \quad (3.4-15b)$$

The eigenvalue λ in this case represents the ratio of actual buckling load to the applied inplane forces:

$$\lambda = \frac{N_1}{\bar{N}_1} = \frac{N_2}{\bar{N}_2} = \frac{N_6}{\bar{N}_6} \quad (3.4-16)$$

Temporal (time) approximation

For the transient response, the time derivatives of the solution in Eq. (3.4-9) should be represented in terms of the solution itself to obtain algebraic equations relating the solution $\{\Delta\}$ at time $(t + \delta t)$ to the solution at time t , where δt is the time step. In the Newmark integration scheme, the solution vectors $\{\Delta\}$ and $\{\dot{\Delta}\}$ at time $t = (s + 1)\delta t$ are approximated by the expressions,

$$\begin{aligned}\{\dot{\Delta}\}_{s+1} &= \{\dot{\Delta}\}_s + [(1 - \alpha)\{\ddot{\Delta}\}_s + \alpha\{\ddot{\Delta}\}_{s+1}]\delta t \\ \{\Delta\}_{s+1} &= \{\Delta\}_s + \{\dot{\Delta}\}_s\delta t + [(1 - \gamma)\{\ddot{\Delta}\}_s + \gamma\{\ddot{\Delta}\}_{s+1}]\frac{(\delta t)^2}{2}\end{aligned}\quad (3.4-17)$$

where s denotes the time step number, α and γ are parameters that control the accuracy and stability of the scheme, and the subscript s indicates that the vectors are evaluated at the s -th time step (*i.e.*, at time, $t = s\delta t$).

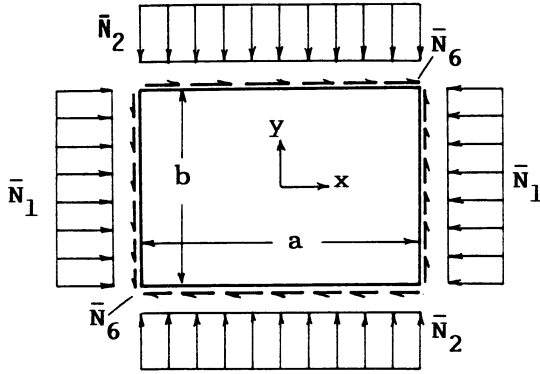


Figure 3.4-2 A rectangular plate element with applied inplane compressive and shear forces.

Using Eqs. (3.4-9) and (3.4-17), we obtain the following algebraic equations for an element (see Reddy [6]),

$$[\hat{K}]_{s+1}\{\Delta\}_{s+1} = \{\hat{F}\}_{s,s+1} \quad (3.4-18a)$$

where $\{\Delta\}_{s+1}$ denotes the value of $\{\Delta\}$ at time $t = (s + 1)\delta t$, and

$$\begin{aligned}[\hat{K}]_{s+1} &= [K]_{s+1} + a_3[M]_{s+1}, \\ \{\hat{F}\}_{s,s+1} &= \{F\}_{s+1} + [M]_{s+1}(a_3\{\Delta\}_s + a_4\{\dot{\Delta}\}_s + a_5\{\ddot{\Delta}\}_s) \\ a_3 &= \frac{2}{\gamma(\delta t)^2}, \quad a_4 = a_3\delta t, \quad a_5 = \frac{1}{\gamma} - 1\end{aligned}\quad (3.4-18b)$$

Once the solution $\{\Delta\}$ is known at time $t_{s+1} = (s+1)\delta t$, the first and second derivatives of $\{\Delta\}$ (velocity and accelerations) at t_{s+1} can be computed from

$$\begin{aligned}\{\ddot{\Delta}\}_{s+1} &= a_3(\{\Delta\}_{s+1} - \{\Delta\}_s) - a_4\{\dot{\Delta}\}_s - a_5\{\ddot{\Delta}\}_s \\ \{\dot{\Delta}\}_{s+1} &= \{\dot{\Delta}\}_s + a_2\{\ddot{\Delta}\}_s + a_1\{\ddot{\Delta}\}_{s+1}\end{aligned}\quad (3.4-19)$$

where $a_1 = \alpha\delta t$, and $a_2 = (1 - \alpha)\delta t$.

Note that Eq. (3.4-18a) is valid for an element. Therefore, operations indicated in Eq. (3.4-18b) are carried out for an element, and $[\hat{K}^e] = \{\hat{F}^e\}$ are assembled as in a static analysis. They are solved, after imposing the boundary conditions, for the solution at time t_{s+1} . This procedure is repeated until for any number of time steps. Thus, Eq. (3.4-18) represents a time-marching scheme.

For the first time step, the initial conditions on $\{\Delta(t)\}$ and $\{\dot{\Delta}(t)\}$ are used to compute $\{\Delta\}_0$ and $\{\dot{\Delta}\}_0$ for each element of the entire mesh. The acceleration vector $\{\ddot{\Delta}\}_0$ is computed from Eq. (3.4-9) at $t = 0$:

$$\{\ddot{\Delta}\}_0 = [M]^{-1}(\{F\}_0 - [K]\{\Delta\}_0) \quad (3.4-20)$$

Often it is assumed that $\{F\}_0 = \{0\}$. If the initial conditions are zero, $\{\Delta\}_0 = \{0\}$, and the applied force is assumed to be zero at $t = 0$, we then take $\{\ddot{\Delta}\}_0 = \{0\}$.

Since Eqs. (3.4-17) represent approximations of the actual solution vectors $\{\Delta\}$ and $\{\dot{\Delta}\}$, error is introduced into the solution $\{\Delta\}_{s+1}$. This error can grow from one time step to the next, as we solve Eq. (3.4-18a) for different times. If the error grows unboundedly with time, such time-approximation schemes are said to be *unstable*. If the error is bounded for all times, the scheme is said to be *stable*. For some schemes, the error is bounded only if certain restrictions on the time step are met. Such schemes are called *conditionally stable schemes*. Of course, we use only either stable or conditionally stable schemes.

The numerical stability of the Newmark scheme depends on the choice of the values for α and γ . Only when $\alpha = \gamma = 1/2$, is the scheme unconditionally stable. For other choices of these parameters, the time increment δt must be chosen such that the following *stability criterion* is met:

$$\delta t < \delta t_{cr} = \left[\frac{1}{2} \omega_{\max}^2 (\alpha - \gamma) \right]^{-\frac{1}{2}}, \text{ for } \alpha \geq \frac{1}{2}, \gamma < \alpha \quad (3.4-21)$$

where $\omega_{\max}$ denotes maximum natural frequency of the structure being analyzed, obtained with the same finite element mesh as that used for the transient analysis.

The parameters α and γ in the Newmark time approximation scheme (3.4-17) can be chosen to yield several well-known finite difference schemes. Some of them are listed here:

1. Constant-average acceleration method (stable scheme): $\alpha = \frac{1}{2}$, $\gamma = \frac{1}{2}$
2. Linear acceleration method (conditionally stable): $\alpha = \frac{1}{2}$, $\gamma = \frac{1}{3}$
3. Central difference method (conditionally stable): $\alpha = \frac{1}{2}$, $\gamma = 0$

(3.4 - 22)

3.4-4 Shear locking

The displacement-based C^0 -plate elements (i.e., those with no derivatives included as generalized displacements) of the first-order shear deformation plate theory are among the simplest available in the literature. Unfortunately, the accuracy of the low-order elements of this type (i.e., those with nine or fewer nodes) is not satisfactory when the span-to-thickness ratio of the elements is large (i.e., thin plates). For thin plates, the shearing strains ϵ_4 and ϵ_5 are required to vanish, and the plate elements based on the first-order theory become excessively stiff, yielding displacements that are too small compared to the true solution. This type of behavior, known as *shear locking*, has plagued shear deformable elements since their inception (see [16]).

Shear locking is due to the inability of shear deformable elements to accurately model the curvatures within an element under a state of zero transverse shearing strain. The various means employed to alleviate shear locking in plate elements include reduced integration [17-19], mixed formulations [20,21], hybrid formulations [22], assumed strain techniques [23], and discrete Kirchhoff methods [24], to name only a few. Of the large number of new displacement-based finite elements for the first-order shear deformation theory, in the literature, only a few have been reasonably successful [19-21], and there is still much room for improvement in terms of accuracy, ease of implementation, and computational efficiency.

A better understanding of locking phenomena is necessary in order to give guidelines for the development of new elements and to allow a more complete analysis of existing elements. When thin plates are modeled by plate finite elements with shear deformation, the energy due to transverse shear strains must vanish. Numerically this is equivalent to a constraint condition which requires that the product of the shear stiffness matrix and the displacement vector be zero. Therefore, in order to obtain a non-trivial solution, the shear stiffness matrix must be singular. One way to achieve the singularity of the shear stiffness matrix is to use an order of numerical integration lower than is necessary to evaluate the integrals exactly. Thus, *reduced integration* of shear stiffnesses is necessary. These

observations led to explanations of locking behavior based upon measures of singularity of the shear stiffness matrix [25,26], the number of independent shear constraints imposed on an element in the thin plate limit, and other criteria.

Recently, Averill and Reddy [16] presented a new simple analytical technique for identifying the exact form of the shear constraints that are imposed on an element when its side-to-thickness ratio is very large. If the form of the shear constraints in an element are known, then by applying the technique to several popular plate elements based on the first-order theory, we can determine their performance *a-priori* when used to model thin structures. Furthermore, by studying the form of the constraints, we can gain a more complete understanding of the locking behavior of plate elements leading to new insights and definitive conclusions regarding the origin of locking phenomena and the effects of reduced integration. For additional details, the reader may consult Reference 16.

3.5 Computational aspects

3.5-1 Evaluation of element matrices and numerical integration

The evaluation of element coefficients, for example the stiffness and mass matrices, require the evaluation of integrals. The integrands of these integrals are often complicated and the domains on which they are defined are irregular (i.e. not rectangular), precluding the evaluation by analytical means. Numerical integration, called *quadrature*, is employed to evaluate such integrals.

There are several numerical methods of evaluating the integrals. All such methods seek to find a polynomial approximation of the integrand because polynomials can be integrated exactly. The Gauss-Legendre quadrature is most commonly used due to the fact that it gives good accuracy with less computational time when compared to other methods (see Reddy [6]). This method is briefly described here to familiarize the reader with the concept of numerical integration.

We begin, for illustrative purposes, with the evaluation of a one dimensional integral. Consider an integral expression of the form,

$$\int_a^b G(x)dx \quad (3.5-1)$$

where $G(x)$ is a function defined in $a \leq x \leq b$. Evaluation of the integral (3.5-1) amounts to determining the area under the curve $G(x)$. In all numerical integration methods, the area under the curve is approximated as the weighted sum,

$$\int_a^b G(x)dx = w_1G(x_1) + w_2G(x_2) + \dots + w_NG(x_N) = \sum_{I=1}^N w_I G(x_I) \quad (3.5-2)$$

where w_I ($I = 1, 2, \dots, N$) are called the *weights* and x_I are referred to as the *sampling points* or *base points*.

In the Gauss quadrature, the base points x_I and the weights w_I are chosen such that the sum of the N appropriately weighted values of the function $G(x)$ yields the exact value of the integral when $G(x)$ is a polynomial of degree $p = (2N - 1)$ or less. The Gauss quadrature is given by

$$\int_a^b G(x)dx = \int_{-1}^1 \hat{G}(r)dr = w_1\hat{G}(r_1) + w_2\hat{G}(r_2) + \dots + w_N\hat{G}(r_N) = \sum_{I=1}^N w_I\hat{G}(r_I) \quad (3.5 - 3)$$

where $\hat{G}(r)$ is the transformed integrand,

$$\hat{G}(r) = G(x(r)).J(r) \quad (3.5 - 4)$$

$J(r)$ is the *Jacobian* of the transformation and N is the number of *Gauss points*. The coordinate transformation between x and r should be such that the interval (a, b) is mapped into the interval $(-1, 1)$. The new coordinate r , $-1 \leq r \leq 1$, is called a *natural* or *normalized coordinate*, and it is defined locally in an element.

In order to integrate a polynomial of degree p exactly, the number of Gauss points N should be taken as equal to the *smallest integer greater than* $(p+1)/2$. The Gauss points and weights can be found in tabular form in any book on numerical analysis (see Reddy [6]).

To numerically evaluate the area integrals that arise in the finite element analysis, such as those found in the element matrices of Eq. (3.3-12) and (3.4-5), the element domains must be transformed to the square domain $-1 \leq (r, s) \leq 1$, where (r, s) is the natural coordinate system, so that the Gauss quadrature can be used. The transformation of an element domain Ω^e to the square domain Ω^M , called the *master element*, requires approximation of the geometry. The approximation of the geometry is accomplished, in much the same way as the solution, by interpolating the geometry of the element domain Ω^e :

$$x = \sum_{j=1}^m x_j \hat{\psi}_j(r, s), \quad y = \sum_{j=1}^m y_j \hat{\psi}_j(r, s) \quad (3.5 - 4)$$

where (x, y) are the global coordinates used in the formulation of the finite element equations. The interpolation functions $\hat{\psi}_j$ used in the approximation of the geometry are, in general, not the same as those ψ_j used in the approximation of the dependent variables (i.e., the solution to the problem). When $\hat{\psi}_j = \psi_j$ (i.e., $m = n$), the formulation is called an *isoparametric formulation*. If $m < n$, it is known as a *subparametric formulation*.

The transformation (3.5-4) allows us to transform integral expressions involving the global coordinates and defined over the element domain Ω^e as those in terms of the local coordinates (r, s) defined over the master element Ω^M :

$$\int_{\Omega^e} G(x, y) dx dy = \int_{\Omega^M} \hat{G}(r, s) dr ds \quad (3.5-5)$$

which can be approximated by the Gauss quadrature formula:

$$\int_{-1}^1 \int_{-1}^1 \hat{G}(r, s) dr ds = \sum_{I, J=1}^N w_I w_J \hat{G}(r_I, s_J) \quad (3.5-6)$$

If $\hat{G}(r, s)$ is a polynomial of highest degree p in r or s , then $N = \left\lceil \frac{p+1}{2} \right\rceil$ to evaluate the integral exactly. When $p+1$ is not an even number, then N should be equal to the nearest integer value greater than $(p+1)/2$. For example, a 2×2 Gaussian quadrature (i.e., $N = 2$) exactly integrates a polynomial of third order and 3×3 quadrature exactly integrates a polynomial of fifth order. Before selecting an integration order, one must examine each term in the expression for the integrand (which includes the Jacobian of the transformation) and determine the polynomial order.

3.5-2 Computation of strains and stresses

Once the nodal values of generalized displacements have been obtained by solving the assembled equations of a problem, the strains are evaluated in each element by differentiating the displacements (3.4-2). Since, in the displacement finite element models, only the displacements are continuous across the element boundaries, strain continuity across the element boundaries is not ensured. That is, along a boundary common to two elements, the strains and hence stresses take different values on the two sides of the interface. However, strains and hence stresses are continuous within an element. Here, a brief discussion on the computation of strains and stresses is presented.

Although the strains can be computed at any point in the element, including the nodes, Barlow [37,38] showed that the strains and stresses are the most accurate if they are computed at the $(N-1) \times (N-1)$ Gauss points, where $N \times N$ is the Gauss quadrature rule used to evaluate the stiffness coefficients. This reduced order integration rule is termed as the *reduced integration*, and the reduced integration points are called the *Barlow points*. For example, the linear rectangular plate bending element of the first-order theory requires 2×2 integration to evaluate the bending stiffnesses exactly. Then the one-point integration (or 1×1) should be used to evaluate the strains and stresses. Similarly, for a quadratic rectangular element the Barlow points are at the Gauss points of the 2×2 Gauss rule.

Note that the displacements in the finite element models are referred to the global coordinates (x, y, z) . Differentiation of these values at the Barlow points yields the strains there in the global coordinates. However, the failure criteria used in laminate analysis (see Chapter 5) require stresses and strains in the lamina (or principal material) coordinates. In order to obtain strains and stresses in the material coordinates, we need to transform the strains and stresses to material coordinates associated with each layer. First the global strains ϵ_i in Eqs. (2.4–10) or (2.5–5) are transformed to the lamina strains $\bar{\epsilon}_i$ using the strain transformation equations similar to Eq. (2.3–3):

$$\begin{Bmatrix} \bar{\epsilon}_1 \\ \bar{\epsilon}_2 \\ \bar{\epsilon}_6 \end{Bmatrix} = \begin{bmatrix} \cos^2 \theta & \sin^2 \theta & \cos \theta \sin \theta \\ \sin^2 \theta & \cos^2 \theta & -\cos \theta \sin \theta \\ -2 \cos \theta \sin \theta & 2 \cos \theta \sin \theta & 2(\cos^2 \theta - \sin^2 \theta) \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{Bmatrix} \quad (3.5-7a)$$

$$\begin{Bmatrix} \bar{\epsilon}_4 \\ \bar{\epsilon}_5 \end{Bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{Bmatrix} \epsilon_4 \\ \epsilon_5 \end{Bmatrix} \quad (3.5-7b)$$

Then the lamina constitutive equations are used to compute the stresses, referred to the material coordinates, at the Barlow points:

$$\begin{Bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \\ \bar{\sigma}_6 \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \bar{\epsilon}_1 \\ \bar{\epsilon}_2 \\ \bar{\epsilon}_6 \end{Bmatrix} \quad (3.5-8a)$$

$$\begin{Bmatrix} \bar{\sigma}_4 \\ \bar{\sigma}_5 \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \begin{Bmatrix} \bar{\epsilon}_4 \\ \bar{\epsilon}_5 \end{Bmatrix} \quad (3.5-8b)$$

Note that $[\bar{Q}]$ matrix represents the matrix of material stiffnesses along the principal coordinates of the layer (lamina) as discussed in Section 2.3–3.

While the inplane stresses ($\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$) can be computed only through stress-strain (i.e., constitutive) relations, the transverse stresses ($\sigma_{xz}, \sigma_{yz}, \sigma_{zz}$) can be computed through either the constitutive equations (3.5–13) or the equilibrium equations of 3-D elasticity,

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0 \quad (3.5-9a)$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0 \quad (3.5-9b)$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0 \quad (3.5-10)$$

Integration of Eqs. (3.5-9) with respect to z gives $(\sigma_{zz}, \sigma_{yz})$, which can be substituted into Eq. (3.5-10) to determine σ_{zz} (see Reddy [2]). It is found that for homogeneous plates and thin laminates the stresses computed from the equilibrium equations (3.5-9) are found to be very accurate.

3.5-3 Solution methods for nonlinear algebraic equations

Note that the element stiffness matrix in Eqs. (3.3-12) (or 3.4-4) is nonlinear and unsymmetric when the nonlinear (von Kármán) strains are included [see Eq. (3.3-15)]. Therefore, the assembled equations will be nonlinear and unsymmetric. The assembled nonlinear equations must be solved, after imposing boundary conditions, by an iterative method, which seeks an approximate solution to the algebraic equations by linearization. Here we discuss two iterative methods that are commonly used.

Suppose that the assembled nonlinear finite element equations are of the form,

$$[K(\{\Delta\})]\{\Delta\} = \{F\} \quad (3.5-11)$$

The stiffness matrix $[K]$ cannot be evaluated because it depends on the unknown solution $\{\Delta\}$. Therefore, we evaluate $[K]$ using an assumed solution vector $\{\Delta\}$ and iteratively solve Eq. (3.5-11) for the solution.

Direct iteration method

In the direct iteration method, also known as the *Picard method*, the solution vector from the previous iteration r is used to evaluate the stiffness matrix, and the solution at the $(r+1)$ -st iteration is determined by solving the equation (see Reddy [6] and Zienkiewicz and Taylor [39]),

$$[K(\{\Delta\}^r)]\{\Delta\}^{r+1} = \{F\} \quad (3.5-12)$$

where $\{\Delta\}^r$ denotes the solution at r -th iteration. Thus, in the direct iteration method, the coefficients K_{ij} (hence $K_{ij}^{\alpha\beta}$) are evaluated using the solution $\{\Delta\}^r$ from the previous iteration, and the solution at $(r+1)$ -st iteration is obtained by solving Eq. (3.5-12):

$$\{\Delta\}^{r+1} = [K(\{\Delta\}^r)]^{-1}\{F\} \quad (3.5-13)$$

At the beginning of the iteration (i.e., $r=0$), we assume a solution $\{\Delta\}^0$, based on our qualitative understanding of the solution behavior. For example, $\{\Delta\}^0 = \{0\}$ for the large-deflection bending would reduce the nonlinear stiffness matrix to a

linear one, and (3.5-13) would yield the linear solution of the problem at the end of the first iteration, $\{\Delta\}^1$. The iteration is continued [i.e., Eq. (3.5-13) is solved in each iteration] until the difference between $\{\Delta\}^r$ and $\{\Delta\}^{r+1}$ reduces to a preselected error tolerance. The *error criterion* is of the form (other error criteria can also be used):

$$\sqrt{\frac{\sum_{I=1}^N |\Delta_I^{r+1} - \Delta_I^r|^2}{\sum_{I=1}^N |\Delta_I^{r+1}|^2}} < \epsilon \quad (\text{say } 10^{-3}) \quad (3.5-14)$$

where N is the total number of primary unknowns (i.e., generalized displacements) in the finite element mesh.

Newton-Raphson iteration method

The Newton-Raphson iterative method is based on the Taylor series expansion of the algebraic equations (3.5-11) about the known solution $\{\Delta\}^r$. To describe the method, we rewrite Eq. (3.5-11) in the form,

$$\{R\} \equiv [K]\{\Delta\} - \{F\} = 0 \quad (3.5-15)$$

where $\{R\}$ denotes the residual. Expanding $\{R\}$ about $\{\Delta\}^r$, we obtain

$$\begin{aligned} \{0\} = \{R\} &= \{R\}^r + \left[\frac{\partial \{R\}}{\partial \{\Delta\}} \right]^r (\{\Delta\}^{r+1} - \{\Delta\}^r) \\ &+ \frac{1}{2!} \left[\frac{\partial^2 \{R\}}{\partial \{\Delta\}^2} \right]^r (\{\Delta\}^{r+1} - \{\Delta\}^r)^2 + \dots \end{aligned} \quad (3.5-16a)$$

or

$$0 \approx \{R\}^r + [K^T]^r \{\delta\Delta\} + O(\{\delta\Delta\}^2) \quad (3.5-16b)$$

where $[K]^T$ is the *tangent stiffness matrix* (or geometric stiffness matrix),

$$[K^T]^r \equiv \left[\frac{\partial \{R\}}{\partial \{\Delta\}} \right] \text{ evaluated at } \{\Delta\} = \{\Delta\}^r \quad (3.5-17)$$

$$\{\delta\Delta\} = \{\Delta\}^{r+1} - \{\Delta\}^r, \quad (3.5-18)$$

For structural problems formulated with variational principles, it can be shown that $[K^T]$ is symmetric even if $[K]$ is not symmetric. From Eq. (3.5-16b), we have

$$\begin{aligned}\{\delta\Delta\} &= -[K^T]^{-1}\{R\}^r \\ &= [K^T(\{\Delta\}^r)]^{-1}(\{F\} - [K(\{\Delta\}^r)]\{\Delta\}^r)\end{aligned}\quad (3.5 - 19a)$$

and the total solution at $(r + 1)$ -st iteration is given by

$$\{\Delta\}^{r+1} = \{\Delta\}^r + \{\delta\Delta\} \quad (3.5 - 19b)$$

The iterative solution of Eq. (3.5-19a) is continued until the convergence criteria in Eq. (3.5-14) is satisfied or the residual $\{R\}$ [measured in the same way as the solution error in Eq. (3.5-14)] in Eq. (3.5-19) is less than a certain preselected value. Note that at the beginning of each iteration the tangent stiffness matrix and residual vector must be updated using the latest available solution $\{\Delta\}$. If the tangent stiffness matrix is kept constant (for a preselected number of iterations) but the residual vector is updated during the iteration, there can be a computational saving. Such an approach is called *modified Newton-Raphson method*. For additional details on iterative methods, (especially for the Riks method) the reader is asked to consult References 15 and 39-41.

3.6 Continuum formulation

In linear analysis, the displacements and rotations are assumed to be small, and the material is treated as linearly elastic. In the nonlinear formulation based on a laminate theory with the von Kármán strains, the geometry of the continuum [structure] is assumed to remain unchanged during the loading. For small strains and moderately large deflections and rotations, these assumptions yield good results. In this study, we shall term the elements based on such assumptions as *laminated elements*.

However, in problems with large displacements and strains, the geometry of the structure changes during the loading. In such problems, the geometrical changes must be taken into account before next increment of load is applied. Thus, the total load on the structure must be applied incrementally, and the geometry updated between load increments. Finite elements based on these considerations will be called *continuum elements*.

There are two incremental formulations that are used to determine the deformation and stress states in continuum problems (see [15]): (1) Total Lagrangian formulation and (2) Updated Lagrangian formulation. In these formulations, the configuration (i.e., geometry) of the structure for the current load increment is determined from a previously known configuration. In the Total Lagrangian formulation, all of the quantities are referred to a fixed configuration, and changes

in the displacement and stress fields are determined with respect to the reference configuration. In the Updated Lagrangian formulation, the geometry of the structure from the previous increment is updated using the solution of the current increment, and the updated configuration is used as the reference configuration for the next increment. The general form of the incremental equations of equilibrium in these two formulations is given by

$$[K_L + K_{NL}]\{\delta U_i\} = \{R^{i+\delta t}\} - \{F_{i-1}^{i+\delta t}\} \quad (3.6 - 1)$$

where i refers to the current iteration number, $\{\Delta U\}$ is the incremental displacement vector, $\{R\}$ is the residual force vector, $\{F\}$ is the external load vector, and $[K_L]$ and $[K_{NL}]$ are the linear and nonlinear stiffness matrices, respectively.

In the Total Lagrangian formulation, the strain displacement relations are written in terms of the Green-Lagrange strain tensor and 2nd Piola-Kirchhoff stress tensor, S . The nonlinear stiffness matrix and nodal point stresses at time t are expressed by the following equations over an element Ω^e in the original (undeformed) configuration:

$$\begin{aligned} [K_L] &= \int_{\Omega^e} [B_L]^T [C] [B_L] dV \\ [K_{NL}] &= \int_{\Omega^e} [B_{NL}]^T [S] [B_{NL}] dV \\ \{F\} &= \int_{\Omega^e} [B_L]^T \{S\} dV \end{aligned} \quad (3.6 - 2)$$

where $[B_L]$ and $[B_{NL}]$ are linear and nonlinear strain-displacement transformation matrices, $[C]$ is the constitutive elasticity matrix, $[S]$ is the matrix of 2nd Piola-Kirchhoff stress components, and $\{S\}$ is the vector of these stresses.

In the Updated Lagrangian formulation, the latest known deformed configuration is used as the reference state to determine the next configuration. The nodal coordinates are updated using the incremental solution to generate the new reference state. Therefore the stresses used in the nonlinear stiffness expression, as well as the nodal stress values, are the Cauchy stresses, τ . The element stiffness and force vectors are defined by

$$\begin{aligned} [K_L] &= \int_{\Omega^e} [B_L]^T [C] [B_L] dV \\ [K_{NL}] &= \int_{\Omega^e} [B_{NL}]^T [\tau] [B_{NL}] dV \\ \{F\} &= \int_{\Omega^e} [B_L]^T \{\tau\} dV \end{aligned} \quad (3.6 - 3)$$

A specific example of the continuum formulation is provided by the Ahmed-Zienkiewicz degenerated shell element (see [44,48]), which is obtained from 3-D elasticity equations by invoking the same assumptions as those used in the first-order shear deformation plate/shell theory (see Sec. 2.5-2). The degenerate 3-D shell element (see Figure 3.6-1) can admit moderately large displacements and rotations. For additional details on the formulation of the element, the reader is referred to [44,47,48]. Applications of the element are included in Section 3.7-3. .

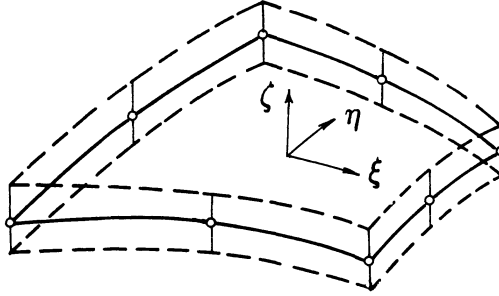


Figure 3.6-1 The degenerated 3-D shell element

Next, we present a continuum element formulation of a second-order plate theory with an eight-noded quadrilateral plate element. The following expressions are assumed for the displacement field of the element (see Ochoa [49]):

$$\begin{aligned} u_1(x, y, z) &= u(x, y) + z\theta_1(x, y) + z^2\phi_1(x, y) \\ u_2(x, y, z) &= v(x, y) + z\theta_2(x, y) + z^2\phi_2(x, y) \\ u_3(x, y, z) &= w(x, y) \end{aligned} \quad (3.6-4)$$

The in-plane displacements (u, v) and the coefficients (ϕ_1, ϕ_2) are expanded with a bilinear polynomial with the coefficients,

$$\{ 1 \ x \ y \ xy \} \quad (3.6-5)$$

where as w and the rotations (θ_1, θ_2) are interpolated with polynomials containing the following coefficients

$$\{ 1 \ x \ y \ x^2 \ xy \ y^2 \ x^2y \ xy^2 \} \quad (3.6-6)$$

The corner nodes are assigned seven degrees of freedom $(u, v, w, \theta_1, \theta_2, \phi_1, \phi_2)$ and the midside nodes are assigned (w, θ_1, θ_2) . The element has 40 degrees of freedom. Therefore, the element will be referred to as QHD40.

To evaluate the volume integrals in the element matrices of Eqs. (3.6-2) and (3.6-3), we employ the Gauss quadrature described earlier. Since we are dealing with laminated composite plate element, the integration through the thickness involves individual lamina. Note that the constitutive matrix $[C]$ varies from layer to layer and is not a continuous function in the thickness direction, and the integration should be performed separately for each layer. This increases the computation time as the number of layers is increased. An alternative is to perform explicit integration through the thickness and reduce the problem to a 2-D one. The Jacobian matrix, in general, is a function of the local coordinates (ξ, η, ζ) , ζ being the thickness coordinate. The terms in ζ may be neglected, provided the thickness to curvature ratios are small. Then the Jacobian matrix $[J]$ becomes independent of ζ and explicit integration can be employed. If ζ terms are retained in $[J]$, Gaussian points through the thickness should be added. In the present study we assume that the Jacobian matrix is independent of ζ in the evaluation of element matrices and the internal nodal force vector.

The displacement field (3.6-4) of the QHD40 element is selected so that a good representation of interlaminar stresses is possible. In this formulation, the inplane stresses ($\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$) are computed from the constitutive equations (3.5-8a), and they are of the form,

$$\sigma_{xx} = f_1(z^2, x^2, y^2) \quad (3.6-7a)$$

$$\sigma_{yy} = f_2(z^2, x^2, y^2) \quad (3.6-7b)$$

$$\sigma_{xy} = f_3(z^2, x^2, y^2) \quad (3.6-7c)$$

The interlaminar stresses ($\sigma_{xz}, \sigma_{yz}, \sigma_{zz}$) are computed from the equilibrium equations (3.5-9) and (3.5-10), and are of the form,

$$\sigma_{xz} = f_4(z^3, x, y) \quad (3.6-8a)$$

$$\sigma_{yz} = f_5(z^3, x, y) \quad (3.6-8b)$$

$$\sigma_{zz} = f_6(z^3) \quad (3.6-8c)$$

In order to avoid 'element locking' for thin shell structures, we use reduced integration in evaluating the stiffness coefficients associated with the transverse shear deformation (i.e., terms involving C_{44} , C_{45} and C_{55}). Hence we split the constitutive matrix $[C]$ into two parts, one without transverse shear moduli $[C]_b$, and the other with only transverse shear moduli $[C]_s$. Full integration is used to evaluate the stiffness coefficients containing $[C]_b$, and reduced integration is used for those containing $[C]_s$ (see [44,48]).

Even though selective reduced integration is used to prevent shear locking in static and dynamic problems, this should not be practiced as a 'blanket rule'. In nonlinear problems, it may be necessary to use higher order quadrature in order to accurately evaluate the nonlinear stiffness coefficients. It is recommended that one should use integration schemes with increasing order until an efficient yet adequate one is selected. Naturally, this is not possible with some of the commercial computer programs that do not give the user any options on integration rules.

3.7 Numerical examples

3.7-1 Preliminary comments

The finite element models reviewed in Sections 3.3 and 3.4 are used to: analyze laminated composite plates; assess the accuracy of various single-layer theories; bring out certain modeling aspects; and assess the importance of geometric nonlinearity. The present discussion is focused on illustrating certain interesting features characteristic of composite laminates. These include: the use of symmetry boundary conditions in quarter plate models, the effect of geometric nonlinearity on deflections, and the determination of critical buckling loads. Two examples of the application of the continuum shell elements [44,48] are included.

The following two sets of material properties, typical of a high-modulus graphite-epoxy are used in the first few examples:

1. $E_1/E_2 = 25$, $G_{12} = G_{13} = 0.5 E_2$, $G_{23} = 0.2 E_2$, $\nu_{12} = 0.25$
2. $E_1/E_2 = 40$, $G_{12} = G_{13} = 0.6 E_2$, $G_{23} = 0.5 E_2$, $\nu_{12} = 0.25$. (3.7-1)

All layers are assumed to have the same thickness and orthotropic material properties in the material principal directions. The shear correction factors ($K_{44}^2 = K_{55}^2$) for the first-order shear deformation theory (FSDT) are taken to be 5/6 [see Eq. (2.5-10)].

We note that for antisymmetric cross-ply laminates, $(0^\circ/90^\circ/0^\circ/90^\circ/\dots)_{\text{even}}$, the following laminate stiffnesses are identically zero:

$$A_{16}, A_{26}, A_{45}, B_{12}, B_{16}, B_{26}, D_{16}, D_{26} = 0, \quad (3.7-2)$$

and for antisymmetric angle-ply laminates, $(-\theta/\theta/\dots)_{\text{even}}$, the following laminate stiffnesses are zero:

$$A_{16}, A_{26}, A_{45}, B_{11}, B_{12}, B_{22}, B_{66}, D_{16}, D_{26}, D_{45} = 0. \quad (3.7-3)$$

3.7-2 Linear bending and vibration analysis

Examples with laminate elements

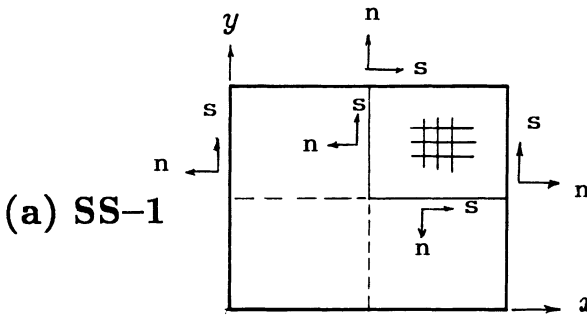
The maximum bending deflections and stresses of antisymmetric cross-ply square laminates with different boundary conditions and subjected to transverse distributed sinusoidal load of intensity q_0 are presented in Table 3.7-1. In all cases, the edges parallel to the x -axis are simply supported (SS). Other two edges are subjected to various types of boundary conditions: S—simply supported; C—clamped; and F—free. In the clamped case, all generalized displacements are set to zero; no generalized displacements are specified on a free edge. The finite element solutions were obtained using a 4×4 mesh of CLPT and 2×2 mesh of nine-node quadratic FSDT elements in the quarter plate (see Figure 3.7-1 for boundary conditions). The analytical solutions were obtained using the Lévy type solution procedure (see [3,4]). The finite element results are in good agreement with the analytical solutions. In general, the classical plate theory underpredicts deflections and stresses.

Table 3.7-1 Comparison of the finite element and analytical solutions of various theories for antisymmetric cross-ply (0/90/...) laminated square plates with various boundary conditions and transverse sinusoidal load ($a/h = 10$; material 1; $N = \text{No. of layers}$).

N	Solution**	Theory	SSSS	SSCC	SSFF
2	$\bar{w}$	FSDT	1.238(1.237)	0.657(0.656)	2.027(2.028)
		CPT	1.043(1.064)	0.417(0.429)	1.786(1.777)
	$\bar{\sigma}_1$	FSDT	6.948(7.157)	4.232(4.450)	2.331(2.442)
		CPT	6.659(7.157)	4.348(4.800)	2.034(2.403)
	$\bar{\sigma}_4$	FSDT	2.652(2.729)	1.459(1.523)	3.814(3.882)
20	$\bar{w}$	FSDT	0.616(0.615)	0.386(0.385)	0.914(0.915)
		CPT	0.444(0.442)	0.169(0.167)	0.686(0.665)
	$\bar{\sigma}_1$	FSDT	4.863(5.009)	2.550(2.692)	1.648(1.723)
		CPT	4.611(5.009)	2.798(3.167)	1.324(1.725)
	$\bar{\sigma}_2$	FSDT	4.863(5.009)	3.031(3.135)	7.384(7.533)
		CPT	4.611(5.009)	1.694(1.911)	7.395(7.480)
	$\bar{\sigma}_4$	FSDT	2.652(2.729)	1.648(1.708)	3.782(3.853)

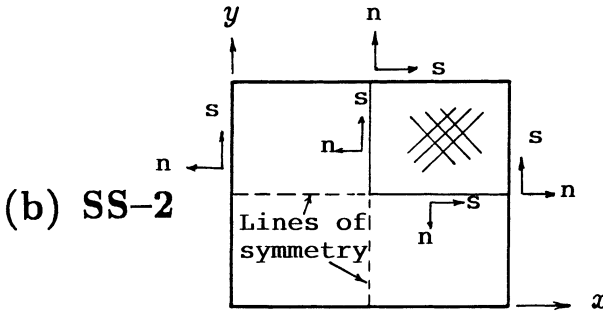
$$**\bar{w} = (wE_2h^3)10^2/(a^4q_0), \quad \bar{\sigma}_1 = \sigma_1\left(\frac{a}{2}, \frac{b}{2}, -\frac{h}{2}\right)h^2 \cdot 10/(a^2q_0),$$

$$\bar{\sigma}_2 = \sigma_2\left(\frac{a}{2}, \frac{b}{2}, \frac{h}{2}\right)h^2 \cdot 10/(a^2q_0), \quad \bar{\sigma}_4 = \sigma_4(0, 0, 0)h \cdot 10/(aq_0)$$



At a line of symmetry with normal $\mathbf{n}$: $u_n = 0$, $\phi_s = 0$

At an edge with normal $\mathbf{n}$: $u_s = 0$, $w = 0$, $\phi_s = 0$



At a line of symmetry with normal $\mathbf{n}$: $u_s = 0$, $\phi_s = 0$

At an edge with normal $\mathbf{n}$: $u_n = 0$, $w = 0$, $\phi_s = 0$

Figure 3.7-1 Boundary conditions for simply-supported cross-ply and anti-symmetric angle-ply laminates. (a) SS-1: Cross-ply laminates. (b) SS-2: Angle-ply laminates.

Because of the bending-stretching coupling ($B_{11} \neq 0$ for antisymmetric cross-ply and $B_{16} \neq 0$ for antisymmetric angle-ply laminates), use of proper 'simply-supported' boundary conditions is important (see Reddy [50]). The 'simply-supported' boundary conditions are different for different lamination schemes. They can be deduced from the exact (i.e., the Navier) solutions (see Fig. 3.7-1). If the SS-1 boundary conditions are used to analyze the angle-ply laminates, or the SS-2 boundary conditions are used to analyze antisymmetric cross-ply laminates the resulting solutions will be different, as can be seen from Table 3.7-2. The bending-stretching coupling coefficients (i.e., B_{ij}) become smaller as the number of layers is increased, and both SS-1 and SS-2 yield almost the same solution.

Table 3.7–2 Effect of boundary conditions on the bending response of two-layer antisymmetric cross-ply and angle-ply square laminates under uniform transverse load ($\bar{w} = \frac{wE_2h^310^2}{p_0a^4}$, $a/h = 10$).

N	$\frac{z}{a}$	Cross-Ply		Angle-Ply	
		SS1*	SS2	SS2*	SS1
2	0.0	1.950	1.648	1.281	1.127
	0.125	1.813	1.531	1.196	1.059
	0.25	1.416	1.189	0.948	0.845
	0.375	0.787	0.654	0.536	0.4800
	0.50	0.000	0.000	0.000	0.0000
10	0.0	0.958	0.955	0.632	0.631
	0.125	0.893	0.890	0.592	0.591
	0.25	0.703	0.701	0.475	0.474
	0.375	0.397	0.396	0.274	0.274
	0.50	0.000	0.000	0.000	0.000

*The first-order theory is used with 2×2 mesh of nine-node quadratic elements in a quarter plate (N = No. of layers).

