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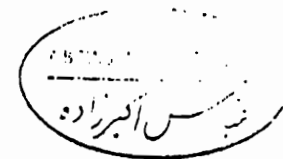
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THE MECHANICS OF CRYSTALS AND TEXTURED POLYCRYSTALS

William F. Hosford

The University of Michigan

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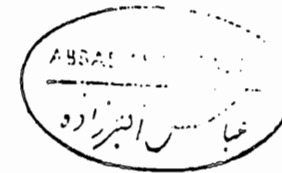
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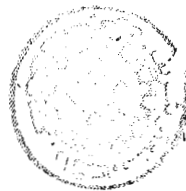
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PREFACE

This book focuses on the mechanics of crystals. This is a level of mechanics between that of dislocation mechanics and continuum mechanics. It is concerned with the phenomena of slip and twinning on specific crystallographic systems (planes and directions), without being involved with the nature of the dislocations that cause these events. The basic concepts are not new. Schmid and Boas wrote a classic book, *Kristallplastizität*, in 1935 dealing with mechanics of slip and twinning in single crystals. Much of the basis for our present understanding of the plastic behavior of polycrystals and how it relates to the behavior of single crystals was laid in 1938 by G. I. Taylor. The development of dislocation theory during the 1930s turned the attention of materials scientists (largely physicists and metallurgists) from crystal mechanics to viewing mechanical behavior on this finer (dislocation) scale. While the study of dislocations has been a fruitful field, it focuses on such a fine scale that it has not been as useful as crystal mechanics in relating polycrystal behavior to slip and twinning phenomena observed in single crystals nor in predicting the effects of texture on plastic anisotropy.

At the other end of the scale, engineers involved with engineering problems have been slow to recognize the limitations of conventional continuum plasticity. Only recently have there been attempts to modify the basic ideas of continuum mechanics so as to minimize the gap between it and crystal mechanics. Yet, in principle, it is possible to bypass continuum mechanics completely and solve engineering problems involving plasticity using crystallographic analyses directly. With our rapidly growing computing power, such an approach to engineering problems may become routine in a few decades.

The treatment here is primarily on plasticity. There is a chapter on elasticity that serves to introduce the reader to transformations of stress and strain from one set of axes to another. The effects of anisotropic thermal expansion on polycrystals is included here. There is no treatment of fracture phenomena. The reason is that anisotropy of fracture behavior usually results from mechanical fibering (alignment of inclusions and grain boundaries) rather than from crystallographic texture, which is the main focus of the book. The concepts of crystal plasticity are extended to predict the behavior of texture polycrystals, and the predicted behavior is related to continuum theories of yielding.

To make the book useful to a readership with diverse backgrounds, example problems have been included in the first few chapters and Appendix I to illustrate points and show approaches to solving problems. Some readers may have strong backgrounds in mechanics but be new to the field of crystals or vice versa. For this reason, a number of appendices have been included to cover fundamentals of geometry, crystallography, and mechanics. It is hoped that their coverage is complete enough to serve as a thorough introduction to those not already familiar with the area.

The choice of topics is admittedly somewhat arbitrary and strongly influenced by the author's own interests. If it seems that the treatment reflects the author's views and past work, the author pleads guilty. Most of the treatment is focused on cubic and hexagonal metals with only casual attention to other crystalline materials.

The author wishes to acknowledge the influence of two inspiring teachers: Walter Hibbard, who introduced the author to the basic concepts of crystal plasticity, and W. A. Backofen, who not only taught him the basic mechanical framework in which these concepts must fit, but also encouraged him to teach a course on textures and anisotropy and cooperatively worked with him in developing an approach to the subject. The author also wishes to thank U. F. Kocks for introducing him to the Taylor and Bishop and Hill papers and to G. Y. Chin for encouragement to write this book. Support of former graduate students, especially R. W. Logan, D. J. Meuleman, C. Vial, and Y. Zhu, is recognized. The help of A. Graf has been appreciated. Finally, the author wishes to acknowledge the influence of R. M. Caddell for an introduction to the drudgery and joy of book writing. Dr. F. Barlat made a large number of constructive criticisms of the original manuscript.

Ann Arbor
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W.F.H

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ELASTICITY

1.1. Isotropic elasticity

The elastic behavior of an isotropic solid is completely described by Hooke's law:

$$e_x = \frac{1}{E} [\sigma_x - \nu(\sigma_y + \sigma_z)], \quad (1.1a)$$

where E is Young's modulus and ν is Poisson's ratio. The only necessary assumption is that second-order effects are negligible, (i.e., the elastic behavior is linear). This is a very good assumption in crystalline materials, because elastic strains are small, but is not good for polymeric materials that can undergo large elastic strains. The x , y , and z axes are interchangeable and need not be principal axes. Therefore, we can also write:

$$e_y = \frac{1}{E} [\sigma_y - \nu(\sigma_x + \sigma_z)] \quad (1.1b)$$

$$e_z = \frac{1}{E} [\sigma_z - \nu(\sigma_x + \sigma_y)]. \quad (1.1c)$$

1.1.1. Shear modulus

Shear strains also must be proportional to shear stresses,

$$\left. \begin{aligned} \gamma_{yz} &= \frac{1}{G} \tau_{yz}, \\ \gamma_{zx} &= \frac{1}{G} \tau_{zx}, \\ \gamma_{xy} &= \frac{1}{G} \tau_{xy}, \end{aligned} \right\} \quad (1.2)$$

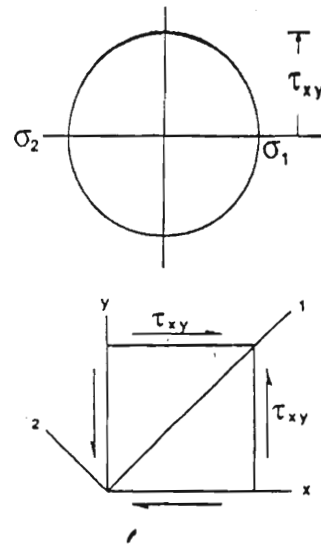


Fig. 1.1. A body under shear stress (above) and the corresponding Mohr's circle diagram (below).

where G is the shear modulus. The engineering shear strain, γ_{xy} , is twice the tensor shear strain, e_{xy} (see Appendix 2).

The modulus, G , is not independent of E and ν . The relation between them can be deduced by considering a body under a state of pure shear, τ_{xy} , as shown in Fig. 1.1. If the stress τ_{xy} is resolved onto the 1 and 2 axes, $\sigma_1 = \tau_{xy}$, $\sigma_2 = -\tau_{xy}$, $\sigma_3 = 0$, and $\tau_{12} = 0$, so 1 and 2 are principal axes. According to Eq. (1.1), $e_1 = (1/E)[\sigma_1 - \nu(\sigma_2 + \sigma_3)] = \tau_{xy}(1 + \nu)/E$, and $e_2 = (1/E)[\sigma_2 - \nu(\sigma_1 + \sigma_3)] = -\tau_{xy}(1 + \nu)/E$.

The shear strain, γ_{xy} , can be expressed in terms of e_1 and e_2 , $\gamma_{xy} = e_1 - e_2 = -\tau_{xy}(1 + \nu)/E$. Comparing this with Eq. (1.2),

$$G = \frac{E}{2(1 + \nu)} \quad (1.3)$$

1.1.2. Bulk modulus

The bulk modulus B relates the volume dilatation, $\delta = \Delta V/V$, to the hydrostatic stress, $-p = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z)$,

$$\delta = \frac{-p}{B} \quad (1.4)$$

but B is also related to E and ν . Consider an element (Fig. 1.2) of dimensions L_x , L_y , and L_z . Its volume is $L_x L_y L_z$, so $dV/V = (L_x/L_x)dL_x + (L_y/L_y)dL_y + (L_z/L_z)dL_z +$

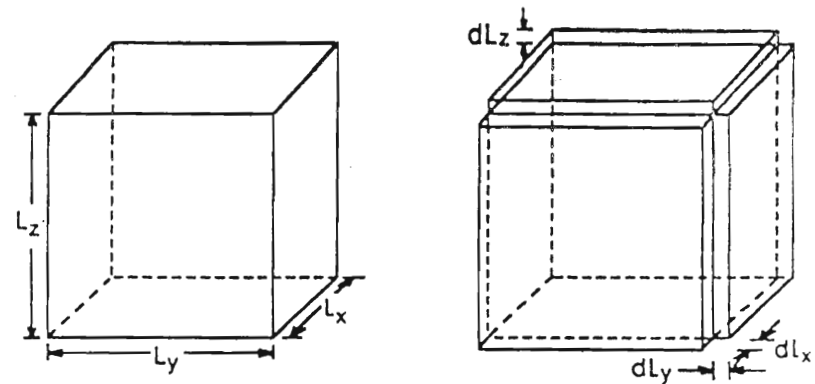


Fig. 1.2. Volume dilation.

$L_x L_y dL_z)/L_x L_y L_z = dL_x/L_x + dL_y/L_y + dL_z/L_z = de_x + de_y + de_z$. Therefore $\delta = e_x + e_y + e_z$. Substituting in Eq. (1.1),

$$\begin{aligned} \delta &= \frac{1}{E} \{ [\sigma_x - \nu(\sigma_y + \sigma_z)] + [\sigma_y - \nu(\sigma_x + \sigma_z)] + [\sigma_z - \nu(\sigma_x + \sigma_y)] \} \\ &= [(1 - 2\nu)/E](\sigma_x + \sigma_y + \sigma_z). \end{aligned}$$

Comparing with Eq. (1.4), and substituting $-3p = \sigma_x + \sigma_y + \sigma_z$,

$$B = \frac{E}{3(1 - 2\nu)} \quad (1.5)$$

Any of the four elastic constants, E , ν , G , and B can be expressed in terms of two others:

$$E = 2G(1 + \nu) = 3B(1 - 2\nu) = \frac{9BG}{G + 3B} \quad (1.6)$$

$$\nu = \frac{E}{2G} - 1 = \frac{1}{2} - E/6B = \frac{1}{2} - 3/2 \frac{G}{3B + G} \quad (1.7)$$

$$G = \frac{E}{3(1 - 2\nu)} = 3EB/(9B - E) = 3/2 \frac{B(1 - 2\nu)}{1 + \nu} \quad (1.8)$$

$$B = \frac{E}{3(1 - 2\nu)} = \frac{EG}{3(1 - 2\nu)} = 2/3 \frac{G(1 + \nu)}{1 - 2\nu} \quad (1.9)$$

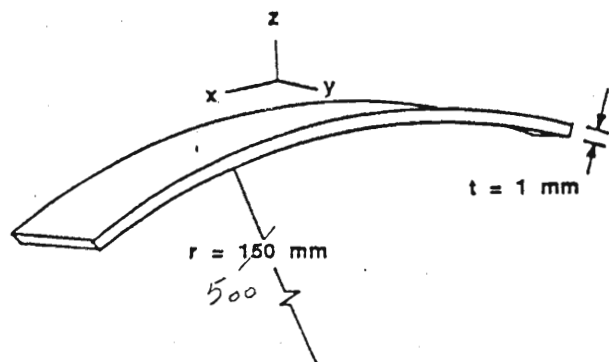


Fig. 1.3. Bent sheet.

To solve elastic problems with Eq. (1.1), three of the six quantities, σ_x , σ_y , σ_z , e_x , e_y , and e_z , must be known in addition to any two elastic constants.

Example 1.1

A wide aluminum sheet of thickness $t = 1.0$ mm is bent to a radius of curvature $r = 500$ mm, as shown in Fig. 1.3. Determine the stress in the surface, assuming elastic deformation and $E = 69$ GPa and $\nu = 0.3$.

Solution: Let x , y , and z be tangential, axial, and radial directions as shown. The stress normal to the free surface $\sigma_z = 0$, and, if the sheet is wide relative to its thickness, $e_y = 0$. Substituting into Eq. (1.1b), $e_y = (1/E)[\sigma_y - \nu(0 + \sigma_x)] = 0$, so $\sigma_y = \nu\sigma_x$.

At the surface in bending $e_x = t/2r = 0.0010$. Now substituting for σ_y , σ_z , and e_x in Eq. (1.1a),

$$\sigma_x = \frac{Ee_x}{1 - \nu^2} = \frac{69 \times 10^9}{1 - 0.09} \times 10^{-3} = 76 \text{ MPa.}$$

$$\sigma_y = \nu\sigma_x = 0.3 \times 76 = 23 \text{ MPa.}$$

1.2. Anisotropic elasticity

In the most general case of Hookean (linear) elasticity, each strain component is a linear function of every stress component. This relation may be written as:

$$e_{ij} = s_{ijmn}\sigma_{mn} \quad (1.10)$$

where a summation is implied. The individual compliances, s_{ijmn} , form a fourth-order tensor. However, this relation may be greatly simplified by taking advantage of the relations $\sigma_{ij} = \sigma_{ji}$ and $\gamma_{ij} = 2e_{ij} = 2e_{ji}$ and by adopting a new subscript convention for the compliances: Two subscripts, i and j , are used to identify the location (row and column) of the compliance in the matrix, rather than the axes of strain and stress components. Because the subscripts identify the location of the compliances, the order of the stress and strain components in the matrix is critical. With this new convention Hooke's laws are written as:

$$\left. \begin{aligned} e_{xx} &= s_{11}\sigma_{xx} + s_{12}\sigma_{yy} + s_{13}\sigma_{zz} + s_{14}\sigma_{yz} + s_{15}\sigma_{zx} + s_{16}\sigma_{xy} \\ e_{yy} &= s_{21}\sigma_{xx} + s_{22}\sigma_{yy} + s_{23}\sigma_{zz} + s_{24}\sigma_{yz} + s_{25}\sigma_{zx} + s_{26}\sigma_{xy} \\ e_{zz} &= s_{31}\sigma_{xx} + s_{32}\sigma_{yy} + s_{33}\sigma_{zz} + s_{34}\sigma_{yz} + s_{35}\sigma_{zx} + s_{36}\sigma_{xy} \\ \gamma_{yz} &= s_{41}\sigma_{xx} + s_{42}\sigma_{yy} + s_{43}\sigma_{zz} + s_{44}\sigma_{yz} + s_{45}\sigma_{zx} + s_{46}\sigma_{xy} \\ \gamma_{zx} &= s_{51}\sigma_{xx} + s_{52}\sigma_{yy} + s_{53}\sigma_{zz} + s_{54}\sigma_{yz} + s_{55}\sigma_{zx} + s_{56}\sigma_{xy} \\ \gamma_{xy} &= s_{61}\sigma_{xx} + s_{62}\sigma_{yy} + s_{63}\sigma_{zz} + s_{64}\sigma_{yz} + s_{65}\sigma_{zx} + s_{66}\sigma_{xy} \end{aligned} \right\} \quad (1.11)$$

Note that the x , y , and z subscripts on the stress and strain terms refer to crystallographic axes of the crystal.

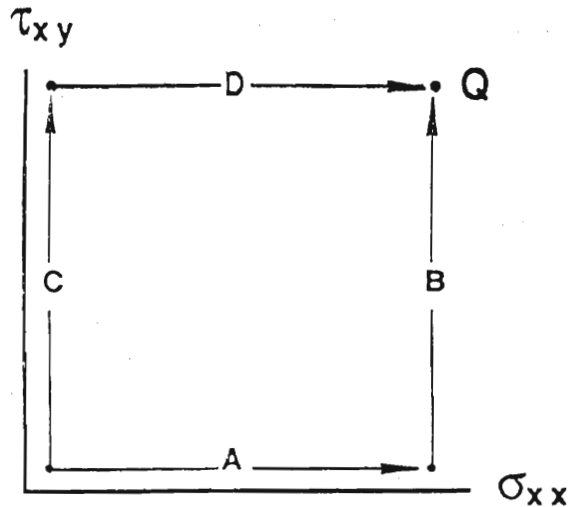
1.2.1. Symmetry of matrix ($s_{ji} = s_{ij}$)

The 36 compliances in Eq. (1.11) can be reduced to 21 independent compliances by utilizing the relation $s_{ij} = s_{ji}$, which will be discussed here. Using this, the matrix of elastic constants becomes:

$$\left. \begin{aligned} &s_{11} \quad s_{12} \quad s_{13} \quad s_{14} \quad s_{15} \quad s_{16} \\ &s_{12} \quad s_{22} \quad s_{23} \quad s_{24} \quad s_{25} \quad s_{26} \\ &s_{13} \quad s_{23} \quad s_{33} \quad s_{34} \quad s_{35} \quad s_{36} \\ &s_{14} \quad s_{24} \quad s_{34} \quad s_{44} \quad s_{45} \quad s_{46} \\ &s_{15} \quad s_{25} \quad s_{35} \quad s_{45} \quad s_{55} \quad s_{56} \\ &s_{16} \quad s_{26} \quad s_{36} \quad s_{46} \quad s_{56} \quad s_{66} \end{aligned} \right\} \quad (1.12)$$

Rather than derive the general relation $s_{ij} = s_{ji}$, a specific case will be treated. To show that $s_{16} = s_{61}$, a calculation will be made of the work required along two different loading paths (Fig. 1.4) to the stress state $Q = \sigma_{11}, \tau_{12}$, with $\sigma_{22} = \sigma_{33} = \tau_{23} = \tau_{31} = 0$. In general, the incremental work per volume, dw , in deforming a body is given by:

$$dw = \sigma_{xx} de_{xx} + \sigma_{yy} de_{yy} + \sigma_{zz} de_{zz} + \tau_{yz} d\gamma_{yz} + \tau_{zx} d\gamma_{zx} + \tau_{xy} d\gamma_{xy}$$


 Fig. 1.4. Loading paths in the proof that $s_{ij} = s_{ji}$.

In this case dw reduces to $dw = \sigma_{xx} d\epsilon_{xx} + \tau_{xy} d\gamma_{xy}$. From the matrix of elastic constants [Eq. (1.11)],

$$\begin{aligned} d\epsilon_{xx} &= s_{11} d\sigma_{xx} + s_{16} d\tau_{xy} \\ d\gamma_{xy} &= s_{61} d\sigma_{xx} + s_{66} d\tau_{xy} \end{aligned}$$

The body can be loaded to Q along paths A and B . In the first step (A), the stress σ_{xx} is applied while $\tau_{xy} = 0$, so the work is $w_A = \int (s_{11}\sigma_{xx}) d\sigma_{xx} = (s_{11}/2)\sigma_{xx}^2$. Then in step B , the stress τ_{xy} is applied while σ_{xx} is held constant; so the additional work is $w_B = \int (s_{16}\sigma_{xx} + s_{66}\tau_{xy}) d\tau_{xy} = s_{16}\sigma_{xx}\tau_{xy} + (s_{66}/2)\tau_{xy}^2$. Hence the total work to load the material to the stress state Q along paths A and B is

$$w_A + w_B = \frac{s_{11}}{2} \sigma_{xx}^2 + s_{16} \sigma_{xx} \tau_{xy} + \frac{s_{66}}{2} \tau_{xy}^2.$$

Alternatively the body could be loaded to Q along paths C and D . In the first step (C), τ_{xy} is applied while holding $\sigma_{xx} = 0$; so $w_C = \int (s_{66}\tau_{xy}) d\tau_{xy} = (s_{66}/2)\tau_{xy}^2$. Then along step D , σ_{xx} is applied while holding τ_{xy} constant; so

$$w_D = \int (s_{11}\sigma_{xx} + s_{61}\tau_{xy}) d\sigma_{xx} = \frac{s_{11}}{2} \sigma_{xx}^2 + s_{61}\tau_{xy}\sigma_{xx}.$$

Therefore the energy associated with loading to Q along paths C and D is

$$w_C + w_D = \frac{s_{11}}{2} \sigma_{xx}^2 + s_{61}\sigma_{xx}\tau_{xy} + \frac{s_{66}}{2} \tau_{xy}^2.$$

The elastic energy is completely reversible; so the energy at Q cannot depend on the path. Hence $w_A + w_B = w_C + w_D$, so $s_{61} = s_{16}$. By analogous treatments it can be shown that $s_{12} = s_{21}$, $s_{32} = s_{23}$, etc. In general

$$s_{ij} = s_{ji}. \quad (1.13)$$

1.2.2. Matrices for several crystal classes

The most general matrix of elastic constants, Eq. (1.12), is appropriate for triclinic crystals that have only a center of symmetry. In materials of higher symmetry, the number of independent elastic constants is further reduced and the corresponding matrix is simplified. Several examples are:

Cubic

(x, y, z are fourfold axes)

$$\begin{array}{cccccc} s_{11} & s_{12} & s_{12} & 0 & 0 & 0 \\ s_{12} & s_{11} & s_{12} & 0 & 0 & 0 \\ s_{12} & s_{12} & s_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{44} \end{array} \quad (1.14)$$

Hexagonal

(x, y are twofold axes, z is sixfold)

$$\begin{array}{cccccc} s_{11} & s_{12} & s_{13} & 0 & 0 & 0 \\ s_{12} & s_{11} & s_{13} & 0 & 0 & 0 \\ s_{13} & s_{13} & s_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(s_{11} - s_{12}) \end{array} \quad (1.15)$$

Orthorhombic

(x, y, z are twofold axes)

$$\begin{matrix}
 s_{11} & s_{12} & s_{13} & 0 & 0 & 0 \\
 s_{12} & s_{22} & s_{23} & 0 & 0 & 0 \\
 s_{13} & s_{23} & s_{33} & 0 & 0 & 0 \\
 0 & 0 & 0 & s_{44} & 0 & 0 \\
 0 & 0 & 0 & 0 & s_{55} & 0 \\
 0 & 0 & 0 & 0 & 0 & s_{66}
 \end{matrix}$$

(1.16)

 Tetragonal¹

 (x and/or y are twofold axes,
z is fourfold)

$$\begin{matrix}
 s_{11} & s_{12} & s_{13} & 0 & 0 & 0 \\
 s_{12} & s_{11} & s_{13} & 0 & 0 & 0 \\
 s_{13} & s_{13} & s_{33} & 0 & 0 & 0 \\
 0 & 0 & 0 & s_{44} & 0 & 0 \\
 0 & 0 & 0 & 0 & s_{44} & 0 \\
 0 & 0 & 0 & 0 & 0 & s_{66}
 \end{matrix}$$

(1.17)

The elastic behavior of isotropic materials can be described in terms of a matrix of elastic constants that is identical to that for cubic crystals except that $s_{44} = 2(s_{11} - s_{12})$. Table 1.1 shows the form of the matrices for 11 symmetry classes.

2.3. Axes of twofold symmetry

In the matrices (1.14–1.17), a number of the elastic constants vanish (=0) because of symmetry. The following example illustrates why $s_{14} = 0$ (and therefore $s_{41} = 0$ also) with twofold rotational symmetry about the z axis. The term s_{14} relates the shear strain, γ_{yz} , to the normal stress, σ_{xx} . Suppose s_{14} is positive. Then a tensile (positive) stress, σ_{xx} , would result in a positive shear strain, γ_{yz} [Fig. 1.5(a)]. But if the z axis is an axis of twofold rotational symmetry, the same behavior should be seen when the crystal is rotated 180° or x is interchanged with -x and y is interchanged with -y. However, if the shear is such that γ_{yz} appears positive to an observer looking from the

Table 1.1. Matrices of elastic constants for different symmetries

triclinic	monoclinic	orthorhombic
tetragonal (422, 4mm, 42m, 4/mmm)	tetragonal (4, 4, 4/m)	trigonal (3, 3)
trigonal (32, 3m, 3m)	hexagonal	cubic
isotropic		

•	$s = 0, c = 0$
•—•	$s \neq 0, c \neq 0$
•—•	equal values of s or c
•—•	$\odot = -\bullet$
•—•	$\odot = 2\bullet$ for s; $\odot = \bullet/2$ for c
+	$2(s_{11} - s_{12})$ or $(c_{11} - c_{12})/2$

¹ There is another subclass of tetragonal crystals in which x and y are not twofold axes. This lower symmetry requires a matrix with more terms. See Nye [1].

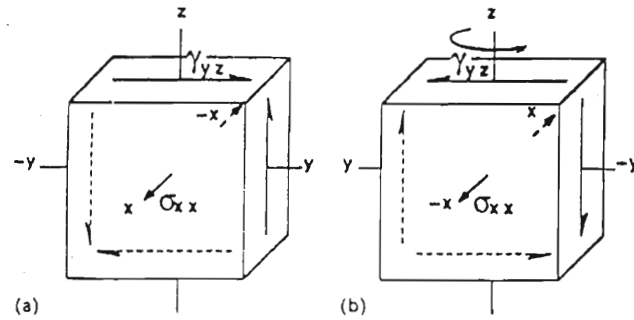


Fig. 1.5. Element loaded under a tensile stress, σ_{xx} . (a) If s_{14} is positive, a positive shear strain, γ_{xy} , would result. (b) However, if there is twofold rotational symmetry about z , the x and $-x$ directions are interchangeable; so an observer behind the element would perceive a negative shear strain, indicating a negative s_{14} . Since s_{14} cannot be both positive and negative, it must be zero.

positive x direction, it would appear negative to an observer looking from the $-x$ direction, as shown in Fig. 1.5(b). This apparent contradiction is resolved only if the stress, σ_{xx} , causes a zero shear strain, γ_{xy} , which in turn implies that s_{14} must be zero. Note that the same conclusion would result if z were an axis of twofold rotational symmetry. All of the other zero values of the elastic constants in the matrices (1.14–1.17) are related to axes that have at least twofold symmetry.

2.4. Hexagonal symmetry

For hexagonal crystals, matrix (1.15) indicates that $s_{22} = s_{11}$, $s_{23} = s_{13}$, and $s_{66} = 2(s_{11} - s_{12})$. These relations are proved by considering the elastic response along two sets of orthogonal axes, 1,2,3 and x,y,z that differ by a 60° rotation. This is illustrated in Fig. 1.6 together with the matrix of direction cosines between the two sets of axes.

Since there is at least two-fold symmetry about each of the axes, the matrix (1.16) of elastic constants for orthorhombic symmetry is applicable. The elastic strain e_x along the x axis can be related to the strains expressed in the 1,2,3 coordinate system:

$$e_x = \frac{1}{4}e_1 + \frac{3}{4}e_2 + (\sqrt{3}/4)\gamma_{12}. \quad (1.18)$$

Substituting the matrix of elastic constants (1.16),

$$e_x = \frac{1}{4}(s_{11}\sigma_1 + s_{12}\sigma_2 + s_{13}\sigma_3) + \frac{3}{4}(s_{12}\sigma_1 + s_{22}\sigma_2 + s_{23}\sigma_3) + (\sqrt{3}/4)s_{66}\sigma_{12}. \quad (1.19)$$

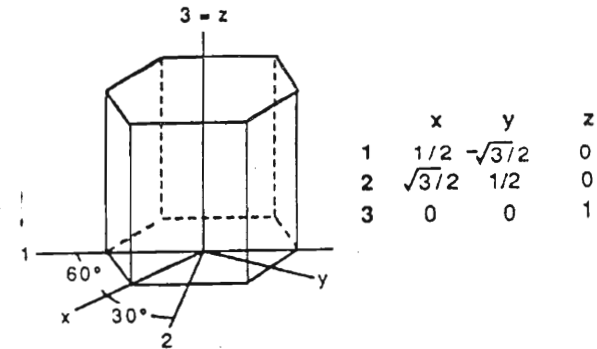


Fig. 1.6. Relation between two possible sets of orthogonal axes for a hexagonal crystal. Note that 1 and x are crystallographically equivalent, as are 2 and y .

Now transforming the stresses,

$$\left. \begin{aligned} \sigma_1 &= \frac{1}{4}\sigma_x + \frac{3}{4}\sigma_y - (\sqrt{3}/2)\sigma_{xy}, \\ \sigma_2 &= \frac{3}{4}\sigma_x + \frac{1}{4}\sigma_y + (\sqrt{3}/2)\sigma_{xy}, \\ \sigma_3 &= \sigma_z, \\ \sigma_{12} &= (\sqrt{3}/4)\sigma_x - (\sqrt{3}/4)\sigma_y - \frac{1}{2}\sigma_{xy}. \end{aligned} \right\} \quad (1.20)$$

Substituting Eq. (1.20) into (1.19),

$$\left. \begin{aligned} e_x &= \frac{1}{16}(s_{11} + 6s_{12} + 9s_{22} + 3s_{66})\sigma_x \\ &+ \frac{1}{16}(3s_{11} + 10s_{12} + 3s_{22} - 3s_{66})\sigma_y \\ &+ \frac{1}{4}(s_{13} + 3s_{23})\sigma_z + (\sqrt{3}/8)(-s_{11} - 2s_{12} + 3s_{22} - s_{66})\sigma_{xy}. \end{aligned} \right\} \quad (1.21)$$

Because the x and 1 axes are crystallographically equivalent, e_x can be found directly from the matrix of elastic constants,

$$e_x = s_{11}\sigma_x + s_{12}\sigma_y + s_{13}\sigma_z + (0)\sigma_{23}. \quad (1.22)$$

Equations (1.21) and (1.22) must be equivalent. Comparing the coefficients of σ_x , σ_y , σ_z , and σ_{xy} ,

$$\begin{aligned}
 s_{11} &= \frac{1}{18}(s_{11} + 6s_{12} + 9s_{22} + 3s_{66}) \\
 s_{12} &= \frac{1}{18}(3s_{11} + 10s_{12} + 3s_{22} - 3s_{66}) \\
 s_{13} &= \frac{1}{4}(s_{13} + 3s_{23}) \\
 0 &= (\sqrt{3}/8)(-s_{11} - 2s_{12} + 3s_{22} - s_{66}).
 \end{aligned}$$

Solving simultaneously,

$$s_{22} = s_{11}, \quad s_{23} = s_{13}, \quad \text{and } s_{66} = 2(s_{11} - s_{12}). \quad (1.23)$$

The matrix of elastic constants for a material with axial or cylindrical symmetry (i.e., all directions normal to the axis are equivalent) has exactly the same form as that for hexagonal crystals. Thus, if the elastic properties of a material have hexagonal symmetry, they also have cylindrical symmetry.

1.2.5. Elastic moduli

Instead of writing the elastic relations in terms of compliances [Eq. (1.11)], they can be expressed in terms of elastic moduli:

$$\left. \begin{aligned}
 \sigma_{xx} &= c_{11}e_{xx} + c_{12}e_{yy} + c_{13}e_{zz} + c_{14}\gamma_{yz} + c_{15}\gamma_{zx} + c_{16}\gamma_{xy} \\
 \sigma_{yy} &= c_{21}e_{xx} + c_{22}e_{yy} + c_{23}e_{zz} + c_{24}\gamma_{yz} + c_{25}\gamma_{zx} + c_{26}\gamma_{xy} \\
 \sigma_{zz} &= c_{31}e_{xx} + c_{32}e_{yy} + c_{33}e_{zz} + c_{34}\gamma_{yz} + c_{35}\gamma_{zx} + c_{36}\gamma_{xy} \\
 \sigma_{yz} &= c_{41}e_{xx} + c_{42}e_{yy} + c_{43}e_{zz} + c_{44}\gamma_{yz} + c_{45}\gamma_{zx} + c_{46}\gamma_{xy} \\
 \sigma_{zx} &= c_{51}e_{xx} + c_{52}e_{yy} + c_{53}e_{zz} + c_{54}\gamma_{yz} + c_{55}\gamma_{zx} + c_{56}\gamma_{xy} \\
 \sigma_{xy} &= c_{61}e_{xx} + c_{62}e_{yy} + c_{63}e_{zz} + c_{64}\gamma_{yz} + c_{65}\gamma_{zx} + c_{66}\gamma_{xy}
 \end{aligned} \right\} \quad (1.24)$$

The matrices of elastic moduli are simplified by symmetry, and their forms are exactly analogous to those for the compliances (1.14–1.17). If all of the compliances s_{ij} for a material are known, the moduli c_{ij} can be calculated and vice versa, but the relations are not simple reciprocals. For example, the moduli for cubic crystals are:

$$\left. \begin{aligned}
 c_{11} &= (s_{11} + s_{12})/(s_{11}^2 + s_{11}s_{12} - 2s_{12}^2) \\
 c_{12} &= -s_{12}/(s_{11}^2 + s_{11}s_{12} - 2s_{12}^2) \\
 c_{44} &= 1/s_{44}
 \end{aligned} \right\} \quad (1.25)$$

The relations for the compliances are analogous:

$$\left. \begin{aligned}
 s_{11} &= \frac{c_{11} + c_{12}}{c_{11}^2 + c_{11}c_{12} - 2c_{12}^2} \\
 s_{12} &= \frac{-c_{12}}{c_{11}^2 + c_{11}c_{12} - 2c_{12}^2} \\
 s_{44} &= \frac{1}{c_{44}}
 \end{aligned} \right\} \quad (1.26)$$

The relations for hexagonal crystals are:

$$\left. \begin{aligned}
 c_{11} + c_{12} &= \frac{s_{33}}{s} \\
 c_{11} - c_{12} &= \frac{1}{s_{11} - s_{12}} \\
 c_{13} &= \frac{-s_{13}}{s} \\
 c_{33} &= \frac{s_{11} + s_{12}}{s} \\
 c_{44} &= \frac{1}{s_{44}}
 \end{aligned} \right\} \quad (1.27)$$

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where $s = s_{33}(s_{11} + s_{12}) - 2s_{13}^2$. Again the relations for the compliances are analogous. Reference [2] gives the relations for other crystal classes.

Tables 1.2 and 1.3 give the elastic constants for some cubic and hexagonal crystals. Most of the data are from a compilation by Simmons and Wang [3].

1.3. Orientation dependence of elastic response

The elastic response of a crystal along any set of directions may be calculated by resolving the stress state onto the crystal axes, using the matrices of elastic constants to find the strains along the crystal axes and then resolving these strains onto the axes of interest.

Example 1.2

Derive expressions for the effective Young's modulus and Poisson's ratio for a cubic crystal stressed along the $[112]$ direction.

Solution: Let $x = [112]$, $y = [1\bar{1}\bar{1}]$, and $z = [1\bar{1}0]$. The direction

Table 1.2. Elastic constants for various cubic crystals

Crystal	c_{11} (GPa)	c_{12} (GPa)	c_{44} (GPa)	s_{11} (TPa ⁻¹)	s_{12} (TPa ⁻¹)	s_{44} (TPa ⁻¹)
Cr	339.8	58.6	99.0	3.10	-0.46	10.10
Fe	231.4	134.7	116.4	7.56	-2.78	8.59
K	3.70	3.14	1.88	1223.9	56.19	53.19
Li	13.50	11.44	8.78	332.8	-152.7	113.9
Mo	441.6	172.7	121.9	2.90	-0.816	8.21
Na	6.15	4.96	5.92	581.0	-259.4	168.9
Nb	240.2	125.6	28.2	6.50	-2.23	35.44
Ta	260.2	154.5	82.6	6.89	-2.57	12.11
W	522.4	204.4	160.8	2.45	-0.69	6.22
Ag	122.2	90.7	45.4	22.26	-9.48	22.03
Al	107.3	60.9	28.3	15.82	-5.73	35.34
Au	192.9	163.8	41.5	23.55	-10.81	24.10
Cu	166.1	199.0	75.6	15.25	-6.39	13.23
Ni	248.1	154.9	124.2	7.75	-2.98	8.05
Pb	49.5	42.3	14.9	94.57	-43.56	67.11
Pd	227.1	176.0	71.7	13.63	-5.95	13.94
Pt	346.7	250.7	76.5	7.34	-3.08	13.07
C	949.0	151.0	521.0	1.10	-1.51	1.92
Ge	128.4	48.2	66.7	9.80	-2.68	15.00
Si	166.2	64.4	79.7	7.67	-2.14	12.54
MgO	287.6	87.4	151.4	4.05	-0.94	6.60
MnO	223.0	120.0	79.0	7.19	-2.52	12.66
LiF	114.0	47.7	63.6	11.65	-3.43	15.71
KCl	39.5	4.9	6.3	26.00	-2.85	158.6
NaCl	49.0	12.6	12.7	22.80	-4.66	78.62
NaBr	40.4	10.1	10.2	27.54	-5.53	98.52
NaI	30.1	9.12	7.33	38.72	-9.011	136.4
NaF	97.0	23.8	28.2	11.41	-2.29	35.43
ZnS	103.2	64.8	46.2	18.77	-7.24	21.65
InP	102.2	57.6	46.0	16.48	-5.94	21.74
GaAs	118.8	53.7	59.4	11.72	-3.65	16.82

Table 1.3. Elastic compliances for various hexagonal crystals, in TPa⁻¹

Material	s_{11}	s_{12}	s_{44}	s_{33}	s_{13}
Cadmium	12.9	-1.5	64.0	36.9	-9.3
Hafnium	4.25	0.398	17.95	6.13	-1.56
Magnesium	22.15	-7.7	60.3	19.75	-4.93
Titanium	4.41	0.507	21.41	6.97	-1.89
Zinc	8.40	1.1	26.4	28.7	-7.75
Zirconium	5.32	0.719	31.25	7.96	-2.39

cosines between these axes and the cubic axes of the crystal are:

$$\begin{array}{ccc}
 x = & y = & z = \\
 [112] & [\bar{1}\bar{1}\bar{1}] & [1\bar{1}0] \\
 1 = [100] & 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\
 2 = [010] & 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\
 3 = [001] & 2/\sqrt{6} & -1/\sqrt{3} & 0
 \end{array}$$

For tension along $[112]$, the only finite stress on the x,y,z axis system is σ_x (i.e., $\sigma_y = \sigma_z = \sigma_{yz} = \sigma_{zx} = \sigma_{xy} = 0$). Transforming to the 1,2,3 axes:

$$\begin{aligned}
 \sigma_1 &= \frac{1}{6}\sigma_x & \sigma_{23} &= \frac{1}{3}\sigma_x \\
 \sigma_2 &= \frac{1}{6}\sigma_x & \sigma_{31} &= \frac{1}{3}\sigma_x \\
 \sigma_3 &= \frac{2}{3}\sigma_x & \sigma_{12} &= \frac{1}{6}\sigma_x
 \end{aligned}$$

From the matrix (1.14) of elastic constants for a cubic crystal, the resulting elastic strains are:

$$\begin{aligned}
 e_1/\sigma_x &= \frac{1}{6}s_{11} + \frac{1}{6}s_{12} + \frac{2}{3}s_{12} = \frac{1}{6}s_{11} + \frac{5}{6}s_{12} \\
 e_2/\sigma_x &= \frac{1}{6}s_{12} + \frac{1}{6}s_{11} + \frac{2}{3}s_{12} = \frac{1}{6}s_{11} + \frac{5}{6}s_{12} \\
 e_3/\sigma_x &= \frac{1}{6}s_{12} + \frac{1}{6}s_{12} + \frac{2}{3}s_{11} = \frac{2}{3}s_{11} + \frac{1}{3}s_{12} \\
 \gamma_{23}/\sigma_x &= \gamma_{31}/\sigma_x = \frac{1}{3}s_{44} \\
 \gamma_{12}/\sigma_x &= \frac{1}{6}s_{44}
 \end{aligned}$$

Transforming these strains onto the x,y,z axes,

$$\begin{aligned}
 e_x &= \frac{1}{6}e_1 + \frac{1}{6}e_2 + \frac{2}{3}e_3 + \frac{1}{3}\gamma_{23} + \frac{1}{3}\gamma_{31} + \frac{1}{6}\gamma_{12} \\
 e_y &= \frac{1}{3}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_3 - \frac{1}{3}\gamma_{23} - \frac{1}{3}\gamma_{31} + \frac{1}{3}\gamma_{12} \\
 e_z &= \frac{1}{2}e_1 + \frac{1}{2}e_2 - \frac{1}{2}\gamma_{23}
 \end{aligned}$$

or

$$\begin{aligned}
 e_x/\sigma_x &= \frac{1}{3}s_{11} + \frac{1}{3}s_{12} + \frac{1}{3}s_{44} \\
 E_{[112]} = \sigma_x/e_x &= \left(\frac{1}{3}s_{11} + \frac{1}{3}s_{12} + \frac{1}{3}s_{44}\right)^{-1}
 \end{aligned}$$

There are two values of Poisson's ratio; $\nu_y = -e_y/e_x$ and $\nu_z = -e_z/e_x$.

$$e_y/\sigma_x = +\frac{1}{3}s_{11} + \frac{2}{3}s_{12} - \frac{1}{6}s_{44}$$

and

$$e_z/\sigma_x = +\frac{1}{6}s_{11} + \frac{2}{3}s_{12} - \frac{1}{12}s_{44}$$

so

$$v_y = -\frac{4s_{11} + 8s_{12} - 2s_{44}}{6s_{11} + 6s_{12} + 3s_{44}}$$

$$v_z = -\frac{2s_{11} + 10s_{12} - s_{44}}{6s_{11} + 6s_{12} + 3s_{44}}$$

1.3.1. Orientation dependence of Young's modulus for cubic crystals

The foregoing treatment can be generalized for finding the effective Young's modulus, E_d , along any direction d . Consider a cubic crystal loaded under a single tensile stress σ_d applied along d . Let the direction cosines of d with the 1,2,3 axes be α , β , and γ , as shown in Fig. 1.7. The stresses along the cubic axes are then:

$$\left. \begin{aligned} \sigma_1 &= \alpha^2 \sigma_d, & \sigma_{23} &= \beta\gamma \sigma_d \\ \sigma_2 &= \beta^2 \sigma_d, & \sigma_{31} &= \gamma\alpha \sigma_d \\ \sigma_3 &= \gamma^2 \sigma_d, & \sigma_{12} &= \alpha\beta \sigma_d \end{aligned} \right\} \quad (1.28)$$

From the matrix of elastic constants (1.14), the corresponding strains are:

$$\left. \begin{aligned} e_1/\sigma_d &= s_{11}\alpha^2 + s_{12}\beta^2 + s_{12}\gamma^2, & \gamma_{23}/\sigma_d &= s_{44}\beta\gamma, \\ e_2/\sigma_d &= s_{12}\alpha^2 + s_{11}\beta^2 + s_{12}\gamma^2, & \gamma_{31}/\sigma_d &= s_{44}\gamma\alpha, \\ e_3/\sigma_d &= s_{12}\alpha^2 + s_{12}\beta^2 + s_{11}\gamma^2, & \gamma_{12}/\sigma_d &= s_{44}\alpha\beta. \end{aligned} \right\} \quad (1.29)$$

Resolving these components onto the d axis,

$$e_d/\sigma_d = \alpha^2 e_1 + \beta^2 e_2 + \gamma^2 e_3 + \beta\gamma\gamma_{23} + \gamma\alpha\gamma_{31} + \alpha\beta\gamma_{12} \quad (1.30)$$

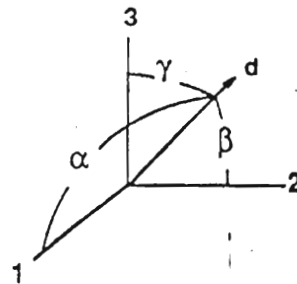


Fig. 1.7. Coordinate system for finding E_d .

and substituting Eq. (1.29) into Eq. (1.30),

$$\begin{aligned} e_d/\sigma_d &= s_{11}(\alpha^4 + \beta^4 + \gamma^4) + 2s_{12}(\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2) \\ &\quad + s_{44}(\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2) \\ &= s_{11}(\alpha^4 + \beta^4 + \gamma^4) + (2s_{12} + s_{44})(\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2). \end{aligned} \quad (1.31)$$

Substituting the identity $(\alpha^2 + \beta^2 + \gamma^2) = 1$ in the form of $(\alpha^4 + \beta^4 + \gamma^4) = 1 - 2(\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2)$,

$$1/E_d = e_d/\sigma_d = s_{11} + (-2s_{11} + 2s_{12} + s_{44})(\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2). \quad (1.32)$$

If d is expressed by direction indices $[hkl]$,

$$\begin{aligned} \alpha &= h/\sqrt{h^2 + k^2 + l^2}, \\ \beta &= k/\sqrt{h^2 + k^2 + l^2}, \\ \gamma &= l/\sqrt{h^2 + k^2 + l^2}, \end{aligned}$$

and Eq. (1.32) can be written as:

$$\frac{1}{E_d} = s_{11} + \frac{(2s_{12} - 2s_{11} + s_{44})(k^2l^2 + l^2h^2 + h^2k^2)}{(h^2 + k^2 + l^2)^2}. \quad (1.33)$$

For an isotropic material, $1/E$ is independent of direction. This requires that in Eqs. (1.32) and (1.33), $(2s_{12} - 2s_{11} + s_{44}) = 0$, or

$$s_{44} = 2(s_{11} - s_{12}). \quad (1.34)$$

According to Eq. (1.33), the extreme values of E in a cubic crystal occur in the $[100]$ and $[111]$ directions. Since $1/E_{[100]} = s_{11}$ and $1/E_{[111]} = 2(s_{11} + 2s_{12} + s_{44})/3$, the ratio is given by

$$\frac{E_{[111]}}{E_{[100]}} = \frac{3s_{11}}{s_{11} + 2s_{12} + s_{44}}. \quad (1.35)$$

This ratio represents the largest possible anisotropy in cubic crystals. Table 1.4 gives its value for several cubic crystals. For most, $E_{[111]} > E_{[100]}$, but there are exceptions (Mo, Nb, Cr, and alkali halides). Tungsten is almost elastically isotropic.

The orientation dependence in Eq. (1.32) can be expressed as

$$\frac{1}{E_d} = \frac{1}{E_{[100]}} + f\left(\frac{1}{E_{[111]}} - \frac{1}{E_{[100]}}\right), \quad (1.36)$$

where $f = 3(\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2)$.

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Table 1.4. Values of $E_{[111]}/E_{[100]}$ for several cubic crystals

Material	$E_{[111]}/E_{[100]}$	Material	$E_{[111]}/E_{[100]}$
Cr	0.76	Fe	2.14
Mo	0.92	Nb	0.52
W	1.01	Al	1.20
Cu	2.91	Ni	2.36
Si	1.44	MgO	1.39
KCl	0.44	NaCl	0.74
ZnS	2.17	GaAs	1.66

Figure 1.8 shows how E_d varies with orientation in the basic triangle. (See Appendix 1 for an explanation of the coordinate system.)

1.3.2. Orientation dependence of E in other crystal classes

Similar expressions [1] can be derived for other crystal classes. For hexagonal crystals,

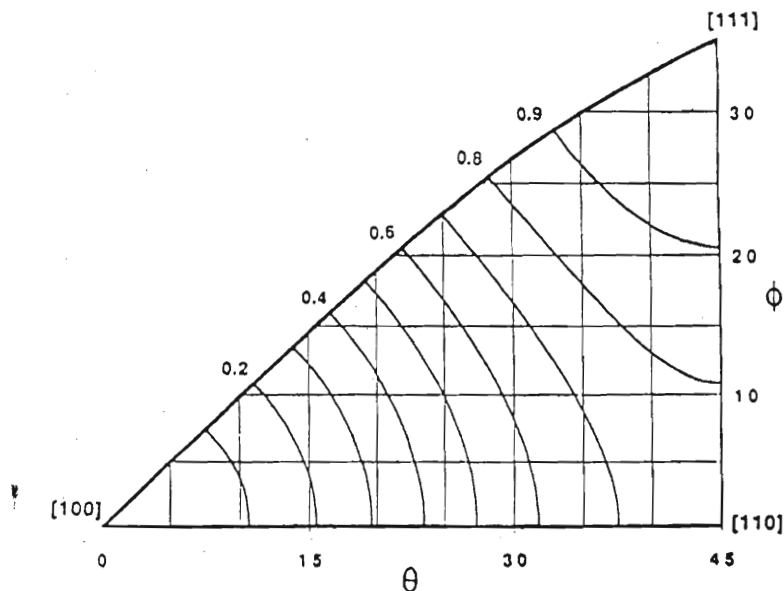


Fig. 1.8. The orientation dependence of $1/E_d$ for cubic crystals. The contours are levels of $f = 3(\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2)$ in the expression $1/E_d = 1/E_{[100]} + f(1/E_{[111]} - 1/E_{[100]})$. The value of f has a maximum value at [111] and a minimum at [100].

$$\frac{1}{E} = (1 - \gamma^2)^2 s_{11} + \gamma^4 s_{33} + \gamma^2(1 - \gamma^2)(2s_{13} + s_{44}), \quad (1.37)$$

where γ is the cosine of the angle between the direction d and the c axis. For orthorhombic crystals,

$$\frac{1}{E} = s_{11}\alpha^4 + s_{22}\beta^4 + s_{33}\gamma^4 + \beta^2\gamma^2(2s_{23} + s_{44}) + \gamma^2\alpha^2(2s_{31} + s_{55}) + \alpha^2\beta^2(2s_{12} + s_{66}). \quad (1.38)$$

1.4. Symmetry in other materials

The basic forms of the matrix of elastic constants can be used for materials that are not single crystals, but which do have similar symmetries of structure and properties. Such materials may be either natural or artificial composites. Examples include:

1. Orthotropic (same as orthorhombic) symmetry:
 - a. Rolled sheets or plates, both before and after recrystallization. The anisotropy is caused by crystallographic texture. The axes are rolling, transverse, and thickness directions.
 - b. Tubes and swaged wire or rod resulting from cylindrical crystallographic textures. The axes are the radial, tangential (hoop), and axial directions.
 - c. Wood. The axes are the radial, tangential, and axial directions. Also paper and cardboard.
2. Tetragonal symmetry:
 - a. Plywood. The anisotropy is from oriented fibers.
 - b. Woven cloth and window screen and composites reinforced with woven material.
 - c. Sheets and plates that have been cross rolled.
3. Axial symmetry (same as hexagonal symmetry):
 - a. Drawn or extruded wire and rod. The anisotropy is caused by a fiber texture, all directions normal to the wire axis being equivalent.
 - b. Electroplates and portions of ingots with columnar grains.
 - c. Uniaxially aligned composites.

1.5. Elastic behavior of a polycrystal as a single crystal average

In principle it should be possible to obtain the elastic modulus for a polycrystal from a weighted average of the elastic behavior of all orientations of crystals.

present in the polycrystal. However, the appropriate way to average is not obvious. Two extreme methods have been proposed for a random polycrystal.

The Voigt average [4] is based on assuming uniform local strains and averages the moduli over all orientations.

$$E_V = (F + G + 3H)(F + 2G)/(2F + 3G + H), \quad (1.39)$$

where

$$F = \frac{1}{3}(c_{11} + c_{22} + c_{33})$$

$$G = \frac{1}{3}(c_{12} + c_{23} + c_{13})$$

$$H = \frac{1}{3}(c_{44} + c_{55} + c_{66}).$$

For cubic crystals this reduces to:

$$E_V = (c_{11} - c_{12} + 3c_{44})(c_{11} + 2c_{12})/(2c_{11} + 3c_{12} + c_{44}). \quad (1.40)$$

The Reuss average [5] assumes uniform local stress and is based on averaging the compliances over all orientations:

$$1/E_R = \frac{1}{3}(3F' + 2G' + H'), \quad (1.41)$$

where

$$F' = \frac{1}{3}(s_{11} + s_{22} + s_{33})$$

$$G' = \frac{1}{3}(s_{12} + s_{23} + s_{13})$$

$$H' = \frac{1}{3}(s_{44} + s_{55} + s_{66}).$$

For cubic crystals this reduces to:

$$E_R^{-1} = \frac{1}{3}(3s_{11} + 2s_{12} + s_{44}). \quad (1.42)$$

Hill [6] showed that the Voigt and Reuss averages correspond to upper and lower bounds to the true behavior. Table 1.5 shows a comparison of the two averages for materials of cubic structure, together with the extreme values and the experimental value for polycrystals. Hill suggested that either the arithmetic or geometric average of the Reuss and Voigt averages is a good approximation for the polycrystal.

Hashmin and Shtrikman [7], however, derived very much better upper and lower bounds for cubic polycrystals. Their upper and lower bounds for

Table 1.5. Comparison of extreme elastic moduli with upper- and lower-bound averages for random polycrystals (GPa)

Material	$E_{(111)}$	$E_{(100)}$	E_R	E_V	E_H	E_S	Expt.
Aluminum	75.6	63.2	70.1	70.7	70.4	70.4	62.0
Copper	191.7	66.9	109.8	145.2	139.0	125.0	128.0
Gold	115.2	42.5	68.4	87.0	80.9	76.7	78.0
α -iron	283.3	132.3	194.5	227.1	214.6	208.8	208.0
Nickel	304.9	129.0	197.3	239.5	223.8	215.4	207.0
Niobium	80.0	153.8	99.0	110.6	105.9	104.2	103.0

the shear modulus (G_H and G_S , respectively) are:

$$G_H = (c_{11} - c_{12})/2 + 3(a_1 + b_1)^{-1}, \quad \text{where} \quad (1.43)$$

$$a_1 = 10/(2c_{44} - c_{11} + c_{12})$$

$$b_1 = \frac{1}{3}(2c_{11} - c_{12})/[(c_{11} - c_{12})(3c_{11} - 2c_{12})]$$

and

$$G_S = c_{44} + 2(A_2 + B_2)^{-1}, \quad \text{where} \quad (1.44)$$

$$A_2 = 10/(c_{11} - c_{12} - 2c_{44})$$

$$B_2 = \frac{2}{3}(c_{11} + c_{12} + 3c_{44})/[c_{44}(c_{11} + c_{12} + 4c_{44})].$$

In all cases the bulk modulus is given by

$$B = (c_{11} + 2c_{12}). \quad (1.45)$$

The corresponding values of Young's modulus can be found from B and G using Eq. (1.6), $E = 9BG/(G + 3B)$. Table 1.5 compares the extreme values of Young's modulus and the upper- and lower-bound averages for several materials. Experimental values from Ref. [9] are also given.

1.6. Thermal expansion

The small dimensional changes caused by temperature changes can be treated as strains, $e_{ij} = \alpha_{ij} \Delta T$, where α_{ij} is the linear coefficient of thermal expansion. Hooke's laws can be generalized to include these strains. In the most general case, the thermal expansion coefficients form a symmetric tensor:

$$\begin{matrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{12} & \alpha_{22} & \alpha_{23} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{matrix} \quad (1.46)$$

Table 1.6. Coefficients of thermal expansion for some noncubic crystals

Material	Struct.	Temp. range (°C)	α_{11} [($\mu\text{m}/\text{m}$)/K]	α_{22} [($\mu\text{m}/\text{m}$)/K]	α_{33} [($\mu\text{m}/\text{m}$)/K]	Ref.
Zirconium	hcp	300-900	5.7	— ^a	11.4	8
Zinc	hcp	0-100	15.0	—	61.5	9
Magnesium	hcp	0-35	24.3	—	27.1	9
Cadmium	hcp	at 0	19.1	—	54.3	9
Titanium	hcp	0-700	11.0	—	12.8	9
Tin	tetr.	at 50	16.6	—	32.9	9
Uranium	ortho.	at 75	20.3	-1.4	22.2	10
Calcite	hex.	0-85	-5.6	—	25.1	10

^a Dash indicates $\alpha_{22} = \alpha_{11}$.

where the subscripts on the α terms refer to the crystal axes x, y, z and are the same as on the corresponding strain component. Using shorter notation,

$$\begin{aligned} e_x &= \alpha_1 \Delta T, & e_y &= \alpha_2 \Delta T, & e_z &= \alpha_3 \Delta T, \\ \gamma_{yz} &= \alpha_4 \Delta T, & \gamma_{zx} &= \alpha_5 \Delta T, & \gamma_{xy} &= \alpha_6 \Delta T, \end{aligned} \quad (1.47)$$

where $\alpha_1 = \alpha_{11}$, $\alpha_2 = \alpha_{22}$, $\alpha_3 = \alpha_{33}$, $\alpha_4 = 2\alpha_{23}$, $\alpha_5 = 2\alpha_{31}$, and $\alpha_6 = 2\alpha_{12}$. There are simplifications for crystals of higher symmetry:

1. Cubic: $\alpha_1 = \alpha_2 = \alpha_3$, $\alpha_4 = \alpha_5 = \alpha_6 = 0$;
2. Hexagonal and tetragonal: $\alpha_1 = \alpha_2 \neq \alpha_3$, $\alpha_4 = \alpha_5 = \alpha_6 = 0$;
3. Orthorhombic crystals: $\alpha_1 \neq \alpha_2 \neq \alpha_3$, $\alpha_4 = \alpha_5 = \alpha_6 = 0$.

Table 1.6 gives the coefficients of thermal expansion for several hexagonal and other noncubic crystals.

1.7. Effects of anisotropy of thermal expansion in polycrystals

The anisotropy of thermal expansion with crystallographic direction in noncubic crystals has significant effects on the behavior of polycrystalline aggregates.

Lord Rayleigh [11] found that the properties of marble (polycrystalline calcite) are significantly affected by heating and cooling. Appreciable effects were found after heating to only 100°C. Increases of porosity and dimensions were observed, as well as decreases of rigidity and tensile strength. Higher temperatures caused greater damage. These changes were attributed to intergranular cracking generated by the different thermal expansion of neighboring grains.

Thermal cycling of polycrystals of the noncubic metals, tin, cadmium, and zinc, has been found to cause plastic deformation, which is revealed by roughening of the surface [12, 13]. The lack of similar observations in magnesium and titanium is consistent with their much smaller anisotropy of thermal expansion coefficients.

Bars of α -uranium undergo a phenomenon called *thermal ratchetting*, whereby thermal cycling causes a permanent increase of length. Cycling between 100 and 570°C causes growth rates up to $\Delta L/L = 600 \times 10^{-6}$ per cycle [14]. This has been explained [15] in terms of:

1. The duplex texture consisting of two components with some grains (A) being oriented with [010] parallel to the rod axis, and others (B) having significantly different orientations;
2. The very low (or slightly negative) coefficient of thermal expansion parallel to [010]; and
3. The easily activated slip system, (010)[100], which for the [010]-oriented grains (orientation A) produces no dimensional changes parallel to the rod axis. On heating, the difference in thermal expansion between the grains is accommodated by grain boundary flow at the higher temperatures. On cooling, tensile stresses in the (B) component cause slip. Thus the length of the B grains is increased during the cycle. A similar explanation was given for the deformation of zinc bicrystals by a cycle of heating and cooling [16]. Similar thermal ratchetting has been observed in tin.

1.8. Residual stresses from anisotropic thermal expansion

It is of interest to estimate the magnitude of the internal stresses that arise from anisotropic expansion in polycrystalline metals. Zinc, which is hexagonal, is an interesting example. Consider the behavior of a bicrystal composed of two crystals: one, A, with [0001] parallel to the external x direction and the other, B, with [0001] perpendicular to the x direction, as shown in Fig. 1.9. They are bonded parallel to x , so that $e_{xA} = e_{xB}$. With

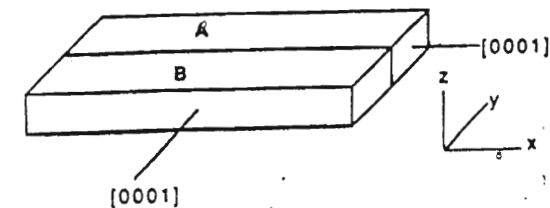


Fig. 1.9. Bicrystal of zinc.

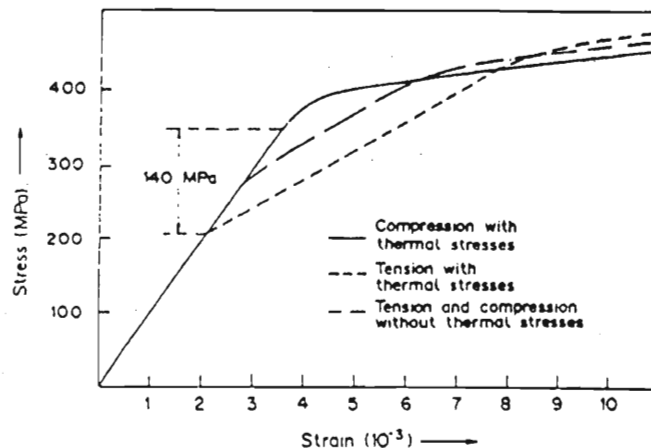
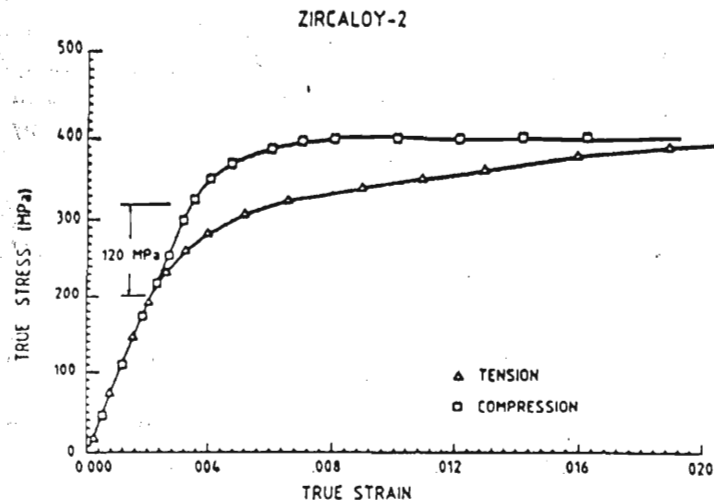


Fig. 1.10. Stress-strain curves for a textured zircaloy-2 rod in tension and compression. The experimental curves are shown above and the calculated ones below. (From MacEwen, Tomé, and Faber [17].)

no external force acting on the bicrystal, $\sigma_{yA} = \sigma_{yB} = \sigma_{zA} = \sigma_{zB} = 0$, and because of the shape $e_{xA} = s_{33}\sigma_{xA} + \alpha_3 \Delta T = e_{xB} = s_{11}\sigma_{xB} + \alpha_1 \Delta T$. If the crystals are the same size and $F_x = 0$, $\sigma_{xA} + \sigma_{xB} = 0$. Substituting and solving for σ_{xA} ,

$$\sigma_{xA} = \left(\frac{\alpha_1 - \alpha_3}{s_{11} - s_{33}} \right) \Delta T.$$

For zinc, $\alpha_1 = 15 \times 10^{-6}$ and $\alpha_3 = 61.5 \times 10^{-6} (^{\circ}\text{C})^{-1}$ and $s_{33} = 28.7 \times 10^{-12}$ and $s_{11} = 8.4 \times 10^{-12} (\text{Pa})^{-1}$. Substituting these values, $\sigma_{xA}/\Delta T = 1.26 \text{ MPa}/^{\circ}\text{C}$. Thus temperature changes of only a few degrees should cause yielding of some grains.

It is impossible to produce stress-free zinc by annealing because of the intergranular stresses created during cooling from any appreciable annealing temperature. During cooling, yielding will occur in many grains, leaving them at or very near their yield strengths. Application of any additional stress, as in a subsequent tension test, will cause yielding in those grains. This is why the stress-strain curves for polycrystalline zinc do not have linear elastic regions.

The residual stresses resulting from cooling after annealing have been studied in textured zirconium rods [17]. Annealing at 900 K was found to produce residual stresses of over 100 MPa. These were explained using a much more detailed model than that presented here. These residual stresses caused differences between the yielding behavior in subsequent tension and compression tests which were successfully predicted by the model as shown in Figure 1.10.

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2

SLIP IN SINGLE CRYSTALS

Slip is the primary deformation mechanism in most crystals. It involves shearing on crystallographic planes in crystallographic directions. The magnitude of the shear displacement is an integral number of interatomic distances; so the lattice is left unaltered when slip occurs across an entire plane. When, however, only part of a plane undergoes slip, there remains a dislocation or boundary between the slipped and unslipped portions of the plane, and this dislocation is the center of a lattice distortion. Slip occurs by movement of dislocations through the lattice, and it is the accumulation of the dislocations left by slip that is responsible for work hardening. Although the study of dislocations and their movement is an important topic, this book is focused on the resulting slip.

Examination of the surface of a deformed crystal indicates that slip is inhomogeneous on an atomic scale. Displacements of hundreds or thousands of atomic diameters must be occurring on discrete or closely spaced planes to create the visible slip lines on the surface. Furthermore, on an atomic scale, the planes of active slip are widely separated. Yet the scale of the slip displacements and distances between slip lines are small compared to most grain sizes; so slip usually can be treated as macroscopically homogeneous shearing.

The slip elements (planes and directions) for several common crystals are summarized in Table 2.1. Almost without exception the slip directions are the crystallographic directions with the shortest repeat distance, and the slip planes are usually densely packed planes.

2.1. Schmid's Law

When a crystal is stressed, slip begins when the shear stress on some slip system reaches a critical value, τ_c (often called the *critical resolved shear stress*). In most crystals² slip can occur with equal ease forward or

² bcc metals are exceptions. They have an easy and a hard direction of slip.

Table 2.1. Slip directions and planes

Structure	Slip direction	Slip planes
fcc	$\langle 110 \rangle$	$\{111\}$
bcc	$\langle 111 \rangle$	$\{110\}, \{112\}, \{123\}$, pencil glide*
hcp	$\langle 11\bar{2}0 \rangle$	$\{0001\}, \{1\bar{1}00\}, \{1\bar{1}01\}$
	$\langle 11\bar{2}3 \rangle^*$	$\{1\bar{1}0\bar{1}\}$
dia. cub.	$\langle 110 \rangle$	$\{111\}$
NaCl	$\langle 110 \rangle$	$\{110\}$
CsCl	$\langle 001 \rangle$	$\{100\}$
Fluorite	$\langle 110 \rangle$	$\{001\}, \{110\}, \{111\}^*$

* With pencil glide, slip is possible on all planes containing the slip direction.
 * Reported in Zn, Zr, and Be crystals under special loading conditions or high temperatures.
 * At elevated temperatures.

backward; so the condition necessary for slip can be written as

$$\tau_{nd} = \pm \tau_c \quad (2.1)$$

Here n and d refer to the *slip plane normal* and the *slip direction*, respectively. This simple yield criterion for crystallographic slip is called *Schmid's Law*. In a uniaxial tension test along x , the shear stress can be found from the equations for stress transformation (see Appendix 2)

$$\tau_{nd} = l_{nx} l_{dx} \sigma_{xx} \quad (2.2)$$

Defining λ as the angle between the slip direction and the tensile axis, and ϕ as the angle between the tensile axis and the slip plane normal, as shown in Fig. (2.1), and substituting in Eq. (2.2), the necessary condition for slip may be expressed by the usual form of Schmid's Law

$$\tau_c = \pm \sigma_x \cos \lambda \cos \phi \quad (2.3)$$

or simply

$$\sigma_x = \pm \tau_c / m_x \quad (2.4)$$

where the Schmid factor $m_x = \cos \lambda \cos \phi$.