Table 3.7–3 Nondimensionalized fundamental frequencies ($\bar{\omega} = \frac{\omega a^2}{h} \sqrt{\rho/E}$) of simply-supported, cross-ply square plates according to the shear deformation theory (material 2).

No. of layers	a/h	Quarter Plate Models			Full Plate (4 × 4)	Analytical solution
		1 × 1	2 × 2	4 × 4		
2	100	45.318	14.350	11.537	14.358	11.300
	10	12.119	10.717	10.594	10.777	10.568
	5	9.676	9.136	9.103	9.210	9.087
4	100	47.462	19.643	17.490	19.647	17.278
	10	16.112	14.943	14.859	15.033	14.846
	5	11.495	11.172	11.176	11.247	11.172
10	100	48.034	20.877	18.817	20.880	18.610
	10	16.897	15.782	15.758	15.911	15.770
	5	11.848	11.614	11.664	11.716	11.673

Table 3.7-4 First four nondimensionalized frequencies, $\bar{\omega} = \omega a^2(\sqrt{\rho/E})/h$, of simply-supported, antisymmetric, cross-ply, square laminates, according to the shear deformation theory (material 2).

N	a/h	Source*	$\bar{\omega}_{11}$	$\bar{\omega}_{12} = \bar{\omega}_{21}$	$\bar{\omega}_{22}$
2	100	4 × 4	14.358	45.669	89.269
		8 × 8	11.538	32.656	48.185
		Exact	11.300	31.447	45.100
	10	4 × 4	10.777	27.582	38.361
		8 × 8	10.610	26.561	36.748
		Exact	10.568	25.301	36.348
4	100	4 × 4	19.647	61.602	105.313
		8 × 8	17.493	50.553	71.650
		Exact	17.278	49.122	68.740
	10	4 × 4	15.033	33.927	45.803
		8 × 8	14.883	33.326	44.913
		Exact	14.846	33.171	44.686
10	100	4 × 4	20.880	65.310	109.267
		8 × 8	18.821	54.478	76.883
		Exact	18.610	53.007	73.994
	10	4 × 4	15.911	35.229	47.393
		8 × 8	15.793	34.819	46.798
		Exact	15.770	34.747	46.692

*Mesh used in the full plate; exact solutions are from [4].

It should be noted that the full plate and quarter plate models with associated symmetry boundary conditions give identical results for linear analysis. It will be shown later in this section that the quarter plate model is not acceptable, even with proper symmetry conditions, in the geometrically nonlinear analysis of two-layer antisymmetric angle-ply laminates.

Table 3.7-3 contains the finite element results for free vibrations of simply-supported (SS-1), antisymmetric cross-ply square plates for the three values of the side-to-thickness ratio, representative of thin, moderately thick and thick plates. The finite element solution converges to the analytical solution [4] as the mesh is refined. It is noted that the full plate model and quarter plate model with proper symmetry conditions (see Fig. 3.7-1) give the same results. A 4×4 mesh in a full plate is equivalent to 2×2 mesh in the quarter plate. Since a larger system of equations is solved in a 4×4 mesh compared to a 2×2 mesh, some small difference is expected in the two sets of values.

Table 3.7-4 contains the first four modes for the same case. It is clear that the finite element model does not predict the higher modes as accurately as the first mode. A refined mesh would increase the accuracy of the higher modes considerably more than the fundamental mode.

Table 3.7-5 contains the fundamental natural frequencies of simply-supported (SS-2), antisymmetric angle-ply ($-45^\circ/45^\circ/\dots$) square laminates as predicted by the three single-layer theories. The finite element solutions were obtained using a 4×4 mesh in a quarter plate. The results obtained using full plate models agreed with those obtained by quarter plate models (with SS-2 boundary conditions), indicating that a quarter plate symmetry exists for antisymmetric angle-ply laminates for the free vibration case. In all cases the classical plate theory overpredicts the frequencies compared to the first-order shear deformation theory (i.e., FSDT).

Table 3.7-5 Nondimensionalized fundamental frequencies, $\bar{\omega} = \frac{\omega a^2}{h} \sqrt{\frac{\rho}{E_2}}$, for simply supported antisymmetric angle-ply ($-45/45/\dots$), square laminates (material 2; *exact solutions are from [4]).

N	a/h	Source*	CLPT	FSDT	HSDT
2	10	FEM	14.731	13.217	13.287
		Exact	14.439	13.044	13.263
	100	FEM	14.811	14.863	14.800
		Exact	14.636	14.618	14.621
10	10	FEM	25.297	19.471	19.273
		Exact	25.052	19.289	19.266
	100	FEM	25.305	25.584	25.318
		Exact	25.264	25.176	25.174

Examples with continuum element QHD40

Next, the performance of the QHD40 continuum element (see Sec. 3.6) is demonstrated by comparing results to those obtained by classical laminated plate theory (CLPT), the first-order shear deformation theory (FSDT), and 3-D elasticity. The material properties (of a lamina) used throughout are those of a high modulus *graphite/epoxy* material:

$$\frac{E_1}{E_2} = 25, \quad \frac{G_{12}}{G_{23}} = 2.5, \quad G_{13} = G_{12}; \quad \nu_{12} = \nu_{13} = 0.25 \quad (3.7-4)$$

All layers are of equal thickness. Cylindrical bending of a plate, and bending of simply-supported square and rectangular plates are considered, with a sinusoidally distributed transverse load. For cylindrical bending the plate is modeled with a row of 20 elements (i.e., 20×1 mesh). For the simply-supported plates, the biaxial symmetry is used and only a quadrant of the plate is modeled. The mesh is refined to demonstrate solution convergence. Additionally, the influence of distorted elements on the solution accuracy is demonstrated. For symmetric laminates, the quadratic terms are restrained; therefore, 32 degree-of-freedom elements are used to obtain these solutions. This is allowable in these particular cases because the quadratic terms do not significantly affect the results. This is not true in the first example considered.

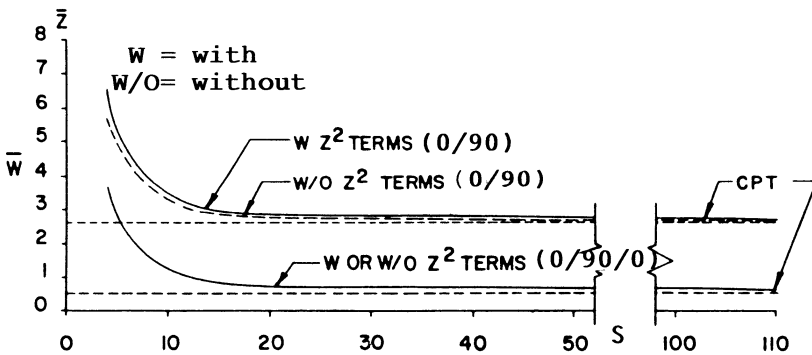


Figure 3.7-2 Normalized transverse displacement versus side-to-thickness ratio for cylindrical bending of (0/90) and (0/90/0) laminates.

Example 1: Cylindrical bending

First consider the cylindrical bending of a (0/90) laminate. In Figure 3.7-2, the calculated maximum normalized (transverse) displacement $\bar{w}$ is presented as a function of the ratio of length to thickness (S) of the plate. Results demonstrate

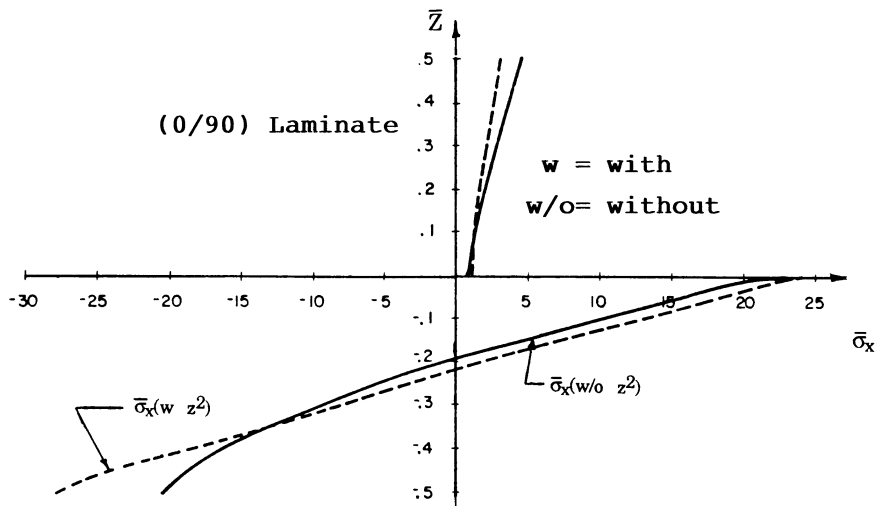


Figure 3.7-3 Normalized inplane stress ($\bar{\sigma}_{xx}$) for cylindrical bending of a (0/90) laminate ($S = 4$).

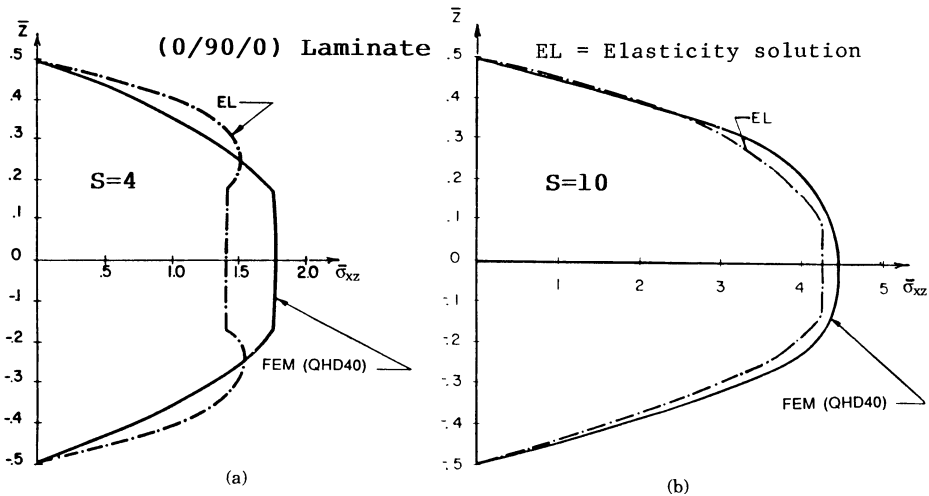


Figure 3.7-4 Normalized transverse shear stress ($\bar{\sigma}_{zz}$) for cylindrical bending of (0/90/0) laminates. (a) $S = 4$. (b) $S = 10$.

differences obtained with and without *quadratic* terms. The 3-D elasticity solution of Pagano [51] gives a nearly quadratic variation of the normalized inplane displacement, $\bar{u}$, with respect to z . The difference is greatest for the lower values of S . For example, for $S = 4$ a difference of 12 per cent is noted. In Figure 3.7-3, the calculated normalized in-plane stress variation is presented for $S = 4$. Note that maximum stresses differ by 35 per cent when computed with, and without, the z^2 terms. The effect of including quadratic terms in the finite element solution is, therefore, much more pronounced in stresses than in displacements.

Next, we consider cylindrical bending of a square (0/90/0) laminate. Once again, the stresses are compared to the elasticity solution of Pagano [51]. Figure 3.7-4 illustrates through the thickness variation of the normalized transverse shear stress (σ_{xz}) at the simply-supported boundary for $S = 4$ and 10. Figure 3.7-5 contains the normalized in-plane stress variation through the thickness at the centre of the laminate. Note that as the side-to-thickness ratio is varied from 4 to 10, the correlation between the finite element and elasticity solution is greatly improved. The lower aspect ratio represents a *thick* plate and, consequently, does not fall within the range of application of the current formulation, which is based on a C^1 -continuous displacement field through the thickness. Thick laminated plates are better modeled by thick plate theories, such as the first order shear deformation theory discussed in Section 3.4.

Example 2: Simply-supported plate under sinusoidal loading

This example deals with a simply-supported, rectangular (0/90/0) laminate. Solutions are obtained for side-to-thickness ratios, $S = 4$ to 100. Transverse shear stress variation σ_{xz} at $(x, y) = (0, a/2)$ and in-plane shear stress variation σ_{xy} at $(x, y) = (0, 0)$ are presented in Figures 3.7-6a and 3.7-6b for a square laminate with $S = 4$. The present finite element results are compared with those obtained by Pagano [52] and the classical plate theory. The correlation between the finite element and elasticity results improves when the plate represents thin to moderately thick geometries. The transverse normal stress variation σ_{zz} through the thickness is presented in Figures 3.7-7a and 3.7-7b for square laminates. The stresses are normalized as $\bar{\sigma}_{zz} = \sigma_{zz}10^{-2}$ at the center of the plate and as $\bar{\sigma}_{zz} = 10\sigma_{zz}$ at the edge of the plate. The present results are compared with the elasticity solution in Table 3.7-6.

The effect of distorted elements is also studied. Results are obtained for the relatively coarse meshes shown in Figure 3.7-8. The stresses and displacements are presented as a function of side-to-thickness ratio for square laminates, and compared to the elasticity solutions in Table 3.7-7. As expected, the values are not as accurately determined as those obtained with the regular meshes. A mesh of distorted elements has a much more adverse effect on the calculated transverse shear stresses than on the calculated inplane stresses and displacements. Since the transverse stresses are based on equilibrium considerations, it seems the mesh

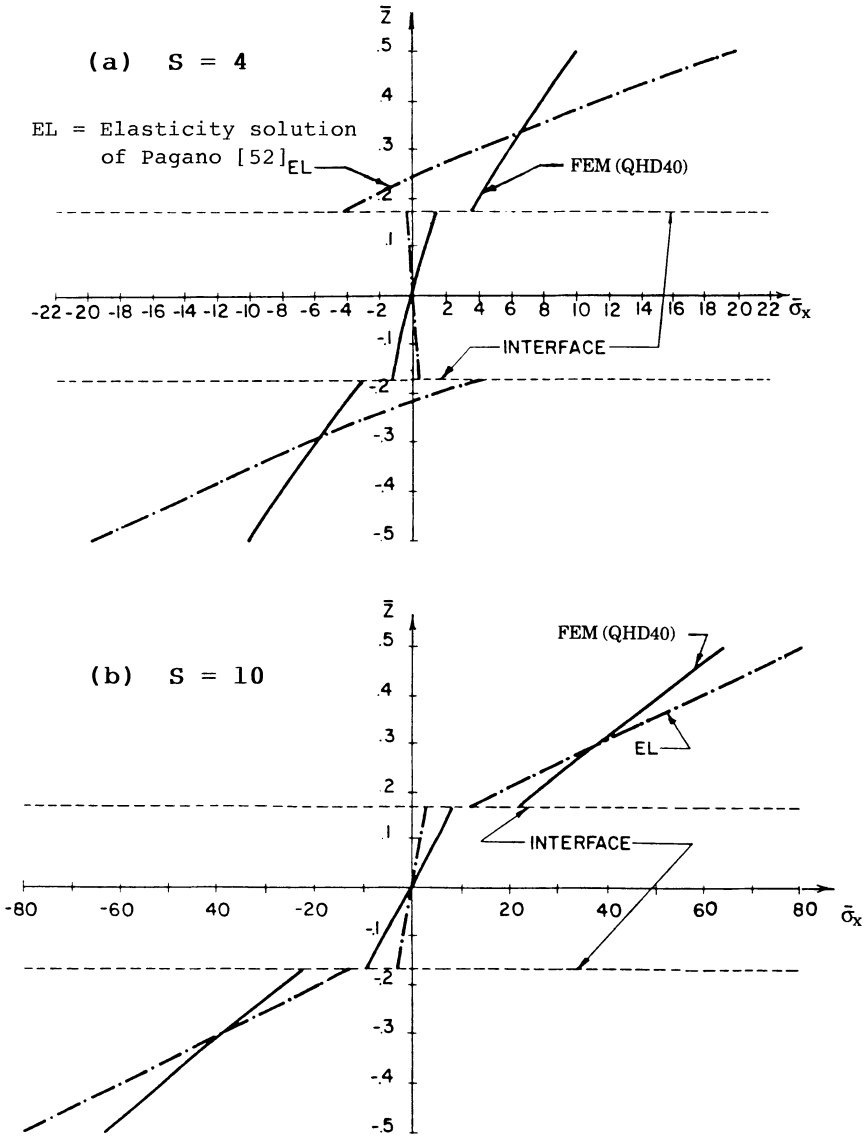


Figure 3.7-5 Normalized inplane stress (σ_x) for cylindrical bending of symmetric cross-ply laminates. (a) $S = 4$. (b) $S = 10$.

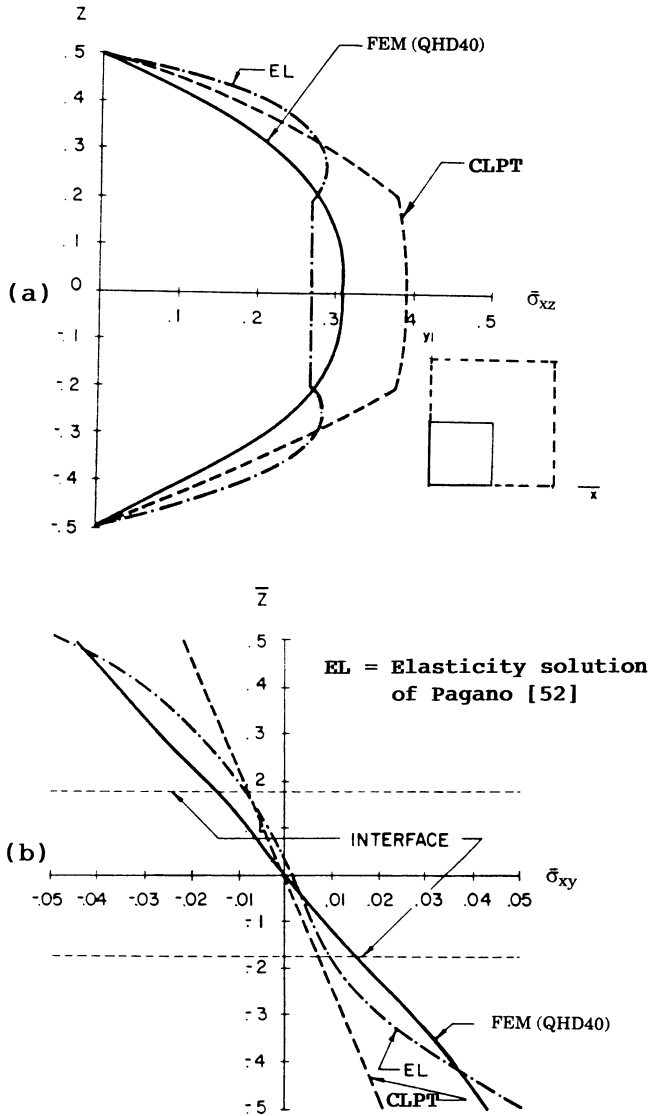


Figure 3.7-6 Normalized transverse and inplane shear stresses ($\bar{\sigma}_{xz}, \bar{\sigma}_{xy}$) of a simply-supported square (0/90/0) laminate. (a) Transverse shear stress, $\bar{\sigma}_{xz}(0, a/2, z)$. (b) Inplane shear stress $\bar{\sigma}_{xy}(0, 0, z)$.

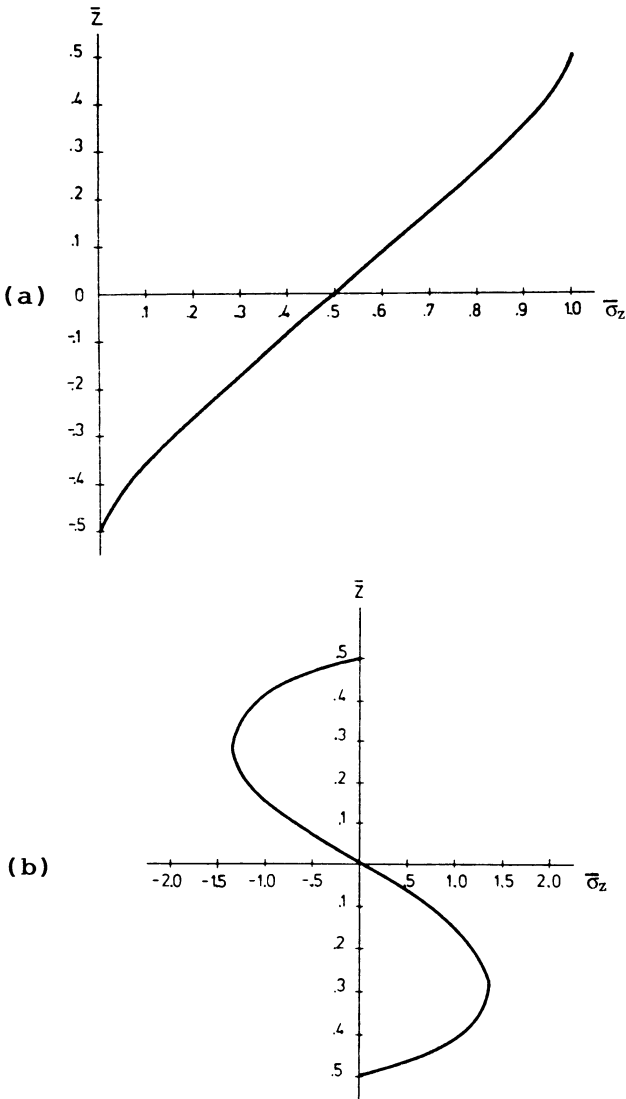
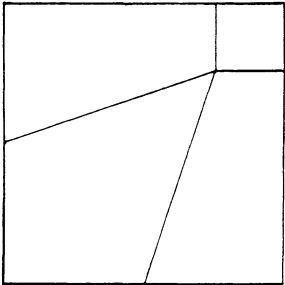


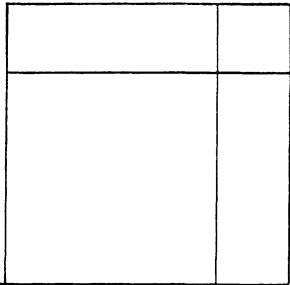
Figure 3.7-7 Normalized transverse normal stress, $\bar{\sigma}_z$, for simply-supported square cross-ply laminate. (a) $\bar{\sigma}_z (\frac{a}{2}, \frac{a}{2}, z)$. (b) $\bar{\sigma}_z (\frac{a}{2}, 0, z)$.

Table 3.7–6 Normalized stresses and displacements for simply-supported square (0/90/0) laminate under transverse sinusoidal load.

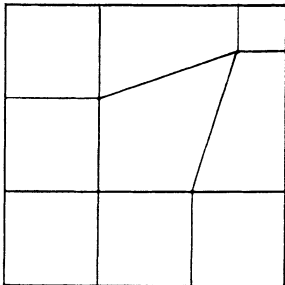
S	Approach	$\bar{\sigma}_x$ $(\frac{a}{2}, \frac{a}{2}, \pm \frac{h}{2})$	$\bar{\sigma}_y$ $(\frac{a}{2}, \frac{a}{2}, \pm \frac{h}{6})$	$\bar{\sigma}_{xy}$ $(0, 0 \pm \frac{h}{2})$	$\bar{\sigma}_{xz}$ $(0, \frac{a}{2}, 0)$	$\bar{\sigma}_{yz}$ $(\frac{a}{2}, 0, 0)$
4	FEM (2 × 2 mesh)	0.414	0.598	0.0465	0.296	0.238
	FEM (6 × 6 mesh)	0.391	0.572	0.0448	0.308	0.251
	Elasticity	0.755	0.556	0.0505	0.282	0.217
10	FEM (2 × 2 mesh)	0.529	0.292	0.0289	0.352	0.123
	FEM (6 × 6 mesh)	0.500	0.279	0.0280	0.369	0.130
	Elasticity	0.590	0.288	0.0289	0.357	0.123
20	FEM (2 × 2 mesh)	0.560	0.206	0.0240	0.367	0.0910
	FEM (6 × 6 mesh)	0.531	0.189	0.0233	0.387	0.0954
	Elasticity	0.552	0.210	0.0234	0.385	0.0938
50	FEM (2 × 2 mesh)	0.567	0.178	0.0226	0.374	0.0812
	FEM (6 × 6 mesh)	0.541	0.164	0.0217	0.392	0.0843
	Elasticity	0.541	0.185	0.0216	0.393	0.0842
100	FEM (2 × 2 mesh)	0.566	0.174	0.0224	0.375	0.0803
	FEM (6 × 6 mesh)	0.542	0.167	0.0215	0.393	0.0827
	Elasticity	0.539	0.181	0.0213	0.395	0.0828
	CLPT	0.539	0.180	0.0213	0.395	0.0823



MESH A



MESH B



MESH C

Figure 3.7–8 FEM meshes of distorted elements.

Table 3.7-7 Normalized stresses and displacements for simply-supported square (0/90/0) laminates with various side-to-thickness ratios (S) and sinusoidal transverseload (distorted meshes).

S	Approach	$\bar{\sigma}_x$ $(\frac{a}{2}, \frac{a}{2}, \pm \frac{h}{2})$	$\bar{\sigma}_y$ $(\frac{a}{2}, \frac{a}{2}, \pm \frac{h}{8})$	$\bar{\sigma}_{xy}$ $(0, 0, \pm \frac{h}{2})$	$\bar{\sigma}_{xz}$ $(0, \frac{a}{2}, 0)$	$\bar{\sigma}_{yz}$ $(\frac{a}{2}, 0, 0)$	$\bar{w}$ $(\frac{a}{2}, \frac{a}{2}, 0)$
4	Mesh A	0.409	0.620	0.0525	0.128	0.122	2.709
	Mesh B	0.393	0.573	0.0485	0.254	0.206	2.473
	Mesh C	0.402	0.596	0.0472	0.290	0.229	2.473
	Elasticity	0.755	0.556	0.0505	0.282	0.217	-
10	Mesh A	0.539	0.292	0.0344	0.141	0.0686	0.866
	Mesh B	0.502	0.279	0.0300	0.293	0.107	0.812
	Mesh C	0.530	0.286	0.0290	0.357	0.114	0.839
	Elasticity	0.590	0.288	0.0289	0.357	0.123	-
20	Mesh A	0.574	0.198	0.0297	0.153	0.0552	0.545
	Mesh B	0.531	0.199	0.0248	0.303	0.0792	0.529
	Mesh C	0.571	0.198	0.0239	0.378	0.0880	0.539
	Elasticity	0.552	0.210	0.0234	0.385	0.0938	-
50	Mesh A	0.585	0.174	0.0300	0.180	0.0516	0.443
	Mesh B	0.533	0.178	0.0226	0.304	0.0701	0.445
	Mesh C	0.587	0.177	0.0221	0.378	0.0055	0.449
	Elasticity	0.541	0.185	0.0216	0.393	0.0842	-
100	Mesh A	0.591	0.190	0.0312	0.200	0.0363	0.418
	Mesh B	0.531	0.176	0.0215	0.302	0.0689	0.432
	Mesh C	0.596	0.180	0.0212	0.359	0.112	0.433
	Elasticity	0.539	0.181	0.0213	0.395	0.0828	-
	CLPT	0.539	0.180	0.0213	0.395	0.0823	-

must be refined enough to approximate equilibrium conditions. This is especially apparent in comparing results obtained for mesh A to those obtained for mesh C. In each of these cases, elements having a taper ratio of 2 to 1 are used. Mesh C gives significantly improved transverse stresses, because the mesh is fine enough to represent the load distribution as well as the kinematics of deformation accurately.

Table 3.7-8 Normalized stresses and displacements for simply supported rectangular (0/90/0) laminates with various side-to-thickness ratios (S) and sinusoidal transverse load.

S	Approach	$\bar{\sigma}_x$ $(\frac{a}{2}, \frac{a}{2}, \pm \frac{h}{2})$	$\bar{\sigma}_y$ $(\frac{a}{2}, \frac{a}{2}, \pm \frac{h}{6})$	$\bar{\sigma}_{xy}$ $(0, 0, \pm \frac{h}{2})$	$\bar{\sigma}_{xz}$ $(0, \frac{a}{2}, 0)$	$\bar{\sigma}_{yz}$ $(\frac{a}{2}, 0, 0)$	$\bar{w}$ $(\frac{a}{2}, \frac{a}{2}, 0)$
4	FEM (2 × 2 mesh)	0.657	0.130	0.0296	0.410	0.0357	3.55
	FEM (6 × 6 mesh)	0.612	0.126	0.0284	0.431	0.0391	3.58
	FEM (Reddy)	-	-	-	-	-	-
	Elasticity	0.726	0.119	0.0281	0.420	0.0334	2.82
10	FEM (2 × 2 mesh)	0.669	0.0431	0.0125	0.414	0.0143	0.995
	FEM (6 × 6 mesh)	0.625	0.0421	0.0121	0.436	0.0160	1.00
	FEM (Reddy)	0.603	0.0364	0.0102	-	-	0.802
	Elasticity	0.725	0.0435	0.0123	0.420	0.0152	0.919
20	FEM (2 × 2 mesh)	0.666	0.0286	0.00959	0.415	0.0112	0.625
	FEM (6 × 6 mesh)	0.628	0.0278	0.00928	0.437	0.0121	0.629
	FEM (Reddy)	0.605	0.0276	0.0086	-	-	0.578
	Elasticity	0.650	0.0299	0.0093	0.434	0.0119	0.610
50	FEM (2 × 2 mesh)	0.659	0.0252	0.00878	0.416	0.0113	0.521
	FEM (6 × 6 mesh)	0.629	0.0237	0.00848	0.437	0.0110	0.524
	FEM (Reddy)	0.604	0.0251	0.0081	-	-	0.515
	Elasticity	0.628	0.0259	0.0084	0.439	0.0110	0.520
100	FEM (2 × 2 mesh)	0.657	0.0259	0.00856	0.416	0.0127	0.507
	FEM (6 × 6 mesh)	0.628	0.0231	0.00837	0.437	0.0108	0.509
	FEM (Reddy)	0.603	0.0253	0.0080	-	-	0.506
	Elasticity	0.624	0.0253	0.0083	0.439	0.0108	0.508
	CLPT	0.623	0.0252	0.0083	0.440	0.0108	0.503

Transverse shear stress variation σ_{yz} with respect to z at the point $(x, y) = (a/2, 0)$ is presented in Figure 3.7-9 for a rectangular laminate ($a \times b$) with $b = 3a$ and $S = 4$. A full range of results is presented in Table 3.7-8. The present method is compared to those obtained using elasticity solution [53], and to those obtained by Reddy [10] using the first-order shear deformation bending element (see Sec. 3.4). Correlation with elasticity is quite good, particularly for ratios of $S \geq 10$.

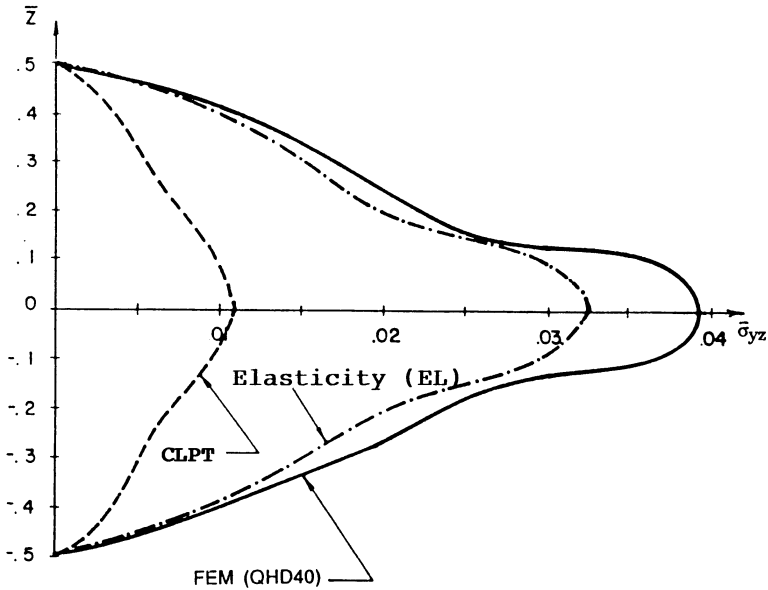


Figure 3.7-9 Normalized transverse shear stress, $\sigma_{yz}(\frac{a}{2}, 0, z)$, for simply-supported cross-ply (0/90/0) rectangular laminate ($b = 3a, S = 4$).

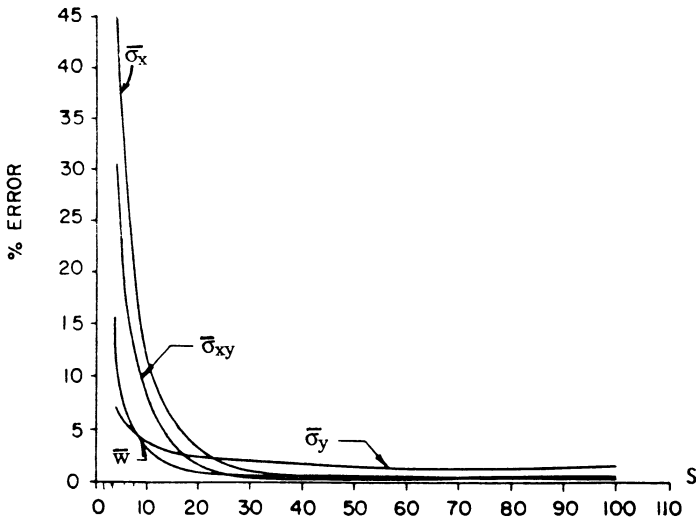


Figure 3.7-10 Error in normalized transverse displacement and inplane stresses vs. side-to-thickness ratio for simply-supported cross-ply (0/90/0) laminates.

As a final consideration, solutions have been obtained for the present geometry using reduced integration and full integration. The results are compared in Figures 3.7-10 and 3.7-11 by giving percentage error in calculated values of displacements and stresses vs. side-to-thickness ratio, S . It is apparent that element locking is not a problem; however, reduced integration is needed to minimize errors in calculated transverse stresses.

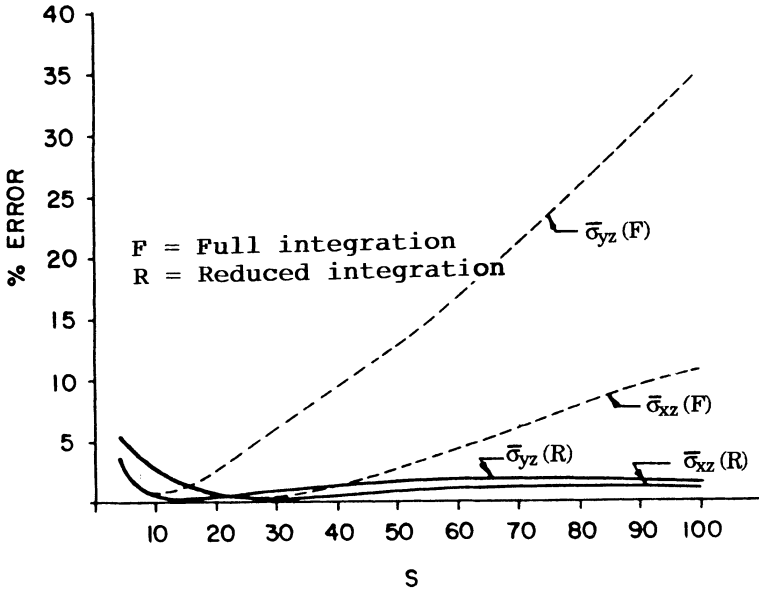


Figure 3.7-11 Error in normalized transverse shear stresses vs. side-to-thickness ratio for simply-supported cross-ply laminates.

An example with layerwise elements

Consider a simply supported, square, cross-ply (0/90/0) laminated plate, subjected to sinusoidally distributed transverse load. For this particular problem, exact solutions exist for the following three theories: 1) 3-D elasticity theory [52], 2) classical laminate theory (denoted CLPT), and 3) first order shear deformation theory (denoted FSDT) [2]. The exact solutions from these three theories will be compared with the layerwise finite element solution of this problem (see [36]). A thick plate ($L/H = 4$) is modeled using two finite element meshes that differ only in the refinement through the thickness. Full integration is used for both meshes (see Fig. 3.7-12):

- Mesh 1: A 2×2 uniform 2-D mesh of eight-node quadratic elements in the xy -plane and 3 quadratic line elements through the thickness (one quadratic subdivision for each material layer); a total of 441 global degrees of freedom.
- Mesh 2: A 2×2 uniform 2-D mesh of eight-node quadratic elements in the xy -plane and six quadratic line elements through the thickness (two quadratic subdivisions for each material layer); a total of 669 global degrees of freedom. .

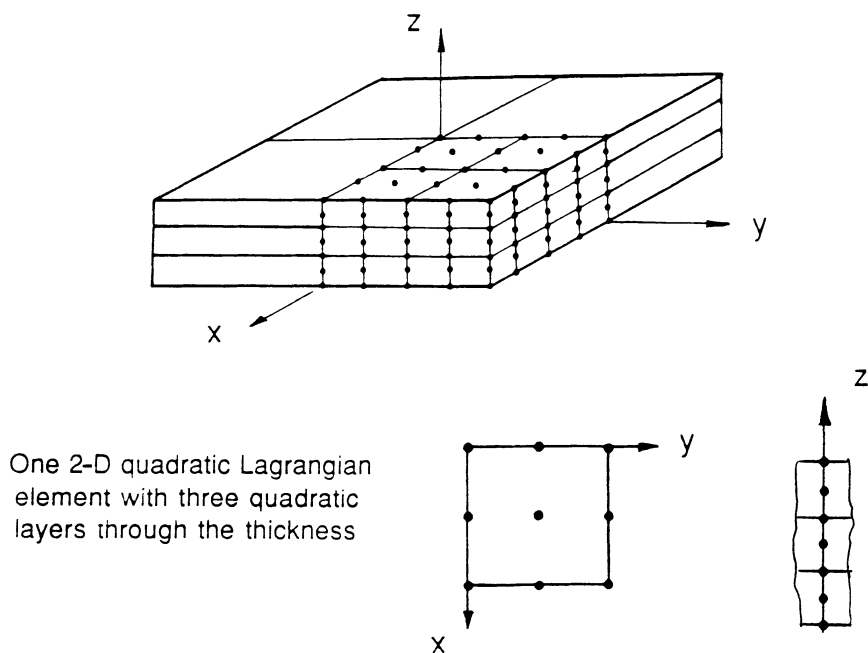


Figure 3.7-12 Finite element meshes of layerwise elements used for the analysis of a simply-supported, square, cross-ply ($0/90/0$) laminate under sinusoidal transverse load. The finite element mesh shown here is Mesh 1; Mesh 2 differs from Mesh 1 in only in doubling the number of through-thickness elements.

The stresses are computed via the constitutive relations at the reduced Gauss points for each finite element. The reduced Gauss points correspond to the $2 \times 2 \times 2$ integration points within the domain of each layer of each element. Figures 3.7-13 through 3.7-16 show the distribution of various stresses ($\bar{\sigma}_x, \bar{\sigma}_y, \bar{\sigma}_{xz}, \bar{\sigma}_{yz}$) through

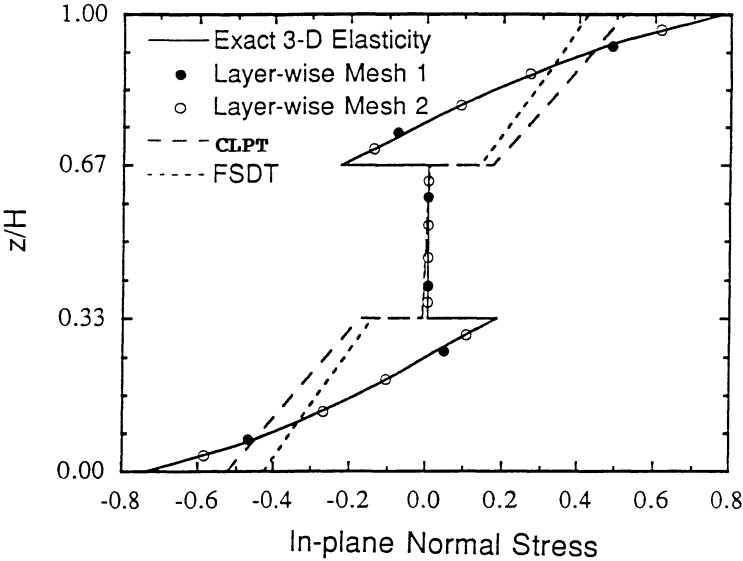


Figure 3.7-13 Distribution of inplane normal stress ($\bar{\sigma}_x$) through the thickness of a simply-supported, square (0/90/0) laminate under sinusoidal transverse load ($L/H = 4$).