In tensor notation the condition for yielding under a more general stress state is

$$\pm \tau_c = l_{nx} l_{dx} \sigma_{xx} + l_{ny} l_{dy} \sigma_{yy} + \dots + (l_{nx} l_{dy} + l_{ny} l_{dx}) \sigma_{xy} \quad (2.5)$$

Example 2.1

An aluminum crystal is tested in uniaxial tension parallel to $[112]$. On which of the twelve $\langle 110 \rangle \{111\}$ slip systems should slip occur? If the crystal yields when the applied tensile stress is 1.08 MPa, what is the value of τ_c ?

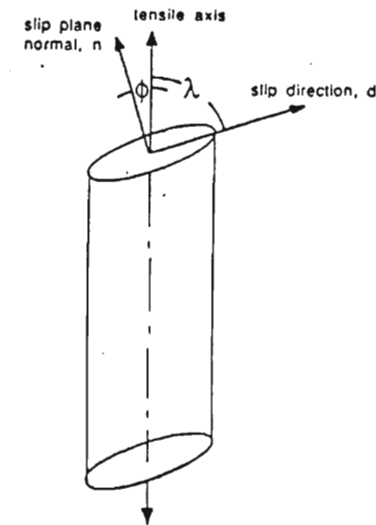


Fig. 2.1. Slip elements in uniaxial tension.

Solution: First write all twelve possible slip systems³ and compute the Schmid factor for each. For example with $[01\bar{1}](111)$,

$$l_{nx} = \frac{1 \times 1 + 1 \times 1 + 1 \times 2}{\sqrt{(1^2 + 1^2 + 1^2)(1^2 + 1^2 + 2^2)}} = \frac{1}{\sqrt{2}}$$

$$l_{dx} = \frac{1 \times 0 + 1 \times 1 + 2 \times \bar{1}}{\sqrt{(0^2 + 1 + (-1)^2)(1^2 + 1^2 + 1^2)}} = -\frac{1}{\sqrt{3}}$$

so

$$m = l_{nx} l_{dx} = \frac{-\frac{1}{\sqrt{2}}}{\sqrt{6}}$$

Similarly, for the other slip systems:

$$[01\bar{1}](111) \quad m = \frac{-\frac{1}{\sqrt{2}}}{\sqrt{6}} \quad [011](\bar{1}\bar{1}\bar{1}) \quad m = \frac{-1}{\sqrt{6}}$$

$$[\bar{1}01](111) \quad m = \frac{\frac{1}{\sqrt{2}}}{\sqrt{6}} \quad [10\bar{1}](\bar{1}\bar{1}\bar{1}) \quad m = \frac{\frac{1}{\sqrt{2}}}{\sqrt{6}}$$

³ Note that in writing the 12 possible slip systems attention must be paid to the requirement that the slip direction must lie in the slip plane.

$$[\bar{1}\bar{1}0](111) \quad m = 0 \quad [\bar{1}\bar{1}0](\bar{1}\bar{1}\bar{1}) \quad m = \frac{3}{\sqrt{6}}$$

$$[01\bar{1}](\bar{1}\bar{1}1) \quad m = -\frac{1}{\sqrt{6}} \quad [011](11\bar{1}) \quad m = 0$$

$$[101](\bar{1}11) \quad m = \frac{1}{\sqrt{6}} \quad [1\bar{1}0](11\bar{1}) \quad m = 0$$

$$[\bar{1}\bar{1}0](\bar{1}11) \quad m = -\frac{3}{\sqrt{6}} \quad [101](11\bar{1}) \quad m = 0$$

The $[101](\bar{1}11)$ and $[011](\bar{1}\bar{1}\bar{1})$ systems have the highest Schmid factors and should be active. The negative value of m for the $[011](\bar{1}\bar{1}\bar{1})$ system means that slip will occur in the $-[011]$ or $[0\bar{1}\bar{1}]$ direction on the $(\bar{1}\bar{1}\bar{1})$ plane or in the $+ [011]$ direction on the (111) plane.

Since slip occurs when $\tau_c = \pm \sigma_x m_x$, $\tau_c = 1.08(1/\sqrt{6}) \approx 0.441$ MPa.

Example 2.2

Consider the yielding of a single crystal with a single slip system (Fig. 2.2). The angles that the x , y , and z axes make with the slip plane normal, φ , and the slip direction, λ , are:

$$\varphi_x = 45^\circ, \quad \varphi_y = 60^\circ, \quad \varphi_z = 60^\circ$$

$$\lambda_x = 45^\circ, \quad \lambda_y = 120^\circ, \quad \lambda_z = 120^\circ.$$

The critical shear stress for slip is 1.40 MPa.

a. If the crystal is loaded under biaxial compression, $\sigma_x = \sigma_y$, with $\sigma_z = \tau_{yz} = \tau_{zx} = \tau_{xy} = 0$, what is the value of σ_x at yielding?

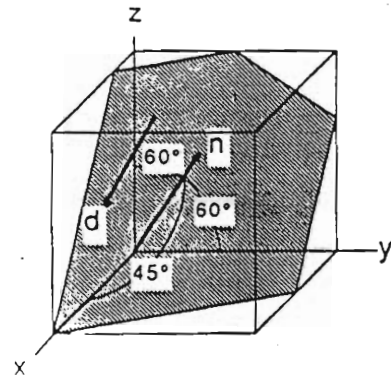


Fig. 2.2 Orientation of a slip system.

b. If the crystal is loaded under a shear stress σ_{yz} , with all other stress components being zero, what is the value of σ_{yz} at yielding?

Solution: Designating the slip plane normal as n , and the slip direction as d , the relevant direction cosines are:

$$l_{nx} = \frac{1}{\sqrt{2}}, \quad l_{ny} = \frac{1}{2}, \quad l_{nz} = \frac{1}{2},$$

$$l_{dx} = \frac{1}{\sqrt{2}}, \quad l_{dy} = -\frac{1}{2}, \quad l_{dz} = -\frac{1}{2}.$$

The shear stress, σ_{nd} , is

$$\sigma_{nd} = l_{nx}l_{dx}\sigma_{xx} + l_{ny}l_{dy}\sigma_{yy} + \dots + (l_{nx}l_{dy} + l_{ny}l_{dx})\sigma_{xy}.$$

a. Since $\sigma_x = \sigma_y$, and $\sigma_{nd} = \tau_c$ at yielding,

$$\tau_c = (l_{nx}l_{dx} + l_{ny}l_{dy})\sigma_x = \left[\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) \right] \sigma_x = \frac{1}{4}\sigma_x$$

$$\sigma_x = 4\tau_c = 5.6 \text{ MPa.}$$

b. For simple shear, σ_{yz} ,

$$\begin{aligned} \tau_c &= (l_{nz}l_{dy} + l_{ny}l_{dz})\sigma_{yz} = \left[\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) \right] \sigma_{yz} \\ &= -\frac{1}{2}\sigma_{yz}, \quad \sigma_{yz} = -2\tau_c = -2.8 \text{ MPa.} \end{aligned}$$

2.2. Strains produced by slip

The incremental strain transformation equations (see Appendix 2) may be used to find the shape change that results from slip when the strains are small (or more precisely, when the lattice rotations are negligible),

$$d\epsilon_{xx} = l_{xn}^2 d\epsilon_{nn} + l_{xd}^2 d\epsilon_{dd} + \dots + l_{xn}l_{xd} d\gamma_{nd}. \quad (2.6)$$

With slip on a single slip system (in the d direction and on the n plane), the only finite strain term on the right-hand side of Eq. (2.6) is γ_{nd} , so we can simply write

$$d\epsilon_{xx} = l_{xn}l_{xd} d\gamma_{nd} \quad (2.7)$$

or in the Schmid notation,

where $d\gamma$ is the shear strain on the slip system. The other strain components referred to the x, y, z axes are similarly:

$$\left. \begin{aligned} d\epsilon_{yy} &= l_{yn} l_{yd} d\gamma \\ d\epsilon_{zz} &= l_{zn} l_{zd} d\gamma \\ d\gamma_{yz} &= (l_{yn} l_{zd} + l_{yd} l_{zn}) d\gamma \\ d\gamma_{zx} &= (l_{zn} l_{xd} + l_{zd} l_{xn}) d\gamma \\ d\gamma_{xy} &= (l_{xn} l_{yd} + l_{xd} l_{yn}) d\gamma. \end{aligned} \right\} \quad (2.9)$$

Example 2.3

Consider the deformation of the crystal in Example 2, and suppose the crystal is deformed until $\epsilon_y = -0.1$. What will be the values of the other strains? Note that it does not matter what stress state caused the slip.

Solution: Since ϵ_y is small, the strains may be transformed by the infinitesimal strain transformation without much error.

$$\epsilon_y = l_{yn} l_{yd} \gamma = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\gamma = -0.1; \quad \gamma = \frac{-0.1}{-\frac{1}{4}} = 0.4$$

$$\epsilon_x = l_{xn} l_{xd} \gamma = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} (0.4) = 0.2$$

$$\epsilon_z = l_{zn} l_{zd} \gamma = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(0.4) = -0.1$$

$$\begin{aligned} \gamma_{yz} &= (l_{yd} l_{zn} + l_{yn} l_{zd}) \gamma \\ &= \left[\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) \right] (0.4) = -0.20, \text{ etc.} \end{aligned}$$

2.3. Tensile deformation of fcc crystals

It is customary to represent the tensile axis in the basic stereographic triangle with $[100]$, $[110]$, and $[111]$ corners, as shown in Fig. 2.3. For all orientations of fcc crystals within this triangle, the Schmid factor for the $[101](11\bar{1})$ slip system is higher than for any other system (a proof is given at the end of the chapter.) Single slip on this *primary slip system* should be expected. If the tensile axis is represented as lying in any other stereographic triangle, the slip elements may be found by simply noting the remote corners of the three adjacent triangles. The $\langle 111 \rangle$ in one is the slip plane normal and the $\langle 110 \rangle$ in another is the slip direction.

Figures 2.4 and 2.5 show the orientation dependence of the Schmid factor m within the basic triangle. The highest value, $m = 0.5$, occurs when the tensile axis lies on the great circle between the slip direction and the slip

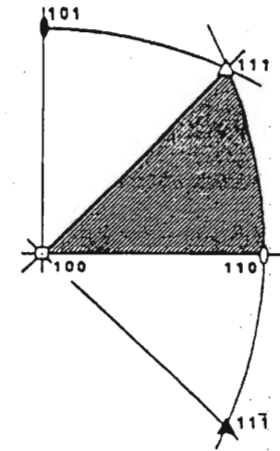


Fig. 2.3. Basic orientation triangle. If the tensile axis of an fcc crystal lies in this triangle, the most heavily stressed slip system is $[101](11\bar{1})$.

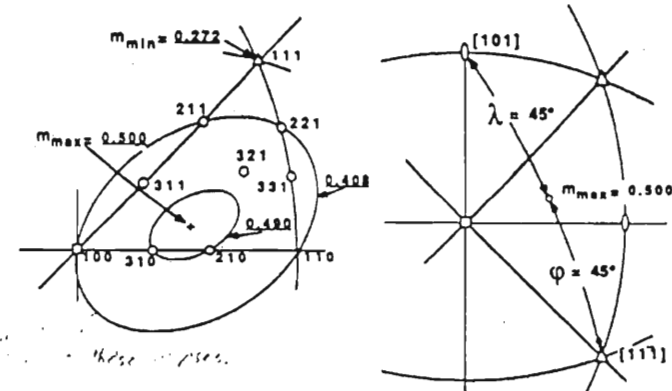


Fig. 2.4. Schematic illustration showing the orientation dependence of the Schmid factor for fcc crystals.

$[110]$, $[211]$, and $[221]$, $m = 1/\sqrt{6} = 0.408$. For other orientations the Schmid factor can be expressed in terms of the latitude θ and longitude ϕ of the tensile axis:

$$\begin{aligned} m &= \cos \lambda \cos \phi \\ &= \left(\frac{1}{\sqrt{6}} \right) (\cos \phi \cos \theta + \sin \phi) [\cos \phi (\cos \theta + \sin \theta) - \sin \phi]. \end{aligned} \quad (2.10)$$

The derivation is given as Example A1.10 in Appendix 1.

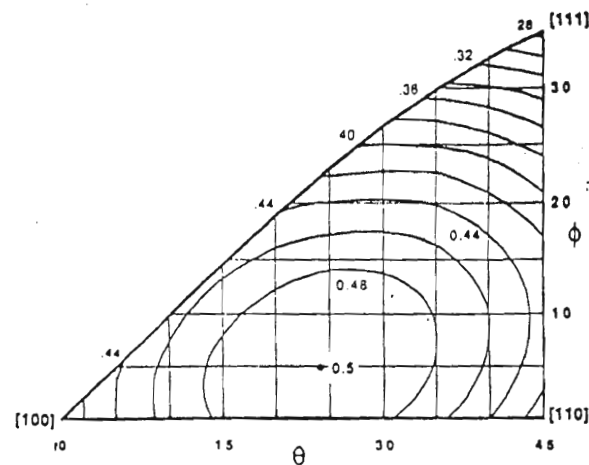


Fig. 2.5. Orientation dependence of the Schmid factor for fcc crystals.

If the tensile axis lies on a boundary of the basic triangle, two slip systems are equally favored. The second system is the one most favored in the adjoining triangle. For example, if the tensile axis is on the $[100]$ - $[111]$ boundary, the second system is $[110](1\bar{1}1)$. It can be seen from the symmetry that this system, which is called the *conjugate system*, has the highest m factors for orientations within the triangle $[100]$ - $[111]$ - $[101]$. Two other systems have names: $[10\bar{1}](111)$ is the *critical slip system*, and the system that shares the slip direction with the primary system, $[101](1\bar{1}\bar{1})$, is called the *cross slip system*. A screw dislocation moving on the primary plane can move onto the cross slip plane to avoid an obstacle and the term *cross slip* is used to describe such a change of slip planes. Slip systems that share the primary plane but have different slip directions are called *coplanar slip systems*. The various named systems are noted in Fig. 2.6 by the triangle in which each is most favored.

2.4. Lattice rotation resulting from slip (tension)

Slip normally causes a gradual lattice rotation or orientation change. Figure 2.7 shows the elongation of a single crystal by slip on a single slip system. As the crystal is extended, the orientation of the longest axis (which is the tensile axis for a crystal being pulled in tension) changes relative to the crystallographic elements. In Fig. 2.7 the slip plane and slip direction of a very long thin crystal are fixed in space, and the tensile axis rotates relative to the crystallographic elements. If the tensile axis would remain vertical

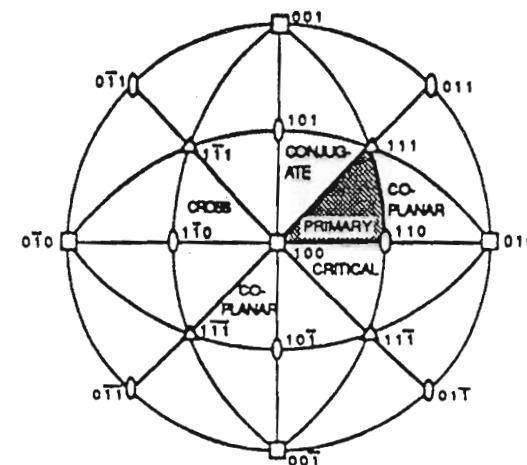
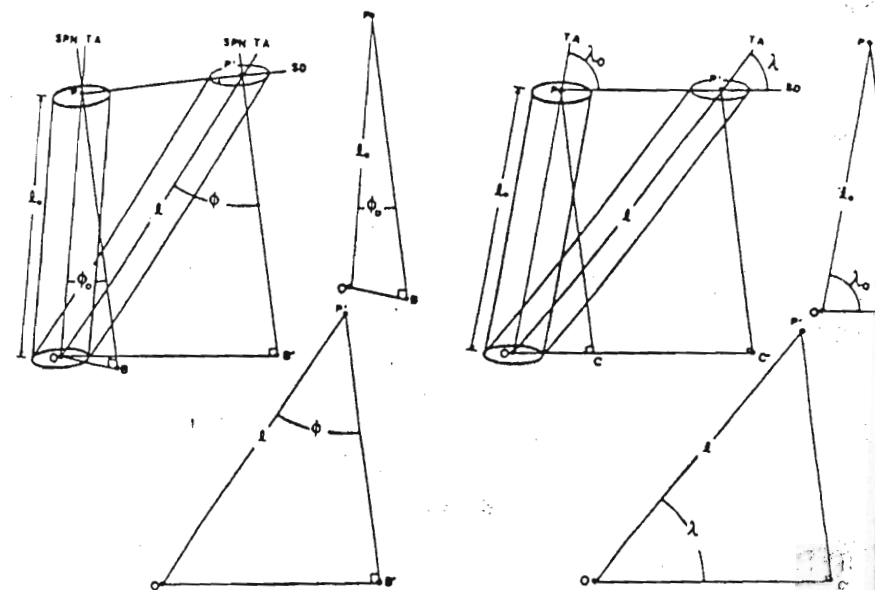


Fig. 2.6. Slip systems for fcc crystals.

Fig. 2.7. Extension of a long thin crystal. Note that λ decreases and ϕ increases as the crystal is extended.

and the crystal elements would rotate. The resulting deformation is much like what occurs when one shears a deck of cards or a stack of poker chips on the top of a table. The following geometric treatment of the orientation change during tension is essentially that of Schmid et al. [1, 2].

Slip causes translation of point P parallel to the slip direction to a new position, P'. Points B and B' are constructed by extending the slip plane normals through P and P' until they intersect the extension of the slip plane through point O. Similarly the points C and C' are formed by extending the slip direction through O and constructing normals from P and P' to it.

The normal distance, $\overline{PB}$, between the slip planes is not changed by slip (just as the normal distance between two cards in a deck is not changed by shearing the deck). Therefore $\overline{PB} = \overline{P'B'}$. Substituting $\overline{PB} = l_0 \cos \varphi_0$ and $\overline{P'B'} = l \cos \varphi$,

$$\frac{\cos \varphi_0}{\cos \varphi} = \frac{l}{l_0} = 1 + e, \quad \text{or} \quad \cos \varphi = \frac{\cos \varphi_0}{1 + e}, \quad (2.11)$$

where $e = (l - l_0)/l_0 = \Delta l/l_0$ is the engineering strain. Similarly slip does not change the distance between two parallel slip directions, so $\overline{PC} = \overline{P'C'}$. Substituting $\overline{PC} = l_0 \sin \lambda_0$ and $\overline{P'C'} = l \sin \lambda$,

$$\sin \lambda = \sin \lambda_0 / (1 + e). \quad (2.12)$$

The shear strain γ associated with the slip is

$$\gamma = \frac{\overline{PP'}}{\overline{PB}} = \frac{\overline{OC'}}{\overline{PB}} - \frac{\overline{OC}}{\overline{PB}}. \quad (2.13)$$

Substituting $\overline{OC'} = l \cos \lambda$, $\overline{OC} = l_0 \cos \lambda_0$ and $\overline{PB} = l_0 \cos \varphi_0 = l \cos \varphi$, Eq. (2.13) becomes

$$\gamma = \frac{l \cos \lambda}{l \cos \varphi} - \frac{l_0 \cos \lambda_0}{l_0 \cos \varphi_0} = \frac{\cos \lambda}{\cos \varphi} - \frac{\cos \lambda_0}{\cos \varphi_0}. \quad (2.14)$$

It can also be shown that

$$l/l_0 = (1 + 2\gamma \cos \varphi_0 \cos \lambda_0 + \gamma^2 \cos^2 \varphi_0)^{1/2}. \quad (2.15)$$

It is clear from Fig. 2.6 and Eqs. (2.11) and (2.12) that φ increases and λ decreases during tensile extension. The relative rates of change of φ and λ can be found by differentiating Eqs. (2.11) and (2.12). Since $l \cos \varphi = l_0 \cos \varphi_0$,

$$\cos \varphi dl - l \sin \varphi d\varphi = 0, \quad \text{or} \quad d\varphi = \cot \varphi \left(\frac{dl}{l} \right) \quad \text{so} \quad \frac{d\varphi}{d\varepsilon} = \cot \varphi. \quad (2.16)$$

Similarly, since $l \sin \lambda = l_0 \sin \lambda_0$,

$$\sin \lambda dl + l \cos \lambda d\lambda = 0 \quad \text{or} \quad d\lambda = -\tan \lambda \left(\frac{dl}{l} \right), \quad \text{so} \quad \frac{d\lambda}{d\varepsilon} = -\tan \lambda. \quad (2.17)$$

If the tensile axis happens to lie in the plane defined by the slip direction and the slip-plane normal, $\lambda + \varphi = 90^\circ$. Otherwise $\lambda + \varphi > 90^\circ$, so $\cot \varphi \leq \cot(90^\circ - \lambda) = \tan \lambda$. Thus, comparing Eqs. (2.16) and (2.17),

$$\left| \frac{d\lambda}{d\varepsilon} \right| \geq \left| \frac{d\varphi}{d\varepsilon} \right|. \quad (2.18)$$

Equation (2.18) shows that the rate of rotation of the slip direction toward the tensile axis ($-d\lambda/d\varepsilon$) is greater than or equal to the rate of rotation of the slip-plane normal away from the tensile axis, and therefore should be regarded as the primary rotation. This is shown in Fig. 2.8.

For an fcc crystal oriented with the tensile axis in the primary triangle, the tensile axis rotates on a great circle toward the primary slip direction, [101], until it reaches the [100]-[111] boundary of the triangle. Along this boundary the primary and conjugate systems are equally favored. Slip on the [110](111) conjugate system alone would cause a rotation of the tensile axis back into the primary triangle toward the [110] conjugate slip direction. The net result is simultaneous slip on these two systems, which causes a slow rotation along the [100]-[111] boundary toward the [211]. This direction lies on the great circle through the two slip directions. If the axis were at [211], the rotations toward the two slip directions would exactly cancel each other. Thus [211] is the stable end orientation. Goler and Sachs [3] showed that the rotation along the [100]-[111] boundary is given by

$$l \sin \theta = l_1 \sin \theta_1, \quad (2.19)$$

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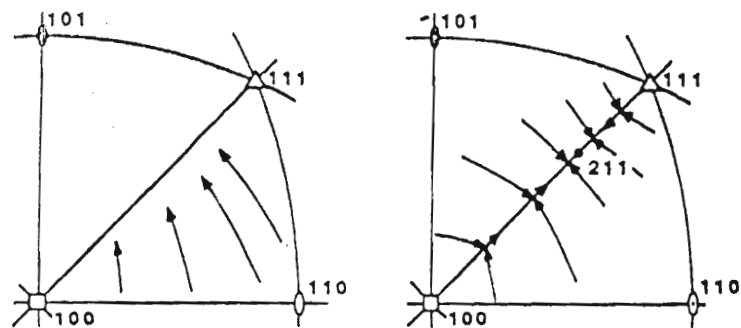


Fig. 2.8. Primary slip on $[101](1\bar{1}1)$ causes the tensile axis to rotate toward the $[101]$ slip direction (left). Once the tensile axis reaches the $[100]$ - $[111]$ symmetry line, simultaneous $[101]$ and $[110]$ slip causes rotation toward $[211]$ (right).

where θ is the angle between the tensile axis and $[211]$, and l is the length of the crystal, θ_1 and l_1 are the values of θ and l when the tensile axis first reaches the $[100]$ - $[111]$ boundary.

The general tendency to rotate toward the stable $[211]$ orientation has been confirmed experimentally. However, with crystals of α -brass and other materials of low stacking fault energy, the tensile axis has a tendency to rotate into the conjugate triangle by a finite angle before conjugate slip commences. This "overshoot" indicates that the shear stress required for slip on the conjugate system must be higher than that for slip on the active system, which in turn suggests that slip on the primary system hardens the latent conjugate system more than the active primary system.

Aluminum, which has a high stacking fault energy, behaves differently. Observed rotations suggest that conjugate slip may start before the tensile axis reaches the $[100]$ - $[111]$ boundary. This is explained by somewhat less hardening on the latent system than the active one.

Example 2.4

Consider uniaxial tension of a bcc crystal that deforms by $\langle 111 \rangle$ -pencil glide. Let the tensile axis be oriented in the basic triangle.

a. What are the most favored slip system(s) in this triangle? Sketch a contour plot of the Schmid factor for the most favored system(s).

b. Indicate the direction of lattice rotation for various orientations in the triangle. What is the stable end orientation?

Solution: Under uniaxial tension, the slip plane associated with each $\langle 111 \rangle$ -slip direction will be oriented so that its normal has as low a value of ϕ as possible (maximum value of $\cos \phi$). For a given slip direction (fixed λ), this corresponds to $\phi = 90^\circ - \lambda$, so $\cos \phi = \sin \lambda$ and the Schmid factor for each slip direction is $m = \sin \lambda \cos \lambda$.

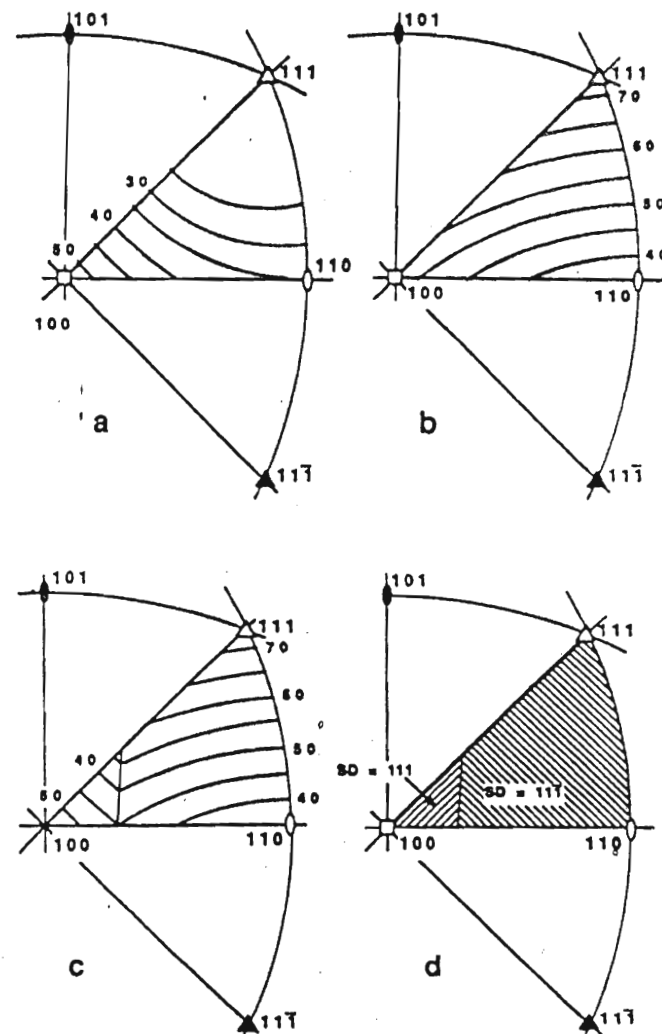


Fig. 2.9. Construction of lines of equal Schmid factor for a $\langle 111 \rangle$ -pencil glide. Note that the $[11\bar{1}]$ slip direction is favored over most of the triangle, but $[111]$ is favored for orientations near $[100]$.

a. The most highly stressed slip system is that one for which $\sin \lambda \cos \lambda$ is largest, that is, the one for which $|45^\circ - \lambda|$ is the least. Inspection shows that $[111]$ is the most favored slip direction for orientations near $[100]$, and that $[11\bar{1}]$ is more favored for other orientations. Figures 2.9(a) and (b) show the loci of points at equal angles from these two directions. These are

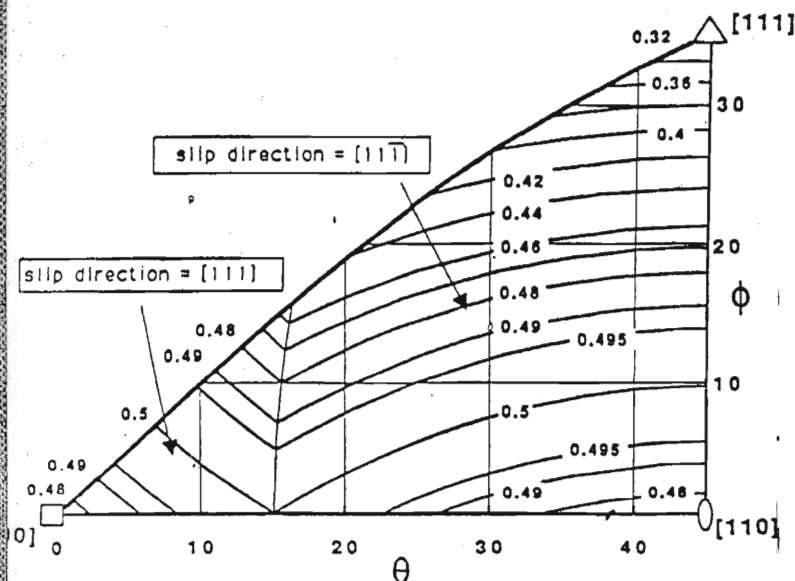


Fig. 2.10. Plot of Schmid factors for $\langle 111 \rangle$ -pencil glide. The basic orientation triangle is divided into two regions with different slip directions, the Schmid factor equals 0.417 at $[100]$ and 0.314 at $[111]$.

contours corresponding to equal values of $|45^\circ - \lambda|$ [Fig. 2.9(c)]. Figure 2.10(d) shows the regions of most favored slip systems. Greater detail is shown in Fig. 2.10.

The rotations toward the active slip direction are indicated in Fig. 2.11. Orientations initially in region B rotate into region A. All of the orientations in region A rotate to the $[100]$ - $[110]$ boundary, where duplex slip causes a net rotation toward $[110]$, which is the stable end orientation.

5. Lattice rotation in compression (Taylor)

The orientation change of a thin flat crystal during compression is different from that during tension. The compression axis, CA, remains normal to the flat surface in contact with the compression platen. Let P and O in Fig. 2.12 be the intersections of the top and bottom surfaces, respectively, with the compression axis. The slip direction through O intersects the top surface at A. PA lies in the top surface and is perpendicular to the compression axis. The angle ϕ is the angle between the slip direction and the compression axis.

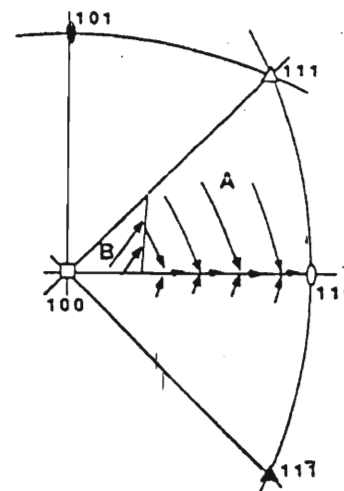


Fig. 2.11. Rotation of the tensile axis for bcc crystals deforming by $\langle 111 \rangle$ -pencil glide. If the tensile axis is in region B, rotation toward $[111]$ will carry it into region A. In region A rotation toward $[111]$ will move it to the $[100]$ - $[110]$ boundary, where duplex slip causes a net rotation toward $[110]$.

triangle and

$$\overline{OP}/\overline{OA} = \cos \lambda \quad \text{or} \quad \overline{OA} = \frac{h}{\cos \lambda} \quad (2.20)$$

The length of a line in the slip plane, such as $\overline{OA}$, is not changed by slip. Therefore, this construction is possible at all stages with only λ and h changing and

$$\frac{h}{\cos \lambda} = \frac{h_0}{\cos \lambda_0} \quad \text{or} \quad \frac{\cos \lambda}{\cos \lambda_0} = \frac{h}{h_0} = 1 + \epsilon \quad (2.21)$$

Now construct the plane through O determined by the slip-plane normal and the compression axis. Let the point on the top surface where this plane intersects the slip plane through O be designated B. Now from the right triangle POB,

$$\cos(90^\circ - \phi) = \frac{\overline{PO}}{\overline{OB}} = \frac{h}{\overline{OB}}; \quad \overline{OB} = \frac{h}{\sin \phi} \quad (2.22)$$

The length, $\overline{OB}$, also does not change during deformation, so

$$\frac{h_0}{\sin \phi_0} = \frac{h}{\sin \phi}; \quad \frac{h}{h_0} = \frac{\sin \phi_0}{\sin \phi} \quad (2.23)$$

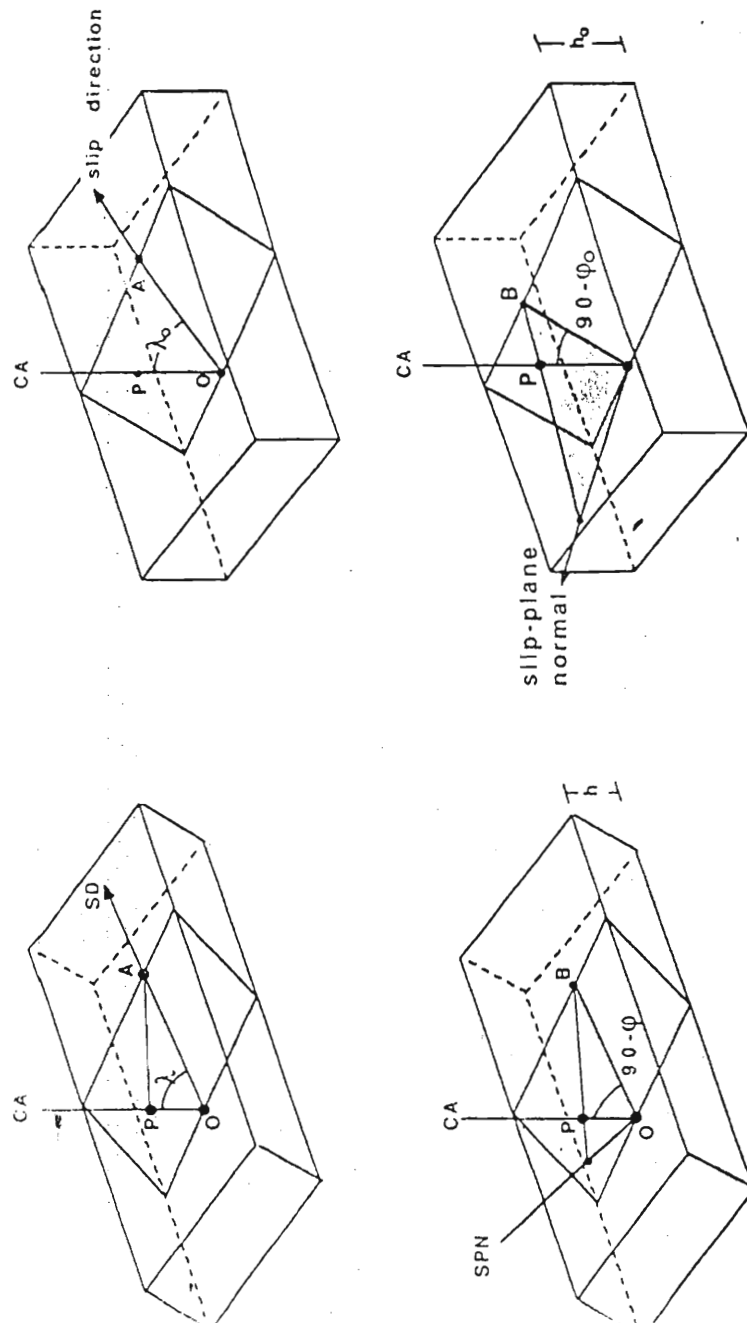
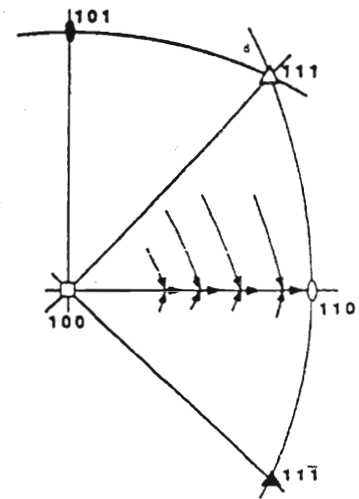


Fig. 2.12. Construction showing the lattice rotation of a thin flat crystal during compression. With compression λ increases and φ decreases.

Fig. 2.13. Rotation of the compression axis during compression of an fcc crystal. The CA rotates toward $[11\bar{1}]$ until it reaches the $[100]$ - $[110]$ boundary. Then duplex slip rotates it toward $[110]$.



With compression h/h_0 decreases, φ decreases, and λ increases. The relative rates of rotation are:

$$\frac{d\varphi}{(-d\varepsilon)} = -\tan \varphi \quad (2.24)$$

and

$$\frac{d\lambda}{(-d\varepsilon)} = \cot \lambda, \quad (2.25)$$

where $-d\varepsilon = -dh/h$ is the compressive strain increment. Since $(\varphi + \lambda) \geq 90^\circ$,

$$\left| \frac{d\varphi}{d\varepsilon} \right| \geq \left| \frac{d\lambda}{d\varepsilon} \right|, \quad (2.26)$$

which means that the rotation of the CA toward the SPN is the primary rotation, with the CA moving along a great circle approaching the SPN.

Consider an fcc crystal whose CA is initially in the $[100]$ - $[110]$ triangle. During compression, the CA rotates directly toward $[111]$. When it reaches the $[100]$ - $[110]$ boundary, as shown in Fig. 2.13, a second rotation on the critical system should commence, and the result is a gradual rotation toward $[110]$, which is the stable end orientation.

Taylor analysis for compression do not predict exactly the opposite rotations. First it should be emphasized that the stress state causing the deformation is not intrinsic to either analysis. Rather it is the geometry of the specimen and its reference system that makes one analysis appropriate rather than the other. The Schmid analysis (tension) can be used to describe the rotation of the lattice relative to *any physical line* in the material, if λ is taken as the angle between that line and the slip direction and de is the incremental strain measured along that line. It could be used in compression, with the initial compression axis being the reference line. In this case it should be noted that a physical line drawn on the crystal so that it is initially perpendicular to the compression surfaces will not continue to be perpendicular to those surfaces as the crystal deforms.

In contrast the Taylor analysis (compression) describes the rotation of any physical plane in the material (e.g., the compression surface). The differences in the reference systems are illustrated in Fig. 2.14. Before deformation the direction d and the normal n to the compression surface are both aligned with some crystallographic direction x . After deformation n and d are no longer parallel, having undergone different rotations.

The tensile axis of a long crystal naturally remains parallel to its longest dimension. During compression of a flat crystal, the planes initially in contact with the compression platens remain normal to the compression axis.

Both analyses are special cases of a general treatment of finite strains by Chin et al. [5], which covers multiple slip as well as single slip. Theories of texture development are concerned with lattice rotation of grains in polycrystals. If the grains are extended in a direction in which they are

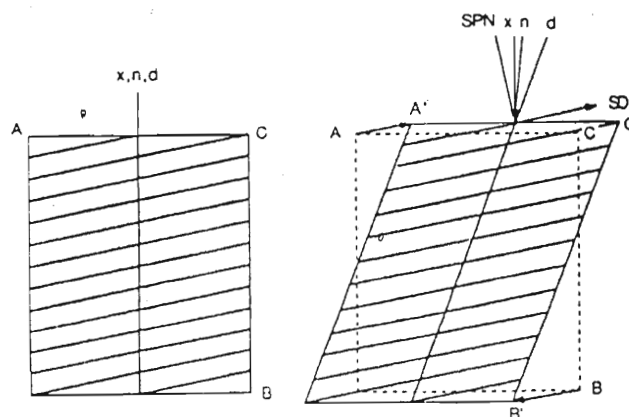


Fig. 2.14. A schematic two-dimensional illustration of the different rotations caused by slip. A physical line initially perpendicular to the compression surfaces will not continue to be perpendicular to those surfaces as the crystal deforms.

already very elongated, the Schmid analysis may seem appropriate. Conversely, if grains that are already pancake shaped are subjected to further compression, the Taylor analysis may be better. In many cases, however, conditions are intermediate, with the grains being constrained on all sides.

2.6. Mathematical analysis of rotation

The mathematical analysis of rotations (see Appendix 2) describes the rotation of particles in a body with infinitesimal strain:

$$d\omega_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right), \quad (2.27)$$

where ω_{ij} is the clockwise rotation (in radians) about an axis k normal to the i and j directions, u_i is the shear displacement parallel to i occurring over a distance x_j measured in the j direction. The shear strain, defined by

$$d\gamma_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \quad (2.28)$$

If i and j are taken as the slip direction and the slip-plane normal, respectively, $\partial u_j / \partial x_i = 0$ and

$$d\omega = \frac{1}{2} d\gamma. \quad (2.29)$$

The rotation occurs about an axis that is normal to both the slip direction and the slip-plane normal. An interesting aspect of the mathematical analysis is that because the rotation is referred to no single external plane or direction, its predictions are between those of the first two analyses, and it appears to be more appropriate for equiaxed grains.

For tension, the axes of rotation predicted by the three analyses are shown in Fig. 2.15 for an fcc crystal deforming by slip on the $(11\bar{1})$ plane in the $[101]$ direction. The Schmid analysis predicts that a direction originally parallel to $[100]$ rotates toward the slip direction, the axis of rotation, a_1 , being $[0\bar{1}0]$. If the Taylor analysis were used for tension it would predict that the normal to a plane initially parallel to (100) rotates away from the slip-plane normal about the axis $a_2 = [0\bar{1}\bar{1}]$. Finally, according to the mathematical analysis, the lattice rotates about $a_3 = [101] \times [11\bar{1}] = [1\bar{2}\bar{1}]$.

Figure 2.16 also compares the rotations caused by $(11\bar{1})[101]$ slip. Again

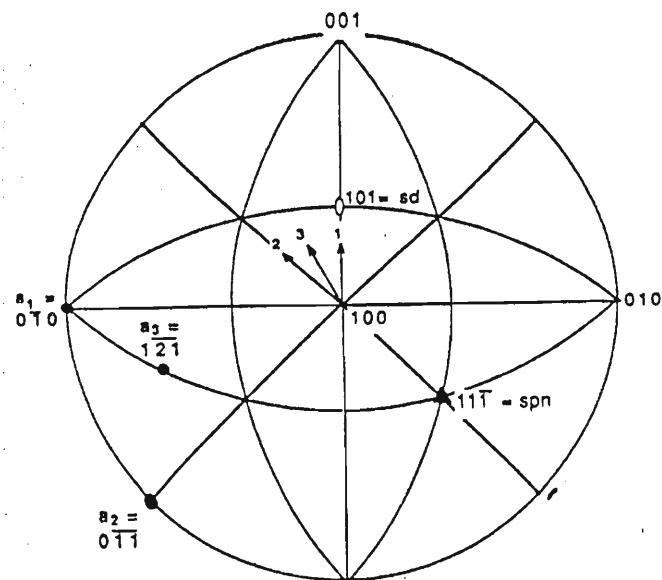


Fig. 2.15. Stereographic representation of the three reference systems used to describe lattice rotation. Slip is on a single system, $(111)[101]$. Schmid's analysis describes rotation 1 of a physical direction initially parallel to $[100]$ about the axis $a_1 = [010]$. Taylor's analysis describes rotation 2 about $a_2 = [011]$ of the normal to the plane initially parallel to (100) . The mathematical analysis describes rotation 3 about an axis $a_3 = [121]$, which is normal to both the slip direction and the slip-plane normal.

$\varphi + \lambda = 90^\circ$, $d\epsilon/d\gamma = \cos \lambda \sin \lambda$, and Eqs. (2.17) and (2.25) reduce to

$$\frac{d\lambda}{d\gamma} = -\sin^2 \lambda$$

$$\frac{d\lambda}{d\gamma} = -\cos^2 \lambda,$$

respectively, for the Schmid and Taylor analyses. The average of these two rotations is $d\lambda/d\gamma = -\frac{1}{2}$, in accordance with Eq. (2.29).

2.7. Multiple slip

For slip to occur simultaneously on more than one crystallographically equivalent slip system, the shear stress on both systems must reach the critical

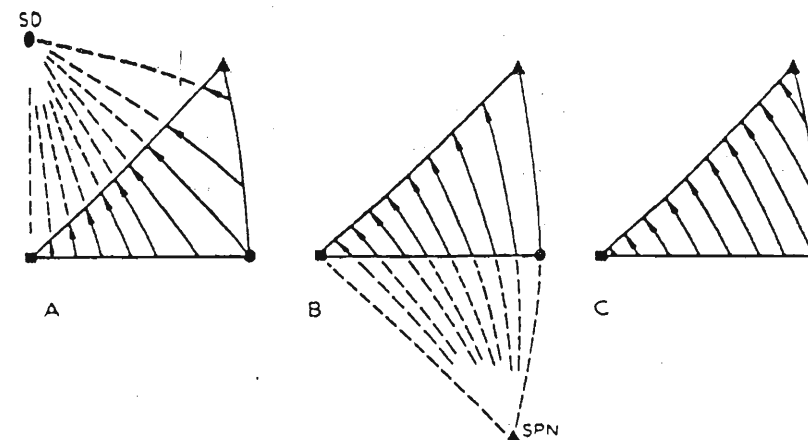


Fig. 2.16. Lattice rotations predicted by the three analyses for single slip: (a) The Schmid rotation of a physical plane toward the slip direction; (b) the Taylor rotation of the normal to a physical plane away from the slip plane normal; and (c) the mathematical rotation. (From Hosford [6].)

value for slip. For simultaneous slip on the A plane in the a direction and on the B plane in the b direction,

$$|\tau_{Aa}| = |\tau_{Bb}| = \tau, \quad (2.30)$$

where τ is the critical shear stress for slip. Furthermore, the critical shear stress cannot be exceeded on any other system. That is,

$$|\tau_i| \leq \tau. \quad (2.31)$$

The external strains are simply the sums of the contributions from the individual slip systems. If slip occurred only on the A plane in the a direction, we could write $d\epsilon_x = l_{xA}l_{xa}d\gamma_{Aa}$, $d\epsilon_y = l_{yA}l_{ya}d\gamma_{Aa}$, etc., but with simultaneous slip on the Aa, Bb, ..., systems,

$$\left. \begin{aligned} d\epsilon_x &= l_{xA}l_{xa}d\gamma_{Aa} + l_{xB}l_{xb}d\gamma_{Bb} + \dots \\ d\epsilon_y &= l_{yA}l_{ya}d\gamma_{Aa} + l_{yB}l_{yb}d\gamma_{Bb} + \dots \\ d\epsilon_z &= l_{zA}l_{za}d\gamma_{Aa} + l_{zB}l_{zb}d\gamma_{Bb} + \dots \\ d\gamma_{yz} &= (l_{yA}l_{za} + l_{ya}l_{zA})d\gamma_{Aa} + (l_{yB}l_{zb} + l_{yb}l_{zB})d\gamma_{Bb} + \dots \end{aligned} \right\} \quad (2.32)$$

Often a material is deformed by imposition of strains rather than

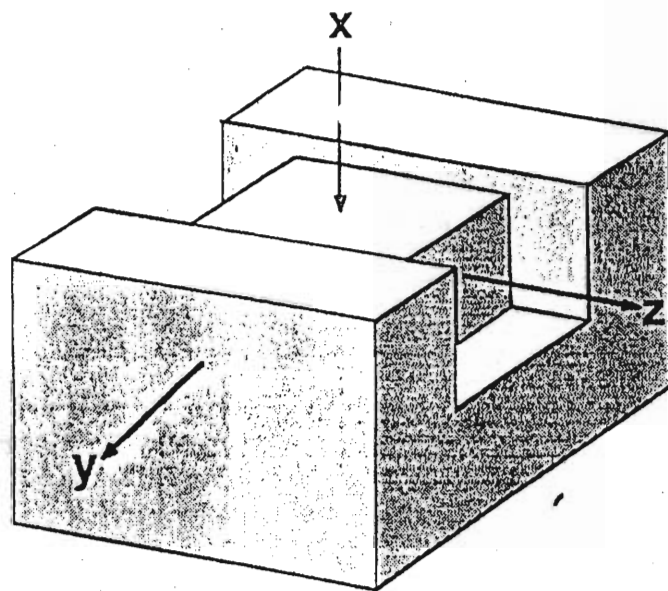


Fig. 2.17. Plane-strain compression in a channel.

stresses. For example, when a crystal is extended in a tension test along the x direction, only the strain ϵ_x is imposed. In this case the shear strain γ_{yz} , on only one of the slip systems need be finite to satisfy Eq. (2.32), so slip is expected on a single slip system (except for special orientations). On the other hand, consider a crystal deformed by plane-strain compression in a channel (Fig. 2.17). The imposed strains are $\epsilon_x, \epsilon_y = 0, \epsilon_z = -\epsilon_x$, and $\gamma_{xy} = 0$. Three of these strains are independent; so, except for special orientations, at least three slip systems must be active; that is, at least three of the γ_{ij} terms must be finite to satisfy Eqs. (2.32). For a completely arbitrary shape change, at least five independent slip systems must be active. (This is a reasonable approximation to what happens in individual grains of polycrystals.)

2.8. Independent slip systems

A group of slip systems is *independent*, if and only if, it is able to produce shape changes that could not be produced if one of the group were omitted. This is illustrated best by considering combinations of slip systems that are

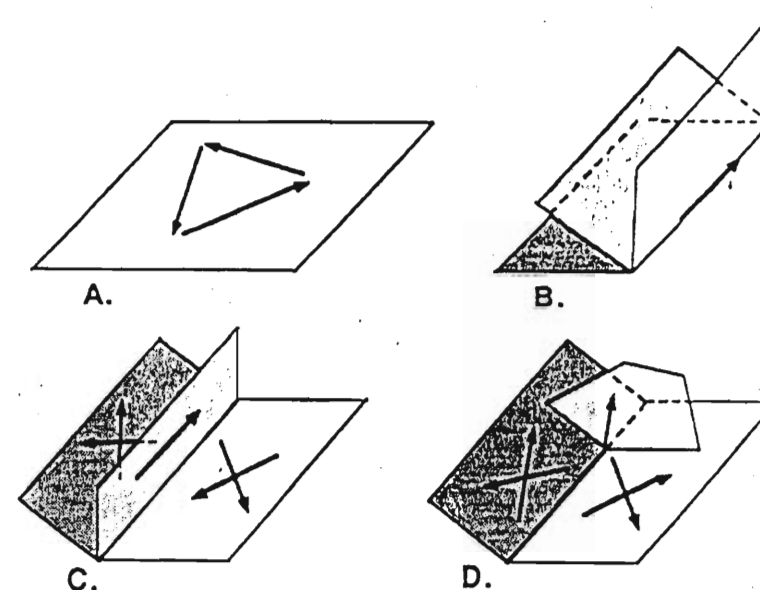


Fig. 2.18. Illustration of combinations of slip systems that are not independent.

1. Any arbitrary shear on a plane can be accommodated by slip in two directions in the plane. Hence three slip systems with a common slip plane are not independent [Fig. 2.18(a)].
2. Likewise, three slip systems sharing a common slip direction are not independent, because any two of them can produce any shape change resulting from the operation of all three [Fig. 2.18(b)].
3. A combination of five slip systems is not independent if it includes two slip planes, each containing two slip directions, with the fifth slip system having its slip direction either [Fig. 2.18(c)] parallel or [Fig. 2.18(d)] normal to the intersection of the first two planes.

Example 2.5

NaCl crystals deform by $\{110\}\langle 1\bar{1}0\rangle$ slip. How many independent $\{110\}\langle 1\bar{1}0\rangle$ slip systems are there?

Solution: Let $[100]$, $[010]$, and $[001]$ be designated by 1, 2, and 3, respectively. The possible slip systems are $a = (110)[1\bar{1}0]$, $b = (1\bar{1}0)[110]$, $c = (101)[10\bar{1}]$, $d = (10\bar{1})[101]$, $e = (011)[01\bar{1}]$, and $f = (01\bar{1})[011]$. However, slip on $(h_1k_1l_1)[h_2k_2l_2]$ produces exactly the same shape change as slip on $(h_2k_2l_2)[h_1k_1l_1]$; so no more than three of the systems can be independent.

of direction cosines, the strains relative to the cubic axes are:

	1 = [100]	2 = [010]	3 = [001]	External strains
$a = [110]$	$1/\sqrt{2}$	$1/\sqrt{2}$	0	$\epsilon_1 = (\frac{1}{2})(\gamma_a + \gamma_c)$
$b = [\bar{1}\bar{1}0]$	$1/\sqrt{2}$	$-1/\sqrt{2}$	0	$\epsilon_2 = (\frac{1}{2})(-\gamma_a + \gamma_c)$
$c = [101]$	$1/\sqrt{2}$	0	$1/\sqrt{2}$	$\epsilon_3 = (\frac{1}{2})(-\gamma_c - \gamma_a)$
$d = [10\bar{1}]$	$1/\sqrt{2}$	0	$-1/\sqrt{2}$	$\gamma_{23} = 0$
$e = [011]$	0	$1/\sqrt{2}$	$1/\sqrt{2}$	$\gamma_{31} = 0$
$f = [01\bar{1}]$	0	$1/\sqrt{2}$	$-1/\sqrt{2}$	$\gamma_{12} = 0$

There are only two independent slip systems because no shear strains along the cube axes can be accommodated and ϵ_3 is fixed once ϵ_1 and ϵ_2 are chosen. It is not surprising that polycrystalline NaCl is brittle.

Example 2.6

Determine the number of independent slip systems in crystals with the following slip systems:

1. $\{100\}\langle 001 \rangle$;
2. $\{100\}\langle 011 \rangle$;
3. $\{110\}\langle 001 \rangle$;
4. All three of the above.

Solution: Let 1 = [100], 2 = [010], and 3 = [001].

1. With $\{100\}\langle 001 \rangle$ slip, $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$, $\gamma_{23} = \gamma_{010 \cdot 001} + \gamma_{001 \cdot 010}$, $\gamma_{31} = \gamma_{001 \cdot 100} + \gamma_{100 \cdot 001}$, and $\gamma_{12} = \gamma_{100 \cdot 010} + \gamma_{010 \cdot 100}$. Three independent strains, γ_{23} , γ_{31} , and γ_{12} , can be imposed; so there are three independent slip systems.
2. With $\{100\}\langle 011 \rangle$ slip, the three shear strains, γ_{23} , γ_{31} , and γ_{12} can be produced by a combination of two slip directions on a single plane. However, the normal strains ϵ_1 , ϵ_2 , and ϵ_3 will not result, because each of the $\{100\}$ slip-plane normals is either parallel or perpendicular to a particular $\langle 100 \rangle$ reference axis. In either case the resolving factor is zero. Thus again there are three independent slip systems.
3. With $\{011\}\langle 100 \rangle$ slip, the slip direction and slip-plane normal are simply interchanged with case 2. Again no normal strains can be produced along the cube axes, and there are three independent slip systems.

4. With all of the systems operating together, there is still no way of generating normal strains along the cube axes; so there are still only three independent slip systems.

2.9. Most favored slip system for fcc crystals under tension

If the crystal is oriented such that the tensile axis falls in the basic orientation triangle [100]-[111]-[110], the most highly stressed slip system is (111)[101]. To prove this, let the direction of the tensile axis be $[hkl]$. Note that $h > k > l > 0$ if $[hkl]$ lies in the basic orientation triangle, [100]-[111]-[110]. (See Fig. 2.19.) The Schmid factor, $m = \cos \lambda \cos \phi$ can be calculated for each of the 12 possible $\{111\}\langle 110 \rangle$ slip systems.

As an example, for (111)[011],

$$\cos \phi = (h + k + l) / \sqrt{3(h^2 + k^2 + l^2)}$$

and $\cos \lambda = (k - l) / \sqrt{2(h^2 + k^2 + l^2)}$; so

$$m = (h + k + l)(k - l) / [\sqrt{6}(h^2 + k^2 + l^2)] = A(h + k + l)(k - l),$$

where the factor $A = \frac{1}{\sqrt{6}(h^2 + k^2 + l^2)}$ is the same for all the slip systems.

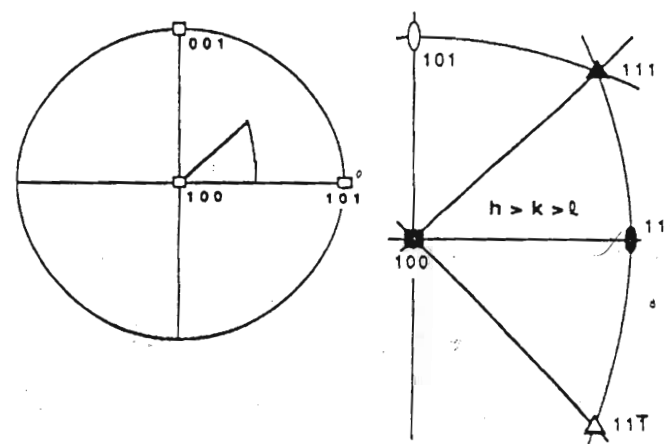


Fig. 2.19. For fcc crystals, if the tensile axis lies in the basic triangle [100]-[110]-[111], (111)[101] is the most favored slip system.

One need only compare the values of m/A . For all 12 systems these are:

No.	System	m/A
1	(111)[01 $\bar{1}$]	$(h + k + l)(k - l)$
2	(111)[10 $\bar{1}$]	$(h + k + l)(h - l)^*$
3	(111)[1 $\bar{1}$ 0]	$(h + k + l)(h - k)$
4	(1 $\bar{1}$ 1)[011]	$(h + k - l)(k + l)$
5	(1 $\bar{1}$ 1)[101]	$(h + k - l)(h + l)^*$
6	(1 $\bar{1}$ 1)[1 $\bar{1}$ 0]	$(h + k - l)(h - k)$
7	(1 $\bar{1}$ 1)[01 $\bar{1}$]	$(h - k - l)(k - l)$
8	(1 $\bar{1}$ 1)[10 $\bar{1}$]	$(h - k - l)(h + l)$
9	(1 $\bar{1}$ 1)[1 $\bar{1}$ 0]	$(h - k - l)(h + l)^*$
10	(1 $\bar{1}$ 1)[011]	$(h - k + l)(k + l)$
11	(1 $\bar{1}$ 1)[10 $\bar{1}$]	$(h - k + l)(h - l)$
12	(1 $\bar{1}$ 1)[110]	$(h - k + l)(h + k)^*$

Knowing that $h > k > l > 0$, the highest factor on each {111} plane is easily identified. These values of m/A are marked with an asterisk. Inspection also reveals that (12) has a higher m than (9). Expanding (2), (5), and (12),

$$(2) \quad \frac{m}{A} = h^2 + hk - kl - l^2$$

$$(5) \quad = h^2 + hk + kl - l^2$$

$$(12) \quad = h^2 - k^2 + hl + kl.$$

Now it is clear that the value of m/A for (5) is larger than that for (2). Furthermore, by comparing (5) and (12), and eliminating the common term $(h^2 + kl)$, it is seen that m/A for (5) is the larger since $hk > hl$ and $(-l^2) > (-k^2)$. Therefore, the Schmid factor for the (1 $\bar{1}$ 1)[101] system, (5), is the highest and slip will preferentially occur on this system.

The most favored {111}<110> slip system for a crystal with its tensile axis in another triangle can be found by symmetry.

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ANALYSES OF TAYLOR AND OF BISHOP AND HILL

The deformation mechanisms in the grains of a polycrystal are the same as those in a single crystal. Therefore it should be possible to relate quantitatively the plastic properties of a polycrystalline material to those of single crystals. The first attempts to do this were by G. Sachs [1] and by H. L. Cox and D. G. Sopwith [2], who proposed that the tensile yield strength of a polycrystal should be the average of the tensile yield strengths of the crystals in it. For a single crystal the tensile yield strength is given by Schmid's law

$$\sigma = \frac{1}{m} \tau, \quad (3.1)$$

where m is the Schmidt factor, $\cos \phi \cos \lambda$, for the most highly stressed slip system and τ is the critical resolved shear stress for slip. For fcc crystals, the average value of $(1/m)$ for all orientations is $(1/m)_{av} = 2.238$; so, according to Sachs, the tensile yield strength of a randomly oriented fcc polycrystal is $\sigma = 2.238\tau$.

Taylor [3, 4] pointed out the weakness of Sachs's model, which tacitly assumes that only one slip system is active in each grain. When a polycrystal deforms, the shape change in each grain must be compatible with that in the neighboring grains. Such shape compatibility cannot be maintained by single slip. With single slip, an initially circular cross section becomes elliptical. In a randomly oriented polycrystal, the major axes of the elliptical cross sections would not be parallel; so the deformation would not be compatible. The Sachs theory is useful only in that it gives a valid lower bound to the strength in uniaxial tension.

3.1. Taylor's analysis

Satisfying the compatibility requirement is the critical point in Taylor's development. He assumed that all grains undergo the same shape change

tension parallel to the x direction, deformation occurs by axially symmetric flow; so

$$d\epsilon_y = d\epsilon_z = -\frac{1}{2} d\epsilon_x \quad \text{and} \quad d\gamma_{yz} = d\gamma_{zx} = d\gamma_{xy} = 0. \quad (3.2)$$

The work per volume, dw , that is internally expended by slip on all active slip systems within a grain is

$$dw = \sum_i \tau d\gamma_i = \tau \sum_i |d\gamma_i|, \quad (3.3)$$

where the magnitude of the critical stress for slip, τ , is assumed to be the same on all slip systems and $d\gamma_i$ is the incremental slip on the individual slip systems. Denoting $\sum_i |d\gamma_i|$ as $d\gamma$, Eq. (3.3) simplifies to $dw = \tau d\gamma$. For uniaxial tension the incremental work per volume supplied by the external stress is $\sigma_x d\epsilon_x$. For a tension test

$$dw = \tau d\gamma = \sigma_x d\epsilon_x, \quad \text{or} \quad \frac{\sigma_x}{\tau} = \frac{d\gamma}{d\epsilon_x} = M. \quad (3.4)$$

Here M is the Taylor factor, which depends on the orientation. By calculating the value of $M = d\gamma/d\epsilon_x$ for all orientations and averaging to find $\bar{M}$, the flow behavior of a random polycrystal can be calculated as

$$\sigma_x = \bar{M}\tau \quad \text{and} \quad d\epsilon_x = \frac{d\gamma}{\bar{M}}. \quad (3.5)$$

3.1.1. Calculation of M for a single orientation

It is convenient first to express the external strains, $d\epsilon_x, d\epsilon_y, \dots, d\epsilon_z$, relative to the 1,2,3 cubic axes of the crystal as $d\epsilon_1, d\epsilon_2, \dots, d\epsilon_3$ and to then relate $d\epsilon_1, d\epsilon_2, \dots, d\epsilon_3$ to the crystallographic shear strains, $d\gamma_i$, on the $\{111\}\langle 1\bar{1}0 \rangle$ slip systems. In the present treatment, the orientation of the tensile axis, x , is described by two angles, θ and ϕ , as shown in Fig. 3.1. Imagine that the x, y , and z axes had initially coincided with the 1, 2, and 3 axes but were first rotated clockwise about the 2 axis through an angle ϕ and then clockwise about the 1 axis through an angle θ . The angles ϕ and θ may be regarded as the latitude and longitude, respectively, of the x axis. Table 3.1 gives the appropriate direction cosines. These are derived with spherical trigonometry in Example A1.10 of Appendix 1.

In a tension test of a polycrystal composed of randomly oriented grains, the deformation is axially symmetric about z , and the rotational position of each grain about the z axis is not of concern. Thus for any orientation described

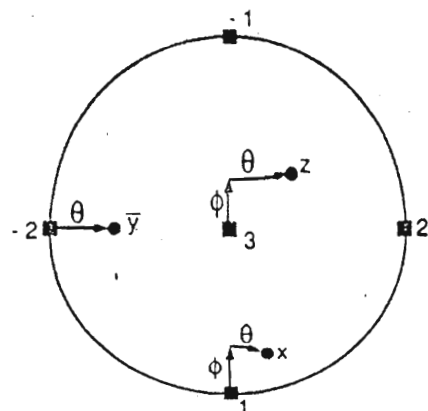


Fig. 3.1. Stereographic representation of the angular relations between the external axes, x, y, z and the crystal axes 1, 2, 3. Negative signs indicate the opposite (negative) end of a direction.

by θ and ϕ , the strains $de_1, de_2, \dots, d\gamma_{23}$ may be expressed in terms of the external strain, de_z .

Now consider how these strains can be produced in an fcc crystal by $\{111\}\langle 1\bar{1}0 \rangle$ slip. To shorten the notation, the slip systems are designated as shown in Table 3.2 and Fig. 3.2, the slip planes being a, b, c , or d and the slip directions I, II, or III. Table 3.3 gives the direction cosines between the slip elements and the cubic axes.

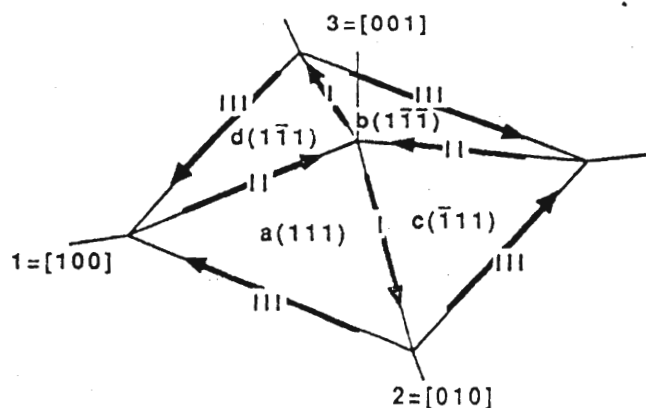


Fig. 3.2. Half of an octahedron representing the fcc slip systems. The faces are $\{111\}$ slip planes and the edges are $\langle 1\bar{1}0 \rangle$ slip directions.

Table 3.1. Direction cosines between the 1, 2, 3 and the x, y, z axes

	1	2	3
x	$\cos \phi$	$\sin \theta \sin \phi$	$\cos \theta \sin \phi$
y	0	$\cos \theta$	$-\sin \theta$
z	$-\sin \phi$	$\sin \theta \cos \phi$	$\cos \theta \cos \phi$

Table 3.2. Slip system designation

Plane	$a = (111)$			$b = (\bar{1}\bar{1}1)$			$c = (\bar{1}11)$			$d = (1\bar{1}\bar{1})$		
Direction	011	101	110	011	101	110	011	101	110	011	101	110
System	aI	aII	aIII	bI	bII	bIII	cI	cII	cIII	dI	dII	dIII

Table 3.3. Direction cosines between slip elements and cube axes

	1 = [100]	2 = [010]	3 = [001]
$a = (111)$	$1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$
I = [011]	0	$1/\sqrt{2}$	$-1/\sqrt{2}$
II = [101]	$-1/\sqrt{2}$	0	$1/\sqrt{2}$
III = [110]	$1/\sqrt{2}$	$-1/\sqrt{2}$	0
$b = (\bar{1}\bar{1}1)$	$-1/\sqrt{3}$	$-1/\sqrt{3}$	$1/\sqrt{3}$
I = [011]	0	$-1/\sqrt{2}$	$-1/\sqrt{2}$
II = [101]	$1/\sqrt{2}$	0	$1/\sqrt{2}$
III = [110]	$-1/\sqrt{2}$	$1/\sqrt{2}$	0
$c = (\bar{1}11)$	$-1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$
I = [011]	0	$1/\sqrt{2}$	$-1/\sqrt{2}$
II = [101]	$1/\sqrt{2}$	0	$1/\sqrt{2}$
III = [110]	$-1/\sqrt{2}$	$-1/\sqrt{2}$	0
$d = (1\bar{1}\bar{1})$	$1/\sqrt{3}$	$-1/\sqrt{3}$	$1/\sqrt{3}$
I = [011]	0	$-1/\sqrt{2}$	$-1/\sqrt{2}$
II = [101]	$-1/\sqrt{2}$	0	$1/\sqrt{2}$
III = [110]	$1/\sqrt{2}$	$1/\sqrt{2}$	0

Using Table 3.3, the strains relative to the cube axes may be expressed as:

$$\begin{aligned} d\epsilon_1 &= (-d\gamma_{a11} + d\gamma_{a12} - d\gamma_{b11} + d\gamma_{b12} - d\gamma_{c11} + d\gamma_{c12} - d\gamma_{d11} + d\gamma_{d12})/\sqrt{6} \\ d\epsilon_2 &= (+d\gamma_{a11} - d\gamma_{a12} + d\gamma_{b11} - d\gamma_{b12} + d\gamma_{c11} - d\gamma_{c12} + d\gamma_{d11} - d\gamma_{d12})/\sqrt{6} \\ d\epsilon_3 &= (-d\gamma_{a11} + d\gamma_{a12} - d\gamma_{b11} + d\gamma_{b12} - d\gamma_{c11} + d\gamma_{c12} - d\gamma_{d11} + d\gamma_{d12})/\sqrt{6} \\ d\gamma_{21} &= (+d\gamma_{a11} - d\gamma_{a12} - d\gamma_{b11} + d\gamma_{b12} + d\gamma_{c11} - d\gamma_{c12} - d\gamma_{d11} + d\gamma_{d12})/\sqrt{6} \\ d\gamma_{31} &= (-d\gamma_{a11} + d\gamma_{a12} + d\gamma_{b11} - d\gamma_{b12} + d\gamma_{c11} - d\gamma_{c12} - d\gamma_{d11} + d\gamma_{d12})/\sqrt{6} \\ d\gamma_{12} &= (+d\gamma_{a11} - d\gamma_{a12} + d\gamma_{b11} - d\gamma_{b12} - d\gamma_{c11} + d\gamma_{c12} - d\gamma_{d11} + d\gamma_{d12})/\sqrt{6}. \end{aligned} \quad (3.6)$$

The first three of these equations are not independent because there is no volume change, so $d\epsilon_1 + d\epsilon_2 + d\epsilon_3 = 0$. Therefore, there are only five independent relationships among Eq. (3.6). To satisfy a general shape change (arbitrary combination of $d\epsilon_1, d\epsilon_2, \dots, d\gamma_{23}$), at least five of the shear terms, $d\gamma_{a1}, d\gamma_{a12}, \dots, d\gamma_{d12}$, must be finite. This means that five active slip systems are required for a general shape change.

Equations 3.6 cannot be solved directly for the values of the crystallographic strains ($d\gamma_{a1}, d\gamma_{a12}, \dots, d\gamma_{d12}$) required to produce a given shape change ($d\epsilon_1, d\epsilon_2, \dots, d\gamma_{23}$) because only five equations are independent and there are 12 unknowns. To circumvent this problem, Taylor assumed that the shape change would occur in such a way as to minimize the energy expended in slip, $\sum_i (\tau_i d\gamma_i)$. He assumed that the critical stress for slip, τ , is the same for all active slip systems and they work harden at the same rate. Thus minimizing $\sum_i (\tau_i d\gamma_i)$ is equivalent to minimizing $\sum_i |d\gamma_i| = d\gamma$. Taylor reasoned, in analogy to frictional systems, that the minimum work would correspond to slip on as few systems as geometrically possible (i.e., 5 systems), with the rest of the slip systems (i.e., 7) being inactive. By setting 7 of the 12 shear terms, $d\gamma_{a1}, d\gamma_{a12}, \dots, d\gamma_{d12}$, in Eq. (3.6) equal to zero, he could solve Eqs. (3.6) and find $d\gamma/d\epsilon_x$. [For a given orientation $d\epsilon_1, d\epsilon_2, \dots, d\gamma_{23}$ are known relative to $d\epsilon_x$ by Eqs. (3.5).] By repeating this calculation for all possible combinations of five active and seven inactive slip systems, he could find the lowest, and hence appropriate, value of $d\gamma/d\epsilon_x = M$.

In the days before computers, making such calculations was a major undertaking. There are $(12 \cdot 11 \cdot 10 \cdot 9 \cdot 8)/(1 \cdot 2 \cdot 3 \cdot 4 \cdot 5) = 792$ different ways of choosing 5 things from a group of 12. Many of the 792 slip system combinations are not possible because they cannot produce 5 independent strains. Such combinations of slip systems are not independent because the shear produced by one of the slip systems can be reproduced by slip on some combination of the other slip systems. Taylor later [5] showed that the number of combinations that must be analyzed can be reduced to 96. In his

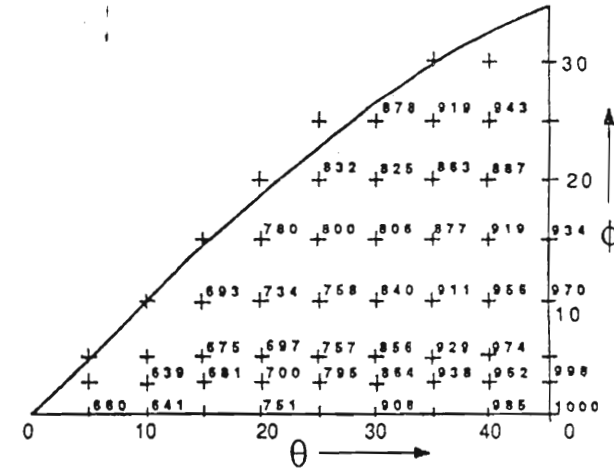


Fig. 3.3. Orientation dependence of M for axisymmetric flow. (From Chin and Mammel [7].)

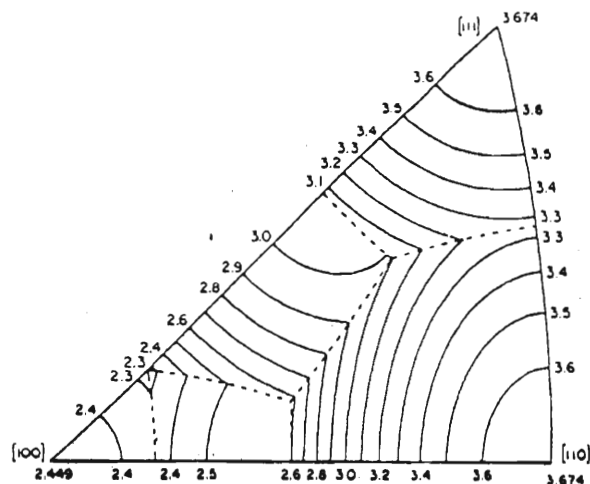
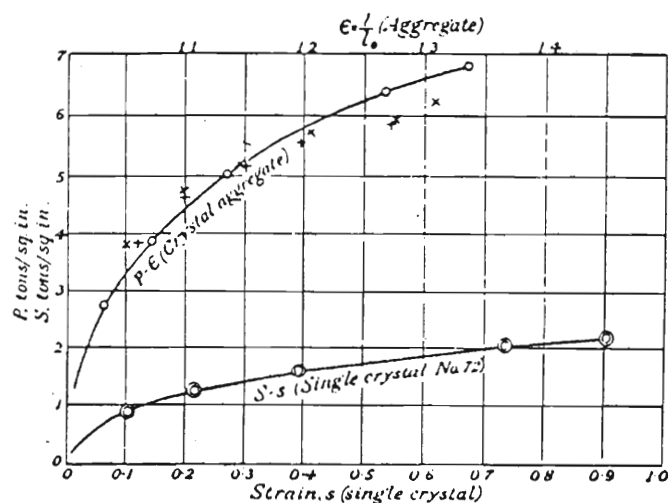
not appear that his minimum values of $M = d\gamma/d\epsilon_x$ were greatly affected by this omission.

Taylor made such calculations and evaluated M for a range of orientations in the basic stereographic triangle (Fig. 3.3). It is interesting to note that for most orientations he found the same minimum M value with several different combinations of slip systems. These calculations have since been repeated [6, 7] with similar results, as shown in Fig. 3.4. Taylor used an average value of $\bar{M} = 3.06$ to calculate the shape of the tensile stress-strain (σ - ϵ) curve for a randomly oriented polycrystal of aluminum from the shape of the shear stress-strain (τ - γ) curve for a single crystal taking points as $\sigma = 3.06\tau$ and $\epsilon = \gamma/3.06$. As shown in Fig. 3.5, the agreement with experiment was quite good.

3.2. Bishop and Hill analysis

Taylor's analysis is based on finding the slip systems in a crystal that can geometrically produce a specified shape change with the least expenditure of energy. No regard is given to whether there is a stress state that could physically activate the required slip systems. In the process of investigating this point, Bishop and Hill [8, 9] developed a different approach to analyzing plastic flow in fcc polycrystals. Its basis and predictions are similar to Taylor's, but the method of calculation is different and simpler.

Bishop and Hill first identified all of the stress states (combinations of

Fig. 3.4. Orientation dependence of M for axisymmetric flow. (From Chin and Mammel [7].)Fig. 3.5. Stress-strain curves for polycrystalline aluminum polycrystal (σ versus ϵ , +x) and for single crystal (τ versus γ , double circle). The polycrystal curve was calculated from the single crystal points using $\sigma = M\tau$ and $\epsilon = \gamma/M$, with $M = 3.06$. (From Taylor [3].)

Then for a given shape change they selected the appropriate stress state and finally calculated the work per unit volume expended by that stress state in producing the shape change,

$$\frac{dw}{d\epsilon_x} = \sigma_1 \frac{d\epsilon_1}{d\epsilon_x} + \sigma_2 \frac{d\epsilon_2}{d\epsilon_x} + \dots \quad (3.7)$$

If a single stress, σ_x , is applied, then the plastic strength is given by $\sigma_x = dw/d\epsilon_x$. Here again calculations of $M = (1/\tau) dw/d\epsilon_x = \sigma_x/\tau$ are made for various orientations of grains. Averaging over all possible orientations gives a value of $\bar{M}$ for relating a random polycrystal to the single crystal.

The slip system notation and direction cosines here are the same as those for the Taylor analysis (Fig. 3.2 and Table 3.2). Using Table 3.3 the shear stress on each system can be related to the stresses on the cube axes, $\sigma_1, \sigma_2, \dots, \sigma_{23}$. For example, the shear stress σ_{a1} on the a plane in the 1 direction is

$$\begin{aligned} \sigma_{a1} &= l_{a1}l_{11}\sigma_{11} + l_{a2}l_{12}\sigma_{22} + \dots + (l_{a1}l_{12} + l_{a2}l_{11})\sigma_{12} \\ &= (1/\sqrt{6})(\sigma_{22} - \sigma_{33} - \sigma_{31} + \sigma_{12}). \end{aligned}$$

Yielding will occur on this system when $\sigma_{a1} = \pm\tau$, where τ is the critical stress for slip; so the corresponding yield criterion is

$$(\sigma_{22} - \sigma_{33} - \sigma_{31} + \sigma_{12})/(\sqrt{6}\tau) = \pm 1.$$

The yield criteria for all of the systems are given below. For simplification, the stress states are expressed in terms of $A, B, \dots, H$,

$$\begin{array}{lll} aI, A - G + H = \pm 1 & aII, B + F - H = \pm 1 & aIII, C - F + G = \pm 1 \\ bI, A + G + H = \pm 1 & bII, B - F - H = \pm 1 & bIII, C + F - G = \pm 1 \\ cI, A + G - H = \pm 1 & cII, B + F + H = \pm 1 & cIII, C - F - G = \pm 1 \\ dI, A - G - H = \pm 1 & dII, B - F + H = \pm 1 & dIII, C + F + G = \pm 1 \end{array} \quad (3.8)$$

where

$$\begin{aligned} A &= \frac{\sigma_{22} - \sigma_{33}}{\sqrt{6}\tau}; & F &= \frac{\sigma_{23}}{\sqrt{6}\tau}; \\ B &= \frac{\sigma_{33} - \sigma_{11}}{\sqrt{6}\tau}; & G &= \frac{\sigma_{31}}{\sqrt{6}\tau}; \\ C &= \frac{\sigma_{11} - \sigma_{22}}{\sqrt{6}\tau}; & H &= \frac{\sigma_{12}}{\sqrt{6}\tau}. \end{aligned} \quad (3.9)$$

The critical value of shear stress, $\pm\tau$, must be reached on 5 slip systems without it being exceeded on any of the others. This restriction may be expressed mathematically as:

$$|A \pm G \pm H| \leq 1, \quad |B \pm F \pm H| \leq 1, \quad |C \pm F \pm G| \leq 1. \quad (3.10)$$

Bishop and Hill showed that there are 28 stress states (i.e., combinations of $A, B, \dots, H$) that satisfy five or more of Eqs. (3.8) while not violating the restrictions [inequalities (3.10)]. (The number of stress states is 56 if the negative of each is counted separately.) Several examples are given here:

Example 3.1

$$A = 1, B = -1, \text{ and } C = F = G = H = 0$$

$$aI, A - G + H = 1; \quad aII, B + F - H = -1; \quad aIII, C - F + G = 0$$

$$bI, A + G + H = 1; \quad bII, B - F - H = -1; \quad bIII, C + F - G = 0$$

$$cI, A + G - H = 1; \quad cII, B + F + H = -1; \quad cIII, C - F - G = 0$$

$$dI, A - G - H = 1; \quad dII, B - F + H = -1; \quad dIII, C + F + G = 0.$$

Eight systems are simultaneously raised to a high enough stress to allow slip while the other systems are unstressed. Such a stress state is produced by compression parallel to 3 = [001].

Example 3.2

$$A = B = C = F = G = 0 \text{ and } H = -1$$

$$aI, A - G + H = -1; \quad aII, B + F - H = -1; \quad aIII, C - F + G = 0$$

$$bI, A + G + H = -1; \quad bII, B - F - H = -1; \quad bIII, C + F - G = 0$$

$$cI, A + G - H = 1; \quad cII, B + F + H = 1; \quad cIII, C - F - G = 0$$

$$dI, A - G - H = 1; \quad dII, B - F + H = 1; \quad dIII, C + F + G = 0.$$

Again the stress on eight systems is raised to the critical stress while the other systems are unstressed. Such a stress state is produced by plane-strain deformation, $\epsilon_{1101} = -\epsilon_{1102}$, $\epsilon_{1001} = 0$.

Example 3.3

$$A = B = C = 0 \text{ and } F = G = H = \frac{1}{2}$$

$$aI, A - G + H = 0; \quad aII, B + F - H = 0; \quad aIII, C - F + G = 0$$

$$bI, A + G + H = 1; \quad bII, B - F - H = -1; \quad bIII, C + F - G = 0$$

$$cI, A + G - H = 0; \quad cII, B + F + H = 1; \quad cIII, C - F - G = -1$$

Table 3.4. Bishop and Hill stress states and corresponding slip systems

	Stress (value of)						Operating slip systems											
	A	B	C	F	G	H	aI	aII	aIII	bI	bII	bIII	cI	cII	cIII	dI	dII	dIII
1	1	-1	0	0	0	0	+	-	0	+	-	0	+	-	0	+	-	0
2	0	1	-1	0	0	0	0	+	-	0	+	-	0	+	-	0	+	-
3	-1	0	1	0	0	0	-	0	+	-	0	+	-	0	+	-	0	+
4	0	0	0	1	0	0	0	+	-	0	-	+	0	+	-	0	-	+
5	0	0	0	0	1	0	-	0	+	+	0	-	+	0	-	-	0	+
6	0	0	0	0	0	1	+	-	0	+	-	0	-	+	0	-	+	0
7	$\frac{1}{2}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	-	+	+	-	0	+	-	0	0	-	+
8	$\frac{1}{2}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	+	-	0	0	-	+	0	-	+	+	-	0
9	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	-	+	0	-	0	+	-	+	0	-	0	+
10	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	-	0	+	-	+	0	-	+	0	-	+	0
11	$\frac{1}{2}$	$\frac{1}{2}$	-1	0	0	$\frac{1}{2}$	+	0	-	+	0	-	0	+	-	0	+	-
12	$\frac{1}{2}$	$\frac{1}{2}$	-1	0	0	$\frac{1}{2}$	0	+	-	0	+	-	+	0	-	+	0	-
13	$\frac{1}{2}$	0	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	+	0	-	+	-	0	0	+	-	0	0	0
14	$\frac{1}{2}$	0	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	+	-	0	+	0	-	0	0	0	0	+	-
15	$\frac{1}{2}$	0	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	+	-	0	0	0	+	0	-	+	-	0
16	$\frac{1}{2}$	0	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	0	+	-	+	-	0	+	-	0
17	0	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	-	+	+	-	0	0	0	0	-	0	+
18	0	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	+	-	0	0	-	+	-	0	+	0	0	0
19	0	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	-	0	+	0	0	0	+	-	0	0	-	+
20	0	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	-	0	+	0	-	+	+	-	0
21	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	-	+	0	0	0	0	0	+	-	-	0	+
22	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	-	0	+	0	+	-	0	0	0	-	+	0
23	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	+	-	-	0	+	-	+	0	0	0	0
24	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	-	+	0	-	0	+	0	+	-
25	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	-1	-	+	0	0	0	0	+	0	-	0	-	+
26	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	-1	+	0	-	0	-	+	-	+	0	0	0	0
27	0	0	0	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	-	+	+	0	-	0	0	0	-	+	0
28	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	+	-	0	0	+	-	-	0	+

Now six slip systems can be simultaneously activated. This stress state is produced by uniaxial tension parallel to [111].

These and the other Bishop and Hill stress states are listed in Table 3.4, together with the slip systems that are activated. The negative of each is also possible and will produce slip of the opposite sense. All of the stress states raise the stress on either six or eight systems to the critical level. This explains why Taylor found several combination of five slip systems produced the same minimum value of M .

Bishop and Hill used the principle of maximum virtual work (see Appendix 5) to determine which of the 28 stress states (or their negatives)

supplied by the external stresses is

$$dw = \sigma_1 d\epsilon_1 + \sigma_2 d\epsilon_2 + \sigma_3 d\epsilon_3 + \sigma_{23} d\gamma_{23} + \sigma_{31} d\gamma_{31} + \sigma_{12} d\gamma_{12} \quad (3.11)$$

Substituting the constant-volume relationship, $d\epsilon_1 + d\epsilon_2 + d\epsilon_3 = 0$,

$$\begin{aligned} dw &= \sigma_1 d\epsilon_1 + \sigma_2 d\epsilon_2 + \sigma_3(-d\epsilon_1 - d\epsilon_2) + \sigma_{23} d\gamma_{23} + \dots \\ &= (\sigma_1 - \sigma_3) d\epsilon_1 + (\sigma_2 - \sigma_3) d\epsilon_2 + \sigma_{23} d\gamma_{23} + \sigma_{31} d\gamma_{31} + \sigma_{12} d\gamma_{12} \end{aligned} \quad (3.12)$$

From the definitions of $A, B, \dots, H$ [Eq. (3.8)], Equation (3.12) becomes

$$\begin{aligned} M = \left(\frac{1}{\sqrt{6}} \right) \frac{dw}{d\epsilon_z} &= \sqrt{6} \left(-B \frac{d\epsilon_1}{d\epsilon_z} + A \frac{d\epsilon_2}{d\epsilon_z} + F \frac{d\gamma_{23}}{d\epsilon_z} \right. \\ &\quad \left. + G \frac{d\gamma_{31}}{d\epsilon_z} + H \frac{d\gamma_{12}}{d\epsilon_z} \right) \end{aligned} \quad (3.13)$$

For axisymmetric flow about z ($d\epsilon_x = d\epsilon_y = -d\epsilon_z/2$ with $d\gamma_{yz} = d\gamma_{zx} = d\gamma_{xy} = 0$) expressions for $d\epsilon_1/d\epsilon_z, d\epsilon_2/d\epsilon_z, \dots, d\gamma_{12}/d\epsilon_z$ can be found using Table 3.1:

$$\begin{aligned} \frac{d\epsilon_1}{d\epsilon_z} &= \frac{1}{2} \sin^2 \phi - \frac{1}{2} \\ \frac{d\epsilon_2}{d\epsilon_z} &= \frac{1}{2} \sin^2 \theta \cos^2 \phi - \frac{1}{2} \\ \frac{d\epsilon_3}{d\epsilon_z} &= \frac{1}{2} \cos^2 \theta \cos^2 \phi - \frac{1}{2} \\ \frac{d\gamma_{23}}{d\epsilon_z} &= 3 \sin \theta \cos \theta \cos^2 \phi \\ \frac{d\gamma_{31}}{d\epsilon_z} &= -3 \cos \theta \cos \phi \sin \phi \\ \frac{d\gamma_{12}}{d\epsilon_z} &= -3 \sin \theta \cos \phi \sin \phi \end{aligned} \quad (3.14)$$

Substituting Eqs. (3.14) into Eq. (3.13),

$$\begin{aligned} M/\sqrt{6} &= +A(3 \sin^2 \theta \cos^2 \phi - 1)/2 + B(-3 \sin^2 \phi + 1)/2 \\ &\quad + 3F \sin \theta \cos \theta \cos^2 \phi - 3G \cos \theta \cos \phi \sin \phi \end{aligned}$$

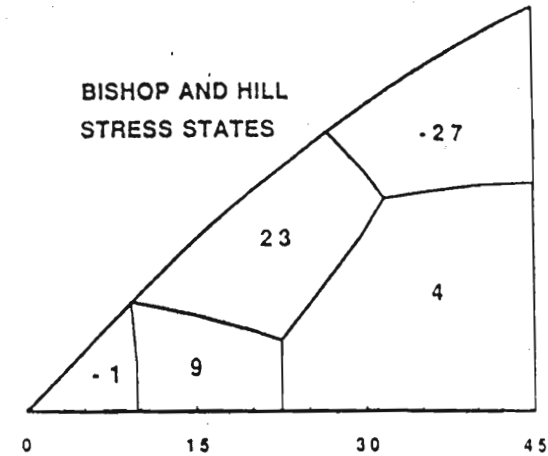


Fig. 3.6. Basic orientation triangle showing the appropriate Bishop and Hill stress states in various regions.

The appropriate value of M is found by substituting into Eq. (3.15) the values of $A, B, \dots, H$ for all of the 28 stress states and selecting the absolutely largest M value. This is the method used to calculate the orientation dependence of M , which was shown in Fig. 3.4 [6]. The basic stereographic triangle is divided into five regions, as indicated in Fig. 3.6, each with its own set of active slip systems. The stress state for each region and expressions for M are given in Table 3.5.

Table 3.5. Stress states and M values for axially symmetric flow

Region	Stress states						$M/\sqrt{6}$
	A	B	C	F	G	H	
-1	-1	1	0	0	0	0	$1 - \frac{1}{2}(\sin^2 \phi + \cos^2 \phi \sin^2 \theta)$
9	-1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2} \cos^2 \phi [1 + 2 \sin \theta (\cos \theta - \sin \theta)]$
4	0	0	0	1	0	0	$3 \cos^2 \phi \cos \theta \sin \theta$
23	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2} - \frac{1}{2}(\cos^2 \phi \sin^2 \theta + \sin^2 \phi)$ $+ \frac{1}{2} \cos \phi \cos \theta (\cos \phi \sin \theta + \sin \phi)$
-27	0	0	0	-1	1	1	$3 \sin^2 \phi \sin^2 \theta$

3.3. Discussion of the Taylor and the Bishop and Hill analyses

These two seemingly divergent approaches to the same problem are mathematically equivalent [10]. They depend on the same assumptions and yield the same results. It should be noted that while they were specifically formulated for fcc metals with $\{111\}\langle 110 \rangle$ slip elements, all of the calculations (except those dealing with lattice rotation) are equally applicable to bcc metals for which the slip elements are $\{110\}\langle 111 \rangle$.

There are several critical assumptions. (See U. F. Kocks [11] for an excellent discussion.) One is that all grains in a polycrystal deform homogeneously and undergo the same shape change. While this ensures compatible deformation, it is not the only way that compatibility can be maintained. Deformation is not always homogeneous. Departure from homogeneous flow should be expected when the energy expended, dw/dc , for nonhomogeneous flow is less than that for homogeneous flow. Such deviations from homogeneity of flow are discussed in Chapter 6.

Another basic assumption in these theories is that the critical shear stress for slip, τ , is the same for all active slip systems. Certainly this is reasonable for the very first increment of slip. However, use of these analyses to predict the shapes of stress-strain curves or even the level of stress after small plastic strains makes it necessary to assume also that the value of τ remains equal for all active slip systems and depends only on the total amount of slip on all slip systems, $\sum |d\gamma_i|$. According to this assumption, work hardening should not depend on which combination of slip systems is active as long as multiple slip occurs. Yet dislocation theory predicts different types of dislocation interactions with different combinations of slip systems, and one might expect different interactions to cause different rates of strain hardening. There have been several experiments designed to measure work hardening during multiple slip.

U. F. Kocks [12, 13] tested aluminum single crystals under loading conditions designed to produce *polyslip* (a term he defined as the simultaneous operation of five or more slip systems). These loading conditions corresponded to four of the five types of Bishop and Hill stress states. For most of these conditions the initial slopes were similar, but the curves departed from one another at shear strains above about 0.05. It is likely that the difference between stress-strain curves was due to a failure to achieve polyslip in all cases. The experiments were designed to impose a stress state that could produce multiple slip, but there was not an imposed shape change that would require five or more systems to be active.

Attempts were made to force axially symmetric flow on a large range of orientations of aluminum single crystals by wire drawing [14, 15]. The results tended to support the Taylor assumption, but complications caused by friction and finite die angles produced considerable scatter. Plane-strain compression tests of single crystals [16-18] have been used to study strain

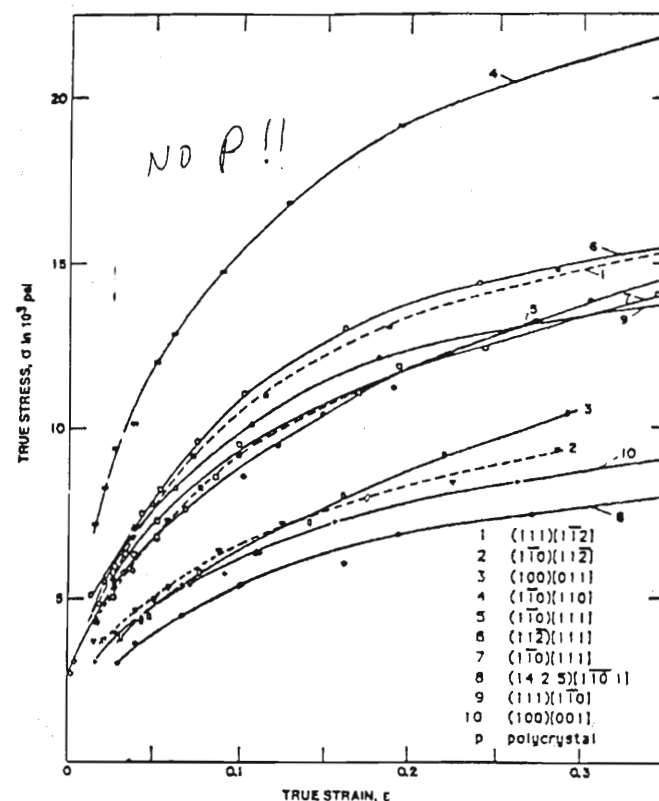


Fig. 3.7. True stress-strain curves for aluminum single crystals tested in plane-strain compression.

hardening under polyslip. By testing a range of orientations, many combinations of slip systems could be activated. Figure 3.7 shows the true stress-strain curves for aluminum single crystals tested in plane-strain compression. Shear-stress-shear-strain curves constructed by taking $\tau = \sigma/M$ and $\gamma = M\epsilon$ are shown in Fig. 3.8. These all fall within a rather narrow scatter band together with the curve for a polycrystal. Plane-strain compression tests on permalloy, copper, and copper-based alloys give similar results. The conclusion is that under polyslip conditions, the strain hardening depends primarily on the total amount of slip and not very greatly on which combination of slip systems is active.

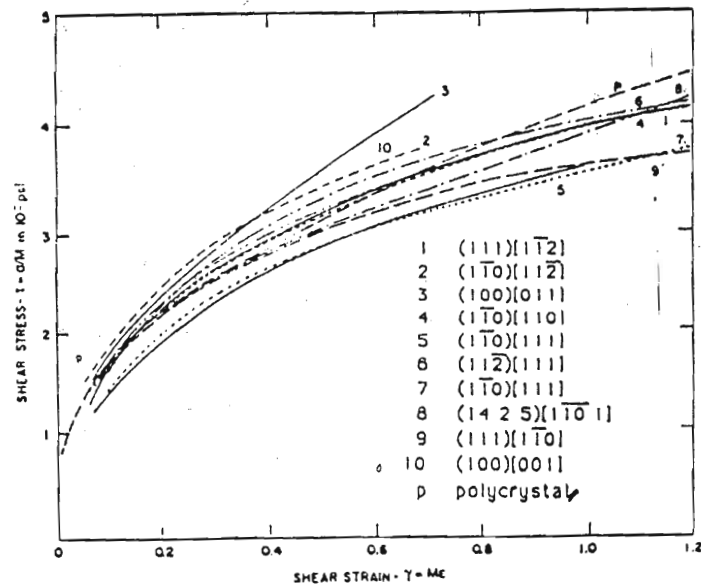


Fig. 3.8. Shear stress-strain curves for aluminum single crystals derived from curves in Fig. 3.7 using $\tau = \sigma/M$ and $\gamma = M\epsilon$.

3.4. Application to textured metals

Although the papers of Taylor and of Bishop and Hill were primarily concerned with explaining the properties of randomly oriented polycrystals, their analyses are adaptable for predicting the anisotropy of crystallographically textured materials [6].

3.4.1. Axially symmetric flow

The calculated orientation dependence of M for axially symmetric flow in Fig. 3.6 can be applied directly to predicting the relative strength of rods or wires with different fiber textures because the flow would also be axially symmetric. The relative strength is indicated by the M value corresponding to the fiber axis. For wires with $\langle 111 \rangle$ or $\langle 110 \rangle$ fiber textures $M = \frac{1}{2}\sqrt{6} = 3.67$ and $M = \sqrt{6} = 2.45$ for wires with $\langle 100 \rangle$ fiber textures, compared with $\bar{M} = 3.06$ for randomly oriented polycrystals. This comparison, however, somewhat underestimates the difference at a constant level of strain, ϵ , for materials that work harden. Assuming that $\tau = f(\gamma)$ and $\gamma = M\epsilon$,

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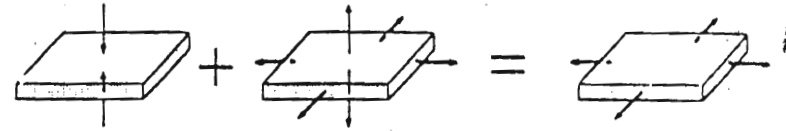


Fig. 3.9. Equivalence of through thickness compression and balanced biaxial tension: The two stress states differ only by a hydrostatic tension.

$\tau = K\gamma^n$, is also assumed, then

$$\tau = K(M\epsilon)^n \quad (3.16)$$

$$\sigma = M\tau = M^{(1+n)}K\epsilon^n. \quad (3.17)$$

If the stress-strain curves for two wires with different textures are compared at the same level of axial strain, ϵ , the ratio of flow stresses is given by

$$\sigma_1/\sigma_2 = \left(\frac{M_1}{M_2}\right)^{1+n}. \quad (3.18)$$

For example, with $n = 0.25$, the strengths of wires with $\langle 111 \rangle$ and $\langle 110 \rangle$ fiber textures should be $(3.67/3.06)^{1.25} = 1.26$ times the strength of a random polycrystal and a $\langle 100 \rangle$ fiber-textured wire would be only 0.76 times as strong.

The yielding behavior in compression should be the same as in tension. Furthermore, the yielding of a sheet under uniaxial compression applied in the thickness direction is equivalent to yielding under balanced biaxial tension applied in the plane of the sheet. These two stress states differ by only a hydrostatic stress which should not affect yielding (Fig. 3.9). Therefore, the M values for axially symmetric flow may also be used to predict the resistance to yielding of sheets with textures that are rotationally symmetric about the sheet normal. Textural components with $\{111\}$ and $\{110\}$ parallel to the plane of the sheet tend to increase the biaxial strength and those with $\{100\}$ decrease it.

3.4.2. Sheet textures

It should be emphasized that the treatment above applies only to axially symmetric flow conditions. When a tensile test is made on a strip tensile specimen cut from a textured sheet, generally different contractile strains are

commonly used measure of the anisotropy is the strain ratio, R ,^{*} defined as

$$R = \frac{d\epsilon_w}{d\epsilon_l}, \quad (3.19)$$

where w and t refer to the width and thickness directions in a sheet tensile specimen extended in the length or l direction. For an isotropic material, $R = 1$. Although the R value is commonly used to describe materials, analysis of possible behavior is often more convenient with a parameter ρ , defined as:

$$\rho = \frac{-d\epsilon_w}{d\epsilon_l}. \quad (3.20)$$

Since the volume is constant ($d\epsilon_l + d\epsilon_w + d\epsilon_t = 0$),

$$\frac{d\epsilon_t}{d\epsilon_l} = \rho - 1; \quad (3.21)$$

so the two parameters are interrelated by

$$\rho = \frac{R}{R+1} \quad \text{and} \quad R = \frac{\rho}{1-\rho}. \quad (3.22)$$

Consider now, the tensile behavior of a textured sheet. Let the tensile axis coincide with the rolling direction, x , and the sheet normal with z . During a tension test the symmetry should insure that $d\epsilon_{yz} = d\epsilon_{zx} = d\epsilon_{xy} = 0$. With constancy of volume, only one independent strain component, $d\epsilon_y$ or $d\epsilon_z$, remains unspecified relative to $d\epsilon_x$. It might at first seem that the Taylor and the Bishop and Hill analyses cannot be used because the shape change is not fully specified. However, the shape change can be expressed in terms of ρ ,

$$\begin{aligned} \frac{d\epsilon_y}{d\epsilon_x} &= -\rho \\ \frac{d\epsilon_z}{d\epsilon_x} &= \rho - 1 \\ d\gamma_{yz} &= d\gamma_{zx} = d\gamma_{xy} = 0, \end{aligned} \quad (3.23)$$

and then M calculated for all possible values of ρ , $0 \leq \rho \leq 1$. Since the

^{*} In practice R is usually taken as the ratio of finite strains, $R = \epsilon_w/\epsilon_l$, rather than using differential strains as indicated by Eq. (3.19). With most materials the change of R with strain, ϵ_l , is negligible. Also it should be noted that the symbol r is often used instead of R .

material will elongate with the least expenditure of energy, the lowest value of M should be the appropriate one for uniaxial tension, and the corresponding value of ρ should describe the expected shape change.

Example 3.4

Predict M and R for a rolling direction tension test on a sheet with an ideal $[112](110)$ texture.

Solution: Let $x = [1\bar{1}2]$, $y = [\bar{1}11]$, and $z = [110]$. The direction cosines are:

	1 = $[100]$	2 = $[010]$	3 = $[001]$
$x = [1\bar{1}2]$	$1/\sqrt{6}$	$-1/\sqrt{6}$	$2/\sqrt{6}$
$y = [\bar{1}11]$	$-1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$
$z = [110]$	$1/\sqrt{2}$	$1/\sqrt{2}$	0

We can write

$$d\epsilon_l = \frac{1}{6} d\epsilon_x + \frac{1}{3} d\epsilon_y + \frac{1}{2} d\epsilon_z + \dots \quad \text{or}$$

$$\frac{d\epsilon_l}{d\epsilon_x} = -\frac{1}{3} + \frac{1}{6}\rho$$

$$\frac{d\epsilon_2}{d\epsilon_x} = -\frac{1}{3} + \frac{1}{6}\rho$$

$$\frac{d\epsilon_3}{d\epsilon_x} = \frac{2}{3} - \frac{1}{3}\rho$$

$$\frac{d\gamma_{23}}{d\epsilon_x} = -\frac{2}{3} - \frac{2}{3}\rho$$

$$\frac{d\gamma_{31}}{d\epsilon_x} = \frac{2}{3} + \frac{2}{3}\rho$$

$$\frac{d\gamma_{12}}{d\epsilon_x} = -\frac{4}{3} + \frac{2}{3}\rho$$

Substituting these into Eq. (3.13),

$$\begin{aligned} \frac{M}{\sqrt{6}} &= A\left(-\frac{1}{3} + \frac{\rho}{6}\right) - B\left(-\frac{1}{3} + \frac{\rho}{6}\right) + F\left(-\frac{2}{3} - \frac{2\rho}{3}\right) \\ &\quad + G\left(\frac{2}{3} + \frac{2\rho}{3}\right) + H\left(-\frac{4}{3} + \frac{5\rho}{3}\right) \\ &= \frac{-A + B - 2F + 2G - 4H}{3} + \frac{\rho(+A - B - 4F + 4G + 10H)}{6} \end{aligned}$$

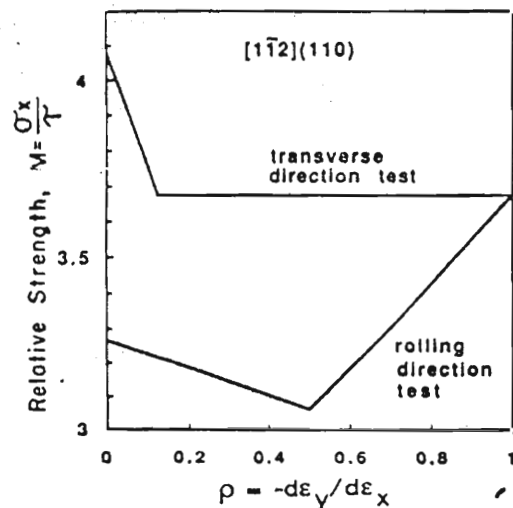


Fig. 3.10. Dependence of M on $\rho = -d\epsilon_y/d\epsilon_x$ for rolling and transverse direction tension tests for an ideal $[1\bar{1}2](110)$ texture. In the rolling direction test $x = [1\bar{1}2]$, and in the transverse test $x = [\bar{1}11]$.

Examination of the Bishop and Hill stress states shows that for $0 \leq \rho \leq \frac{1}{2}$, stress state -26 ($F = -\frac{1}{2}$, $G = \frac{1}{2}$, $H = -\frac{1}{2}$, $A = B = C = 0$) gives the largest value of M , $M = \sqrt{6} (\frac{2}{3} - \rho/6)$ and is therefore appropriate. For $\frac{1}{2} \leq \rho \leq 1$, stress state 22 ($A = -\frac{1}{2}$, $B = \frac{1}{2}$, $F = -\frac{1}{2}$, $G = \frac{1}{2}$, $H = 0$) is appropriate; so $M = \sqrt{6} (1 + \rho/2)$. A plot of M versus ρ (Fig. 3.10) has a minimum of $M = \frac{2}{3}\sqrt{6}$ at $\rho = \frac{1}{2}$ (i.e., $R = 1$).

A similar analysis for a tension test in the transverse direction results in a plot, also shown in Fig. 3.10, with a flat minimum for $\frac{1}{2} \leq \rho \leq 1$. In this case it is not possible to predict a unique value of ρ (or R). For a sheet with an ideal $[1\bar{1}0](111)$ texture, an infinite value of R ($\rho = 1$) is predicted for a rolling direction tension test.

3.4.3. Mixed textures

These predictions are for ideal textures. Even a small amount of other textural components would modify the results, probably creating a unique minimum for the $[1\bar{1}2](110)$ transverse-direction test and shifting the minimum to finite R values in the $[1\bar{1}0](111)$ rolling-direction test.

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components is simply the weighted average of the M versus ρ curves for each component

$$M = f_a M_a + f_b M_b + \dots \quad (3.24)$$

where $f_a, f_b, \dots$, are the volume fractions of components $a, b, \dots$. The minimum corresponds to the condition

$$\frac{dM}{d\rho} = f_a \frac{dM_a}{d\rho} + f_b \frac{dM_b}{d\rho} + \dots = 0. \quad (3.25)$$

A randomly oriented texture component can be treated as an isotropic material. If the von Mises yield criterion⁵ and associated flow rules are assumed for the randomly oriented grains, it can be shown that

$$\frac{dw}{d\epsilon_x} = Y[\frac{2}{3}(\rho^2 - \rho + 1)]^{1/2}, \quad (3.26)$$

where Y is the yield strength under uniaxial tension. Substituting equation Eq. (3.4) into Eq. (3.26),

$$M = \frac{Y}{\tau} [\frac{2}{3}(\rho^2 - \rho + 1)]^{1/2}. \quad (3.27)$$

Using the average value for a random polycrystal, $Y/\tau = 3.067$,

$$M = 3.54(\rho^2 - \rho + 1)^{1/2}, \quad (3.28)$$

which should approximate the shape of the M versus ρ curve for the randomly oriented grains. The presence of randomly oriented grains in a textured material will move the minimum toward $\rho = \frac{1}{2}$ or $R = 1$. This is illustrated in Fig. 3.11 for the ideal $[1\bar{1}0](111)$ texture, which has a minimum M at $\rho = 1$ ($R = \infty$).

3.4.4. Rotationally symmetric sheet textures

A similar analysis has been used to predict R values for sheets with textures that are rotationally symmetric about the sheet normal. Many grains are assumed to have the same (hkl) parallel to the plane of the sheet with their orientations differing by rotation about the sheet normal, making different

⁵ The calculated shape of the yield surface for randomly oriented fcc metals is discussed in Chapter 4. The resulting shape is a better description of the randomly oriented component and can be used instead of the von Mises criterion for this type of calculation.

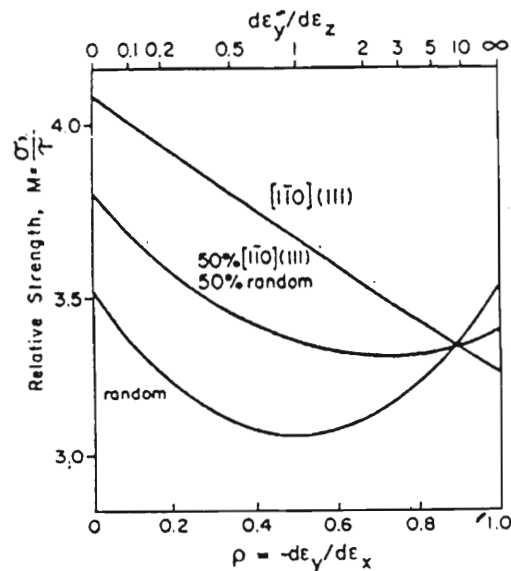


Fig. 3.11. Dependence of M on ρ for rolling direction tests of sheets with ideal $\langle 112 \rangle \{111\}$ texture, randomly oriented grains and equal volume of the $\langle 112 \rangle \{111\}$ ideal texture and randomly oriented grains. Here $x = [112]$ and $y = [1\bar{1}0]$.

directions parallel to the tensile axis. For the purpose of analysis, such a texture with rotational symmetry can be approximated by a number of components, each sharing a common $\{hkl\}$ plane, but differing from one another by 5° rotations about the $[hkl]$ sheet normal. First one value of ρ is assumed, and the values of M calculated for all components and averaged. This is repeated for other values of ρ until an M versus ρ curve can be generated. Figures 3.12–3.15 show the results of such calculations for textures with $\{111\}$, $\{110\}$, and $\{100\}$ in the plane of the sheets. Although these calculations predict that pure $\{111\}$ and $\{110\}$ textures should give infinite R values, a small amount of another component would lead to a finite R .

3.5. Upper-bound pencil glide model

The general form of the Taylor and Bishop and Hill analyses can be extended to other crystal structures. Of these, the most important is bcc. The slip direction is definitely $\langle 111 \rangle$, but different planes or combinations of planes serve as the slip planes. The most commonly reported is $\{110\}$. In this case

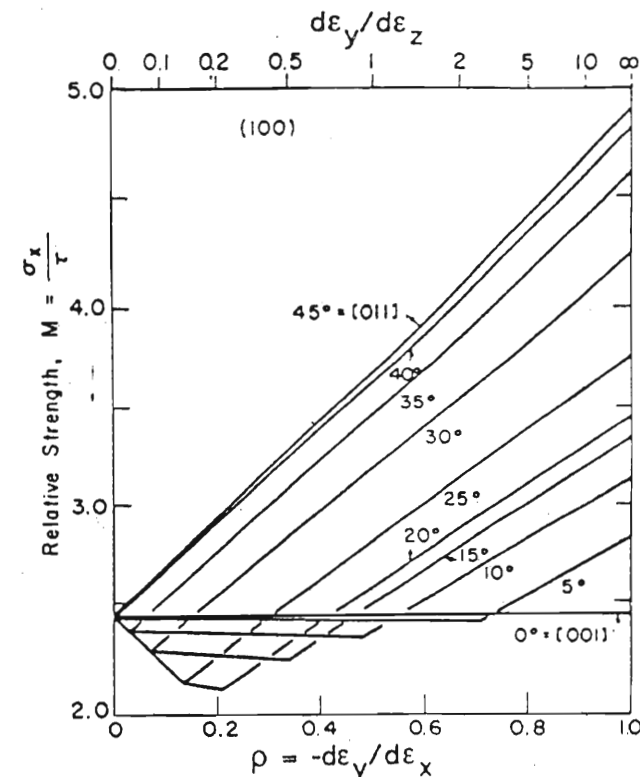


Fig. 3.12. M versus ρ curves for various components of a $\{100\}$ sheet texture. The angles between the tensile axis and $[001]$ are indicated. (Adapted from [6].)

those for fcc, and all calculations of the effects of texture on strength and R value for fcc are applicable to bcc metals. The lattice rotations accompanying deformation, however, are the reverse of those for fcc.

Bcc metals have also been reported to deform by slip on $\{112\}$ and $\{123\}$ planes and on mixed $\{110\}$, $\{112\}$, and $\{123\}$ planes. It has also been proposed that $\langle 111 \rangle$ -pencil glide⁶ is a reasonable model for slip in bcc. Kocks [11] pointed out that $\langle 111 \rangle$ -pencil glide is equivalent to slip occurring in any direction on a $\{111\}$ plane. Any shear strain in a $\{111\}$ plane could

⁶ The term $\langle 111 \rangle$ -pencil glide means that slip is restricted to $\langle 111 \rangle$ directions, but that any plane containing a $\langle 111 \rangle$ direction can act as the slip plane. The term was coined in analogy to shearing a stack of pencils. The shear direction must be parallel to the pencils, but the plane of shearing can be any of the faces of the stack.

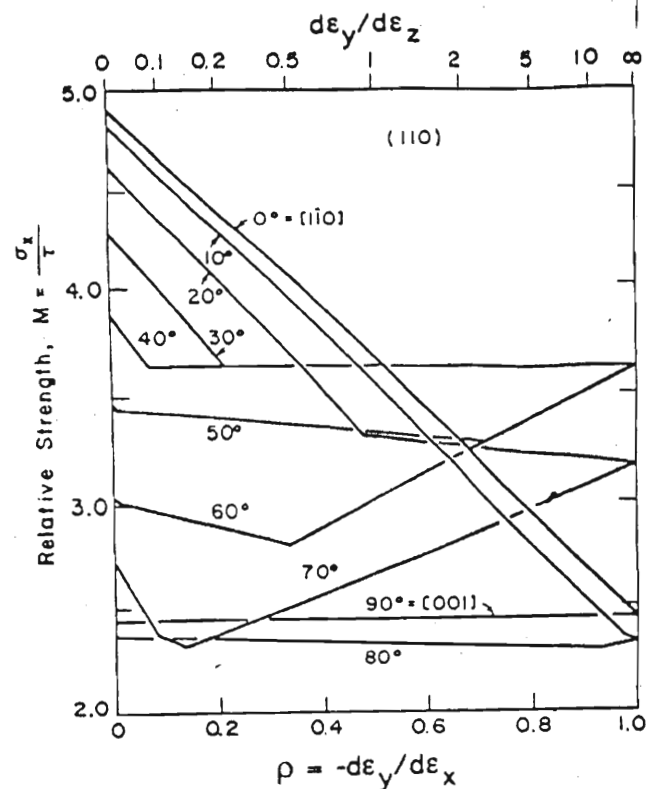


Fig. 3.13. M versus ρ curves for various components of a $\langle 110 \rangle$ sheet texture. The angles between the tensile axis and $[1\bar{1}0]$ are indicated. (From [6].)

be geometrically duplicated by shear strains in two $\langle 110 \rangle$ directions in that plane. The ratio of the sum of the two $\langle 110 \rangle$ shears to the arbitrary shear is greatest when the direction of shear is along a $\langle 112 \rangle$ direction. In this case the ratio is $2/\sqrt{3}$. Therefore the amount of shear required with pencil glide can never be less than $\sqrt{3}/2 = 0.866$ times the amount of shear with slip in $\langle 110 \rangle$ directions. Hence it can be concluded that with $\langle 111 \rangle$ -pencil glide, the M factors cannot be less than 0.866 times those for $\{111\}\langle 110 \rangle$ slip.

Chin and Mammel [7] used a modified Taylor approach for $\langle 111 \rangle$ slip on various planes and combinations of planes. Their calculations for

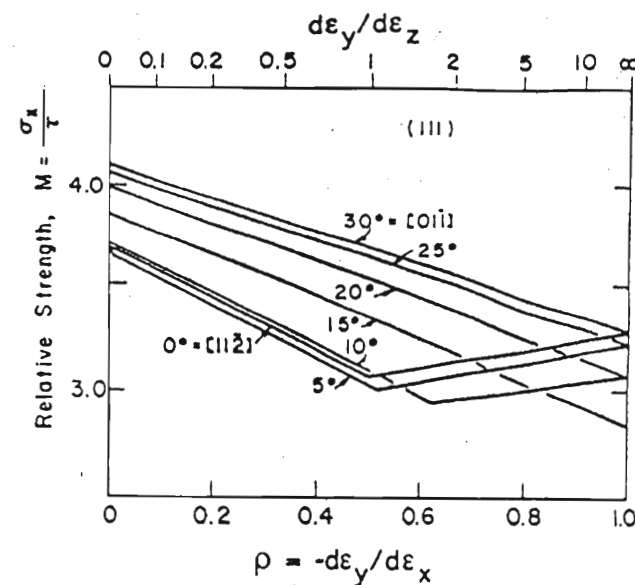


Fig. 3.14. M versus ρ curves for various components of a $\langle 111 \rangle$ sheet texture. The angles between the tensile axis and $[11\bar{2}]$ are indicated. (From [6].)

[19] approximated pencil glide by allowing slip on 24 equally rotated planes for each $\langle 111 \rangle$ direction.

A minimum of five independent slip variables is required to produce an arbitrary shape change. With fixed slip planes and directions, the five variables are the amounts of shear on each of five slip systems. With $\langle 111 \rangle$ -pencil glide there are eight variables, namely the amount of slip and the orientation (or angular rotation) of the plane of slip for each of the four $\langle 111 \rangle$ directions. The requirement of five independent slip variables means that at least three slip directions must be active.

Penning [20] described an exact solution for pencil glide that treats both the three and four active slip direction cases and is based on a Taylor-type least-work analysis. It was used for calculation of yield loci from orientation distribution functions. Parniere and Sauzay [21] also used a similar analysis for yield locus calculations.

Piehl et al. [22, 23] approached the problem in a way analogous to the Bishop and Hill analysis for fcc slip. He derived general expressions of the stress states for the four active slip direction case, but he was unable to derive such expressions for the three slip system cases. Instead he used a limited number of specific solutions. For each shape change, the stress state

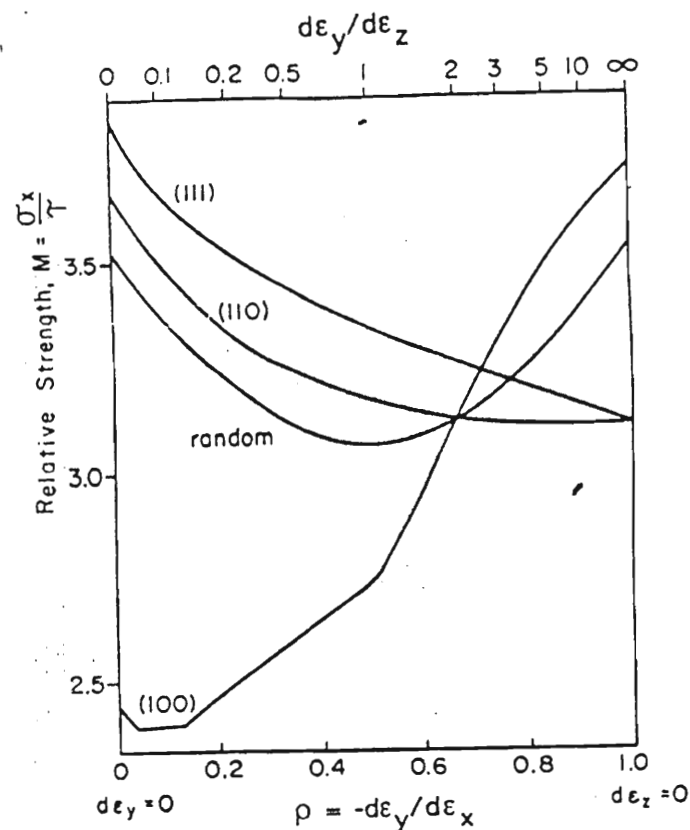


Fig. 3.15. M versus ρ curves for sheets with (111), (110), (100) and random textures which are rotationally symmetric about the sheet normal. (From [6].)

Logan et al. [24] combined the two methods. Pichler's exact stress states were used for maximum virtual work solution for cases in which they were appropriate. The validity of these solutions was checked by calculating the internal work implied by the shear strains on the four systems and determining whether it was equal to the external work. Taylor's least-work approach was used for the three slip system cases. First one slip system was assumed inactive, leaving six independent variables (the orientation of the slip planes for the other three directions and the shear strain on each). With one of the systems inactive, the orientation of one slip plane was assumed (leaving five independent variables); so the total amount of slip could be

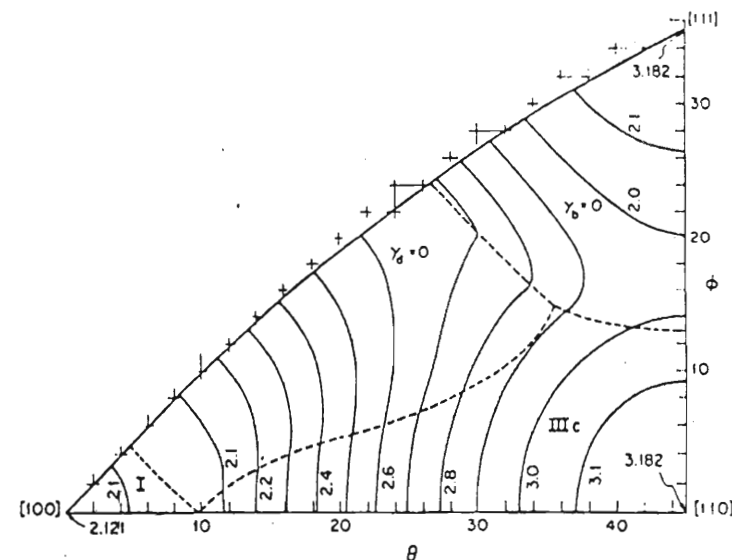


Fig. 3.16. Orientation dependence of M for axisymmetric flow by $\langle 111 \rangle$ -pencil glide. Four slip directions are active in regions I and IIIc. Only three slip systems are active in the regions labeled $\gamma_0 = 0$ and $\gamma_2 = 0$. Here $b = [111]$ and $d = [1\bar{1}\bar{1}]$. (From [24].)

was found. Then the procedure was repeated with another of the four $\langle 111 \rangle$ directions inactive and the lowest minimum selected as appropriate. The three slip system solution was then checked to assure that it did not correspond to a stress state that would activate slip on the fourth system. The calculations for the orientation dependence of M for axisymmetric deformation are shown in Fig. 3.16.

3.6. Calculations for other deformation modes

A Taylor-type analysis has been used for $\{111\}\langle 112 \rangle$ and $\{111\}\langle 123 \rangle$ slip [7]. The calculated orientation dependence of M for $\{111\}\langle 112 \rangle$, $\{111\}\langle 123 \rangle$, and for mixed $\langle 111 \rangle$ slip on $\{110\}$, $\{112\}$, and $\{123\}$ planes is shown in Figs. 3.17, 3.18, and 3.19, respectively. Deformation by $\{112\}\langle 111 \rangle$ and $\langle 112 \rangle\{111\}$ twinning was treated with a Bishop and Hill-type analysis [25]. The values of M calculated for various deformation modes for axisymmetric deformation of randomly oriented material are summarized in Table 3.6.

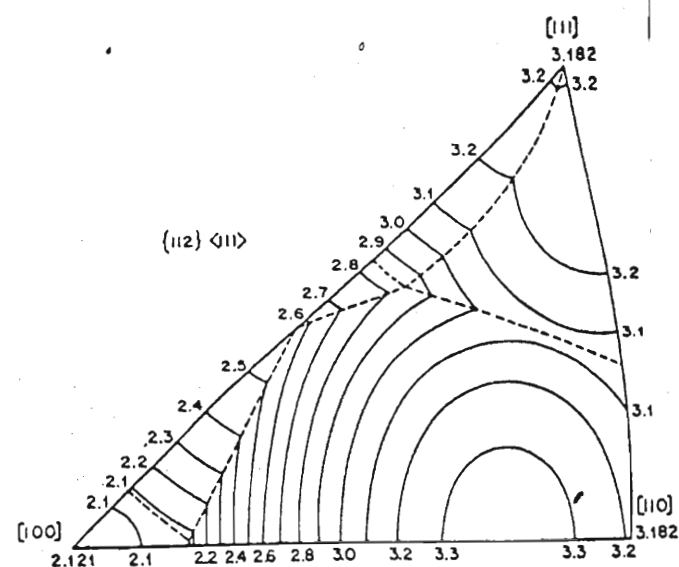


Fig. 3.17. Orientation dependence of M for axisymmetric flow $\langle 111 \rangle \{112\}$ slip. (From Chin and Mammel [7].)

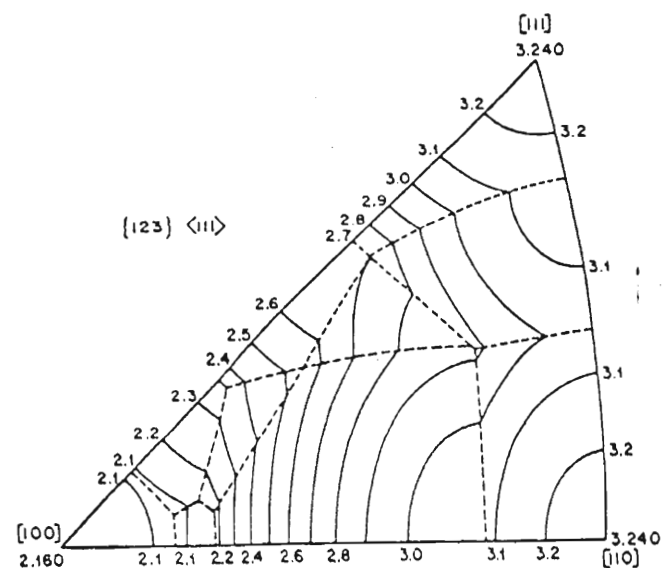


Fig. 3.18. Orientation dependence of M for axisymmetric flow $\langle 111 \rangle \{123\}$ slip. (From Chin and Mammel [7].)

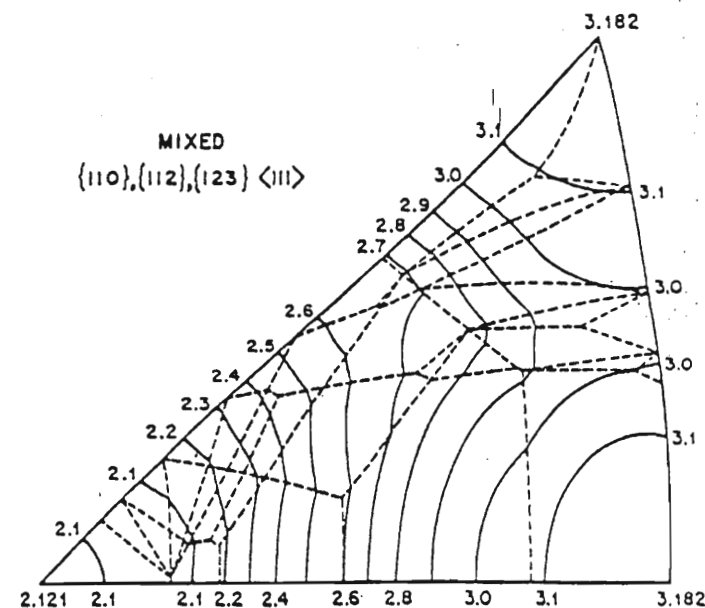


Fig. 3.19. Orientation dependence of M for axisymmetric flow with mixed $\langle 111 \rangle \{110\}$, $\langle 111 \rangle \{112\}$, and $\langle 111 \rangle \{123\}$ slip. The same critical stress τ was assumed on all systems. (From Chin and Mammel [7].)

Table 3.6. Average M values for axisymmetric flow of randomly oriented polycrystals

Slip systems	M^*	Ref.
$\langle 111 \rangle \{110\}$	3.06	3, 4
	3.072	19
	3.067	7
	3.067 ± 0.001	6
$\langle 111 \rangle \{112\}$	2.954	7
$\langle 111 \rangle \{123\}$	2.803	7
mixed $\{110\}$, $\{112\}$, and $\{123\}$	2.754	7
approx. $\langle 111 \rangle$ -pencil glide	2.748	19
$\langle 111 \rangle$ -pencil glide	2.733	23
	2.740 ± 0.002	24
$\{111\} \langle 112 \rangle$ twinning	3.632 ± 0.002	25
$\{112\} \langle 111 \rangle$ twinning	4.643 ± 0.007	25

* + Indicates 90% confidence limits.

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UPPER-BOUND YIELD LOCUS CALCULATIONS

Because it is based on assumptions that ensure strain compatibility, the Bishop and Hill analysis can be used to calculate upper-bound yield loci for textured metals. (The necessary assumptions for upper-bound solutions are discussed in Chapter 5.)