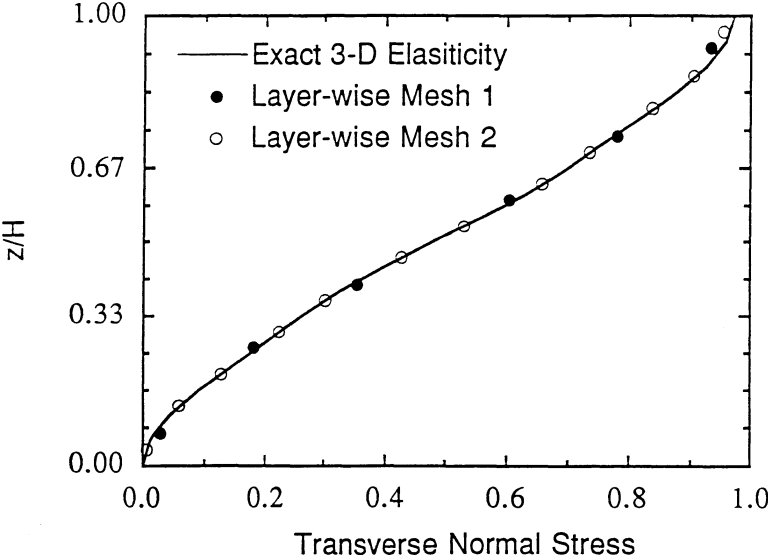


Figure 3.7-14 Distribution of transverse normal stress (σ_z) through the thickness of a simply-supported, square (0/90/0) laminate under sinusoidal transverse load ($L/H = 4$).

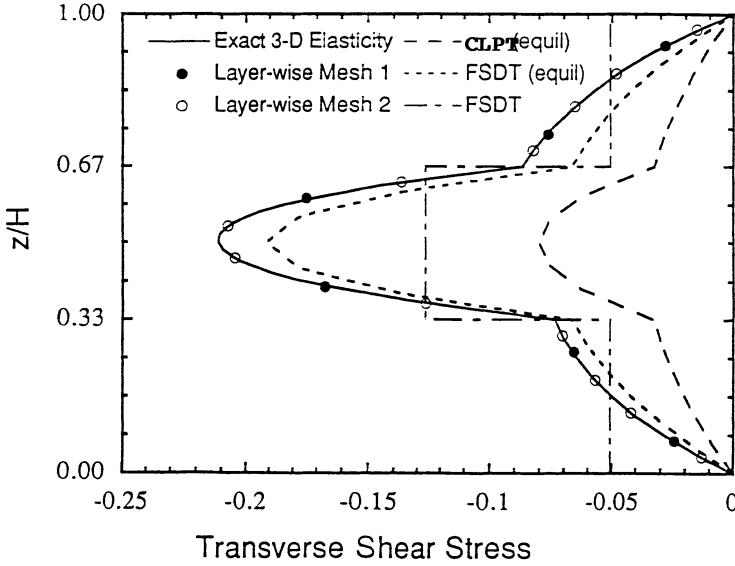


Figure 3.7-15 Distribution of transverse shear stress (σ_{yz}) through the thickness of a simply-supported, square (0/90/0) laminate under sinusoidal transverse load ($L/H = 4$).

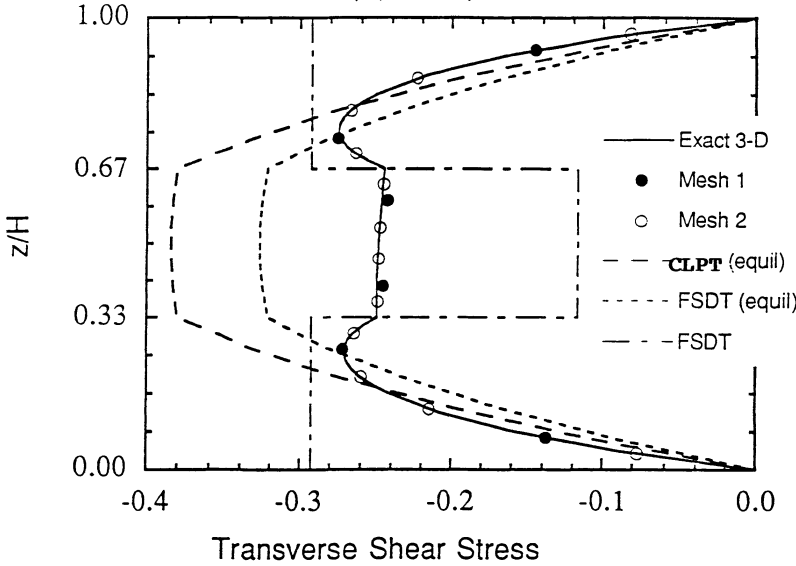


Figure 3.7-16 Distribution of transverse shear stress (σ_{xz}) through the thickness of a simply-supported, square (0/90/0) laminate under sinusoidal transverse load ($L/H = 4$).

the thickness of the plate at the reduced Gauss points closest to the position where each stress is maximum. Thus for mesh 1, the stresses are computed at six different points through the thickness, while for mesh 2, the stresses are computed at 12 different points through the thickness. The nondimensionalized stresses, along with the corresponding reduced Gauss points, are given by

$$\begin{aligned}\bar{\sigma}_x &= \sigma_x(A, A, z) \frac{H^2}{q_0 L^2}, \quad \bar{\sigma}_z = \sigma_z(A, A, z) \frac{1}{q_0} \\ \bar{\sigma}_{yz} &= \sigma_{yz}(A, B, z) \frac{H}{q_0 L}, \quad \bar{\sigma}_{xz} = \sigma_{xz}(B, A, z) \frac{H}{q_0 L} \\ A &= 1.105662\left(\frac{L}{2}\right), \quad B = 1.894338\left(\frac{L}{2}\right)\end{aligned}\quad (3.7-5)$$

where H is the total thickness and L is the length of the laminate, and q_0 is the intensity of the sinusoidally distributed transverse load. The material properties used here are the same as those listed in Eq. (3.7-4):

$$\frac{E_1}{E_2} = 25, \quad E_3 = E_2, \quad \frac{G_{12}}{G_{23}} = 2.5, \quad G_{13} = G_{12}; \quad \nu_{12} = \nu_{23} = \nu_{13} = 0.25$$

Excellent agreement between the layerwise finite element solution and the exact three-dimensional elasticity solution is found as shown in Figures 3.7-13 through 3.7-16. All stress distributions predicted by the single-layer theories (CLPT and FSDT) show considerable error for this thick plate. Note that the transverse shear stresses in CLPT and FSDT are computed via integration of the equilibrium equations. The equilibrium shear stresses can also be computed for the layerwise model, thus yielding continuous, layerwise smooth shear stresses using relatively coarse refinement through the thickness; however, only the constitutive transverse shear stresses are shown here.

3.7-3 Geometric nonlinear analyses

Examples with the laminate elements

Unlike isotropic metallic plates, composite plates exhibit quite different nonlinear behavior. For example, the geometric nonlinear effects could be very significant even at small loads and deflections, depending on the lamination scheme and boundary conditions (see [53,35]). To illustrate the point we analyze an anti-symmetric cross-ply square laminate ($90^\circ/0^\circ$) with two opposite edges pinned or hinged and the other two edges free, and subjected to uniformly distributed transverse load. The geometry and boundary conditions for the pinned and hinged cases are shown in Fig. 3.7-17. The material properties used are the same as those used by Sun and Chin [53]:

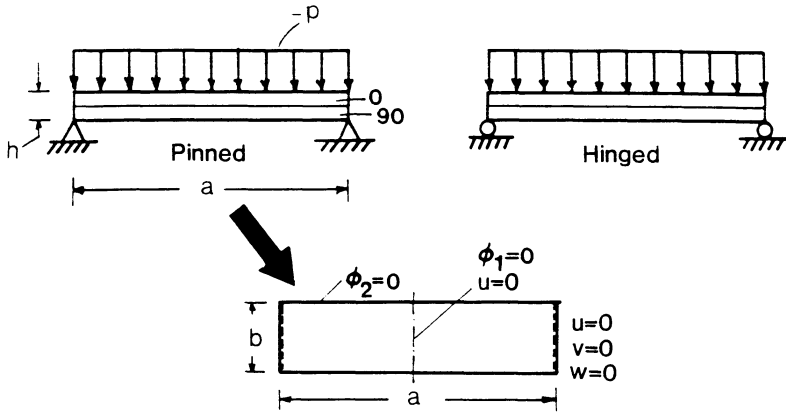


Figure 3.7-17 Geometry and boundary conditions for a (90/0) laminated plate strip under uniformly distributed transverse load.

$$E_1 = 20 \text{ Msi}, E_2 = 1.4 \text{ Msi}, \nu_{12} = 0.3$$

$$G_{12} = G_{13} = G_{23} = 0.7 \text{ Msi}$$

$$a = 9 \text{ in.}, b = 1.5 \text{ in.}, h = 0.4 \text{ in.} \quad (3.7 - 6)$$

The results of the nonlinear bending analysis using the first-order shear deformable elements are presented in Table 3.7-9. For hinged cases, the plate strip is essentially in pure bending and the axial force N_1 is approximately equal to zero. Therefore, the solution is independent of the sign of the applied load. For a pinned plate strip, the axial force N_1 is not zero; it is given by

$$N_1 = A_{11} \left[\frac{du}{dx} + \frac{1}{2} \left(\frac{dw}{dx} \right)^2 \right] + B_{11} \frac{d\phi_1}{dx}, B_{11} < 0 \quad (3.7 - 7)$$

For small values of the positive load, the expression containing A_{11} coefficient is small compared to the expression containing B_{11} coefficient, which is negative for $0 < x < 4.5$. Hence, N_1 is compressive and increases the transverse deflection — analogous to the transverse deflection of a plate strip under an axial compressive load and a transverse load. Thus, the nonlinear solution is larger than the linear solution for small values of the load. As the load is increased, the A_{11} -expression

becomes larger than the B_{11} -expression, and N_1 becomes positive. This stiffens the structure and the nonlinear solution becomes smaller than the linear solution. The load deflection curves for the first few load steps are shown in Fig. 3.7-18. For a negative load, $B_{11} \frac{d\phi_1}{dx}$ is positive, and the two terms in N_1 add; this yields a larger axial force and therefore a stiffer structure than for the positive load case. Therefore the deflection is lower than that for the case of positive load.

Table 3.7-9 Nonlinear cylindrical bending of a (90/0) laminate under uniformly distributed transverse load.

$\bar{w} = w/h$					
Load P_o	Pinned Linear*	Pinned Nonlinear*	Pinned Nonlinear‡	Hinged	
				Case 1*	Case 2‡
0.005	-0.235	-0.159	0.475	-0.429	0.429
0.01	-0.470	-0.255	0.673	-0.858	0.858
0.02	-0.940	-0.386	0.847	-1.710	1.710
0.03	-1.41	-0.480	0.954	-2.550	2.550
0.04	-1.88	-0.555	1.034	-3.370	3.370
0.05	-2.35	-0.618	1.100	-4.190	4.190
0.10	-4.70	-0.845	1.327	-7.920	7.920
0.25	-11.75	-1.233	1.705	-16.16	16.16
0.50	-23.50	-1.609	2.075	-24.82	24.82
0.75	-35.25	-1.870	2.332	-30.87	30.87
1.0	-47.00	-2.078	2.532	-35.69	35.69
2.0	-94.00	-2.665	3.117	-49.56	49.56
3.0	-141.00	-3.075	3.525	-59.65	59.65
4.0	-188.00	-3.402	3.850	-68.00	68.00
5.0	-235.00	-3.675	4.125	-75.33	75.33

*For negative load values; ‡ For positive load values.

The behavior described above does not occur for simply supported (on all edges) laminates with SS-1 or SS-2 boundary conditions or for laminates without bending-stretching coupling.

As noted earlier, the SS-2 boundary conditions must be used to analyze anti-symmetric angle-ply laminates. The quarter plate symmetry associated with SS-2 holds for linear analysis but does not hold for geometrically nonlinear analysis. To illustrate this point, a two-layer (45/ - 45) square plate ($a = 1000$ mm, $h = 2$

mm), with uniform transverse load and the following layer properties is analyzed:

$$\begin{aligned} E_1 &= 250 \text{ GPa}, E_2 = 20 \text{ GPa}, G_{12} = G_{13} = 10 \text{ GPa}, \\ G_{23} &= 4 \text{ GPa}, \nu_{12} = \nu_{13} = \nu_{23} = 0.25 \end{aligned} \quad (3.7-8)$$

The load-deflection curves obtained from the quarter-plate and full-plate analyses are shown in Fig. 3.7-19. Meshes of 2×2 and 4×4 nine-node quadratic elements based on the first-order shear deformation plate theory are used to model the quarter and full plates, respectively. It is clear from the results that the use of quarter plate model with the symmetry conditions of SS-2 (see Fig. 3.7-1) yields larger deflections than those obtained from the full plate model. The discrepancy increases with the intensity of the transverse load. This discrepancy can be explained (see [53]) in the light of the symmetry conditions used to model the quarter plate. As noted earlier, the symmetry conditions are derived from the Navier solution for the linear theory. For the angle-ply case, the assumed solution is of the form

$$\begin{aligned} u &= U \sin \frac{\pi x}{a} \cos \frac{\pi y}{b} \\ v &= V \cos \frac{\pi x}{a} \sin \frac{\pi y}{b} \\ w &= W \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \\ \phi_1 &= S^1 \cos \frac{\pi x}{a} \sin \frac{\pi y}{b} \\ \phi_2 &= S^2 \sin \frac{\pi x}{a} \cos \frac{\pi y}{b}, \end{aligned} \quad (3.7-9)$$

where the coordinate system is fixed at the lower left corner of the laminate (see Fig. 3.7-1). The solution in Eq. (3.7-9) implies the following conditions along the symmetry lines (*i.e.*, $x = \frac{a}{2}$, $y = \frac{b}{2}$) for the linear problem:

$$v(a/2, y) = \phi_1(a/2, y) = 0 ; N_1(a/2, y) = M_2(a/2, y) = 0 \quad (3.7-10a)$$

$$u(x, b/2) = \phi_2(x, b/2) = 0 ; N_2(x, b/2) = M_1(x, b/2) = 0 \quad (3.7-10b)$$

However, the boundary conditions on N_1 and N_2 for the nonlinear theory indicate that

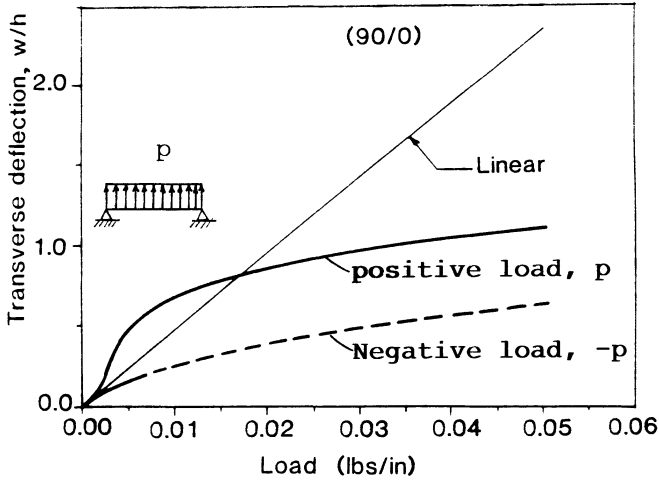


Figure 3.7-18 Load-deflection curves for the bending of an antisymmetric cross-ply laminated plate strip with pinned edges.

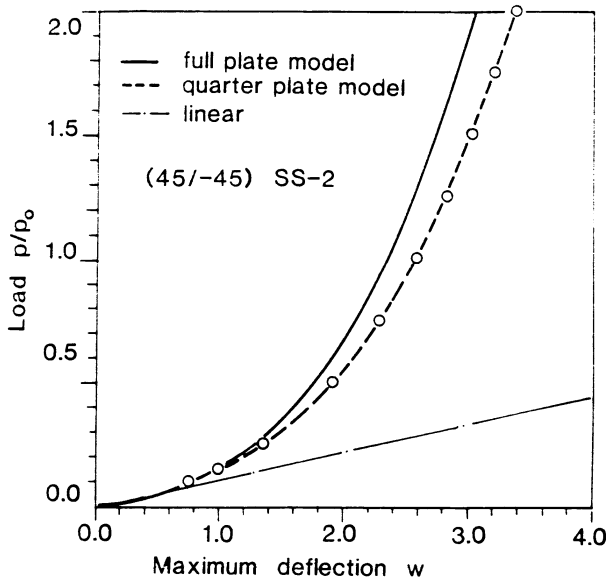


Figure 3.7-19 Effect of full and quarter plate modeling on the nonlinear response of a simply-supported square laminate under uniformly distributed transverse load.

$$\begin{aligned}
 N_1(a/2, y) &= A_{12} \frac{\pi^2 W^2}{2b^2} \cos^2(y\pi/b) \neq 0 \\
 N_2(x, b/2) &= A_{12} \frac{\pi^2 W^2}{2a^2} \cos^2(x\pi/a) \neq 0
 \end{aligned}
 \tag{3.7 - 11}$$

When a quarter plate model is used without specifying N_1 and N_2 on the $x = \frac{a}{2}$ and $y = \frac{b}{2}$ lines, it is implied in the finite-element analysis that they are (being the natural boundary conditions) specified to be zero because $u(\frac{a}{2}, y)$ and $v(x, \frac{b}{2})$ are not specified. The quarter plate with zero inplane forces N_1 and N_2 is more flexible than the one with nonzero inplane forces N_1 and N_2 . It should be noted that this phenomenon is not observed in cross-ply laminates, *i.e.*, the quarter and full plate models give the same results for the cross-ply laminates (with SS-1 boundary conditions).

In the above examples, we considered the nonlinear bending of rectangular laminated plates with two opposite edges supported and other two edges free. The next two examples deal with the nonlinear bending of square plates with all four edges either simply supported or clamped.

Figures 3.7-20 and 3.7-21 contain the load-deflection curves obtained by applying the nonlinear laminate element to a simply-supported orthotropic plate and a clamped cross-ply (0/90/90/0) laminate, respectively. In both plates a uniform transverse load is used. The geometry and material properties used are given below.

Orthotropic plate:

$$\begin{aligned}
 a &= b = 12 \text{ in.}, \quad h = 0.138 \text{ in.} \\
 E_1 &= 3 \times 10^6 \text{ psi}, \quad E_2 = 1.28 \times 10^6 \text{ psi}, \\
 G_{12} &= G_{23} = G_{13} = 0.37 \times 10^6 \text{ psi}, \\
 \nu_{12} &= \nu_{23} = \nu_{13} = 0.32
 \end{aligned}
 \tag{3.7 - 12}$$

Cross-ply laminate:

$$\begin{aligned}
 a &= b = 12 \text{ in.}, \quad h = 0.096 \text{ in.} \\
 E_1 &= 1.8282 \times 10^6 \text{ psi}, \quad E_2 = 1.8315 \times 10^6 \text{ psi}, \\
 G_{12} &= G_{13} = G_{23} = 0.3125 \times 10^6 \text{ psi}, \\
 \nu_{12} &= \nu_{23} = \nu_{13} = 0.2395.
 \end{aligned}
 \tag{3.7 - 13}$$

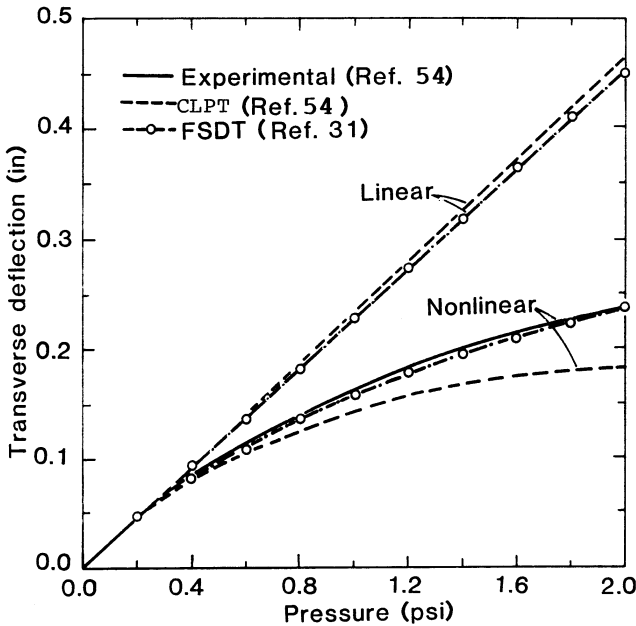


Figure 3.7-20 Load-deflection curves for the bending of a simply-supported, square, orthotropic plate under uniform transverse load.

For the simply supported orthotropic plate, the shear deformation element yields results that are in excellent agreement with the experimental data of Zaghoul and Kennedy [54], while the classical plate bending element underpredicts deflections even for this thin plate (S is about 120). For the clamped, cross-ply laminate, the finite element results are not in as good agreement with the experimental data. This is attributed to inaccuracies in the representation of the material stiffnesses as well as the actual boundary conditions of the experiment.

Examples with continuum elements

The isotropic spherical shell panel shown in Fig. 3.7-22 was analyzed for its large displacement response with four 9-node elements and sixteen 4-node elements, in one-quarter of the shell (see [47]). The following geometric and material parameters are used:

$$R = 100 \text{ in.}, a = 30.9017 \text{ in.}, h = 3.9154 \text{ in.}$$

$$E = 10^4 \text{ psi}, \nu = 0.3. \quad (3.7 - 14)$$

The boundary conditions used are:

$$u = \phi_1 = 0 \text{ on } x_1 = 0 ; u = v = w = \phi_2 = 0 \text{ on } x_1 = a \quad (3.7 - 15a)$$

$$v = \phi_2 = 0 \text{ on } x_2 = 0 ; u = v = w = \phi_1 = 0 \text{ on } x_2 = a \quad (3.7 - 15b)$$

Figure 3.7-22 contains plots of the load-deflection curves obtained using the shell element. The modified Riks method, which automatically determines the load increments, is used. Figure 3.7-22 also includes the results of Bathe and Ho [53].

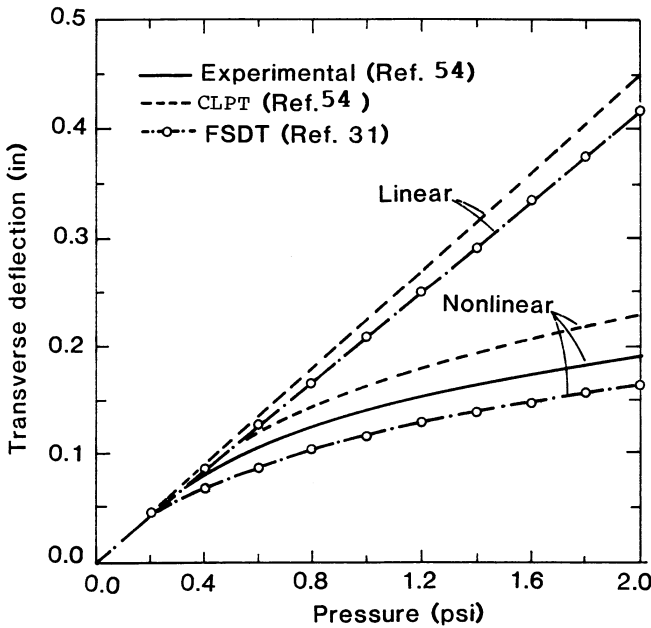


Figure 3.7-21 Load-deflection curves for the bending of a clamped, square, symmetric cross-ply (0/90/90/0) plate under uniform transverse load.

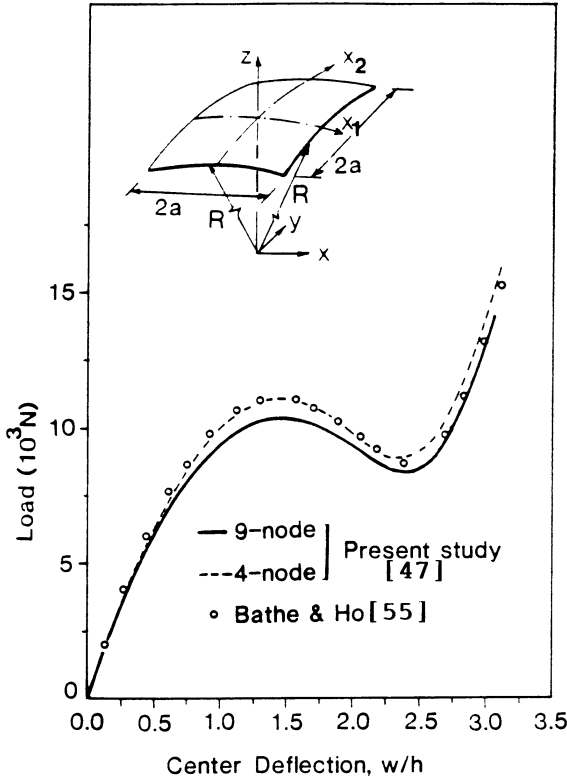


Figure 3.7-22 Load-deflection curve for the bending of a simply-supported, isotropic, spherical shell panel subjected to a point load at the center.

Next, we consider a two-layer ($-45/45^\circ$) laminated spherical shell panel, simply supported at its edges and subjected to uniform internal pressure (see [47]). The material and geometric parameters used are:

$$E_1/E_2 = 25, \quad E_2 = 10^6 \text{ psi}, \quad \nu_{12} = 0.25, \quad G_{12}/E_2 = 0.5,$$

$$G_{13} = G_{12}, \quad G_{23}/E_2 = 0.2, \quad \nu_{13} = \nu_{23} = \nu_{12}$$

$$R = 1000 \text{ in.}, \quad h = 1 \text{ in.}, \quad a = 50 \text{ in.} \quad (3.7 - 16)$$

One quarter of the shell panel was modelled with four 9-node shell elements. The geometric nonlinear response of the shell panel is shown in Fig. 3.7-23.

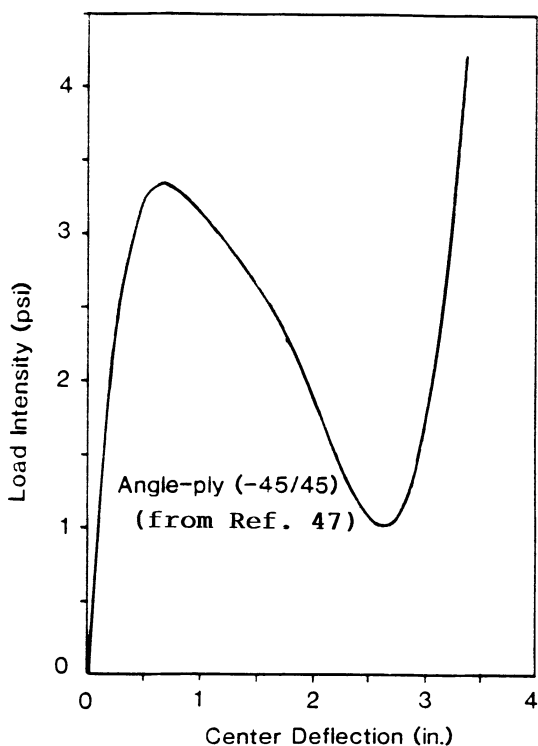


Figure 3.7-23 Load-deflection curve for the bending of a simply-supported, angle-ply ($-45/45$) spherical shell panel under transverse pressure load.

3.7-4 Buckling analysis

Another interesting characteristic of composite laminates is their behavior under compressive loads. Most often the critical buckling loads are determined through an eigenvalue analysis. The critical buckling loads can also be determined from geometric nonlinear analysis, where the critical buckling load is taken to be the limit load.

The effect of orthotropy and number of layers on the nondimensional buckling loads of simply-supported square laminates ($0^\circ/90^\circ/0^\circ/\dots$) is investigated using the finite element model based on the first-order theory (FSDT). The results are

compared in Table 3.7–10 with the analytical solutions [4]. The following notation is used: S = simply supported; F = free; C = clamped. The first two letters in the label denote the boundary conditions on the edges, $y = 0, b$, the third letter on the edge $x = a/2$ and the fourth on the edge $x = -a/2$. The agreement between the analytical and finite element results is excellent.

Table 3.7–10 The effect of side to thickness ratio on the dimensionless critical buckling loads ($\bar{N}_2 = N_2 b^2 / (E_2 h^3, N_1 = N_6 = 0)$) of antisymmetric cross-ply square laminates including various boundary conditions.

No. of layers	b/h	Theory	SSFF	SSFS	SSFC	SSSS	SSSC	SSCC
2	5	FSDT	3.719	4.094	4.667	8.328	9.650	9.949
		CLPT	5.616	6.292	7.203	14.520	23.869	37.106
	10	FSDT	4.916	5.420	6.234	11.485	18.338	21.916
		CLPT	5.616	6.292	7.203	14.520	23.869	37.106
10	5	FSDT	6.791	7.064	8.174	11.172	11.181	11.216
		CLPT	16.457	17.141	19.422	36.384	60.406	90.833
	10	FSDT	12.226	12.661	14.480	25.647	33.970	36.129
		CLPT	16.457	17.141	19.422	36.384	60.406	90.833

The same geometry as in the example of the nonlinear analysis (Sec. 3.7–3) ($a = b = 1000$ mm, $h = 2$ mm) is used to analyze a simply-supported laminate (45/–45) under uniform inplane load $N_2 = 10.85 \lambda$ (N/m), where λ is the nondimensional load. Both the eigenvalue and nonlinear equilibrium approaches are used. The material properties used are:

$$\begin{aligned}
 E_1/E_2 &= 40 ; E_2 = 6.25 \text{ GPa} ; G_{12}/E_2 = 0.82 ; G_{13} = G_{12} ; \\
 G_{23}/E_2 &= 0.52 ; \nu_{12} = \nu_{13} = 0.24 ; \nu_{23} = 0.49
 \end{aligned}
 \tag{3.7 – 17}$$

Although a closed-form solution of the buckling equations exists [4] for this case, the SS-2 boundary conditions [see Eq. (3.7–10)] used to obtain that solution cannot be used for the nonlinear analysis without modification. For the nonlinear

analysis, the boundary conditions have to allow for the applied loads, $N_1 = 0$ and $N_2 = N$. Both the eigenvalue analysis and nonlinear analysis are performed using 4×4 mesh of nine node elements in full plate models, and the results are presented in Fig. 3.7-24. In the full plate model, u and v must be constrained at two points to remove the rigid body modes in the nonlinear analysis. The critical buckling loads predicted by both the nonlinear and eigenvalue analyses agree with each other.

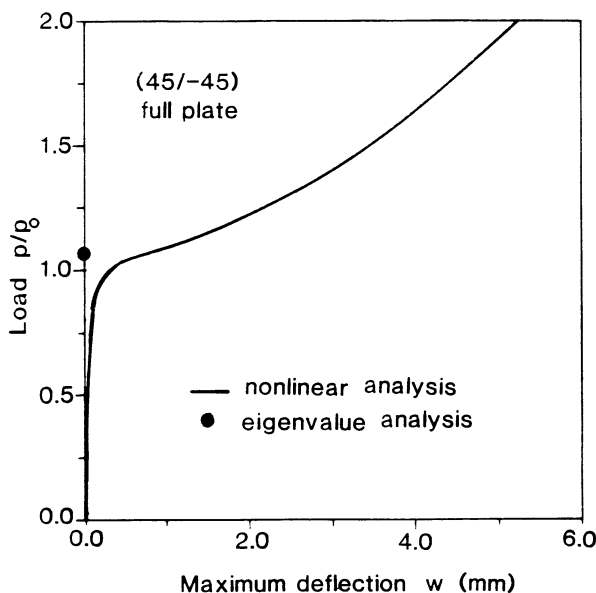


Figure 3.7-24 The post-buckling response of a simply-supported (SS-2) square, antisymmetric angle-ply (-45/45) laminate under uniform in-plane force, $p = N_2$.

The results of a nonlinear analysis of anti-symmetric cross-ply (0/90/...) laminated plates under inplane load $N_2 = 6.25 \lambda$ (N/m) are shown in Fig. 3.7-25. The geometry and material properties used are the same as in the previous example. The simply-supported boundary conditions of SS-1 type are used with a 2×2 mesh of nine-node elements in a quarter plate. In this case, the full plate and quarter plate models give the same results. In order to assess the effect of number of layers, four laminates are analyzed. The values of the critical buckling load given by the closed-form solutions based on the eigenvalue analysis are shown on the

load-deflection curve for comparison. It is clear that in this case the eigenvalues are not representative of the critical buckling load; in fact, there are no apparent limit points for the laminates analyzed.

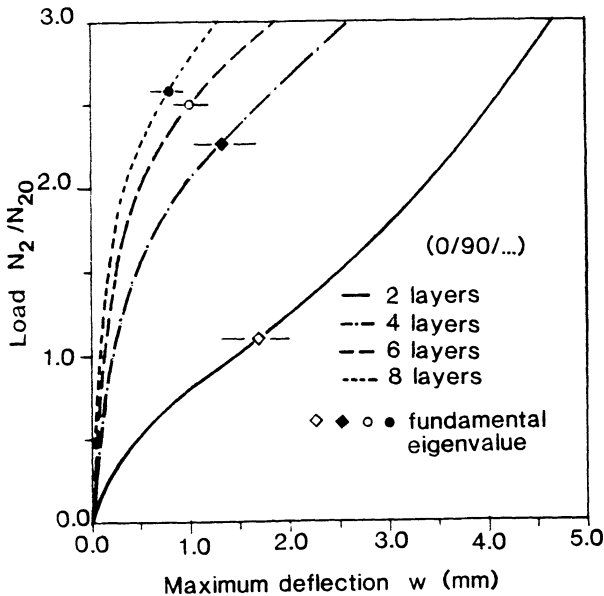


Figure 3.7-25 Load-deflection curves for the bending of a simply-supported, square, antisymmetric cross-ply laminated plate under uniform compressive inplane force, N_2 .

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Chapter 4

Failure: Damage Initiation and Progression

4.1 Introduction

Under service conditions, laminated composite structures develop matrix cracks, fiber-matrix debonds, fiber fractures, and delaminations. These effects, which cause permanent loss of integrity within the laminate, are termed *damage*, and they result in the loss of stiffness and strength of the material. As a result, the load-carrying capacity and the service life of the structure is reduced. When a structure or a component ceases to carry out its intended function, it is said to have ‘failed’. For example, the microcracks observed within a layer constitute damage. As these microcracks grow in size and number, they coalesce and develop into debonds, resulting in a reduction of the load-carrying capacity of the laminate. In order to determine the load carrying capacity and service life of a composite structure, it is necessary to predict the initiation and evolution of damage.

In the last decade, considerable progress has been made in understanding the initiation and evolution of various damage modes. But this understanding has been limited to simple geometry and loading conditions. Besides, much of the research has been directed towards understanding the effect of an isolated damage mode such as matrix cracking or delamination on the stiffness, strength and life of a laminate. A brief review of these developments is presented here.

Masters and Reifsnider [1] and Highsmith and Reifsnider [2] performed both experimental and analytical work to understand the effect of matrix cracking on the stiffness reduction of composite laminates. They developed a one dimensional shear-lag model, which as a first step determines the inplane stress normal to the cracked surface in the cracked ply of a laminate. Then, the reduction in the load carrying capacity of the cracked ply is computed and the elastic modulus of the cracked ply is reduced correspondingly. The classical laminate analysis is used to calculate overall stiffness properties of the laminate.

Talreja [3,4] used a continuum approach in which the cracks are considered as microstructural entities with tensorial characteristics. He developed a set of four equations for the four inplane stiffness coefficients, i.e., E_1 , E_2 , G_{12} , ν_{12} (after matrix cracking), in terms of the stiffness coefficients of the un-cracked lamina and four unknown constants. The unknown constants are determined experimentally.

Hashin [5] used a variational approach to predict the stiffness reduction due to matrix cracking in a symmetric cross-ply laminate under tensile and inplane shear loading. Admissible stress systems which satisfy equilibrium and all boundary and interface conditions are constructed, and the principle of minimum complementary energy is employed to find an optimal approximation. The stiffness reduction is computed from the two dimensional stress fields obtained by this procedure.

Daniel et al. [6,7] investigated the progressive matrix cracking of cross ply composite laminates both analytically and experimentally. They developed closed form solutions for the matrix crack density, stress distributions and reduced stiffnesses of damaged plies, as well as of the entire laminate, as a function of applied load and lamina properties.

The effect of delamination on stiffness and strength is not as well understood as the effect of matrix cracking. Most of the effort in the area of delamination is directed towards predicting the initiation and growth of delamination. O'Brien [8,9] analyzed local delaminations and studied their influence on tensile stiffness of composite laminates. He also studied [10] the effect of delamination on the strength of unnotched quasi-isotropic graphite/epoxy laminates. Chan and Ochoa [11,12] presented a two-dimensional finite element model to calculate interlaminar stresses and strain energy release rates for the study of free-edge delamination in composite laminates subjected to tension, bending and torsional loads. Reifsnider et al. [13] studied the relationship of delamination to other damage modes such as matrix cracking and fiber breakage in quasi-isotropic graphite/epoxy laminates. They found that delamination usually nucleates from the tip of a transverse crack and spreads along the length in both directions to join an adjacent crack.

Most of the above analyses are based on a two-dimensional state of stress. However, damage initiation and evolution in composite laminates is governed by a three dimensional state of stress and complex interaction of various damage modes. Hence, in order to predict the failure behavior of composite laminates accurately, we require a failure analysis technique that can capture both inplane and interlaminar stress components and allow for the interaction of various damage modes.

However, the present day analytical techniques cannot handle the combined effect of various damage modes acting concurrently on composite structures with complex configuration subjected to combined loading conditions. The task becomes even more complicated if one has to take into account stresses induced due

to processing, thermal loading and moisture absorption. Therefore, one has to resort to numerical techniques, such as the finite element method to be able to predict the failure behavior of composite structures accurately.

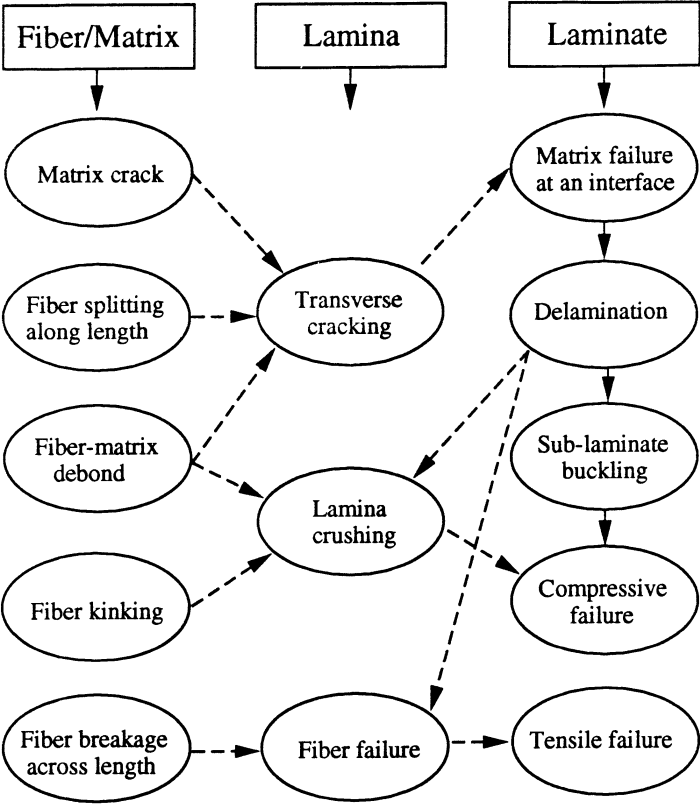


Figure 4.1–1 Various failures at different scales.

The main difficulty in mathematical modeling of damage is due to the different geometric scales that are involved in the initiation and progression of damage. At the micro-level the focus is on the fiber and the matrix and fiber/matrix interface, at macro-level the focus is on the global response of the structure. Figure 4.1-1 shows various failures at different scales. The first failure always initiates at a micro-level and gradually evolves into a macro defect or damage mode. However, when the objective is to predict the failure behavior of a laminate, it becomes prohibitively difficult to focus attention on the events taking place at the micro-level. Therefore, in the failure analysis of laminated composite structures, attention is focussed on ply-level failures.