4.1. Single component textures

Example 4.1

The method is illustrated by an example for a single orientation of crystal. Consider a sheet with a $[1\bar{1}2](110)$ texture and let $x = [1\bar{1}2]$, $y = [\bar{1}11]$, and $z = [\bar{1}\bar{1}0]$ as illustrated in Fig. 4.1. It will be assumed that the sheet has twofold rotational symmetry about x , y , and z typical of rolled sheets,⁷ and that loading is restricted to σ_x and σ_y ; that is, $\sigma_z = \tau_{yz} = \tau_{zx} = \tau_{xy} = 0$. Under such loading the symmetry will ensure that $\gamma_{yz} = \gamma_{zx} = \gamma_{xy} = 0$. The calculation procedure is to assume several external shape changes, $d\epsilon_x$, $d\epsilon_y$, and $d\epsilon_z$, transform these strains onto the crystallographic axes and then use the Bishop and Hill analysis to find the appropriate stress states, $A, B, \dots, H$. Finally the stresses are transformed to the external coordinate system. Using the direction cosines (Table 4.1), the assumed strains are transformed onto the cubic axes [Eq. 4.1].

$$\left. \begin{aligned} d\epsilon_1 &= \frac{1}{6} d\epsilon_x + \frac{1}{3} d\epsilon_y + \frac{1}{2} d\epsilon_z \\ d\epsilon_2 &= \frac{1}{6} d\epsilon_x + \frac{1}{3} d\epsilon_y + \frac{1}{2} d\epsilon_z \\ d\epsilon_3 &= \frac{2}{3} d\epsilon_x + \frac{1}{3} d\epsilon_y \\ d\gamma_{23} &= -\frac{2}{3} d\epsilon_x + \frac{2}{3} d\epsilon_y \\ d\gamma_{31} &= \frac{2}{3} d\epsilon_x - \frac{2}{3} d\epsilon_y \\ d\gamma_{12} &= -\frac{1}{3} d\epsilon_x - \frac{2}{3} d\epsilon_y + d\epsilon_z \end{aligned} \right\} \quad (4.1)$$

⁷ Actually this requires the presence of a second orientation: $x = [1\bar{1}2]$, $y = [\bar{1}11]$, and $z = [\bar{1}\bar{1}0]$.

UPPER-BOUND YIELD LOCUS CALCULATIONS

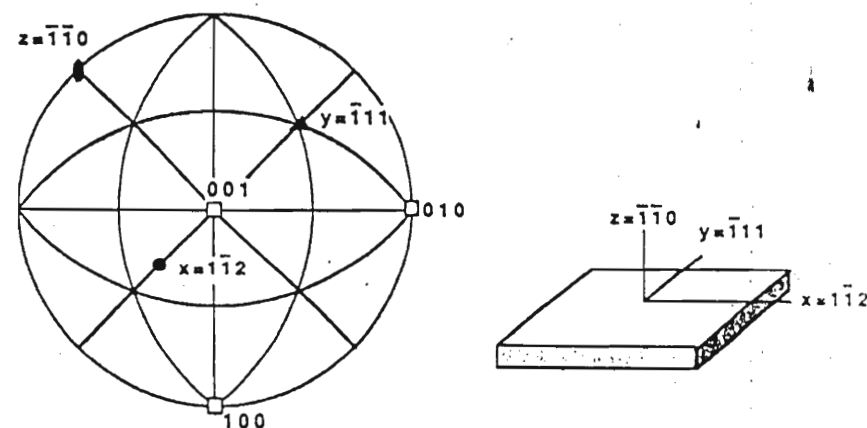


Fig. 4.1. Orientations in a $\{110\}\langle 112 \rangle$ texture.

Now assuming several strain states, Eqs. (4.1) can be substituted into the basic Bishop and Hill relation,

$$\frac{dw}{\tau} = \sqrt{6}(A d\epsilon_2 - B d\epsilon_1 + F d\gamma_{23} + G d\gamma_{31} + H d\gamma_{12}). \quad (4.2)$$

The resulting equations are given in Table 4.2. For each strain state the values of dw are calculated for all permissible Bishop and Hill stress states,

Table 4.1. Direction cosines

	$x = [1\bar{1}2]$	$y = [\bar{1}11]$	$z = [\bar{1}\bar{1}0]$
1 = $[100]$	$1/\sqrt{6}$	$-1/\sqrt{3}$	$-1/\sqrt{2}$
2 = $[010]$	$-1/\sqrt{6}$	$1/\sqrt{3}$	$-1/\sqrt{2}$
3 = $[001]$	$2/\sqrt{6}$	$1/\sqrt{3}$	0

Table 4.2. Shape change

Case	$d\epsilon_x$	$d\epsilon_y$	$d\epsilon_z$	$dw/(\sqrt{6}\tau)$
I	1	-1	0	$\frac{1}{6}(B-A) - \frac{1}{3}F + \frac{1}{3}G + \frac{1}{3}H$
II	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{6}(B-A) - F + G - \frac{1}{3}H$
III	1	0	-1	$\frac{1}{6}(B-A) - \frac{1}{3}F + \frac{1}{3}G - \frac{1}{3}H$
IV	$\frac{1}{2}$	$\frac{1}{2}$	-1	$\frac{1}{6}(B-A) - \frac{1}{3}H$
V	0	1	-1	$\frac{1}{6}(B-A) + \frac{1}{3}F - \frac{1}{3}G - \frac{1}{3}H$

Table 4.3. Calculated values of $dw/\sqrt{6}\tau$ for several shape changes

No.	Stress state					Calculated $dw/\sqrt{6}\tau$				
	A	B	F	G	H	I	II	III	IV	V
1	1	-1	0	0	0	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$
4	0	0	1	0	0	$-\frac{1}{3}$	-1	$-\frac{1}{3}$	0	$\frac{1}{3}$
5	0	0	0	1	0	$\frac{1}{3}$	1	$-\frac{1}{3}$	0	$-\frac{1}{3}$
6	0	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$
13	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{12}$	$-\frac{1}{12}$	-1	$-\frac{1}{12}$	$-\frac{1}{12}$
18	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{12}$	$-\frac{1}{12}$	$-\frac{1}{12}$	$-\frac{1}{12}$	$-\frac{1}{12}$
19	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{12}$	$-\frac{1}{12}$	$-\frac{1}{12}$	$-\frac{1}{12}$	$\frac{1}{12}$
22	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{12}$	$\frac{1}{12}$	1	$\frac{1}{12}$	$-\frac{1}{12}$
26	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$
27	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$	0	$-\frac{1}{6}$	$-\frac{1}{6}$
28	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$

using the permissible combinations of $A, B, \dots, H$ from Table 3.4. The results are summarized in Table 4.3. For brevity, calculations are shown only for 11 of the 28 stress states. The absolutely largest (and therefore appropriate) values of $dw/\sqrt{6}\tau$ are underlined.

Only 4 stress states have been found. Using the direction cosines in Table 4.1, the external stresses are:

$$\left. \begin{aligned}
 \sigma_x &= \frac{1}{6}\sigma_1 + \frac{1}{6}\sigma_2 + \frac{2}{3}\sigma_3 - \frac{1}{3}\sigma_{23} + \frac{1}{3}\sigma_{31} - \frac{1}{3}\sigma_{12} \\
 \sigma_y &= \frac{1}{3}\sigma_1 + \frac{1}{3}\sigma_2 + \frac{1}{3}\sigma_3 + \frac{2}{3}\sigma_{23} - \frac{1}{3}\sigma_{31} - \frac{1}{3}\sigma_{12} \\
 \sigma_z &= \frac{1}{3}\sigma_1 + \frac{1}{3}\sigma_2 + \sigma_{12} \\
 \sigma_{yz} &= -\frac{1}{6}\sigma_1 + \frac{1}{6}\sigma_2 - \frac{1}{6}\sigma_{23} - \frac{1}{6}\sigma_{31} \\
 \sqrt{3}\sigma_{zx} &= -\frac{1}{2}\sigma_1 + \frac{1}{2}\sigma_2 - \sigma_{23} - \sigma_{31} \\
 \sqrt{2}\sigma_{xy} &= -\frac{1}{3}\sigma_1 - \frac{1}{3}\sigma_2 + \frac{2}{3}\sigma_3 + \frac{1}{3}\sigma_{23} - \frac{1}{3}\sigma_{31} + \frac{2}{3}\sigma_{12}
 \end{aligned} \right\} \quad (4.3)$$

The brevity, $\sigma_x, \sigma_y, \dots, \sigma_{xy}$, in the following calculations are expressed in units of $\sqrt{6}\tau$. For example, $\sigma_x = 1$ means $\sigma_x = \sqrt{6}\tau$.

For stress state 27 ($F = -\frac{1}{2}, G = H = \frac{1}{2}, A = B = 0$),

$$A = \sigma_2 - \sigma_3 = 0 \text{ and } B = \sigma_3 - \sigma_1 = 0 \text{ means that } \sigma_1 = \sigma_2 = \sigma_3.$$

Also $G = H = \sigma_{31} = \sigma_{12} = \frac{1}{2}$ and $F = \sigma_{23} = -\frac{1}{2}$.

Substituting and setting $\sigma_z = 0$ in Eqs. (4.3), $\sigma_1 = \sigma_2 = \sigma_3 = -\frac{1}{2}$, and $\sigma_x = 0, \sigma_y = -\frac{1}{2}, \sigma_{yz} = \sigma_{xy} = 0$.

For stress state -6, ($A = B = F = G = 0, H = -1$), similarly, $\sigma_{23} = \sigma_{31} = 0$ and $\sigma_{12} = -1$. Setting $\sigma_z = 0$ leads to

$$\sigma_1 = \sigma_2 = \sigma_3 = 1, \text{ so } \sigma_x = \frac{4}{3}, \sigma_y = \frac{2}{3}, \sigma_{yz} = \sigma_{zx} = 0, \text{ and } \sigma_{xy} = \frac{-\sqrt{2}}{3}.$$

For stress state -26 ($A = B = 0, F = H = -\frac{1}{2}, G = \frac{1}{2}$),

$$\sigma_x = \frac{4}{3}, \sigma_y = \frac{1}{6}, \sigma_{yz} = \sigma_{zx} = 0, \text{ and } \sigma_{xy} = \frac{-\sqrt{2}}{3}.$$

For stress state 22 ($A = F = -\frac{1}{2}, B = G = \frac{1}{2}, H = 0$),

$$\sigma_x = 1, \sigma_y = -\frac{1}{2}, \text{ and } \sigma_{yz} = \sigma_{zx} = \sigma_{xy}.$$

Figure 4.2 is a plot of σ_y versus σ_x , with each stress state indicated as a point. In addition, lines are drawn through each point with slopes of $d\sigma_y/d\sigma_x = -d\epsilon_x/d\epsilon_y$ for the ratios of $d\epsilon_x/d\epsilon_y$ assumed in calculating that point. For example stress state -6 was found appropriate for shape changes III ($d\epsilon_x/d\epsilon_y = \infty$), IV ($d\epsilon_x/d\epsilon_y = 1$), and V ($d\epsilon_x/d\epsilon_y = 0$). Each stress state corresponds to a corner of the yield locus. When two stress states can produce the same shape change (e.g., -6 and -26 are both appropriate for $d\epsilon_y = 0$), the locus must be a straight line between them with a slope corresponding to the shape change.

These calculations thus establish most of the locus, but there is an uncertainty between stress states -6 and -27. There are two possibilities:

1. The locus is a straight line connecting -6 and -27; or
2. One or more additional stress states may be appropriate in this region.

If the first possibility is correct, stress states -6 and -27 would both be appropriate for the shape change corresponding to the slope between them, namely $d\epsilon_y/d\epsilon_x = -d\sigma_x/d\sigma_y = -8$. Now, repeating the foregoing procedure with $d\epsilon_x = -\frac{1}{8}, d\epsilon_y = 1$, and $d\epsilon_z = -\frac{1}{8}$, it is found that stress states -6 and -27 are both appropriate; so there is a single straight line in this region. The active slip systems can also be determined at each point on the yield locus. At each corner there are either 6 or 8 potential slip systems, but on the straight segments only the slip systems common to both stress states at the adjacent corners will be operative, as illustrated in Table 4.4. The values

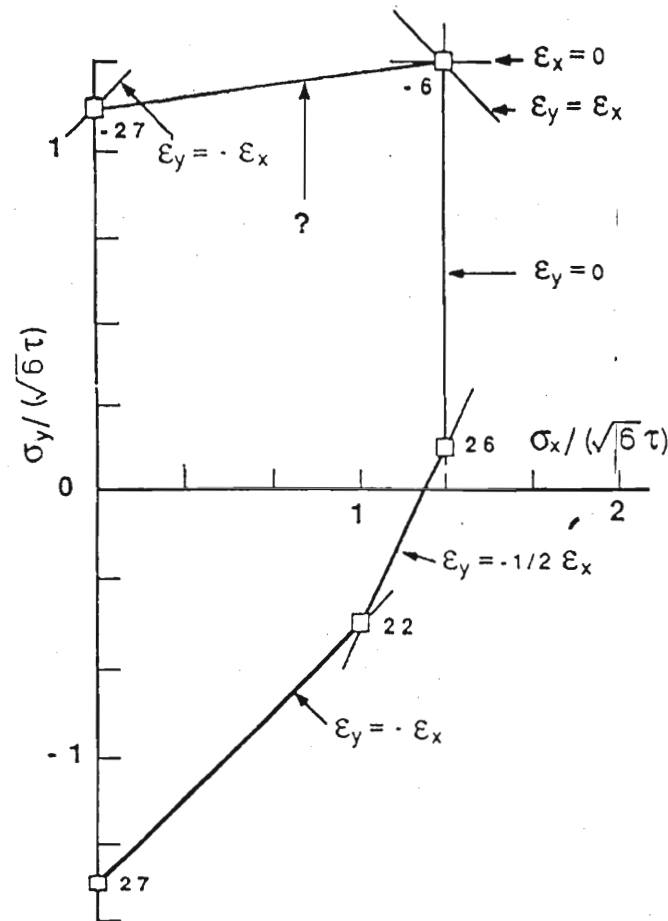


Fig. 4.2. Yield locus calculated for a $[1\bar{1}2](110)$ texture. Here $x = [1\bar{1}2]$ and $y = [\bar{1}11]$. The points are the appropriate Bishop and Hill stress states and the tangents correspond to the assumed shape changes.

appropriate stress states, as shown in Fig. 4.3. This is because the appropriate stress states must correspond to the largest values of dw . This suggests an alternative method of constructing yield loci. The values of σ_x and σ_y could be calculated and plotted for all of the stress states, and the locus drawn so that it circumscribed all the points.

It may be noted that for this yield locus, $\sigma_{yz} = \sigma_{zx} = 0$ for all stress states, but σ_{xy} is finite for stress states -6 and -26 . If a similar calculation had been made for the mirror image orientation with $y = [1\bar{1}\bar{1}]$, the resulting

Table 4.4. Active slip systems

Stress state	Active slip systems
corner -27	$+a_2, -a_3, -b_1, +b_3, +d_1, -d_2$
between -27 and -6	$-a_2, -b_1, +d_1, -d_2$
corner -6	$-a_1, +a_2, -b_1, +b_2, +c_1, -c_2, +d_1, -d_2$
between -6 and -26	$-a_1, +b_2, +c_1, -c_2$
corner -26	$-a_1, +a_3, +b_2, -b_3, +c_1, -c_2$
between -26 and $+22$	$-a_1, +a_3, +b_2, -b_3$
corner $+22$	$-a_1, +a_3, +b_2, -b_3, -d_1, +d_2$
between $+22$ and $+27$	$+a_3, -b_3, -d_1, +d_2$
corner $+27$	$-a_2, +a_3, +b_1, -b_3, -d_1, +d_2$

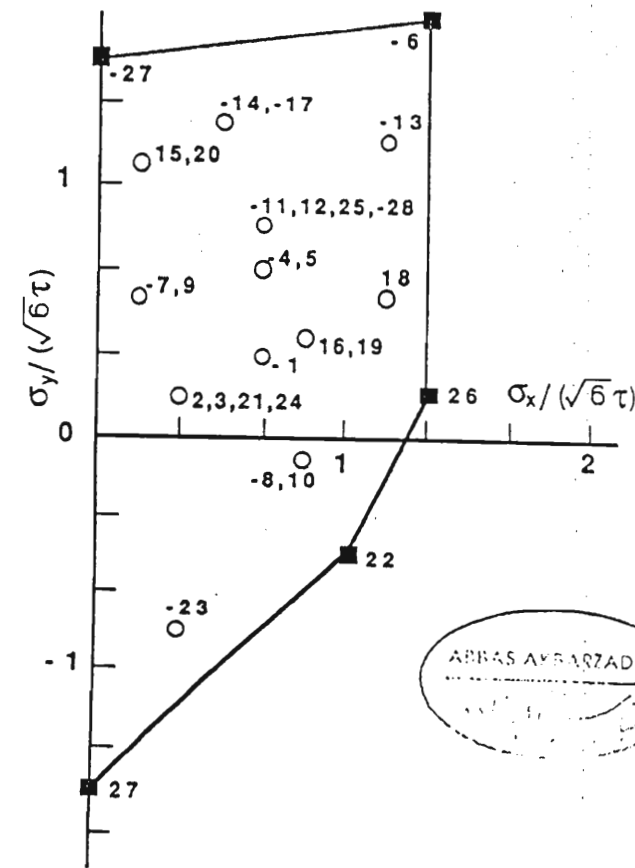


Fig. 4.3. Plot of all of the Bishop and Hill stress states (with $\sigma_z = 0$) for the $[1\bar{1}2](110)$ texture. The inappropriate stress states fall inside the locus drawn through the outermost points.

locus would have the same shape but the signs of the shear stress terms σ_{xy} would have been opposite. In a real sheet with texture, both orientations are present in equal amounts; so there would be no external shear stresses, σ_{xy} , even though the shear stress would be finite in individual grains.

4.2. Yield loci for general orientations

The orientations of crystals usually cannot be expressed by simple Miller indices, but any orientation can be fully specified by three angles. The relation between the 1,2,3 crystal axes and an x,y,z set of external axes can be described by three rotations, as indicated in Fig. 4.4. First a clockwise rotation ϕ about the 2 axis followed by a clockwise rotation θ about the 1 axis form an axis set x',y',z' . Finally a clockwise rotation α about the $z' = z$ axis produces the x,y,z axis set. Transformations of strains between the 1,2,3 and x,y,z axes are easily made in two steps using the direction cosines in Tables 4.5 and 4.6. Other systems of describing orientations in terms of three angles are described in Appendix 3.

4.3. Yield loci for multiple orientations

Upper-bound yield loci can be computed for polycrystals consisting of more than one textural component by averaging the behavior of the components. The averaging must be made for a specific shape change because all grains

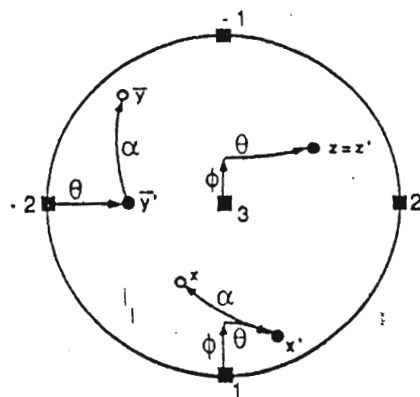


Fig. 4.4. Stereographic representation of the angular relations between the external axes x,y,z and the crystal axes 1,2,3. Negative signs indicate the opposite (negative) end of the direction.

Table 4.5. Direction cosines between 1,2,3 and x',y',z' axes

	1	2	3
x'	$\cos \phi$	$\sin \theta \sin \phi$	$\cos \theta \sin \phi$
y'	0	$\cos \theta$	$-\sin \theta$
z'	$-\sin \phi$	$\sin \theta \cos \phi$	$\cos \theta \cos \phi$

Table 4.6. Direction cosines between x',y',z' and x,y,z axes

	x'	y'	z'
x	$\cos \alpha$	$-\sin \alpha$	0
y	$\sin \alpha$	$\cos \alpha$	0
z	0	0	1

are assumed to deform with the same strains.

$$\begin{aligned}\bar{\sigma}_x &= f_a(\sigma_x)_a + f_b(\sigma_x)_b + \dots \\ \bar{\sigma}_y &= f_a(\sigma_y)_a + f_b(\sigma_y)_b + \dots\end{aligned}\quad (4.4)$$

where $f_a, f_b, \dots$ are the volume fractions of orientations $a, b, \dots$ and $(\sigma_x)_a, (\sigma_y)_a$, etc. are the calculated stresses in orientation a .

To illustrate this, consider a material with two orientations: a , with $x = [001]$ and $y = [110]$; and b , with $x = [110]$ and $y = [00\bar{1}]$ present in equal amounts. For simplicity, it is assumed that $\sqrt{6}\tau = 1$. The individual yield loci are shown in Fig. 4.5. The shape change ($0 \geq d\epsilon_y/d\epsilon_x \geq -1$) requires the stresses $\sigma_x = 2, \sigma_y = 1$ in orientation a and $\sigma_x = 1, \sigma_y = 2$ in orientation b . For the polycrystal, this shape change requires the average of those, $\sigma_x = \frac{3}{2}$ and $\sigma_y = \frac{3}{2}$. It is interesting to note that this stress state lies outside both individual loci.

More complex mixtures can be handled by computer calculations. The stress state for uniaxial tension has to be found by an iterative process since the shape change $\rho = -d\epsilon_y/d\epsilon_x$ is not known at the onset. Calculations are repeated for different assumed ratios of ρ until σ_y reaches zero. The R value equals $\rho^*/(1 - \rho^*)$, where ρ^* is this value of ρ .

With the assumptions used, the calculated yield loci for individual orientations are always bounded by straight sides meeting at corners. With polycrystals, the number of straight sides and corners increases as the number of orientations increases.

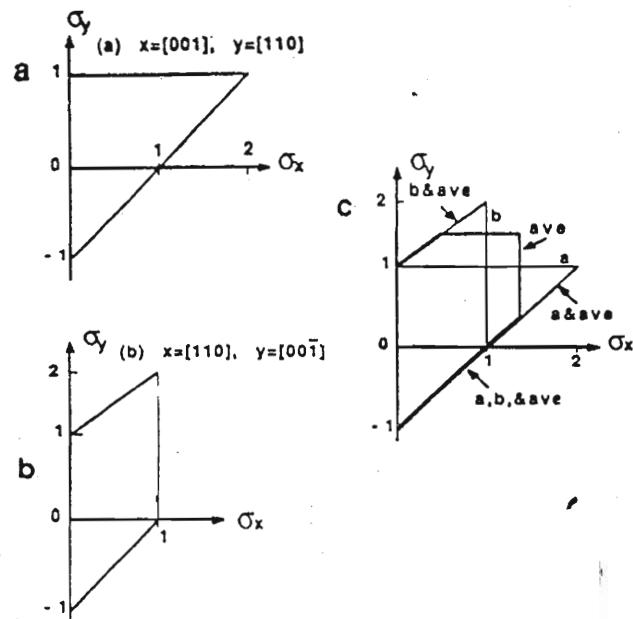


Fig. 4.5. Yield loci calculated for orientations with (a) $x = [100]$, $y = [110]$ and (b) $x = [110]$, $y = [001]$ (left). At the right is the yield locus for a texture with both components. It is assumed that $\sqrt{6}\tau = 1$.

becomes very large can the yield locus shape be approximated by a smooth curve.

4.4. Yield loci for rotationally symmetric textures

It is of interest to consider yield loci for sheets with textures that are rotationally symmetric about the sheet normal. Such textures may be approximated for a given sheet normal (fixed values of θ and ϕ) by considering orientations differing by enough small increments of rotation, $\Delta\alpha$, about the sheet normal to represent all orientations. In general 180° of rotation is required, but this is reduced by symmetry for special directions normal to the sheet (45° is required for $[100]$, 30° for $[111]$, and 90° for $[110]$). Figure 4.6 shows the upper-bound yield loci calculated for fcc metals with textures that are rotationally symmetric about $[100]$, $[110]$, and $[111]$ sheet normals. In each case there is a corner at biaxial tension. Figure 4.7

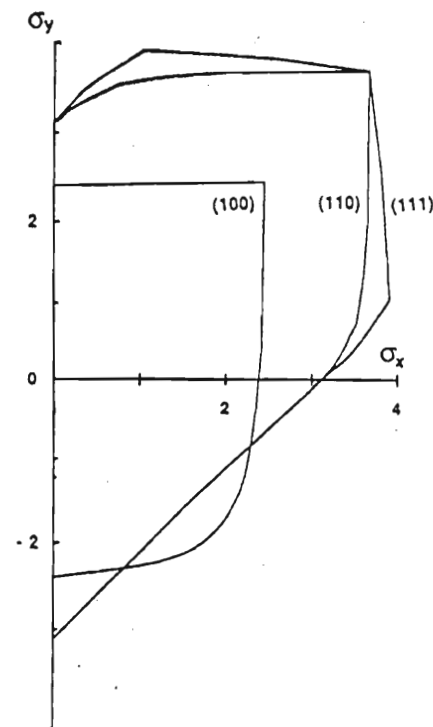


Fig. 4.6. Upper-bound yield loci calculated for fcc metals having rotationally symmetric textures with $[100]$, $[110]$, and $[111]$ sheet normals. The values of σ_x and σ_y are normalized by τ .

$\langle 111 \rangle$ -pencil glide. Note that there are corners at $\sigma_x = \sigma_y$ for fcc slip but not for pencil glide.

4.5. Yield loci for randomly oriented material

An isotropic material can be approximated by a large number of randomly oriented grains. However, some care must be taken to assure a representative sample of orientations. As already noted, any orientation can be described by three angles, θ , ϕ , and α . For a cubic crystal a random orientation can be found by selecting θ randomly between 0 and 45° , α randomly between 0 and 180° , and $\sin \phi$ randomly between 0 and 1 . This will cover three basic stereographic triangles. (Alternatively, for coverage of a single stereographic triangle, $\sin \phi$ can be chosen randomly between 0 and 1 with the restriction that $\tan \phi \leq \sin \theta$, so that whenever this restriction is violated the orientation

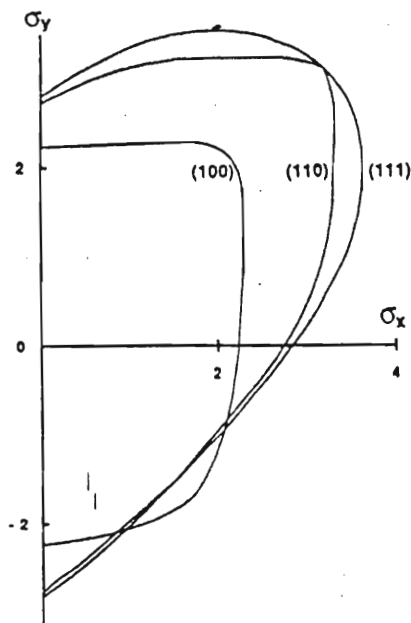


Fig. 4.7. Upper-bound yield loci for rotationally symmetric textures with [100], [110], and [111] sheet normals calculated assuming $\langle 111 \rangle$ -pencil glide. The values of σ_x and σ_y are normalized by τ .

Note that if a number of ϕ values were randomly chosen between 0 and 90° , the density of poles would be much higher near the north pole ($\phi = 90^\circ$) than near the equator ($\phi = 0^\circ$), whereas choosing $\sin \phi$ randomly eliminates this bias. Consider a hemisphere of unit radius as shown in Fig. 4.8. The circumference of a line of latitude at ϕ is $2\pi \cos \phi$; so the surface area between latitudes of ϕ and $\phi + d\phi$ is $2\pi \cos \phi d\phi$. The spherical area between latitudes of ϕ_1 and ϕ_2 is

$$\Delta A = \int \cos \phi d\phi = \sin \phi_2 - \sin \phi_1. \quad (4.5)$$

Therefore, choosing the latitude intervals so that each interval corresponds to the same $\Delta \sin \phi$ increment results in equal increments of spherical area.

There are several possible strategies for approximating an isotropic material:

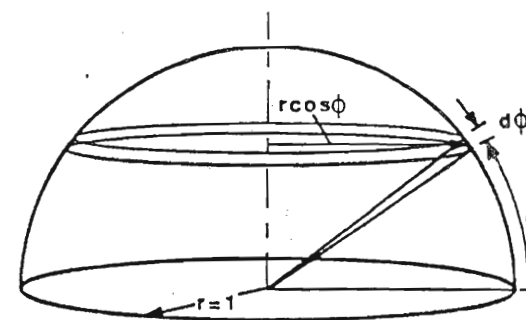


Fig. 4.8. Hemisphere of unit radius. The area between ϕ and $\phi + d\phi$ equals $2\pi \cos \phi d\phi$.

1. Randomly select θ , $\sin \phi$, and α as described here. The disadvantage is that a very large number of orientations are needed to ensure a reasonable sampling of orientations.
2. Use small increments of θ (e.g., 5°) between 0 and 45° . For each θ -value increment $\sin \phi$ from 0 to 1 by small increments (e.g., 0.1) and for each θ - ϕ combination increment α from 0 to 180° (e.g., by 18°) so that the orientations chosen cover the entire range. The disadvantage of this procedure is that the results will depend on the first orientation and the weight given to the extreme orientations.
3. Choose each value of θ , $\sin \phi$, and α randomly within a small grid element [e.g., of size $\Delta\theta = 5^\circ$, $\Delta(\sin \phi) = 0.1$, and $\Delta\alpha = 18^\circ$] and use the incrementation scheme in (2) to choose the bounds of the grid elements. This procedure will assure sampling over all of the spherical area of interest without being biased by an arbitrary starting point. Furthermore, the error can be estimated by repeating the calculations with new random numbers.

Table 4.7 and Fig. 4.9 give the isotropic upper-bound yield loci calculated by strategy (3) for $\{111\}\langle 110 \rangle$ fcc slip [1] and for $\langle 111 \rangle$ -pencil glide in bcc metals [2]. Both loci lie between those for the von Mises and Tresca criteria. Calculations were made only for imposed strain paths $0 \leq \rho \leq 0.5$, but because of isotropy, the behavior along other loading paths can be deduced from these. Each loading path in the range $-1 \leq \rho \leq 0$ has an equivalent loading path, $\rho' = \rho + 1$ in Table 4.7. The corresponding stresses, σ_x and σ_y , on the yield locus are given by $\sigma_x = \sigma'_x$ and $\sigma_y = \sigma'_y - \sigma'_x$, where σ'_x and σ'_y are the stresses listed in Table 4.5. For example, consider the path, $\rho = -0.75$. The equivalent path is $\rho' = +0.25$, and for the fcc case, $\sigma_x = \sigma'_x = 3.228\tau$ and $\sigma_y = \sigma'_y - \sigma'_x = (3.228 - 0.439)\tau = 2.789\tau$. Similarly,

Table 4.7. Calculated yield loci for fcc and bcc metals ($\sigma_z = 0$)

Strain path $p = -d\epsilon_z/d\epsilon_x$	fcc [111]<110> slip			bcc <111>-pencil glide	
	σ_x/τ	σ_y/τ	σ_z/τ	σ_x/τ	σ_y/τ
0.500	3.067 ± 0.001	0.000	2.740 ± 0.002	0.000	2.283
0.400	3.156 ± 0.004	0.207 ± 0.006	2.834 ± 0.012	0.468	0.728
0.375	3.228 ± 0.004	0.439 ± 0.011	2.928 ± 0.007	1.115	0.000
0.300	3.284 ± 0.002	0.748 ± 0.011	3.040 ± 0.016	3.0647 ± 0.009	1.539 ± 0.015
0.200	3.313 ± 0.006	1.080 ± 0.036	1.651 ± 0.021		
0.125					
0.100					
0.050					
0.000					

are given by $p' = 1 - p$, and the yield loci stresses by $\sigma'_x = \sigma_x - \sigma_y$ and $\sigma'_y = -\sigma_y$.

4.6. Calculations including strain hardening

It is possible to include strain hardening in the calculations [3] by assuming that the shear stress for slip depends on the prior strain. For example, with power-law strain hardening,

$$\tau_i = k\gamma_i^n \quad (4.6)$$

The work per volume to deform an individual grain, w_i , is

$$w_i = \int \tau_i d\gamma_i = \frac{k\gamma_i^{n+1}}{n+1} \quad (4.7)$$

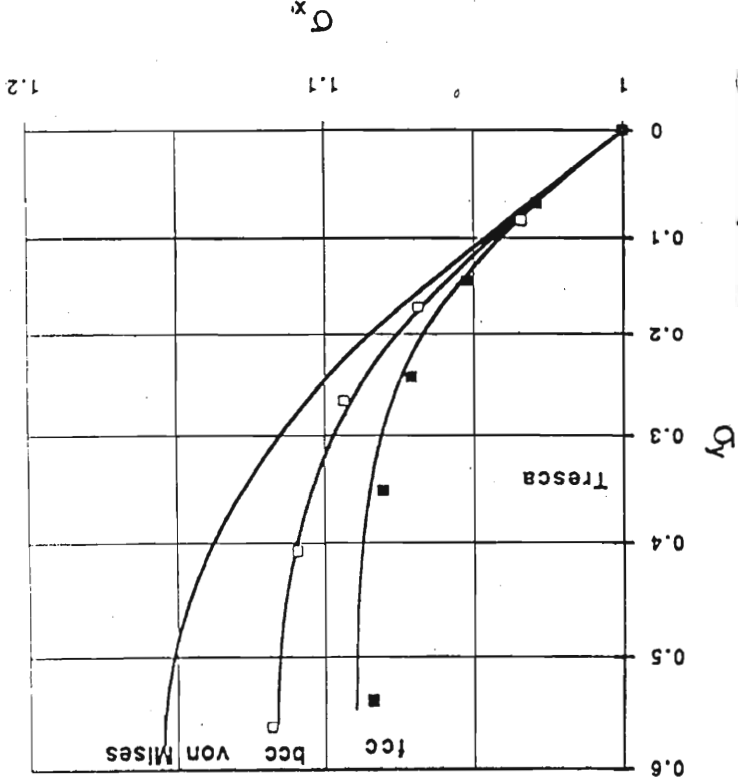
Substituting $\gamma_i = M_i \epsilon_x$,

$$w_i = \frac{kM_i^{n+1} \epsilon_x^{n+1}}{(n+1)} \quad (4.8)$$

For a polycrystal containing g grains, all deformed to the same strain, ϵ_x , the average work per volume is

$$w = \frac{1}{g} \sum w_i = \frac{kM^{n+1} \epsilon_x^{n+1}}{n+1} \quad (4.9)$$

Fig. 4.9. Calculated upper-bound yield loci for randomly oriented fcc metals deforming by <110><111> slip and bcc metals deforming by <111>-pencil glide. The von Mises and Tresca criteria are shown for comparison.



UPPER-BOUND YIELD LOCUS CALCULATIONS

The values of $(\sigma_x/\tau)_i$ and $(\sigma_y/\tau)_i$ for plane stress ($\sigma_z = 0$) can be found for each shape change and orientation using the BH analysis.⁸

$$\epsilon_x = \left(\frac{\sum w_i^{1/(n+1)}}{w^*(g/k)(n+1)} \right)^{1/(n+1)} \quad (4.10)$$

where the summation is over all g grains. Yielding can be defined by w reaching a critical value, w^* . The strain at yielding is then

Note that M_i is defined here as $M_i = (1/\tau_i)(dw_i/d\epsilon_x) = \gamma_i/\epsilon_x$. In general M_i does not equal σ_i/τ_i and $\sum M_i \neq 1$.

$$\begin{aligned}\sigma_{xi} &= \left(\frac{\sigma_x}{\tau}\right)_i \tau_i = \left(\frac{\sigma_x}{\tau}\right)_i k M_i^n \dot{\epsilon}_x^n \\ &= \left(\frac{\sigma_x}{\tau}\right)_i k M_i^n \left(\frac{(g/k)(n+1)w^*}{\sum_i M_i^{n+1}}\right)^{n/(n+1)},\end{aligned}\quad (4.11)$$

and the average value of σ_x is given by

$$\sigma_x = A \sum_i \frac{(\sigma_x/\tau)_i M_i^n}{\left(\sum_i M_i^{n+1}\right)^{n/(n+1)}}, \quad (4.12)$$

where A is the same for all grains

$$A = (k/g)^{1/(n+1)} [w^*(n+1)]^{n/(n+1)}.$$

Similarly,

$$\sigma_y = A \sum_i \frac{(\sigma_y/\tau)_i M_i^n}{\left(\sum_i M_i^{n+1}\right)^{n/(n+1)}}. \quad (4.13)$$

Note that the shape of the yield locus is independent of the level of w^* chosen to define yielding. An interesting feature calculated this way (with strain hardening) is that the corners of the yield locus are rounded. This is illustrated in Fig. 4.10 by the example of a texture composed of two orientations. With no strain hardening ($n=0$), there are sharp corners at $\sigma_x = 1.5$, $\sigma_y = 1.0$ and $\sigma_x = 1.5$, $\sigma_y = 0.5$, and these corners become more rounded as n is increased.

For $n=0$ (no strain hardening) Eqs. (4.12) and (4.13) reduce to the usual

$$\sigma_x = A \sum_i \left(\frac{\sigma_x}{\tau}\right)_i \quad (4.14)$$

$$\sigma_y = A \sum_i \left(\frac{\sigma_y}{\tau}\right)_i. \quad (4.15)$$

Assuming a strain-rate dependence of τ (i.e., $\tau = k\dot{\epsilon}^m$) has a similar effect in that otherwise sharp corners become rounded with finite values of m

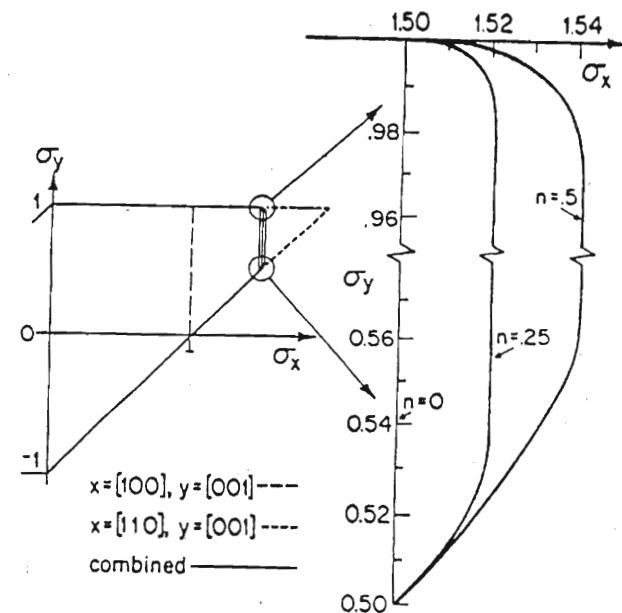


Fig. 4.10. The effect on strain hardening on the corners of a yield locus of a polycrystal composed of two orientations. The enlarged view of the two corners (illustrated at the right) shows that the corners become rounded with strain hardening.

4.7. A rate-sensitive model

Asaro and Needleman have developed a rate-sensitive model based on the Taylor assumptions of homogeneous strain. Every slip system is assumed to be active, with a shear strain rate that depends on the current stress, $\dot{\epsilon} = c\tau^m$, and a hardness parameter that is strain dependent. With even a low level of m (0.005 or 0.01), the ambiguities of the Bishop-Hill and Taylor models caused by the nonuniqueness of active slip systems are absent, so that rotations can be predicted without ambiguity. The model includes elastic deformation, which allows the stress within a crystal to be related to the strain. The calculations iterate so as to give stresses and strains that are self-consistent. It would be interesting to check how closely the stress states in the grains, predicted by this analysis, compare with the Bishop-Hill stress states. The author expects that the comparison would be close. In that case, an alternative would be to assume the Bishop-Hill stress states, and use them to determine the relative operation of each slip system by assuming

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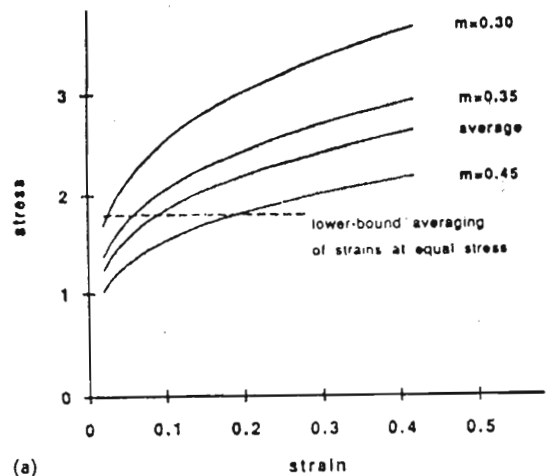
LOWER-BOUND YIELD LOCUS CALCULATIONS

5.1. Introduction

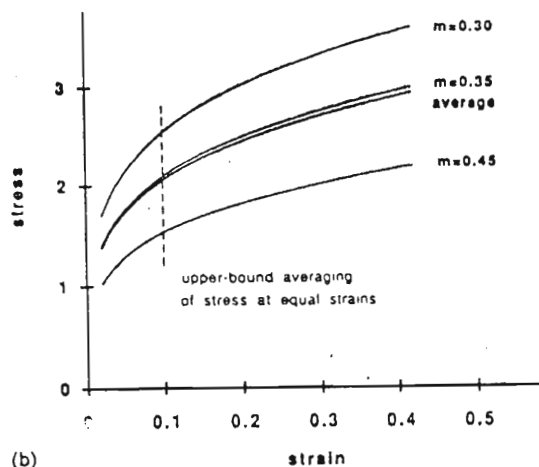
Upper- and lower-bound analyses are based on well-established principles [1, 2]. For a valid upper bound, a *geometrically* self-consistent internal deformation field (one that is *kinematically* admissible) must be assumed. The external stresses are calculated from the internal work necessary to produce such deformation. The Taylor and Bishop and Hill model is an upper-bound model in which the geometrically self-consistent deformation field is assured by assuming that every grain undergoes the same shape change. The stress is averaged over all grains. However, this model is not exact because it implies different stresses in neighboring grains and hence unbalanced forces acting across grain boundaries.

Lower-bound analyses are quite different. The lower-bound theorem requires a self-consistent internal *stress* field (one that is *statically* admissible). However, no regard need be taken of strain compatibility. The simplest way to assure self-consistent stresses is to assume that every grain is loaded under exactly the same stress state and that the mean (overall) strain is the average of the strains in the individual grains. Figures 5.1(a) and (b) illustrate the difference between a lower-bound average and an upper-bound average for a polycrystal containing three grains. (These figures are based on a hypothetical material for which $\tau = \gamma^{0.25}$, $\sigma = \tau/m$, and $\epsilon = m\gamma$, and the grains are assumed to differ in their values of m .)

The lower-bound model is reasonable only if work hardening is assumed. Otherwise (with no work hardening) the stress cannot rise above the level required for slip in the most favorably oriented grain (highest m); so the calculated strength of the polycrystal is identical to its strength. This is illustrated for uniaxial tension in Fig. 5.2. Clearly this is unreasonable; the softest grain may deform more than the others, but not infinitely more. Strain hardening can be incorporated into upper-bound Taylor and Bishop and Hill-type calculations [3], but it does not have a large effect on the results because the averaging is done at the same level of strain in each grain.



(a)



(b)

Fig. 5.1. (a) Stress-strain curves for three orientations of crystals with the lower-bound average for a polycrystal composed of these three orientations. (b) Stress-strain curves for three orientations of crystals with the upper-bound average for a polycrystal composed of these three orientations.

5.2. Calculation method

Calculations have been made [4] assuming power-law work hardening in each grain with

LOWER-BOUND YIELD LOCUS CALCULATIONS

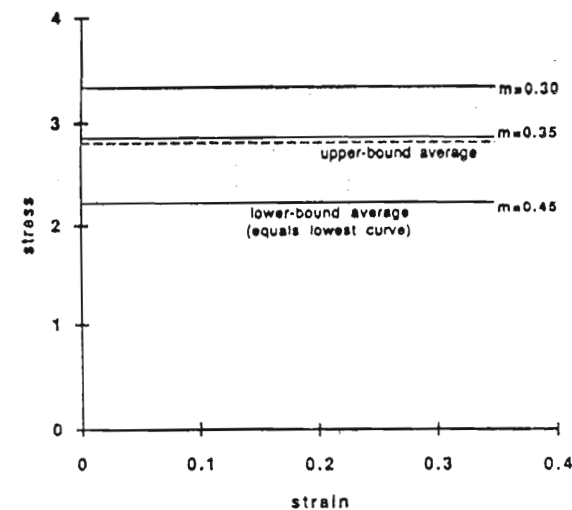


Fig. 5.2. Lower-bound average for the three orientations in Fig. 5.1 with no strain hardening.

where τ_i is the shear stress necessary to cause slip on the active slip system in that grain and γ_i is the shear strain on the slip system. With this assumption all grains make some contribution to the overall strain and therefore have some effect on the calculated yielding behavior.

Yielding is defined to occur when the plastic work per volume, w , of the polycrystal reaches a critical level, w^* . This is equivalent to defining yielding in terms of an effective strain, $\bar{\epsilon}$, since $\bar{\sigma}$ and $\bar{\epsilon}$ are constant everywhere on the yield surface and $w = \int \bar{\sigma} d\bar{\epsilon}$. This definition of yielding avoids the complication that would arise if yielding were defined as occurring when a critical level of effective strain is reached. The difficulty with that approach is that the appropriate effective strain function is not known until after the yield locus shape (and therefore effective stress function) has been determined.

The basic assumptions are:

1. A plane-strain stress state, σ_x and $\sigma_y = \alpha\sigma_x$ with $\sigma_z = 0$, $\tau_{yz} = \tau_{zx} = \tau_{xy} = 0$.
2. Slip in each grain occurs only on the system with the highest resolved shear stress with no restriction on the resulting shape change.
3. Strain hardening is described by a power law [Eq. (5.1)].
4. The textures have rotational symmetry about z .

The plastic work per volume, w_i , in an individual grain, i , is given by

$$w_i = \int \tau_i d\gamma_i = \frac{k\gamma_i^{n+1}}{n+1} = \frac{k^{-1/n}\tau_i^{(n+1)/n}}{n+1}, \quad (5.2)$$

where τ_i and γ_i are the shear stress and shear strain on the most heavily stressed slip system in each grain. For a polycrystal consisting of g grains of equal size, the overall plastic work per volume, w , is simply

$$w = \frac{1}{g} \sum_i w_i = \frac{q}{g} \sum_i \tau_i^{(n+1)/n}, \quad (5.3)$$

where $q = 1/[(n+1)k^{1/n}]$. At yielding $w = w^*$.

For a given stress ratio α , the shear stress τ_i is given by

$$\tau_i = m_{xi}\sigma_x + m_{yi}\sigma_y = \sigma_x(m_{xi} + \alpha m_{yi}). \quad (5.4)$$

Here the resolving factor is $m_{xi} = \cos \lambda_{xi} \cos \phi_{xi}$, where λ_{xi} and ϕ_{xi} are the angles from the x direction to the slip direction and slip-plane normal, respectively, of the most heavily stressed slip system. The resolving factor m_{yi} is defined similarly. Substitution of Eq. (5.4) into Eq. (5.3) gives

$$w^* = \frac{q}{g} \sum_i [\sigma_x(m_{xi} + \alpha m_{yi})]^{(n+1)/n}. \quad (5.5)$$

Because σ_x is assumed to be the same in all grains, it can be removed from the summation. Taking $w^*/q = w^*k^{1/n}(n+1)$ as unity and solving for σ_x ,

$$\sigma_x = \left[\frac{g}{\sum_i (m_{xi} + \alpha m_{yi})^{(n+1)/n}} \right]^{n/(n+1)}, \quad (5.6)$$

$$\sigma_y = \alpha \sigma_x.$$

The ratio of the resulting strains, ϵ_y/ϵ_x , can be found in a similar manner. In each grain $\epsilon_{xi} = m_{xi}\gamma_i$ and $\epsilon_{yi} = m_{yi}\gamma_i$, where $\gamma_i = (\tau_i/k)^{1/n}$ and $\tau_i = (m_{xi} + \alpha m_{yi})\sigma_x$. Substituting,

$$\epsilon_{xi} = m_{xi} \left(\frac{\sigma_x(m_{xi} + \alpha m_{yi})}{k} \right)^{1/n} \quad (5.7)$$

$$\epsilon_{yi} = m_{yi} \left(\frac{\sigma_x(m_{xi} + \alpha m_{yi})}{k} \right)^{1/n}.$$

Therefore the ratio of the external strains, $\epsilon_x = (1/g) \sum_i \epsilon_{xi}$ and $\epsilon_y = (1/g) \sum_i \epsilon_{yi}$, is

$$\frac{\epsilon_y}{\epsilon_x} = \frac{\sum_i \epsilon_{yi}}{\sum_i \epsilon_{xi}} = \frac{\sum_i [m_{yi}(m_{xi} + \alpha m_{yi})^{1/n}]}{\sum_i [m_{xi}(m_{xi} + \alpha m_{yi})^{1/n}]} \quad (5.8)$$

5.3. Results of calculations

Lower-bound calculations were made for both $\{111\}\langle 110 \rangle$ slip, which is characteristic of fcc metals, and $\langle 111 \rangle$ -pencil glide, which is characteristic of bcc metals. Both sets of calculations were made for 100 different textures with rotational symmetry about the sheet normal, assuming strain-hardening exponents of $n = 0.1, 0.25$, and 0.5 . As with the upper-bound analysis, the yielding behavior was calculated for the four characteristic loading paths illustrated in Fig. 5.3: (1) uniaxial tension ($\sigma_y = 0$), (2) plane strain ($\epsilon_y = 0$), (3) biaxial tension ($\epsilon_y = \epsilon_x$ and $\sigma_y = \sigma_x$), and (4) plane strain ($\epsilon_z = 0$, $\sigma_y = -\sigma_x$). For uniaxial tension (path 1) the strain ratio $R = \epsilon_y/\epsilon_x$ was determined as $R = \rho^*/(1 - \rho^*)$, where ρ^* is the value of $-(\epsilon_y/\epsilon_x)$ in Eq. (5.8) with $\alpha = 0$. For plane strain ($\epsilon_y = 0$) (path 2) the corresponding stress ratio, α^* , was found by iteration. Various values of α were assumed

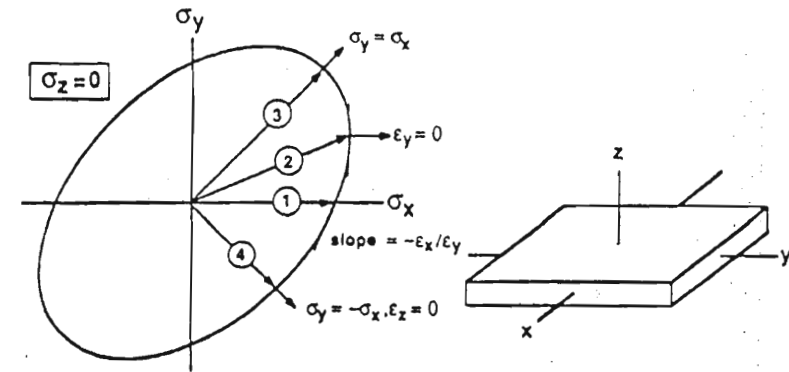


Fig. 5.3. Yield locus showing loading paths and the coordinate system.

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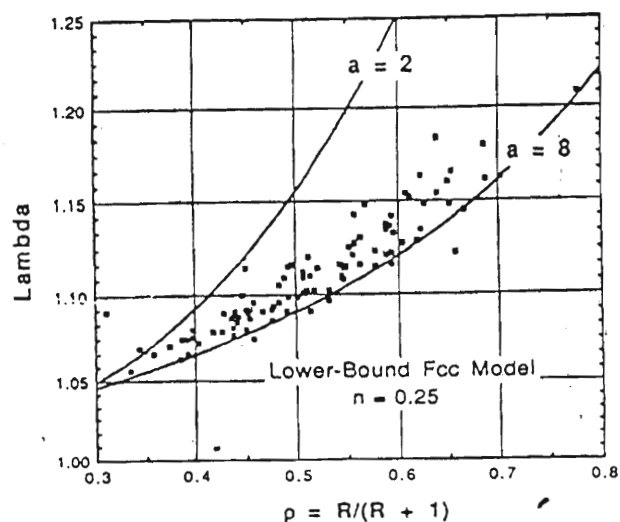


Fig. 5.4. Calculated values of λ and R for the lower-bound model with fcc assumptions and $n = 0.25$. (From [4].)

until Eq. (5.8) resulted in $\epsilon_y/\epsilon_x = 0$. For each of the assumed textures, three ratios of the yield stresses along the several loading paths, namely,

$$\beta = \frac{\sigma_{x(\text{path } 2)}}{2\sigma_{x(\text{path } 4)}},$$

$$\lambda = \frac{\sigma_{x(\text{path } 2)}}{\sigma_{x(\text{path } 1)}},$$

$$\chi = \frac{\sigma_{x(\text{path } 3)}}{\sigma_{x(\text{path } 1)}}.$$

were calculated together with the R value in uniaxial tension. Figure 5.4 is a plot of the calculated values of λ and R for the assumed textures with fcc slip and $n = 0.25$. The correlation between the calculated R values and the yield strength ratios were compared with the trends predicted by the nonquadratic yield criterion,

$$\sigma_y^a + \sigma_x^a + R(\sigma_x - \sigma_y)^a = (R+1)X^a. \quad (5.9)$$

As with the corresponding upper-bound models, the yield loci are best fit by values of a of 6 to 10 (instead

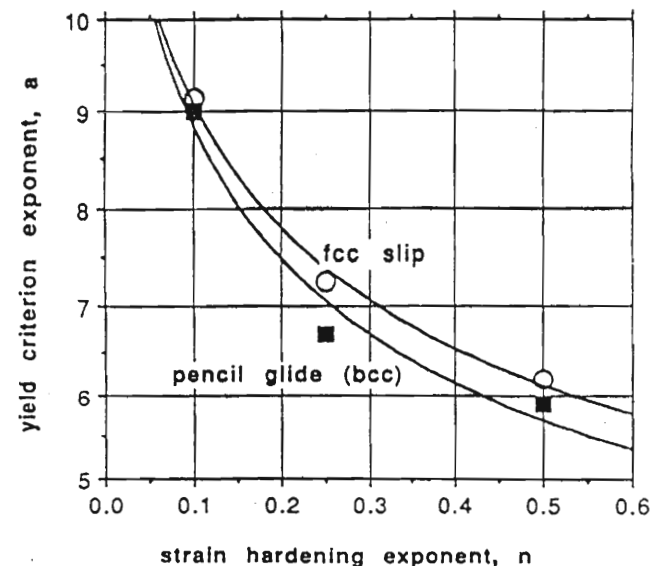


Fig. 5.5. The yield criterion exponents a that result in the best fit of calculated values of λ to those predicted by Eq. (5.9).

of 2 as in the 1948 Hill theory; see Chapter 8). The values of the exponent a were varied to find the values that gave the best fit of the parameters β , λ , and χ to those predicted from Eq. (5.9). Figure 5.5 shows that the exponents a that give the best fit for λ decrease for higher strain-hardening exponents for both the fcc and pencil-glide models. Similar trends were found for the other parameters.

5.4. Other models

Other models of polycrystalline strength have incorrectly been described as lower-bounds. The Sachs model [5] averages the reciprocal Schmid factors, $1/m$ (or axial stress level) in the various grains. This is a lower bound only for uniaxial tension of a polycrystal in which all of the grain boundaries are parallel to the stress axis. Attempts have been made to extend the Sachs model by assuming the same stress ratio, $\alpha = \sigma_y/\sigma_x$ (but not the same magnitude of the stresses) in each grain [6-10]. This assumption does not fulfill the lower-bound requirement for a statically admissible stress field; so such calculations are not lower bounds. Furthermore, normality is not assured. Figure 5.6 shows an example of how normality is violated for a polycrystal consisting of two grains: A with $x = [001]$, $y = [110]$, $z = [\bar{1}\bar{1}0]$

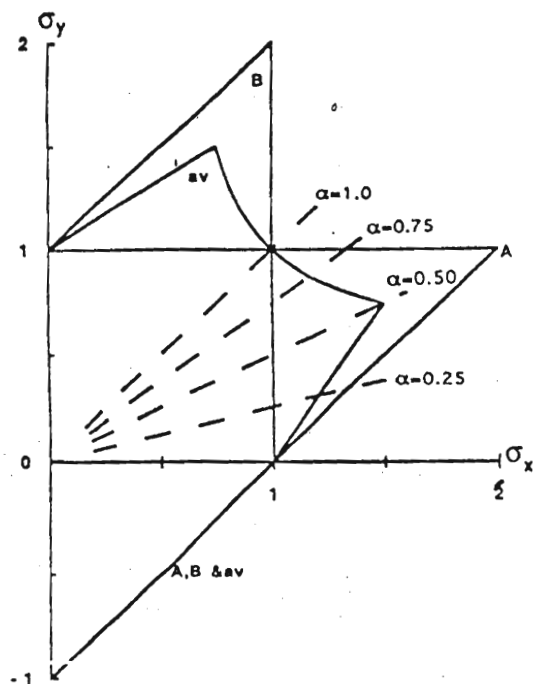


Fig. 5.6. Example showing that averaging along paths of constant stress ratio α can result in a yield locus that is outwardly concave and therefore violates the normality principle.

and B with $x = [110]$, $y = [00\bar{1}]$, $z = [\bar{1}10]$. The locus resulting from averaging the individual loci along paths of constant stress ratio is outwardly concave, which violates normality. Compare this figure with Fig. 4.5.

References

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6 RELAXED CONSTRAINTS

6.1. Number of slip systems

The Taylor assumption that five active slip systems are required in each grain of a polycrystal was first suggested by von Mises [1]. It has been argued [2, 3] that this assumption is too severe and is not necessary for compatibility. Kocks pointed out that in hcp metals twinning can contribute to achieving imposed shape changes, although because it is unidirectional a twinning system cannot substitute for a slip system in all the grains. However, in a random polycrystal, half the grains should be oriented so the direction of twinning is correct to provide the needed deformation, and deformation in these grains can accommodate the neighboring grains.

Fleischer [3, 4] studied the problem experimentally by sectioning and etching deformed brass to determine the active slip planes in each grain. From these measurements, he concluded that fewer than 5 independent systems were active in 40% of the grains.

6.2. Constraints at grain boundaries

At grain boundaries, compatibility and equilibrium impose certain constraints. Consider a grain boundary between two grains, A and B (Fig. 6.1). Let the normal to the grain boundary be z , so x and y lie in the boundary. Force equilibrium across the boundary requires that

$$\sigma_{xA} = \sigma_{xB}, \quad \tau_{yzA} = \tau_{yzB}, \quad \tau_{zxA} = \tau_{zxB},$$

and unless there is grain boundary sliding, strain compatibility requires that

$$\epsilon_{xA} = \epsilon_{xB}, \quad \epsilon_{yA} = \epsilon_{yB}, \quad \gamma_{xyA} = \gamma_{xyB}.$$

σ_x , σ_y , and τ_{xy} need not be the same in each grain and the strains γ_{yz} and γ_{zx} may be different. The strain ϵ_z is not independent unless there is a volume

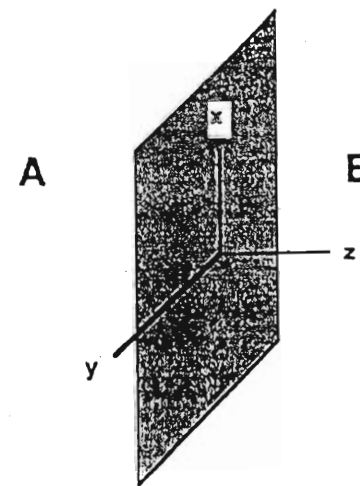


Fig. 6.1. Schematic of a grain boundary between two grains, A and B.

6.3. Deformation bands

Not all regions within a grain need deform with the same slip systems. The deformation bands form when different slip systems are active in different regions.

Dunn [5] had found that a $\{011\}\langle 100 \rangle$ single crystal of iron-3.5% silicon divided into two layers of different orientations during rolling. Chin and Wonsiewicz [6] showed that $\langle 100 \rangle$ - and $\langle 111 \rangle$ -oriented crystals in fiber-textured fcc metals may break up into regions with different combinations of slip systems. They found similar behavior during plane-strain compression of single crystals of copper and silver alloys [7], with different regions of the crystal deforming by slip on different systems so that differently oriented regions resulted.

6.4. Free surface effect

Taylor's assumption that all grains deform with the same shape change is clearly violated at and near the free surface. This is indicated by the orange peel effect, which is the development of a surface roughness on the scale of the grain size. At the free surface the compatibility requirement normal to the surface is relaxed; so fewer slip systems need operate. In uniaxial tension, those slip systems that are least efficient in producing elongation (lowest Schmid factor, $m = \cos \lambda \cos \phi$) are those that are most likely not to operate.

Therefore, the tensile stress required to cause deformation will be lower at the surface than in the interior.

One consequence of this is a dependence of the flow stress or yield strength on the ratio of grain size to specimen size [8]. For a fixed specimen size, this will result in an apparent dependence of the flow stress on grain size. Certainly this relaxation of constraint in grains at a free surface is not the only cause of the grain size effect, but it may make an important contribution when the grain size is large relative to the specimen size. Kocks [9] has pointed out that in a cylindrical tensile specimen with 100 grains per cross section, about $\frac{1}{3}$ of the grains are at the free surface. For the number of free surface grains to be reduced to 10%, the specimen diameter must be 30 to 40 times the grain diameter.

Another result of the difference between the flow stress of surface and interior grains is the development of residual stresses on unloading [10, 11]. If the surface and interior could deform independently, the lower stress in the surface would, on unloading after tensile extension, cause it to contract less elastically than the interior. Strain compatibility, however, requires that the elastic contraction of surface and center be the same; so the surface grains are left under a state of residual compression.

The phenomenon of *ridging* (also called *roping*) of ferritic stainless steels is related to the orange peel effect [12-15]. The effect is attributed to the existence of two different textural components (or groups of textural components) with grains of similar orientation being grouped in regions that are elongated during rolling. At a free surface during the deformation of stainless steel sheet, the different regions deform differently, leaving a surface roughness in the form of ridges. In ferritic steels containing about 17% Cr, the two textural component groups are orientations described as "cube-on-face" and "cube-on-corner." The "cube-on-face" components are those with $\{100\}$ parallel or nearly parallel to the plane of the sheet: $\{001\}\langle 110\rangle$, $\{117\}\langle 110\rangle$, $\{115\}\langle 110\rangle$, and $\{113\}\langle 110\rangle$, while the "cube-on-corner" components are those orientations with $\{111\}$ parallel or nearly parallel to the plane of the sheet: $\{111\}\langle 110\rangle$, $\{111\}\langle 112\rangle$, and $\{554\}\langle 225\rangle$. The cube-on-corner orientations have much higher R values, so in tension undergo much more lateral contraction than the cube-on-face orientations. This results in elongated surface roughness parallel to the prior rolling direction.

6.5. Axisymmetric flow

In a polycrystal subjected to axisymmetric flow, the tendency of an individual crystal to depart from axisymmetric flow depends on its orientation. For grains with $\langle 100\rangle$ and $\langle 111\rangle$ parallel to the extension axis, the Taylor factor for axisymmetric flow, M , is the same as the reciprocal of the Schmid factor, $1/m$, so axisymmetric flow is expected. For other orientations the ratio of $M/(1/m)$ is greater than one, as shown in Fig. 6.2. The highest ratio occurs

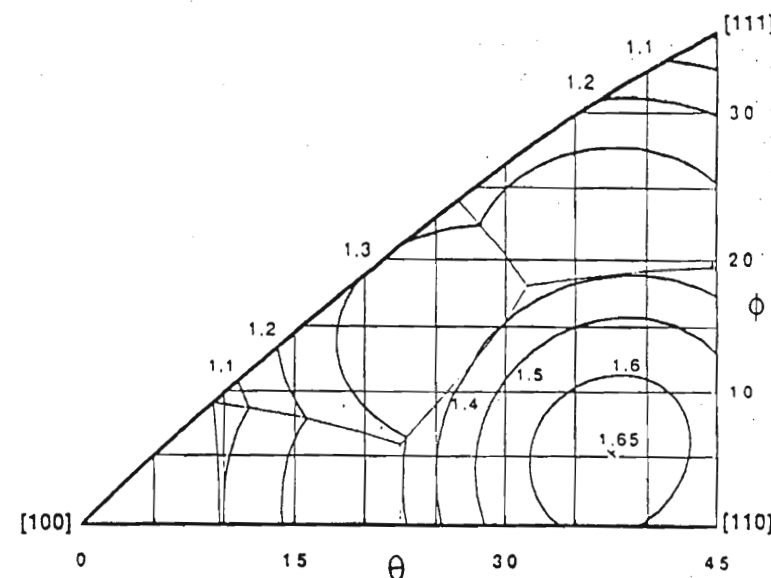


Fig. 6.2. Ratio, $M/(1/m)$, of the Taylor factor for axisymmetric flow to the reciprocal of the Schmid factor. The tendency of a grain to depart from axisymmetric flow increases with this ratio, which is 1.0 at $\langle 100\rangle$ and $\langle 111\rangle$ but is over 1.6 near $\langle 110\rangle$.

for orientations near $\langle 110\rangle$; so it is expected that these orientations would have the largest tendency to depart from axisymmetric flow.

6.6. Curly microstructures in $\langle 110\rangle$ -fiber textures

One would expect that initially equiaxed grains would become cigar shaped during axisymmetric elongation such as wire drawing. However, drawn wires of bcc metals develop a grain shape that departs significantly from this expectation. Peck and Thomas [16] found that the grains are shaped more like ribbons that are curled by bending about the wire axis. Figure 6.3 shows the microstructure of a tungsten wire viewed on a section parallel to the wire axis and on the cross section perpendicular to the wire axis. Figure 6.4 is a schematic representation of the grain shape.

These observations have been explained [17] in terms of the strong $\langle 110\rangle$ -fiber texture that is formed in wires of bcc metals and the difficulty in achieving axisymmetric flow about $\langle 110\rangle$. Let the wire axis $z = [011]$, $x = [100]$, and $y = [0\bar{1}\bar{1}]$, as shown in Fig. 6.5. For this orientation, the Taylor factor for axisymmetric elongation of the grains along the wire axis,

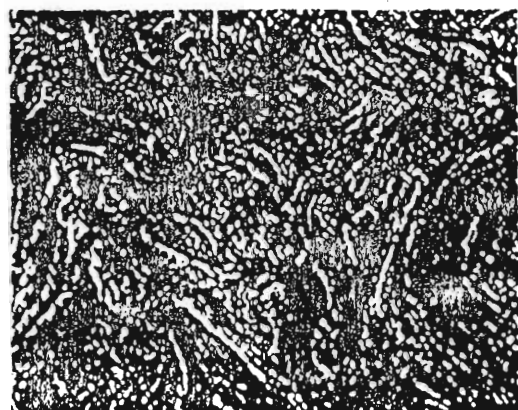


Fig. 6.3. Microstructure of a tungsten wire. A longitudinal section is shown at the above and the cross section at the below. (From Peck and Thomas [16].)

$\epsilon_x = \epsilon_y = -\frac{1}{2}\epsilon_z$, is 1.5 times the Taylor factor for plane-strain ($\epsilon_y = 0$) extension. The values of M are $\frac{1}{2}\sqrt{6}$ versus $\sqrt{6}$ for $\{111\}\langle 110 \rangle$ slip or $\frac{1}{2}\sqrt{2}$ versus $\frac{1}{2}\sqrt{2}$ for $\langle 111 \rangle\{112\}$ slip or $\langle 111 \rangle$ -pencil glide. This difference causes a strong propensity for plane-strain flow within individual grains, each differing in orientation by rotation about z . Neighboring grains can maintain compatibility as they deform by plane strain by curling about one another, as shown in Fig. 6.4. Although this curling requires some additional work, the amount of required bending work decreases as the grains develop a

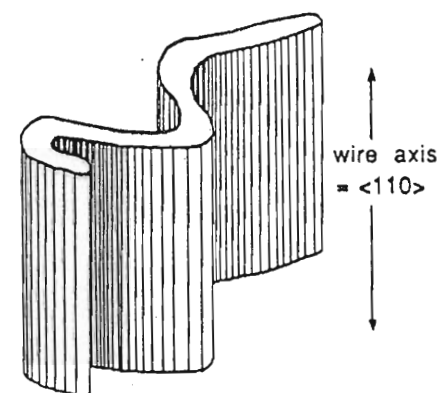


Fig. 6.4. The shapes of grains in a drawn bcc metals can be described as ribbons aligned and folded parallel to the wire axis.

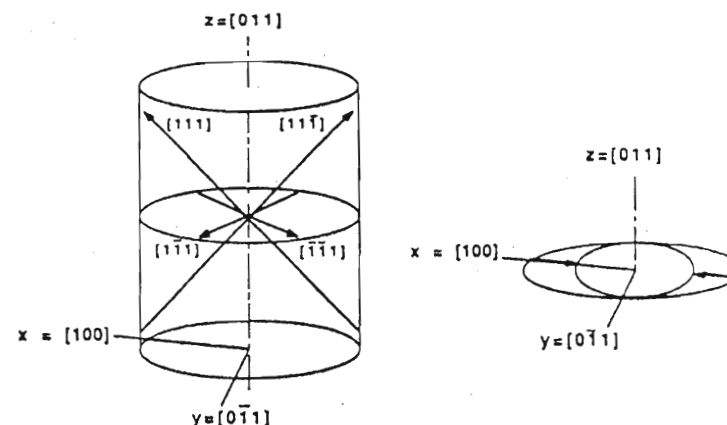


Fig. 6.5. Arrangement of slip directions in bcc crystal oriented for extension parallel to $[011]$. With slip on the two favorably oriented $\langle 111 \rangle$ directions, there is contraction parallel to $[100]$ but not parallel to $[0\bar{1}1]$.

allow for the bending work. He found that after a strain of $\epsilon = 2$, the Taylor factor for plane-strain flow with curling is about 0.7 (in comparison with $\frac{1}{2}$) that for axisymmetric flow of the grains.

This model predicts that the thin transverse direction of a grain should correspond to a $\langle 001 \rangle$ and the long one to a $\langle 110 \rangle$, and this was confirmed by Meieran and Thomas [19]. It has been found [20] that even the structure



Fig. 6.6. Cross section of an iron wire swaged to an 87% reduction of area. Note the swirled microstructure. (From Meieran and Thomas [19].)

of pearlitic steels develop similar structure because the orientation within the ferrite plates of the pearlite is such that $\{001\}$ is parallel to the plates. Similar microstructures should be expected in wires of hcp metals having a fiber texture.

An analogous microstructure develops in fcc metals during severe compression [17] because of the strong $\langle 110 \rangle$ compression texture. As a crystal is compressed parallel to $\langle 110 \rangle$, the orientation of the $\{111\}$ slip planes is such that it will readily spread in the lateral $\langle 001 \rangle$ direction but not in the lateral $\langle 1\bar{1}0 \rangle$ direction. Again the overall compatibility can be maintained by curling of the grains about the $\langle 110 \rangle$ compression direction.

It is of interest to note that Peck and Thomas [16] found that bcc wires formed by swaging, instead of wire drawing, form a swirled microstructure (see Fig. 6.6) with grains oriented so that $\langle 110 \rangle$ is parallel to the wire axis and $\langle 001 \rangle$ oriented in the radial direction. As the grains elongate, thinning occurs preferentially in the $\langle 001 \rangle$ radial direction. Again the M is reduced to $\frac{1}{2}$ that for axisymmetric flow.

6.7. Relaxed constraints model

This modification of the Taylor model was first suggested by Honneff and Mecking [21] and later developed by Kocks, Jonas, Canova, and others [22-24]. It is assumed that when the deformation is great enough so that grains are no longer equiaxed, there is a partial relaxation of the Taylor requirement for equal strains throughout a material. In material further than

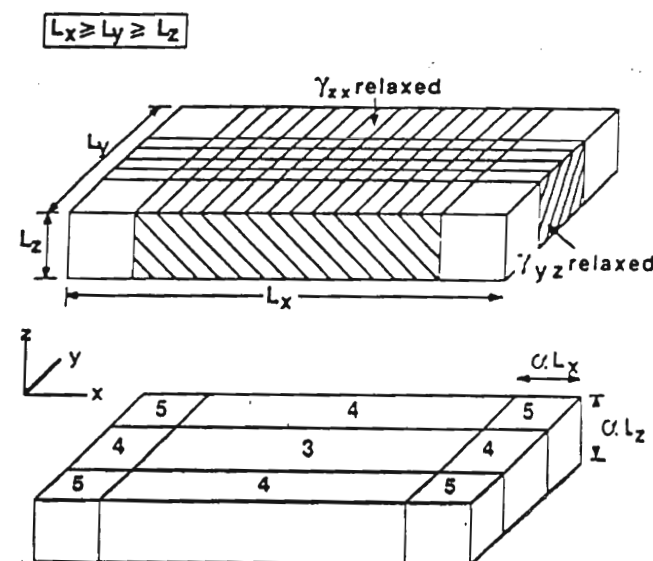


Fig. 6.7. Schematic diagram of a distorted grain showing regions in which the constraints are assumed to be relaxed.

a characteristic distance from a grain boundary, the shear strain causing displacement normal to that boundary is relaxed. For example, if the dimension of the grain were L_x, L_y, L_z with $L_x > L_y > L_z$, the requirement for a fixed level of the shear strain, γ_{yx} , is relaxed except within a characteristic distance, αL_x , of the grain boundaries normal to y , as shown in Fig. 6.7. Likewise the shear strain, γ_{zx} , should be relaxed except within a distance αL_z , of the grain boundaries normal to x . Thus a grain is divided into a central region where only three independent strains are prescribed (requiring slip on only three slip systems), edge regions where four independent strains are prescribed (requiring four slip systems), and fully constrained corner regions in which five slip systems must operate. In this development α is a geometric parameter between 0.5 and 1.5.

The relaxation of this model gradually becomes more important as the grain shape becomes more and more distorted with deformation. Figures 6.8, 6.9, and 6.10 show how the volume fractions of the material requiring 3, 4, and 5 active slip systems are predicted to change with strain for axisymmetric elongation, axisymmetric compression, and plane-strain compression. These figures are based on calculations assuming that the grains are initially cubic and that $\alpha = 1$.

For the axisymmetric elongation along z (Fig. 6.8), the shear strains γ_{yz} ,

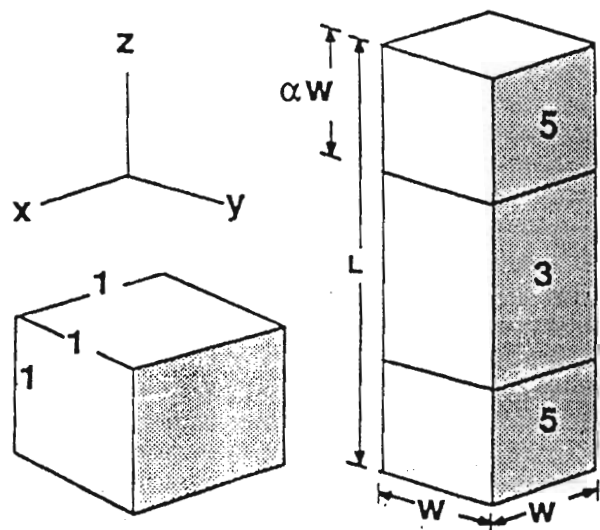
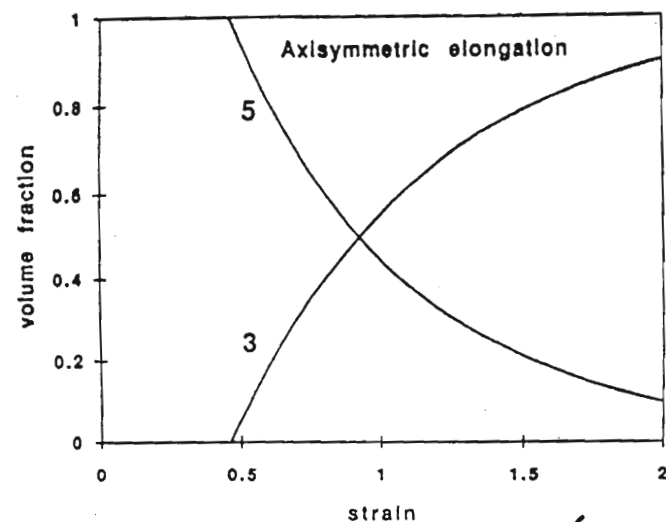


Fig. 6.8. Changes in the volume fraction of material requiring 3 and 5 active slip systems with strain during axisymmetric extension according to the relaxed constraints model. Calculated from Eq. (6.1) assuming initially cubic grains and $\alpha = 1$. (Adapted from [22].)

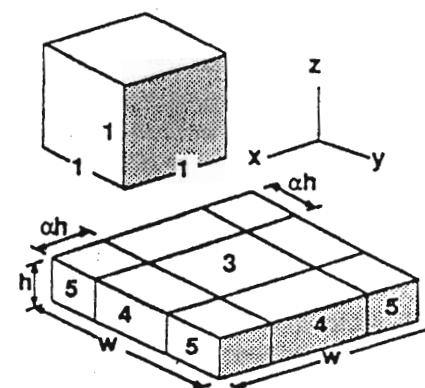
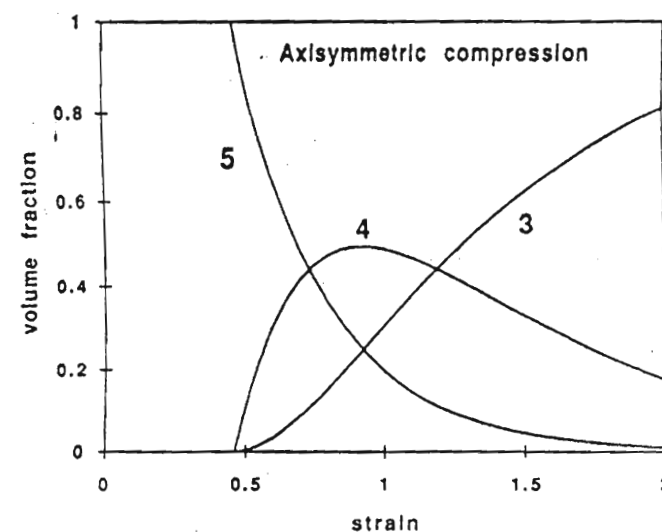


Fig. 6.9. Changes in the volume fraction of material requiring 3, 4, and 5 active slip systems during axisymmetric compression according to the relaxed constraints model. Calculated from Eq. (6.2) assuming initially cubic grains and $\alpha = 1$. (Adapted from [22].)

γ_{xx} and γ_{yy} are relaxed except at the ends. The volume fraction in which five slip systems operate, f_5 , is unity for $L \leq 2\alpha w$. When $L \geq 2\alpha w$, f_5 is given by $f_5 = 2\alpha w^3/(w^2 L) = 2\alpha w/L$. In the remaining material ($f_3 = 1 - f_5$), γ_{xy} is relaxed so three slip systems are required.⁹ Both L and w are functions of the axial

⁹ This relaxation to three slip systems is not in agreement with ref. 24 which proposes that only γ_{xx} is relaxed in the central region. However, relaxation of γ_{xx} and γ_{yy} is in agreement with the present model.

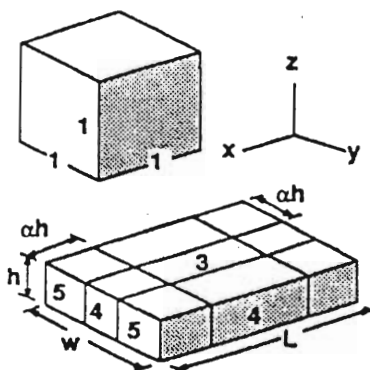
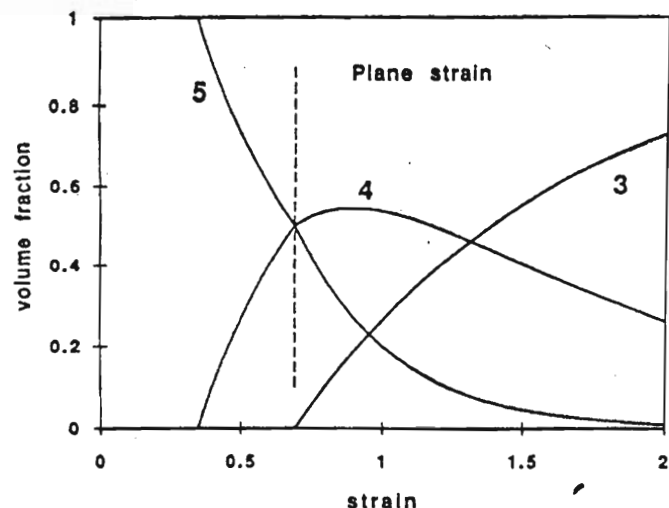


Fig. 6.10. Changes in the volume fraction of material requiring 3, 4, and 5 active slip systems during plane-strain extension according to the relaxed constraints model. Calculated from Eq. (6.3) assuming initially cubic grains and $\alpha = 1$. (Adapted from [22].)

strain, ϵ . With axisymmetric flow, $L = L_0 \exp(\epsilon)$ and $w = w_0 \exp(-\epsilon/2)$. Taking L_0 and w_0 as unity and substituting,

$$f_5 = 2\alpha \exp(-3\epsilon/2) \quad \text{and} \quad f_4 = 1 - f_5. \quad (6.1)$$

It is interesting to compare the strain relaxation of this model with relaxation in bcc wires discussed in the previous section. In that case

one of the strains normal to the wire axis, ϵ_x or ϵ_y , is relaxed as well as γ_{yz} .

For the axisymmetric compression about z (Fig. 6.9), $h = \exp(-\epsilon)$ and $w = \exp(\epsilon/2)$, where here ϵ is $\ln(h_0/h)$. The volume fraction in which five slip systems operate, f_5 , is unity for $2\alpha h \geq w$, which corresponds to $\epsilon \leq \frac{2}{3} \ln(2\alpha)$. For higher strains,

$$f_5 = \frac{4\alpha^2 h^3}{w^2 h} = \frac{4\alpha^2 h^2}{w^2} = 4\alpha^2 \exp(-3\epsilon), \quad (6.2)$$

$$f_4 = \frac{4\alpha h^2(w - 2\alpha h)}{w^2 h} = 4\alpha h/w - 8\alpha^2 \left(\frac{h}{w}\right)^2$$

$$= 4\alpha \exp\left(-\frac{3\epsilon}{2}\right) - 8\alpha^2 \exp(-3\epsilon)$$

$$f_3 = 1 - f_4 - f_5.$$

For the plane-strain compression (Fig. 6.10), the volume fraction in which five slip systems operate, f_5 , is unity for $L \leq 2\alpha h$, which corresponds to $\epsilon \leq \frac{1}{2} \ln(2\alpha)$. Here $L = \exp \epsilon$ and $h = \exp(-\epsilon)$ and $w = w_0 = 1$. In the interval $\frac{1}{2} \ln(2\alpha) \leq \epsilon \leq \ln(2\alpha)$,

$$f_5 = 2\alpha h^2 = 2\alpha^2 \exp(-2\epsilon), \quad f_4 = 1 - f_5. \quad (6.3a)$$

For higher strains $\epsilon \geq \ln(2\alpha)$

$$\left. \begin{aligned} f_5 &= 4\alpha^2 h^3 = 4\alpha^2 \exp(-3\epsilon), \\ f_4 &= 2\alpha \exp(-\epsilon) - 4\alpha^2 \exp(-3\epsilon), \\ f_3 &= 1 - f_5 - f_4 = 1 - 2\alpha \exp(-\epsilon). \end{aligned} \right\} \quad (6.3b)$$

Similar calculations have been made for torsion. In each case, as the average number of required slip systems decreases with the relaxation of the constraints, the Taylor factor, and therefore the relative strength, decreases below what it would be for fully constrained flow. The basic reason for introducing this relaxed model was to help explain the discrepancies between stress-strain curves in tension, compression, and torsion at high strains [24]. The difference between effective stress-strain curves in tension, compression, and torsion was explained partially by the different textures developed and partially by the different rate of decrease of average number of required slip systems, with torsion having the greatest decrease and tension the least.

6.8. Strengthening from second phases

In many precipitation hardening systems, the precipitates form as needles or platelets aligned with certain crystallographic directions or planes in the matrix. It is because these needles and platelets are plastically strong that they harden the crystal. If there is a texture in the matrix, these oriented precipitates will cause an anisotropic hardening of the matrix. The precipitate must deform in a way that is compatible with the matrix. At very low strains the compatibility can be accommodated elastically. One might assume that at higher strains, plastic deformation of the crystal would require the particles to undergo the same strains as the overall crystal. However, because of their shape, some of the strain components in the matrix may be accommodated by rotation. This is illustrated in Fig. 6.11, for a platelet oriented perpendicular to the 1 axis. When the matrix deforms, it must undergo the same normal strains, ϵ_1 , ϵ_2 , and ϵ_3 and one of the same shear strain components (γ_{23}), but two of the shear strain components (γ_{31} and γ_{12}) can be satisfied by a rigid body rotation instead of deformation. These are the same strain components that would be relaxed in the centers of the platelets according to the relaxed constraints model of the previous section.

The anisotropic precipitation hardening of Al-4% Cu has been analyzed

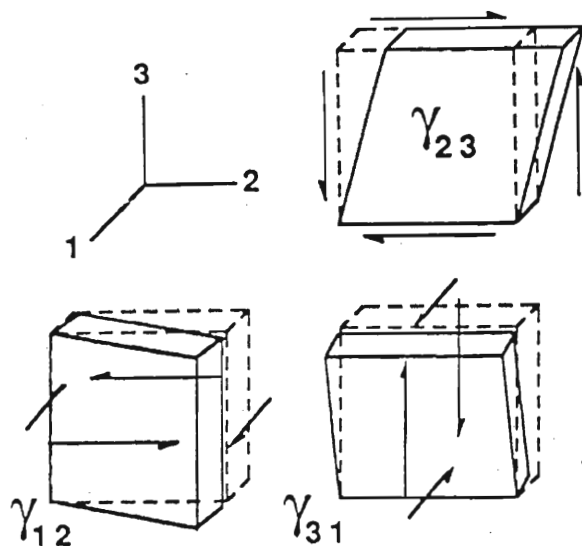


Fig. 6.11. For a platelet, compatibility for two of the three shear strains can be satisfied by body rotation.

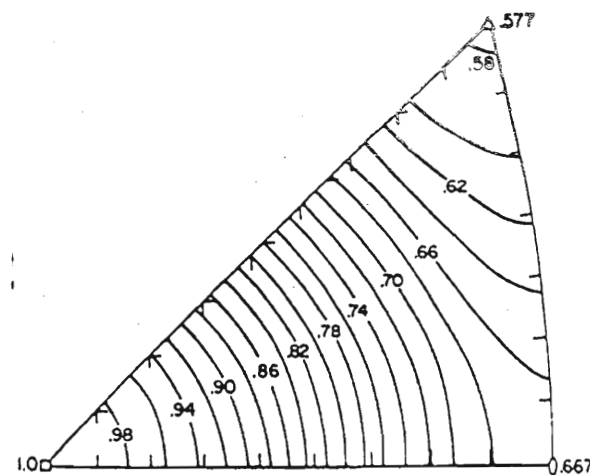


Fig. 6.12. The orientation dependence of N_v for axisymmetric flow of crystals having platelets parallel to {100} planes. (From [25].)

this way [22]. In this alloy the precipitates are platelets parallel to the {100} planes of the matrix.

The incremental work to deform the platelets oriented perpendicular to the 1 direction is $dw_1 = f_1 \bar{\sigma} d\bar{\epsilon}_1$, where f_1 is the volume fraction particles parallel to (100), $\bar{\sigma}$ is the effective stress and $d\bar{\epsilon}_1$ is the incremental effective strain in the platelets. Although the plastic properties of the platelets may not be known, the anisotropy caused by them may be approximated if it is assumed that they are, themselves, plastically isotropic and the von Mises effective strain function is appropriate. In that case, $N_1 = d\bar{\epsilon}_1/d\epsilon_x$ is given by

$$N_1 = \left\{ \frac{2}{3} \left[\left(\frac{d\epsilon_1}{d\epsilon_x} \right)^2 + \left(\frac{d\epsilon_2}{d\epsilon_x} \right)^2 + \left(\frac{d\epsilon_3}{d\epsilon_x} \right)^2 \right] + \frac{1}{3} \left(\frac{d\gamma_{23}}{d\epsilon_x} \right)^2 \right\}^{1/2}, \quad (6.4)$$

where the terms describe the strain in the matrix. The corresponding expressions for platelets oriented perpendicular to the 2 and 3 axes are identical to Eq. (6.4) except that the term $d\gamma_{23}$ is replaced by $d\gamma_{31}$ and $d\gamma_{12}$, respectively. If the three orientations of platelets are present in equal amounts, the strength contribution of the platelets will be

$$\frac{dw_1}{d\epsilon_x} = f \bar{\sigma} \frac{d\bar{\epsilon}_{av}}{d\epsilon_x} = f \bar{\sigma} N_{av}, \quad (6.5)$$

where $N_{av} = (N_1 + N_2 + N_3)/3$.

Calculations of N_{sl} for axisymmetric flow in crystals of different orientations are plotted in Fig. 6.12. Here it can be seen that the greatest platelet hardening (highest values of N_{sl}) occur near $\langle 100 \rangle$ and the least hardening (lowest values of N_{sl}) occur near $\langle 111 \rangle$. (This is almost opposite to the effect of orientation on the slip in the matrix, for which the Taylor analysis predicts the highest strength at $[111]$ and the lowest near $[100]$). In age-hardened Al-4% Cu alloys these two different anisotropic effects partially cancel, resulting in only a small amount of anisotropy. This strong decrease in the degree of anisotropy with precipitation hardening has been documented in extruded bars of an aluminum alloy containing magnesium and silicon [26].

Much the same relaxation can occur for rodlike particles. For rods oriented parallel to the 1 axis, the particles can accommodate γ_{12} and γ_{31} strain components by rotation. This analysis is based on infinitesimal strains, whereas finite shear strains can be only partially accommodated by rotation.

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DEFORMATION TEXTURES

Barrett and Massalski [1] and Hu [2] have published excellent reviews of experimentally determined crystallographic textures in metals. Textures are formed in metals during solidification, deformation, recrystallization, grain growth, and phase transformations. The treatment here is concerned only with deformation textures and summarizes some of the experimental observations and recent computer predictions.

7.1. Observed deformation textures

7.1.1. Wire textures

The textures of fcc metals after large axisymmetric elongation as in wires or rods formed by drawing or extrusion have two major components: one with $\langle 100 \rangle$ approximately aligned with the wire axis and the other with $\langle 111 \rangle$ so aligned. Figure 7.1 is the inverse pole figure [3] of an extruded aluminum rod. The reduction was 92% at 24°C. The relative amounts of the $\langle 100 \rangle$ and $\langle 111 \rangle$ components vary from one metal to another. English and Chin [4] showed that this variation correlates well with the stacking fault energy of the metal, as shown in Fig. 7.2. The large amount of $\langle 100 \rangle$ component for alloys with very low stacking fault energies is probably caused by the high frequency of twinning in these alloys during the deformation. The increase of $\langle 111 \rangle$ component with stacking fault energy in the other materials must be related to the slip behavior. The relative amount of the two components also varies with temperature.

All bcc metals and alloys develop strong $\langle 110 \rangle$ fiber textures after drawing or extrusion. Leber [5] has shown that for bcc metals often the texture should be described as "cylindrical texture" in which there is a characteristic $\langle 110 \rangle$ axial direction with an $\langle 001 \rangle$ radial direction. Such a texture is easily mistaken for a fiber texture in which there is only a characteristic $\langle 110 \rangle$ wire axis with random rotation about it. Peck and Thomas [6] showed that this cylindrical texture arises during swaging, a process that is often carried out prior to wire drawing, and that the texture

DEFORMATION TEXTURES

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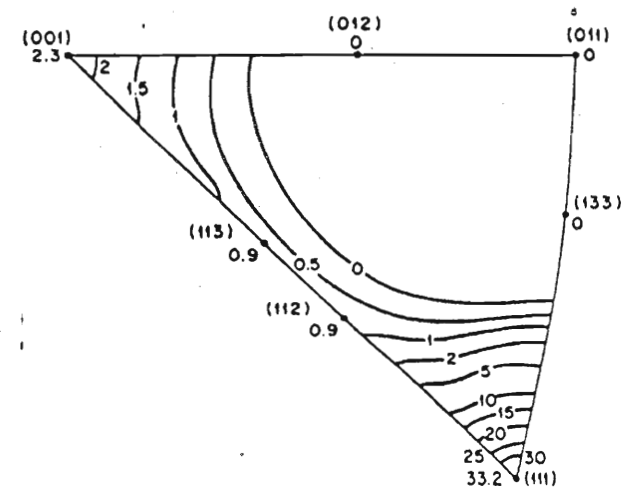


Fig. 7.1. Inverse pole figure of an extruded aluminum rod (92% reduction). (From McHargue et al. [3].)

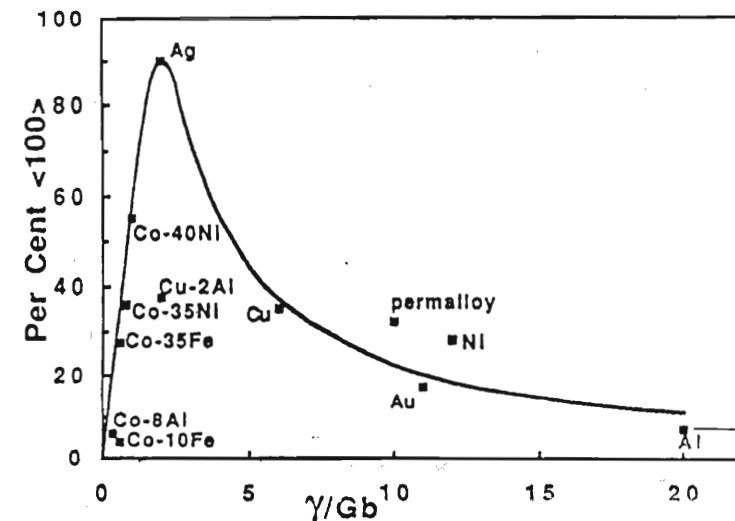


Fig. 7.2. Correlation of wire textures of fcc metals with stacking fault energy. (Adapted from English and Chin [4].)

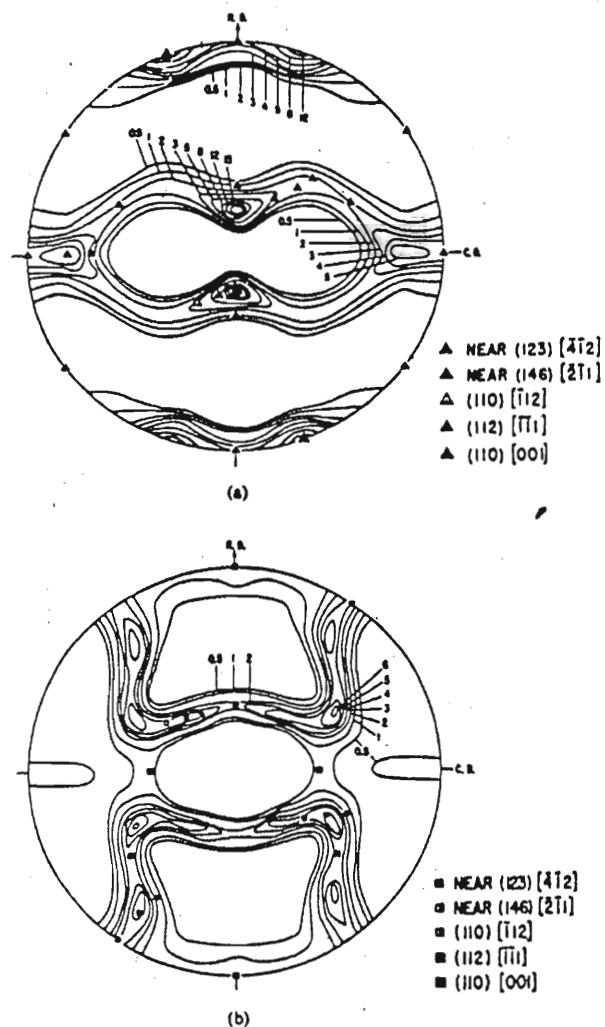


Fig. 7.3. Rolling texture of copper after 96.5% reduction at 25°C. (111) pole figure (top) and (200) pole figure (below). (From Hsu and Goodman [7].)

is associated with a swirled microstructure in contrast to the true $\langle 110 \rangle$ -fiber texture that is associated with the curly microstructure.

The wire texture of zirconium and other hcp metals can be described as the c axis being oriented normal to the wire axis. It is likely that cylindrical textures may develop after swaging.

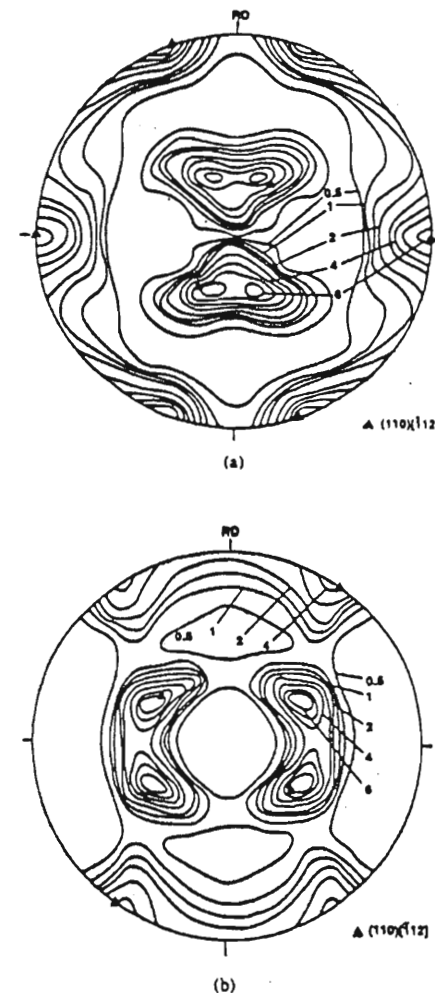


Fig. 7.4. Rolling texture of silver after 91% reduction at 20°C. (111) pole figure (above) and (200) pole figure (below). (From Hsu and Cline [8].)

7.1.2. Compression textures

After heavy compression, fcc metals form textures that can be roughly described as $\langle 110 \rangle$ aligned with the compression axis. Again there is some variation among the fcc metals. In aluminum there is some spread from this ideal orientation, particularly toward $\langle 113 \rangle$, while in brass some grains are oriented near $\langle 111 \rangle$.

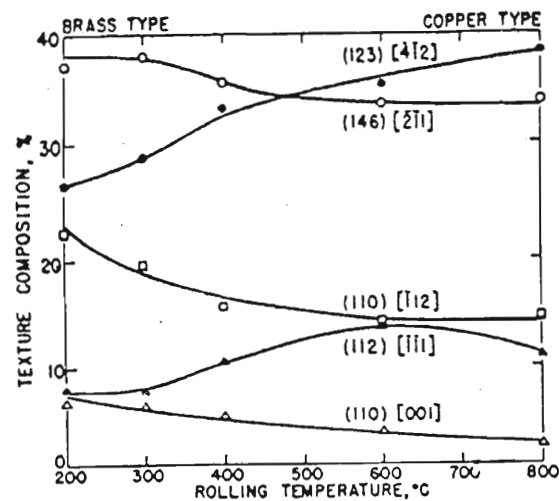


Fig. 7.5. The texture transition in 18-8 stainless steel. (From Hsu and Goodman [9].)

In body-centered cubic metals there is a duplex texture with a strong $\langle 111 \rangle$ component and a weaker $\langle 100 \rangle$ component.

For zirconium and other hcp metals the compression texture is one with the c axis aligned with the compression axis.

7.1.3. Rolling textures

The rolling textures of aluminum, copper, nickel, and austenitic iron-nickel alloys are very similar (Fig. 7.3), but the pole figures for brass, austenitic stainless steel, and silver are somewhat different (Fig. 7.4). It is interesting to note that copper rolled at low temperatures will develop a brasslike texture [7, 8], and stainless steel rolled at elevated temperatures develops a copper texture [9]. The texture changes in stainless steel are shown as a function of the rolling temperature in Fig. 7.5. Thus the rolling textures in fcc metals appear to depend on both stacking fault energy and the deformation temperature.

The rolling textures of all bcc metals (iron, tantalum, vanadium, tungsten, β -titanium alloys, and β -brass) are all similar. The pole figures for rolled molybdenum [10] are shown in Fig. 7.6.

In the rolling textures of hcp metals with $c/a < 1.63$ (Ti, Zr, Hf, Be), $\langle 10\bar{1}0 \rangle$ is aligned with the rolling direction (RD) and the $[0001]$ tends to be rotated about the rolling direction (RD) with maxima at ± 30 – 40° from the normal to the rolling plane (ND). Figure 7.7 shows the pole figures for rolled titanium. For magnesium the rotation of maxima from ND is about the

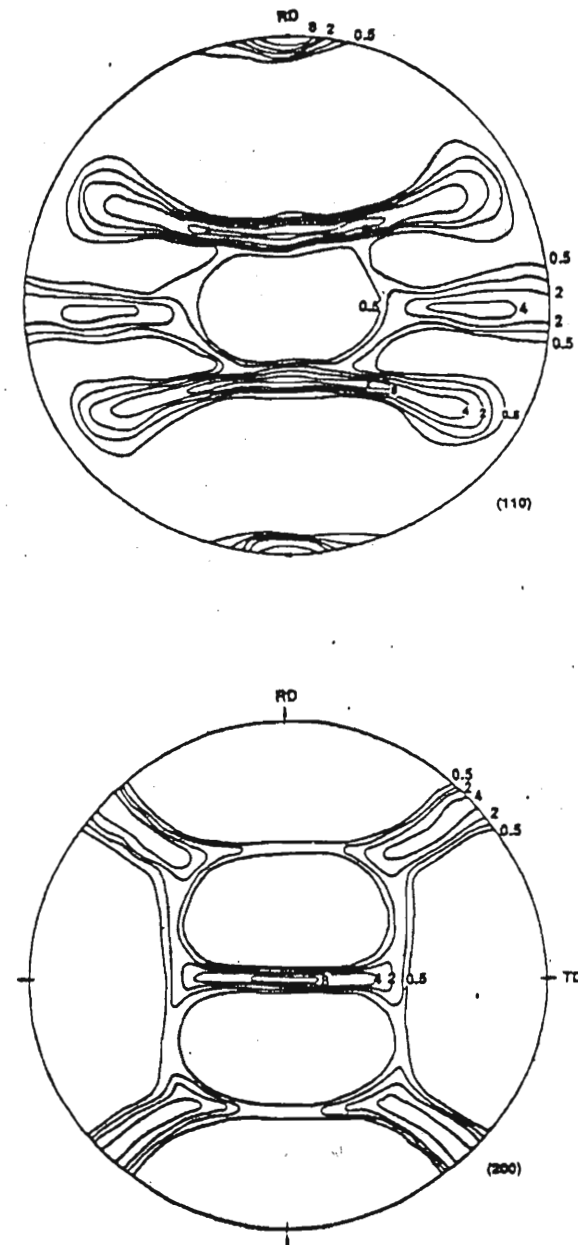


Fig. 7.6. Rolling texture of molybdenum. (110) pole figure (above) and (200) pole figure (below). (From Segmüller and Wassermann [10].)

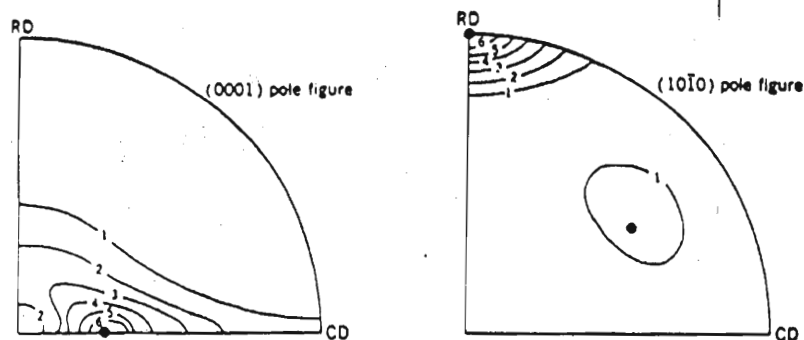


Fig. 7.7. Pole figures for α -titanium cold-rolled 97%. The (0001) maxima are rotated $\pm 30^\circ$ from the sheet normal. (From Williams and Eppelsheimer [11].)

transverse direction. With zinc and cadmium, twinning causes large rotations of the [0001] maxima about the transverse direction (TD).

7.2. Texture predictions for fcc metals

Lattice rotations were predicted as part of Taylor's early work [12] relating the axisymmetric deformation of fcc polycrystals to single crystal behavior. Figure 7.8(a) shows the predictions for a compression of 2.37%. (For extension the arrows would be reversed.) For many orientations, two arrows are shown. In these cases, Taylor found more than one combination of five slip systems that could produce the shape change with the same minimum energy, and the arrows correspond to these combinations. Any rotation between these is also possible by operation of both slip-system sets. For tension, his predictions show that orientations initially near $\langle 100 \rangle$ tend to rotate toward $\langle 100 \rangle$, while those initially near $\langle 111 \rangle$ tend to rotate toward $\langle 111 \rangle$, which is in qualitative agreement with the observation of a duplex $\langle 100 \rangle$ - $\langle 111 \rangle$ wire texture in fcc metals. It is now known as a result of Bishop and Hill's work that six or eight slip systems are equally favored for each orientation. Taylor's calculations inadvertently omitted a number of possible slip system combinations; so an even greater range of possible rotations should have been predicted.

Barrett and Levenson [13] experimentally studied rotations during axisymmetric compression of coarse-grained polycrystalline aluminum. A cylindrical compression specimen was sectioned normal to its axis, and the orientation of each grain on the surface of the section was determined before and after compression. The orientation changes were compared with the negative of Taylor's predictions for tension [Fig. 7.8(b)]. Only about half of

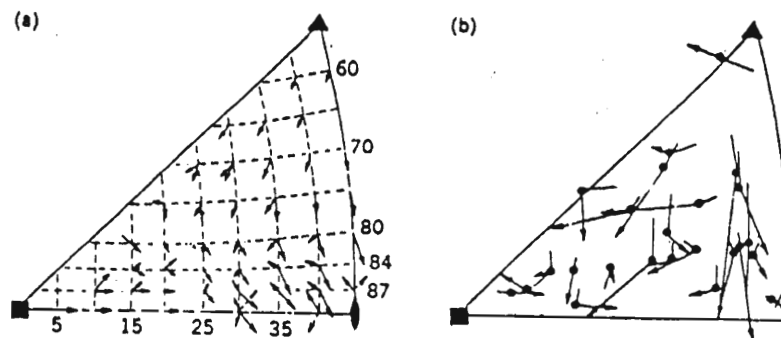


Fig. 7.8. (a) The lattice rotations predicted by Taylor for axisymmetric compression of fcc crystals. (b) The lattice rotations found experimentally in aluminum after 11 and 31% compression. (From Barrett and Levenson [13].)

the measured rotations were in agreement with the theory. Had Taylor included all of the possible orientations in his calculations, he would have predicted a greater spread of possible rotations; so, of course, the agreement of theory with experiment would have appeared better.

It is this ambiguity in predicting the active slip systems that is the greatest weakness of Taylor's model. While this ambiguity causes no difficulty in strength predictions, it does prevent useful predictions of texture formation. Various attempts have been made to develop rules for selecting the active slip systems from the six or eight corresponding to the least energy.

Most of the recent work has been a modification of the Taylor model in ways that remove the ambiguities about the active slip systems or by substituting some form of relaxed constraints instead of full constraints. Methods of selecting the active slip systems for computer codes have been reviewed [14-16]. These include: (1) assuming that latent hardening is less than active hardening; (2) distributing slip as evenly as possible or as unevenly as possible between systems; (3) choosing the combination of slip systems that will cause the smallest increase or greatest decrease in the Taylor factor, M ; (4) averaging over all extreme solutions; and (5) choosing the various extremes at random. Another approach is the rate-sensitive model of Needleman and Asaro [17]. Figure 7.9 (from [15]) shows the rotations in axisymmetric extension predicted by four different models based on different selection rules. The pole figures for rolling textures predicted by these models (Fig. 7.9) are all similar to the experimentally determined pole figures for copper [16] (Fig. 7.2). In general the agreement is quite good.

Kocks [16] found some differences between the pole figures predicted with a relaxed constraints model and with a full constraints model, as shown in Fig. 7.10.

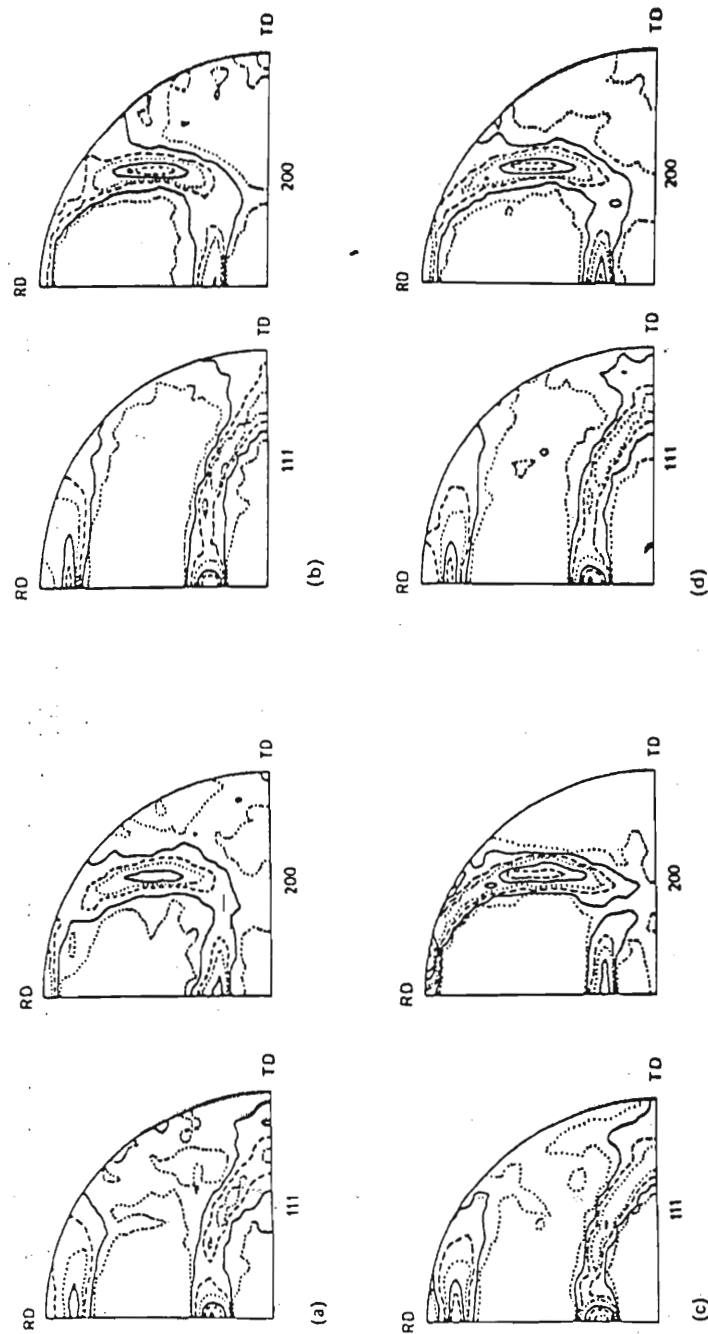


Fig. 7.9. The (111) and (200) pole figures predicted for fcc after 50% rolling reduction using four different numerical models. (From [15].) Note the relatively small difference between these models and the general agreement with the experimental pole figures for copper in Fig. 7.2 (a) "Bacroix average Taylor"; (b) "Lower and Asaro"; (c) Bacroix-Renouard-Wintenberger-Taylor; and (d) "Leflers".

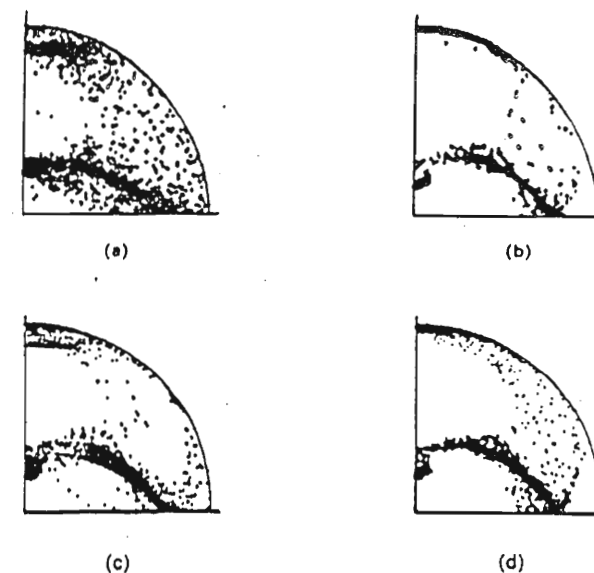


Fig. 7.10. The (111) pole figures predicted for fcc metals using (a) the full constraints model and (b) the relaxed constraints model (93% reduction). Textures predicted by (c) averaging extreme Taylor solutions and (d) a rate-sensitivity mode after 89% reduction. (From Kocks [16].)

Reasonable simulations have been made of the texture development in hcp metals during rolling [18] and axial compression [19] using models that allow for some relaxations of the constraints.

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8

CONTINUUM ANALYSES OF ANISOTROPIC PLASTICITY

The anisotropy of plastic properties has important effects in the formability of metals and the strength of formed parts. Although plastic anisotropy is a result of crystallographic texture, it is not at present practical for most engineering calculations to assess the anisotropic behavior of a particular piece of metal by first determining its crystallographic texture and then making upper- or lower-bound analyses based on the texture. Instead analyses are made with continuum anisotropic yield criteria. To be useful, an anisotropic yield criterion should describe material behavior with reasonable accuracy and also involve only a small number of material parameters that can easily be evaluated by simple tests (e.g., tension tests).

8.1. 1948 Hill criterion

Hill [1, 2] proposed a generalization of the von Mises criterion,

$$2f(\sigma_{ij}) = F(\sigma_y - \sigma_x)^2 + G(\sigma_x - \sigma_z)^2 + H(\sigma_z - \sigma_y)^2 + 2L\tau_{yz}^2 + 2M\tau_{zx}^2 + 2N\tau_{xy}^2 = 1, \quad (8.1)$$

where F , G , H , L , M , and N are material constants. The x , y , and z axes, which are called the *principal axes of anisotropy*, are assumed to be axes of twofold rotational symmetry. An equivalent assumption is that the yz , xz , and xy planes are planes of mirror symmetry. This form of anisotropy is common among rolled sheets, extruded bars, pipes, etc. For rolled sheets and plates it is conventional to take the x , y , and z directions to be the rolling, transverse, and thickness directions, respectively. Because Eq. (8.1) involves only quadratic terms, it predicts that yielding is insensitive to the sign of the applied stress.

The parameters F , G , and H may be evaluated from a few tension tests. Consider an x -direction tension test and let X be the yield strength. By substituting $\sigma_x = X$ and $\sigma_y = \sigma_z = \tau_{yz} = \tau_{zx} = \tau_{xy} = 0$ into Eq. (8.1), $X^2 = 1/(G + H)$. Similarly defining Y and Z as the yield strengths in y and

z-direction tension tests,

$$\left. \begin{aligned} X^2 &= \frac{1}{G+H}, \\ Y^2 &= \frac{1}{H+F}, \\ Z^2 &= \frac{1}{F+G}. \end{aligned} \right\} \quad (8.2)$$

Solving simultaneously,

$$\left. \begin{aligned} 2F &= \frac{1}{Y^2} + \frac{1}{Z^2} - \frac{1}{X^2}, \\ 2G &= \frac{1}{Z^2} + \frac{1}{X^2} - \frac{1}{Y^2}, \\ 2H &= \frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2}. \end{aligned} \right\} \quad (8.3)$$

The flow rules associated with this criterion follow directly from applying the general form $d\epsilon_{ij} = d\lambda \partial f(\sigma_{ij}) / \partial \sigma_{ij}$ to Eq. (8.1).

$$\left. \begin{aligned} d\epsilon_x &= d\lambda [H(\sigma_x - \sigma_y) + G(\sigma_x - \sigma_z)] \\ d\epsilon_y &= d\lambda [F(\sigma_y - \sigma_z) + H(\sigma_y - \sigma_x)] \\ d\epsilon_z &= d\lambda [G(\sigma_z - \sigma_x) + F(\sigma_z - \sigma_y)] \\ d\epsilon_{yz} &= d\epsilon_{zy} = d\lambda L\tau_{yz} \\ d\epsilon_{zx} &= d\epsilon_{xz} = d\lambda M\tau_{zx} \\ d\epsilon_{xy} &= d\epsilon_{yx} = d\lambda N\tau_{xy}. \end{aligned} \right\} \quad (8.4)$$

In deriving the flow rules for the shear strains, $d\epsilon_{yz}$, $d\epsilon_{zx}$, and $d\epsilon_{xy}$, the yield criterion, Eq. (8.1), must be written with the shear stress terms as $L(\tau_{yz}^2 + \tau_{zy}^2) + M(\tau_{zx}^2 + \tau_{xz}^2) + N(\tau_{xy}^2 + \tau_{yx}^2)$ before the partial differentiation.)

Often, as with sheets or thin plates, z-direction tension tests are impractical, but measurements of the ratios R of the contractile strains during tension tests of specimens cut from the sheet can be used to deduce the strength in the thickness direction. Here the strain ratio is defined as

$$R = \epsilon_w / \epsilon_t \quad \left(\text{or, more properly, } \frac{d\epsilon_w}{d\epsilon_t} \right), \quad (8.5)$$

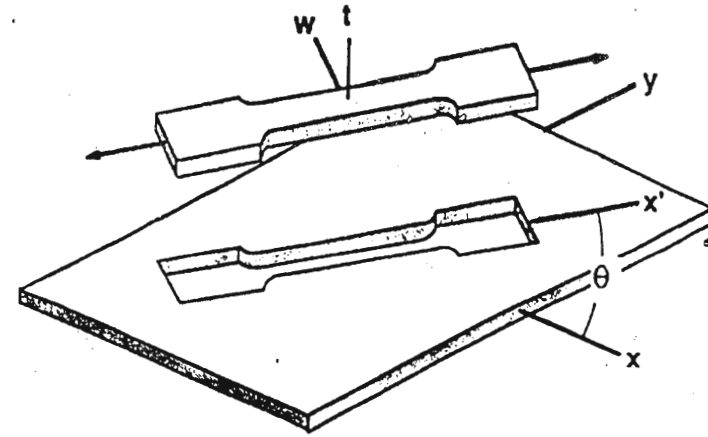


Fig. 8.1. Schematic drawing of a sheet tensile specimen showing the coordinate axes of the sheet (x , y , and z) and of the specimen (t , w , and x'). The strain ratio R is defined as the ratio of the ϵ_w/ϵ_t in an x' direction tension test.

where w and t refer to the width and thickness directions of the specimen, as shown in Fig. (8.1). Usually a subscript is used to denote the angle θ between the tensile axis and the rolling direction x . Thus R_θ indicates the strain ratio measured when the tension axis is at an angle θ to the x direction. However, to simplify the notation, R_θ will be denoted here by R , R_{90} by P , and R_{45} by Q . For an x -direction test ($\sigma_y = \sigma_z = \tau_{yz} = \tau_{zx} = \tau_{xy} = 0$), the flow rules give

$$R = \frac{d\epsilon_y}{d\epsilon_x} = \frac{H}{G}. \quad (8.5a)$$

Similarly, in a y -direction test,

$$P = \frac{d\epsilon_x}{d\epsilon_z} = \frac{H}{F}. \quad (8.5b)$$

As will be shown in section 8.1.3, in a 45° tension test,

$$Q = \frac{d\epsilon_{135}}{d\epsilon_x} = \frac{N}{F+G} - \frac{1}{2}. \quad (8.5c)$$

Now the z -direction yield strength, Z , can be found from Eq. (8.2) by

dividing by G and substituting Eqs. (8.5a) and (8.5b).

$$\left. \begin{aligned} \left(\frac{Z}{X}\right)^2 &= \frac{G+H}{F+G} = \frac{1+H/G}{1+F/G} = \frac{1+R}{1+R/P} \\ Z &= X \left[\frac{P(1+R)}{P+R} \right]^{1/2} = Y \left[\frac{R(1+P)}{R+P} \right]^{1/2} \end{aligned} \right\} \quad (8.6)$$

If $\tau_{yz} = \tau_{zx} = 0$, the z axis is an axis of principal stress. Then, with substitution of Eqs. (8.2), (8.5a), and (8.5b), the yield criterion [Eq. (8.1)] may be written as

$$R(\sigma_y - \sigma_z)^2 + P(\sigma_z - \sigma_x)^2 + RP(\sigma_x - \sigma_y)^2 + (2Q+1)(R+P)P\tau_{xy}^2 = P(1+R)X^2. \quad (8.7)$$

Thus, if both R and P are greater than unity, the through-thickness yield strength, Z , is greater than both X and Y , which is what one might intuitively expect because high strain ratios suggest a high resistance to thinning.

For plane stress loading ($\sigma_z = \tau_{yz} = \tau_{zx} = 0$), Eq. (8.1) reduces to:

$$A\sigma_1^2 + B\sigma_2^2 + 2C\sigma_1\sigma_2 = 1, \quad (8.8)$$

where σ_1 and σ_2 are principal stresses in the x - y plane, θ is the angle between the l and x axes, and

$$\begin{aligned} A &= F \sin^4 \theta + G \cos^4 \theta + H(\cos^2 \theta - \sin^2 \theta) + 2N \cos^2 \theta \sin^2 \theta \\ B &= F \cos^4 \theta + G \sin^4 \theta + H(\cos^2 \theta - \sin^2 \theta) + 2N \cos^2 \theta \sin^2 \theta \\ C &= (F+G) \cos^2 \theta \sin^2 \theta - H(\cos^2 \theta - \sin^2 \theta) - 2N \cos^2 \theta \sin^2 \theta. \end{aligned}$$

Substitution of the flow rules allows A , B , and C to be expressed in terms of the strain ratios [Eqs. (8.5)] and X :

$$\begin{aligned} A &= [R \sin^4 \theta + P \cos^4 \theta + RP(\cos^2 \theta - \sin^2 \theta) \\ &\quad + (2Q+1)(R+1) \cos^2 \theta \sin^2 \theta] / [P(R+1)X^2] \\ B &= [R \cos^4 \theta + P \sin^4 \theta + RP(\cos^2 \theta - \sin^2 \theta) \\ &\quad + (2Q+1)(R+1) \cos^2 \theta \sin^2 \theta] / [P(R+1)X^2] \\ C &= [(R+P) \cos^2 \theta \sin^2 \theta - RP(\cos^2 \theta - \sin^2 \theta) \\ &\quad - (2Q+1)(R+1) \cos^2 \theta \sin^2 \theta] / [P(R+1)X^2]. \end{aligned}$$

8.1.1. Simplifications for planar isotropy

The mathematical treatment can be greatly simplified if it can be assumed that the properties are the same in all directions in the plane of the sheet while properties normal to the sheet may be different. This is often referred to as *planar isotropy with normal anisotropy*. For these conditions $G = F$, $N = 2H + F$, $R_0 = P = R$, and $Y_0 = Y = X$. If z is a direction of principal stress ($\tau_{yz} = \tau_{zx} = 0$), the x and y axes may be oriented so they are also. Then, the yield criterion [Eq. (8.6)] further simplifies to

$$(\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + R(\sigma_x - \sigma_y)^2 = (R+1)Y^2. \quad (8.9)$$

Figure 8.2 shows that Eq. (8.7) plots as ellipses. The greater R is, the more the yield loci extend into the first quadrant, indicating an increased resistance to yielding under biaxial tension. The dashed line shows the stress states corresponding to plane strain ($\epsilon_y = 0$). Note that with increasing R , the stress state required for plane strain approaches balanced biaxial tension.

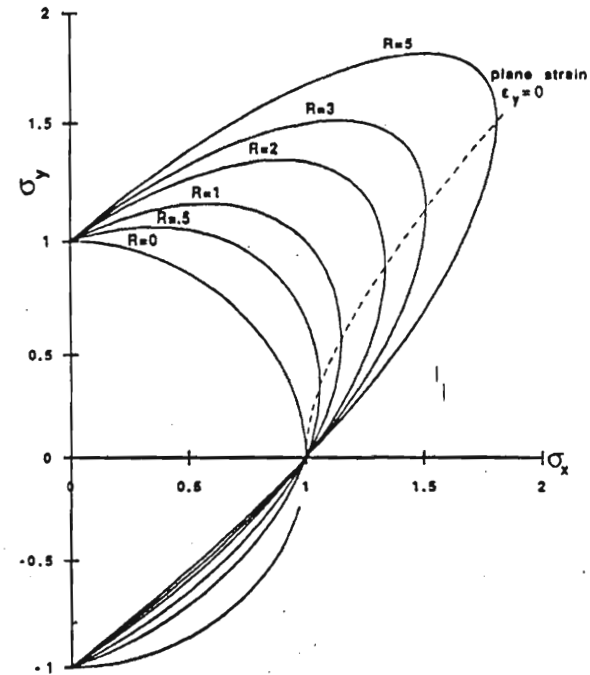


Fig. 8.2. Yield loci corresponding to the Hill 1948 criterion with planar isotropy. The dashed line indicates the locus of stress states for plane strain ($\epsilon_y = 0$).

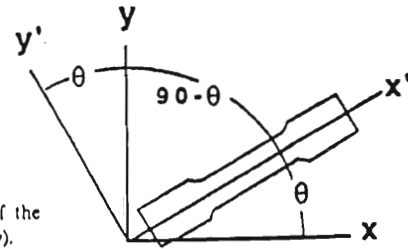


Fig. 8.3. Relation of the axes, x' and y' , of the tensile specimen to the material axes (x and y).

For planar isotropy the flow rules can be expressed as

$$d\epsilon_x : d\epsilon_y : d\epsilon_z = R(\sigma_x - \sigma_y) + \sigma_x : \sigma_y + R(\sigma_y - \sigma_x) : -\sigma_x - \sigma_y. \quad (8.10)$$

8.1.2. Angular variation of properties in the xy plane

The variation of properties in the plane of the sheet can be determined by considering a tension test along a direction x' at an angle θ to the x axis (Fig. 8.3). The stresses expressed on the x , y , and z axes are $\sigma_x = \sigma_{x'} \cos^2 \theta$, $\sigma_y = \sigma_{x'} \sin^2 \theta$, and $\tau_{xy} = \sigma_{x'} \cos \theta \sin \theta$ with $\sigma_z = \tau_{yz} = \tau_{zx} = 0$. Substituting into Eq. (8.1) with $\sigma_{x'} = Y_\theta$,

$$Y_\theta = [F \sin^4 \theta + G \cos^4 \theta + H(\cos^4 \theta - 2 \cos^2 \theta \sin^2 \theta + \sin^4 \theta) + 2N \sin^2 \theta \cos^2 \theta]^{-1/2}. \quad (8.11)$$

Equation (8.11) can be simplified by substituting the identities

$$\begin{aligned} \sin^4 \theta &= \sin^2 \theta (1 - \cos^2 \theta) \\ \cos^4 \theta &= \cos^2 \theta (1 - \sin^2 \theta) \\ Y_\theta &= [H + F \sin^2 \theta + G \cos^2 \theta \\ &\quad + (2N - F - G - 4H) \cos^2 \theta \sin^2 \theta]^{-1/2}. \end{aligned} \quad (8.12)$$

The nature of the maxima and minima of Y_θ permitted by the Hill criterion can be found by differentiating Eq. (8.12) and setting it to zero. Maxima or minima will be found at

$$\theta = 0, \theta = 90, \text{ and } \theta^* = \arctan[(N - G - 2H)/(N - F - 2H)]^{1/2}. \quad (8.13)$$

There are four possibilities as shown in Fig. (8.4):

1. If $N > G + 2H$ and $N > F + 2H$, maxima occur at 0° and 90° and a minimum at θ^* ;

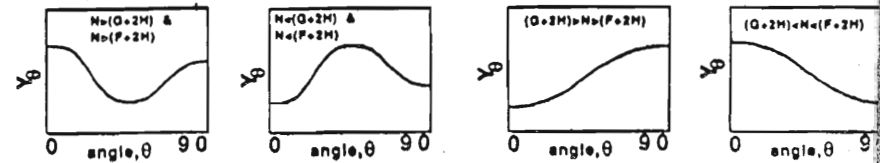


Fig. 8.4. Possible variations of yield strength Y_θ with angle θ .

2. If $N < G + 2H$ and $N < F + 2H$, minima occur at 0° and 90° and a maximum at θ^* ;
3. If $G + 2H > N > F + 2H$, θ^* is imaginary and there is a maximum at 0° and a minimum at 90° ;
4. If $G + 2H < N < F + 2H$, θ^* is imaginary and there is a minimum at 0° and a maximum at 90° .

If $F = G = N - 2H$, the yield strength is the same for all directions in the plane. If $F = G = H$ and $L = M = N = 3F$, there is complete isotropy and Eq. (8.1) reduces to the von Mises criterion.

The strain ratio, $R_\theta = d\epsilon_y/d\epsilon_x$, can also be expressed in terms of the anisotropic parameters. Substituting $\sigma_x = \sigma_{x'} \cos^2 \theta$, $\sigma_y = \sigma_{x'} \sin^2 \theta$, and $\tau_{xy} = \sigma_{x'} \cos \theta \sin \theta$ with $\sigma_x = \tau_{yz} = \tau_{zx} = 0$ into the flow rules,

$$\begin{aligned} \epsilon_x &= \lambda \sigma_{x'} [(H + G) \cos^2 \theta - H \sin^2 \theta] \\ \epsilon_y &= \lambda \sigma_{x'} [(F + H) \sin^2 \theta - H \cos^2 \theta] \\ \epsilon_z &= \lambda \sigma_{x'} [-F \sin^2 \theta - G \cos^2 \theta] \\ \gamma_{xy} &= \lambda \sigma_{x'} [2N \sin \theta \cos \theta]. \end{aligned} \quad (8.14)$$

The strain ratio, R_θ , can be found using Eqs. (8.14) with the strain transformation $\epsilon_y = \epsilon_x \sin^2 \theta + \epsilon_z \cos^2 \theta - \gamma_{xy} \sin \theta \cos \theta$ and the identity $\cos^4 \theta + \sin^4 \theta = 1 - 2 \sin^2 \theta \cos^2 \theta$:

$$R_\theta = \frac{H + (2N - F - G - 4H) \sin^2 \theta \cos^2 \theta}{F \sin^2 \theta + G \cos^2 \theta}. \quad (8.15)$$

The permissible variation of R_θ with θ is similar to the variation of Y_θ with a maximum or minimum at 0 and 90° and only one maximum or minimum possible between 0 and 90° . Figure 8.5 shows how R_θ and Y_θ might vary. If there is a minimum in R between 0 and 90° , there will be a maximum in Y and vice versa (although the extrema do not occur at the same angles). However, if $P > R$, $Y_{90} > Y_0$.