4.2 Damage mechanisms

4.2-1 Intra-ply damage

The common damage modes that one observes within a ply are:

- Fiber-matrix debonding (see Fig. 4.2-1)
- Fiber fracture (see Fig. 4.2-1)
- Fiber pull-out (see Fig. 4.2-1)
- Matrix cracks (see Fig. 4.2-2)

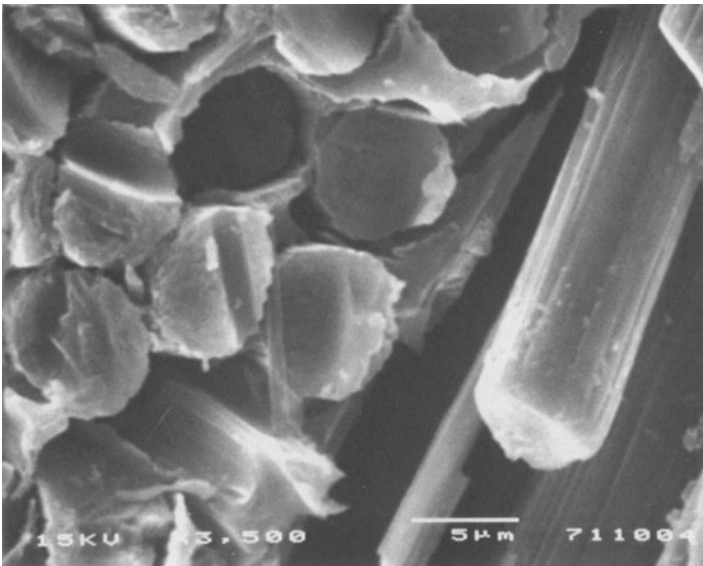


Figure 4.2-1 Fiber matrix debonding, fiber pull out, and fiber fracture.

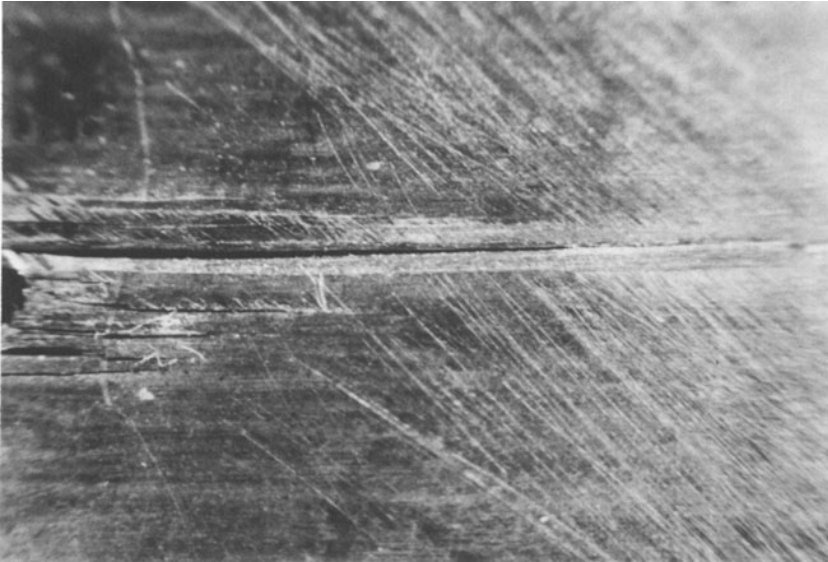


Figure 4.2–2 Matrix cracks.

Matrix cracks are the most common of these local failures because the strength of the matrix is considerably lower than that of the reinforcing fibers. The next common occurrence is the fiber-matrix debond, where a debond parallel to the direction of the fibers separates the constituents from each other, thus eliminating the support provided by the matrix to the fibers in that region. The fiber fracture and pull out modes, as highlighted in Figure 4.2–1, may be localized. However, in general, they lead to catastrophic failures of the laminate. Figure 4.2–3 illustrates the simultaneous formation of matrix cracks and fiber-matrix debonding around a hole drilled in a graphite epoxy coupon. Note that these modes occur simultaneously or within seconds of each other. Figures 4.2–1 to 4.2–3 illustrate that inplane events are not confined to the plane in which they originate. The failure modes begin at a ply level; however, as they initiate, all the layers in the laminate readjust the load/strain they carry in order to adequately redistribute the stress released by the damage modes.

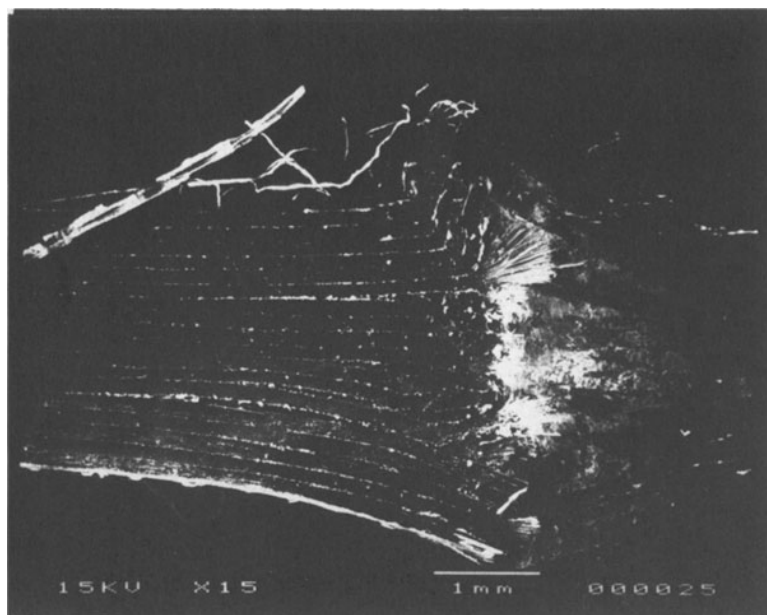


Figure 4.2-3 Through-the-thickness delaminations (drilled hole cross section).

4.2-2 Interply damage: delaminations

As a layer weakens with the onset of the damage modes discussed earlier, the adjacent layers take on additional loads and go through damage initiation and evolution processes. First, matrix cracks form randomly, then they coalesce and lead to delaminations at an interface between layers (see Fig. 4.2-4). To model delaminations, one needs to know their geometry (i.e., length and width) which is usually not known for interior delaminations.

For an edge delamination problem, the width of delamination is given (e.g., the *X-ray* diffraction technique combined with an enhancer may give an indication about the width). The length of the delamination is assumed to be equal to the dimension of the entire free edge (see Fig. 4.2-5). This assumption is valid for laminates with straight edges. When a circular free edge is encountered, the description of the delamination geometry becomes cumbersome. In general, if multiple delaminations exist, any symmetry present due to the stacking sequence, loading, and geometry is lost.

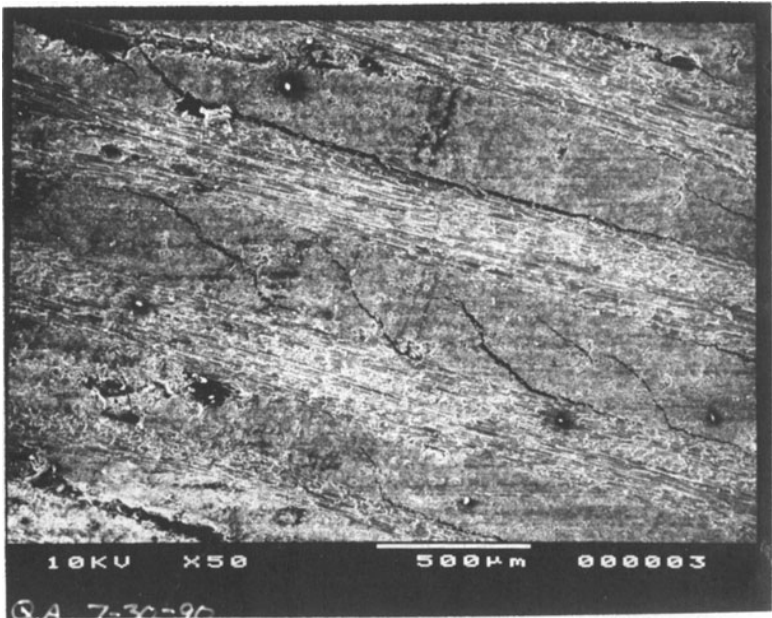


Figure 4.2–4 Transverse cracks and delamination.

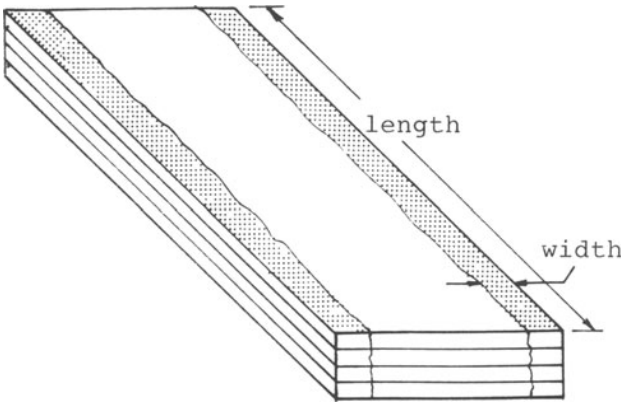


Figure 4.2–5 Edge delamination.

It is essential to integrate explorative testing with numerical simulation in these problems. For example, experiments should be designed to observe which interface fails first and how the progression unfolds. Such an approach may provide additional understanding about the extent of the delamination zone, and with such an understanding, a realistic mathematical model may be built for analysis. Tools such as the scanning electron microscope (SEM) and optical microscope are invaluable in providing information about damage under applied loads. By using loading chambers to test *in-situ* thin laminates with SEM, one can observe and take photographs of the progression of damage within a ply or through the laminate thickness, as appropriate. Figure 4.2-6 depicts an SEM photograph of woven carbob-carbon composite subjected to in-situ three point bending load. The multiple delaminations observed are primarily due to the low interlaminar shear and normal strengths associated with brittle composites (see [14]). Delaminations also form in laminates subjected to inplane tensile and compressive loads. Usually, in these cases Poisson's ratio mismatch or the presence of a geometric discontinuity are the culprits.



Figure 4.2-6 In-situ three point bend loading of a carbon-carbon laminate in SEM chamber.

Extreme care should be exercised in the analytical modeling of the delamination front. If the number of finite elements used in the vicinity of the delamination front are inadequate, the predictions of delamination growth will be inaccurate. The rule of thumb is to keep the element size about a quarter to one-half the thick-

ness of the layer. Our experience indicates that singularity (i.e., quarter-point) elements as well as a refined mesh of shear deformable-non-singular elements work equally well [11]. In the majority of cases, delaminations initiate at a free edge such as a bolt hole or an unloaded edge and propagate inward. Invariably, most studies address two dimensional characterization of this growth. However, delaminations are usually not self-similar crack growths. In composite materials, they grow along weakest directions, and the delamination surface is two dimensional. When a delamination front crosses a ply, delamination geometry becomes three dimensional. It may be appropriate to look at trends of strain energy release rates as a function of delamination size when the delamination growth is within a single interface. However, when the delamination crosses a ply, the strain energy release rates do not give meaningful trends. It is more meaningful to determine the change in the strain energy of the total laminate with respect to the load.

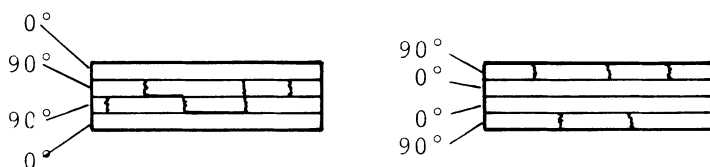


Figure 4.2-7 Damage accumulation in symmetric cross-ply laminates.

A typical example of damage accumulation is provided by a symmetric cross-ply laminate. This stacking sequence illustrates brilliantly the transverse crack formation, although this laminate is not recommended in real structures since its load carrying capacity is limited under general loading. Figure 4.2-7 depicts a four-layer $(0/90)_s \equiv (0/90/90/0)$ laminate pulled along the zero degree laminae. Since the modulus of the 90° -layer in the load direction is so small that matrix microcracks form first in the 90° layers. These matrix cracks coalesce, form transverse cracks, and propagate to the $(0/90)$ interface, causing delaminations, which are the failure modes in tension for these laminates. These observations for the four-layer $(0/90)_s$ laminate cannot be generalized to other laminates of arbitrary orientation. The presence of many 90° layers next to each other, versus 90° layers separated by 0° layers, has a profound difference on how the damage progresses and the failure load is reached. For example, a laminate with 90° layers separated by 0° layers will fail at a higher load than a laminate with 90° layers grouped

together. In the case where all 90° layers are placed together, the laminate has a much softer region in the center and cannot carry as much load.

Laminates subjected to inplane compressive loads exhibit further complications in the form of sublaminate buckling (see Fig. 4.2–8). In this case, a collection of laminae separate from the laminate and undergo buckling. Eventually, these layers fracture after buckling (see [15]). This phenomena is also observed in laminates subjected to impact loads. The impact loads usually create a sublaminate at the opposite side from the load application. An excellent example is the case of a tube shot with a bullet (see [16]). The side from which the bullet exits always displays simultaneous presence of sublaminate formation and debonding around the *damaged/penetrated* site. In general, panels and tubes do not exhibit typical structural buckling behavior since they tend to fail at a more localized region through the initiation of sublaminate failure modes.

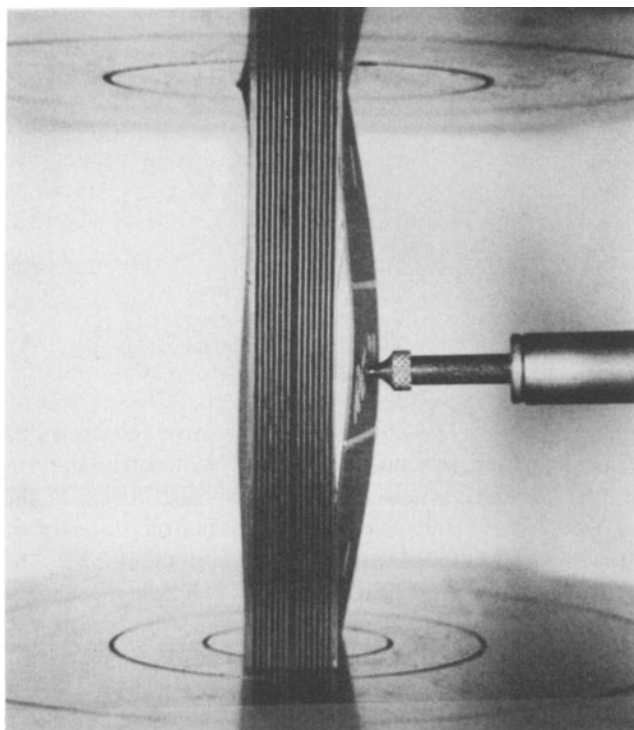


Figure 4.2–8 The $[0_4/(0/45/90/-45)_{7s}/0_4]$ laminate of IM6–3501 under compressive load: sublaminate buckling.

4.3 Failure criteria

4.3-1 General comments

As discussed in Chapter 2, a lamina is assumed to be homogeneous and the mechanical behavior is characterized by a set of equivalent or effective moduli and strength properties. These properties can be obtained either by a micro-mechanics approach or by a phenomenological approach. In the micro-mechanics approach, the lamina properties are predicted in terms of the constituent (fiber, matrix) properties by using a mathematical model; in the phenomenological approach, the lamina properties are determined experimentally by conducting tests on a single lamina or a laminate.

Once the mechanical properties of the ply are known the initial failure of the ply within a laminate or structure can be predicted by applying an appropriate failure criterion. The subsequent failure prediction requires an understanding of damage modes and damage accumulation. A brief discussion on various damage modes, damage accumulation and failure criteria is presented in the sequel.

As discussed in Section 4.2, damage modes are dependent on loading, stacking sequence, and specimen geometry. There are many proposed theories to predict the on-set of failures and their progression (see [17-25]). The selection from such an abundant menu seems impossible, yet there are some salient features that one may look for rather than treating these options as potential "curve fitting the data" exercises. Most of the failure criteria are based on the stress state in a lamina. Hence, an accurate kinematic model of the laminate is necessary to determine three-dimensional stress/strain fields. Ideally speaking, a 3-D or layerwise model is desirable.

The failure criteria, for example, the quadratic polynomial criteria consist of parameters that must be experimentally determined. Often, these parameters are difficult to determine with certainty. Since the failure criteria are intended to predict the macroscopic failures, the microscale event of the split of a single fiber from the surrounding matrix cannot be determined. Rather, at this stage we are interested in a given layer, if the stress in a loading direction versus the transverse is substantial enough to exceed the allowable stress. If the allowable stress/strain is not exceeded, then we proceed to check the other layers. When an allowable is exceeded in a given layer, the engineering constants responsible for the particular mode of failure are degraded (i.e., reduced) by a predetermined magnitude.

The reduction of stiffnesses is another area where care and compromise should support the reality of the situation. When an allowable is exceeded, we do not know the extent of the damage and exact mode of failure at that stage. Thus by reducing the material stiffnesses by a certain amount, we account for the loss of stiffness. The stiffness reduction is carried out within a layer and within a

single finite element at selected integration points. Thus, stiffness reduction is made point wise, and it is not as drastic as discarding an entire layer. Additional discussion of stiffness reduction procedure is given in Chapter 5.

A few comments are in order on strain versus stress based criteria. If experimental data is available, the strain criterion is a more accurate measure since it is directly obtained from the strain gages. The stress based data is difficult to generate, due to the statistical variations in obtaining stresses from strains. The major problem is that the constitutive matrix is not known precisely at the strain level. Since damage accumulation is cumulative, and failure is usually progressive rather than catastrophic and instantaneous, error introduced through assumed material stiffnesses can grow with local steps. Hence, calculation of stresses from strains of a laminate requires further study.

It should be noted that the failure criterion is used only to check whether allowables are exceeded. In the case of delaminations, the failure criterion can only determine the on-set of delaminations. However, the growth of delaminations cannot be modeled unless the kinematic model is three-dimensional. In addition, a criterion for the growth of delamination is required. Often such a criterion is expressed in terms of the strain energy release rates, using fracture mechanics concepts. The fracture mechanics laws to study crack growth are primarily developed for homogeneous isotropic material systems. Thus the fracture mechanics concepts without any homogenization do not hold for composite material systems.

Failure criteria for composite materials can be classified into two groups (see [18]), namely: (i) independent failure criteria, and (ii) polynomial failure criteria. These are discussed below.

4.3-2 Independent failure criteria

The maximum stress and maximum strain criteria belong to this category. In the maximum stress criterion, failure is assumed to occur if any one of the following conditions are satisfied:

$$\begin{aligned}\sigma_1 &\geq X_T, \sigma_4 \geq R \\ \sigma_2 &\geq Y_T, \sigma_5 \geq S \\ \sigma_3 &\geq Z_T, \sigma_6 \geq T\end{aligned}\tag{4.3-1}$$

where $(\sigma_1, \sigma_2, \sigma_3)$ are the normal stress components, $(\sigma_4, \sigma_5, \sigma_6)$ are shear stress components, (X_T, Y_T, Z_T) are the lamina normal strengths in tension (T) along the (1,2,3) directions, and (R, S, T) are the shear strengths in the (23, 13, 12) planes, respectively. When $(\sigma_1, \sigma_2, \sigma_3)$ are of compressive nature, they should be compared with (X_C, Y_C, Z_C) , which are normal strengths in compression (C) along the (1,2,3) directions, respectively.

In the maximum strain criterion, failure is assumed to occur if any of the following conditions are satisfied.

$$\begin{aligned}\epsilon_1 &\geq X_{\epsilon T}, \quad \epsilon_4 \geq R_{\epsilon} \\ \epsilon_2 &\geq Y_{\epsilon T}, \quad \epsilon_5 \geq S_{\epsilon} \\ \epsilon_3 &\geq Z_{\epsilon T}, \quad \epsilon_6 \geq T_{\epsilon}\end{aligned}\tag{4.3-2}$$

where $(\epsilon_1, \epsilon_2, \epsilon_3)$ are the normal strain components, $(\epsilon_4, \epsilon_5, \epsilon_6)$ are shear strain components, $(X_{\epsilon T}, Y_{\epsilon T}, Z_{\epsilon T})$ are the lamina normal strain strengths in tension along the (1,2,3) directions, and $(R_{\epsilon}, S_{\epsilon}, T_{\epsilon})$ are the shear strain strengths in the (23, 13, 12) planes, respectively. When $(\epsilon_1, \epsilon_2, \epsilon_3)$ are negative, they should be compared with $(X_{\epsilon C}, Y_{\epsilon C}, Z_{\epsilon C})$, which are normal strain strengths in compression along the (1,2,3) directions.

4.3-3 Polynomial failure criteria

The most general polynomial failure criterion for composite materials is the tensor polynomial criterion proposed by Tsai [21]. All other polynomial failure criteria are degenerate cases of this criterion. In index notation, the tensor polynomial failure criterion is expressed as:

$$F_i \sigma_i + F_{ij} \sigma_i \sigma_j + F_{ijk} \sigma_i \sigma_j \sigma_k + \dots \geq 1,$$

or, in more explicit form,

$$\begin{aligned}F_1 \sigma_1 + F_2 \sigma_2 + F_3 \sigma_3 + 2F_{12} \sigma_1 \sigma_2 + 2F_{13} \sigma_1 \sigma_3 + 2F_{23} \sigma_2 \sigma_3 + F_{11} \sigma_1^2 \\ + 2F_{22} \sigma_2^2 + F_{33} \sigma_3^2 + F_{44} \sigma_4^2 + F_{55} \sigma_5^2 + F_{66} \sigma_6^2 + \dots \geq 1\end{aligned}\tag{4.3-3}$$

Particular cases of the above criterion are discussed below.

Maximum stress (polynomial) criterion The maximum stress criterion can also be expressed as a tensor polynomial criterion as:

$$\begin{aligned}(\sigma_1 - X_T)(\sigma_1 + X_C)(\sigma_2 - Y_T)(\sigma_2 + Y_C)(\sigma_3 - Z_T)(\sigma_3 + Z_C) \\ (\sigma_4 - R)(\sigma_4 + R)(\sigma_5 - S)(\sigma_5 + S)(\sigma_6 - T)(\sigma_6 + T) = 0\end{aligned}\tag{4.3-4}$$

Comparing Eq. (4.3-4) with Eq. (4.3-3) and ignoring the higher order terms in both the equations, we obtain

$$F_1 = \frac{1}{X_T} - \frac{1}{X_C}, \quad F_2 = \frac{1}{Y_T} - \frac{1}{Y_C}, \quad F_3 = \frac{1}{Z_T} - \frac{1}{Z_C};$$

$$\begin{aligned}
F_{11} &= \frac{1}{X_T X_C} ; F_{22} = \frac{1}{Y_T Y_C} ; F_{33} = \frac{1}{Z_T Z_C} ; \\
F_{44} &= \frac{1}{R^2} ; F_{55} = \frac{1}{S^2} ; F_{66} = \frac{1}{T^2} ; \\
F_{12} &= -\frac{F_1 F_2}{2} ; F_{13} = -\frac{F_1 F_3}{2} ; F_{23} = -\frac{F_2 F_3}{2}
\end{aligned} \tag{4.3-5}$$

The remaining strength constants are zeros.

Maximum strain (polynomial) criterion The maximum strain criterion can be expressed as a tensor polynomial criterion:

$$\begin{aligned}
&(\epsilon_1 - X_{\epsilon T})(\epsilon_1 + X_{\epsilon C})(\epsilon_2 - Y_{\epsilon T})(\epsilon_2 + Y_{\epsilon C})(\epsilon_3 - Z_{\epsilon T})(\epsilon_3 + Z_{\epsilon C}) \times \\
&(\epsilon_4 - R_{\epsilon})(\epsilon_4 + R_{\epsilon})(\epsilon_5 - S_{\epsilon})(\epsilon_5 + S_{\epsilon})(\epsilon_6 - T_{\epsilon})(\epsilon_6 + T_{\epsilon}) = 0
\end{aligned} \tag{4.3-6}$$

Using the compliance matrix and the constitutive relation, we can express Eq. (4.3-6) in the form of equation (4.3-3) as:

$$\begin{aligned}
F_1 &= F_1^A + \frac{S_{12}}{S_{22}} F_2^A + \frac{S_{13}}{S_{33}} F_3^A \\
F_2 &= \frac{S_{12}}{S_{11}} F_1^A + F_2^A + \frac{S_{23}}{S_{33}} F_3^A \\
F_3 &= \frac{S_{13}}{S_{11}} F_1^A + \frac{S_{23}}{S_{22}} F_2^A + F_3^A \\
F_{11} &= S_x + \left(\frac{S_{12}}{S_{22}}\right)^2 S_y + \left(\frac{S_{13}}{S_{33}}\right)^2 S_z - \frac{S_{13}}{S_{33}} F_1^A F_3^A - \frac{S_{12}}{S_{22}} F_1^A F_2^A - \frac{S_{12} S_{13}}{S_{22} S_{33}} F_2^A F_3^A \\
F_{22} &= S_y + \left(\frac{S_{12}}{S_{11}}\right)^2 S_x + \left(\frac{S_{23}}{S_{33}}\right)^2 S_z - \frac{S_{12}}{S_{11}} F_1^A F_2^A - \frac{S_{23}}{S_{33}} F_2^A F_3^A - \frac{S_{12} S_{23}}{S_{11} S_{33}} F_1^A F_3^A \\
F_{33} &= S_z + \left(\frac{S_{13}}{S_{22}}\right)^2 S_x + \left(\frac{S_{23}}{S_{22}}\right)^2 S_y - \frac{S_{13}}{S_{11}} F_1^A F_3^A - \frac{S_{23}}{S_{22}} F_2^A F_3^A - \frac{S_{13} S_{23}}{S_{11} S_{22}} F_1^A F_2^A \\
S_x &= \frac{1}{X_T X_C} ; S_y = \frac{1}{Y_T Y_C} ; S_z = \frac{1}{Z_T Z_C} \\
F_{44} &= \frac{1}{R^2} ; F_{55} = \frac{1}{S^2} ; F_{66} = \frac{1}{T^2} \\
F_{12} &= \frac{S_{12}}{S_{11}} \frac{1}{X_T X_C} + \frac{S_{12}}{S_{22}} \frac{1}{Y_T Y_C} + \frac{S_{13} S_{23}}{S_{33}^2} \frac{1}{Z_T Z_C} - \frac{1}{2} \left(\frac{S_{12}^2}{S_{11} S_{22}} + 1 \right) F_1^A F_2^A \\
&\quad - \frac{1}{2} \left(\frac{S_{13} S_{12}}{S_{11} S_{33}} \right) F_1^A F_3^A - \frac{1}{2} \left(\frac{S_{12} S_{23}}{S_{22} S_{33}} + \frac{S_{13}}{S_{33}} \right) F_2^A F_3^A
\end{aligned}$$

$$\begin{aligned}
F_{13} &= \frac{S_{13}}{S_{11}} \frac{1}{X_T X_C} + \frac{S_{13}}{S_{33}} \frac{1}{Z_T Z_C} + \frac{S_{12} S_{23}}{S_{22}^2} \frac{1}{Y_T Y_C} - \frac{1}{2} \left(\frac{S_{13}^2}{S_{11} S_{33}} + 1 \right) F_1^A F_2^A \\
&\quad - \frac{1}{2} \left(\frac{S_{12} S_{13}}{S_{11} S_{22}} + \frac{S_{23}}{S_{22}} \right) F_1^A F_2^A - \frac{1}{2} \left(\frac{S_{13} S_{23}}{S_{22} S_{33}} + \frac{S_{12}}{S_{22}} \right) F_2^A F_3^A \\
F_{23} &= \frac{S_{23}}{S_{22}} \frac{1}{Y_T Y_C} + \frac{S_{23}}{S_{33}} \frac{1}{Z_T Z_C} + \frac{S_{12} S_{13}}{S_{11}^2} \frac{1}{X_T X_C} - \frac{1}{2} \left(\frac{S_{23}^2}{S_{22} S_{33}} + 1 \right) F_1^A F_2^A \\
&\quad - \frac{1}{2} \left(\frac{S_{12} S_{23}}{S_{11} S_{22}} + \frac{S_{13}}{S_{11}} \right) F_1^A F_2^A - \frac{1}{2} \left(\frac{S_{23} S_{13}}{S_{11} S_{33}} + \frac{S_{12}}{S_{11}} \right) F_2^A F_3^A \quad (4.3-7)
\end{aligned}$$

In the above expression S_{11} , S_{12} , etc. are the components of compliance matrix; X_T , Y_T , Z_T are the tensile strengths in the 1,2,3 directions; X_C , Y_C , Z_C are the compressive strengths in the 1,2,3 directions; R , S , T are the shear strengths in the 23, 13, 12 planes, respectively; and F_1^A , F_2^A , F_3^A are the expressions given for F_1 , F_2 , F_3 in the maximum stress criterion. All other strength tensor terms are zero.

Tsai-Hill's criterion Hill's criterion can be expressed as

$$\begin{aligned}
\left(\frac{\sigma_1}{X} \right)^2 + \left(\frac{\sigma_2}{Y} \right)^2 + \left(\frac{\sigma_3}{Z} \right)^2 - \left(\frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2} \right) \sigma_1 \sigma_2 - \left(\frac{1}{Y^2} + \frac{1}{Z^2} - \frac{1}{X^2} \right) \sigma_2 \sigma_3 \\
- \left(\frac{1}{Z^2} + \frac{1}{X^2} - \frac{1}{Y^2} \right) \sigma_1 \sigma_3 + \left(\frac{\sigma_4}{R} \right)^2 + \left(\frac{\sigma_5}{S} \right)^2 + \left(\frac{\sigma_6}{T} \right)^2 \geq 1 \quad (4.3-8)
\end{aligned}$$

Hoffman's criterion Hoffman's criterion is expressed as

$$\begin{aligned}
\frac{1}{2} (S_y + S_z - S_x) (\sigma_2 - \sigma_3)^2 + \frac{1}{2} (S_z + S_x - S_y) (\sigma_3 - \sigma_1)^2 \\
+ \frac{1}{2} (S_x + S_y - S_z) (\sigma_1 - \sigma_2)^2 + \left(\frac{1}{X_T} - \frac{1}{X_C} \right) \sigma_1 + \left(\frac{1}{Y_T} - \frac{1}{Y_C} \right) \sigma_2 \\
- \left(\frac{1}{Z_T} - \frac{1}{Z_C} \right) \sigma_3 + \left(\frac{\sigma_4}{R} \right)^2 + \left(\frac{\sigma_5}{S} \right)^2 + \left(\frac{\sigma_6}{T} \right)^2 \geq 1 \quad (4.3-9)
\end{aligned}$$

where

$$S_x = \frac{1}{X_T X_C}; \quad S_y = \frac{1}{Y_T Y_C}; \quad S_z = \frac{1}{Z_T Z_C}$$

Tsai-Wu's criterion The Tsai-Wu criterion is given by

$$F_i \sigma_i + F_{ij} \sigma_i \sigma_j \geq 1 \quad (4.3 - 10)$$

where

$$\begin{aligned} F_1 &= \frac{1}{X_T} - \frac{1}{X_C} ; F_2 = \frac{1}{Y_T} - \frac{1}{Y_C} ; F_3 = \frac{1}{Z_T} - \frac{1}{Z_C} ; \\ F_{11} &= \frac{1}{X_T X_C} ; F_{22} = \frac{1}{Y_T Y_C} ; F_{33} = \frac{1}{Z_T Z_C} ; \\ F_{44} &= \frac{1}{R^2} ; F_{55} = \frac{1}{S^2} ; F_{66} = \frac{1}{T^2} ; \\ F_{12} &= -\frac{1}{2} \frac{1}{\sqrt{X_T X_C Y_T Y_C}} ; \\ F_{13} &= -\frac{1}{2} \frac{1}{\sqrt{X_T X_C Z_T Z_C}} ; \\ F_{23} &= -\frac{1}{2} \frac{1}{\sqrt{Y_T Y_C Z_T Z_C}} \end{aligned} \quad (4.3 - 11)$$

It should be noted that in Hill's criterion the linear terms in σ_i do not appear; therefore, the values of F_1 , F_2 and F_3 are zero. The values of X , Y , Z are taken as either X_T , Y_T , Z_T or as X_C , Y_C , Z_C depending upon the sign of σ_1 , σ_2 and σ_3 , respectively. Strength tensors for this criterion are:

$$\begin{aligned} F_i &= 0 \\ F_{11} &= \frac{1}{X^2} ; F_{22} = \frac{1}{Y^2} ; F_{33} = \frac{1}{Z^2} ; \\ F_{44} &= \frac{1}{R^2} ; F_{55} = \frac{1}{S^2} ; F_{66} = \frac{1}{T^2} ; \\ F_{12} &= -\frac{1}{2} \left(\frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2} \right) ; \\ F_{13} &= -\frac{1}{2} \left(\frac{1}{Z^2} + \frac{1}{X^2} - \frac{1}{Y^2} \right) ; \\ F_{23} &= -\frac{1}{2} \left(\frac{1}{Y^2} + \frac{1}{Z^2} - \frac{1}{X^2} \right) \end{aligned} \quad (4.3 - 12)$$

All other strength tensor terms are zero.

Hoffman's criterion is a special case of the tensor polynomial criterion for the following choice of the parameters F_i and F_{ij} :

$$\begin{aligned}
 F_1 &= \frac{1}{X_T} - \frac{1}{X_C} ; F_2 = \frac{1}{Y_T} - \frac{1}{Y_C} ; F_3 = \frac{1}{Z_T} - \frac{1}{Z_C} \\
 F_{11} &= \frac{1}{X_T X_C} ; F_{22} = \frac{1}{Y_T Y_C} ; F_{33} = \frac{1}{Z_T Z_C} \\
 F_{44} &= \frac{1}{R^2} ; F_{55} = \frac{1}{S^2} ; F_{66} = \frac{1}{T^2} ; \\
 F_{12} &= -\frac{1}{2} \left(\frac{1}{X_T X_C} + \frac{1}{Y_T Y_C} - \frac{1}{Z_T Z_C} \right) \\
 F_{13} &= -\frac{1}{2} \left(\frac{1}{X_T X_C} + \frac{1}{Z_T Z_C} - \frac{1}{Y_T Y_C} \right) \\
 F_{23} &= -\frac{1}{2} \left(\frac{1}{Z_T Z_C} + \frac{1}{Y_T Y_C} - \frac{1}{X_T X_C} \right)
 \end{aligned} \tag{4.3 - 13}$$

Other strength tensor terms are zero.

Hashin's criteria The criteria reviewed above were originally developed to predict initial failures, without regard to the mode of failure. However, in recent years these criteria have been also used to identify damage modes (see [20]). Hashin [23] suggested a set of failure criteria that have the ability to predict modes of failures. These are reviewed here.

Fiber failure

Tensile:

$$\left(\frac{\sigma_1}{X_T} \right)^2 + \frac{\sigma_5^2 + \sigma_6^2}{T^2} \tag{4.3 - 14}$$

Compressive:

$$\sigma_1 = \frac{G_r}{1 - k} \tag{4.3 - 15}$$

where G_r is resin modulus and k is volume fraction ratio.

Matrix failure

Tensile:

$$\frac{(\sigma_2 + \sigma_3)^2}{R^2} + \frac{(\sigma_4^2 + \sigma_5^2 + \sigma_6^2 - \sigma_2 \sigma_3)}{T^2} = 1 \tag{4.3 - 16}$$

Compressive:

$$\frac{1}{Y_C} \left[\left(\frac{Y_C}{2T} \right)^2 - 1 \right] (\sigma_2 + \sigma_3) + \frac{(\sigma_2 + \sigma_3)^2}{4T^2} + \frac{(\sigma_4^2 + \sigma_5^2 + \sigma_6^2 - \sigma_2\sigma_3)}{T^2} = 1 \quad (4.3 - 17)$$

Delamination [through-the-thickness]

$$\left(\frac{\sigma_3}{Z_T} \right)^2 + \frac{\sigma_6^2 + \sigma_5^2}{R^2} = 1 \quad (4.3 - 18)$$

Recently, Reddy and his colleagues [17-19] studied first-ply failures in composite laminates using the first order shear deformation theory with von Kármán nonlinearity [18]. The results of these studies will be discussed in the sequel. Application of the failure criteria described here for the prediction of first-ply-failure and progressive failure loads are presented in Chapter 5.

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Chapter 5

Case Studies

5.1 Introduction

In this chapter we discuss the results of specific composite structural problems that are analyzed using the theories and computational models developed in the previous chapters. The objective of these case studies is to bring out certain physical features that influence design of composite structures. While the conclusions drawn in each case apply to that particular problem, they are valuable in the design of similar laminates.

5.2 Modeling of delaminations around loaded holes

5.2–1 Preliminary comments

Advanced composite materials are prone to delamination during manufacturing operations such as drilling. Quite often drilling causes delaminated regions around the circumference of the hole. A typical form of these defects is interlaminar delamination, as shown in Figure 5.2–1. These delaminations are generally worse on the drill exit side of the laminate, and the damage often extends into many layers of the laminate. The propagation of these delaminations is one of the most serious problems in failure of composite parts [1].

Here we illustrate a simple simulation of this practically important problem with two selected laminates, as shown in Table 5.2–1. The laminates are manufactured from different types of prepregs, i.e., from a tape and fabric. This difference in the lamina architecture influences failure progression and therefore the load varying ability of the laminate. A single 3/16 in (4.76 mm) diameter hole is drilled in each of the 9 in $\times$ 1.5 in (22.86 cm $\times$ 3.81 cm) test coupons. In some specimens the holes are drilled to perfection and they are referred to as “control” holes. The holes in the remaining specimens exhibited induced backside delami-

nations. The average size is the delaminations are controlled to be twice the hole diameter ($2D$) and one and half times the hole diameter ($1.5D$), where D denotes the diameter of the hole.

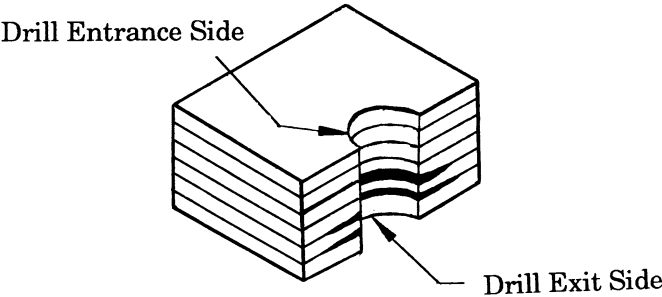


Figure 5.2–1 Drilling induced fastener hole delamination.

Table 5.2–1 Laminates used in the drilling study.

Properties	18 Ply Tape [0/45/0/ – 45/0/45/90/ – 45/90] _s	10 Ply Fabric [0/(±45) ₂] _s
E_1	24 Msi (165.5 GPa)	5.14 Msi (35.4 GPa)
E_2	1.38 Msi (35.4 GPa)	5.14 Msi (35.4 GPa)
G_{12}	0.80 Msi (5.5 GPa)	0.8 Msi (5.5 GPa)
ν_{12}	0.3	0.03
thickness	0.0055 in (.139 mm)	0.007 in (.178 mm)

Table 5.2–2 Pin Bearing Static Test Results for Control Coupons.

Configuration	Load at Failure	Bearing Stress in Pin	Maximum Fatigue Load
18 Ply Tape	5300 lbs (23.6 kN)	268 Ksi (185 MPa)	3500 lbs (15.6 kN)
10 Ply Fabric	6380 lbs (28.4 kN)	245 Ksi (169 MPa)	4200 lbs (18.7 kN)

5.2-2 Loading

The pin loaded coupon under tension is simulated with a finite element model. A photograph of the loading mechanism (tension-bearing) is presented in Figure 5.2-2. As the testing frame pulls the coupon and frame assembly in tension, the pin applies a compressive force on the coupon. At higher loads, the pin penetrates into the coupon, compressing the material along its path. Experimentally obtained loads for a pin loaded coupon are presented in Table 5.2-2. The results reported are an average of nine tests for each configuration with a control hole.

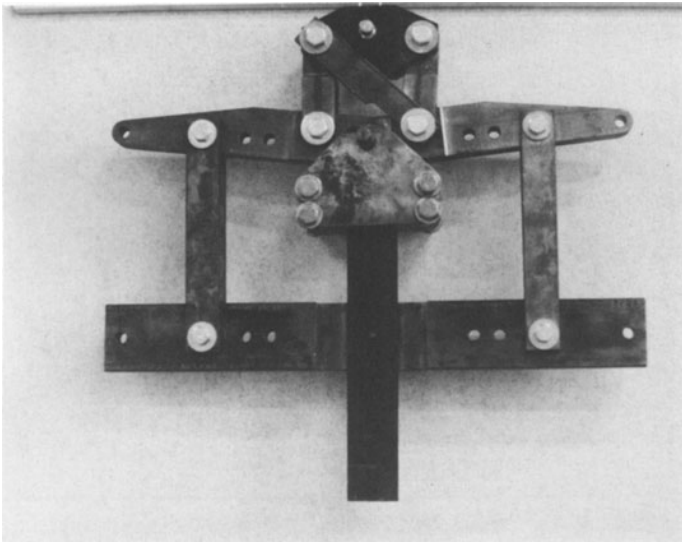


Figure 5.2-2 Bearing-tension fatigue test frame.

There are several options in modeling the load transfer between the pin and hole. Some of the more elaborate methods include physically modeling the actual pin, in addition to the coupon, and others replace the pin with various load distributions, such as a sinusoidal one, around the circumference of the hole [2,3]. The action of the pin is simulated by a distributed load applied to its contact surface. Thus, rather than modeling the pin itself, the effects of the pin contact are represented as an evenly distributed pressure, as shown in Figure 5.2-3. This assumption of half pin contact leads to a uniform pressure distribution in front of the pin and no load behind the pin. In the testing configuration, the test frame exerts 30% of the total tension load at the pin. Thus nodal loads that represent 70% tension load are applied along the right end of the coupon to represent the top grip of the testing frame. A schematic of the test coupon and its loads is shown in Fig. 5.2-4.

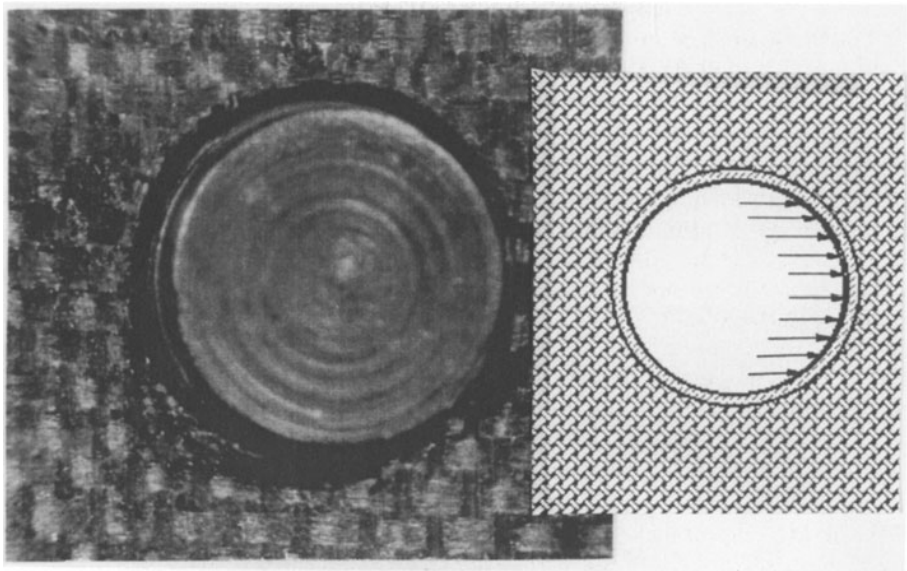


Figure 5.2-3 Actual pin loading and its numerical simulation.

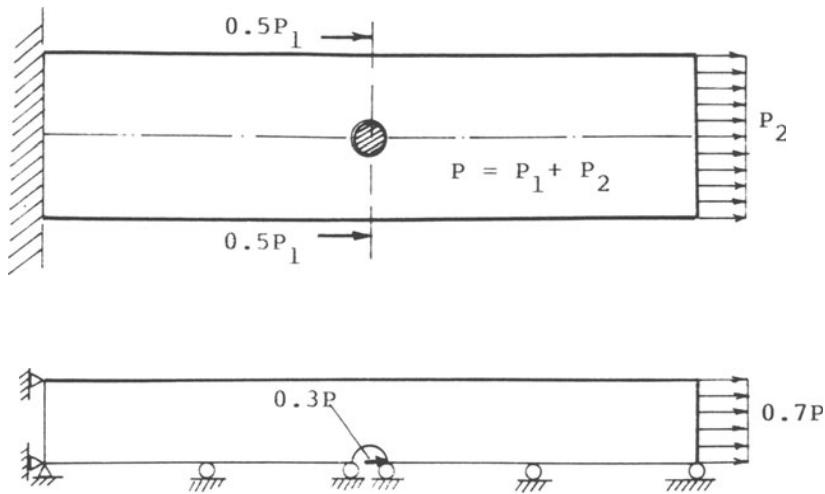


Figure 5.2-4 A schematic of loads on test coupon.

5.2-3 Computational models

Finite element analysis is performed on a microcomputer with a MS DOS-based program, Algor [4]. A four node composite plate element based on the assumptions of classical plate theory is used (see Sec. 3.3-2). Each node has five degrees of freedom (see Fig. 3.3-1a): two in-plane displacements (u, v); one out-of-plane displacement (w); and two rotations, ($\partial w / \partial x, \partial w / \partial y$), about the y and x axes, respectively. The element formulation is based on the classical plate theory assumptions: (1) thickness is small in comparison to plane dimensions; (2) all displacements are small in comparison to the plate thickness; (3) a straight line normal to surface remains straight after deformation; (4) interlaminar transverse normal and shear stresses are neglected; (5) each material layer is linearly elastic.

Since the coupon geometry, assumed load distribution, and assumed delamination shape are symmetric about the long axis of the specimen, a half symmetry model, as shown in Figure 5.2-5, is used as the computational domain. The model has 378 elements, 430 nodes and 2150 degrees of freedom before the application of boundary conditions. Displacements u and v are constrained at the fixed end of the model. Symmetry boundary conditions (SS-1; see Fig. 3.7-1) are applied along the bottom line of the computational domain (i.e., $y = 0$ line). To remove the rigid body modes, a single node on the left end is fully constrained (i.e., all five degrees of freedom are set to zero).

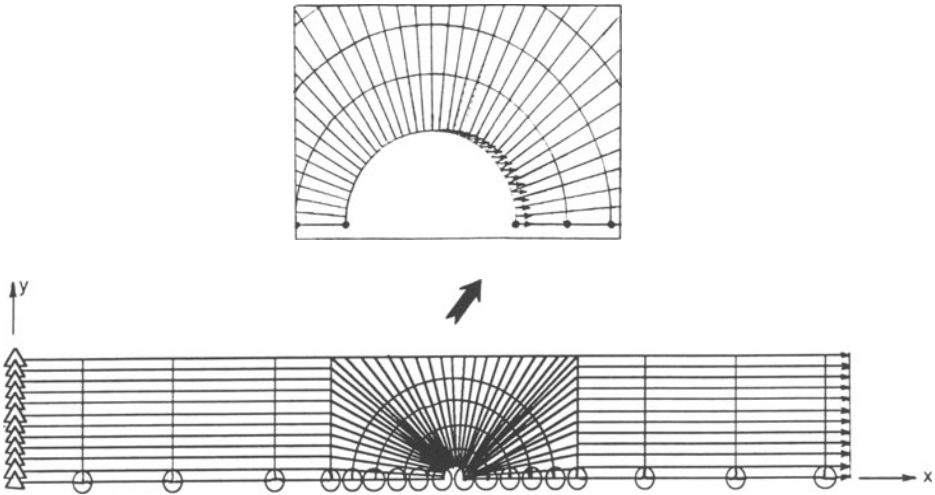


Figure 5.2-5 The finite element mesh used in the study.

5.2-4 Simulation of damage sites

The actual damage zones caused by drilling have been documented through the use of both X - ray and ultrasonic C -scans, and they are found to be of different size, shape and length. This creates complications in the development of numerical models. As an approximation, the damage region at an interface is assumed to be annular, with the inner diameter being the diameter D of the hole. Two effective delamination zone diameters of $1.5D$ and $2D$ are studied. From the study of optical micrographs, it is observed that 30% to 40% of the coupon thickness from the drill exit side contained delaminated layers. In the numerical model, the damaged region on the bottom side, or drill exit side, of the coupon are given reduced properties. Under the assumption that a delaminated layer is much less stiffer than a bonded or undamaged layer, the moduli E_1 , E_2 , and G_{12} in damaged layers are reduced to low values. Since the actual magnitudes of the moduli of the damaged material are not known, a value of 1000 psi ($6.89 \times 10^6 N/m^2$) is used to approximate zero stiffness. As shown in Figure 5.2-6, zone C represents the $1.5D$ diameter region, while zone B represents the $2.0D$ region. Zone A represents the region of the coupon which, in this ideal case, remains unchanged for all magnitudes of delamination. Thus by varying the properties in these three zones, we can use one mesh to represent the damaged and undamaged coupons. Zones A , B , and C are given undamaged material properties in the models representing coupons with control holes. Models that represent $1.5D$ delamination regions are created by assigning undamaged material properties to zones A and B and reduced material properties to zone C . Similarly, the $2.0D$ delamination region is created by assigning undamaged material properties to zone A and reduced properties to zones B and C .

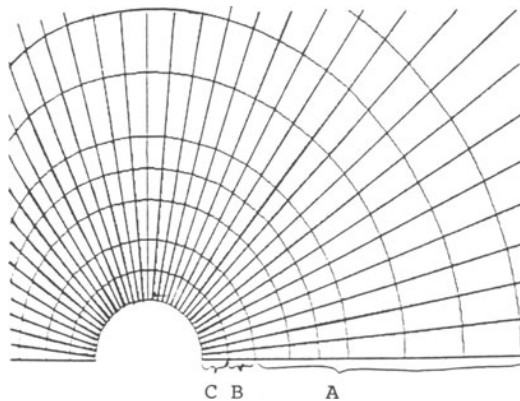


Figure 5.2-6 Definition of delamination zones.

5.2-5 Results and discussion

The models presented above are two-dimensional linear approximations of the states of stress in the coupons. It is noted that the inhomogeneity of the delaminated material in the damaged zones around the hole is not accurately represented by an element under the assumption of linear elastic behavior. Secondly, the through-the-thickness transverse stresses, which could be critical near the holes edge, are not included in this planar model. Despite these shortcomings, these two-dimensional models give insight into the conditions surrounding the hole.

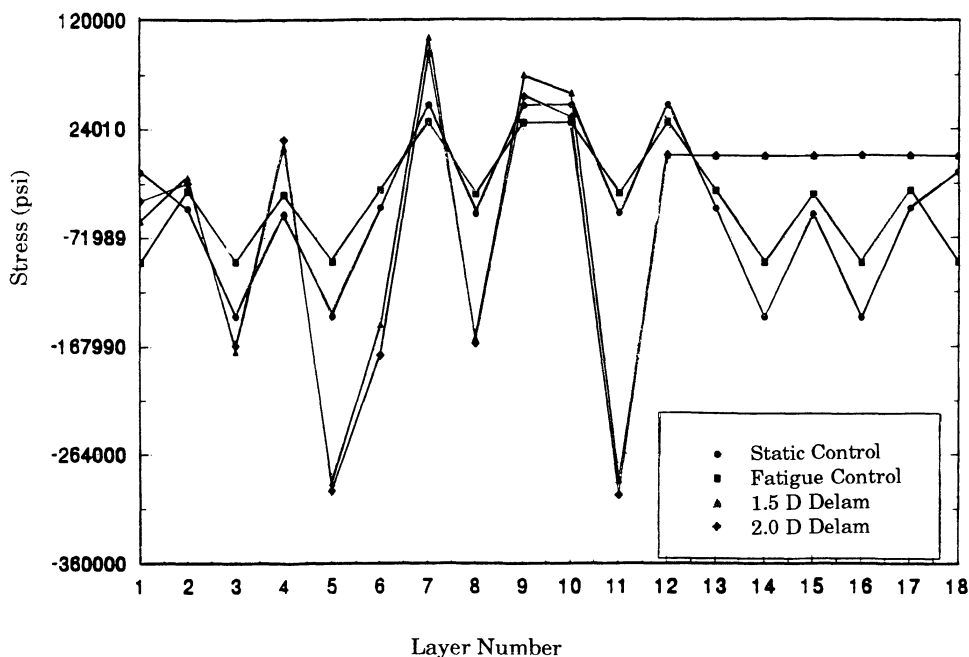


Figure 5.2-7 Inplane normal stress, σ_{11} , contours through the thickness for 18 ply tape laminate (evaluated in element 2 located at 0° around the hole).

Figure 5.2-7 illustrates the through-the-thickness variations of the inplane normal stress σ_{11} . The corresponding stress contour plots for the element located on the hole boundary directly in front of the pin in the 18-ply-tape (0/45/0/ - 45/0/45/90/ - 45/90), specimen are shown in Figs. 5.2-8 and 5.2-9. The four curves of Figure 5.2-7 show the stress distributions of coupons with control hole and, 1.5D and 2.0D delaminations during fatigue load. The stress distribution of a control coupon subjected to static failure load is also shown. Note that the stress is zero in laminae 12-18 for 1.5D and 2.0D holes. This location is asso-

ciated with the drill exit side with delaminations where very low moduli values are used. An increase in the magnitudes of the compressive and tensile stresses is observed in laminae 5, 7 and 11 in coupons with delamination. Figures 5.2-8 and 5.2-9 illustrate the distribution of stress around the hole for a specimen with control hole and 1.5D delaminated zone, respectively. The difference in the contour plots are attributed to the influence of the simulated backside damage. Similar results are presented in Figure 5.2-10 for the 10-ply-fabric $(0/(\pm 45)_2)_s$ coupons. Note that the inplane stresses are much smaller in the fabric coupons versus the tape coupons. This confirms the experimentally observed phenomena that fabric configuration provides more resistance to delamination (see Table 5.2-2).

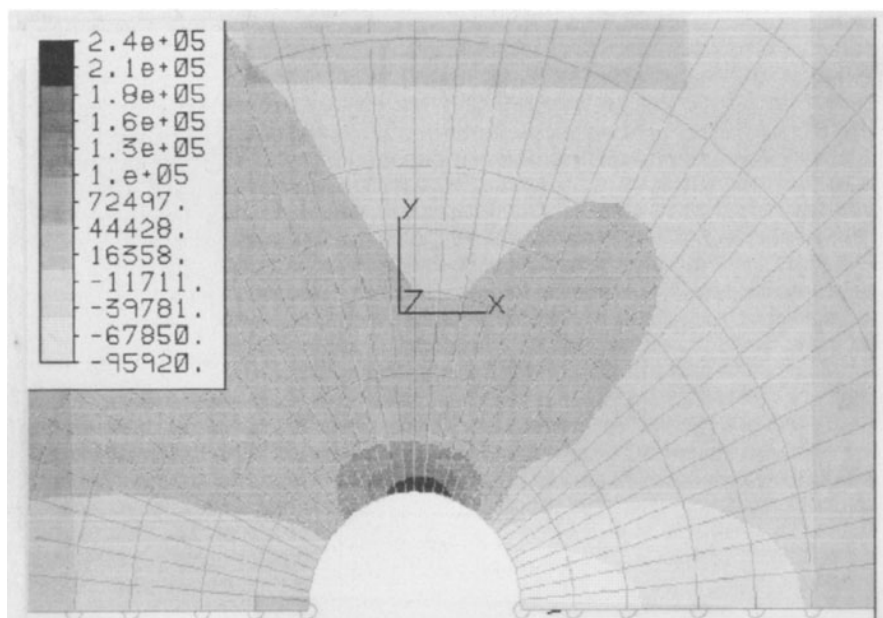


Figure 5.2-8 Control hole: stress (σ_{11}) contours in the top layer (0°) of 18 ply tape laminate.

5.3 Cutouts in composites

5.3-1 Introduction

An area of critical stress fields for load-carrying members is near cutouts. A cutout may provide a passage for hydraulic lines, avionic harnesses, and on a larger scale, an access door in an aircraft fuselage. The presence of multiple layers as well as the anisotropy of the material system establishes a new challenge in terms of optimizing the cutout geometry. The earlier research activities evaluated the effects of single circular and elliptical cutouts [5-11].

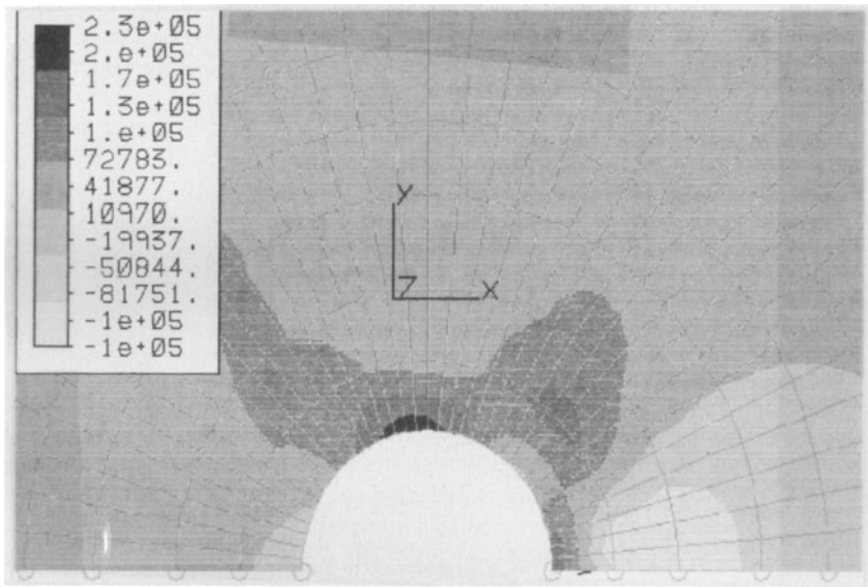


Figure 5.2-9 1.5D hole: stress (σ_{11}) contours in the top layer (0°) of 18 ply tape laminate.

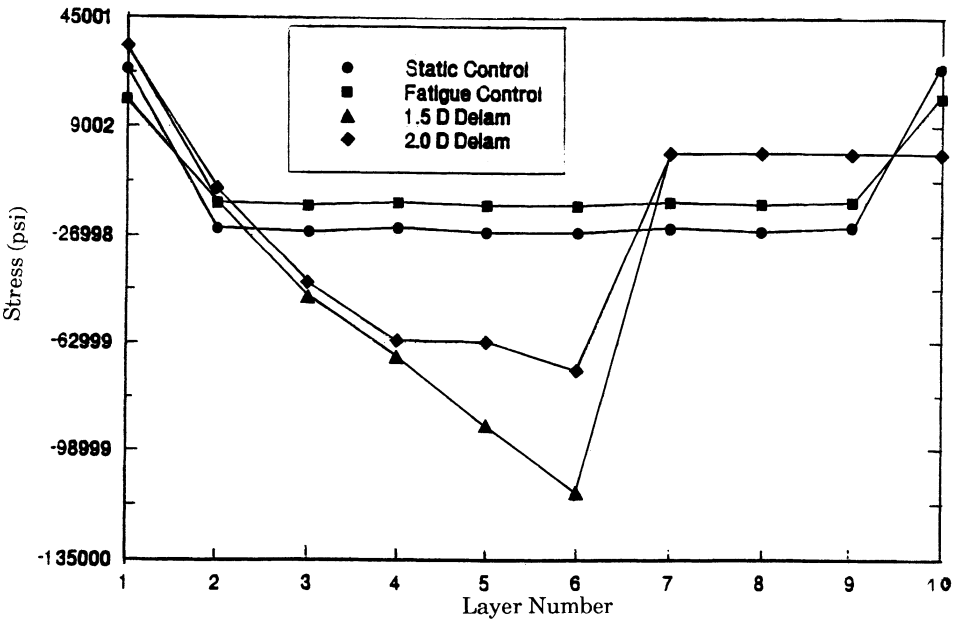


Figure 5.2-10 Through-the-thickness variation of (σ_{11}) for 10 ply fabric laminate (evaluated in element 2 located at 0° around the hole).

The present discussion will focus on the study of flat tensile coupons with multiple cutouts and a hat stiffened panel with a cutout in the web section [12,13]. The material system used is IM7/977-2, with the following properties:

$$E_1 = 160.6 \text{ GPa}, E_2 = 8.48 \text{ GPa}, \nu_{12} = 0.325, G_{12} = 4.69 \text{ GPa}.$$

The selected stacking sequences for the 9 in $\times$ 1 in coupons are:

$$(0/45/-45/90)_{2s}, (0/45/-45/90)_{2s}, (45/-45/0/90)_{2s}.$$

5.3-2 Computational models

Parametric studies are undertaken to reflect the differences in the selection of stacking sequences, cutout size, and geometry (see [12,13]). Primary emphasis is on the identification of localized high stress and strain regions in order to warn the designer of potential hazards. Patran II software [14] is used to develop the nodal and element geometry, then converted to a neutral file. The neutral file is used as input to ABAQUS, a general purpose FEM code for structural analysis (see [15]).

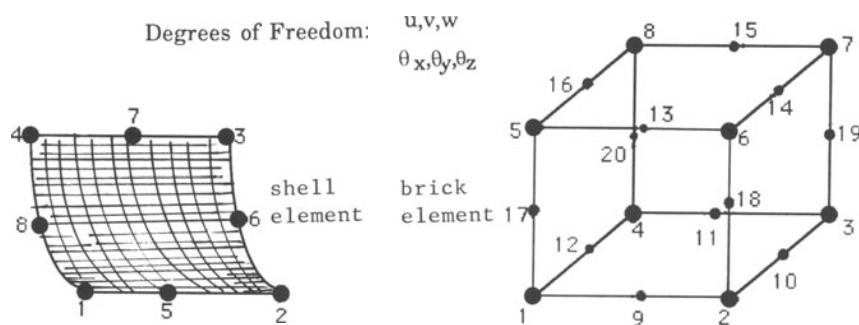


Figure 5.3-1 The eight-noded shell and twenty-noded brick elements.

Eight-noded shell elements with reduced integration, S8R5, and solid elements with twenty nodes, C3D20R, are used from the ABAQUS element library. The shell element has five degrees of freedom per node, namely, three displacements and two rotations as presented in Figure 5.3-1 (also see Fig. 3.6-1). The isoparametric interpolation function used for the displacements is based on the corner and mid-side nodes. Reduced integration is used with 2×2 Gaussian quadrature points. The values of stress and strain are extrapolated from the integration points to the nodes. Effects of transverse shear are included in the formulation in the same way as in the first order shear deformation plate theory. The solid element presented in Figure 5.3-1 is a quadratic isoparametric serendipity element. Once again,

reduced integration is employed, and thus, only eight integration points for the element are used.

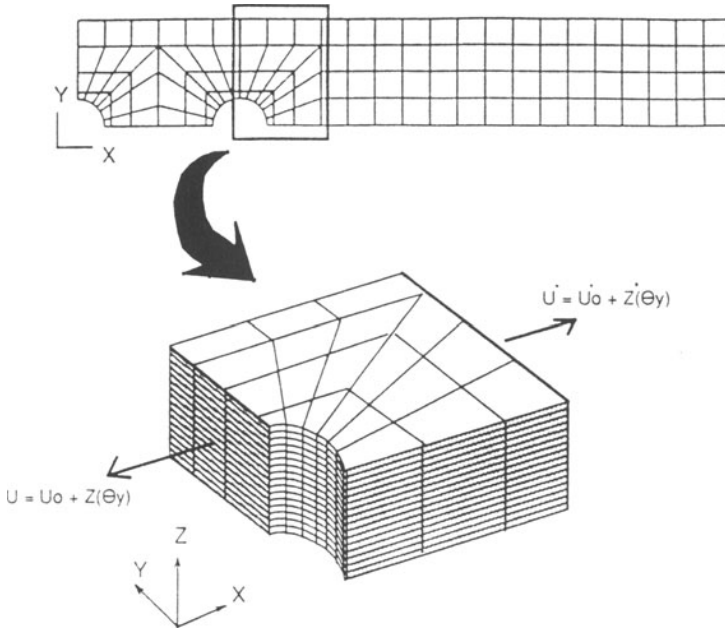


Figure 5.3-2 Global to local transformation.

Two-dimensional, eight-noded elements with reduced integration are used to evaluate the in-plane stress and strain response of the test coupons with three aligned cutouts that are subjected to axial tension loads. Due to the biaxial symmetry of the problem, only one quarter of the specimen is modeled. Symmetry boundary conditions are imposed along the x and y axes. The tensile load is simulated as a distributed load applied along the x -axis at the edge of the specimen. The distributed edge load is then translated to consistent nodal loads using Eq. (3.3-14):

$$F_i^1 = \int_0^b q \psi_i dy$$

where q is the distributed edge load, b is the length of the element side along the y -axis, and ψ_i is the interpolation function at element node i . Evaluating the integral for a uniformly distributed load along the width of a quadratic element, we obtain $(\frac{qb}{6}, \frac{4qb}{6}, \frac{qb}{6})$. That is, the corner nodes carry $qb/6$ and the midside nodes take on $4qb/6$, where q is the magnitude of the uniformly distributed load and b is the width of the element.

When high stress regions are identified at a particular location, then that region is remodeled with three-dimensional twenty-noded brick elements for further study. This approach increases the efficiency of the simulation since only a part of the two dimensional model is considered for the three dimensional analysis. The refinement allows the calculation of both the inplane and interlaminar stress fields, which may help interpret potential failure modes. Figure 5.3-2 illustrates the global-to-local remodeling schematically.

The boundary conditions for the three dimensional region are obtained from the displacements of the two-dimensional global model. However, one should use caution in converting the shell displacement field to boundary conditions for the solid model. In a first-order shell kinematics, a straight line before deformation remains straight after deformation. If the laminate is modeled with a solid element per lamina, then the constraints imposed by shell kinematics are no longer valid.

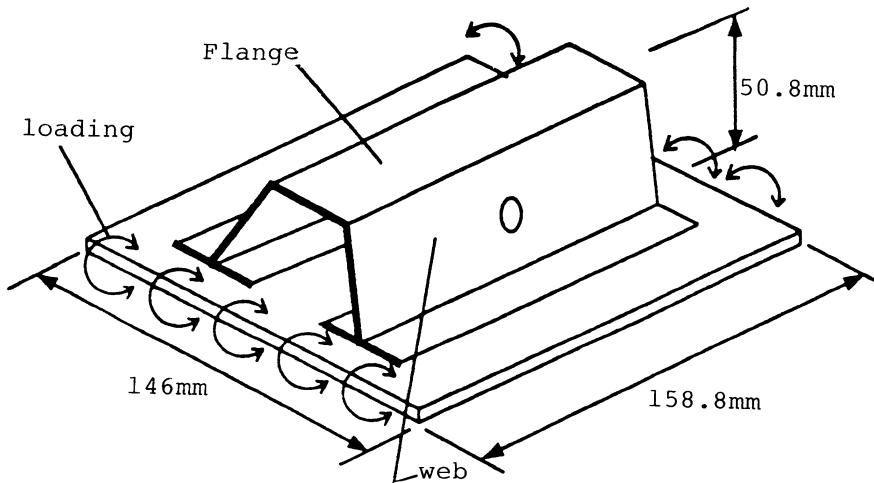


Figure 5.3-3 Hat stiffened panel.

Next we consider the response of a hat-stiffened panel of Figure 5.3-3 with a cutout in the web section subjected to bending loads. This loading is representative of the in-flight loading incurred by a stiffened wing skin. Shell elements are used in the modeling throughout the structure. The skin configuration is a $(45/-45/90/-45/0/45/0/-45/0/45/0)$ laminate. The stiffener web is three layers of $\pm 45^\circ$ fabric, with a 12.7 mm diameter cutout through each web center. The cutout is positioned 79.4 mm from each panel edge, and is located 25.4 mm vertical from the panel. The flange is reinforced with 0° plies; in other words, a layup of $(45/0_6/45/0_6/45)$ is used.

5.3-3 Results

The following observations are based on investigation of flat coupons with cutouts. The in-plane stress fields are altered by varying the stacking sequence of a laminate. For example, the magnitudes of σ_{11} in the 0° layer in different laminates are illustrated in Figure 5.3-4 as a function of the angular position, θ , around the cutout. The angular position is measured counter-clockwise from the loading axis. In analyzing the specimens with circular cutouts and subjected to an inplane tensile load, it is observed that the layup $(0/45/-45/90)_2s$ results in a decrease of σ_{11} by 7.2% for the 0° ply. Similarly, the transverse stress, σ_{22} , shows a decrease of peak value by 10% in the 90° ply for both the $(0/45/-45/90)_2s$ and the $(45/-45/0/90)_2s$ layups.

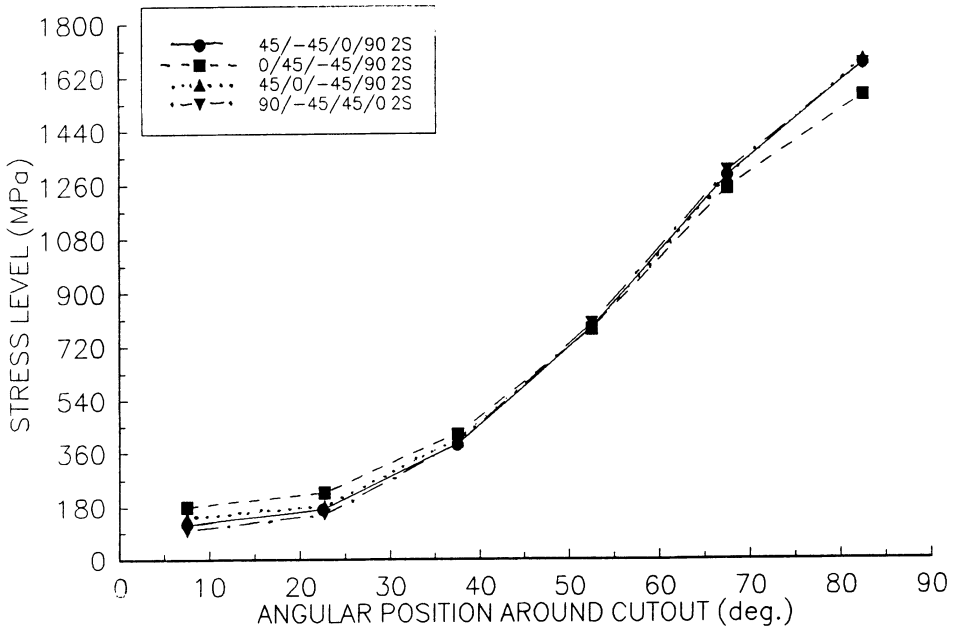


Figure 5.3-4 Variation of stress σ_{11} around the cutout on 0° ply for various laminates and 0.25in. circular cutout.

It is observed that the stress intensity around the cutout within each ply increases with the size of the cutout. For the 50% cutout size (increase from 6.35 mm to 9.5 mm diameter), the stress increase is 27.9% for σ_{11} in the 0° ply and 17.3% for σ_{22} in the 90° ply. This response is due to the fact that the cutouts are closer to each other as well as to the edges of the specimen. Stress contour plots in Figs. 5.3-5a and 5.3-5b indicate the influence of the diameter change.

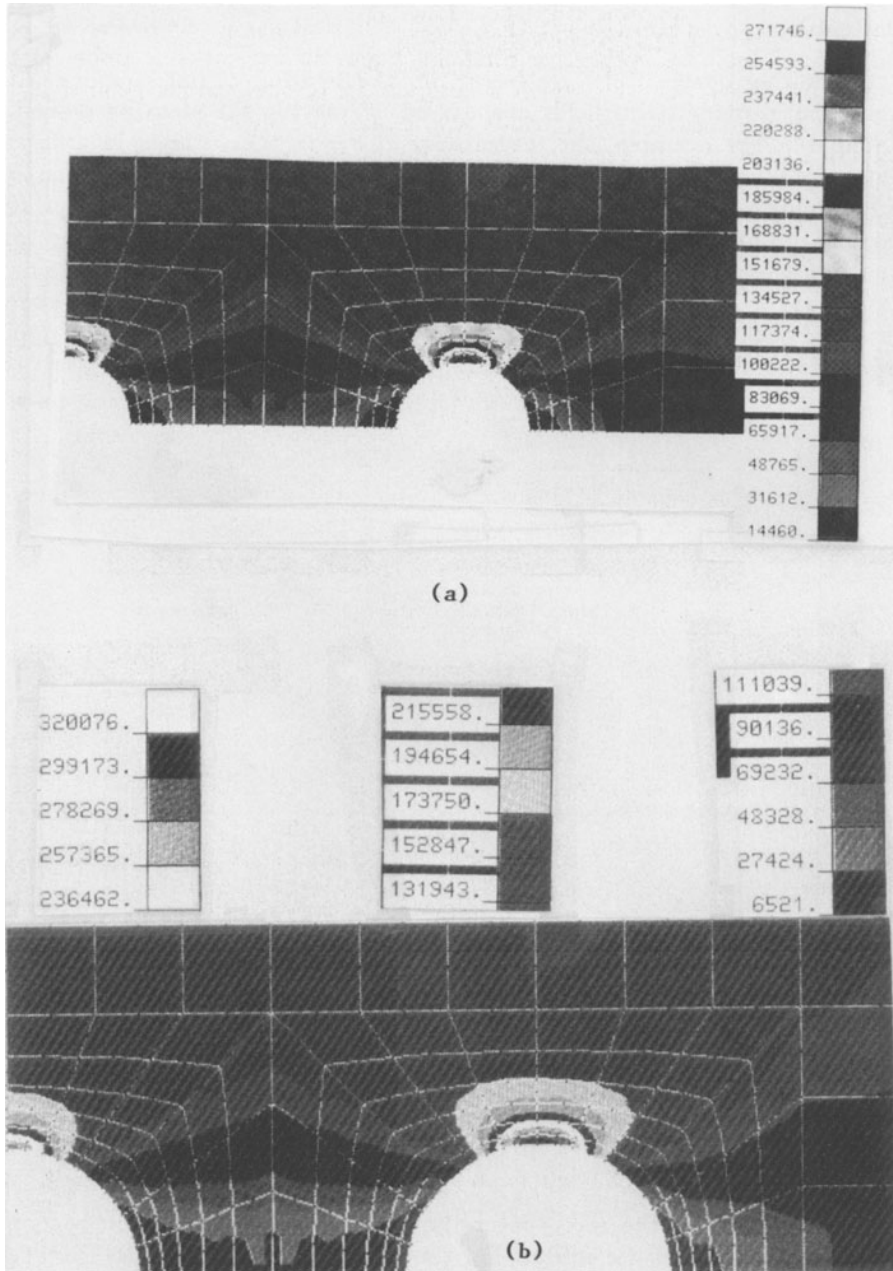


Figure 5.3-5 Stress (σ_{11}) contours for circular cutouts in $(0/45/-45/90)_2$ s laminates under tensile load. (a) 0.25in. cutout. (b) 0.375in. cutout.

The peak σ_{11} levels in the 0° , 45° , -45° plies and σ_{22} levels in the 90° ply are normalized by their respective far field values to represent a global stress concentration factor, K_T , by which a comparison of the various geometries is made. The stress concentration results are tabulated for the 0° , 45° , -45° , and 90° plies in Table 5.3-1. It is interesting to note the K_T values in the $\pm 45^\circ$ plies are significantly higher. The peak σ_{11} stresses in the $\pm 45^\circ$ plies are actually one-half the values of σ_{11} in the 0° ply. Therefore, even though the stress concentration factor is higher, the critical design criterion is still based on the σ_{11} stresses in 0° ply and σ_{22} stresses in the 90° ply. In all plies, it is apparent that the best choice for a cutout shape would be a square.

Table 5.3-1 Values of $(\sigma_{11}/\sigma_{xx})$ in 0° , 45° , and -45° layers and $(\sigma_{22}/\sigma_{yy})$ in 90° layer of the $(0/45/-45/90)_2$ laminate.

Cutout Shape	0	45	-45	90
Circular	2.5	4.08	4.34	2.56
Square	2.15	2.92	3.21	2.22
Diamond	3.15	7.53	8.99	3.26

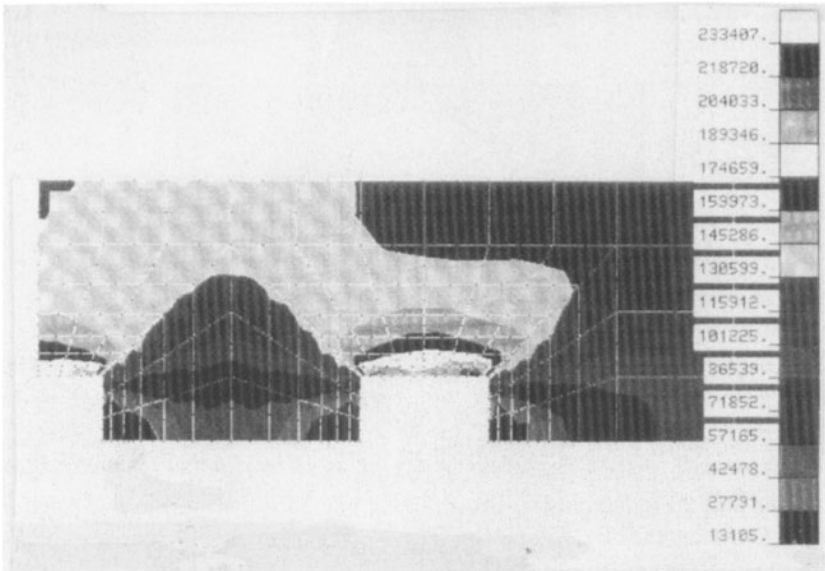


Figure 5.3-6 Stress (σ_{11}) contours of 0.25 in. square cutout for $(0/45/-45/90)_2$ laminate under tensile load.

The fact that stresses at the corners of the square are not a major concern is apparent when studying the stress contours around the square cutout shown in Figure 5.3-6. Rather than giving rise to dominant stress singularities at the corners of the cutout, stress is distributed between the singularities along the 0° ply. There are still concentrations of stresses at the corners, but the 0° ply redistributes the stress such that these singularities are reduced.

Another failure mode of interest is delamination caused by interlaminar shear stresses, σ_{xz} and σ_{yz} , and normal stress, σ_{zz} . For the specimen geometry and loading, the shear stresses are an order of magnitude less than the normal stress. Therefore, σ_{zz} is of primary concern. In analyzing σ_{zz} near the free edge of the cutout, one should note that it has a singularity there. Therefore, stress levels of σ_{zz} are reported at one ply thickness ($187 \mu\text{m}$) and two ply thicknesses ($374 \mu\text{m}$). Peak values of σ_{zz} near the free edge are evaluated in each of the four layups with 6.35 mm circular cutout. Unique to each stacking sequence is the angle α , shown in Figure 5.3-7, at which the peak interlaminar normal stress occurs. For the $(0/45/-45/90)_{2s}$ laminate, peak σ_{zz} occurs at $\alpha = 45^\circ$. However, for $(45/-45/0/90)_{2s}$ and $(45/0/-45/90)_{2s}$ laminates, peak σ_{zz} occurs at $\alpha = 74^\circ$, while the angle for $(90/-45/45/0)_{2s}$ is 90° . These peak values are plotted through the thickness for each stacking sequence and respective angle α in Figures 5.3-8. It is interesting to note that the $(0/45/-45/90)_{2s}$ laminate has an overall reduced stress σ_{zz} through the thickness, while the $(90/-45/45/0)_{2s}$ laminate has σ_{zz} stress that is large.

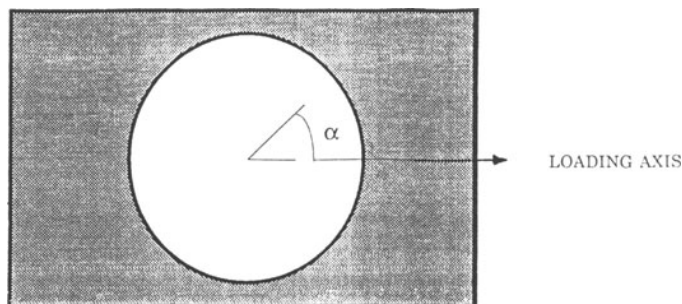


Figure 5.3-7 Cutout reference angle.

In the hat stiffened panel, the cutouts are located in the web section which contains three layers of $\pm 45^\circ$ cloth. The web section mostly carries shear stresses throughout. Stress contours in the 45° ply are shown in Figure 5.3-9 and 5.3-10 for the circular and diamond cutout, respectively. Although stress levels are low due to low input loads, the evidence supports the use of diamond shaped cutouts in regions of a structure which are primarily in shear loading, and contain primarily $\pm 45^\circ$ fibers.