If F , G , and H are known from x - and y -direction tension tests, N may

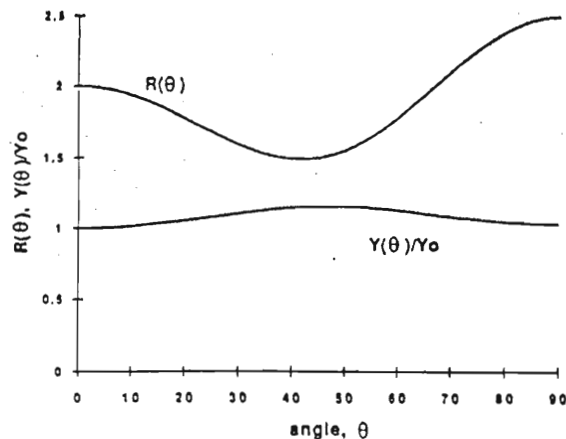


Fig. 8.5. Angular variation of R , for $R = 2$, $Q = 1.5$, and $P = 2.5$ calculated from Eqs. (8.12) and (8.15).

be determined by measuring R_θ in another direction. For example,

$$\left. \begin{aligned} R_{45} = Q = \frac{N}{F+G} - \frac{1}{2}, \quad \text{so} \quad N = (Q + \frac{1}{2})(F+G) \\ \text{or} \quad N = \frac{(2Q+1)(R+P)}{2(R+1)X^2} \end{aligned} \right\} \quad (8.16)$$

The flow rules [Eqs. 8.4] can now be rewritten in terms of the strain ratios by substituting Eqs. (8.5) and (8.16).

$$\begin{aligned} d\epsilon_x : d\epsilon_y : d\epsilon_z : d\gamma_{xy} &= RP(\sigma_x - \sigma_y) + P(\sigma_x - \sigma_z) : \\ R(\sigma_y - \sigma_z) + RP(\sigma_y - \sigma_x) : P(\sigma_z - \sigma_x) + R(\sigma_z - \sigma_y) : \\ P(2Q+1)(R+P)\tau_{xy}. \end{aligned} \quad (8.17)$$

8.1.3. *Effective stress and strain for normal loading (i.e., loading restricted to $\sigma_x, \sigma_y, \sigma_z$ with $\tau_{yz} = \tau_{zx} = \tau_{xy} = 0$)*

If the effective stress $\bar{\sigma}$ and strain $\bar{\epsilon}$ are written so as to reduce to σ_x and ϵ_x in x -direction tension, for the loading conditions

$$\bar{\sigma} = \{ [P(\sigma_x - \sigma_y)^2 + P(\sigma_x - \sigma_z)^2 + RP(\sigma_y - \sigma_z)^2] / [P(R+1)] \}^{1/2} \quad (8.18)$$

and the corresponding effective strain¹⁰ as

$$\bar{\epsilon} = C [P(\epsilon_y - R\epsilon_z)^2 + R(P\epsilon_z - \epsilon_x)^2 + (R\epsilon_x - P\epsilon_y)^2]^{1/2}, \quad (8.19)$$

where $C = [(R+1)/R]^{1/2}(R+P+1)$. This expression can be derived by setting $\bar{\sigma}\bar{\epsilon} = \sigma_x\epsilon_x + \sigma_y\epsilon_y + \sigma_z\epsilon_z$. By considering plane stress ($\sigma_z = 0$) and substituting $\sigma_y = \alpha\sigma_x$, and $\epsilon_y = -\rho\epsilon_x$, $\epsilon_z = -(1-\rho)\epsilon_x$, this can be written as $\bar{\epsilon}/\epsilon_x = [1/(\bar{\sigma}/\sigma_x)](1-\alpha\rho)$. With the flow rules, α can be expressed in terms of ρ . Substituting for $\alpha = f(\rho)$ into Eq. (8.18) for $(\bar{\sigma}/\sigma_x)$, as well as in $(1-\alpha\rho)$, $\bar{\epsilon}/\epsilon_x$ can be expressed in terms of ρ .

8.2. A nonquadratic yield criterion

Although the 1948 Hill criterion is simple and self-consistent, it is not in good agreement with either experiments or the yield loci calculated using crystallographic analyses. Calculated yield loci tend not to be elliptical. Rather it has been shown [3-5] that, for normal loading, they can be much better approximated by

$$F|\sigma_y - \sigma_x|^a + G|\sigma_z - \sigma_x|^a + H|\sigma_x - \sigma_y|^a = 1, \quad (8.21)$$

where a is a constant much larger than 2. This is a special case of a very general anisotropic yield criterion proposed by Hill [6] in 1979. Upper-bound calculations based on {111}<110> slip for fcc metals [3] and <111>-pencil glide for bcc metals [4] are fitted quite well using exponents of $a = 8$ and 6, respectively. For even-integer exponents the absolute magnitude signs are not necessary, so for plane stress loading ($\sigma_z = 0$) the criterion may be written as

$$F\sigma_y^a + G\sigma_x^a + H(\sigma_x - \sigma_y)^a = 1. \quad (8.22)$$

The flow rules are

$$\begin{aligned} d\epsilon_x : d\epsilon_y : d\epsilon_z &= G\sigma_x^{a-1} + H(\sigma_x - \sigma_y)^{a-1} : \\ F\sigma_y^{a-1} + H(\sigma_y - \sigma_x)^{a-1} : -(G\sigma_x^{a-1} + F\sigma_y^{a-1}). \end{aligned} \quad (8.23)$$

¹⁰ Hill [2] defined $\bar{\sigma}$ and $\bar{\epsilon}$ with different constants. For an x -direction tension test, his expressions reduce to

$$\bar{\sigma} = K\sigma_x \quad \text{and} \quad \bar{\epsilon} = \frac{\epsilon_x}{K}, \quad \text{where} \quad K = \left[\frac{1}{R+P+RP} \right]^{1/2}. \quad (8.20)$$

Therefore his definitions of $\bar{\sigma}$ and $\bar{\epsilon}$ differ from Eqs. (8.18) and (8.19) by factors of K and $1/K$, respectively.

The values of R and P are thus

$$R = H/G \quad \text{and} \quad P = H/F; \quad (8.24)$$

Substituting these into Eq. (8.21), the yield criterion becomes

$$P\sigma_x^a + R\sigma_y^a + RP(\sigma_x - \sigma_y)^a = P(R+1)X^a, \quad (8.25)$$

and substituting into Eq. (8.23), the flow rules become

$$\begin{aligned} d\epsilon_x: d\epsilon_y: d\epsilon_z &= P\sigma_x^{a-1} + RP(\sigma_x - \sigma_y)^{a-1}: \\ R\sigma_y^{a-1} + RP(\sigma_y - \sigma_x)^{a-1}: &-(P\sigma_x^{a-1} + R\sigma_y^{a-1}). \end{aligned} \quad (8.26)$$

The expressions for effective stress and strain are

$$\bar{\sigma} = \left\{ \frac{P\sigma_y^a + R\sigma_x^a + RP(\sigma_x - \sigma_y)^a}{P(R+1)} \right\}^{1/a}, \quad (8.27)$$

$$\bar{\epsilon} = \epsilon_x(1 - \alpha\rho)(\sigma_x/\bar{\sigma}), \quad (8.28)$$

where $\alpha = \sigma_y/\sigma_x$ and

$$-\rho = \frac{\epsilon_y}{\epsilon_x} = \left(\frac{R}{P} \right) \frac{\alpha^{a-1} - P(1-\alpha)^{a-1}}{1 + R(1-\alpha)^{a-1}}. \quad (8.29)$$

If radial isotropy is assumed, the yield criterion and flow rules further simplify to

$$\sigma_y^a + \sigma_x^a + R(\sigma_x - \sigma_y)^a = (R+1)X^a \quad (8.30)$$

$$\begin{aligned} d\epsilon_x: d\epsilon_y: d\epsilon_z &= \sigma_x^{a-1} + R(\sigma_x - \sigma_y)^{a-1}: \\ \sigma_y^{a-1} + R(\sigma_y - \sigma_x)^{a-1}: &-(\sigma_x^{a-1} + \sigma_y^{a-1}). \end{aligned} \quad (8.31)$$

Figure 8.6 shows the shapes of the yield loci for several values of a and R . Note that as $a \rightarrow \infty$, the loci approach Tresca for all values of R . As the exponent a is increased, there is much less effect of R on the ratio of biaxial to uniaxial strength, and the stress ratio for plane strain $\epsilon_y = 0$.

8.2.1. Determination of the exponent a

In the upper-bound (Bishop-Hill-type) yield locus calculations for fcc metals [3], various textures characterized by rotational symmetry about the sheet

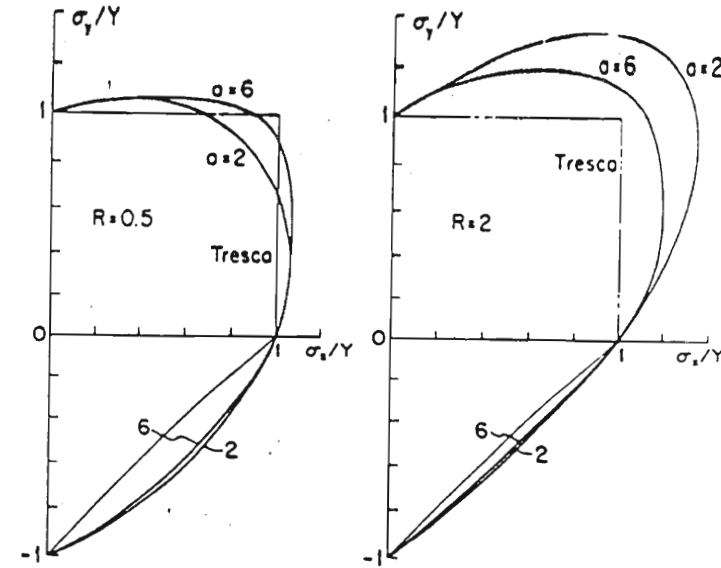


Fig. 8.6. Yield loci predicted by Eq. (8.30) for $R = 0.5$ and 2 and for several levels of the exponent a . Hill's theory [Eq. (8.7)] corresponds to $a = 2$. The Tresca locus ($a = \infty$) is shown for comparison.

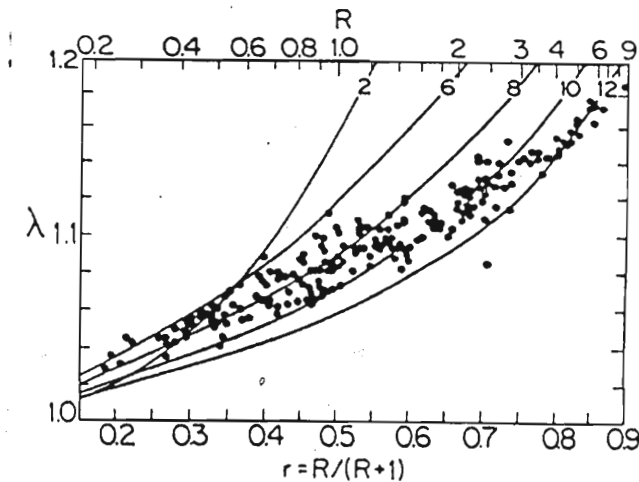


Fig. 8.7. Computed value of λ and R for various textures. The solid lines are the predictions.

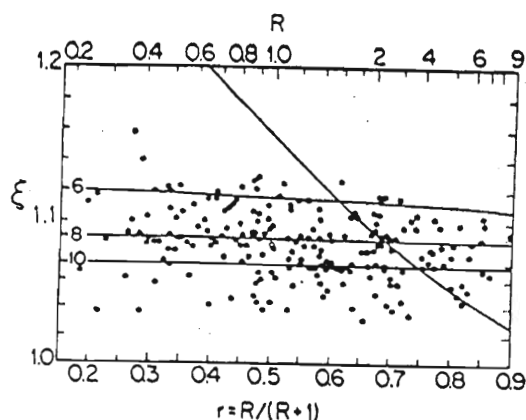


Fig. 8.8. Computed values of ξ and R for various textures. The solid lines are the predictions of Eqs. (8.33) and (8.34) for several levels of α . (From [3].)

loading paths were calculated for each assumed texture. These ratios were:

$$\left. \begin{aligned} \chi &= \sigma_x(\text{biaxial tension, } \sigma_x = \sigma_y) / \sigma_x(\text{uniaxial tension}) \\ \lambda &= \sigma_x(\text{plane strain, } \epsilon_y = 0) / \sigma_x(\text{uniaxial tension}) \\ \psi &= 2\sigma_x(\text{plane strain, } \epsilon_x = 0) / \sigma_x(\text{uniaxial tension}) \\ \xi &= \sigma_x(\text{plane strain, } \epsilon_y = 0) / \sigma_x(\text{biaxial tension, } \sigma_x = \sigma_y) = \lambda / \chi \\ \beta &= \sigma_x(\text{plane strain, } \epsilon_y = 0) / 2\sigma_x(\text{plane strain, } \epsilon_x = 0) = \lambda / \psi. \end{aligned} \right\} \quad (8.32)$$

The results of these calculations are summarized in Figs. 8.7–8.11. Each point represents the calculated ratio and the calculated R value for a single assumed texture. For comparison are shown the values of the ratios according to the yield criterion [Eq. (8.30)] and flow rules [Eqs. (8.31)] for several levels of the exponent α :

$$\left. \begin{aligned} \chi &= \left(\frac{R+1}{2} \right)^{1/\alpha} \\ \lambda &= \left\{ \frac{R+1}{\alpha^2 + 1 + R(1-\alpha)^2} \right\}^{1/\alpha} \\ \psi &= 2 \left[\frac{R+1}{2+2^\alpha R} \right]^{1/\alpha} \\ \xi &= \lambda / \chi = \left\{ \frac{2}{\alpha^2 + 1 + R(1-\alpha)^2} \right\}^{1/\alpha} \\ \beta &= \lambda / \psi = \frac{1}{2} \left\{ \frac{2+2^\alpha R}{\alpha^2 + 1 + R(1-\alpha)^2} \right\}^{1/\alpha} \end{aligned} \right\} \quad (8.33)$$

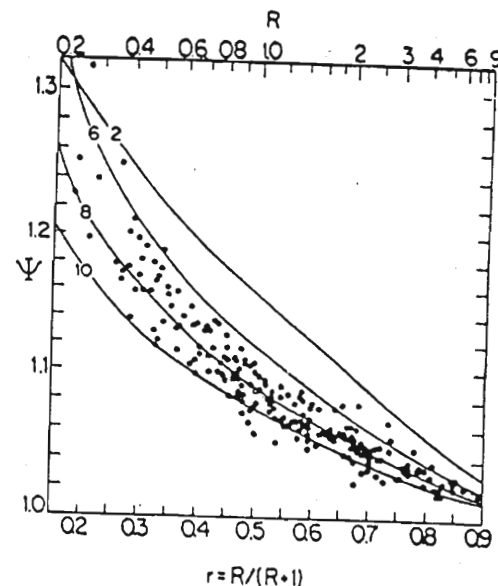


Fig. 8.9. Computed values of ψ and R for various textures. The solid lines are the predictions of Eq. (8.33) for several levels of α . (From [3].)

where α , the stress ratio for plane strain, is given by

$$\alpha = \frac{R^{1/(a-1)}}{1 + R^{1/(a-1)}}. \quad (8.34)$$

It can be seen from Figs. 8.7–8.11 that the agreement between the continuum yield criterion and the upper-bound calculations is best for values of $\alpha \approx 8$. Similar calculations [4] for the $\langle 111 \rangle$ -pencil glide model for bcc metals indicate an exponent of $\alpha \approx 6$. Lower-bound calculations (see Chapter 5) give similar results.

For all finite values of the exponent α , Eq. (8.24) predicts $\chi < 1$ if $\bar{R} < 1$, though with high values of α , χ is not much greater than unity. The experimental observation of $\chi > 1$ with $R < 1$ has been termed *anomalous behavior* [5]. While not permitted by Eq. (8.9), the crystallographic calculations (Fig. 8.7) show that this is predicted for some textures even though the general trend of Fig. 8.7 is best fitted by an equation that predicts $\chi < 1$ if $\bar{R} < 1$.

It is not possible to use this criterion for loading conditions for which the axes of principal stress do not coincide with the principal axes of anisotropy, that is, for loading conditions in which $r \neq 1$ or $\alpha \neq 0$.

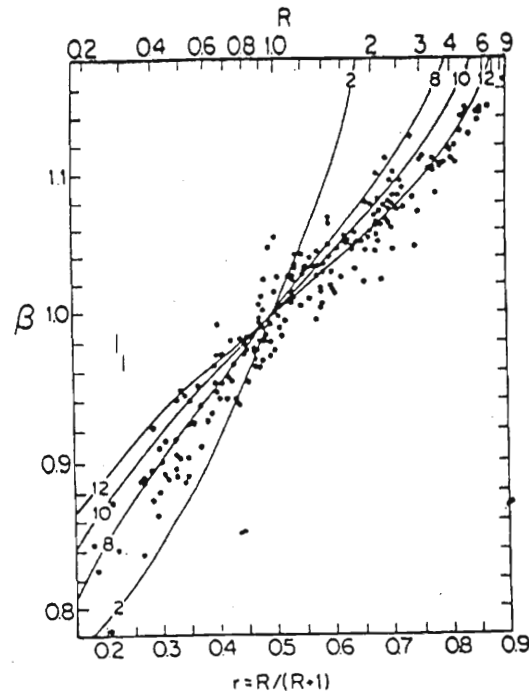


Fig. 8.10. Computed values of β and R for various textures. The solid lines are the predictions of Eqs. (8.33) and (8.34) for several levels of α . (From [3].)

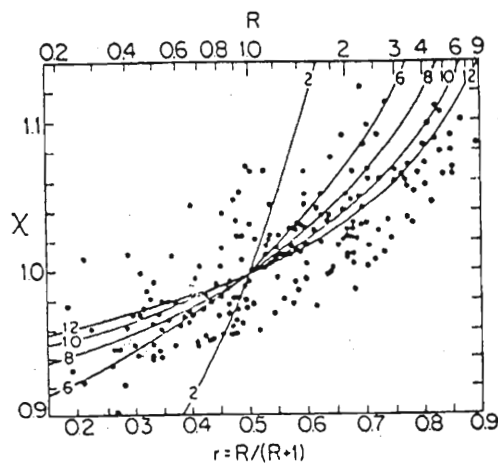


Fig. 8.11. Computed values of X and R for various textures. The solid lines are the predictions of Eqs. (8.33) and (8.34) for several levels of α . (From [3].)

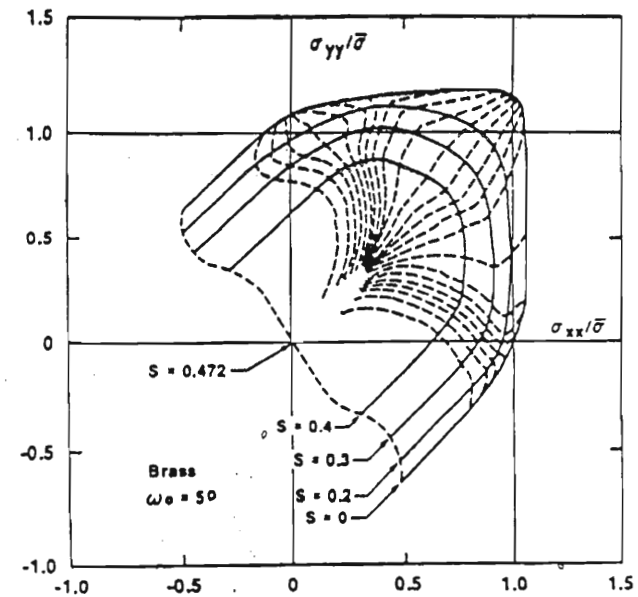


Fig. 8.12. Plane stress yield surfaces for a textured aluminum alloy sheet computed using the Bishop and Hill model. The stresses are normalized to the x -direction yield strength. S indicates the normalized level of τ_{xy} . (From Barlat and Lian [8].)

Furthermore the criterion cannot be modified by simply adding τ_{ij}^2 terms [7], but Barlat and Lian [8] have demonstrated a way to account for shear stress terms. They proposed a generalization of Eq. (8.21) that can fully accommodate planar anisotropy:

$$2\bar{\sigma}^M = a|K_1 + K_2|^M + a|K_1 - K_2|^M + (2-a)|2K_2|^M. \quad (8.35)$$

where $K_1 = (\sigma_x + h\sigma_y)/2$ and $K_2 = \{[(\sigma_x - h\sigma_y)/2]^2 + p^2\tau_{xy}^2\}^{1/2}$. For isotropy, $h = p = 0$. With an exponent $M = 8$, they found nearly perfect coincidence with the locus determined for isotropic fcc metals using the Bishop and Hill model. Lege et al. [9] have shown that this criterion accurately described the yield locus of an aluminum alloy sheet. Figures 8.12 and 8.13 show the yield loci calculated from the texture with the Bishop and Hill model and fitted to Eq. (8.35). More recently, Barlat and co-workers [10] have proposed a six-component yield criterion for a still better fit. An exponent higher than 8 was required for very strong textures.

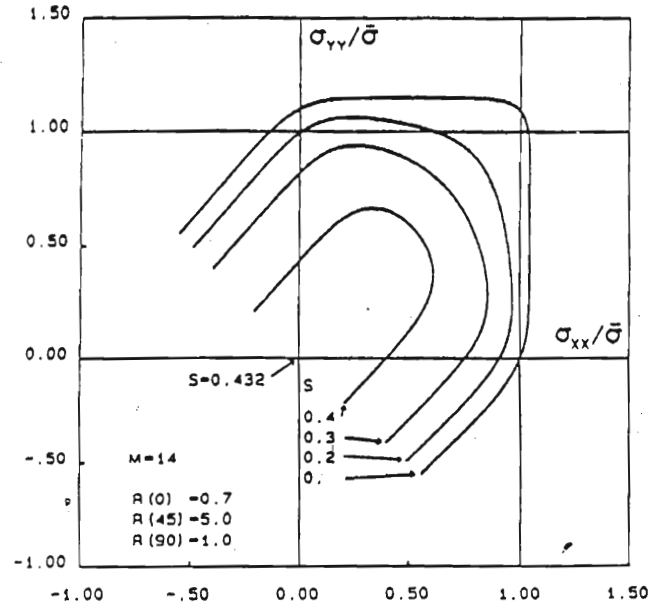


Fig. 8.13. Plane stress yield surfaces for a textured aluminum alloy sheet fitted to Eq. (8.35) with $m = 8$, $a = 1.240$, $h = 1.150$, and $p = 1.02$. The stresses are normalized to the x-direction yield strength. S indicates the normalized level of τ_{xy} . (From Barlat and Lian [8].) Note the agreement with Fig. 8.12.

8.3. Hill's general nonquadratic yield criterion

An experimental observation [11, 12] of a biaxial-uniaxial yield stress ratio greater than one for a material with an average strain ratio $\bar{R}$ less than 1 is not consistent with either the 1948 Hill criterion or the nonquadratic form suggested previously. To accommodate this "anomalous" behavior, Hill [6], in 1979, proposed a new general nonquadratic criterion of the form

$$f|\sigma_2 - \sigma_3|^m + g|\sigma_3 - \sigma_1|^m + h|\sigma_1 - \sigma_2|^m + a|2\sigma_1 - \sigma_2 - \sigma_3|^m + b|2\sigma_2 - \sigma_3 - \sigma_1|^m + c|2\sigma_3 - \sigma_1 - \sigma_2|^m = 1, \quad (8.36)$$

where 1, 2, and 3 are both axes of symmetry and principal stress. Hill proposed four simplifications of this general criterion, in all of which rotational symmetry about the 3 axis is assumed, ($f = g$ and $a = b$). For these cases there is only one strain ratio, R , and one tensile yield strength, X . However, it has been shown [13] that the first three of these special cases

are outwardly concave for many combinations of strain ratio R and m , and there are even combinations of m and R for which the loci are unbounded. Only case 4 ($a = b = 0$, $f = g = 0$),

$$(1 + 2R)|\sigma_1 - \sigma_2|^m + |2\sigma_3 - \sigma_1 - \sigma_2|^m = 2(R + 1)X^m, \quad (8.37)$$

is well behaved in this respect. It has been widely used, but there is little or no experimental or theoretical foundation for its use. It should be noted that the 1948 Hill criterion [Eq. (8.1)] is a special case of Eq. (8.36) with $m = 2$ and $a = b = c = 0$, and Eq. (8.20) is a special case with $a = b = c = 0$.

8.4. Other nonquadratic yield criteria

Other nonquadratic criteria have been suggested, but they require more experimentally determined constants. Bassani [14] proposed for a material with planar isotropy under plane stress ($\sigma_3 = 0$)

$$f = |(\sigma_1 + \sigma_2)/(2\sigma_b)|^n + |(\sigma_1 - \sigma_2)/(2\tau)|^m - 1, \quad (8.38)$$

where σ_b is the yield strength under balanced biaxial tension, τ is the yield strength under pure shear, and n and m are independent exponents. This criterion can be expressed in terms of the R value as;

$$|\sigma_1 + \sigma_2|^n + (n/m)(1 + 2R)X^{n-m}|\sigma_1 - \sigma_2|^m = X^n[1 + (n/m)(1 + 2R)], \quad (8.39)$$

where X is the yield strength in uniaxial tension. Two tests, in addition to a tension test, are required to determine the constants experimentally. Bassani showed that yield loci calculated using the Bishop and Hill method can be approximated by this equation with the appropriate exponents.

Gotoh [15] proposed the plane stress ($\sigma_3 = 0$) criterion,

$$f = A_0(\sigma_x + \sigma_y)^2 + A_1\sigma_x^4 + A_2\sigma_x^2\sigma_y^2 + A_3\sigma_y^4 + A_4\sigma_x\sigma_y^3 + A_5\sigma_y^4 + (A_6\sigma_x^2 + A_7\sigma_x\sigma_y + A_8\sigma_y^2)\tau_{xy}^2 + A_9\tau_{xy}^4, \quad (8.40)$$

which has ten independent constants. The first term allows for compressibility; so for incompressible materials $A_0 = 0$. This criterion does not require the assumption of planar isotropy. It can accommodate a wider range of planar anisotropy than Hill's 1948 criterion [Eq. (8.1)], and therefore it can be used to analyze problems of earing in deep drawing that are not treatable with Hill's criterion.

In an attempt to accommodate planar anisotropy and loading without

principal stress axes coinciding with symmetry axes, it was suggested that the yield criterion be expressed in terms of the principal stress axes rather than stress components along the symmetry axes [7]. Modifying Eq. (8.24) accordingly,

$$R_\theta \sigma_\theta + \sigma_{\theta+90} + R_\theta R_{\theta+90} (\sigma_\theta - \sigma_{\theta+90})^2 = R_{\theta+90} (R_\theta + 1) X_\theta^2. \quad (8.41)$$

where σ_θ and $\sigma_{\theta+90}$ are the principal stresses, R_θ and $R_{\theta+90}$ are the strain ratios, measured in tension tests in these directions, and X_θ is the yield strength in the θ -direction tension test. For this criterion the dependence of R_θ and X_θ on θ must be determined experimentally. This criterion was used to compare experimentally measured ratios of X_{45}/X_0 and X_{90}/X_0 with the strain ratios measured in 0° , 45° , and 90° tension tests.

Zhou [16] proposed yet a different nonquadratic formulation of an anisotropic yield criterion:

$$\bar{\sigma}^m = \left[\frac{1}{2} (F + G + H) \right] \{ [F(\sigma_x^2 + 3\tau_{xy}^2)^{m/2} + G(\sigma_x^2 + 3\tau_{xy}^2)^{m/2} + H[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]^{m/2} + 2N(\tau_{xy}^2)^{m/2} \}, \quad (8.42)$$

which reduces to Hill's 1948 criterion [Eq. (8.1)] for $m = 2$, and to Eq. (8.24) for $\tau_{xy} = 0$. He found that the experimental variation of R and yield strength in several titanium and aluminum alloys could be fitted with values of $m = 8$. The value of m can be found from measurements of the yield strength and R values in three tension tests if there is planar anisotropy. With planar anisotropy, the exponent must either be taken from crystallographic calculations (e.g., 6 or 8) or found experimentally in another type of test. The yield strength in a 45° test can be found from

$$(\sigma_{45}/\sigma_0)^m = \frac{2P(R+1)}{(R+P)(Q+1)}. \quad (8.43)$$

Still another nonquadratic criterion has been proposed [17] to accommodate both the "anomaly" and planar anisotropy:

$$c|\alpha_1\sigma_1 + \alpha_2\sigma_2|^m + h|\sigma_1 - \sigma_2|^m + 2n|\tau_{12}|^m = \sigma_0. \quad (8.44)$$

where it is convenient to take σ_0 as the yield strength in uniaxial tension in the 1 direction. Here 5 constants are independent.

In 1990, Hill [18] proposed a criterion that both allows for in-plane anisotropy and accommodates the biaxial "anomaly"

$$|\sigma_1 + \sigma_2|^m + k^m |\sigma_1 - \sigma_2|^m + |\sigma_1 + \sigma_2|^{(m/2)-1} [-2a(\sigma_1^2 - \sigma_2^2) + b(\sigma_1 - \sigma_2)^2 \cos 2\alpha] \cos 2\alpha = (2\sigma_b)^m. \quad (8.45)$$

where $a = (F - G)/(F + G) \geq 0$, $b = (F + G + 4H - 2N)/(F + G)$, σ_b is the yield strength in equibiaxial tension, and k is the ratio of the yield strength in equibiaxial tension to that in pure shear. Here σ_1 and σ_2 are the principal stresses, and α is the angle between the 1 and x axes, the x axis being a symmetry axis.

This criterion can be rewritten in terms of the stresses along the symmetry axes:

$$|\sigma_x + \sigma_y|^m + k^m [(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]^{m/2} + (\sigma_x^2 + \sigma_y^2 + 4\tau_{xy}^2)^{(m/2)-1} \times [-2a(\sigma_x^2 - \sigma_y^2) + b(\sigma_x - \sigma_y)^2] = (2\sigma_b)^m. \quad (8.46)$$

The constants a , b , k , σ_b , and m can be evaluated from tensile tests in the 0° , 45° , and 90° directions together with the yield strength in equibiaxial tension:

$$a = \frac{1}{4} [(2\sigma_b/\sigma_{90})^m - (2\sigma_b/\sigma_0)^m] \\ b = \frac{1}{2} [(2\sigma_b/\sigma_{90})^m + (2\sigma_b/\sigma_0)^m] - (2\sigma_b/\sigma_{45})^m.$$

k can be found from

$$k = (1 + 2R_{45})^{1/m}.$$

Finally,

$$m = \ln(2 + 2R_{45}) / \ln(2\sigma_b/\sigma_{45}).$$

The various yield criteria are summarized in Table 8.1.

Lequeu et al. [19] have proposed an approach they call "computer modeling of the plastic properties of textured polycrystals" (CMTP). It is a technique in which an analytic yield function (quadratic or nonquadratic) is fitted to the crystallographic yield locus of a single crystal oriented to approximate the texture.

8.5. Significance of yield locus parameters in metal forming

Some of the parameters that describe the shape of the yield locus are of practical significance in metal forming. In deep drawing or cupping, the ratio of the blank diameter to the cup diameter is called the *drawing ratio*, and the largest drawing ratio that can be successfully drawn without failure of the cup walls is called the *limiting drawing ratio* (LDR). The limiting drawing ratio of a material is largely governed by β . For a non-work-hardening

Table 8.1. Comparison of continuum anisotropic yield criteria simplified for plane stress loading ($\sigma_z = \tau_{yz} = \tau_{zx} = 0$)

Name	Eq.	Ref.	Assume planar isotropy?	Accommodate σ_y/σ_x anomaly?	Number of constants	Tests required to evaluate constants
Hill (1948)	(8.1)	[1, 2]	no	no	4	3 tension tests
Hill (1948) (simplified)	(8.7)	[1, 2]	yes	no	2	1 tension test
Hill (1979) Case 4	(8.37)	[5]	yes	yes	3	1 tension test and 1 other test
Bassani (1977)	(8.38)	[14]	yes	yes	4 (3) ^a	1 tension test and 2 (or 1) other tests ($m = 1$ is suggested)
Gotouh (1977)	(8.40)	[15]	no	no	9	4 tension tests and 1 other test
Hosford (1979)	(8.24)	[3, 4]	no ^b	no	3 ^c	2 tension tests (with suggested exponent)
(simplified)	(8.30)	[3, 4]	yes	no	2 ^c	1 tension test (with suggested exponent)
Hosford (1985)	(8.41)	[7]	no	no	— ^d	several tension tests (with suggested exponent)
Burlat and Lian	(8.35)	[8]	no	no	4 ^c	3 tension tests (with suggested exponent)
Zhou Montheillet et al. (1991)	(8.42) (8.44)	[16] [17]	no	no	4	3 tension tests
Hill (1990)	(8.45)	[18]	no	yes	5	3 tension tests and biaxial tension

^a If $m = 1$ as suggested, there are only three independent constants.

^b Different yielding behavior in orthogonal directions permitted but principal stress axes must coincide with symmetry axes.

^c If the exponent is assumed to be 6 for bcc or 8 for fcc.

^d 3 constants for each orientation of principal stress axes with assumed exponent of 6 or 8.

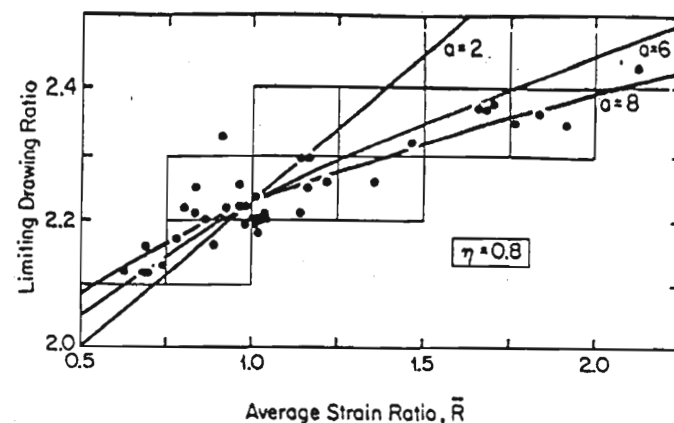


Fig. 8.14. Effect of $\bar{R}$ on the limiting drawing ratio. The experimental points are for 41 different materials including many grades of steel, 4 aluminum alloys, 6 copper alloys, and nickel. The curves are from Eq. (8.45) with values of β calculated from Eqs. (8.33) and (8.34). The level of $\eta = 0.8$ was arbitrarily selected to represent the data. (From [20].)

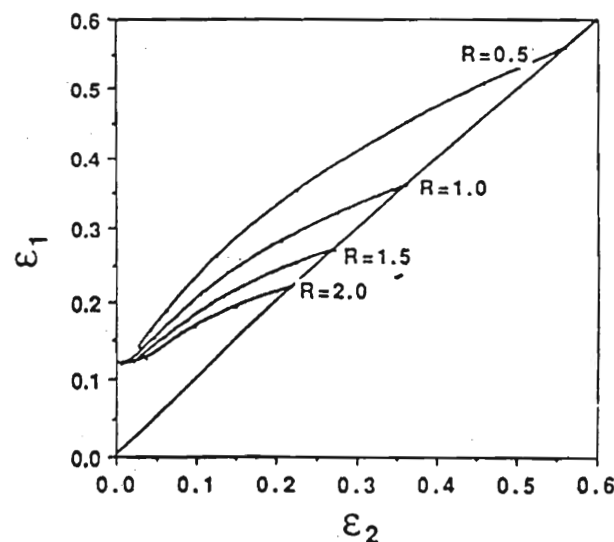


Fig. 8.15. Forming limit diagrams calculated for three values of R , using the Hill criterion [Eq. (8.7) or (8.30) with $n = 2$]. The large R dependence is not observed experimentally. (From [24].)

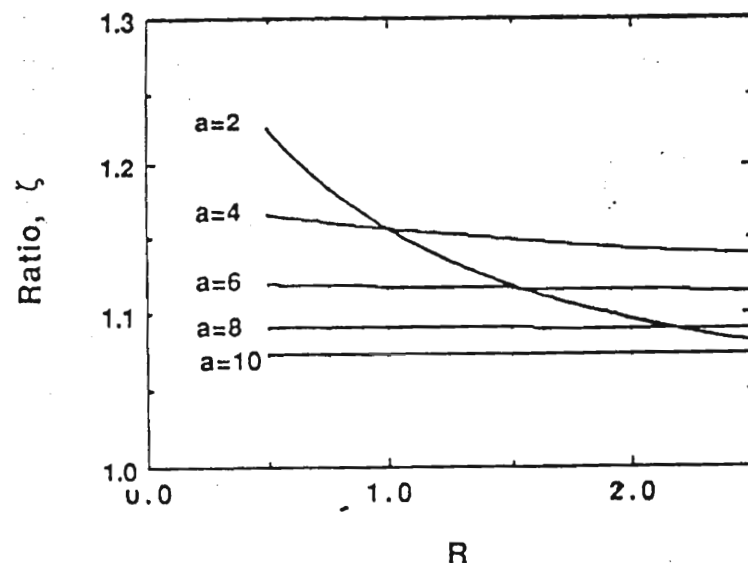


Fig. 8.16. Variation of the ratio ξ with R according to Eqs. (8.33) and (8.34). Note that ξ is almost independent of R for $a \geq 6$. (From [25].)

material, it can be shown that

$$\text{LDR} = \exp(\eta/\beta), \quad (8.47)$$

where η is a mechanical efficiency [20]. Figure 8.14 shows good agreement of this equation with experimental dependence of LDR on $\bar{R}$ when β is calculated from experimental $\bar{R}$ values using Eq. (8.33) with $a = 6$ or 8, but not when β is calculated using $a = 2$ [21].

Similarly it has been shown that the shape of the right-hand side of the forming limit diagram is very sensitive to ξ (plane-strain/biaxial strengths) [22–26]. With $a = 2$, ξ is quite dependent on $\bar{R}$ (Fig. 8.15); so FLDs calculated using $a = 2$ predict a large drop of forming limits for high values of $\bar{R}$ (Fig. 8.16). This R -dependence is not observed experimentally. In contrast, with $a = 6$ or 8, ξ is almost independent of R . When these higher exponents are used in the calculations, the shapes of FLDs calculated are virtually independent of $\bar{R}$ (Fig. 8.17) and are in good agreement with

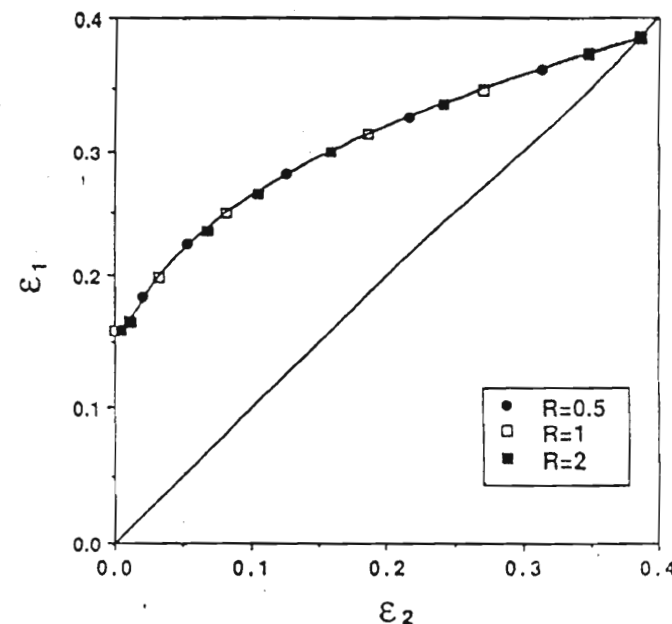


Fig. 8.17. Forming limit diagrams calculated for three values of R using Eq. (8.30) with $a = 6$. Note that the curves almost coincide, indicating a negligible effect of R , in contrast to Fig. 8.15. (From [25].)

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9

MECHANICAL TWINNING

9.1. General

A twin is a region of a crystal in which the lattice orientation is a mirror image of the orientation of the rest of the crystal. Normally the boundary between the twin and the matrix lies in or near the mirror plane. Twins may form during crystal growth (annealing twins), but the concern here is formation of mechanical twins by uniform shearing, as illustrated in Fig. 9.1. In this figure, plane 1 undergoes a shear displacement relative to plane 0 (the mirror plane), plane 2 undergoes the same shear relative to plane 1, plane 3 relative to plane 2, etc. The net effect of the shears is to reproduce the lattice, but with the new (mirror-image) orientation.

Both slip and twinning are deformation mechanisms involving shear displacements on specific crystallographic planes and in specific crystallographic directions; however, there are important differences:

1. With slip, the magnitude of the shear displacement on a plane is variable, but it is always an integral number of interatomic repeat distances (i.e., nb , where b is the Burgers vector). Slip occurs on only a few of the parallel planes (i.e., the planes on which the slip occurs are separated by relatively large distances). With twinning, the shear displacement is a fraction of an interatomic repeat distance and involves shear between every successive plane.
2. The twinning shear is always directional in the sense that shear in one direction is not equivalent to shear in the opposite direction (e.g., twinning in fcc crystals occurs by shear on the (111) plane in the $[1\bar{1}2]$ direction but not by shear in the $[\bar{1}12]$ direction). In fcc metals slip can occur on a (111) plane in either the $[\bar{1}10]$ or the $[1\bar{1}0]$ directions.
3. Slip leaves the lattice virtually unchanged, and lattice rotation is gradual. Twinning reorients the lattice abruptly.

There are several very good general reviews of mechanical twinning [1-10].

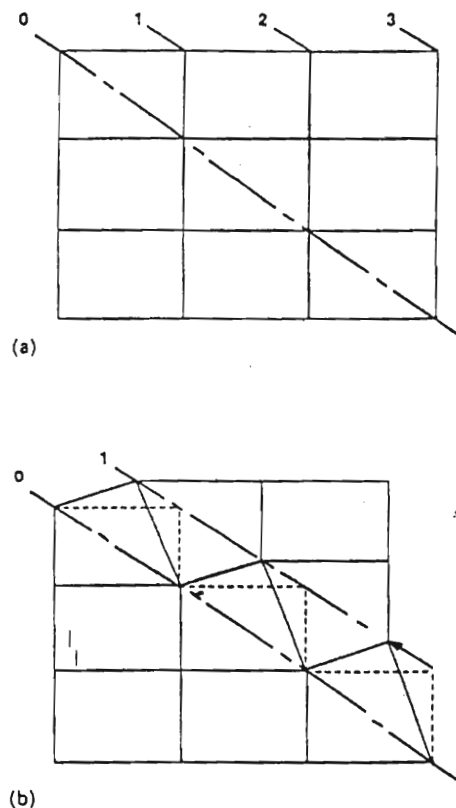


Fig. 9.1. Formation of a twin by shearing between each parallel plane of atoms.

9.2. Formal notation

There is a formal notation system that describes twinning. Figure 9.2 illustrates a sphere of material, the top half having undergone a twinning shear. There are two material planes that remain undistorted. The first undistorted plane, K_1 , is the "twinning" plane (i.e., the mirror plane and the plane of shear). The second undistorted plane is denoted by K_2 . The direction of shear is η_1 . The second characteristic direction, η_2 , lies in K_2 and is perpendicular to the intersection of K_1 and K_2 . The plane containing η_1 , η_2 , and the normals to K_1 and K_2 is the plane of shear.

All of the twinning elements need not be rational, and in crystals of low symmetry classes they are not. There are several possibilities:

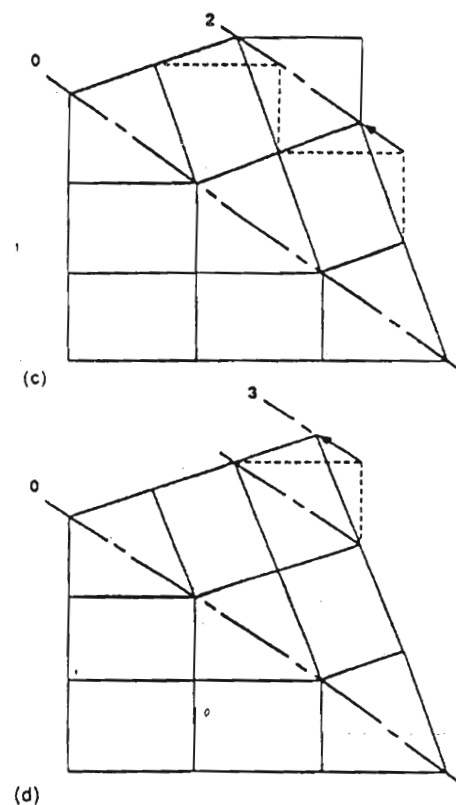


Fig. 9.1. (Continued).

Type I twins: K_1 and η_2 are rational but K_2 and η_1 are irrational;
 Type II twins: K_2 and η_1 are rational but K_1 and η_2 are irrational;
 Compound (rational) twins: K_2 , η_1 , K_1 , and η_2 are all rational.

Only rational twins will be considered here. Table 9.1 gives the twinning elements reported for rational twins in several metals.

9.3. Twinning shear

For all of the twins listed in Table 9.1, K_2 and η_2 have the same forms as K_1 and η_1 , and twinning can be regarded as a shear creating a mirror image in an orthorhombic cell, as shown in Fig. 9.3. These cases can be analyzed

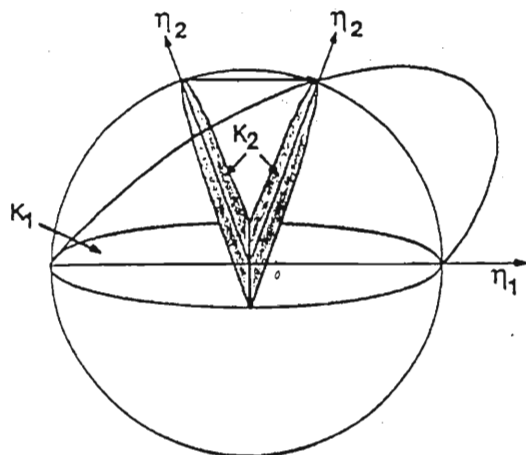


Fig. 9.2. A homogeneously sheared hemisphere. The two undistorted planes are K_1 and K_2 . The shear direction is η_1 , and η_2 is the direction in the K_2 plane that is perpendicular to the intersection of K_2 and K_1 .

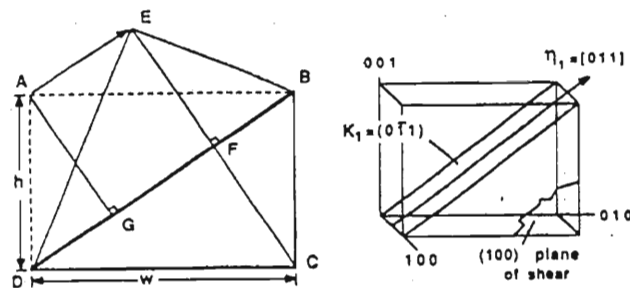


Fig. 9.3. Sketch of a simple orthorhombic cell undergoing twinning.

Table 9.1. Twinning elements for simple twins

Structure	K_1	K_2	η_1	η_2	h	w
fcc and dia. cubic	$\{111\}$	$\{1\bar{1}\bar{1}\}$	$\langle 11\bar{2} \rangle$	$\langle 112 \rangle$	a	$\sqrt{2}a$
bcc	$\{112\}$	$\{1\bar{1}\bar{2}\}$	$\langle 11\bar{1} \rangle$	$\langle 111 \rangle$	a	$a/\sqrt{2}$
hcp	$\{10\bar{1}2\}$	$\{101\bar{2}\}$	$\langle 10\bar{1}1 \rangle$	$\langle 1011 \rangle$	c	$\sqrt{3}a$

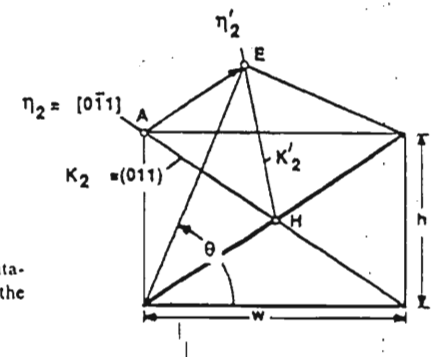


Fig. 9.4. Twinned cell showing the reorientation angle θ of the $[010]$ direction and the twinning elements K_2 and η_2 .

as follows: Let the twinning plane, K_1 , be $(10\bar{1})$ and the twinning direction, η_1 , be $[101]$. The shear plane, (100) , is projected in the figure. When the upper left-hand portion of the crystal undergoes the twinning shear, point A moves to a new position, E . The shear strain, γ , is defined as $\overline{AE}/\overline{EF}$. The length $\overline{EF}$ can be related to a and to c by noting that triangle EFB is similar to triangle DCB , so that $\overline{EF}/\overline{DC} = \overline{EB}/\overline{DB}$. Substituting $\overline{DC} = w$, $\overline{EB} = \overline{BC} = h$, and $\overline{DB} = \sqrt{w^2 + h^2}$, $\overline{EF} = wh/\sqrt{w^2 + h^2}$. The length $\overline{AE} = \overline{DB} - 2\overline{FB}$, and $\overline{FB}$ can be found by again noting that triangle EFB is similar to triangle DCB , so $\overline{FB}/\overline{EB} = \overline{BC}/\overline{DC}$, or $\overline{FB} = h^2/\sqrt{w^2 + h^2}$. Therefore, $\overline{AE} = \sqrt{w^2 + h^2} - 2h^2/\sqrt{w^2 + h^2}$. Substituting for $\gamma = \overline{AE}/\overline{EF} = (\sqrt{w^2 + h^2} - 2h^2/\sqrt{w^2 + h^2}) / (wh/\sqrt{w^2 + h^2}) = (wh/\sqrt{w^2 + h^2})$. Simplifying,

$$\gamma = \frac{w}{h} - \frac{h}{w}. \quad (9.1)$$

If the center of the cell (point H) is connected to points A and E , as shown in Fig. 9.4, it is clear that line $\overline{HA}$ in the parent is equivalent to line $\overline{HE}$ in the twin. Therefore, $K_2 = (011)$ and $\eta_2 = [0\bar{1}1]$. The angle of reorientation, θ , of the $[010]$ direction and (001) plane is given by

$$\tan\left(\frac{\theta}{2}\right) = \frac{h}{w}. \quad (9.2)$$

It should be noted that the analysis here ignores the geometric possibility of forming the twin by a shear in the opposite direction, as shown in Fig. 9.5. If the twin were formed by the opposite shear, K_1 , η_1 , and θ would be the same, but K_2 , η_2 , and the shear strain γ would be different. However, consideration of the atom movements involved in this proposed reverse shear suggests that the atom at B would interfere with the movement of the atom at J to position E , while there would not be similar interference with the

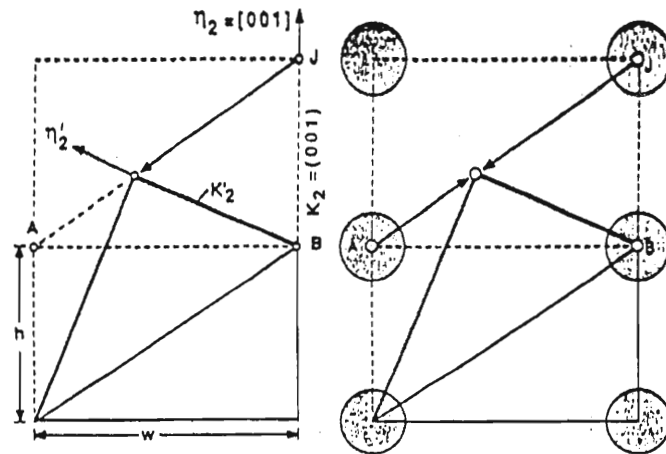


Fig. 9.5. Sketch showing that it is geometrically possible to form a twin by shear in the opposite direction (left) but that there would be more interference by other atoms (right).

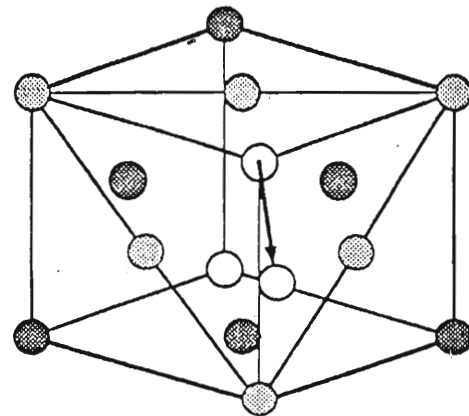


Fig. 9.6. Atom displacement in $\{111\}\langle 11\bar{2}\rangle$ twinning of fcc crystals. Note that the shear direction shown is possible, but the shaded atom would interfere with shear in the opposite direction.

atom at A moving to position E . It has been suggested that for a given twin, the twinning shear is always the smallest possible shear that can produce the twin. This is in agreement with the conclusion presented here, and for the other twins we will consider, but may not be true in some complicated structures.

9.4. Twinning in fcc metals

Twinning in fcc metals occurs on $K_1 = \{111\}$ planes in $\eta_1 = \langle 11\bar{2}\rangle$ directions. The atomic movements are illustrated in Fig. 9.6, where it is evident that the sign of the shear is correct. The shaded atom can move easily in the $[11\bar{2}]$ direction but would be hindered from moving in the $[\bar{1}\bar{1}2]$ direction by the atom centering the top face.

The shear strain can be calculated from Eq. (9.1) by substituting $w = \sqrt{2}a$ and $h = a$,

$$\gamma = \frac{\sqrt{2}a}{a} = \sqrt{2} = 1/\sqrt{2} = \sqrt{2}/2 = 0.707.$$

The reorientation of the $[001]$ direction is [from Eq. (9.2)],

$$\theta = 2 \arctan\left(\frac{h}{w}\right) = 2 \arctan\left(\frac{1}{\sqrt{2}}\right) = 70.5^\circ.$$

Inspection of Fig. 9.7 shows that $K_2 = \{11\bar{1}\}$ and $\eta_2 = \langle 112\rangle$.

Figure 9.8 compares the stacking of close-packed $\{111\}$ planes in an fcc crystal with the stacking of the close-packed $\{0001\}$ planes in an hcp crystal.

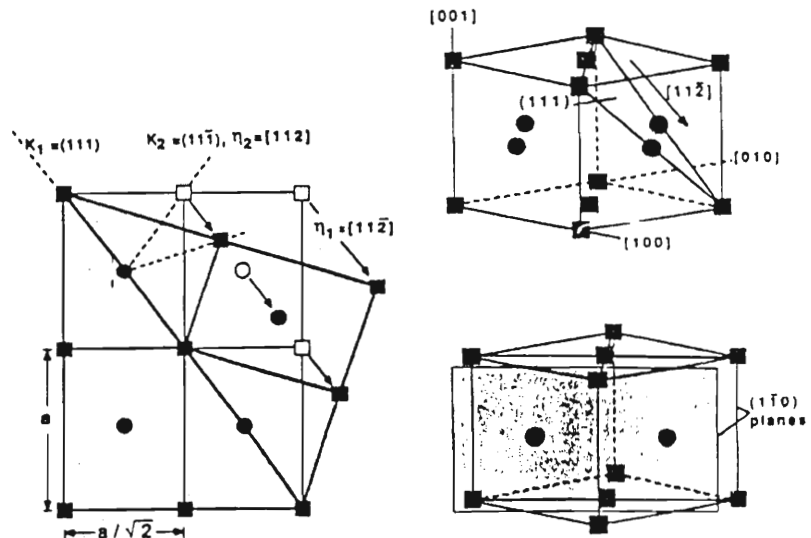


Fig. 9.7. Plan view of the (110) plane of an fcc crystal showing $\{111\}[112]$ twinning

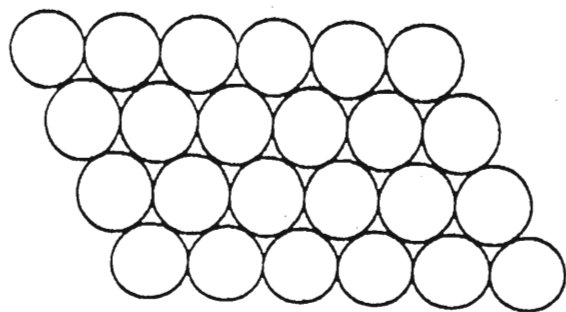
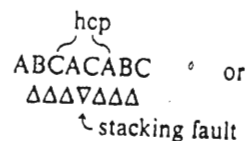


Fig. 9.8. Close-packed plane in an fcc or hcp crystal.

In both structures the close-packed planes have twice as many "holes" for the next plane as atoms. The holes fall into two groups, symbolized by Δ and ∇ . The atoms on the next plane will rest on one of these sets of holes. (It does not matter which, since the two sets differ only by a 60° rotation.) Suppose the Δ set is filled by the second layer. In the hcp structure the atoms in the third layer rest in the ∇ holes of the second layer directly above the occupied hole as $\Delta\nabla\Delta\nabla$ or alternatively by the atom positions $ABABAB$.

In the fcc structure the third layer rests in the Δ holes of the second layer and are not directly over the atoms in the first layer. Only in the fourth layer are the atom positions directly over the first layer. The stacking sequence can be symbolized by either $\Delta\Delta\Delta\Delta\Delta\Delta$ or $ABCABC$.

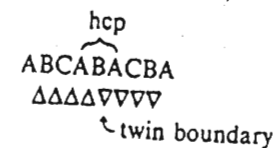
Slip in an fcc crystal displaces a layer in the B (or Δ) positions to equivalent positions. However, the entire displacement does not occur simultaneously. A partial dislocation will first move the plane to C (∇) positions. Later a second partial dislocation may complete the move to the B (Δ) positions. Between the two partial dislocations the stacking sequence is



A narrow region, three planes thick, has the hcp stacking. Since the metal normally crystallizes in an fcc structure, this thin layer increases the energy; so there is a specific surface energy (the *stacking fault energy*) associated with this defect.

A twin boundary in an fcc metal

packing. The stacking sequence is mirrored across the twin boundary



There is a two-plane-thick layer of hcp packing. Thus the twin boundary energy is closely related to the SFE and is about one-half to two-thirds as great. The twinning shear has the same direction and magnitude, $(a/6)\langle 11\bar{2} \rangle$, as the Burgers vectors of the partial dislocations formed by dissociation of $(a/2)[01\bar{1}]$ slip dislocations. Indeed, the movement of an $(a/6)\langle 11\bar{2} \rangle$ partial dislocation on every successive $\{111\}$ plane would create a twin.

9.5. Twinning in bcc metals

Figure 9.9 is a plan view of the (110) plane in a bcc metal. The upper right half has undergone the twinning shear on the $(\bar{1}12)[1\bar{1}1]$ system. The shear strain is $\gamma = \sqrt{2}/2$ and produces a tensile elongation parallel to $[001]$. Thus the deformation is equal in magnitude to that in fcc twinning but of the opposite sign. There is another large difference, however. In an fcc crystal

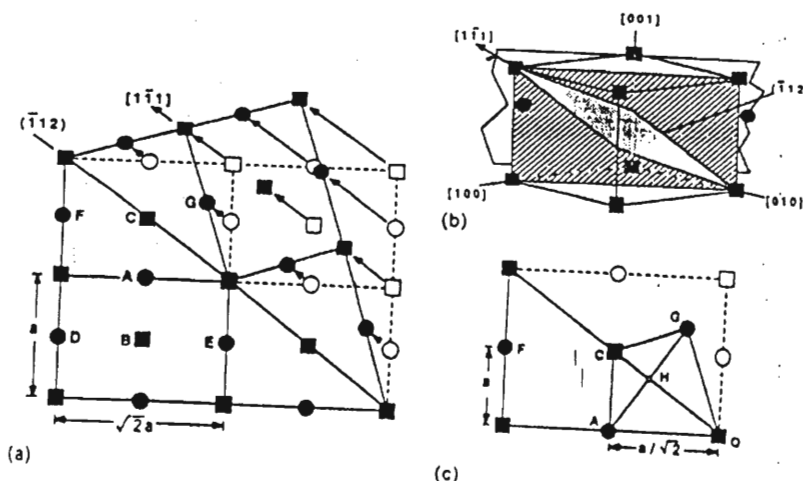


Fig. 9.9. (a) Plan view of the (110) plane in a bcc metal showing the upper right half twinned on the $(\bar{1}12)[1\bar{1}1]$ system. The atoms in one (110) plane are shown as squares, while those in the (110) planes one plane forward and one plane back are indicated by circles. (b) Perspective sketch showing the twinning elements. (c) Nearest neighbor distances.

all of the nearest-neighbor distances near a twin boundary are correct so that every atom near the boundary has 12 near neighbors at the same distance. In bcc crystals, however, some near-neighbor distances between atoms near the boundary are not correct. In fact, the distance $\overline{AG}$ is a little smaller than the usual distance (for example, $\overline{CQ}$), as shown in Fig. 9.9(c). Consider the atom at A . The nearest neighbors at B and C (two each) and at D , E , and F (one each) are at the correct distance from A (i.e., $a\sqrt{3}/2$). However, the atom at G is not at exactly the correct distance.

The distance $\overline{AG}$ can be found by realizing that $\overline{AG} = 2\overline{AH}$ and that triangle AHQ is similar to triangle CAQ , so $\overline{AH}/\overline{AC} = \overline{AQ}/\overline{QC}$.

Therefore, $\overline{AH} = \overline{AC} \times \overline{AQ}/\overline{QC} = (a/2)(a/\sqrt{2})/[(a/2)^2 + (a/\sqrt{2})^2]^{1/2} = [a^2/(2\sqrt{2})]/[a\sqrt{1/4 + 1/2}] = a/\sqrt{6}$. The distance, $\overline{AG} = 2\overline{AH} = 2a/\sqrt{6} = 0.8165a$. Comparing this with the correct "near-neighbor" distance of $a\sqrt{3}/2 = 0.8662a$, there is a difference of $(0.8165 - 0.8662)/0.8660 = 0.0572$ or 5.7%.

9.6. Twinning in hcp metals

The number of slip systems is limited in the hcp metals. All of the easy slip directions, $\langle 11\bar{2}0 \rangle$, are perpendicular to the c axis, and therefore slip does not produce any elongation or shortening parallel to the c axis. This is in contrast to the cubic metals (both fcc and bcc), in which slip can produce any shape change. Therefore, twinning is much more important in the hcp metals than in the cubic metals.

The most common twinning system in hcp metals is $\{10\bar{1}2\}\langle\bar{1}011\rangle$. This is a simple mode of the form in Fig. 9.3 and Eqs. (9.1) and (9.2). The shear strain is $\gamma = (c/a)/\sqrt{3} - \sqrt{3}/(c/a)$ and $\theta = 2 \arctan[(c/a)/\sqrt{3}]$. Figure 9.10

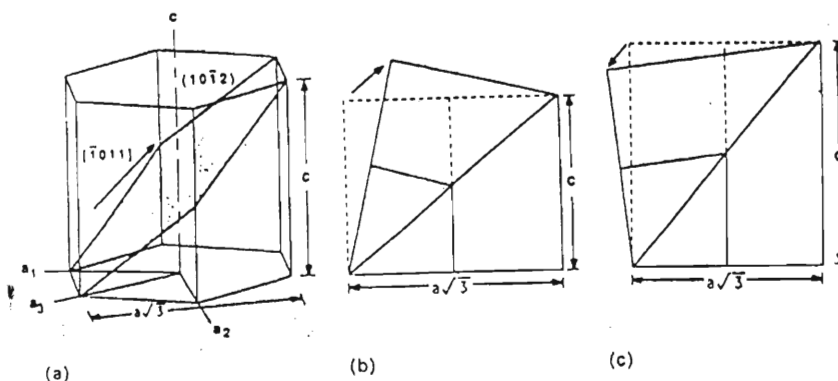


Fig. 9.10. (a) $\{10\bar{1}2\}\langle\bar{1}011\rangle$ twinning in hcp crystals. (b) For $c/a < \sqrt{3}$, the direction of shear is $[1011]$, and twinning will occur under tension parallel to the c axis. (c) For $c/a > \sqrt{3}$, the direction of shear is $[1011]$ and twinning will occur under compression parallel to the c axis.

Table 9.2. The c/a ratios of various hcp metals

Metal	c (nm)	a (nm)	c/a
Be	0.3584	0.2286	1.568
Cd	0.5617	0.2979	1.886
Hf	0.5042	0.3188	1.582
Mg	0.5210	0.3209	1.624
Ti	0.4683	0.2590	1.587
Zn	0.4947	0.2665	1.856
Zr	0.5148	0.3231	1.593
Spherical atoms			$\sqrt{8/3} = 1.633$

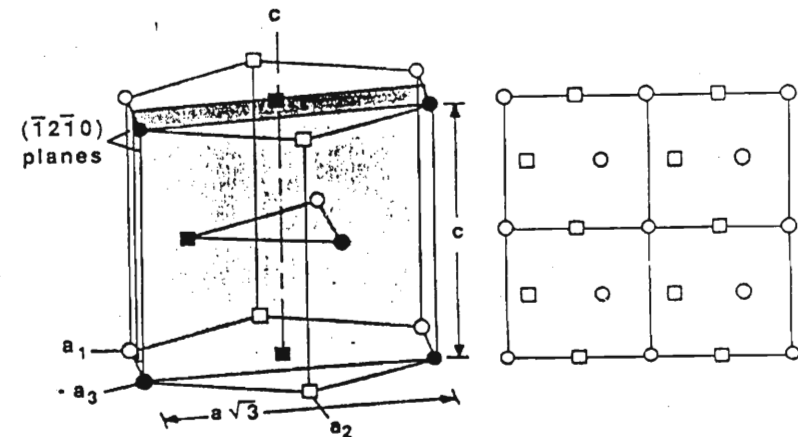


Fig. 9.11. Hexagonal cell (left) showing atom positions in adjacent $\{10\bar{1}0\}$ planes and a plan view of several cells (right). Squares indicate one $\{10\bar{1}0\}$ plane, and circles indicate the adjacent $\{10\bar{1}0\}$ planes.

shows that the direction of shear associated with this twinning system depends on the c/a ratio. These are given in Table 9.2 for a number of hcp metals. Only zinc and cadmium have c/a ratios greater than $\sqrt{3}$, and therefore only these metals twin when compression is applied parallel to the c axis. For all of the others $c/a < \sqrt{3}$, so twins are formed by compressive stresses parallel to the c axis (or tension perpendicular to it). It has been found [11] that the frequency of twins in Cd-Mg solid solution alloys decreases as the composition approaches that for which $c/a = \sqrt{3}$.