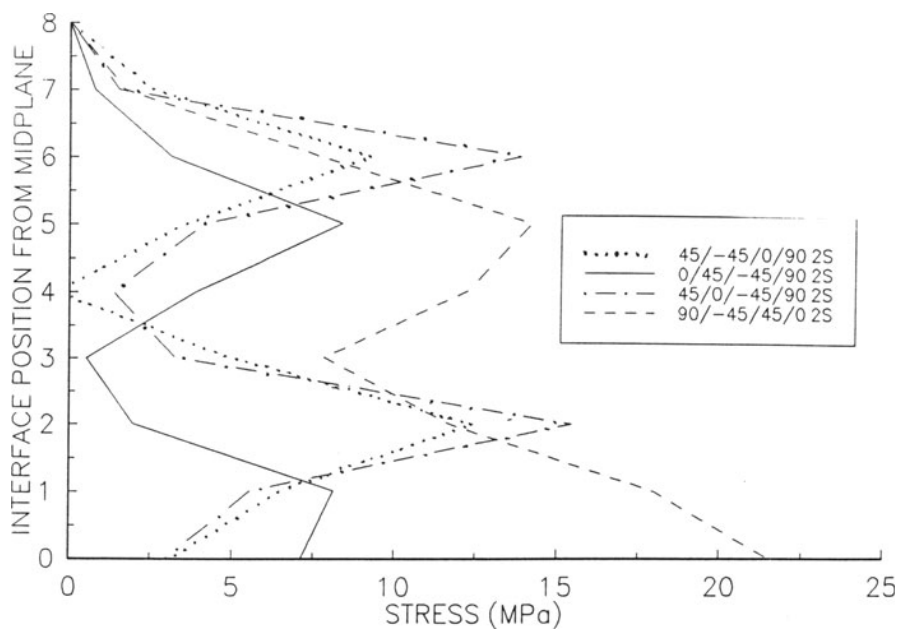


Figure 5.3-8 Peak (σ_{33}) stresses at one ply thickness from cutout edge for various laminates.

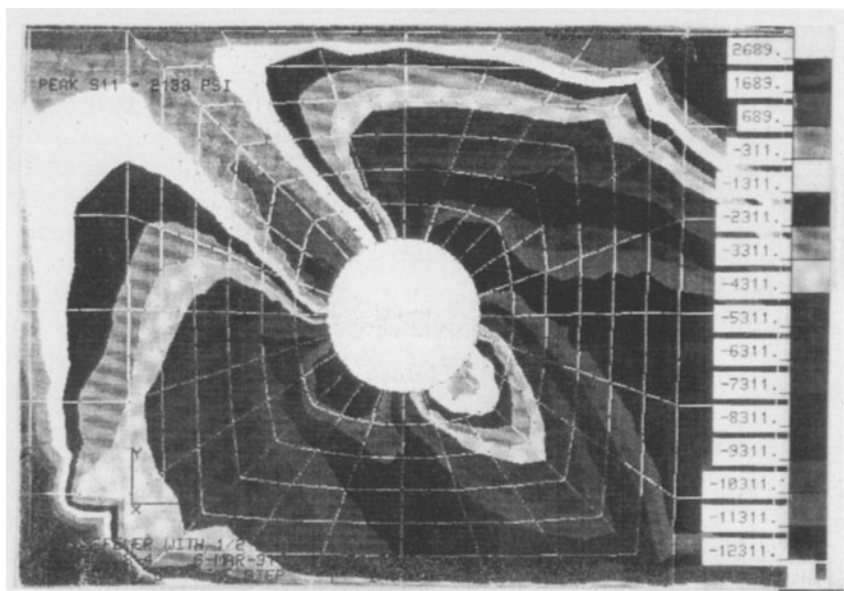


Figure 5.3-9 Stress (σ_{11}) contours in 45° ply of 0.5 in. circular cutout in hat stiffener web.

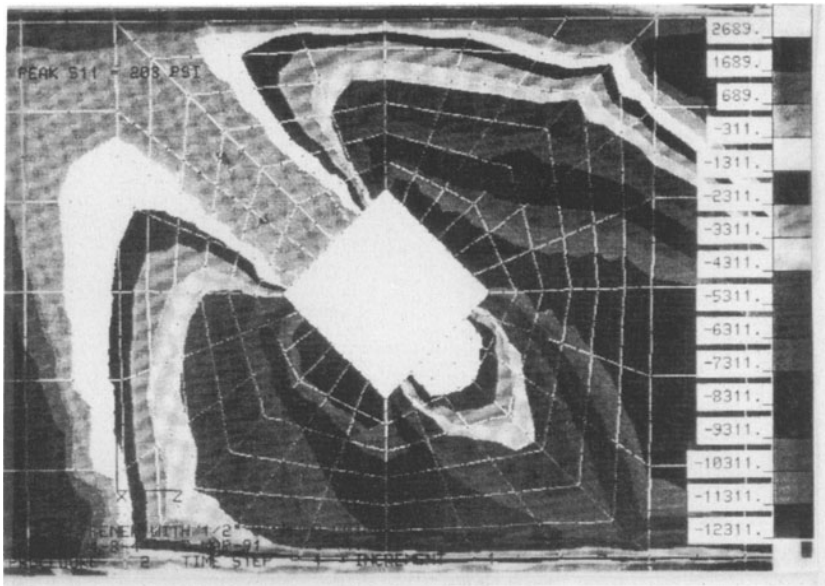


Figure 5.3-10 Stress (σ_{11}) contours in 45° ply of 0.5 in. diamond cutout in hat stiffener web.

5.3-4 Observations

It can be deduced that in a composite structure, it is not always best to choose a circular cutout. For an effective composite design, a dominant percentage of fibers should be aligned with the principal stress directions. In addition, if a cutout is required in the design, it is most desirable to use a rectangular geometry with sides parallel and perpendicular to the principal stress directions. For the demonstrated case of tensile loading, this corresponds to a square cutout oriented 0° from the material axis. As illustrated in the specific case of a panel with an attached stiffener to sustain out-of-plane loads, the stress in the web is predominantly shear. For this scenario, the $\pm 45^\circ$ fiber orientation with a diamond cutout (a square oriented 45° to the material axis) proves to be an efficient cutout.

5.4 Woven preregs

5.4-1 Preliminary comments

In brittle material systems, such as 2-D woven carbon-carbon composites, parameters of importance in modeling constitutive behavior are largely dictated by processing methods. In general, the processing of carbon-carbon composites starts with a regular cure cycle of graphite reinforced thermoset matrix, called a

pregreg. Then the laminate is heated (pyrolysis) to high temperatures of about 2800° F at which the matrix material converts to char. In order to increase the density, further infiltration with resin is required. This process is repeated until a desired density of the structural component is achieved. Shrinkage observed during carbonization of the matrix, and residual stresses produced during several cycles of infiltration and heating result in a laminate which exhibits both matrix cracks and weak interfacial bonds. There are many interfaces in a woven composite that are affected, namely, between the fiber and matrix in a bundle, between bundles, and between layers, and the presence of an oxidation protection coating adds an additional interface (see [16,17]). Therefore, it is important to assess the status of a laminate before mechanical and thermal service loads are applied. It is improper to assume that the above cited features are all detrimental. In fact, weak bonds may be beneficial because they resist crack propagation and thus delay fracture. On the other hand, these weak bonds result in a low shear modulus of the laminate.

For brittle material systems, strong emphasis should be placed on obtaining statistical data to support the numerical modeling of possible defects within a unit cell (or a representative volume) of the laminate. Since the experimental observations using microscopy are restricted to one plane at a time, many details through the depth cannot be observed. Therefore a two-dimensional unit cell may be developed based on SEM observations in two orthogonal planes, the plane of and the plane through the thickness (see [18,19]) of the laminate.

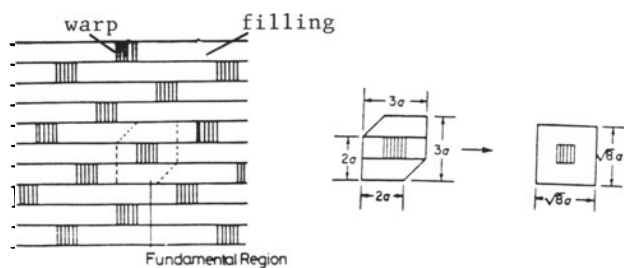
The two-dimensional unit cell used here reflects the results of substantial statistical data obtained from scanning electron microscope (SEM) observations to generate realistic dimensions for voids, debonds and weave misalignment, as illustrated in Figure 5.4-1. Note that these architectural defects were briefly discussed in Chapter 4. Since these anomalies affect the mechanical and material response, we will use numerical simulations of the unit cell to illustrate the changes in the moduli in the presence of these defects.

5.4-2 Numerical models and discussion of the results

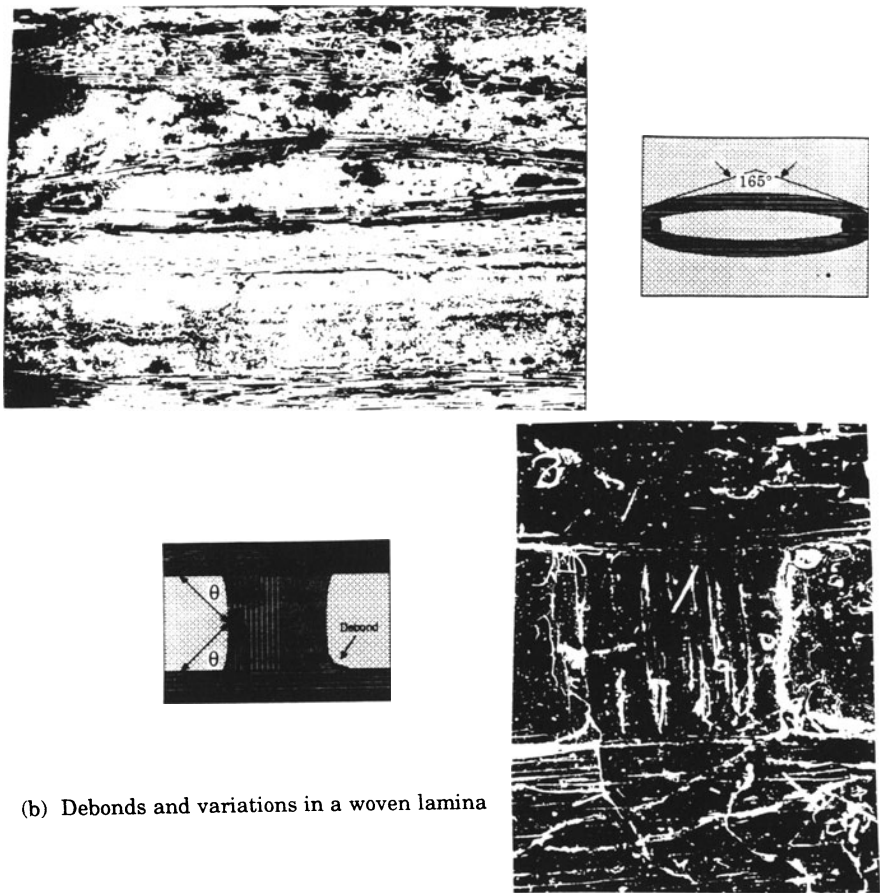
The analytical techniques presented here address the constituent interactions, in an average sense, in the presence of cross-over debonds and crimp angles within a unit cell. The defect features, which are observed experimentally and interpreted statistically, are accommodated in the computational models of the unit cell.

The ingenuity of the project lies in the idealization of the geometry and the constituent behavior of a repetitive unit cell of an *eight harness satin weave*. The statistical characteristics generated through images taken from SEM, and processed through an image analyzer, are built into the finite element model of the unit cell, which has a thickness of a single layer (see [20]). The unit cell is approximated by a square with a side of 2300 μ m. A quarter model of the cell is presented in Figure 5.4-2. As can be seen, a bundle crossover region of 800 μ m

is modeled with an adjacent debond of length $240\ \mu\text{m}$.



(a) Schematic of a woven unit cell



(b) Debonds and variations in a woven lamina

Figure 5.4–1 Woven lamina parameters.

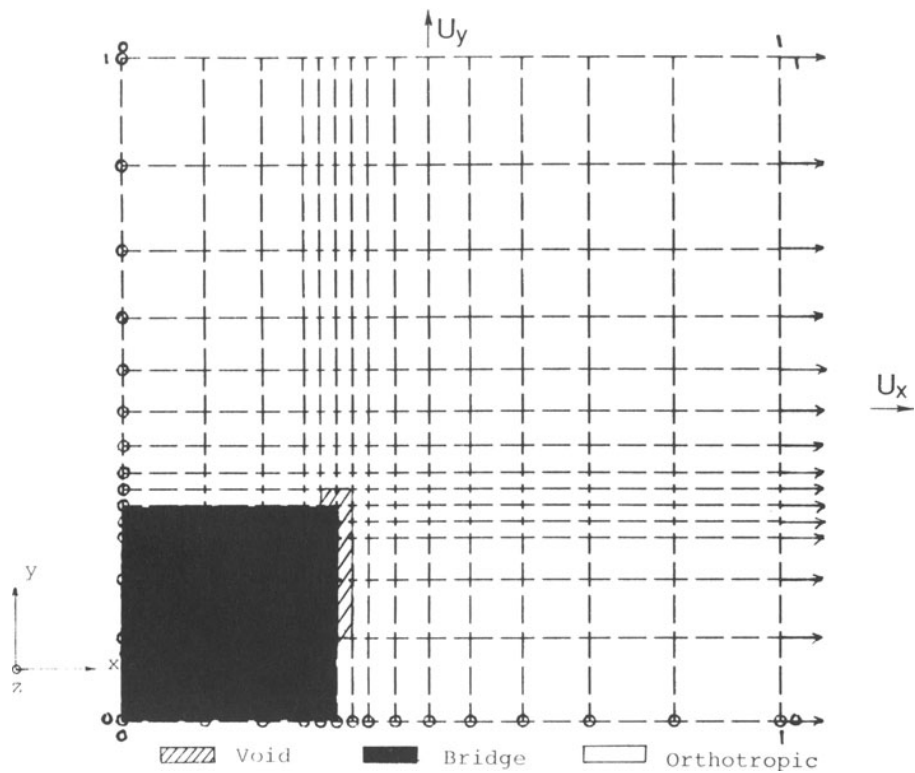


Figure 5.4-2 Finite element mesh of the unit cell.

Since the material near the crossover region consists of warp and fill yarns, it is treated as a homogeneous, specially orthotropic material. The material properties of the bundle crossover section are taken to be those of a specially orthotropic material. However, these values account for the influence of the crimp angle (see [21]). The debond in the initial calculations is treated as an isotropic material with a very low elastic modulus. The material properties used in the analysis are given in Table 5.4-1.

Table 5.4-1 Material properties used in the unit cell.

	E_{11}, Msi	E_{22}, Msi	G_{12}, Msi	ν_{12}
Cell	14.1	14.1	1.3	.01
Debond	0.001	0.001	0.0006	0.2

The four-node transverse shear deformation shell element, S4R of ABAQUS program library, is selected for the models. The element has six degrees of freedom, three displacements and three rotations, per node. Bilinear interpolation functions are used for the displacements. Reduced integration (i.e., one point rule) is employed to compute the strains and stresses. The nonlinear geometric option with a special user-developed material subroutine is used. This subroutine enables the incremental analysis to check for failure strain in each element, at each integration point and at each load level. The failure criterion based on allowable strain is used. In the event of failure, appropriate material coefficients are reduced and residual stresses are calculated. Equilibrium iteration at each load increment is carried out prior to load incrementation.

For tensile loads, the biaxial symmetry is used to model a quarter of the unit cell with 196 elements and 225 nodes. Uniform displacements in the x -direction are used to simulate tensile loads at the right edge. The stress-strain curves are presented in Figure 5.4-3. As can be observed, the load carrying ability is significantly reduced due to the presence of a debond for strains greater than 0.002 in/in.

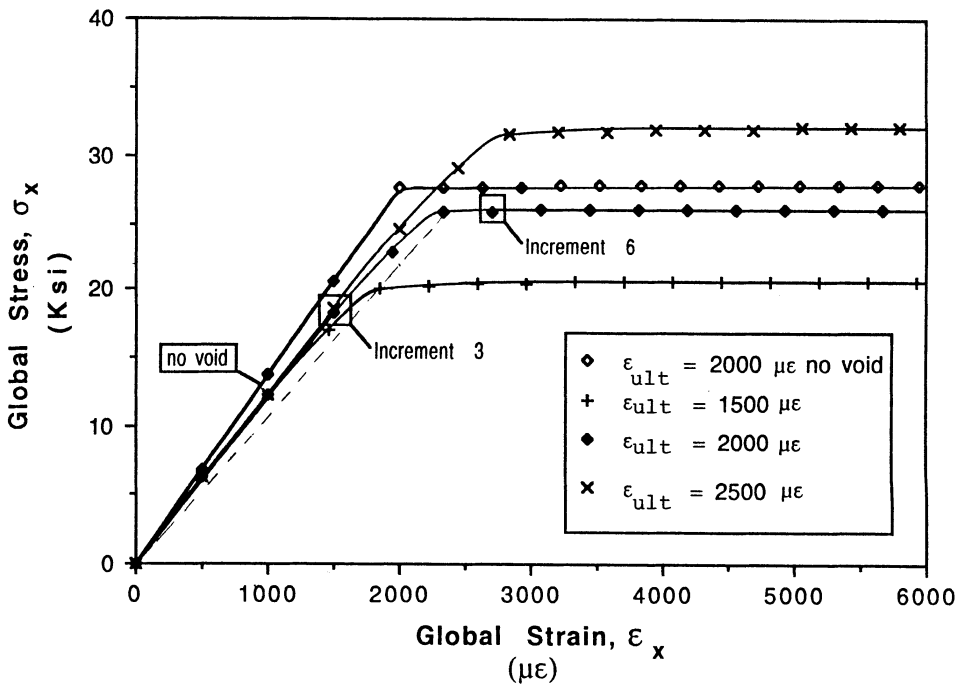


Figure 5.4-3 Global stress-strain response at different strain allowables in the unit cell.

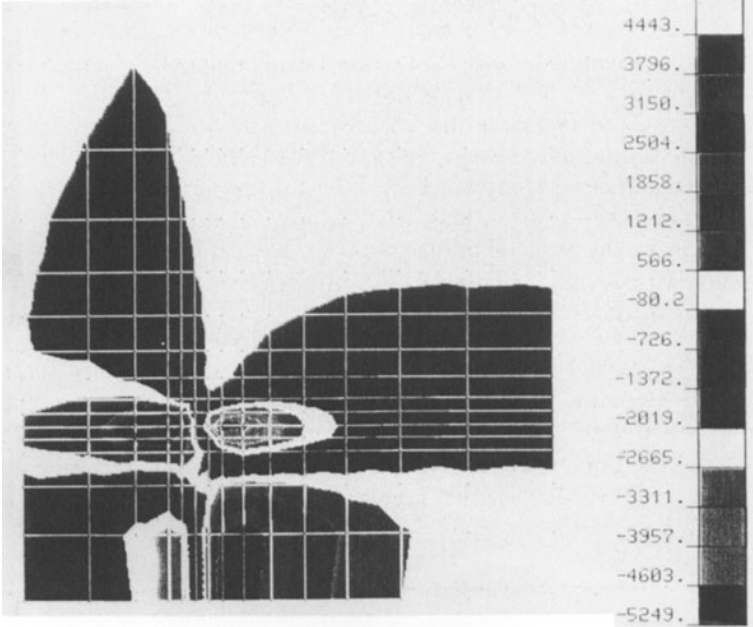


Figure 5.4-4 Stress (σ_x) in unit cell subjected to tension.

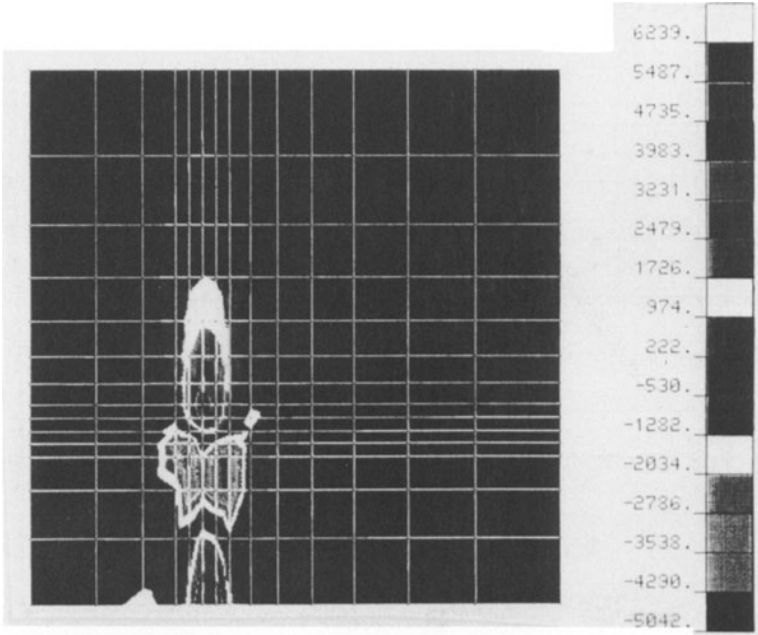


Figure 5.4-5 Stress (σ_y) in unit cell subjected to tension.

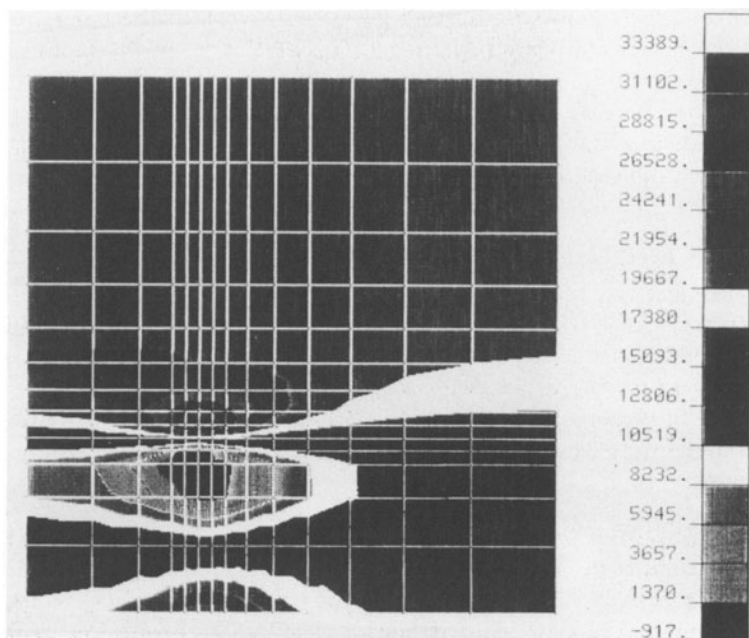


Figure 5.4-6 Stress (σ_{xy}) in unit cell subjected to tension.

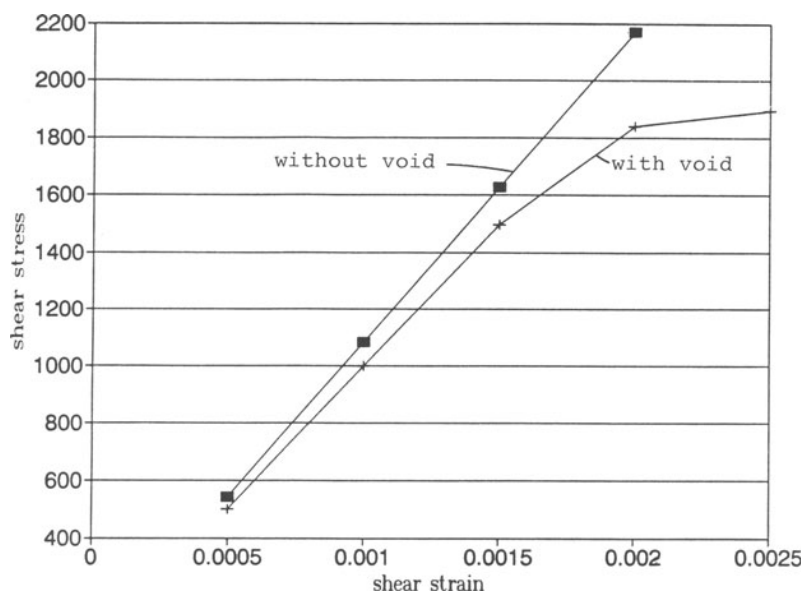


Figure 5.4-7 Shear stress-strain response with and without void.

Since the actual laminate consists of many layers of the woven fabric with random defects, it is appropriate to calculate an effective elastic modulus that can be used as a layer modulus for laminate analysis. Thus, the effective modulus from Figure 5.4-3 can be interpreted as 10 Msi. Typical stress contours of σ_{11} , σ_{12} and σ_{22} are presented in Figures 5.4-4 through 5.4-6. For the in-plane normal stresses, the influence of the debond on the stress distribution is easily observed. Note that the inplane shear stress distribution around the debond is antisymmetric.

In order to capture the response under shear load, the full unit cell is modeled. The shear load is simulated by applying displacements in the x -direction to the top surface and fixing the bottom surface. Multipoint constraints are used on the remaining sides so that they deform as a straight line. The shear stress-strain response to the pure shear loading condition is shown in Figure 5.4-7. Similarly, the reduction in stress magnitudes in the presence of a debond is quite prevalent. Once again, the use of an effective shear moduli of 1.16 Msi is recommended (see Figure 5.4-7).

The significance of processing related defects is illustrated clearly in the evaluation of elastic moduli for the unit cell. The approach brings out the importance of “scaling” in composite structures. The processing anomalies, which are measured in microns, have significant impact on the lamina properties, which in turn will affect the structural response of a component. Extreme care should be taken to understand a prepreg architecture and processing quality before any structural modeling is attempted.

5.5 Environmental effects

5.5-1 Introduction

Here we review some results for unsymmetric composite laminates subjected to mechanical, thermal, and hygroscopic loads (see [22]). The ambient (room) conditions consist of a temperature of 22° C and 70% relative humidity (RH). The *hot/wet* (service) conditions are 82° C and 90% relative humidity. An unsymmetric $(0_4/45_4)_T$ laminate is considered, and the symmetric $(0_2/45_2)_s$ laminate is used as the baseline. In order to observe the effects of hygrothermal gradients on the deformed shape, failure modes, and moduli of laminates, tension and torsion tests were performed under different environmental conditions.

5.5-2 Numerical models

A quasi-three dimensional finite element model developed by Chan and Ochoa [23] is used to calculate interlaminar stresses. In this work, a rectangular laminate, long in the x -direction (see Figure 5.5-1) is used. Hence, the displacement gradients are neglected with respect to the x -direction. The cross-section of the laminate in the $y - z$ plane is modeled using an eight-node isoparametric, quadrilateral element with three degrees of freedom (u, v, w) per node. A uniform strain

in the x -direction is applied to simulate the tension load. Torsion loads are applied as a constant twisting curvature, K_{xy} .

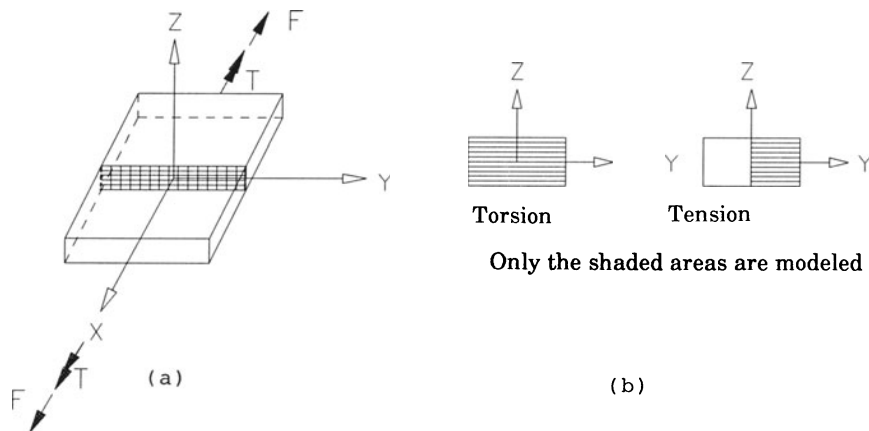


Figure 5.5-1 Schematic of the finite element model of a rectangular laminate.

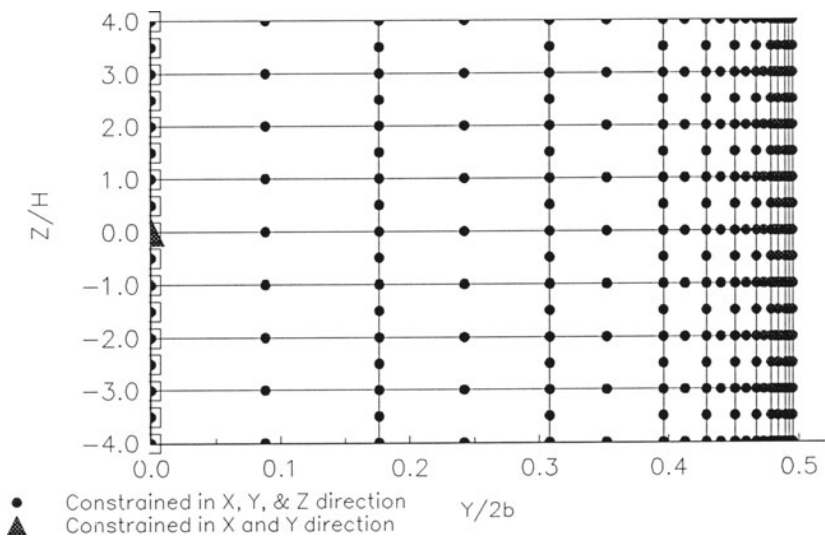


Figure 5.5-2 Finite element mesh for tension model (see Fig. 5.5-1).

When simulating a tension load on symmetric laminates, we assume that the stress distributions are symmetric about the midplane. Therefore, a quarter of the laminate can be modeled. However, for unsymmetric laminates, the symmetric half of the $y-z$ section should be modeled. Regardless of laminate symmetry, the full cross-section should be modeled for torsion loads. The mesh used to model the unsymmetric laminate subjected to tension is presented in Figure 5.5-2. It should be noted that unsymmetric laminates are not straight along the length (i.e., they have initial curvature). Each layer is modeled with a single element through its thickness. Note that the origin is located at the center. In all cases, the mesh is refined at the free edge where delamination is most likely to start. The finite element mesh used in the quarter model consists of 44 elements with 156 nodes. The finite element meshes used for the half and full models consist of 88 elements with 303 nodes and 176 elements with 589 nodes, respectively. The boundary conditions for the three models are shown in Fig. 5.5-2.

The following material properties for the IM7/977-2 graphite epoxy are used in the analysis.

$$\begin{aligned}
 E_1 &= 3.38 \times 10^6 \text{ GPa} & E_2 &= 1.78 \times 10^5 \text{ GPa} \\
 G_{12} = G_{23} = G_{13} &= 9.86 \times 10^4 \text{ GPa} & \nu_{12} = \nu_{23} = \nu_{13} &= 0.325 \\
 \alpha_1 &= 9.52 \times 10^{-7} (\text{m/m})/^{\circ}\text{C}, & \alpha_2 = \alpha_3 &= 1.4392 \times 10^{-4} \\
 \beta_1 &= 0.00 (\text{m/m})/\% \text{M}, & \beta_2 = \beta_3 &= 0.146 \times 10^{-2}
 \end{aligned}
 \tag{5.5-1}$$

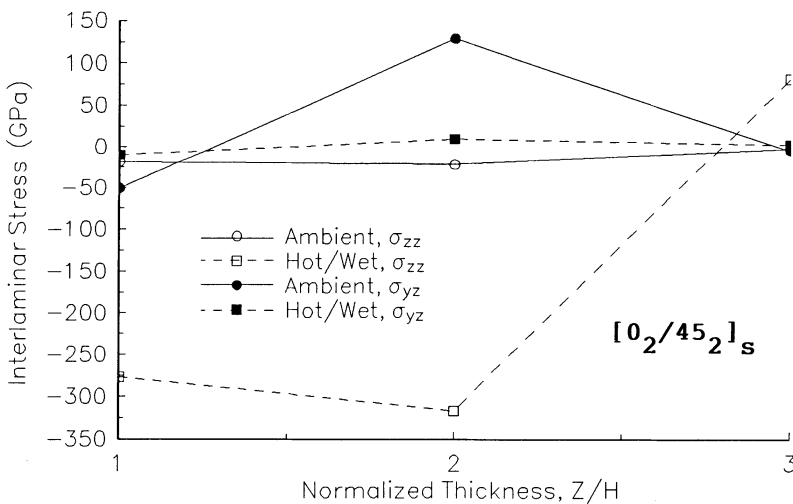


Figure 5.5-3 Interlaminar stress distribution for $(0_2/45_2)_s$ laminate subjected to a tension load.

5.5-3 Discussion of the results

Tension

The interlaminar stresses for the symmetric laminate are presented in Figure 5.5-3. The largest interlaminar normal stress occurs at the 45/0 interface under ambient conditions. Under *hot/wet* conditions, the largest tensile interlaminar stress is observed at 0/0 interface and the largest compressive interlaminar stress occurs at 45/0. The highest interlaminar shear stress occurs at the 0/45 interface for both ambient and hot/wet conditions.

Figures 5.5-4 and 5.5-5 present the stress distributions for the unsymmetric laminate subjected to an applied strain of 0.0001 in/in. At this strain level, the laminate is nearly straight. The laminate is predicted to be completely straight at a strain level of 0.000123 in/in. Note that unlike the previous stress distributions, these stress distributions are not symmetric. Under hot/wet conditions, the normal stress of largest magnitude occurs at the 45/45 interface nearest the center of the cross-section (see Fig. 5.5-4). At the 0/0 interface nearest to the center of the cross-section, there is a positive σ_{zz} , which identifies this region as a possible delamination site. Under hot/wet conditions, the maximum interlaminar shear occurs at the 0/45 interface (see Fig. 5.5-5). Note that the magnitudes of the interlaminar stresses are greatly increased when the hygrothermal effects are included.

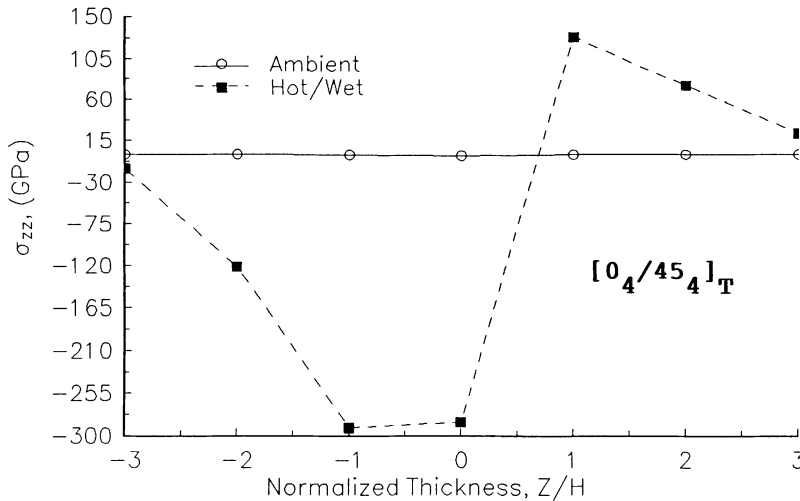


Figure 5.5-4 Interlaminar normal stress distribution for $(0_4/45_4)_T$ laminate subjected to a tension load.

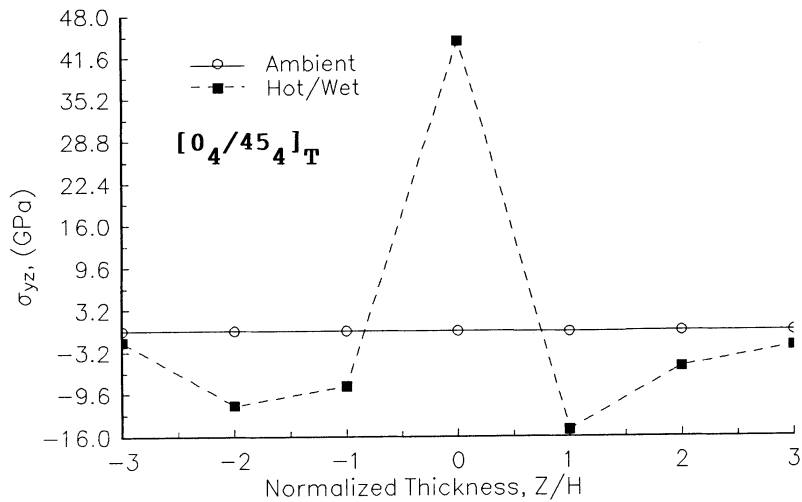


Figure 5.5-5 Interlaminar shear stress distribution for $(0_4/45_4)_T$ laminate subjected to a tension load.

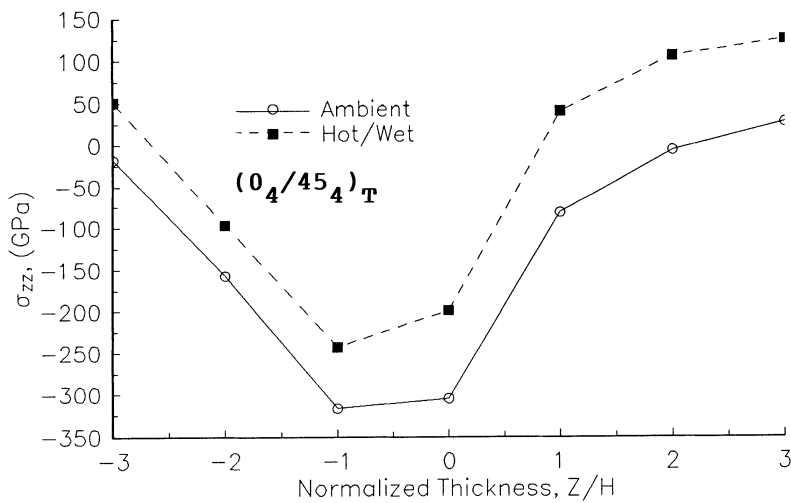


Figure 5.5-6 Interlaminar normal stress distribution for $(0_4/45_4)_T$ laminate subjected to a torsion load.

Torsion

Figures 5.5-6 and 5.5-7 contain plots of the interlaminar transverse normal and shear stress distributions for the unsymmetric laminate. The highest positive interlaminar normal stress occurs at the 0/0 interface nearest to the top surface. Under the hot/wet conditions there is a general increase in this stress. The highest interlaminar shear stress occurs at the 0/45 interface. A significant increase in this stress is observed upon addition of the hygrothermal gradients. These interlaminar stresses can cause delamination and fiber-matrix interface damage. Therefore, the decrease in the torsional resistance, observed during testing, when the hygrothermal gradients are applied, can be attributed to this increase in interlaminar stress.

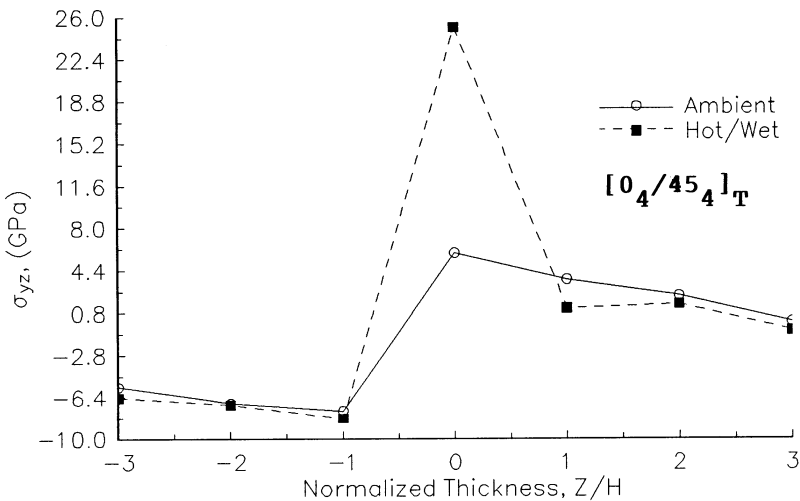


Figure 5.5-7 Interlaminar shear stress distribution for $(0_4/45_4)_T$ laminate subjected to a torsion load.

Figure 5.5-8 presents the interlaminar stresses for the symmetric laminate. The largest interlaminar normal stress occurs at the 45/45 interface away from the midplane. Addition of the hot/wet condition results in an upward shift of the curve. The maximum interlaminar shear occurs at the 45/0 interfaces. This stress is increased at one of these 45/0 interfaces and decreased at the other 45/0 interface when the hygrothermal gradients are applied. Note that these stress distributions are symmetric for the interlaminar shear stress and antisymmetric for the interlaminar normal stress. This symmetry is lost when the hot/wet conditions are applied.