The atomic motions in twinning of the hcp lattice are very complex because the atoms do not all move in the direction of shear. Figures 9.11, 9.12, and 9.13 illustrate this for a hypothetical hcp metal with a c/a ratio of

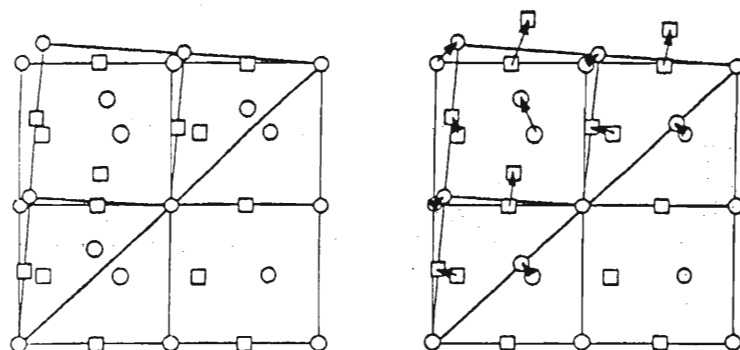


Fig. 9.12. Plan view of $(\bar{1}2\bar{1}0)$ plane in a twinned hcp crystal. These drawings assume that the mirror plane passes through the corners of the unit cell. The atom positions at the left are drawn to mirror those in the untwinned part of the crystal. The necessary movements of the individual atoms are shown at the right. It is assumed that the atoms near the interface take up positions in the interface that are intermediate between those in the twin and untwinned region.

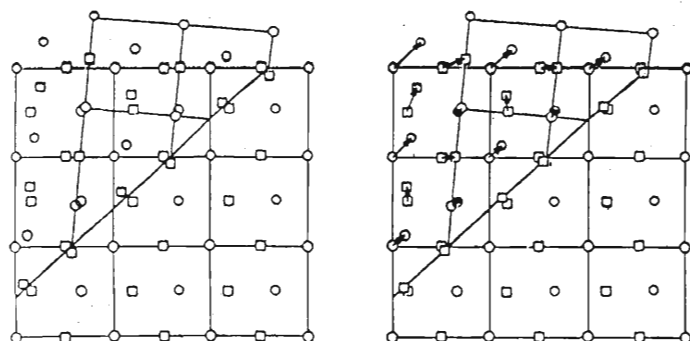


Fig. 9.13. Plan view of $(\bar{1}2\bar{1}0)$ plane in a twinned hcp crystal showing an alternative possibility for the position of the mirror plane. If the mirror plane passes through points midway between atoms, as shown, the "shuffle" of atoms is much reduced.

$\frac{1}{\sqrt{3}} = 1.588$. Figure 9.11 shows the atom positions in two adjacent $(\bar{1}2\bar{1}0)$ planes, which are the planes in which the atoms move. Figure 9.12 shows the atom movements that would result in a twin if the twin-matrix interface were to go through the corners of the unit cell. Figure 9.13 shows the twin boundary passing between the atom positions. In either case it can be seen that the shear is not homogeneous. Different atoms move different amounts, and there are components of motion toward and away from the mirror plane. These atomic movements have been described as a shear plus a *shuffle*. The

Table 9.3. Other twinning elements in hcp metal

Metal	K_1	K_2	η_1	η_2	γ
Mg	$\{10\bar{1}1\}$	$\{\bar{1}013\}$	$\langle 10\bar{1}2 \rangle$	$\langle 30\bar{3}2 \rangle$	— ^a
Mg	$\{10\bar{1}3\}$	$\{\bar{1}011\}$	$\langle 30\bar{3}2 \rangle$	$\langle 10\bar{1}2 \rangle$	— ^a
Zr, Ti	$\{11\bar{2}1\}$	$\{0001\}$	$\langle 11\bar{2}6 \rangle$	$\langle 11\bar{2}0 \rangle$	— ^b
Zr, Ti	$\{11\bar{2}2\}$	$\{11\bar{2}4\}$	$\langle 11\bar{2}3 \rangle$	$\langle 22\bar{4}3 \rangle$	— ^c
	$\{11\bar{2}4\}$	$\{11\bar{2}2\}$			— ^c

$$^a \gamma = \frac{c}{\sqrt{3}a} - \frac{1}{\sqrt{3}a/c}$$

$$^b \gamma = \frac{a}{c}$$

$$^c \gamma = \frac{2}{3} \left(\frac{c}{a} - \frac{2a}{c} \right) (=0.224 \text{ for Zr, and } 0.218 \text{ for Ti}).$$

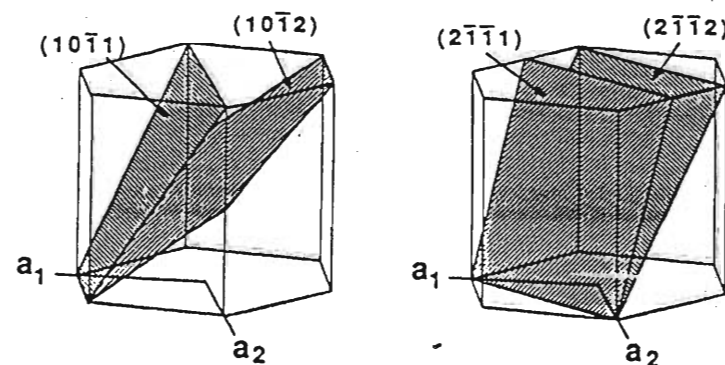


Fig. 9.14. Other twinning planes in hcp crystals.

$\{10\bar{1}2\}$ twinning, considered here, does not seem to require any component of movement perpendicular to the $(\bar{1}2\bar{1}0)$ plane of projection, but such motions may be required in other twinning modes.

The misfit near the mirror plane appears to be quite high. In an hcp lattice each atom has 12 nearest neighbors at approximately the same distance. However, many of the near-neighbor distances are perturbed near a twin boundary. It may well be that the apparent atomic misfit near the boundary is lessened by translation of the twinned region relative to the matrix either parallel or perpendicular to the boundary or both. Such a translation would not be detectable by x-ray diffraction.

There are other twinning modes reported in some of the hcp metals that are not of the simple type described by Eqs. (9.1) and (9.2) and illustrated in Fig. 9.3. Some of these are listed in Table 9.3. The twin planes are shown in Fig. 9.14.

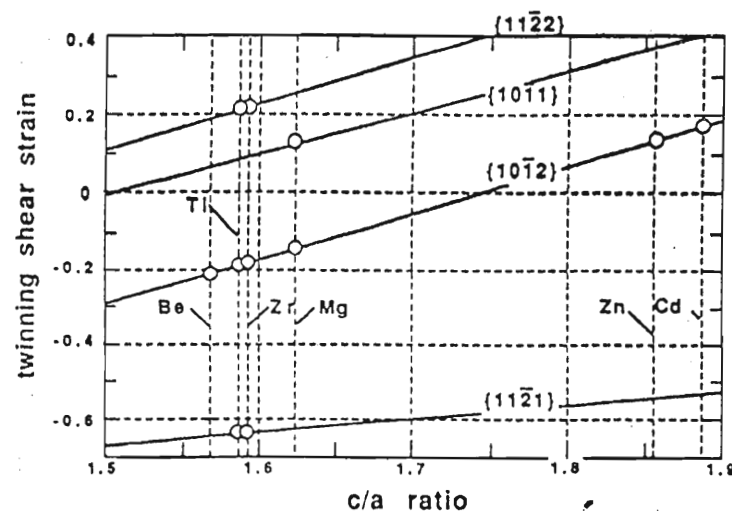


Fig. 9.15. Variation of the shear strain, γ , with the c/a ratio for several twinning modes in hcp metals. The sign convention is that a positive shear strain causes elongation parallel to the c axis. For $\{11\bar{2}1\}$ twinning γ is negative, while for $\{11\bar{2}2\}$ and $\{10\bar{1}1\}$ twinning, γ is positive, and for $\{10\bar{1}2\}$ and $\{10\bar{1}1\}$ twinning, γ is positive, and for $\{10\bar{1}2\}$ twinning the sign of γ depends on c/a . (Adapted from Yoo [3].)

The twinning shear strain for the various modes depends on the c/a ratio, as shown in Fig. 9.15 but only changes sign for $\{10\bar{1}2\}\langle 10\bar{1}1 \rangle$.

9.7. Other twinning modes

Some of the twinning modes reported in other metals are summarized in Table 9.4. These cannot be described by the shears in Fig. 9.3, which lead to Eq. (9.1). The lattice shear for $\{103\}\langle \bar{1}03 \rangle$ twinning in tin is shown in

Table 9.4. Twinning elements in other metals

Metal	K_1	K_2	η_1	η_2	γ
β -tin	$\{301\}$	$\{\bar{1}01\}$	$\langle \bar{1}03 \rangle$	$\langle 101 \rangle$	0.119*
	$\{101\}$	$\{\bar{1}01\}$	$\langle 10\bar{1} \rangle$	$\langle 101 \rangle$	—
rhombohedral (Sb, As, Bi)	$\{110\}$	$\{001\}$	$\langle 00\bar{1} \rangle$	$\langle 110 \rangle$	—

* $\gamma = \frac{1}{2} \left(\frac{a}{c} - \frac{c}{a} \right)$.

$\gamma = \frac{1}{2} \cos \alpha$ where α is the angle between the axes.

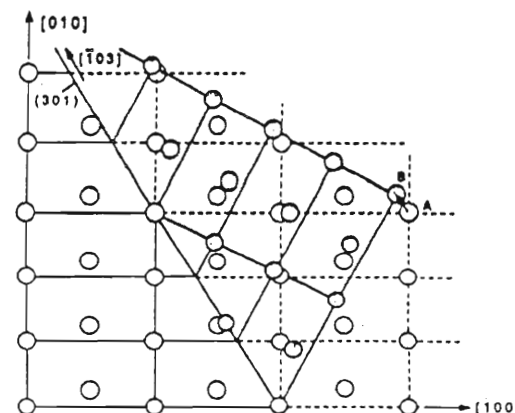


Fig. 9.16. Plan view of the (010) plane in tin, showing $\{103\}\langle \bar{1}03 \rangle$ twinning.

Fig. 9.16. The mirror image lattice is formed by a small shear accompanied by a shuffle.

9.8. Pseudotwinning in ordered solutions

A crystal that is a random solid solution can undergo the same twinning as pure metals. However, the situation is quite different for ordered solid solutions. A shear strain that would produce a twin in a random solid solution now causes a new form of ordering, and hence in a strict sense should be considered as a martensitic transformation rather than twinning. Such shears do occur and are sometimes called *pseudotwins*. Consider as an example a solution of A and B , as shown in Fig. 9.17. This would be regarded as a bcc crystal if the A and B atoms were randomly distributed. As a $B2$ (or $CsCl$) ordered structure, A atoms occupy the corners of the unit cell, and B atoms occupy the center (or vice versa, depending on what one calls the corner). This structure can be thought of as consisting of $\{100\}$ planes of A atoms alternating with $\{100\}$ planes of B atoms.

Figure 9.17 shows what happens if this structure undergoes the $\{112\}\langle \bar{1}\bar{1}1 \rangle$ shear characteristic of twinning in bcc metals. In the "twinning" region the A and B atoms are arranged on alternating (110) planes rather than alternating (001) planes. Furthermore, each B atom has four A atoms and four B atoms as nearest neighbors instead of being surrounded by 8 A atoms. Clearly the order has been changed mechanically, and a metastable phase has been formed from the stable phase. This then is a diffusionless (or athermal) mechanical transformation.

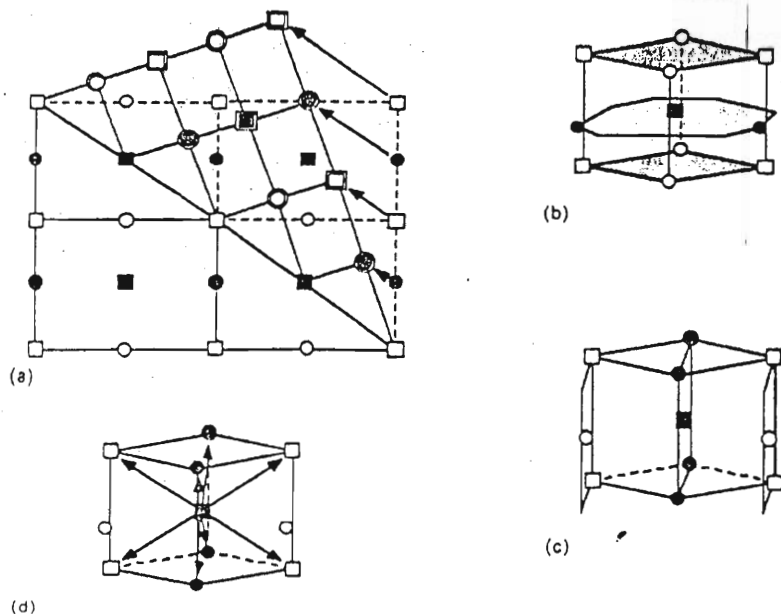


Fig. 9.17. Change of order that accompanies pseudotwinning in an ordered bcc ($B2$) structure. A atoms are indicated as open circles and squares, B atoms as filled circles and squares. Squares are positions in one $\{110\}$ plane and circles are positions in parallel $\{110\}$ planes. The atom movements shown in (a) change the original order of alternating (001) planes (b) to a new order of alternating (110) planes (c). Now each B atom has 4 B and 4 A atoms as nearest neighbors, as shown in (d).

Iron-beryllium alloys of about 25 at. % Be (near the composition Fe_3Be) order in a DO_3 or "pseudo- $B2$ " structure (with Be in half of the body-centering positions). When deformed in tension at room temperature (Fig. 9.18), almost all of the deformation occurs by the type of shear discussed previously [12]. When unloaded, they will "untwin" to the stable structure, reversing the strain, but with a great deal of hysteresis. The memory effect observed in ordered TiNi , CuZn , and other alloy systems may be related to this.

9.9. The shapes of twins

Deformation twins are generally lenticular (lens shaped), as shown in Fig. 9.19(a). [This is in contrast to twins that are formed during recrystallization. These annealing twins are bounded by almost perfectly flat planes parallel to K_1 , as shown in Fig. 9.19(b).] Because deformation twins are lenticular, the

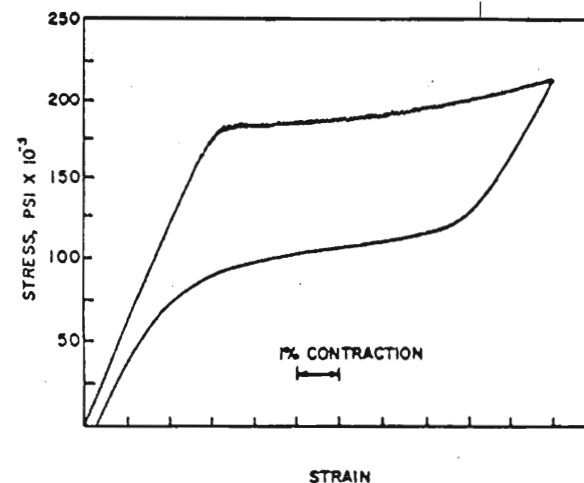


Fig. 9.18. Stress-strain curve in compression for an ordered single crystal of Fe_3Be . (From Richman [12].)

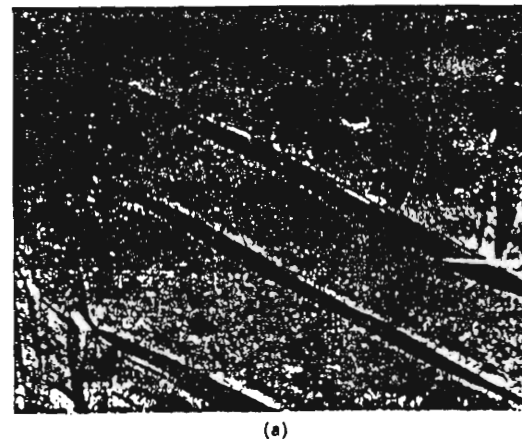


Fig. 9.19. Microstructures illustrating the different appearance of deformation and annealing twins. (a) Deformation twins in magnesium ($400\times$).

boundaries between the twinned and untwinned regions do not coincide exactly with the twinning plane (K_1). Their central plane is, however, approximately parallel to K_1 . In fcc and bcc metals twins are very narrow and in fact usually look like parallel lines. In contrast, $\{10\bar{1}2\}\langle 10\bar{1}1 \rangle$ twins in hcp metals are relatively fat. The shape of the deformation twins is

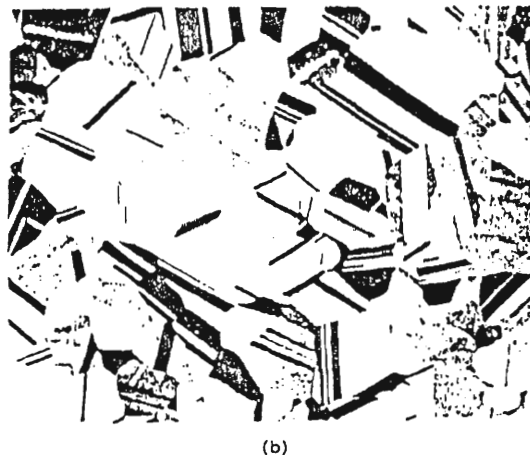


Fig. 9.19. Microstructures illustrating the different appearance of deformation and annealing twins. (b) Annealing twins in 70/30 brass (200 $\times$) (courtesy of A. Graf).

probably related to the overall energy change as a twin forms. There are two main contributions to this overall energy change: One is from the introduction of a new surface (the twin boundary), which has a surface energy; the other is the strain energy that results when one portion of material (the twinned region) undergoes a plastic shear while the surrounding material does not. There is mismatch between the twins and the matrix that must be accommodated by elastic distortion of the parent and/or slip in the parent. This accommodation involves a considerable energy that is minimized when the twins are long and narrow. In other words, the accommodation energy is low when the aspect ratio (ratio of length to thickness) of the twin is very large. On the other hand, the surface energy is minimized when the aspect ratio approaches unity (a spherical shape). For twins with a high shear strain, the strain energy term should predominate so that a high aspect ratio would be expected. Figure 9.20 shows that there is a correlation of twin shape with the twinning shear strain, high aspect ratios correlating with high shear strains. This figure was constructed from measurements made on micrographs in a few published papers with no attempt to correct for how the section of polish intersects K_1 .

The correlation in Fig. 9.20 may be aided by the fact that the surface energies of the high-shear-strain twins tend to be lower than the surface energies of the low-shear-strain twins. At the twin boundaries of fcc metals all of the nearest-neighbor distances are correct, and in bcc metals only a few are wrong. The twin boundaries in hcp metals are much more complicated.

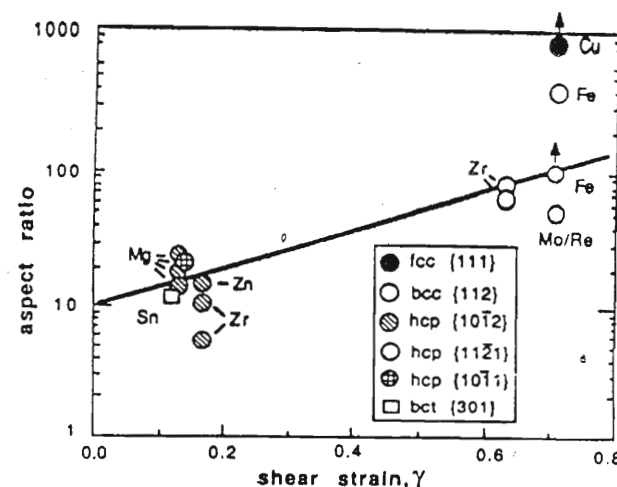


Fig. 9.20. Correlation of the aspect ratio with the shear strain of the twin. The aspect ratio was measured from published micrographs. The surface energy also affects the aspect ratio.

9.10. Mechanism of twin formation

Deformation twins form with such extreme rapidity that their formation is often accompanied by audible sounds. The sounds emitted by tin as it twins are called "tin cry." In a tension test, load drops occur as a twin is formed and cause the stress-strain curve to be serrated. Figure 9.21 is the stress-strain curve for a cadmium single crystal.

Twinning, like slip, must occur by the passage of dislocations. The Burgers vector of these dislocations must have the magnitude (and direction) of the shear between successive planes. In the case of fcc metals, the Burgers vector for twinning is $(a/6)\langle 211 \rangle$, which is the same as the partial dislocations formed by dissociation of $(a/2)\langle 110 \rangle$ slip dislocations. However in twinning, unlike slip, a single $(a/6)\langle 211 \rangle$ dislocation must pass on every successive {111} plane. Cottrell and Bilby [13] proposed a mechanism to explain how this might occur. In their "pole" mechanism (Fig. 9.22) there is a screw dislocation with a $(a/3)\langle 111 \rangle$ Burgers vector normal to the {111} plane containing the $(a/6)\langle 211 \rangle$ partial dislocation. When the moving partial encounters this pole, one-half of it winds its way upward around the pole while the other half winds its way downward. In this way the same partial dislocation passes on every successive {111} plane and at the same time expands outward.

It is not really clear what yield criterion is appropriate for twinning. It is tempting to assume a critical shear stress criterion similar to Schmid's law for slip. Indeed this has been assumed in most analytic treatments. However

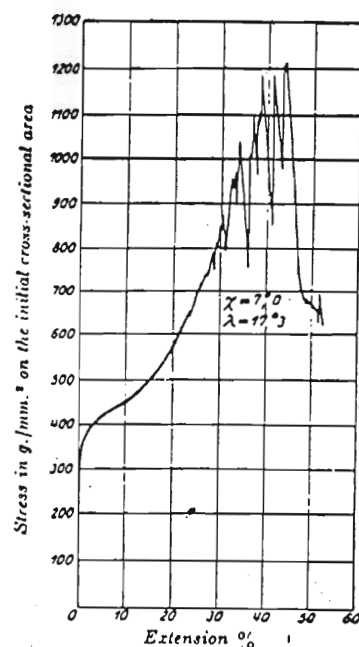


Fig. 9.21. Stress-strain curve of a cadmium single crystal. Each serration corresponds to the formation of a twin. (From Schmid and Boas [1].)

several investigations have questioned the validity of a critical shear stress criterion. Deviations from a critical shear stress criterion may be due to the necessity for some slip to nucleate twinning, or to a sensitivity of twinning to the stress normal to the twin plane.

In discussing this problem, it is important to recognize that the stress necessary to nucleate a twin may be very different from that for propagation of the twin. The jerkiness of the stress-strain curves when deformation is by twinning suggests this. The reason for a high nucleation stress is not hard to rationalize. As a twin is initiated, the ratio of its surface area to volume is very high, and the work to produce the surface must come from the mechanical work expended. On a per-volume basis this can be expressed as

$$dw = \sigma dc = \tau \gamma df, \quad (9.3)$$

where dw is the incremental work per volume expended in causing an incremental volume fraction, df , to undergo twinning, γ is the shear strain associated with twinning, and τ is the required shear stress. For small twins, dw/df will be high; so $\tau = (1/\gamma) dw/df$ must also be high.

It is possible that the nucleation of a twin is sensitive to the normal stress across the twin plane. According to the principle of normality, if the

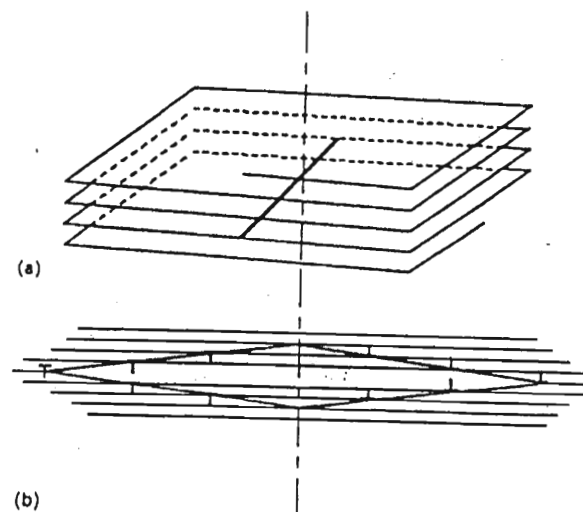


Fig. 9.22. Pole mechanism for twin formation. (a) A partial dislocation approaches a screw dislocation normal to the plane on which it is moving. Part of the partial dislocation runs up the spiral ramp and part runs down it both spiraling outward. In this way the same partial dislocation will move on each successive plane. (b) A cross section shows the lens-shaped twin formed.

shear stress required for twinning depends on the normal stress, there must be a plastic strain normal to the twin plane. Conversely, a strain normal to the twin plane would imply that twin formation is sensitive to the normal stress across the plane. The atomic misfit at a twin boundary may produce a small dilation. However, as the twin thickens, the dilated region simply moves with the boundary without increasing the macroscopic dilation strain. This implies that the stress for nucleation of a twin may depend on the normal stress, but the stress for propagation of the twin should not.

If dilation does occur at a twin boundary, the size of the affected region would be so small that the dilation would not be detectable by macroscopic measurements. It is even possible that the atomic misfit near the boundary may be relieved by a small translation in the twin plane so that the positions on opposite sides of the boundary do not exactly mirror one another. Such a shift, only at the boundary, would not be detectable by x-ray or electron diffraction.

In a polycrystal, the grain size has two effects on twinning. The size of the largest (and first) twins formed in a grain is limited by the grain size. With decreasing grain size, the average twin size must also decrease, and with smaller twins the ratio of surface area to volume increases. Therefore, more energy (higher average stress) is required to produce the same volume

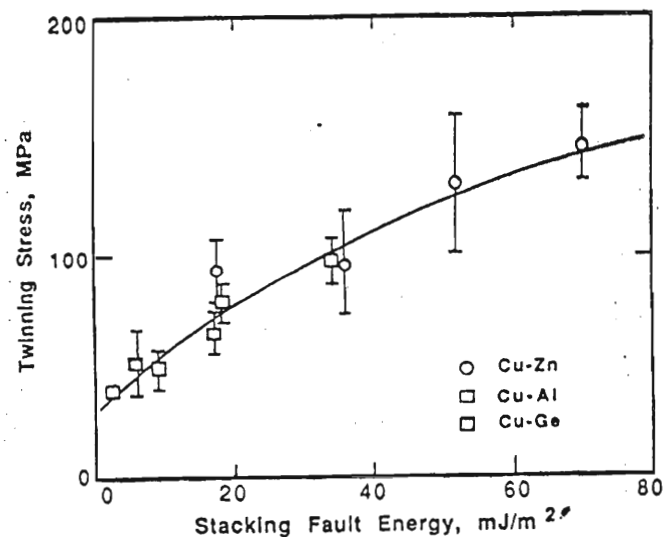


Fig. 9.23. Dependence of twinning stress on stacking fault energy in copper-based alloys. (Adapted from [14].)

fraction twin in fine-grained polycrystals. Decreasing grain size also tends to lessen the jerkiness of the macroscopic stress-strain curves. The reason is that because of stress concentrations only a small region may need be raised to the stress required for twin nucleation. In other grains, already nucleated twins may be growing. The level of the macroscopic stress-strain curve is the average of the stress in all regions. If measurements of σ and an orientation factor are used to deduce τ in Eq. (9.3), the level of τ will depend more on twin growth than twin nucleation.

In bcc and fcc metals, slip can produce the same shape change as twinning. Whether slip or twinning occurs as a load is applied depends on which requires the lower stress. Mechanical twinning is rare in most fcc and bcc metals except at low temperatures or high strain rates. This implies that decreasing temperature and increasing strain rates tend to increase the stresses required for slip more than the stresses required for twinning.

Composition is also important. In fcc metals, twinning is more common in metals and alloys of low stacking fault energy. Figure 9.23 shows how the twinning stress increases with stacking fault energy (SFE) in copper-based alloys. Silver and gold which have very low SFEs twin at room temperature, whereas mechanical twinning is not observed in metals like aluminum, lead, or nickel, which have high SFEs.

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EFFECTS OF TWINNING ON YIELD LOCUS SHAPE

The shape of the yield loci of textured metals in which twinning is an important deformation mechanism can be substantially affected by twinning. Because twinning is sensitive to changes in the sign of the stress, yield loci need not be centered on the origin. This is most apparent in hcp metals.

10.1. Hcp metals—experimental observations

For textured hcp metals, twinning causes different yielding properties in tension and compression. These differences are quite large in wrought magnesium alloys. Table 10.1 lists the tensile and compressive yield strengths of several product shapes of magnesium alloy AZ31B. These differences are explained by the textures and $\{10\bar{1}2\}\langle 10\bar{1}1 \rangle$ twinning. The textures in the wrought magnesium products are such that the c axis is approximately normal to the prior rolling or extrusion direction. Yield strengths are normally measured in tests with the tension and compression axes aligned with the prior working directions. In the compression tests, yielding can occur by twinning. In tension twinning is not possible; so yielding must occur by slip, and for magnesium easy slip is limited to the (0001) plane, which is unfavorably oriented. The tensile yield strengths of α -titanium and α -zirconium alloys are also higher than the compressive yield strengths, but the difference is much less because in these metals there are other twinning

Table 10.1. Yield strengths (MPa)*

Condition	Tension	Compression
Annealed	152	110
Hard sheet	221	179
Extruded bars	200	97
Extruded tubes	165	83

* Data from *Metals Handbook* (ASM, vol 1, 8th ed. p. 1107) converted from 10^3 psi to MPa.

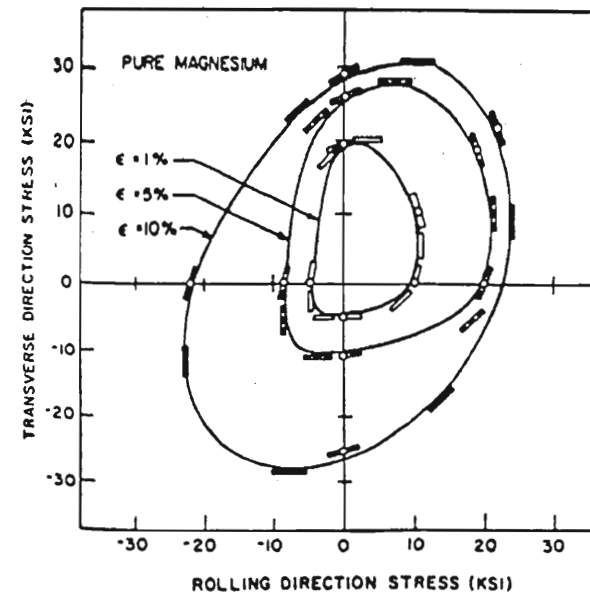


Fig. 10.1. Yield locus for textured magnesium. (From [1].)

systems that operate under compression parallel to the c axis (or tension perpendicular to it) and because easy slip is not restricted to the (0001) plane.

Figures 10.1 and 10.2 are yield loci for sheets of magnesium [1] and titanium alloys [2]. Yield loci for other titanium alloys are similar [2, 3].

10.2. Hcp metals—theory

Calculation of yield loci for textured hcp metals is far more complex than for cubic metals because of the number of different deformation modes that must be considered. Figure 10.3 shows the yield locus for an hcp metal with an ideal (0001) $\langle 10\bar{1}0 \rangle$ texture (single crystal) calculated using very simple lower-bound assumptions and arbitrarily assumed ratios of the critical shear stresses for $\{1\bar{1}00\}\langle 11\bar{2}0 \rangle$ slip and $\{11\bar{2}2\}$ and $\{10\bar{1}2\}$ twinning [4]. More sophisticated analyses have been proposed by Thornburg and Piehler [5] and by Tomé and Kocks [6], but these too must assume ratios of critical shear stresses.

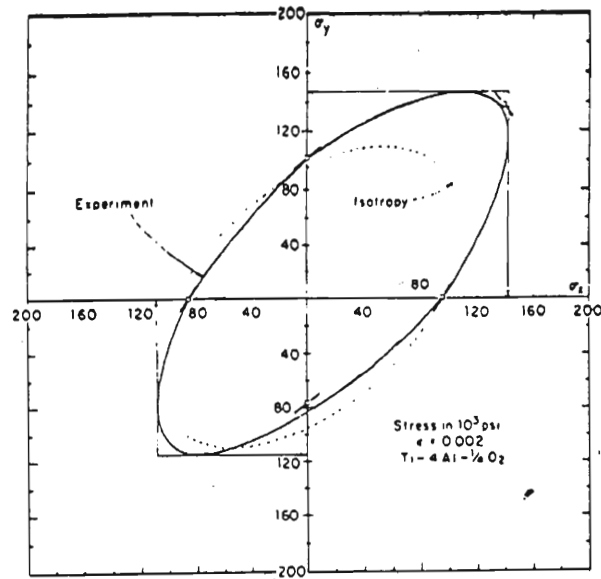
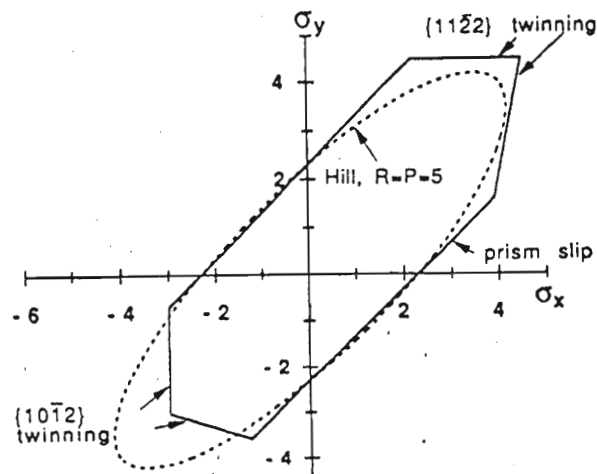
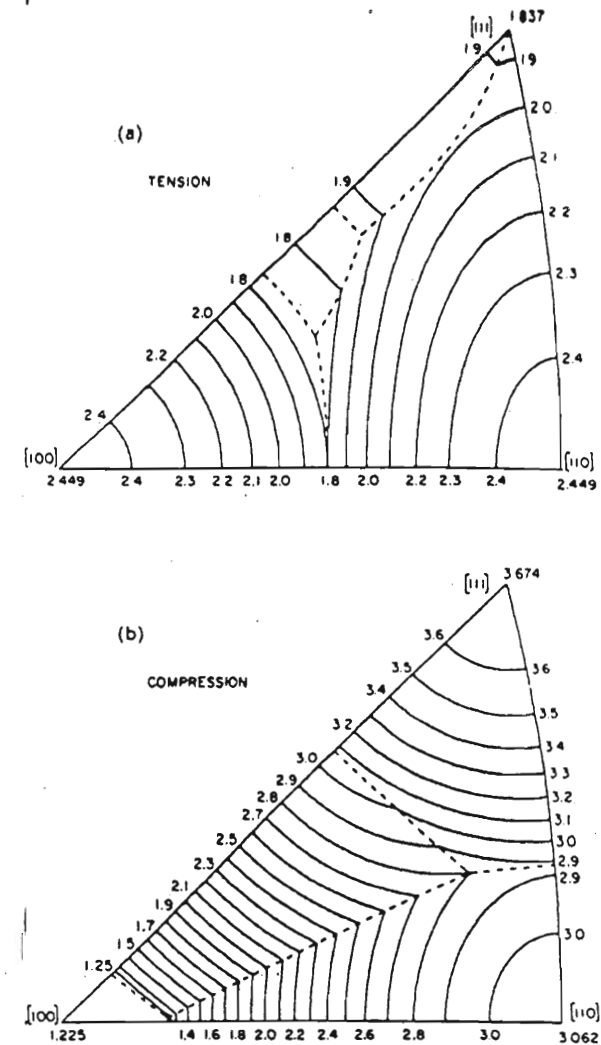


Fig. 10.2. Yield locus for textured titanium alloy Ti-4Al-1O. (From Lee and Backofen [2].)

Fig. 10.3. Yield locus for an ideally textured hcp sheet, calculated on the assumption that the critical shear stresses to active $\{10\bar{1}0\}$ slip, $\{10\bar{1}2\}$ twinning, and $\{11\bar{2}2\}$ twinning are in the ratio 1:1.5:2. The locus predicted by the Hill criterion for $R = P = 5$ is shown for comparison. (From Lee and Backofen [2].)Fig. 10.4. Orientation dependence of M for axisymmetric deformation by $\{1\bar{1}1\}\langle 112 \rangle$ twinning in fcc crystals. (a) Tension and (b) compression. (From Chin *et al.* [8].)

10.3. Cubic metals—theory

It has been shown [7] that the Taylor analysis can be generalized to include the effects of twinning. The generalized Taylor factor to be minimized is $M = (\sum_i s_i + \alpha \sum_i t_i)/\epsilon$, where s_i and t_i are the shear strains resulting from

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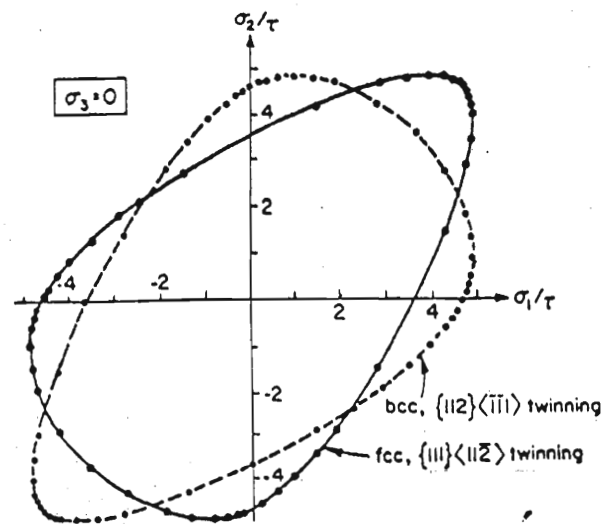


Fig. 10.5. Calculated yield loci for randomly oriented polycrystals deforming by $\langle \bar{1}\bar{1}1 \rangle \{112\}$ twinning (bcc) and $\langle 11\bar{2} \rangle \{111\}$ twinning (fcc). (From [11].)

slip (s_i) and twinning (t_i), α is the ratio of the critical shear stress for twinning to that for slip, and ε is the external strain. Chin et al. [8] used a Taylor-type analysis to calculate the orientation dependence of M for axisymmetric flow for fcc crystals deforming by $\{111\} \langle 112 \rangle$ twinning. Their results for tension and compression are reproduced in Figs. 10.4(a) and (b). (These calculations are equally applicable for $\{112\} \langle \bar{1}\bar{1}1 \rangle$ twinning in bcc crystals with reversed signs.)

Comparison of the two figures reveals that the average M value for fcc compression is much higher than for fcc tension. This implies that randomly oriented fcc metals should have higher yield strengths in compression than in tension if twinning plays an important role in the deformation. Figure 10.5 shows the forms of the yield loci calculated using a Bishop and Hill-type analysis [9] from the stress states capable of activating multiple twinning [10, 11]. Again the conclusion is that for fcc metals the yield strengths should be higher in compression than in tension (and the converse for bcc metals), the average M factors being 3.63 for uniaxial tension and 4.64 for compression.

Richards and Reid [12] studied the yielding behavior of pure iron at 77 K in compression and tension. In randomly oriented material, they observed twins in 57% of the grains after a strain of 8% in compression and in only 24% of the grains after the same strain in tension. This difference was rationalized in terms of the Schmid factors for twinning on the most

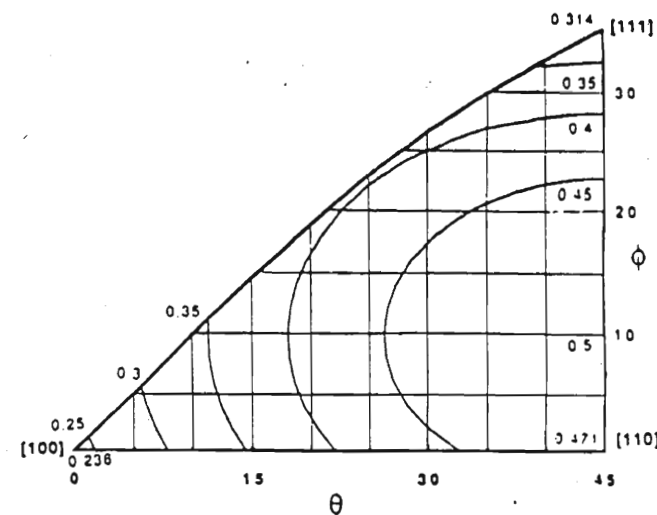


Fig. 10.6. Maximum Schmid factor for $\langle 11\bar{1} \rangle \{112\}$ twinning of bcc single crystals in uniaxial compression. The system $[11\bar{1}]\{112\}$ is most favored for all orientations in the basic triangle.

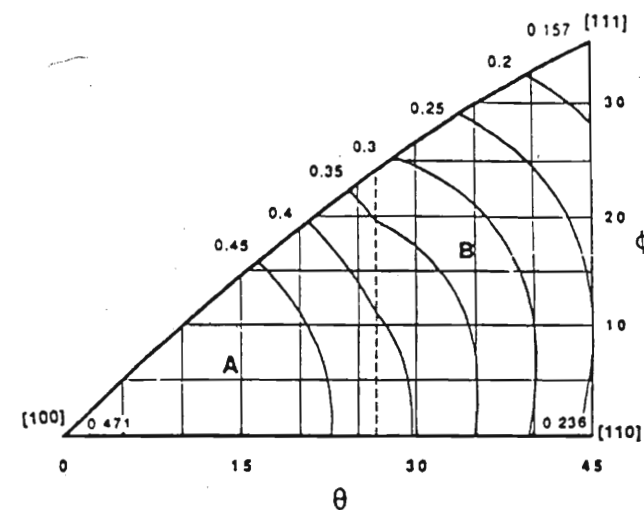


Fig. 10.7. Maximum Schmid factor for $\langle \bar{1}\bar{1}1 \rangle \{112\}$ twinning of bcc single crystals in uniaxial tension. The system $[111](2\bar{1}\bar{1})$ is most favored in region A, and $[11\bar{1}](2\bar{1}1)$ is most favored in region B.

avored systems. For compression, the average Schmid factor (Fig. 10.6) is very much higher than for tension (Fig. 10.7). However, the difference in measured yield strengths was very small (97,800 psi for compression versus 100,800 psi for tension). In iron with a $\langle 110 \rangle$ -fiber texture, the values were 94,100 and 100,500 psi, respectively.

Qualitatively the same strength differential (SD) effect should be predicted for directional slip in bcc metals since the direction of easy slip is the "forward" twinning direction [13]. Yet the SD effect reported in high-strength steels is of the opposite sign (higher yield strength in compression) [11] and therefore is probably due to a pressure sensitivity of yielding.

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11

ORIENTATIONS OF LINES

11.1. Random orientations

Consider the possible orientations of directions relative to another fixed direction, for example, the x axis. Directions can be represented as points on a hemisphere. With a large number of randomly oriented directions, there is an equal (or approximately equal) density of points over the hemispherical surface. The fraction of directions that lie within an angle ψ of the x axis is the same as the fraction of the hemispherical surface area that lies within the angle ψ of x (Fig. 11.1). To calculate this area, first consider the differential surface area, dA , between ψ and $d\psi$. The radius of this element is $r' = r \sin \psi$; so its circumference is $2\pi r \sin \psi$ and its area is $dA = (2\pi r \sin \psi)(r d\psi) = 2\pi r^2 \sin \psi d\psi$. Integrating between 0 and ψ gives the area,

$$A = \int_0^\psi 2\pi r^2 \sin \psi d\psi = 2\pi r^2(1 - \cos \psi). \quad (11.1)$$

Since the entire area of the hemisphere is $2\pi r^2$, the fraction of the total hemispherical area within an angle ψ of x is

$$f_\psi = 1 - \cos \psi \quad (11.2)$$

and the probability of a randomly oriented direction at an angle $\leq \psi$ from x is

$$p = 1 - \cos \psi. \quad (11.3)$$

The probability of a direction lying between ψ and $\psi + \Delta\psi$ is given by $\Delta p = \cos \psi - \cos(\psi + \Delta\psi)$. Using the relation $\cos(x + y) = \cos x \cos y - \sin x \sin y$, $\cos(\psi + \Delta\psi) = (\cos \psi \cos \Delta\psi - \sin \psi \sin \Delta\psi)$. For small $\Delta\psi$, $\cos \Delta\psi \rightarrow 1$ and $\sin \Delta\psi \rightarrow \Delta\psi$; so

$$\Delta p = \Delta\psi \sin \psi \quad \text{or} \quad \frac{dp}{d\psi} = \sin \psi. \quad (11.4)$$

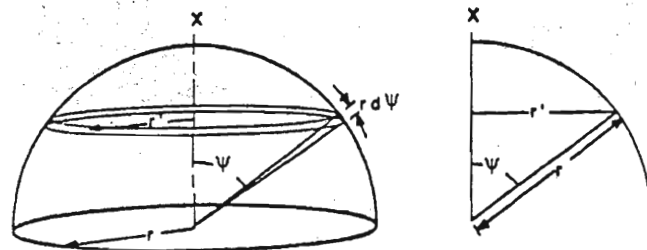


Fig. 11.1. Hemisphere representing all possible directions.

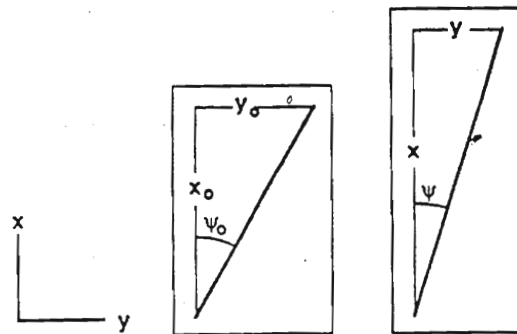


Fig. 11.2. Reorientation of a line during axisymmetric elongation.

A corollary is that to sample a function that depends on ψ , intervals of ψ should be taken so that the probability in each sampling increment is equal. In this case, $p_{n,n+1}$, or $\cos \psi_n - \cos \psi_{n+1}$ should be independent of n so equal increments of $\Delta(\sin \psi)$ should be chosen.

It should be realized that $\psi = 90^\circ - \theta$, where θ is the angle in the notation of Appendix 1 and elsewhere, which defines orientations in terms of θ and ϕ . With this notation, equal intervals of $\Delta(\sin \theta)$ are appropriate for random sampling.

11.2. Reorientation during deformation

11.2.1. Axisymmetric extension

Now consider how a line in a material is rotated by the deformation of the material. Let the initial length of the line be l_0 , and let it be initially oriented at an angle ψ_0 to the axis of axisymmetric extension, x (Fig. 11.2). Before

extension $x_0 = l_0 \cos \psi_0$ and $y_0 = l_0 \sin \psi_0$; so

$$\tan \psi_0 = y_0/x_0. \quad (11.5)$$

After axisymmetric extension with $\epsilon_x = -\frac{1}{2}\epsilon_x$,

$$x = x_0 \exp(\epsilon_x) \quad \text{and} \quad y = y_0 \exp(-\epsilon_x/2); \quad (11.6)$$

so the new angle, ψ , between the line and the x axis is such that

$$\tan \psi = \frac{y}{x} = (y_0/x_0) \exp[-\frac{3}{2}\epsilon_x] \quad (11.7)$$

or

$$\tan \psi = \tan \psi_0 \exp[-\frac{3}{2}\epsilon_x]. \quad (11.8)$$

Now consider the redistribution of a large number of randomly oriented lines. Let $f_0 = 1 - \cos \psi_0$ be the fraction of these lines initially oriented within an angle ψ_0 of the x axis. The same fraction will be oriented within a smaller angle ψ of the x axis after extension. If, for example, 75% of the lines are initially within an angle

$$\psi_0 = \arccos(1 - f_0) = \arccos(0.25) = 75.5^\circ,$$

after a strain of $\epsilon_x = 1$, 75% of the lines will be within

$$\psi = \arctan\{\tan \psi_0 \exp[-\frac{3}{2}]\} = 40.8^\circ,$$

and after a strain of $\epsilon_x = 2$, 75% of the lines will be within

$$\psi = \arctan[\tan \psi_0 \exp(-3)] = 10.9^\circ.$$

11.2.2. Axisymmetric compression

The same equations can be used for compression (Fig. 11.3), but now ϵ_x is negative. If, for example, before compression, 75% of the lines are oriented at angles greater than ψ_0 of x , after a strain of 1 ($\epsilon_x = -1$), 75% of the lines will be oriented at angles greater than

$$\psi = \arctan\{\tan(41.4^\circ) \exp[\frac{3}{2}]\} = 75.8^\circ,$$

and, after a strain of $\epsilon_x = -2$,

$$\psi = \arctan\{\tan(41.4^\circ) \exp(3)\} = 86.7^\circ.$$

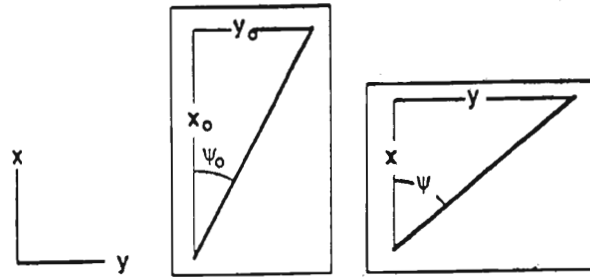


Fig. 11.3. Reorientation of a line during axisymmetric compression.

11.2.3. Change of $dp/d\psi$ with extension

If the orientation were initially random, $dp/d\psi_0 = \sin \psi_0$. Since $dp/d\psi = (dp/d\psi_0)(d\psi_0/d\psi)$,

$$\frac{dp}{d\psi} = \sin \psi_0 \left(\frac{d\psi_0}{d\psi} \right). \quad (11.9)$$

With appropriate trigonometric identities, Eq. (11.8) can be expressed as

$$\sin \psi_0 = \left[\frac{\tan^2 \psi_0}{(1 + \tan^2 \psi_0)} \right]^{1/2} = \frac{c \tan \psi}{(1 + c^2 \tan^2 \psi)^{1/2}}, \quad (11.10)$$

where

$$c = \left(\frac{L}{L_0} \right)^{3/2} = \exp\left[\frac{3}{2}\epsilon\right], \quad (11.11)$$

$$\frac{d\psi_0}{d\psi} = \exp\left[\frac{3}{2}\epsilon\right] \left(\frac{\cos \psi_0}{\cos \psi} \right)^2$$

Now substituting Eqs. (11.10) and (11.11) into $dp/d\psi = \sin \psi_0 (d\psi_0/d\psi)$,

$$\begin{aligned} \frac{dp}{d\psi} &= \left(\frac{c^2}{(1 + c^2 \tan^2 \psi)} \right)^{1/2} (\cos \psi_0 / \cos \psi)^2 \quad \text{or} \\ &= \frac{c^2 \sin \psi}{(\cos^2 \psi + c^2 \sin^2 \psi)^{3/2}}. \end{aligned} \quad (11.12)$$

Figure 11.4 shows how the orientation dependence of $dp/d\psi$ changes with increasing extension. Lines that were initially random in orientation tend to approach the axis of extension as the material is elongated. A peak of $dp/d\psi$ forms at a low angle. With increasing extension the angle at which the peak occurs decreases, and its intensity increases.

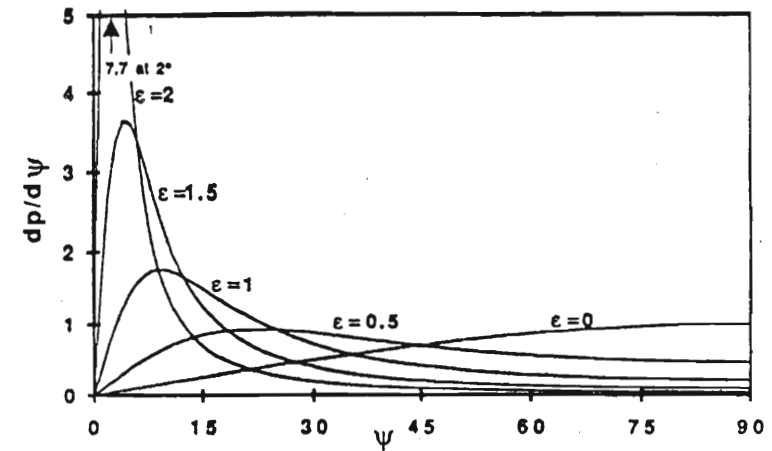


Fig. 11.4. Changes of the orientation distribution with axisymmetric elongation for lines that were initially randomly oriented.

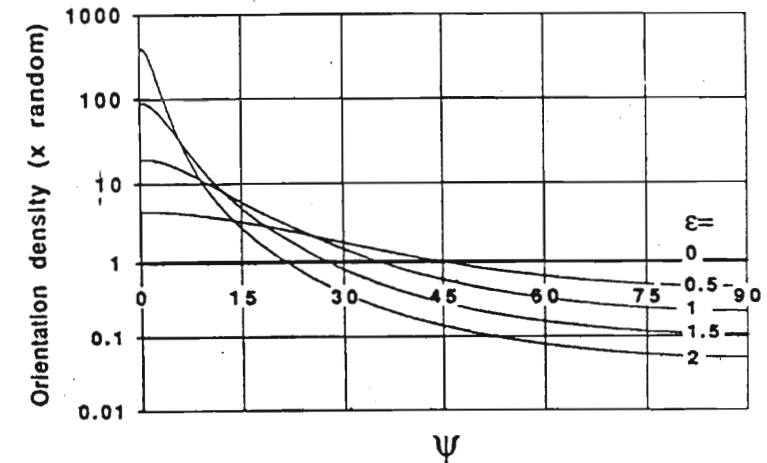


Fig. 11.5. Orientation density after axisymmetric elongation.

The density of orientations, per spherical angle, is given by

$$\frac{dp/d\psi}{\sin \psi} = \frac{c^2}{(\cos^2 \psi + c^2 \sin^2 \psi)^{3/2}}. \quad (11.13)$$

Figure 11.5 is a plot of this orientation density versus ψ .

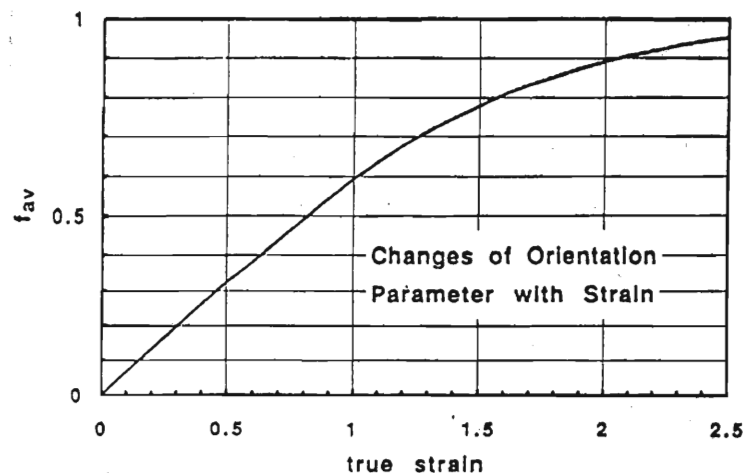


Fig. 11.6. Changes of f_{av} for initially random lines during axisymmetric extension.

11.3. Statistical orientation parameter

Polymer scientists frequently use an orientation parameter¹¹ f to characterize the statistical orientation of molecules. The parameter f is defined by

$$f = (3 \cos^2 \psi - 1)/2, \quad (11.14)$$

where ψ is the angle between a molecule segment and the axis of alignment. For perfect alignment, $\psi = 0$, so $\cos \psi = 1$ and $f_{av} = f = f(0^\circ) = 1$.

For a large number of lines, the average value of f is given by

$$f_{av} = \int_0^{\pi/2} f(\psi) \left(\frac{dp}{d\psi} \right) d\psi, \quad (11.15)$$

where $dp/d\psi$ is the orientation density. Substituting $dp/d\psi = \sin \psi$ for randomly oriented material and integrating, $f_{av} = \frac{1}{2} \int_0^{\pi/2} (3 \cos^2 \psi - 1) \sin \psi d\psi = \frac{1}{2} (-\cos^3 \psi + \cos \psi)$. Evaluating, for integration limits of 0 and $\pi/2$, $f_{av} = 0$.

¹¹ In linear polymers, it may be used to describe the orientation of a line connecting the ends of a molecule or the orientation of any segment of a molecule. Many properties, including Young's modulus and the thermal expansion coefficient, correlate with f_{av} . This parameter may also be used to describe the orientations of lines, fiber axes in composites, normals to grain boundaries, or any other microstructural

11.3.1. Variation of f with extension¹²

Consider now how f_{av} changes with axisymmetric extension for initially random material. Substituting Eq. (11.12) for $dp/d\psi$,

$$f_{av} = \frac{1}{2} \int_0^{\pi/2} c^2 \sin \psi (3 \cos^2 \psi - 1) (\cos^2 \psi + c^2 \sin^2 \psi)^{-3/2} d\psi. \quad (11.16)$$

Integrating from 0 to $\pi/2$,

$$f_{av} = a[b - (3c^2/2)] \left[\frac{\pi}{2} + \arcsin\left(\frac{1}{c^2}\right) \right], \quad (11.17)$$

where $a = \frac{1}{2}(c^2 - 1)^{3/2}$ and $b = (2c^2 + 1)(c^2 - 1)^{1/2}$.

Figure 11.6 shows the variation of f_{av} with $\epsilon = \frac{2}{3} \ln c$. Initially f_{av} is nearly proportional to ϵ ($f_{av} \approx 0.6\epsilon$), but $df_{av}/d\epsilon$ decreases with f_{av} , reaching 0.9 after a tensile strain of about 1.5.

¹² R. L. Curl provided needed assistance with the mathematics.

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APPENDIX 1

ANGULAR RELATIONS

Several methods of treating problems involving angular relationships are outlined in this appendix. Which is most appropriate for a given problem depends on the type of data and the accuracy desired.

A1.1. Vector algebra and Miller indices

Any vectors $\mathbf{a}$ and $\mathbf{b}$ may be represented by their components along three orthogonal axes

$$\left. \begin{aligned} \mathbf{a} &= a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} \\ \mathbf{b} &= b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k} \end{aligned} \right\} \quad (\text{A1.1})$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors along these axes. The absolute magnitudes of $\mathbf{a}$ and $\mathbf{b}$ are then

$$\left. \begin{aligned} |\mathbf{a}| &= (a_1^2 + a_2^2 + a_3^2)^{1/2} \\ |\mathbf{b}| &= (b_1^2 + b_2^2 + b_3^2)^{1/2} \end{aligned} \right\} \quad (\text{A1.2})$$

The dot product of $\mathbf{a}$ and $\mathbf{b}$ is defined as

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta, \quad (\text{A1.3})$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. Therefore,

$$\cos \theta = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{[(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)]^{1/2}} \quad (\text{A1.4})$$

A1.1.1. Cubic crystals

In cubic crystals the Miller direction indices, $[uvw]$, can be considered as representing the vector $\mathbf{d} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$. Hence the angle θ between

$[u_1v_1w_1]$ and $[u_2v_2w_2]$ is given by

$$\cos \theta = \frac{u_1u_2 + v_1v_2 + w_1w_2}{[(u_1^2 + v_1^2 + w_1^2)(u_2^2 + v_2^2 + w_2^2)]^{1/2}} \quad (\text{A1.5})$$

Two directions, 1 and 2, are normal if $d_1 \cdot d_2 = 0$ or

$$u_1u_2 + v_1v_2 + w_1w_2 = 0. \quad (\text{A1.6})$$

The cross product of two vectors a and b is a third vector c that is normal to both a and b .

$$\begin{aligned} c &= a \times b, \\ &= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= (a_2b_3 - a_3b_2)i + (a_3b_1 - a_1b_3)j + (a_1b_2 - b_1a_2)k. \end{aligned} \quad (\text{A1.7})$$

A direction normal to both $[u_1v_1w_1]$ and $[u_2v_2w_2]$ in a cubic crystal therefore has indices

$$[u_3v_3w_3] = [(v_1w_2 - v_2w_1), (w_1u_2 - w_2u_1), (u_1v_2 - u_2v_1)]. \quad (\text{A1.8})$$

A quick check for arithmetic errors can be made to verify that c is normal to both a and b by using Eq. (A1.6) to confirm that $a \cdot c = 0$ and $b \cdot c = 0$.

Example A1.1

What is the angle between $[111]$ and $[110]$ in a cubic crystal?

$$\begin{aligned} \cos \theta &= \frac{1 \cdot 1 + 1 \cdot 1 + 1 \cdot 0}{[(1^2 + 1^2 + 1^2)(1^2 + 1^2 + 0^2)]^{1/2}} \\ &= \frac{2}{\sqrt{6}} \end{aligned}$$

so $\theta = 35^\circ 15'$.

Example A1.2

Find the indices of the plane that contains both the $[11\bar{2}]$ and $[112]$ directions in a cubic crystal.

The plane must be perpendicular to both directions and

is therefore given by

$$[112] \times [11\bar{2}] = \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ 1 & 1 & -2 \end{vmatrix} = -4i + 4j + 0k = [\bar{4}40] = [\bar{1}10].$$

In a cubic crystal a plane and its normal have the same indices, so the plane is the $(\bar{1}10)$.

Example A1.3

What is the angle between the $[111]$ and the intersection of the (112) and (001) planes?

The intersection of the (112) and (001) planes is the direction normal to both; that is, $[112] \times [001] = [1\bar{1}0]$. Let ϕ be the angle between $[111]$ and $[1\bar{1}0]$. Then

$$\cos \phi = \frac{1 \cdot 1 + 1 \cdot \bar{1} + 1 \cdot 0}{[(1^2 + 1^2 + 1^2)(1^2 + \bar{1}^2 + 0^2)]^{1/2}} = 0$$

so $\phi = 90^\circ$.

Example A1.4

Fcc crystals slip on $\{111\}$ planes and $\langle 110 \rangle$ directions.¹³ Is it physically possible for slip to be in the $[110]$ direction if the plane is (a) (111) ? (b) $(\bar{1}\bar{1}1)$?

(a) The slip direction must lie in the slip plane. Therefore it must be perpendicular to the slip plane normal. Using Eq. (A1.6), the numerator of the dot product of $[110]$ and $[111]$ is $1 \cdot 1 + 1 \cdot 1 + 0 \cdot 1 \neq 0$; so $[110]$ does not lie in (111) and $[110](111)$ is not a possible slip system.

(b) With $[110]$ and $(\bar{1}\bar{1}1)$, the numerator is $1 \cdot 1 + 1 \cdot \bar{1} + 1 \cdot 0 = 0$; so $[110]$ and $(\bar{1}\bar{1}1)$ are perpendicular. This is a possible slip system.

A1.1.2. Orthorhombic crystals

The relationships for orthorhombic crystals are more complicated. The Miller direction indices $[uvw]$ can be represented by a vector, $d_1 = uai + vbj + wck$, where a , b , and c are the dimensions of the unit cell measured along its axes. The angle θ between the $[u_1v_1w_1]$ and $[u_2v_2w_2]$ directions is

¹³ The notations $\{hkl\}$ and $\langle uvw \rangle$ are used to indicate specific crystallographic planes and directions, while $\{hkl\}$ and $\langle uvw \rangle$ denote families of crystallographically equivalent planes and directions. Thus $\langle 110 \rangle$ denotes any or all of $[110]$, $[1\bar{1}0]$, $[101]$, $[10\bar{1}]$, etc.

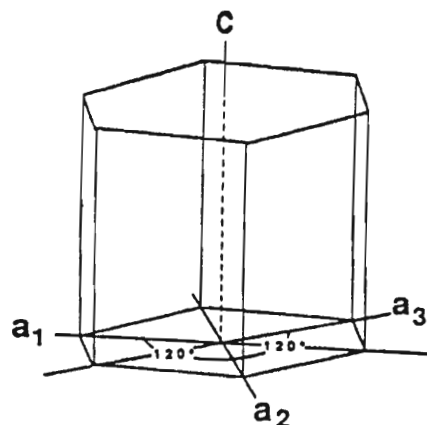


Fig. A1.1. Hexagonal axes.

given by

$$\cos \theta = \frac{u_1 u_2 a^2 + v_1 v_2 b^2 + w_1 w_2 c^2}{[(u_1^2 a^2 + v_1^2 b^2 + w_1^2 c^2)(u_2^2 a^2 + v_2^2 b^2 + w_2^2 c^2)]^{1/2}} \quad (\text{A1.9})$$

Furthermore the normal to the (hkl) plane is the $[h/a, k/b, l/c]$ direction (not $[hkl]$), so Eq. (A1.9) does not apply to the dihedral angle between two planes. Rather the dihedral angle ϕ between $(h_1 k_1 l_1)$ and $(h_2 k_2 l_2)$ is given by

$$\cos \phi = \frac{h_1 h_2 / a^2 + k_1 k_2 / b^2 + l_1 l_2 / c^2}{\left\{ \left[\left(\frac{h_1}{a} \right)^2 + \left(\frac{k_1}{b} \right)^2 + \left(\frac{l_1}{c} \right)^2 \right] \left[\left(\frac{h_2}{a} \right)^2 + \left(\frac{k_2}{b} \right)^2 + \left(\frac{l_2}{c} \right)^2 \right] \right\}^{1/2}} \quad (\text{A1.10})$$

Equations (A1.9) and (A1.10) also apply to tetragonal crystals except that $a = b \neq c$.