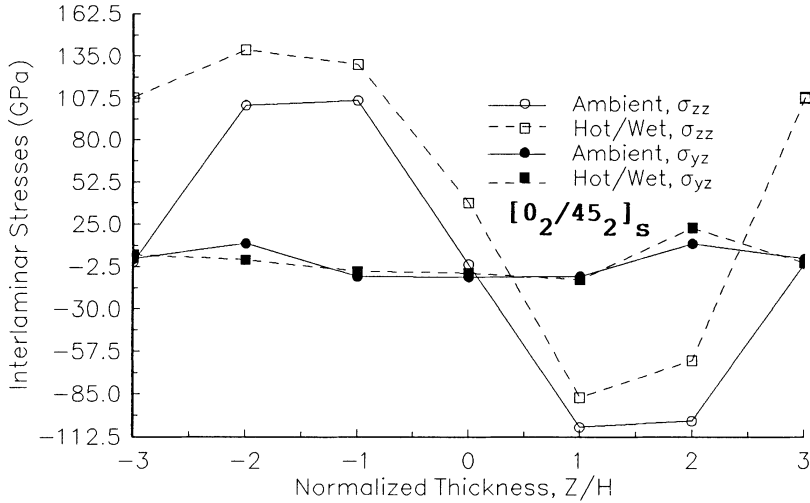


Figure 5.5-8 Interlaminar stress distribution for $(0_2/45_2)_T$ laminate subjected to a torsion load.

From these finite element results we can observe several trends. For a symmetric laminate subjected to either tension or torsion, the stress distributions are either symmetric or antisymmetric; for an unsymmetric laminate they are unsymmetric. Under tension, the hygrothermal loads cause an increase in the interlaminar shear stress. The high interlaminar normal stress in the unsymmetric laminates subject to tension leads to catastrophic brittle failure. Under torsion, the hygrothermal loads cause an increase in the interlaminar normal stresses for the symmetric and unsymmetric laminates. Note that these trends focus on the material system and loading selected in the present examples.

5.6 First-ply failure analysis of composite laminates

5.6-1 Introduction

The design of composite structures requires prediction of the maximum load that a laminate can withstand before it fails. Although the literature is rich in linear elastic analysis of composite structures, not much progress has been reported on analytical or numerical techniques for the failure analysis of composite structures while accounting for geometric nonlinearity. In this section, we present the first-ply failure analysis results of composite laminates subjected to inplane and bending loads. The geometric nonlinearity in the sense of von Kármán [see Sec. 2.4, Eqs. (2.4-9)] and transverse shear deformation effect in the sense of the first order kinematics are included. The literature reviewed in the next paragraph provides a background for this study.

The first series of analytical solutions for first ply failure loads were presented by Turvey [24-29] for symmetric and antisymmetric composite laminates with simply supported boundary conditions. The analytical solutions are based on Reddy's third-order small deflection laminated plate theory [30]. The first finite element procedure for the prediction of first ply failure loads of composite laminates subjected to transverse and in plane (tensile) loading was presented by Reddy and Pandey [31]. In that study, the effect of geometric nonlinearity was not considered. Turvey and Osman [32] presented results for the first-ply failure loads of composite laminates subjected to uniform transverse load, including geometric nonlinearity. In their analysis, they have made use of the Dynamic Relaxation (DR) technique.

A first-ply failure analysis is carried out by Reddy and Reddy [33] and the displacements and stresses are computed using the finite element method. The results of this study are discussed here.

5.6-2 Problem description

Flat rectangular laminates of dimensions 9 in. $\times$ 5 in. and made of T300/5208 graphite epoxy material are analyzed for first-ply failure loads. A total of four laminates with different lamination schemes are analyzed for three different types of loads, namely: (1) uniformly distributed transverse load, (2) concentrated transverse load acting at the center of the plate, and (3) in plane tensile load acting on the shorter edge of the laminate. Both simply supported and clamped boundary conditions are used for the failure analysis. In addition to the failure criteria presented in Chapter 4 (Sec. 4.3), the following three failure criteria, namely (1) maximum 2-D strain (independent) (2) maximum strain (polynomial) with 3-dimensional state of strain, and (3) maximum strain (independent) with 3-dimensional state of strain are used (see [33]). The 3-dimensional state of strain is achieved by relaxing the condition of inextensibility of transverse normals, which was assumed in deriving the first order shear deformation theory [34] (see Sec. 2.4-3). The present discussion is based on the work reported in [33].

The displacement finite element model of the first order shear deformation plate theory developed in Chapter 3 (also see [34-37]), is used in the analysis. It consists of Lagrange isoparametric rectangular elements with five degrees of freedom (u, v, w, ϕ_1, ϕ_2) per node. Reduced integration is used for the shear stiffnesses to avoid shear locking. The Newton-Raphson iteration procedure is used to solve the nonlinear algebraic equations.

5.6-3 Procedure for first ply failure analysis

The first-ply failure analysis is based on the assumption that a given ply will fail if the failure index, $\mathcal{F}$, at any point within the ply reaches a value of unity. The failure index is defined as $\mathcal{F} = F_i \sigma_i + F_{ij} \sigma_i \sigma_j$ (see Sec. 4.3-2). The first-ply failure load computation in the linear formulation does not require an iterative

procedure. In the nonlinear failure analysis, equilibrium iteration is required. The steps used in the iterative procedure are listed here:

- Step 1 Find the displacement field $(u, v, w, \phi_1, \phi_2)$ for a small initial load
- Step 2 Find the stresses in the global coordinates $(\sigma_x, \sigma_y, \sigma_{yz}, \sigma_{xz}, \sigma_{xy})$
- Step 3 Transform the global stresses to the stresses $\bar{\sigma}_{ij}$ in material coordinates [see Eq. (2.3-2)]
- Step 4 Find the maximum failure index, $\mathcal{F}$
- Step 5 Check whether the laminate has failed or not (*i.e.*, $\mathcal{F} \geq 1$)
- Step 6 If $\mathcal{F} < 1$, increase the load; and if $\mathcal{F} \geq 1$, decrease the load proportionately, to make $\mathcal{F} = 1$
- Step 7 Repeat Steps 1 to 6 until the laminate fails

The maximum failure index is determined by carrying out a sequential search at certain chosen points within the laminate. The search sequence is as follows:

- (1) Consider the first element of the finite element mesh
- (2) Consider the first Gauss point within the element
- (3) Consider the first ply (from bottom to top) of the laminate
- (4) Consider the bottom of the current ply
- (5) Find the failure index at that location
- (6) Check whether this is greater than the previous value
- (7) If yes, then store the element number, Gauss point number, ply number, location in the ply and the failure index, and go to the next location (middle of the ply, top of the ply, so on.)
- (8) If answer in (6) is no, then do not store the failure data, but go to the next location and repeat (5) to (8)
- (9) Go to the next ply
- (10) Go to the next Gauss point
- (11) Go to the next element
- (12) Continue the search among all the elements in the finite element mesh for the maximum failure index.

5.6-4 Numerical models and discussion of results

The laminates considered for first ply failure analysis are made of T300/5208 graphite-epoxy (pre-preg) material, whose properties are taken as follows:

$$\begin{aligned}
 E_1 &= 19.20 \times 10^6 \text{ psi}, & E_2 &= 1.56 \times 10^6 \text{ psi}, & E_3 &= 1.56 \times 10^6 \text{ psi}, \\
 G_{12} &= 0.82 \times 10^6 \text{ psi}, & G_{13} &= 0.82 \times 10^6 \text{ psi}, & G_{23} &= 0.49 \times 10^6 \text{ psi}, \\
 \nu_{12} &= 0.24, & \nu_{13} &= 0.24, & \nu_{23} &= 0.49, \\
 X_T &= 219.50 \times 10^3 \text{ psi}, & X_C &= 246.00 \times 10^3 \text{ psi}, & Y_T &= Z_T = 6.35 \times 10^3 \text{ psi}, \\
 Y_C &= Z_C = 6.35 \times 10^3 \text{ psi}, & R &= 9.80 \times 10^3 \text{ psi}, & S &= T = 12.60 \times 10^3 \text{ psi}, \\
 && \text{Ply thickness, } h &= 0.005 \text{ in.} & & (5.6-1)
 \end{aligned}$$

Four lamination schemes are used in the present first ply failure analysis. The individual laminates are labelled *A* – *D* for future reference in this section. The following four laminated plates from [33] are selected for discussion:

- A* : symmetric angle-ply laminate, $(45/-45/45)_T$
- B* : symmetric quasi-isotropic laminate, $(45/-45/90/0/45/90/-45/0)_s$
- C* : symmetric cross-ply laminate, $(0/90/90/0)_T$
- D* : antisymmetric cross-ply laminate, $(0/90/0/90)_T$

where *t* denotes total laminate and *s* denotes symmetric laminate. The geometry, coordinate system, and the ply numbering scheme within the laminate are shown in Figure 5.6-1a. Two types of finite element meshes are used in the analysis: (i) 6×3 (Figure 5.6-1b) and (ii) 9×5 (Figure 5.6-1c). The first mesh is used in a quarter of the plate for the failure analysis of cross-ply laminates *C* and *D*. The second mesh is for a full plate and is used for the failure analysis of laminates *A* and *B*.

Laminates with all edges simply supported or all edges clamped are used:

$$SS1 : u = w = \phi_1 = 0 \text{ at } y = 0, b; \quad v = w = \phi_2 = 0 \text{ at } x = 0, a \quad (5.6-2)$$

$$CC1 : u = v = w = \phi_1 = \phi_2 = 0 \text{ at } x = 0, a \text{ and } y = 0, b \quad (5.6-3)$$

For quarter symmetry, the following boundary conditions along the lines of symmetry are used (see Fig. 5.6-2):

$$v = \phi_2 = 0 \text{ at } y = \frac{b}{2}; \quad u = \phi_1 = 0 \text{ at } x = x = \frac{a}{2} \quad (5.6-4)$$

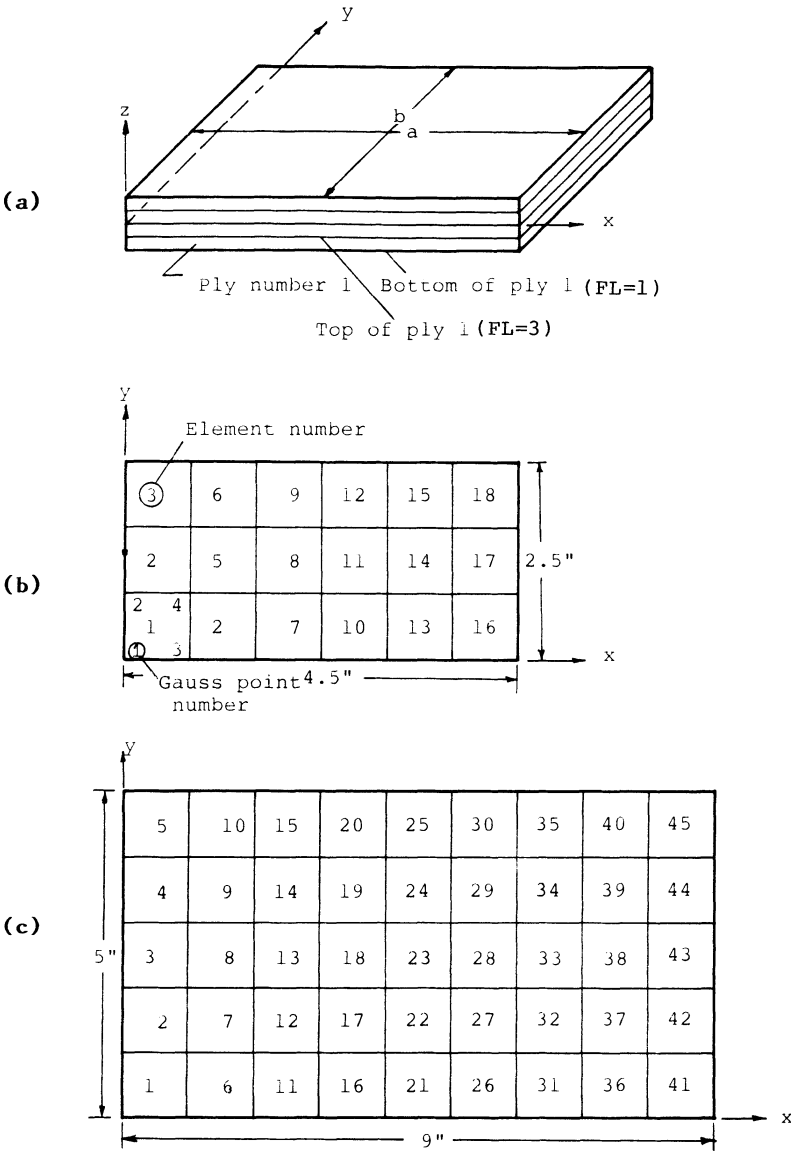


Figure 5.6-1 The geometry, coordinate system and ply numbering scheme used.

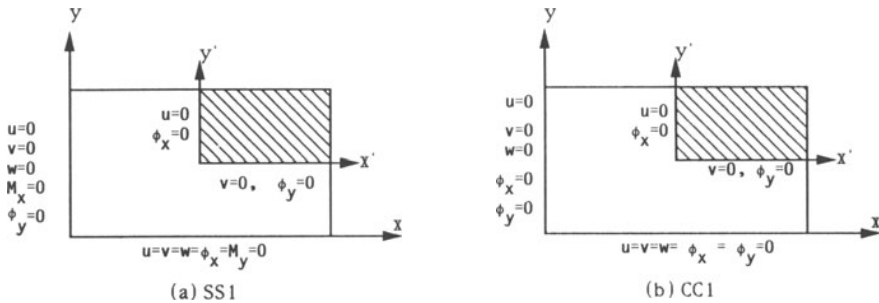


Figure 5.6-2 Boundary conditions for full and quarter plates subjected to transverse load. (a) SS1; (b) CC1.

For the in-plane loading case, two types of boundary conditions are used: (1) hinged and (2) clamped on the left edge. The other edge is on a roller support in both cases. The details of these support conditions are shown in Figure 5.6-3.

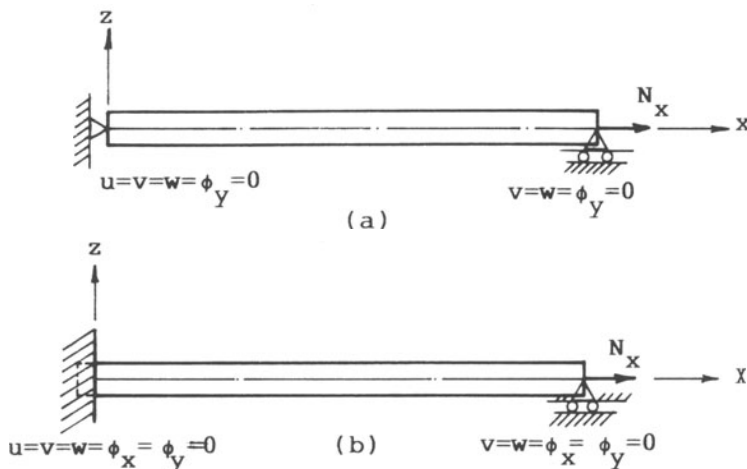


Figure 5.6-3 Boundary conditions for full and quarter plates subjected to in-plane (tensile) loading. (a) Hinged-end boundary conditions. (b) Clamped-end boundary conditions.

Table 5.6–1 Linear and nonlinear first ply failure loads for laminates *A* and *B* under uniformly distributed transverse load with *SS1* boundary condition.

Failure criterion	FLD	CDF	FEL	FGP	FPL	FL
Laminate A (45/-45/45)						
Maximum stress (p)	24158.1(a)	57.307	45	4	1	1
	2690141.4(b)	14.187	35	3	3	3
Maximum strain (p)	21660.4	51.382	45	4	1	1
	3418133.4	15.366	11	2	3	3
Tsai-Hill	24285.3	57.609	45	4	1	1
	2742860.1	14.279	11	2	3	3
Hoffman	24131	57.245	45	4	1	1
	2695775.0	14.196	11	2	3	4
Tsai-Wu	23248.9	55.151	45	4	1	1
	2915620.3	14.572	11	2	3	3
Maximum stress (i)	24719.1	58.638	45	4	3	3
	3045155.7	14.785	39	2	3	3
Maximum strain(i)	22108.6	52.446	45	4	1	1
	3835494.7	15.968	7	1	3	3
Maximum strain(p3D)	19800.5	46.971	45	4	1	1
	2764268.8	14.316	11	2	3	3
Maximum strain(i3D)	22108.6	52.446	45	4	1	1
	3835494.7	15.968	7	1	3	3
Laminate B (45/-45/90/0/45/90/-45/0)						
Maximum stress(p)	2182.3	4.020	23	1	16	3
	9689.1	1.956	7	4	16	3
Maximum strain(p)	1974.1	3.636	45	4	1	1
	9952.8	1.974	1	4	16	3
Tsai-Hill	2212.8	4.076	23	1	16	3
	9883.7	1.969	7	1	16	3
Hoffman	2185.7	4.026	23	1	16	3
	9710.8	1.957	39	1	16	3
Tsai-Wu	2245.2	4.135	45	4	1	1
	10047.3	1.980	45	1	16	3
Maximum stress(i)	2352.5	4.333	28	2	1	1
	9897.6	1.970	7	1	16	3
Maximum strain(i)	2083.7	3.838	45	4	1	1
	9885.4	1.969	1	4	16	3
Maximum strain(p3D)	1820.1	3.352	45	4	1	1
	8251.8	1.850	45	1	16	3
Maximum strain(i3D)	2083.7	3.838	45	4	1	1
	9885.4	1.969	1	4	16	3

Table 5.6–2 Linear and nonlinear first ply failure loads for laminates *C* and *D* uniformly distributed transverse load with *SS1* boundary condition.

Failure criterion	FLD	CDF	FEL	FGP	FPL	FL
Laminate C (0/90/90/0)						
Maximum stress(p)	11363.3	49.710	1	1	4	3
	948395.9	8.558	1	1	4	3
Maximum strain(p)	12212.6	53.425	1	1	4	3
	1051787.7	8.858	3	1	4	3
Tsai-Hill	11436.8	50.031	1	1	4	3
	961543.7	8.597	1	1	4	3
Hoffman	11370.8	49.742	1	1	4	3
	949771.2	8.562	1	1	4	3
Tsai-Wu	11646.5	50.948	1	1	4	3
	1001900.2	8.716	1	1	4	3
Maximum stress(i)	11480.6	50.223	1	1	1	1
	964904.2	8.608	1	1	4	3
Maximum strain(i)	12339.1	53.978	1	1	1	1
	1058146.3	8.875	3	1	4	3
Maximum strain(p3D)	10800.4	47.247	1	1	4	3
	871633.2	8.321	3	1	4	3
Maximum strain(i3D)	12339.1	53.978	1	1	1	1
	1058146.3	8.875	3	1	4	3
Laminate D (90/0/90/0)						
Maximum stress(p)	19834.3	45.730	1	1	4	3
	789853.1	8.155	3	2	4	3
Maximum strain(p)	18592.9	42.868	1	1	4	3
	671425.5	7.732	3	2	4	3
Tsai-Hill	19932.9	45.958	1	1	4	3
	791457.2	8.160	3	2	4	3
Hoffman	19844.1	45.753	1	1	4	3
	790006.5	8.156	3	2	4	3
Tsai-Wu	20204.0	46.583	1	1	4	3
	795511.8	8.174	3	2	4	3
Maximum stress(i)	19970.4	46.044	1	1	4	3
	791487.4	8.161	3	2	4	3
Maximum strain(i)	21071.5	48.583	1	1	4	3
	807807.4	8.216	3	2	4	3
Maximum strain(p3D)	18592.9	42.868	1	1	4	3
	21071.5	48.583	1	1	4	3
Maximum strain (i3D)	21071.5	48.583	1	1	4	3
	807807.4	8.216	3	2	4	3

The results for the linear and nonlinear first ply failure analysis of laminates *A – D* are presented for three different kinds of loads: (1) uniformly distributed transverse load; (2) concentrated transverse load acting at the center of the laminate; (3) uniformly distributed in plane (tensile) load acting on the shorter edge of the laminate. Two different kinds of boundary conditions are used. The significant features of the results specific to the type of loading and boundary conditions are discussed here.

Uniformly distributed transverse load with SS1 boundary condition

The results for the uniformly distributed transverse load with SS1 boundary condition are shown in Tables 5.6-1 and 5.6-2. In these tables the following notation is used:

CDF = Non-dimensionalized center deflection ($CDF = w/h$)

F = First-ply failure load (psi)

FEL = Failed element

FL = Failed location (see Fig. 5.6-1a)

FLD = Non-dimensionalized first-ply failure load [$FLD = (F/E_2)(a/h)^4$]

FGP = Failed gauss point

FPL = Failed ply

w = Center deflection (in.)

(a) = Linear failure load

(b) = Nonlinear failure load

(p) = Polynomial criterion

(i) = Independent criterion

(p3D) = Polynomial criterion with 3-Dimensional state of strain

(i3D) = Independent criterion with 3-Dimensional state of strain

The nonlinear failure load is found to be 173 times the linear failure load for laminate *A* (3 layers), 85 times for laminate *C* (4 layers), 40 times for laminate *D* (4 layers), and only 4.7 times for laminate *B* (16 layers). This shows that the difference between the linear and nonlinear failure loads decreases sharply as the thickness of the laminate is increased. However, it should be noted here that the non-dimensionalized center deflection of thin laminates (3, 4 layers) is so large (see Tables 5.6-1 and 5.6-2) that the large deflection formulation is no longer valid in

this regime. A formulation based on large deflection-large rotation theory might describe the failure behavior of such thin laminates more accurately.

As far as the thicker laminates (16 layers) are concerned, the difference between the linear and non-linear failure loads stems from the fact that the non-linear formulation accounts for the actual stiffening of the plate as it deflects. Hence, for a given load the non-linear model predicts much smaller deflections than the linear model, and smaller deflections in turn give rise to less severe stress fields and large first ply failure loads.

The failure locations do not show any definite trend, but the most frequently occurring locations are: (1) the center of the plate and (2) the corners of the plate. For thin laminates and linear analysis, the failure takes place mostly at the bottom of the laminate; and for thick and nonlinear analysis it is within the laminate, or at the top of the laminate where the external load is applied.

Uniformly distributed transverse load with CC1 boundary condition

The results for the uniformly distributed transverse load with CC1 boundary condition are shown in Table 5.6-3 and 5.6-4. The nonlinear failure load is found to be 18 times the linear failure load for laminate *A* (symmetric angle-ply), 15 times for laminate *C* (symmetric cross-ply), 10 times for laminate *D* (antisymmetric cross-ply), and only 1.4 times for laminate *B* (symmetric quasi-isotropic), which are small compared to the SS1 boundary condition. This shows that the difference between the linear and nonlinear failure loads is smaller for clamped boundary conditions than for the simply supported boundary conditions. The most frequently occurring failure locations are the midpoint of the longer side of the laminate, at the bottom of the laminate, or within the laminate.

Uniformly distributed inplane tensile edge load

The results for the uniformly distributed in plane (tensile) load acting on the smaller edge of the laminated with hinged end boundary condition are shown in Tables 5.6-5 and 5.6-6. The nonlinear failure load is found to be 1.5 times the linear failure load for laminate *D*. The linear and nonlinear failure loads are exactly the same for laminates *B* and *C*, and they differ slightly for laminate *A*. This indicates the difference between the linear and the nonlinear failure loads is small for inplane tensile load. The failure locations are at the corners for all types of laminates, mostly at the bottom for thin (3 or 4 layer) laminates, and within the laminate for thick (16 layer) laminates, for both linear and nonlinear failure analysis. The results for the uniformly distributed in plane tensile load acting on the smaller edge of the laminate with clamped-end boundary condition show that (the results are not included here; see [33]) the linear and nonlinear failure loads are found to be almost the same for all laminates. The failure location follows the same trend as for hinged-end boundary conditions.

Table 5.6–3 Linear and nonlinear first ply failure loads for laminates *A* and *B* under uniformly distributed transverse load with *CC1* boundary condition.

Failure criterion	FLD	CDF	FEL	FGP	FPL	FL
Laminate C (0/90/90/0)						
Maximum stress(p)	75488.5(a)	51.175	30	2	1	1
	1361521.2(b)	11.034	1	4	3	3
Maximum strain(p)	88692.5	60.126	30	2	1	1
	1741423.3	11.981	1	4	3	3
Tsai-Hill	76509.7	51.867	30	2	1	1
	1388666.1	11.107	1	4	3	3
Hoffman	75597.4	51.249	30	2	1	1
	1364452.0	11.042	45	1	3	3
Tsai-Wu	79809.2	54.104	30	2	1	1
	1478786.2	11.344	1	4	3	3
Maximum stress(i)	85507.9	57.967	30	2	1	1
	1426237.2	11.207	1	4	3	3
Maximum strain(i)	101374.6	68.723	45	1	3	3
	1845059.2	12.215	45	1	3	3
Maximum strain(p3D)	77741.0	52.702	30	2	1	1
	1346536.2	10.993	45	1	3	3
Maximum strain(i3D)	101374.6	68.723	45	1	3	3
	1845059.2	12.215	45	1	3	3
Laminate B (45/-45/90/0/45/90/-45/0)						
Maximum stress(p)	3576.2	1.836	21	3	1	1
	5144.9	1.283	21	3	1	1
Maximum strain(p)	3630.1	1.863	21	1	13	3
	4554.6	1.209	21	1	4	1
Tsai-Hill	3611.8	1.854	21	3	1	1
	5248.4	1.296	21	3	1	1
Hoffman	3581.5	1.838	21	3	1	1
	5155.9	1.285	21	3	1	1
Tsai-Wu	3766.0	1.933	21	3	1	1
	5291.2	1.301	21	1	4	1
Maximum stress(i)	4048.1	2.078	21	1	4	1
	5307.7	1.303	21	1	4	1
Maximum strain(i)	4064.1	2.086	21	1	4	1
	5336.0	1.306	21	1	4	1
Maximum strain(p3D)	3630.1	1.863	21	1	13	3
	4554.6	1.209	21	1	4	1
Maximum strain(i3D)	4064.1	2.086	21	1	4	1
	5336.0	1.306	21	1	4	1

Table 5.6–4 Linear and nonlinear first ply failure loads for laminates *C* and *D* uniformly distributed transverse load with *CC1* boundary condition.

Failure criterion	FLD	CDF	FEL	FGP	FPL	FL
Laminate C (0/90/90/0)						
Maximum stress(p)	19025.1	19.314	3	2	1	1
	283271.8	5.636	3	2	1	1
Maximum strain(p)	19103.7	19.394	3	2	4	3
	287298.2	5.663	3	2	1	1
Tsai-Hill	19032.2	19.321	3	2	4	3
	283649.3	5.638	3	2	1	1
Hoffman	19025.8	19.315	3	2	1	1
	283307.7	5.636	3	2	1	1
Tsai-Wu	19050.9	19.340	3	2	1	1
	284591.0	5.645	3	2	1	1
Maximum stress(i)	19031.8	19.321	3	2	1	1
	281201.0	5.622	3	2	1	1
Maximum strain(i)	19105.5	19.396	3	2	1	1
	284937.7	5.647	3	2	1	1
Maximum strain(p3D)	17142.2	17.403	3	2	4	3
	231578.3	5.261	3	2	1	1
Maximum strain(i3D)	19105.5	19.396	3	2	1	1
	284937.7	5.647	3	2	1	1
Laminate D (90/0/90/0)						
Maximum stress(p)	30366.5	16.433	3	2	4	3
	312154.2	5.739	16	3	1	1
Maximum strain(p)	27354.2	14.803	3	2	4	3
	256184.1	5.357	16	3	1	1
Tsai-Hill	30358.4	16.428	3	2	4	3
	312428.6	5.740	16	3	1	1
Hoffman	30367.8	16.433	3	2	4	3
	312180.2	5.739	16	3	1	1
Tsai-Wu	30412.3	16.457	3	2	4	3
	313110.2	5.745	16	3	1	1
Maximum stress(i)	30359.2	16.429	3	2	4	3
	311331.7	5.731	16	3	1	1
Maximum strain(i)	30490.3	16.500	3	2	4	3
	314040.5	5.749	16	3	1	1
Maximum strain(p3D)	27354.2	14.803	3	2	4	3
	256184.1	5.357	16	3	1	1
Maximum strain(i3D)	30490.3	16.500	3	2	4	3
	314040.5	5.749	16	3	1	1

Table 5.6–5 Linear and nonlinear first ply failure loads for laminates *A* and *B* under inplane tensile load with hinged boundary condition.

Failure criterion	FLD	CDF	FEL	FGP	FPL	FL
Laminate A (45/-45/45)						
Maximum stress(p)	91705025.8(a)	2.954	5	2	1	1
	91705025.8(b)	2.954	5	2	1	1
Maximum strain(p)	94086883.6	3.031	5	2	1	1
	94086883.6	3.031	5	2	1	1
Tsai-Hill	91954605.1	2.962	5	2	1	1
	91954605.1	2.962	5	2	1	1
Hoffman	91726868.1	2.955	5	2	1	1
	91726868.1	2.955	5	2	1	1
Tsai-Wu	92514644.4	2.980	5	2	1	1
	92514644.4	2.980	5	2	1	1
Maximum stress(i)	105177573.7	3.388	5	2	1	1
	105177573.7	3.388	5	2	1	1
Maximum strain(i)	108726302.2	3.502	5	2	1	1
	108726302.2	3.502	5	2	1	1
Maximum strain(p3D)	86127951.1	2.775	5	2	1	1
	86127951.1	2.775	5	2	1	1
Maximum strain(i3D)	108726302.2	3.502	5	2	1	1
	108726302.2	3.502	5	2	1	1
Laminate B (45/-45/90/0/45/90/-45/0)						
Maximum stress(p)	1179619.2	0.427	1	1	3	1
	1179619.2	0.427	1	1	3	1
Maximum strain(p)	1145218.3	0.414	1	1	3	1
	1145218.3	0.414	1	1	3	1
Tsai-Hill	1176280.9	0.426	1	1	3	1
	1176280.9	0.426	1	1	3	1
Hoffman	1179279.2	0.427	1	1	3	1
	1179279.2	0.427	1	1	3	1
Tsai-Wu	1167367.2	0.422	1	1	3	1
	1167367.2	0.422	1	1	3	1
Maximum stress(i)	1185182.2	0.429	1	1	3	1
	1185182.2	0.429	1	1	3	1
Maximum strain(i)	1150134.5	0.416	1	1	3	1
	1150134.5	0.416	1	1	3	1
Maximum strain(p3D)	1034938.0	0.375	1	1	3	1
	1034938.0	0.375	1	1	3	1
Maximum strain(i3D)	1150134.5	0.416	1	1	3	1
	1150134.5	0.416	1	1	3	1

Table 5.6–6 Linear and nonlinear first ply failure loads for laminates *C* and *D* under inplane tensile load with hinged boundary condition.

Failure criterion	FLD	CDF	FEL	FGP	FPL	FL
Laminate C (0/90/90/0)						
Maximum stress(p)	111920523.5	1.840	18	3	2	1
	111920523.5	1.840	18	3	2	1
Maximum strain(p)	111508402.3	1.833	18	3	2	1
	111508402.3	1.833	18	3	2	1
Tsai-Hill	111877188.5	1.839	18	3	2	1
	111877188.5	1.839	18	3	2	1
Hoffman	111916225.3	1.840	18	3	2	1
	111916225.3	1.840	18	3	2	1
Tsai-Wu	111763556.2	1.837	18	3	2	1
	111763556.2	1.837	18	3	2	1
Maximum stress(i)	111886430.3	1.839	18	3	2	1
	111886430.3	1.839	18	3	2	1
Maximum strain(i)	111439506.6	1.832	18	3	2	1
	111439506.6	1.832	18	3	2	1
Maximum strain(p3D)	100204590.9	1.647	18	3	2	1
	100204590.9	1.647	18	3	2	1
Maximum strain(i3D)	111439506.6	1.832	18	3	2	1
	111439506.6	1.832	18	3	2	1
Laminate D (90/0/90/0)						
Maximum stress(p)	58876296.1	1.119	18	3	1	1
	85868715.5	1.415	18	3	1	1
Maximum strain(p)	58860816.4	1.119	18	3	1	1
	85704764.0	1.412	18	3	1	1
Tsai-Hill	58872816.4	1.119	18	3	1	1
	85849814.9	1.415	18	3	1	1
Hoffman	58875952.3	1.119	18	3	1	1
	85866845.4	1.415	18	3	1	1
Tsai-Wu	58863722.0	1.119	18	3	1	1
	85800367.9	1.414	18	3	1	1
Maximum stress(i)	58873448.9	1.119	18	3	1	1
	85874167.6	1.415	18	3	1	1
Maximum strain(i)	58837572.2	1.119	18	3	1	1
	85679276.1	1.412	18	3	1	1
Maximum strain(p3D)	52860675.0	1.005	18	3	1	1
	76301306.0	1.258	18	3	1	1
Maximum strain(i3D)	58837572.2	1.119	18	3	1	1
	85679276.1	1.412	18	3	1	1

5.6–5 Summary and conclusions

The results discussed in the previous section show that different failure criteria may predict different failure loads for a given laminate. This is in part due to the difference in the formulation of the criteria, which are purely phenomenological in nature. The difference between the linear and the nonlinear failure loads is found to be large for transverse loading, considerably less for inplane loading with hinged end, and negligible for inplane loading with clamped end. It is also found that the difference is large for thin (3 or 4 layer) laminates and simply supported boundary conditions and small for thick (16 layer) laminates and clamped boundary conditions.

5.7 Postbuckling response and progressive failure

5.7–1 Introduction

The classical lamination theory, in which the transverse shear effects are neglected, is often used to analyze laminated composite structures. Because of low transverse moduli and strengths relative to inplane ones, composite laminates may fail due to transverse stresses. Indeed, it is found that (see [38,39]) composite laminates loaded in compression fail due to high interlaminar stresses. Therefore, shear deformation plate and shell formulations are needed to provide information regarding the through-thickness strength of composite structures. Various shear deformation theories and their computational models have been discussed in Chapter 3. Insight gained by using these formulations may aid in the characterization of failure modes of composite panels.

In this section, we discuss a recent study by Engelstad, Reddy and Knight [39,41] of the postbuckling response of several graphite-epoxy panels loaded in axial compression. The study makes comparisons between the experimentally-obtained and analytically-determined postbuckling response of composite panels.

5.7–2 Experimental study

The postbuckling and failure characteristics of flat, rectangular graphite-epoxy panels, with and without holes, and loaded in axial compression have been examined in an experimental study by Starnes and Rouse [38]. The panels were fabricated from commercially available unidirectional Thornel 300 graphite-fiber tapes pre-impregnated with 450° K cure Narmco 5208 thermosetting epoxy resin. Typical lamina properties for this graphite-epoxy system are 131.0 GPa (19,000 ksi) for the longitudinal Young's modulus, 13.0 GPa (1,890 ksi) for the transverse Young's modulus, 6.4 GPa (930 ksi) for the in-plane shear modulus, 0.38 for the major Poisson's ratio, and 0.14 mm (0.0055 in.) for the lamina thickness. Each panel was loaded in axial compression using a 1.33 MN (300 kips) capacity hydraulic testing machine. The loaded ends of the panels were clamped by fixtures during testing and the unloaded edges were simply supported by knife-edge restraints to

prevent the panels from buckling as wide columns. Most panels exhibited post-buckling strength and failed along a nodal line of the buckling mode in a transverse shear failure mode (see [38]). However, a different failure mode was observed for some of the 24-ply panels with holes. These panels failed along a transverse line passing through the hole, and failed soon after buckling.

The following three panels, denoted C4, C10 and H4 in Reference 38, are analyzed, and the present analytical results compared with the experimental results of Reference 38. Panel C4 is 50.8 cm by 17.8 cm (20.0 in. long and 7.0 in. wide), 24-ply orthotropic laminate, $(\pm 45/0_2/\pm 45/0_2/\pm 45/0/90)_s$. Panel C4 was observed in the test to buckle into two longitudinal half-waves and one transverse half-wave. Panel C10 is 50.8 cm by 14.0 cm (20.0 in. long by 5.5 in. wide), 24-ply quasi-isotropic laminate, $(\pm 45/0/90)_{3s}$. Panel C10 was observed in the test to buckle into four longitudinal half-waves and one transverse half-wave. The third panel, Panel H4 is identical to Panel C10 except for a 1.91 cm diameter (0.75 in. diameter) hole located 19.1 cm (7.5 in.) from one of the loaded edges and along the panel centerline. Panel H4 was observed in the test to buckle into four longitudinal half-waves and one transverse half-wave with the hole located near the buckle crest of the second longitudinal half-wave.

5.7-3 Numerical models

Finite element models of these panels were developed using 9-node quadrilateral shell elements. The finite element model used in [39] is a continuum-based formulation of a laminated 3-D degenerated shell element, 9CR, of Chao and Reddy [40]. The incremental equations of a continuous medium are formulated on the principle of virtual displacements and the total Lagrangian description (see Sec. 3.6). The shell element is obtained from the three-dimensional solid element by imposing two constraints: (1) straight lines normal to the midsurface before deformation remain straight, but not normal, after deformation; (2) the transverse normal components of strain, and hence stress, are ignored in the development. These assumptions turn a 3-D theory into a 2-D theory, and the resulting shell element is called a 3-D degenerated shell element. The nonlinear formulation admits large displacements and rotations of the shell element and small strains since the thickness does not change and the normal does not distort.

The final incremental equilibrium equations for an element are of the form [cf. Eq. (3.6-1)]

$$([K_L] + [K_{NL}])\{\delta U\} = \{R\} - \{F\} \quad (5.7-1)$$

where $\{\delta U\}$ is the vector of nodal incremental displacements in an element, and $[K_L]$, $[K_{NL}]$, and $\{F\}$ are defined by Eqs. (3.6-2)–(3.6-4):

$$\begin{aligned}
[K_L] &= \int_{\Omega^e} [B_L]^T [C] [B_L] dV \\
[K_{NL}] &= \int_{\Omega^e} [B_{NL}] [C] [B_{NL}] dV \\
\{F\} &= \int_{\Omega^e} [B_L] \{\hat{S}\} dV
\end{aligned} \tag{5.7-2}$$

In these equations, $[B_L]$ and $[B_{NL}]$ are linear and nonlinear strain-displacement transformation matrices, $[C]$ is the constitutive elasticity matrix, $\{\hat{S}\}$ is a vector of 2nd Piola-Kirchhoff stresses, and $\{R\}$ is the external load vector. All matrix elements refer to the deformed state and are measured with respect to the original undeformed configuration.

In the process of evaluating the integrals in equation (5.7-1) (e.g., $[K_L]$, $[K_{NL}]$, $\{F\}$), we use Gauss quadrature in the membrane directions of the shell, but explicit integration in the thickness direction. Thus the thickness direction integration for matrices $[K_L]$ and $[K_{NL}]$ gives the following laminate stiffnesses [see Eq. (2.4-24)],

$$\begin{aligned}
[A] &= \sum_{k=1}^P [C']_k (\zeta_{k+1} - \zeta_k) \\
[B] &= \frac{1}{2} \sum_{k=1}^P [C']_k (\zeta_{k+1}^2 - \zeta_k^2) \\
[D] &= \frac{1}{3} \sum_{k=1}^P [C']_k (\zeta_{k+1}^3 - \zeta_k^3)
\end{aligned} \tag{5.7-3}$$

Here ζ_k is the thickness coordinate of the bottom of the k th lamina, P is the number of laminae, $[C']_k$ is the constitutive matrix for the k th lamina in the principal material coordinates, which has the form,

$$[C'] = f([\bar{Q}], \theta), \tag{5.7-4}$$

$\bar{Q}_{ij}$ are the plane stress-reduced elastic coefficients in the material coordinates, and θ is the lamina orientation angle (see Fig. 2.4-1). For an orthotropic lamina, $\bar{Q}_{ij}$ can be expressed as [see Eq. (2.3-2)]

$$\bar{Q}_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, \quad \bar{Q}_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}}, \quad \bar{Q}_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}$$

$$\bar{Q}_{44} = (G_{13})K, \bar{Q}_{55} = (G_{23})K, \bar{Q}_{66} = G_{12} \quad (5.7-5)$$

where E_1 , ν_{12} , ν_{21} , E_2 , G_{13} , G_{23} , G_{12} are the engineering lamina properties and K is the shear correction factor taken to be 5/6.

The modeling approach is based on using six elements per buckle half-wave in each direction. Hence, the finite element model of Panel C4 consists of 12 nine-node quadrilateral elements along the panel length, while that of Panel C10 has 24 along the length. Thus both panels have 6 elements across the panel width. Figure 5.7-1a shows the model used for the C4 specimen. The finite element model of Panel H4 is different, due to the presence of the hole. This model has four "rings" of elements around the hole with each "ring" subdivided into 16 elements, as shown in Figure 5.7-1b. The total numbers of 9-node quadrilateral elements in the finite element models of Panels C4, C10, and H4 are 72, 144, and 124, respectively.

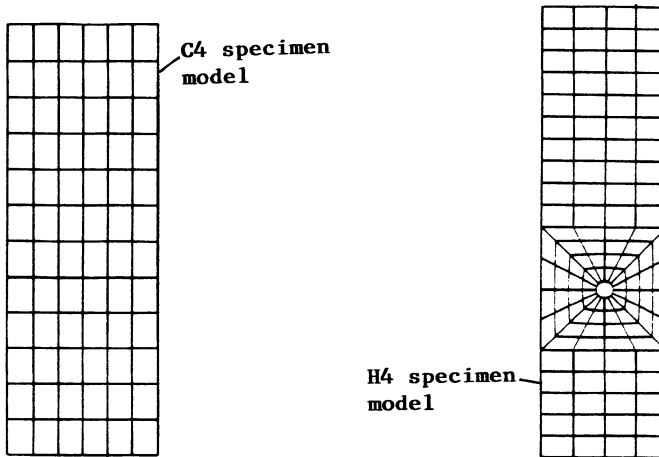


Figure 5.7-1 Finite element meshes used for panels C4 and H4.

In order to proceed beyond the critical buckling point in the analysis of each panel, we assumed an initial geometric imperfection, typically the same shape as the first linear buckling mode. The amplitude of each mode was selected to be 1-5% of the total laminate thickness. This allows efficient progress past the critical buckling point, but does not affect the results in the postbuckling range.

5.7-4 Failure analysis

Two failure criteria, the maximum stress and Tsai-Wu, are used [see Eqs. (4.3-1) and (4.3-10)]. Any other closed form criteria could have been used.

In reality, laminate failure occurs due to propagation of damage as the load is increased. To model this effect, an analytical progressive failure technique is included in the nonlinear analysis. At each load step, Gauss point stresses are used in the selected failure criterion. If failure occurred at a Gauss point, a reduction in lamina stiffness was applied to that point which changed the resultant $A - B - D$ laminate stiffnesses of equation (5.7-3). For example, for the maximum stress criterion, if the σ_1 stress exceeds the longitudinal tensile strength X_T , then the longitudinal modulus E_1 at that point is reduced to zero. For the Tsai-Wu criterion, if failure occurs, then the following expressions are used to determine the failure mode:

$$\begin{aligned} H_1 &= F_1\sigma_1 + F_{11}\sigma_1^2 \\ H_2 &= F_2\sigma_2 + F_{22}\sigma_2^2 \\ H_4 &= F_{44}\sigma_4^2, \quad H_5 = F_{55}\sigma_5^2, \quad H_6 = F_{66}\sigma_6^2 \end{aligned} \quad (5.7-6)$$

The largest H_i term is selected as the dominant failure mode and the corresponding modulus is reduced to zero. Thus H_1 corresponds to the modulus E_1 , H_2 to E_2 , H_4 to G_{23} , H_5 to G_{13} , and H_6 to G_{23} . As a consequence of this reduction, engineering material properties are updated as failure progresses. An outline of the steps used is given below:

1. After nonlinear iterative displacement convergence is achieved, calculate stresses in the global (x, y, z) coordinates at the middle of each layer at each Gauss point.
2. Transform the stresses to the material coordinates.
3. Compute the failure index, $\mathcal{F}$ (see Sec. 5.6).
4. If failure occurs (i.e., $\mathcal{F} \geq 1$), identify the maximum H_i , reduce the appropriate lamina moduli at that Gauss point, recompute laminate stiffnesses, and restart the nonlinear analysis at the same load step.
5. If no failure occurs, proceed to the next load step.

The end shortening of the panel is monitored as in a compression test. The failure load is defined to be that load for which the panel undergoes large end shortening for small increments of load.

5.7-5 Results for C4 panel

Comparison between test results from Reference 38 and finite element results for Panel C4 are shown in Figure 5.7-2. The figure shown end shortening u , normalized by the analytical end shortening u_{cr} at buckling (Fig. 5.7-2a); out-of-plane deflection w near a point of maximum deflection, normalized by the panel

thickness t (Fig. 5.7-2b); the longitudinal surface strains e near a point of maximum out-of-plane deflection, normalized by the analytical buckling strain e_{cr} (Fig. 5.7-2c). These are all shown as a functions of the applied load P , normalized by the analytical buckling load P_{cr} . The circles in the figure represent test data, and the curves denote the results of the nonlinear finite element analyses. These experimental and finite element results agree well up to failure of the panel. The postbuckling response exhibits large out-of-plane deflections (nearly three times the panel thickness, see Fig. 5.7-2b) and high longitudinal strains from front and back surfaces (nearly three times the analytical buckling strain, see Fig. 5.7-2c).

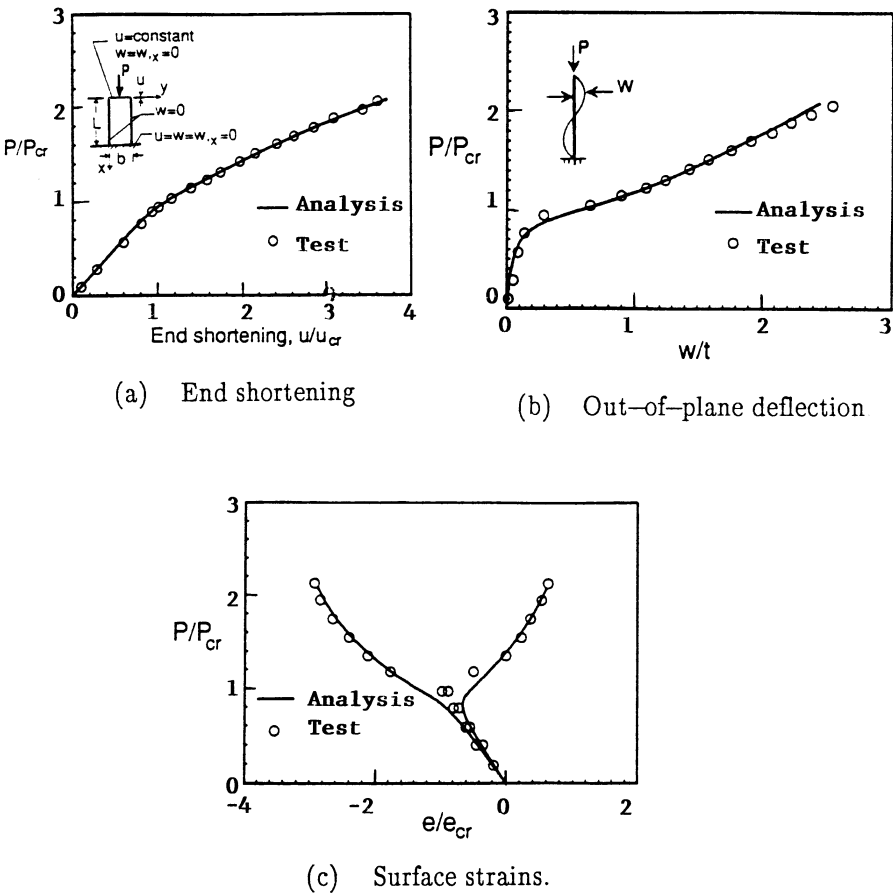


Figure 5.7-2 Postbuckling response characteristics of specimen C4.

Figure 5.7-3a shows a contour plot of the out-of-plane deflections generated from the nonlinear analysis at an applied load of $2.1P_{cr}$. Figure 5.7-3b shows a photograph of the Moiré-fringe pattern from Reference 38 corresponding to the out-of-plane deflections observed during the testing of Panel C4 at the same load. These results indicate that the out-of-plane deflections from both test and analysis have the same pattern over the entire panel. Both patterns indicate two longitudinal half-waves with a buckling-mode nodal line at panel midlength.

Stress distributions in each layer of the laminate are analytically evaluated using the nonlinear results in order to determine the failure loads. The stresses are determined using the constitutive relations for both the in-plane and transverse components. In addition, the transverse shear stress distributions are obtained by integrating the equilibrium equations and using the in-plane stresses from the constitutive relations.

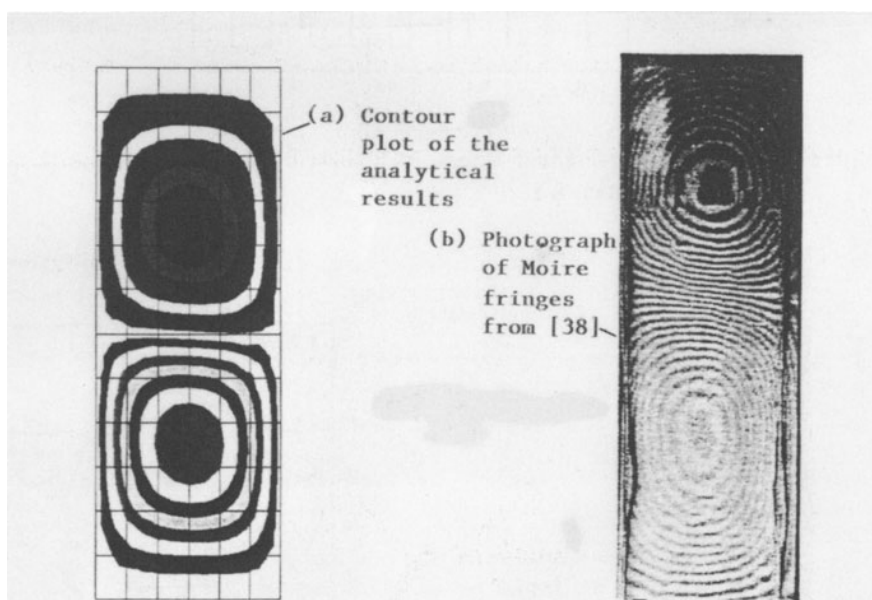


Figure 5.7-3 Comparison of experimental (Moiré) and analytical out-of-plane deflection patterns for specimen C4.

Figure 5.7-4 shows the distribution of the peak σ_{zz} stress through the thickness direction z , normalized by the laminate thickness h for $P = 2.1 P_{cr}$. It is clear that the 0° layers carry the highest transverse shear load. Figure 5.7-5a shows the distribution axial stress σ_{xx} over the entire panel in the third layer of the laminate (a 0° ply) for an applied load of $2.1P_{cr}$. This distribution indicates that

high compressive axial stresses occur along the longitudinal edges of the panel. Figure 5.7-5b indicates the redistribution of the axial stress in this 0° ply at panel midlength for three values of the applied load. At the buckling load, the axial stress is nearly uniform across the panel. Although the axial stress is large, the values are well below the material allowable values [$X_T = 1400$ MPa (203 ksi) in tension; $X_C = 1138$ MPa (165 ksi) in compression].

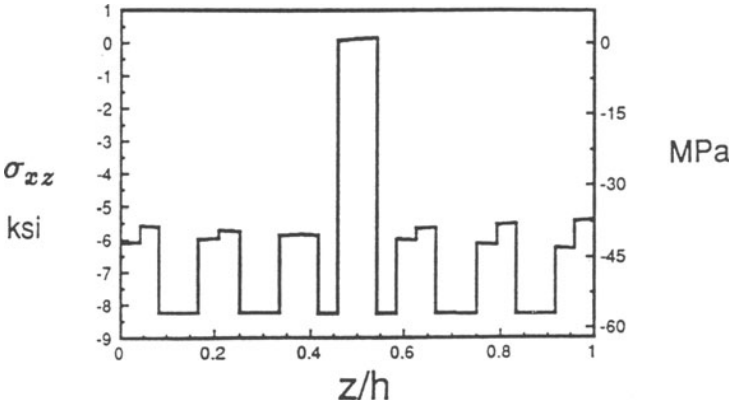
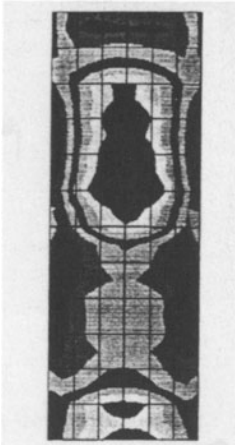
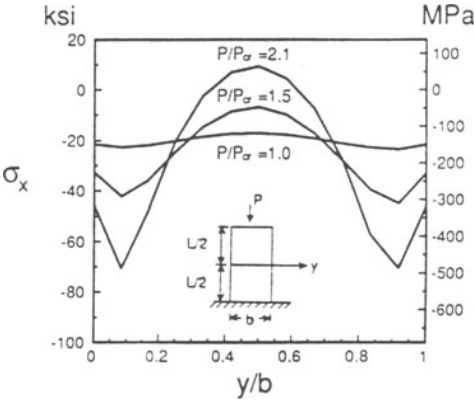


Figure 5.7-4 Transverse shear stress, σ_{xz} , distribution through the thickness of specimen C4.



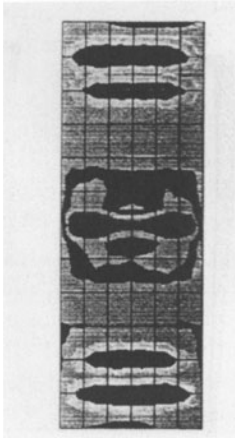
(a) Contour plot of axial stress distributions.



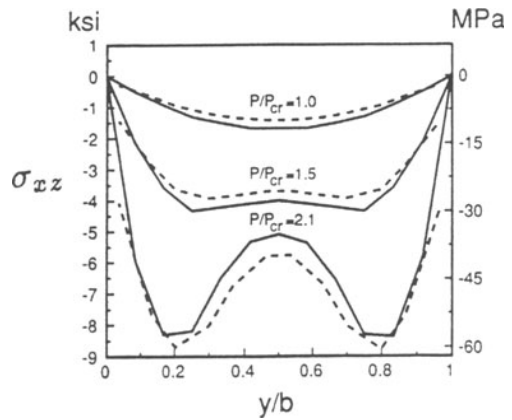
(b) Stress distributions across panel midlength.

Figure 5.7-5 Axial stress, σ_x , distributions in the third layer from the surface (0° ply) of the C4 specimen.

Figure 5.7-6a shows the distribution of the transverse shear stress σ_{xz} over the entire panel in the third layer of the laminate (a 0° ply) for an applied load of $2.1P_{cr}$. This distribution indicates that high transverse shear stresses occur along the buckling-mode nodal line. Figure 5.7-6b indicates the redistribution of this transverse shear stress in this 0° ply at panel midlength for three values of the applied load. The solid curves represent the transverse shear stress distributions obtained using the constitutive relations, and the dashed curves denote the transverse shearing stress distributions obtained from the equilibrium equations. Both methods give very similar results. At the buckling load, the peak transverse shear stress occurs near the center of the panel. After buckling, the transverse shear stresses σ_{xz} redistribute towards the edges of the panel. The peak values of the transverse shear stress σ_{xz} approach the material allowable value of $S = T = 62$ MPa (9ksi) for $P = 2.1 P_{cr}$, indicating the panel fails due to transverse shear stress.



(a) Contour plot



(b) Stress distributions across panel midlength.

Figure 5.7-6 Transverse shear stress, σ_{xz} , distributions in the third layer from the surface (0° ply) of the C4 specimen.

This failure mode can be further explained through a close examination of the Green-Lagrange strain component ϵ_{zz} ,

$$\epsilon_{zz} = \frac{1}{2} \left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} + \frac{\partial u_1}{\partial x} \frac{\partial u_1}{\partial z} + \frac{\partial u_2}{\partial x} \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial x} \frac{\partial u_3}{\partial z} \right) \quad (5.7-7)$$

in conjunction with the displacement field of the first-order shear deformation theory [cf. Eq. (2.5-1)]

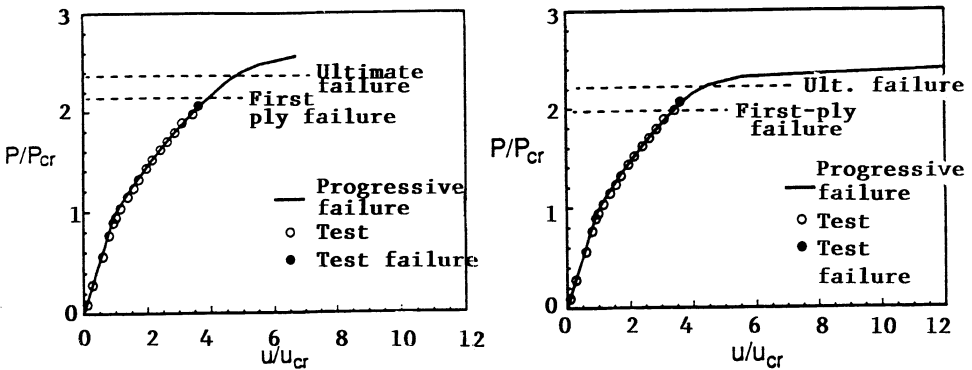
$$u_1(x, y, z) = u(x, y) + z\phi_1, \quad u_2(x, y, z) = v(x, y) + z\phi_2, \quad u_3(x, y, z) = w(x, y) \quad (5.7-8)$$

Substituting equations (5.7-8) into equation (5.7-7) and noting that $\phi_2 = 0$ along a buckling-mode nodal line, we obtain

$$\epsilon_{xz} = \frac{1}{2} \left(\phi_1 + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial x} \phi_1 \right) \quad (5.7-9)$$

The quantity $\frac{\partial w}{\partial x}$ (out-of-plane deflection gradient) is largest along a buckling-mode nodal line and the quantity $\frac{\partial u}{\partial x}$ (related to the membrane strain) is largest along the panel edges. A similar examination of the other transverse shearing strain ϵ_{yz} leads to the conclusion that the transverse shearing strain ϵ_{xz} is the dominant one.

Figures 5.7-7a and 5.7-7b present the progressive failure results for Panel C4, using the maximum stress and Tsai-Wu failure criteria, respectively. In addition to the strengths already mentioned, the other allowables are as follows: transverse tensile strength of $X_T = 80.9$ MPa (11.7 ksi), transverse compressive strength of $X_C = 189.0$ MPa (27.4 ksi), and in-plane shear strength of $T = 69.0$ MPa (10.0 ksi). At some point in the analysis a dramatic change in slope indicates an inability to support additional load. This location is identified as the failure load. Figures 5.7-7a and 5.7-7b show that the Tsai-Wu criterion estimates the experimental failure more closely than the maximum stress criterion. This is attributed to the presence of stress interaction terms in the Tsai-Wu criterion failure index.



(a) Maximum stress failure criterion. (b) Tsai-Wu failure criterion.

Figure 5.7-7 Progressive failure results of specimen C4.

5.7-6 Results for C10 panel

Figures 5.7-8a and 5.7-8b present comparisons of numerical and experimental values of end shortening u and out-of-plane deflection w . In these figures, E is the laminate longitudinal modulus, A is the cross-sectional area, and L is the specimen length. Agreement is not as good as in the C4 specimen because extracting non-normalized experimental data from Reference 38 was difficult. A linear buckling analysis shows that the first mode has three longitudinal half-waves and the second mode four. The Moiré fringe results from Reference 38 in the postbuckling range exhibited four half-waves. Thus a linear combination of 1.5% panel thickness times mode 1 and mode 2 eigenvectors was used to assist in the change that occurs at the critical buckling load from the three half-wave to the four half wave mode.

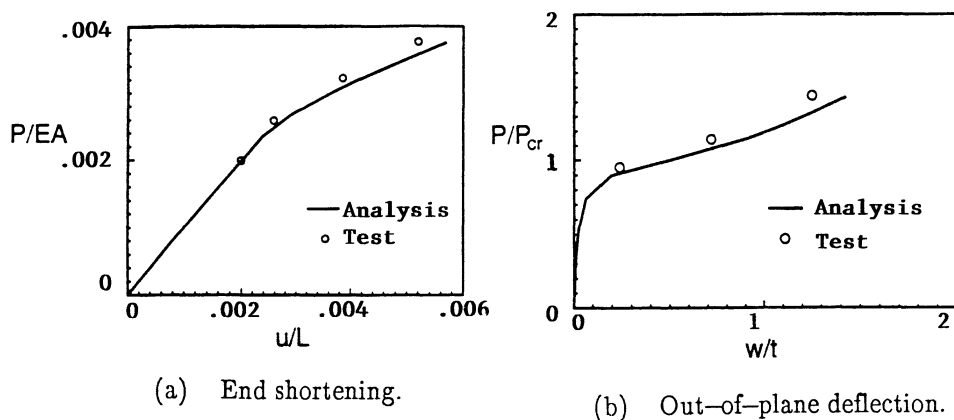


Figure 5.7-8 Postbuckling response characteristics of specimen C10.

A transverse shear failure mechanism develops for this panel, as for Panel C4. Each nodal line in the buckling mode develops the high transverse shear stresses, with the peak at the midlength of the panel. Figure 5.7-9a shows the distribution of σ_{xz} in layer 22 (a 0° ply), where the three peak σ_{xz} nodal line concentrations can be seen. Figure 5.7-9b illustrates the σ_{xz} distribution at the experimental failure load ($1.44P_{cr}$) across the panel width. The results are similar to those obtained for Panel C4.

The progressive failure results for Panel C10 are shown in Figures 5.7-10a and 5.7-10b. For this panel the Tsai-Wu results agree extremely well with the experimental results. However, as seen previously for Panel C4, the maximum stress criterion results do not agree as well.

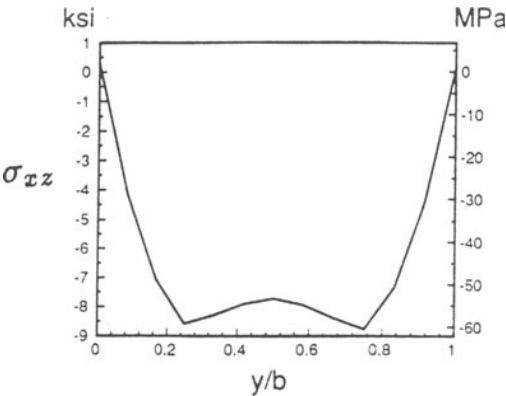
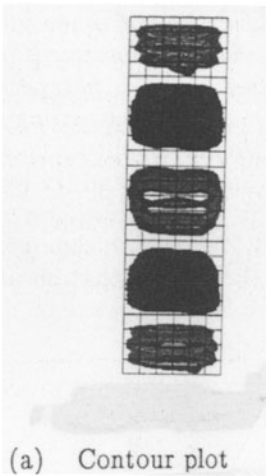


Figure 5.7-9 Transverse shear stress, σ_{xz} , distributions in the twenty-second layer from the surface (0° ply) of the C10 specimen.

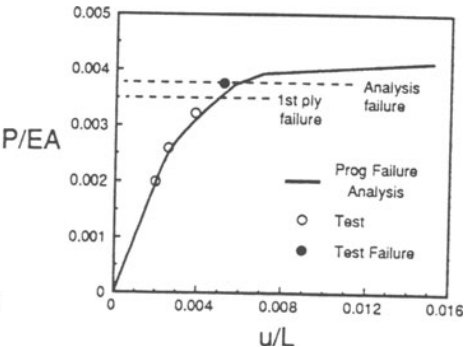
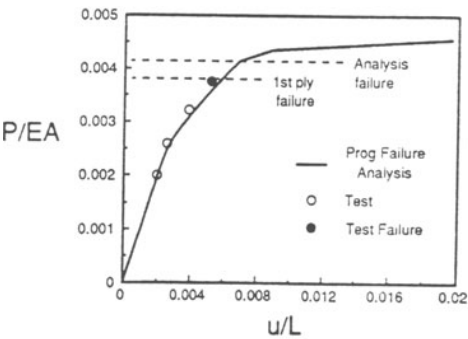
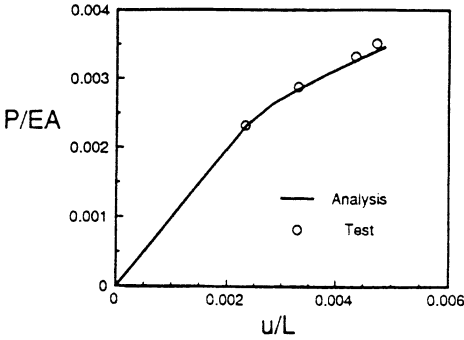


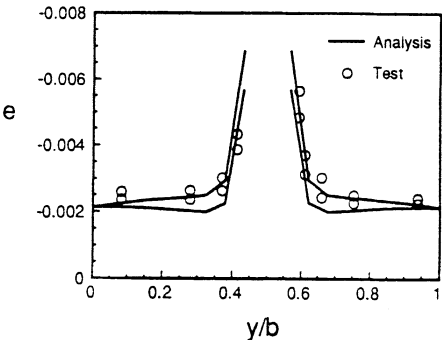
Figure 5.7-10 Progressive failure results of specimen C10.

5.7-7 Results for H4 panel

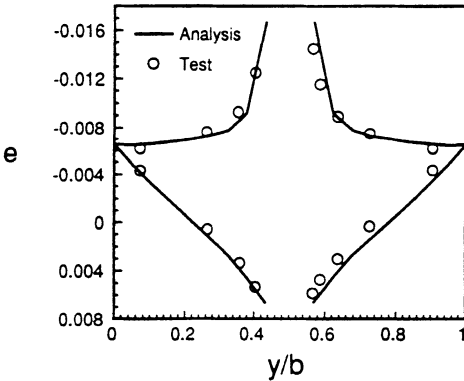
The H4 specimen was analyzed to investigate deformation and failure of a panel with a hole. Figure 5.7-11a contains a comparison of end shortening obtained numerically and experimentally. Figures 5.7-11b and 5.7-11c show the longitudinal surface strains ϵ (both top and bottom surfaces) across the panel at the hole for a load of $0.90P_{cr}$ and $1.39P_{cr}$, respectively. These results are in good agreement with experimental results from Reference 38. Linear buckling and postbuckling out-of-plane deflection mode shapes for this panel are shown in Figures 5.7-12a and 5.7-12b. An imperfection of 1% of panel thickness times mode 1 was used to proceed into the postbuckling range. One can see that four longitudinal half-waves develop in the postbuckled range.



(a) End shortening.



(b) Surface strains at $P/P_{cr} = 0.90$.



(c) Surface strains
at $P/P_{cr} = 1.39$.

Figure 5.7-11 Postbuckling response characteristics of specimen H4.

Analysts should note that if uniformly reduced or selective reduced integration were used in the analysis of this panel, it would predict spurious modes. These occur because of zero energy modes, and lack of restraint of the model around the

hole. Element distortion around the hole could be another contributing factor. It is necessary to use full integration to alleviate this problem, and the mesh should be sufficiently refined so that locking effects are negligible.

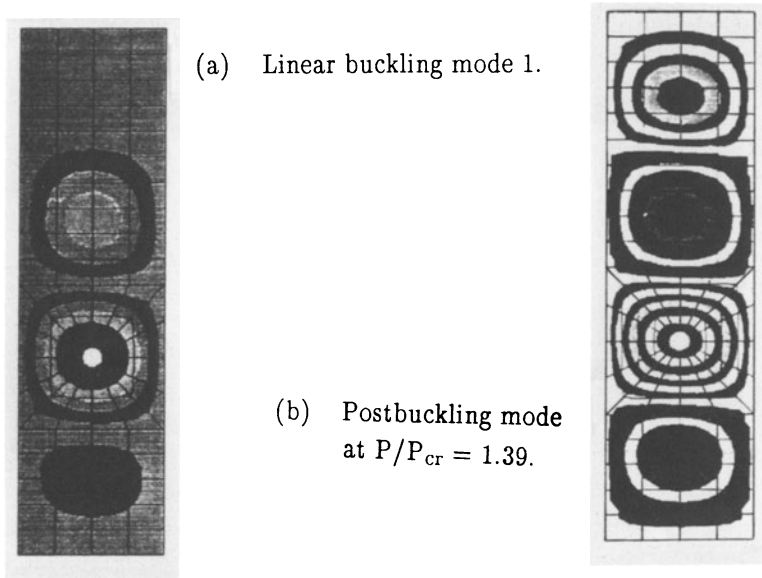


Figure 5.7-12 Out-of-plane deflection patterns of specimen H4.

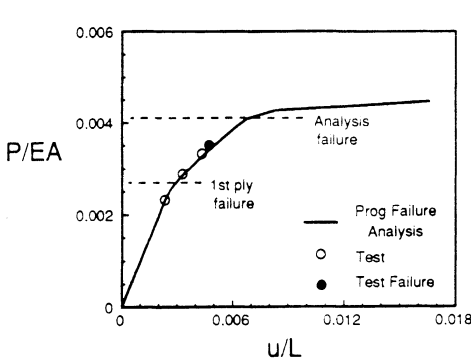
For the H4 panel, once again, a transverse shear mechanism develops along nodal lines away from the hole. However, the peak stress only approaches 48.3 MPa (7.0 ksi) at the experimental failure load. At this load the in-plane shear stress approaches its allowable around the hole. Simultaneous first-ply failure occurs due to σ_{xy} and σ_{xx} components around the hole edge. Thus the failure mode is not a dominant transverse shear mode as for Panels C4 and C10, but a more complex interacting mode with a dominant in-plane shear component. Progressive failure results are shown in Figures 5.7-13a and 5.7-13b. The Tsai-Wu criterion is, once again, in better agreement with experimental results.

5.8. Progressive failure of laminates in bending

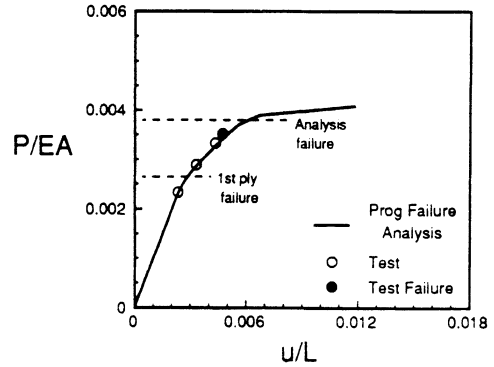
5.8-1 Background

The eight-noded quadrilateral plate element, QHD40 (see Sec. 3.6), based on a second-order plate theory is to demonstrate the use of the quadratic failure criteria [42-44] (see Sec. 4.3). The problem selected here is a unidirectional laminate subjected to four-point bending. For a given load, the total and incremental stresses

are calculated at each Gauss point in each element. In-plane stresses are obtained from the constitutive equations; the interlaminar normal and shear stresses are calculated from the equilibrium equations, as discussed in Chapter 3. The stresses are transformed to the material coordinates and the failure criteria of Hashin [42] are used. When failure is detected, the material properties corresponding to that failure mode are reduced (to a small value) at that location. Equilibrium iterations are performed before incrementing the load.



(a) Maximum stress failure criterion.



(b) Tsai-Wu failure criterion.

Figure 5.7–13 Progressive failure results of specimen H4.

5.8–2 Computational model

A laminate with twenty four layers is modeled. The elastic constants and strengths used are:

$$E_{11} = 190 \text{ GPa}, E_{22} = E_{33} = 11 \text{ GPa}, G_{12} = G_{13} = 7.2 \text{ GPa}$$

$$\nu_{12} = \nu_{13} = \nu_{23} = 0.38$$

$$X_T = X_C = 1502 \text{ MPa}, Y_T = 41 \text{ MPa}, Y_C = 250 \text{ MPa}$$

$$R = S = T = 67.5 \text{ MPa and } Z_T = Z_C = 41 \text{ MPa} \quad (5.8 - 1)$$

The criteria of Hashin [42] are selected for their ability to distinguish between fiber, matrix, and through-the-thickness failure modes [see Eqs. (4.3–14)–(4.3–18)]. Furthermore, they can identify whether failure occurred in tension or in

compression. These criteria are repeated here for the reader's convenience [see Eqs. (4.3-14)-(4.3-18)]:

Fiber failure

Tensile:

$$\left(\frac{\sigma_1}{X_T}\right)^2 + \frac{(\sigma_5^2 + \sigma_6^2)}{T^2} \quad (5.8-2)$$

Compressive:

$$\sigma_1 = \frac{G_r}{1-k} \quad (5.8-3)$$

where G_r is resin modulus and k is volume fraction ratio.

Matrix failure

Tensile:

$$\frac{(\sigma_2 + \sigma_3)^2}{R^2} + \frac{(\sigma_4^2 - \sigma_2\sigma_3)}{T^2} + \left(\frac{\sigma_6^2 + \sigma_5^2}{T^2}\right) = 1 \quad (5.8-4)$$

Compressive:

$$\begin{aligned} \frac{1}{Y_C} \left[\left(\frac{Y_C}{2T}\right)^2 - 1 \right] (\sigma_2 + \sigma_3) + \frac{1}{4T^2} (\sigma_2 + \sigma_3)^2 \\ + \frac{(\sigma_4^2 - \sigma_2\sigma_3)}{T^2} + \frac{1}{T^2} (\sigma_6^2 + \sigma_5^2) = 1 \end{aligned} \quad (5.8-5)$$

Delamination [through-the-thickness]

$$\left(\frac{\sigma_3}{Z_T}\right)^2 + \frac{(\sigma_6^2 + \sigma_5^2)}{R^2} = 1 \quad (5.8-6)$$

The stresses in material coordinates (1,2,3) are computed from the relations [see Eq. (2.3-3)]

$$\sigma_1 = \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + \sigma_{xy} \sin 2\theta$$

$$\sigma_2 = \sigma_x \sin^2 \theta + \sigma_y \cos^2 \theta - \sigma_{xy} \sin 2\theta$$

$$\sigma_3 = \sigma_z$$

$$\sigma_4 = -\sigma_{xz} \sin \theta + \sigma_{yz} \cos \theta$$

$$\begin{aligned}\sigma_5 &= \sigma_{xz} \cos \theta + \sigma_{yz} \sin \theta \\ \sigma_6 &= (\sigma_y - \sigma_x) \sin \theta \cos \theta + \sigma_{xy} \cos 2\theta\end{aligned}\quad (5.8-7)$$

where $(\sigma_x, \sigma_y, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}, \sigma_z)$ are the stresses in the global coordinates. The stresses (5.8-7), referred to the material coordinates, are used when checking the failure criteria presented in this section. When fiber failure occurs, $(\sigma_1, \sigma_6, \sigma_5)$ are assumed to be zero. Thus the only nonzero terms of the constitutive matrix are C_{22} , C_{23} , C_{33} and C_{44} :

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix} \quad (5.8-8)$$

After checking for all the possible failures, we use the modified form of the material matrix, $[C^*]$, to calculate the new total stresses in the lamina:

$$\{\sigma\}^r = \{\sigma\}^{(r-1)} + [C^*]\{\Delta\epsilon\} \quad (5.8-9)$$

After the stresses are updated, the failure criteria are checked once again to ensure that no further failure has occurred in that load increment. Only after this, is the load incremented and the entire procedure repeated until final failure is reached. Usually, the user defines what constitutes a final failure.

5.8-3 Results and discussion

Tables 5.8-1 and 5.8-2 illustrate the type and sequence of damage that has occurred in thin and moderately thick unidirectional laminates under four-point-bending loads. As shown in Table 5.8-1, at load increment 13, tensile fiber failure is observed in layers 21-24, and this mode was accompanied by tensile matrix failure at the same load level in layers 19-23. Furthermore, compression fiber failure mode is observed in layers 1-4 for the same load increment. Note that the fiber failure at the outermost laminate proceeds rapidly toward the center and the ultimate failure occurs within 4% of the initiation load. Table 5.8-2 illustrates the damage progression in a thicker laminate (side-to-thickness ratio is 8). Note that the progression is from the center toward the outer layers of the laminate. Figure 5.8-1 is quite revealing. For the same configuration of Table 5.8-1, the total strain energy of an element is displayed as a function of load increments throughout the laminate. Note the discontinuity in the curve, which is due to the damage observed in the layers within a given increment. The damaged layers are the ones with the very low strain energy magnitudes, as illustrated in this figure.

Table 5.8-1 Damage progression* in a 24-ply laminate $[0]_{24}$ with side-to-thickness ratio $S = 100$ under four-point bending load ($P_{14}/P_{13} = 1.04$).

Ply No.	TF	CF	TM	CM	DL
1	0	13	0	0	0
2	0	13	0	0	0
3	0	13	0	0	0
4	0	13	0	0	0
5	0	14	0	0	0
6	0	14	0	0	0
7	0	14	0	0	0
8	0	14	0	0	0
9	0	14	0	0	0
10	0	0	0	0	0
11	0	0	0	0	0
12	0	0	0	0	0
13	0	0	0	0	0
14	0	0	0	0	0
15	0	0	0	0	0
16	14	0	14	0	0
17	14	0	14	0	0
18	14	0	14	0	0
19	14	0	13	0	0
20	14	0	13	0	0
21	13	0	13	0	0
22	13	0	13	0	0
23	13	0	13	0	0
24	13	0	0	0	0

*The numbers in the table denote the load increment number for various failures. TF = Tensile fiber failure; CF = Compressive fiber failure; TM = Tensile matrix failure; CM = Compressive matrix failure; DL = Delamination.

Table 5.8-2 Damage progression* in a 24-ply laminate $[0]_{24}$ with side-to-thickness ratio $S = 8$ under four-point bending load ($P_{11}/P_6 = 1.20$).

Ply No.	TF	CF	TM	CM	DL
1	0	0	0	0	0
2	0	0	0	0	0
3	0	0	0	0	0
4	0	0	0	0	0
5	0	0	0	0	0
6	0	0	0	0	0
7	0	0	0	0	11
8	0	0	0	10	10
9	0	0	0	9	8
10	0	0	0	7	6
11	0	0	0	6	6
12	0	0	0	6	6
13	6	0	6	6	6
14	6	0	6	6	6
15	6	0	6	6	6
16	7	0	6	7	7
17	7	0	7	8	8
18	8	0	8	9	9
19	10	0	9	10	10
20	0	0	11	0	0
21	0	0	0	0	0
22	0	0	0	0	0
23	0	0	0	0	0
24	0	0	0	0	0

*see the foot note of Table 5.8-1.

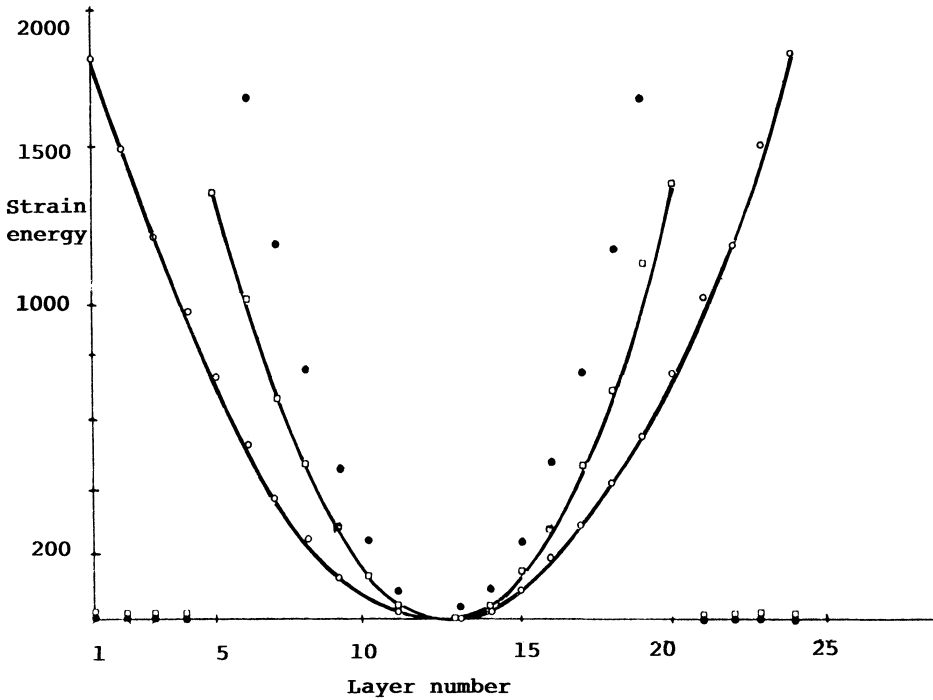


Figure 5.8-1 Failure progression of four-point bend test ($S = 100$).

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