A1.1.3. Direction and planar indices for hexagonal crystals

With hexagonal crystals, a four-axis (Miller-Bravais) system is convenient. Three axes, a_1 , a_2 , and a_3 lie in the basal plane, 120° from each other, as shown in Fig. A1.1. The fourth axis, c , is the axis of hexagonal symmetry.

Planar indices. Indices of planes are described by the same system as used for other crystal classes. The reciprocals of the intercepts on the four axes, a_1 , a_2 , a_3 , and c , are written in order and reduced to the lowest set of integers of the same ratio, as shown in Fig. A1.2(a). Note that in each case the first three indices sum to zero. This is always true. If the general indices

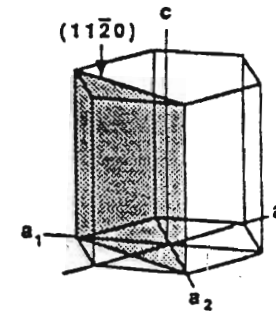
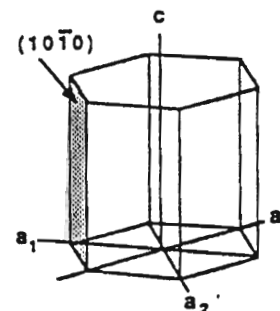
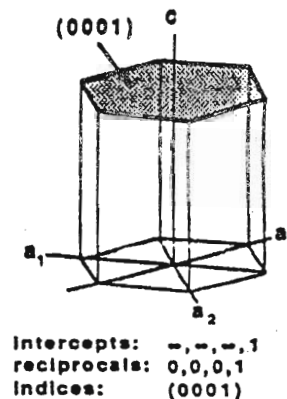
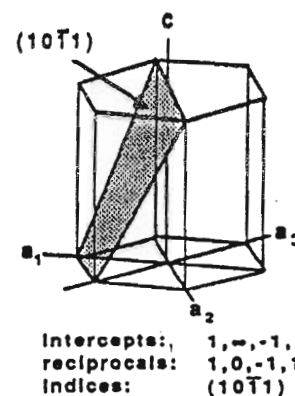


Fig. A1.2. Examples of planar indices for hexagonal crystals.

are designated as h, k, i, l ,

$$h + k + i = 0. \quad (\text{A1.11})$$

Direction indices. The direction indices are the translations parallel to the four axes that produce the direction under consideration. The first three indices must be chosen so that they sum to zero and are the smallest set of integers that will express the direction. They are enclosed without commas in brackets $[uvw]$. Examples are shown in Fig. A1.3.

Comparison of four- and three-digit systems. In the four-digit system, the third digit, i for directions (i for planes) can always be deduced from the

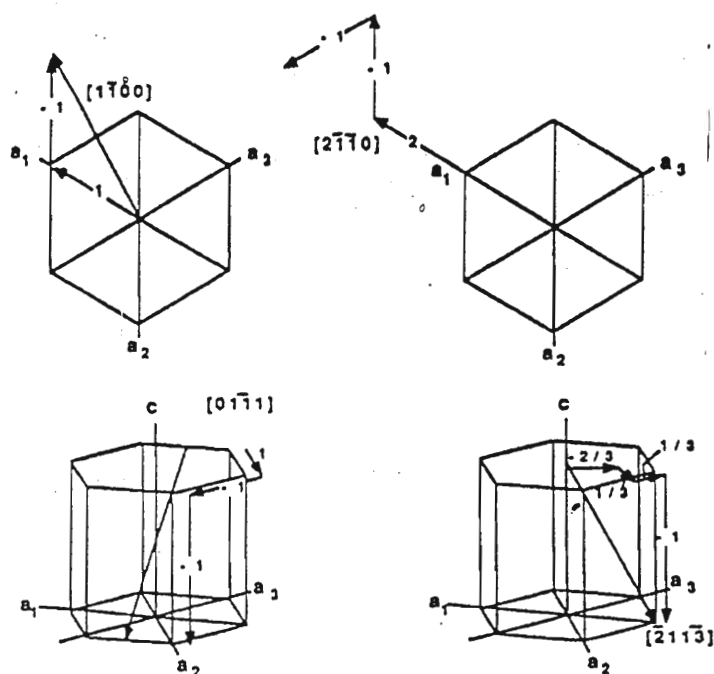


Fig. A1.3. Examples of direction indices with the Miller-Bravais system.

first two, $t = -u - v$ ($i = -h - k$ for planes) and is therefore redundant. With the three-digit systems, it may either be replaced by a dot, $[uvw]$ or (hkl) or omitted entirely, $[uv]$ or (hk) . However, the disadvantage of omitting the third index is that the hexagonal symmetry is not apparent. In the four-digit (Miller-Bravais) system a family of planes is apparent from the indices. For example;

$$\langle 01\bar{1}0 \rangle = [01\bar{1}0], [10\bar{1}0], \text{ and } [1\bar{1}00].$$

The equivalence of the same family is not so apparent in the three-digit system, $\langle 010 \rangle = [010], [100], \text{ and } [1\bar{1}0]$. Also compare $\{2110\} = (\bar{2}110), (1\bar{2}10), \text{ and } (11\bar{2}0)$, with $\{210\} = (\bar{2}10), (1\bar{2}0), (110)$.

Vector representation of directions. A direction $[uvw]$ may be expressed as a vector, d , in an orthogonal coordinate system x, y, z (see Fig. A1.4):

$$d = a \left\{ \left(u - \frac{v+t}{2} \right) i + \left[\frac{\sqrt{3}(v-t)}{2} \right] j + w \gamma k \right\},$$

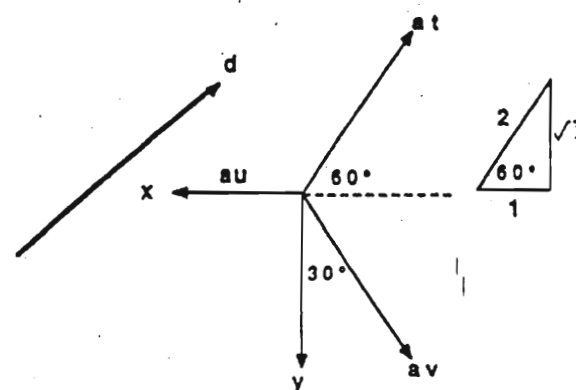


Fig. A1.4. Conversion of hexagonal indices to Cartesian coordinates.

where $\gamma = c/a$. If $t = -u - v$ is substituted,

$$d = \frac{3}{2}uai + \frac{\sqrt{3}}{2}(2v+u)aj + \gamma wk.$$

The magnitude of d is then

$$d = a[3(u^2 + uv + v^2) + \gamma^2 w^2]^{1/2}.$$

Angles between directions and planes in hexagonal crystals. The angle θ between two directions, $[u_1 v_1 t_1 w_1]$ and $[u_2 v_2 t_2 w_2]$, is given by

$$\cos \theta = \frac{u_1 u_2 + v_1 v_2 + (u_1 v_2 + u_2 v_1)/2 + w_1 w_2 (c/a)^2/3}{\left[\frac{u_1^2 + v_1^2 + u_1 v_1 + w_1^2 (c/a)^2}{3} \right] \left[\frac{u_2^2 + v_2^2 + u_2 v_2 + w_2^2 (c/a)^2}{3} \right]^{1/2}} \quad (\text{A1.12a})$$

The normal to (hkl) is $[h/a, (2k+h)/(\sqrt{3}a), l/c]$; so the dihedral angle, φ , between two planes, $(h_1 k_1 l_1)$ and $(h_2 k_2 l_2)$ is given by

$$\cos \varphi = \frac{h_1 h_2 + k_1 k_2 + (h_1 k_2 + h_2 k_1)/2 + \frac{1}{2}(a/c)^2 l_1 l_2}{\left[h_1^2 + k_1^2 + h_1 k_1 + \frac{1}{2} \left(\frac{a}{c} \right)^2 l_1^2 \right] \left[h_2^2 + k_2^2 + h_2 k_2 + \frac{1}{2} \left(\frac{a}{c} \right)^2 l_2^2 \right]^{1/2}} \quad (\text{A1.12b})$$

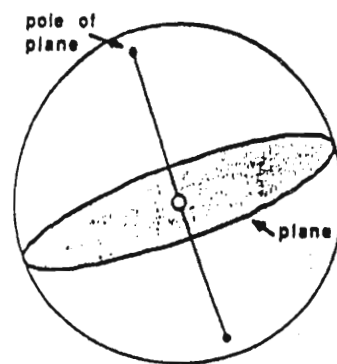


Fig. A1.5. Spherical projection of a plane and its pole.

A1.2. Graphical methods

Many problems involving angular relationships between planes and directions can be solved graphically. To understand the graphical methods, it is necessary to first consider the principles of graphical representation.

A1.3. Spherical projection

All planes and directions can be represented on a spherical surface by letting them pass through the center of a reference sphere and extending them until they intersect the reference sphere. Thus a direction is represented by two diametrically opposite points at which it intersects the sphere. A plane is represented by the great circle formed by its intersection with the reference sphere. A plane is also represented by its pole (normal to the plane through the center of the sphere), which is plotted like a direction. The spherical projection is illustrated in Fig. A1.5. Note that it does not matter whether the physical planes and directions of interest do meet at a common point (center of the reference sphere). Even if they do not, parallel elements with the same angular relations can be constructed through a common point.

A1.4. Stereographic projection

Since plotting and measurement on a spherical surface would be cumbersome, it is convenient to map the sphere onto a plane. The most common mapping system is the stereographic projection. To visualize the stereographic projection, visualize a glass sphere (Fig. A1.6) resting on a flat plane (plane of projection) with a tangent at C and a light source (point of projection) at S on the spherical surface diametrically opposite the point of tangency with the flat surface. Planes and directions are projected onto the

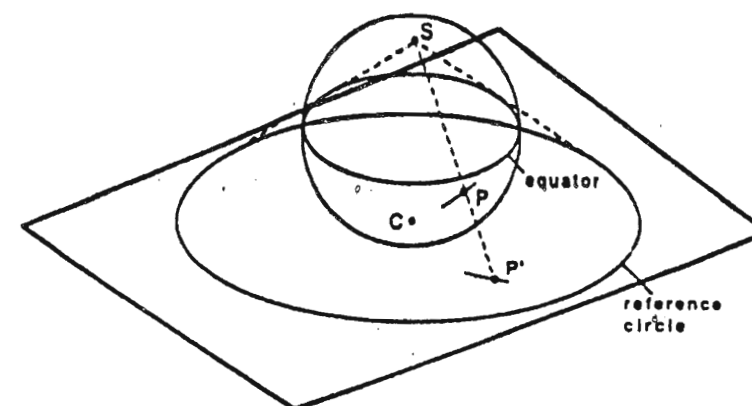


Fig. A1.6. The stereographic projection. Features on the lower hemisphere are projected onto the reference plane.

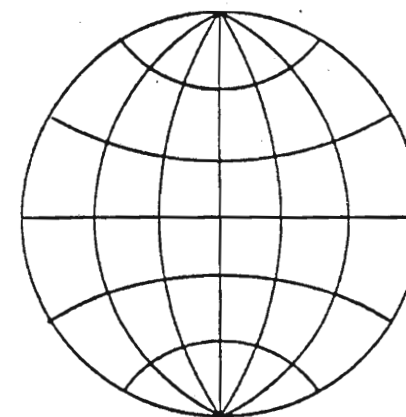


Fig. A1.7. Wulff net (30%).

plane of projection. Point P projects to P' . By custom only the features on the hemisphere nearest the plane of projection are mapped. The great circle that bounds the hemisphere is called the *reference circle*.

Graphical construction and angular measurements may be made with the aid of a Wulff net, which is a stereographic projection of a reference sphere on which lines of longitude (meridians) and lines of latitude (parallels) have been scribed. The point of projection is located on the equator so that the north and south poles lie on the reference circle (Fig. A1.7). Meridians

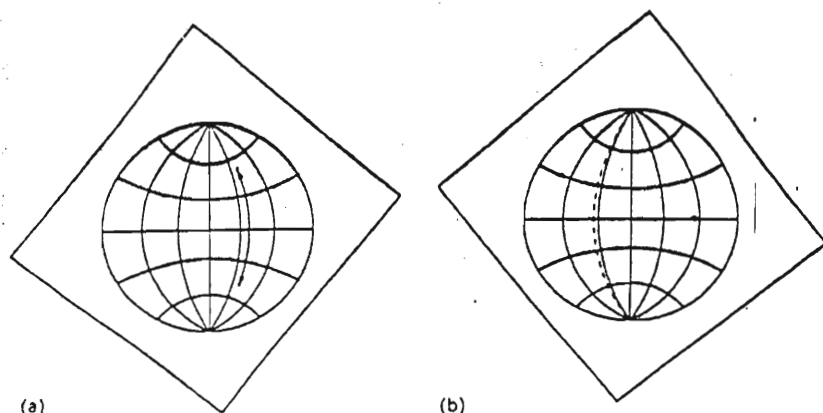


Fig. A1.8. (a) The angular distance between two directions is measured along a great circle. (b) The great circle corresponding to a plane with a known pole is constructed by rotating that pole to the equator.

project as great circles. The equator is also a great circle. Angles between directions are true if measured on great circles. It is convenient to do the actual plotting on a piece of transparent paper laid over a Wulff net with a pin through the center of the paper and net. This allows the paper to be rotated relative to the net about an axis through the reference point.

To measure the angle between two directions (or poles), rotate the paper until both lie on a common meridian. The angle between the directions is the difference in latitude on that meridian [Fig. A1.8(a)].

To construct a great circle representing the plane that contains both directions, rotate the paper until both lie on a common meridian and simply trace that meridian. The pole of that plane lies on the equator 90° from the point where the trace crosses the equator.

To construct the great circle representing a plane, given its pole (normal), rotate the paper until the pole lies on the equator and then trace the meridian which crosses the equator 90° from the pole [Fig. A1.8(b)].

Rotation of all points about a common axis does not change their positions relative to one another. There are two axes about which such rotations can be easily made:

1. To rotate about the center of the projection, rotate the paper relative to the underlying Wulff net as mentioned above.
2. To rotate about the north-south axis of the Wulff net, translate all points along latitude lines by the same change in longitude. If an axis is rotated off the net, its opposite end appears on the reference circle at a point diametrically opposite to that at which it disappeared. See

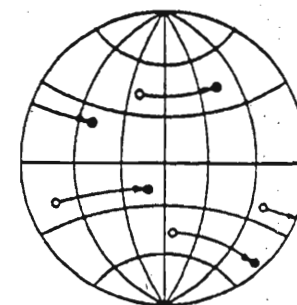


Fig. A1.9. Rotation of points about the north-south axis.

Fig. A1.9. Rotation may be made about other axes using a combination of the two rotations above:

3. To rotate about any axis on the reference circle, turn the plotting paper about its center (step 1) until that axis is at the north or south pole. Then rotate about that pole (step 2).
4. To rotate about any axis, first rotate that axis to the reference circle as described in step 2, then proceed according to step 3.

Example A1.5

What is the great circle distance between Juneau (lat. = 58.3°N , long. = 134.3°W) and Rio de Janeiro (lat. = 22.9°S , long. = 43.2°W)?

Plot the location of the two cities on tracing paper over a Wulff net [Fig. A1.10(a)]. Rotate the paper until both cities lie over a common

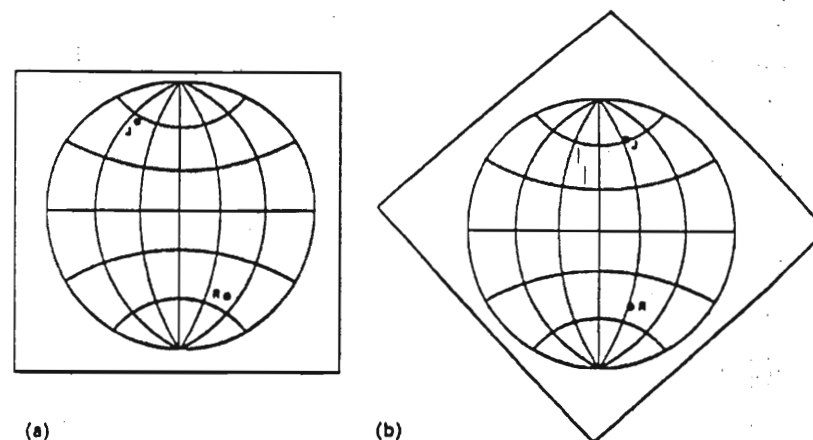


Fig. A1.10. The spherical angle between Juneau and Rio de Janeiro (a) is found by rotating the projection so that both are on the same meridian and noting the difference in latitude (b).

meridian [Fig. A1.10(b)]. Now the difference in latitude, $\phi = 110^\circ = 1.92$ rad, represents the angular length of the great circle. The arc length is $r\phi$, where r is the earth's radius (6371 km) so the distance is 12,230 km.

A1.5. Construction of the standard projection for cubic crystals

The standard projection of a cubic crystal has $[100]$ at the center and $[001]$ at the north pole and shows the $\{100\}$, $\{110\}$, and $\{111\}$ poles and the traces of the $\{100\}$ and $\{110\}$ poles, as illustrated in Figure A.13.

The trace of the $\{100\}$ plane is the reference circle. The $\{010\}$ and $\{001\}$ poles must lie 90° apart on the reference circle. The $\{010\}$ pole is plotted at the intersection of the reference circle and the equator; its trace is the meridian through $[100]$. The $\{001\}$ pole then coincides with the north pole, and its trace is the equator [Fig. A1.11(a)].

The $[110]$ direction is 90° from $[001]$, since $[110] \cdot [001] = 0$, and therefore lies on the equator. It is also 45° from $[100]$ and 45° from $[010]$ since $[100] \cdot [110] = 1/\sqrt{2}$; so it is 45° right of $[100]$ on the equator, and the $\{110\}$ plane is the meridian that cuts the equator 45° left of $[100]$. The $\{1\bar{1}0\}$ pole similarly lies on the equator 45° left of $[100]$, and its trace is the meridian through the $\{110\}$ pole [Fig. A1.11(b)].

The $\{101\}$ pole is 90° from $[010]$ and 45° from $[100]$ and $[001]$. To plot it and its trace, the paper is rotated 90° and the appropriate meridians traced. The $\{10\bar{1}\}$ and its trace are similarly constructed. The $\{011\}$ and $\{0\bar{1}1\}$ poles are 90° from $[100]$ and therefore lie on the reference circle. They are also $\pm 45^\circ$ from $[001]$ and plotted as such. The $\{011\}$ and $\{0\bar{1}1\}$ planes can be constructed by rotating the paper until their poles lie on the equator and then tracing the meridian through $[100]$.

The $\{111\}$ pole is the intersection of the $\{1\bar{1}0\}$, $\{10\bar{1}\}$, and $\{0\bar{1}1\}$ traces; all three have a zero dot product with $[111]$. Similarly the intersection of the $\{1\bar{1}0\}$, $\{101\}$, and $\{011\}$ traces is the $\{11\bar{1}\}$ pole. The other $\{111\}$ poles are

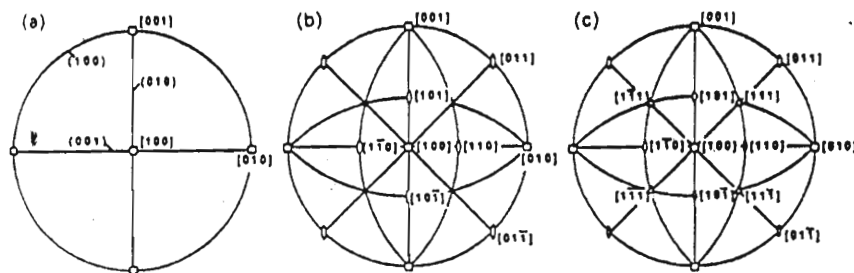


Fig. A1.11. Construction of a standard cubic projection.

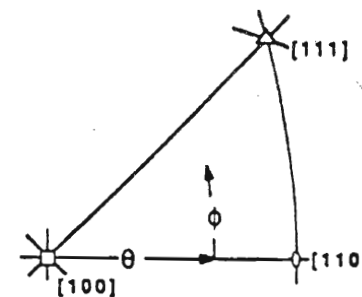


Fig. A1.12. Basic orientation triangle for cubic crystals.

similarly found as intersections of the $\{110\}$ traces. The $\{100\}$ poles are represented by squares, which indicate the fourfold symmetry about $\langle 100 \rangle$ directions. Likewise, triangles are used for the $\{111\}$ poles because of their threefold symmetry, and ovals represent the $\{110\}$ poles, which have twofold symmetry [Fig. A1.11(c)].

A1.6. Basic orientation triangle for cubic crystals

Note that the $\{100\}$ and $\{110\}$ traces break up the hemisphere into 24 equivalent spherical triangles, each having $\langle 100 \rangle$, $\langle 110 \rangle$, and $\langle 111 \rangle$ directions as corners. Because all of these triangles are equivalent, the orientation of a single direction may be represented as a point in any triangle. In this treatment, the triangle with corners at $[100]$, $[110]$, and $[111]$ is chosen as the basic triangle (see Fig. A1.12). The variation of properties with crystallographic direction may be shown by contours within the basic triangle. The triangle may be plotted on a stereographic projection or on cartesian coordinates. In either case it is convenient to describe the orientation of a direction in the triangle by its longitude relative to the center, θ , and its latitude, ϕ .

Example A1.6

Rotate the standard cubic projection so that $[111]$ lies in the center.

See Fig. A1.13. First rotate the standard projection over a Wulff net so that $[111]$ lies on the equator. To bring $[111]$ into the center, it must be rotated by $54^\circ 40'$ along the equator. Translate the other poles along latitude lines by the same amount and construct the appropriate great circles.

A1.7. Spherical trigonometry

Often angular problems can be solved analytically with greater ease and accuracy than is possible with graphical methods. The basic equations of spherical trigonometry can be applied if one can visualize a spherical

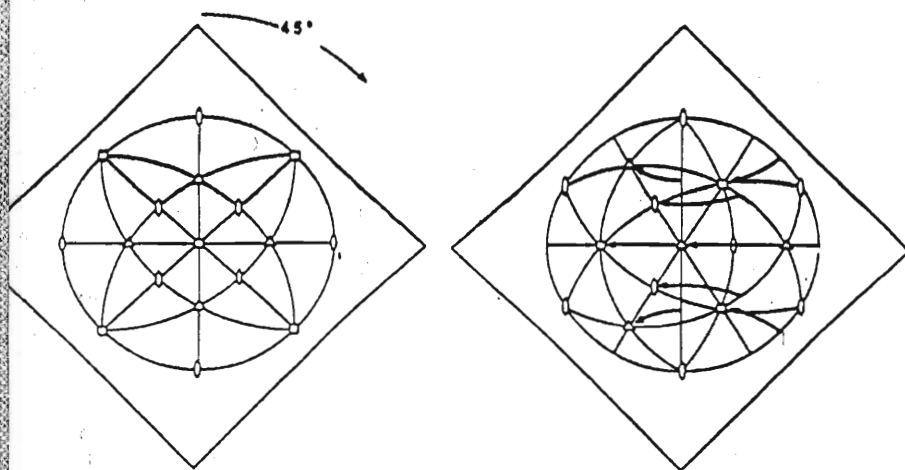


Fig. A1.13. Rotation of a {111} pole to the center of a cubic projection. First the paper is rotated so that the {111} pole lies on the equator (a), then the features are rotated about the north-south axis.

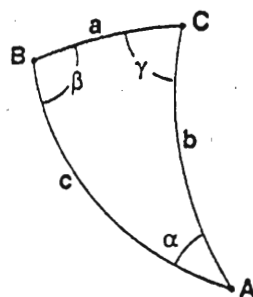


Fig. A1.14. A spherical triangle. a , b , and c are segments of great circles.

projection of the planes and directions of interest. The key is to work with spherical triangles (e.g., triangles formed by the intersection of three great circles). The sides of such triangles are arcs, whose lengths are equal to the angle between the two directions that represent the vertices at the end of the side. (The reference sphere has a unit radius, $r = 1$, so the arc length $s = r\phi = \phi$). For example, the side a of triangle ABC (Fig. A1.14) equals the angle between the directions B and C .

An interior angle of the triangle is the dihedral angle between the two planes whose traces form the adjacent sides. In figure A1.14, α is the dihedral angle between the planes whose traces are b and c . There are several useful

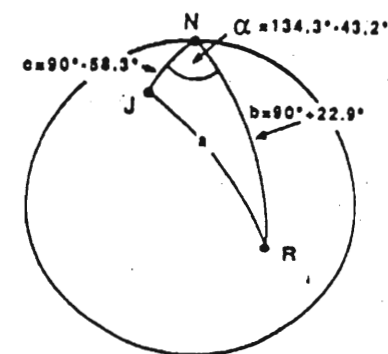


Fig. A1.15. A spherical triangle for calculating distance between Rio (R) and Juneau (J).

relations between the sides and interior angles of a spherical triangle:

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha \quad (\text{A1.13})$$

$$\cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a \quad (\text{A1.14})$$

and

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma} \quad (\text{A1.15})$$

In the special case of the right spherical triangle, these relations simplify to

$$\cos a = \cos b \cos c, \quad \text{if } \alpha = 90^\circ, \quad (\text{A1.16})$$

$$\cos \alpha = -\cos \beta \cos \gamma, \quad \text{if } a = 90^\circ, \quad (\text{A1.17})$$

and

$$\sin a = \sin b / \sin \beta = \sin c / \sin \gamma, \quad \text{if } \alpha = 90^\circ. \quad (\text{A1.18})$$

Example A1.7

What is the great circle distance between Juneau (lat. = 58.3°N , long. = 134.3°W) and Rio de Janeiro (lat. = 22.9°S , long. = 43.2°W)?

Consider the spherical triangle JNR in Fig. A1.15, where J = Juneau, N = north pole and R = Rio. The great circle arc, a , from Juneau to Rio is given by Eq. (A1.13), where $b = 90^\circ + 22.9^\circ = 112.9^\circ$, $c = 90^\circ - 58.3^\circ = 31.7^\circ$, and $\alpha = 134.3^\circ - 43.2^\circ = 91.1^\circ$. Substituting into Eq. (A1.14), $a = 109.9^\circ$. Using the earth's radius of 6371 km, the great circle distance is $6371 \times 109.9\pi/180 = 12,220$ km.

Example A1.8

Calculate the latitude and longitude of the (211) pole of a cubic crystal on a standard projection with $[001]$ at the north pole and $[100]$ at the center on the equator on the Greenwich meridian.

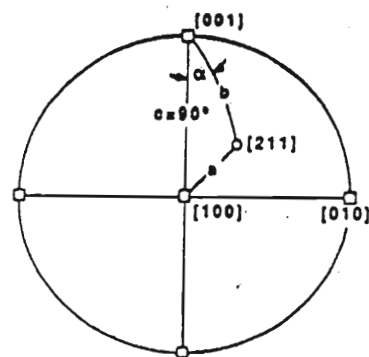


Fig. A1.16. Spherical triangle for calculating the latitude and longitude of [211].

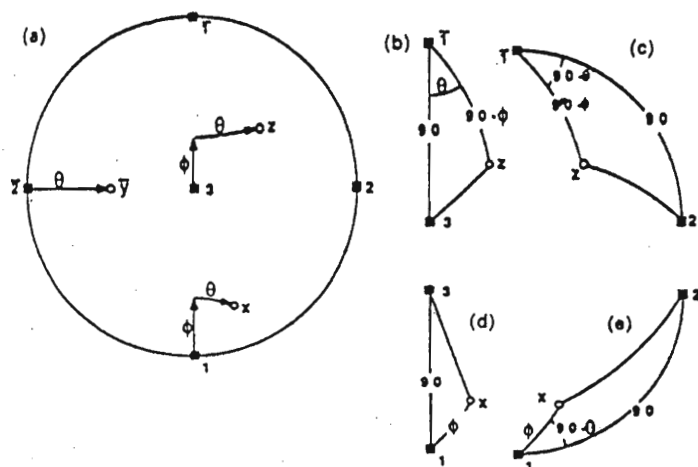


Fig. A1.17. (a) Relation between the x, y, z axes and the $1, 2, 3$ axes. The spherical triangles used in the solution are shown in (b), (c), (d), and (e).

In Fig. A1.16, the latitude $= 90^\circ - b$. $\cos b = (2 \cdot 0 + 1 \cdot 0 + 1 \cdot 1) / \sqrt{6} \cdot 1 = 1/\sqrt{6}$, so $b = 65.9^\circ$ and the latitude $= 24.1^\circ$.

The longitude is the angle, α , in the triangle formed by [001], [100], and [211]. From Eq. (A1.14) with $c = 90^\circ$, $\cos \alpha = \cos a / \sin b$. Substituting $\cos a = (2 \cdot 1 + 1 \cdot 0 + 1 \cdot 0) / \sqrt{6}$ and $\sin b = \sqrt{1 - \cos^2 b} = \sqrt{5/6}$, $\cos \alpha = 2/\sqrt{5} = 0.894$. The longitude, α , is 26.6° .

Example A1.9

Derive the table of direction cosines (Table A1.1) between the x, y, z and $1, 2, 3$ axes for the orientation of crystal defined θ and ϕ as described in Chapter

Table A1.1. Direction cosines between the $1, 2, 3$ and the x, y, z axes

	1	2	3
x	$\cos \phi$	$\sin \theta \sin \phi$	$\cos \theta \sin \phi$
y	0	$\cos \theta$	$-\sin \theta$
z	$-\sin \phi$	$\sin \theta \cos \phi$	$\cos \theta \cos \phi$

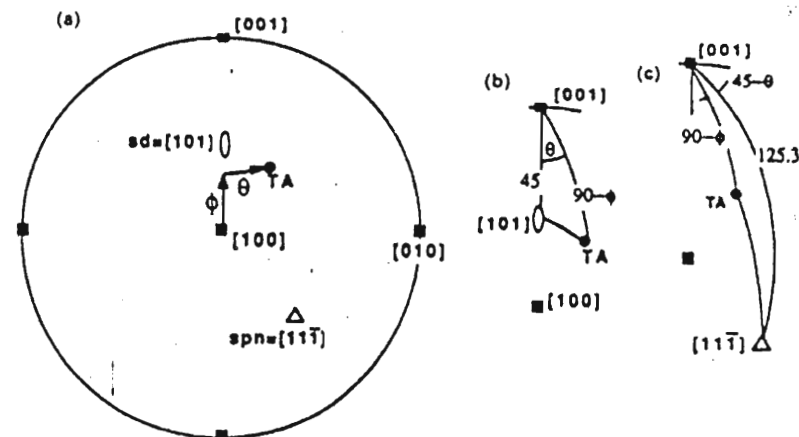


Fig. A1.18. (a) Angular relations for deriving the Schmid factor as a function of orientation. The spherical triangles used to find $\cos \lambda$ and $\cos \phi$ are shown in (b) and (c).

By inspection, $l_{1x} = \cos \phi$, $l_{1y} = 0$, and $l_{1z} = \cos(90^\circ + \phi) = -\sin \phi$. To find l_{2x} , consider the triangle (e) with corners at 1, 2, and x. Applying Eq. (A1.13),

$$l_{2x} = \cos 90^\circ \cos \phi + \sin 90^\circ \sin \phi \cos(90^\circ - \theta) = \sin \phi \sin \theta.$$

By inspection $l_{2y} = \cos \theta$.

To find l_{2z} , consider the triangle (c) with corners at -1, 2, and x. Applying Eq. (A1.13),

$$l_{2z} = \cos(90^\circ - \phi) \cos 90^\circ + \sin(90^\circ - \phi) \sin 90^\circ \cos(90^\circ - \theta) = \cos \phi \sin \theta.$$

To find l_{3x} , consider the triangle (d) with corners at 1, 3, and x. Applying Eq. (A1.13), $l_{3x} = \cos 90^\circ \cos \phi + \sin 90^\circ \sin \phi \cos \theta = \sin \phi \cos \theta$. By inspection,

... To find l_{3z} , consider the triangle (b) with corners at -1 , 3 , and z . Applying Eq. (A1.13),

$$l_{3z} = \cos 90^\circ \cos(90^\circ - \phi) + \sin 90^\circ \sin(90^\circ - \phi) \cos \theta = \cos \phi \cos \theta.$$

Example A1.10

Derive an expression for the Schmid factor m in terms of the latitude ϕ and the longitude θ of the tensile axis for an fcc crystal (Fig. A1.18).

Let the tensile axis lie in the basic triangle with corners at $[100]$, $[110]$, and $[111]$ so the most favored slip system is $[101](11\bar{1})$. Consider first the angle λ . Applying Eq. (A1.13) to the spherical triangle with corners at $[101]$, $[001]$, and TA,

$$\begin{aligned} \cos \lambda &= \cos 45^\circ \cos(90^\circ - \phi) + \sin 45^\circ \sin(90^\circ - \phi) \cos \theta \\ &= \frac{1}{\sqrt{2}} (\sin \phi + \cos \phi \cos \theta). \end{aligned} \quad (\text{A1.19})$$

$\cos \phi$ can be found using the triangle with corners at TA, $[001]$, and $[11\bar{1}]$.

$$\begin{aligned} \cos \phi &= \frac{-1}{\sqrt{3}} \cos(90^\circ - \phi) + \frac{\sqrt{3}}{3} \sin(90^\circ - \phi) \cos(45^\circ - \theta) \\ &= \frac{-1}{\sqrt{3}} \sin \phi + \frac{1}{\sqrt{3}} \cos \phi \cos(45^\circ - \theta). \end{aligned} \quad (\text{A1.20})$$

Substituting $\cos(45^\circ - \theta) = (1/\sqrt{2})(\cos \theta + \sin \theta)$,

$$\cos \phi = \frac{1}{\sqrt{3}} [-\sin \phi + \cos \phi (\cos \theta + \sin \theta)].$$

Combining,

$$\begin{aligned} m &= \cos \lambda \cos \phi \\ &= \frac{1}{\sqrt{6}} (\sin \phi + \cos \phi \cos \theta) [-\sin \phi + \cos \phi (\cos \theta + \sin \theta)]. \end{aligned} \quad (\text{A1.21})$$

References

1. C. Barrett and T. B. Massalski, *Structure of Metals*, 3rd rev. ed., Pergamon Press, Oxford, 1980.
2. B. D. Cullity, *Elements of X-ray Diffraction*, Addison-Wesley, Reading, MA, 1956.

APPENDIX 2

STRESS, STRAIN, AND TRANSFORMATIONS



A2.1. Stress

Stress is defined as force per area on which the force acts, F/A . With a normal stress (compressive or tensile) the force acts on an area normal to it, whereas with a shear stress the force is parallel to the area on which it acts. Nine components of stress are needed to define fully a state of stress at a point, as shown in Fig. A2.1. Two subscripts are required for each component: The first indicates the normal to the plane, and the second indicates the direction of the force. Repeated subscripts (e.g., σ_{xx} , σ_{yy} , ...) denote normal stresses, and different subscripts (e.g., σ_{xy} , σ_{yx} , ...) denote shear stresses. In tensor notation the state of stress is expressed as

$$\sigma_{ij} = \begin{vmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix} \quad (\text{A2.1})$$

For equilibrium, there must be a moment balance about each of the axes. A moment balance about the z axis implies that $\sigma_{xy} = \sigma_{yx}$. Similarly, $\sigma_{yz} = \sigma_{zy}$ and $\sigma_{zx} = \sigma_{xz}$, so

$$\sigma_{ij} = \sigma_{ji} \quad (\text{A2.2})$$

and the stress tensor is symmetric.

Except where tensor notation is required, the notation is often simplified by using a single subscript for normal stresses and denoting shear stresses by τ ; for example,

$$\sigma_x = \sigma_{xx}, \quad \tau_{xy} = \sigma_{xy}. \quad (\text{A2.3})$$

A2.1.1. Transformation of axes

A stress state expressed along one set of orthogonal axes (e.g., m , n , and p) may be expressed along a different set of orthogonal axes (e.g., i , j , and k).

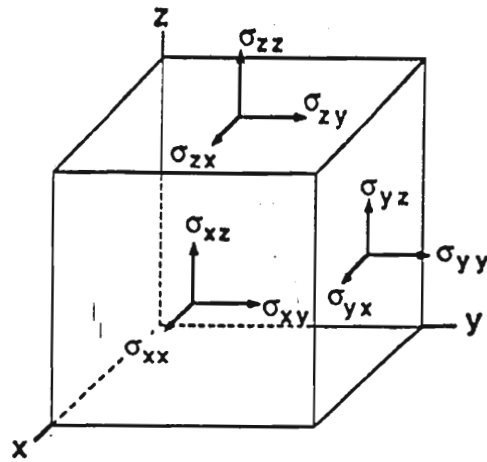


Fig. A2.1. The nine components of stress.

The general form of the transformation is

$$\sigma_{ij} = \sum_n \sum_m l_{in} l_{jn} \sigma_{mn}, \quad (\text{A2.4})$$

where the summations are over the three possible values of n and m , namely m , n , and p . This is often written as

$$\sigma_{ij} = l_{im} l_{jn} \sigma_{mn}, \quad (\text{A2.5})$$

with the summation implied. The terms l_{im} and l_{jn} are the direction cosines between the axes of the two coordinate systems. For example, the stresses along the x, y, z coordinate system may be transformed onto the x', y', z' coordinate system by

$$\begin{aligned} \sigma_{x'x'} = & l_{x'x} l_{x'x} \sigma_{xx} + l_{x'y} l_{x'x} \sigma_{xy} + l_{x'z} l_{x'x} \sigma_{xz} \\ & + l_{x'x} l_{y'x} \sigma_{yx} + l_{x'y} l_{y'x} \sigma_{yy} + l_{x'z} l_{y'x} \sigma_{yz} \\ & + l_{x'x} l_{z'x} \sigma_{zx} + l_{x'y} l_{z'x} \sigma_{zy} + l_{x'z} l_{z'x} \sigma_{zz} \end{aligned} \quad (\text{A2.6a})$$

$$\begin{aligned} \sigma_{y'y'} = & l_{y'x} l_{y'x} \sigma_{xx} + l_{y'y} l_{y'x} \sigma_{xy} + l_{y'z} l_{y'x} \sigma_{xz} \\ & + l_{y'x} l_{z'y} \sigma_{yx} + l_{y'y} l_{z'y} \sigma_{yy} + l_{y'z} l_{z'y} \sigma_{yz} \\ & + l_{y'x} l_{z'y} \sigma_{zx} + l_{y'y} l_{z'y} \sigma_{zy} + l_{y'z} l_{z'y} \sigma_{zz} \end{aligned} \quad (\text{A2.7a})$$

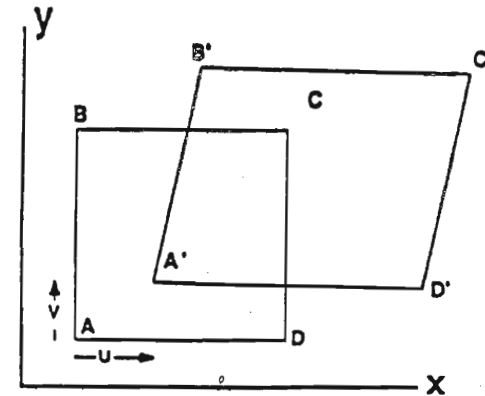


Fig. A2.2. Translation, rotation, and distortion of a two-dimensional body.

These may be simplified using Eq. (A2.2) and the notation in Eq. (A2.3)

$$\begin{aligned} \sigma_{x'} = & l_{x'x}^2 \sigma_x + l_{x'y}^2 \sigma_y + l_{x'z}^2 \sigma_z \\ & + 2l_{x'y} l_{x'z} \tau_{yz} + 2l_{x'z} l_{x'x} \tau_{zx} + 2l_{x'x} l_{x'y} \tau_{xy} \end{aligned} \quad (\text{A2.6b})$$

$$\begin{aligned} \tau_{y'z'} = & l_{y'x} l_{z'x} \sigma_{xx} + l_{y'y} l_{z'y} \sigma_{yy} + l_{y'z} l_{z'z} \sigma_{zz} \\ & + (l_{y'y} l_{z'z} + l_{y'z} l_{z'y}) \tau_{yz} + (l_{y'x} l_{z'x} + l_{y'x} l_{z'z}) \tau_{zx} \\ & + (l_{y'x} l_{z'y} + l_{y'y} l_{z'x}) \tau_{xy}. \end{aligned} \quad (\text{A2.7b})$$

A2.2. Small strains

When a body deforms it often undergoes translation and rotation as well. Strains must be defined in such a way as to exclude these effects. Consider the two-dimensional body in Fig. A2.2. Normal strains are defined as the fractional extensions (tensile) or contractions (compressive), so

$$\epsilon_{xx} = \frac{A'D'}{AD} - 1,$$

which for a small strain is

$$\epsilon_{xx} = \frac{\partial u}{\partial x}. \quad (\text{A2.8})$$

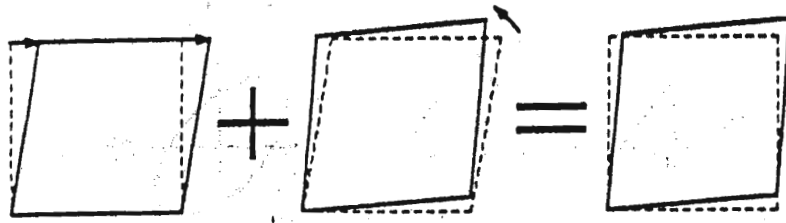


Fig. A2.3. Illustration of shear and rotation.

Similarly,

$$\epsilon_{yy} = \frac{\partial v}{\partial y}.$$

Shear strains are similarly defined in terms of the angles between AD and $A'D'$ and between AB and $A'B'$, which are, respectively,

$$\frac{(\partial v/\partial x) dx}{dx} = \frac{\partial v}{\partial x} \quad \text{and} \quad \frac{(\partial u/\partial y) dy}{dy} = \frac{\partial u}{\partial y}.$$

The total engineering shear strain, γ_{xy} , is defined by the sum of these angles,

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \gamma_{yx}. \quad (\text{A2.9})$$

Figure A2.3 shows that this definition excludes the effects of rotation for small strains.

For a three-dimensional body with displacements w in the z direction, $\epsilon_{zz} = \partial w/\partial z$, $\gamma_{yz} = \gamma_{zy} = \partial w/\partial y + \partial v/\partial z$ and $\gamma_{zx} = \gamma_{xz} = \partial w/\partial x + \partial u/\partial z$.

A2.2.1. Strain tensor

Small strains can be treated as tensors,

$$\epsilon_{ij} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{yx} & \epsilon_{zx} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{zy} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{bmatrix} \quad (\text{A2.10})$$

where the mathematical shear strains, ϵ_{ij} , are one-half of the engineering

shear strains. That is,

$$\left. \begin{aligned} \epsilon_{yz} = \epsilon_{zy} = \frac{1}{2}\gamma_{yz} &= \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \\ \epsilon_{zx} = \epsilon_{xz} = \frac{1}{2}\gamma_{zx} &= \frac{1}{2} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \\ \epsilon_{xy} = \epsilon_{yx} = \frac{1}{2}\gamma_{xy} &= \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \end{aligned} \right\} \quad (\text{A2.11})$$

A2.2.2. Transformation of axes

Small strains may be transformed from one set of axes to another in a manner completely analogous to the transformation of stresses [Eq. (A2.5)]

$$\epsilon_{ij} = l_{im} l_{jn} \epsilon_{mn}, \quad (\text{A2.12})$$

where double summation is implied. For example,

$$\begin{aligned} \epsilon_{x'x'} &= l_{x'x} l_{x'x} \epsilon_{xx} + l_{x'y} l_{x'x} \epsilon_{yx} + l_{x'z} l_{x'x} \epsilon_{zx} \\ &\quad + l_{x'x} l_{x'y} \epsilon_{xy} + l_{x'y} l_{x'y} \epsilon_{yy} + l_{x'z} l_{x'y} \epsilon_{zy} \\ &\quad + l_{x'x} l_{x'z} \epsilon_{xz} + l_{x'y} l_{x'z} \epsilon_{yz} + l_{x'z} l_{x'z} \epsilon_{zz} \end{aligned} \quad (\text{A2.13a})$$

$$\begin{aligned} \epsilon_{x'y'} &= l_{x'x} l_{y'x} \epsilon_{xx} + l_{x'y} l_{y'x} \epsilon_{yx} + l_{x'z} l_{y'x} \epsilon_{zx} \\ &\quad + l_{x'x} l_{y'y} \epsilon_{xy} + l_{x'y} l_{y'y} \epsilon_{yy} + l_{x'z} l_{y'y} \epsilon_{zy} \\ &\quad + l_{x'x} l_{y'z} \epsilon_{xz} + l_{x'y} l_{y'z} \epsilon_{yz} + l_{x'z} l_{y'z} \epsilon_{zz}, \end{aligned} \quad (\text{A2.13b})$$

which can be written more simply in terms of engineering strains,

$$\begin{aligned} \epsilon_{x'} &= l_{x'x}^2 \epsilon_x + l_{x'y}^2 \epsilon_y + l_{x'z}^2 \epsilon_z \\ &\quad + l_{x'y} l_{x'z} \gamma_{yz} + l_{x'z} l_{x'x} \gamma_{zx} + l_{x'x} l_{x'y} \gamma_{xy} \end{aligned} \quad (\text{A2.14a})$$

$$\begin{aligned} \gamma_{x'y'} &= 2(l_{x'x} l_{y'x} \epsilon_x + l_{x'y} l_{y'x} \epsilon_y + l_{x'z} l_{y'x} \epsilon_z) \\ &\quad + (l_{x'y} l_{y'z} + l_{x'z} l_{y'y}) \gamma_{yz} + (l_{x'z} l_{y'x} + l_{x'x} l_{y'z}) \gamma_{zx} \\ &\quad + (l_{x'x} l_{y'y} + l_{x'y} l_{y'x}) \gamma_{xy}. \end{aligned} \quad (\text{A2.14b})$$

A2.3. Body rotations

If the displacement gradients $\partial u/\partial y$ and $\partial v/\partial x$ are not equal, there will be a body rotation about the z axis,

$$\omega_z = \frac{1}{2} \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right), \quad (\text{A2.15})$$

where ω is the clockwise positive rotation in radians. Similarly, the rotations about the x and y axes are given by

$$\omega_x = \frac{1}{2} \left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \right) \quad \text{and} \quad \omega_y = \frac{1}{2} \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \right).$$

A2.4. Large strains

The preceding treatment of strains is limited to cases where the strains are small enough so that rotation of axes relative to one another may be neglected. Chin *et al.* [1] have developed an analysis for large strains resulting from slip.

Reference

1. G. Y. Chin, R. N. Thurston, and E. A. Nesbitt, "Finite Plastic Deformation Due to Crystallographic Slip," *Trans. TMS-AIME* 236 (1966) 69-76.

APPENDIX 3

TEXTURE REPRESENTATION

The orientation relation between a single crystal and an external coordinate system may be represented in one of two ways. The first possibility is to plot the orientation of the crystal axes on a projection where the external axes form the reference system. For example (Fig. A3.1), the cubic axes, 1,2,3, of a crystal may be plotted on a stereographic projection where an external set of axes, x , y , and z are the center, the equator-reference-circle intersection, and north-south poles. Alternatively, the external set of axes, x , y , and z may be plotted on a projection with the 1,2,3 cubic axes at the center, the equator-reference-circle intersection, and north-south poles (Fig. A3.2). These two plotting systems may also be used for polycrystals to show the texture.

A3.1. Pole figures

With the first procedure, a single orientation can be fully represented by plotting only one set of poles (e.g., only $\{100\}$ or only $\{111\}$ poles). In principle, a plot of this type can be used to represent the orientations of a large number of grains. If $\{100\}$ poles are plotted, it is called a $\{100\}$ pole

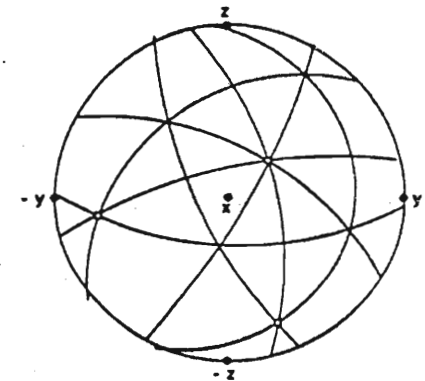


Fig. A3.1. Crystal axes plotted relative to external axes.

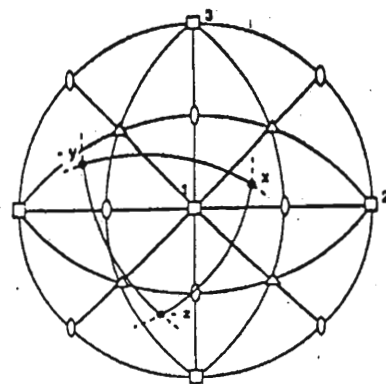


Fig. A3.2. External axes plotted relative to crystal axes.

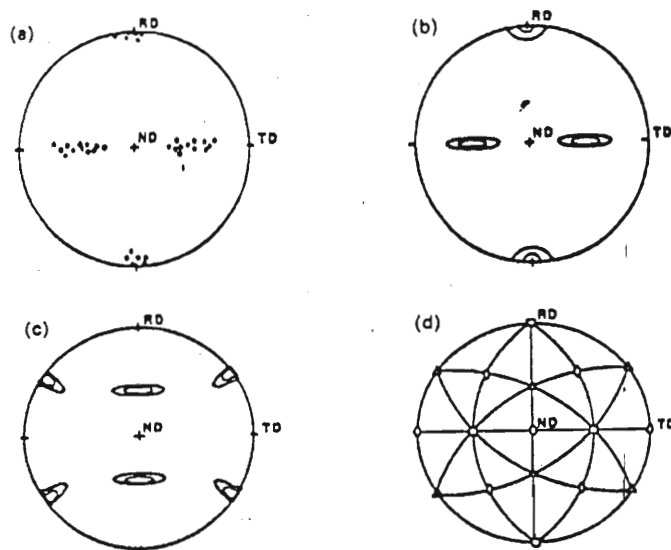


Fig. A3.3. Hypothetical pole figures. (a) {100} poles of a few grains (b) {100} pole figure for a polycrystal (c) {111} pole figure for same material (d) Ideal orientation for the texture.

figure. Figure A3.3(a) is a hypothetical pole figure for a rolled sheet. The natural reference axes are the rolling direction (RD), transverse direction (TD), and the normal to the rolling plane, or normal direction (ND). Here the distribution of {100} poles is not random. If the orientations of a very large number of grains were plotted, contours could be drawn of equal density of {100} per solid angle [Fig. A3.3(b)]. The same texture could also be represented by a {111} pole figure [Fig. A3.3(c)]. Often textures are

described by one or more *ideal orientations*. These are orientations that approximate the texture. Figure A3.3(d) shows that the pole figures are consistent with $\langle 001 \rangle$ parallel to RD and $\langle 1\bar{1}0 \rangle$ parallel to ND, so the ideal orientation is $\langle 001 \rangle \{1\bar{1}0\}$. In this case a better description would be "an ideal $[001](1\bar{1}0)$ rotated $\pm 25^\circ$ about the rolling direction."

For most textures there is orthotropic symmetry; so equivalent orientations of mirror symmetry are present. For example, the ideal orientation $\{hkl\}\langle uvw \rangle$ includes $(hkl)[uvw]$, $(h\bar{k}l)[u\bar{v}w]$, $(hkl)[u\bar{v}\bar{w}]$, and $(h\bar{k}l)[u\bar{v}w]$.

Many real textures are so complex that a large number of ideal orientations are required, and identification of these from pole figures is very difficult.

Usually pole figures are experimentally determined by x-ray diffraction. The Bragg angle, 2θ , between the incident beam and the detector is preset for reflections from a specific set of planes (e.g., {200}, {111}, or {220}). The specimen orientation is then varied and the intensity of x-ray reflection recorded as a function of angle, corrected for variations of absorption with position and plotted to form a pole figure.

A3.2. Inverse pole figures

As indicated, orientations may also be represented by plotting the external reference axes on a standard crystallographic projection. Because of crystal symmetry, a single reference direction may be plotted in any one of the crystallographically equivalent triangles. For cubic crystals the basic standard triangle has {100}, {110}, and {111} poles at the corners. The orientation of a single reference direction (e.g., tensile axis of a single crystal) can be represented by a point in the basic triangle. For a polycrystalline wire, a number of points are required to represent the orientation of the wire axis relative to the crystal axes of different grains. With a sufficiently large number of grains, contours of equal orientation density can be constructed. Such a representation is called an *inverse pole figure*.

In many cases there is more than one external reference axis of importance; so more than one inverse pole figure is required to describe an orientation. For a rolled sheet separate inverse pole figures are needed for the RD, ND, and TD. Figures A3.4(a)–(c) are the inverse pole figures for the textures in Fig. A3.3.

A3.3. Orientation distribution functions

Unless the texture of a material is very simple, it is very difficult to determine quantitatively from pole figures or inverse pole figures the amounts of all the orientations present. Only a three-dimensional plot of orientations can fully describe a texture. Such plots are often called *orientation distribution*

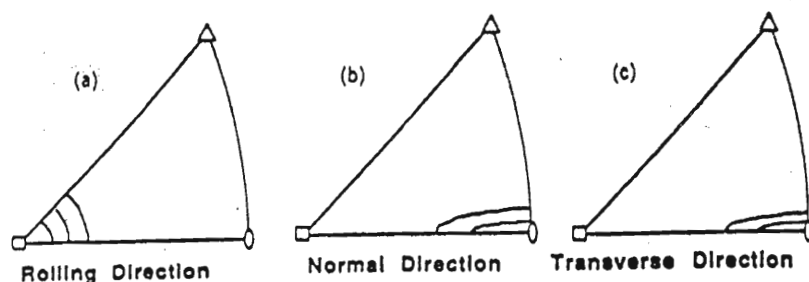
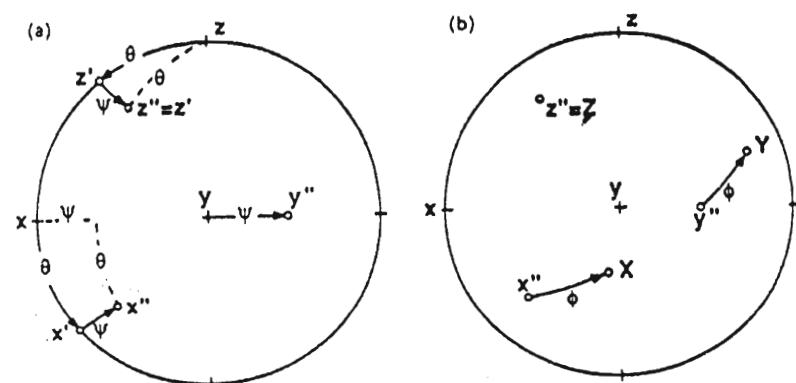


Fig. A3.4. Inverse pole figures representing the same texture as in Fig. A3.3.

Fig. A3.5. The rotations corresponding to Roe's notation system. (a) The first two rotations are by θ and ψ about the y and z axes establish the location of z'' . (b) The third rotation by ϕ about z'' fixes X and Y .

function plots (ODFs) because they can be determined by mathematically representing the texture by spherical harmonics.

Three angles (Euler angles) are required to describe fully a single orientation. There are two different systems of angles commonly used to describe the orientation of the crystal axes ($[100]$, $[010]$, $[001]$) relative to the external coordinate axes (x, y, z). The two most commonly used systems are those proposed by Bunge [1, 2] and Roe [3, 4].

In Roe's system, the crystallographic axes, $[100]$, $[010]$, and $[001]$, are designated X, Y, Z , and the external axes, RD, TD, ND are designated x, y, z . The X, Y, Z axes can be generated from the x, y, z axes by three rotations, θ , ψ , and ϕ , as shown in Fig. A3.5, and to be described here.

Starting with the x, y, z axis system:

Table A3.1. Direction cosines between x, y, z and x'', y'', z'' axes in Roe's system

	x''	y''	z''
x	$\cos \theta \cos \psi$	$-\sin \psi$	$\sin \theta \cos \psi$
y	$\cos \theta \sin \psi$	$\cos \psi$	$\sin \theta \sin \psi$
z	$-\sin \theta$	0	$\cos \theta$

Table A3.2. Direction cosines between x'', y'', z'' and XYZ axes in Roe's system

	$X = 1$	$Y = 2$	$Z = 3$
x''	$\cos \phi$	$-\sin \phi$	0
y''	$\sin \phi$	$\cos \phi$	0
z''	0	0	1

First, rotate the x and z axes counterclockwise about y through an angle θ to form a set of axes, $x', y' = y$, and z' ; $x \rightarrow x', z \rightarrow z'$.

Second, rotate the x', y' , and z' axes counterclockwise about z' through an angle ψ to form a set of axes, x'', y'' , and z'' ; $x' \rightarrow x'', y' \rightarrow y'', z' \rightarrow z'' = Z$. (Thus $90^\circ - \theta$ and ψ can be considered the north latitude and east longitude of Z on a globe with z being the North Pole and x on the equator at the meridian through Greenwich.)

Third, rotate the x'' and y'' axes counterclockwise about $z'' = Z$ through an angle ϕ to form the X, Y, Z axes; $x'' \rightarrow X, y'' \rightarrow Y$.

The angular relations between the axes are described by the direction cosines in Tables A3.1 and A3.2.

For cubic crystals an ideal orientation, $(hkl)[uvw]$ is represented [4] by

$$\begin{aligned} \cos \psi &= w(h^2 + k^2 + l^2)^{1/2} / [(u^2 + v^2 + w^2)(h^2 + k^2)]^{1/2}, \quad h \neq 0, k \neq 0 \\ \cos \theta &= \frac{l}{(h^2 + k^2 + l^2)^{1/2}} \\ \cos \phi &= -h(h^2 + k^2)^{1/2}, \quad h \neq 0, k \neq 0. \end{aligned} \quad (\text{A3.1})$$

For the special case in which $h = k = 0$,

$$\cos(\psi + \phi) = u/(u^2 + v^2 + w^2)^{1/2}.$$

Table A3.3. Equivalence of the Bunge and Roe notation systems

Roe	Bunge
ψ	$\varphi_1 - \pi/2$
θ	Φ
φ	$\varphi_2 + \pi/2$
$\psi + \pi/2$	φ_1
θ	Φ
$\varphi - \pi/2$	φ_2

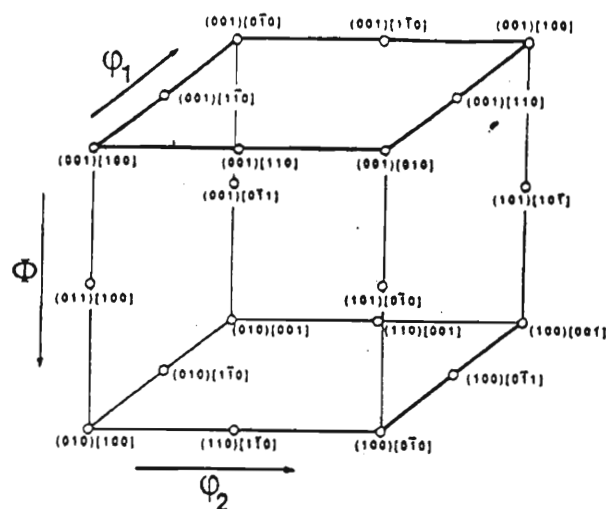


Fig. A3.6. Orientation cube, using Bunge's notation, showing a number of orientations.

Alternatively, the ideal orientation can be found using

$$h = -\sin \theta \cos \varphi, \quad k = \sin \theta \sin \varphi, \quad l = \cos \theta \quad (\text{A3.2})$$

$$\left. \begin{aligned} u &= \cos \psi \cos \theta \cos \varphi - \sin \psi \sin \varphi, \\ v &= -\cos \psi \cos \theta \sin \varphi - \sin \psi \cos \varphi, \\ w &= +\cos \psi \sin \theta. \end{aligned} \right\} \quad (\text{A3.3})$$

Bunge's system [1, 2], which uses different angles, φ_1 , φ_2 , and Φ , has been

shown [5] to be equivalent to Roe's system. Table A3.3 gives the relationships between the angles of the two systems.

Figure A3.6 is a plot, using the Bunge notation, showing a number of orientations. An ODF for a material represents the texture with a series of contours in the cube.

Other systems for describing the orientation relations have been proposed [6, 7]. It should be noted that the orientation system used in Chapters 3 and 4 and Appendix 1 of this book uses three angles, θ , ϕ , and α to describe the rotations used to generate the external coordinate system x, y, z from the crystallographic axes, while the systems described here generate the crystal axes starting with the external coordinates.

References

1. H.-J. Bunge, "Zur Darstellung allgemeiner Texturen," *Z. Metallk.* **56** (1965) 872-74.
2. H.-J. Bunge, "Textures in Three-Dimensional Pole Figures," *Textures in Research and Practice*, eds. J. Grewen and G. Wassermann, Springer-Verlag, Berlin, 1969, 24-35.
3. Ryong-Joon Roe, "Description of Crystallite Orientation in Polycrystalline Materials. III. General Solution to Pole Figure Inversion," *J. Appl. Phys.* **36** (1965) 2024-31.
4. Ryong-Joon Roe, "Inversion of Pole Figures for Materials Having Cubic Symmetry," *J. Appl. Phys.* **37** (1966) 2069-72.
5. G. J. Davies, D. J. Goodwill, and J. S. Kallend, "Charts for Analysing Crystallite Distribution Function Plots for Cubic Metals," *J. Appl. Cryst.* **4** (1971) 67-70.
6. R. O. Williams, "The Representation of Textures of Rolled Copper, Brass, and Aluminum by Biaxial Pole Figures," *Trans. TMS-AIME* **242** (1968) 105-15.
7. H. R. Wenk and U. F. Kocks, "The Representation of Orientation Distribution Functions," *Metall. Trans.* **18A** (1987) 1083-92.

APPENDIX 4

YIELD CRITERIA

A4.1. General

A yield criterion is simply a mathematical expression of the stress states that cause yielding or plastic deformation. In the most general form it may be expressed as:

$$f(\sigma_{ij}) = f(\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{zx}, \tau_{xy}) = C, \quad (\text{A4.1})$$

where C is a critical value. If $f(\sigma_{ij}) < C$, yielding will not occur. For most materials, some simplifications are possible. If the deformation is by shear alone (e.g., slip or twinning), there will be no volume change, and the yield criterion is independent of the level of mean or hydrostatic stress, $\sigma_m = (\sigma_x + \sigma_y + \sigma_z)/3$. In this case the criterion can be expressed as

$$f(\sigma_{ij}) = f((\sigma_y - \sigma_z), (\sigma_z - \sigma_x), (\sigma_x - \sigma_y), \tau_{yz}, \tau_{zx}, \tau_{xy}) = C. \quad (\text{A4.2})$$

If the deformation is reversible in the sense that shear occurs in one direction at the same level of shear stress, τ , as it does in the opposite direction, then only the absolute magnitudes of the terms in Eq. (A4.2) are of importance.

In the special case that x , y , and z are principal stress axes (e.g., $\tau_{yz} = \tau_{zx} = \tau_{xy} = 0$),

$$f((\sigma_y - \sigma_z), (\sigma_z - \sigma_x), (\sigma_x - \sigma_y)) = C, \quad (\text{A4.3})$$

so the yield criterion may be represented in principal stress space as a cylinder or prism with an axis $\sigma_x = \sigma_y = \sigma_z$, as shown in Fig. A4.1(a).

A yield locus is simply a graphical representation of a yield criterion. For convenience the representation is usually a two-dimensional cut of the three-dimensional cylinder. Two types of cut are common: One is a cut on the plane of zero hydrostatic stress ($\sigma_x + \sigma_y + \sigma_z = 0$) [Fig. A4.1(b)], called the π plane; the other is a cut on the $\sigma_z = 0$ plane [Fig. A4.1(c)]. However, in the most general case all terms in Eq. (A4.2) are finite; so five-dimensional stress space is required to represent a yield locus.

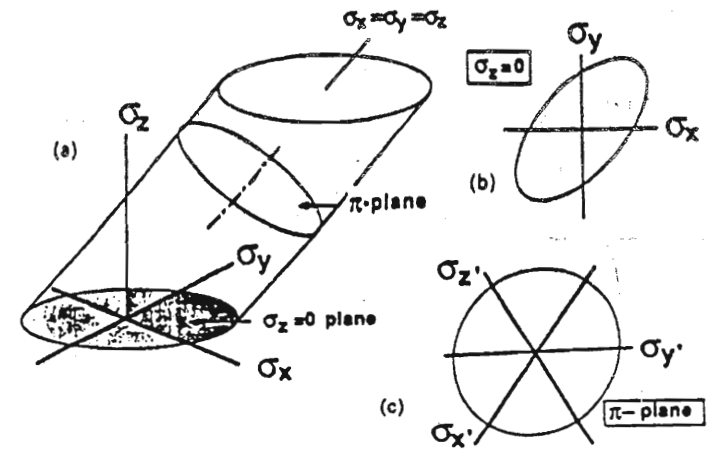


Fig. A4.1. Three-dimensional yield locus (a) and two-dimensional sections corresponding to (b) $\sigma_z = 0$ and (c) $\sigma_x + \sigma_y + \sigma_z = \text{a constant}$ (π plane).

The ratio of the plastic strains resulting from yielding are related to the stress state and the yield criterion by the *principle of normality*, which may be stated mathematically as

$$d\epsilon_{ij} = d\lambda \left(\frac{\partial f(\sigma_{ij})}{\partial \sigma_{ij}} \right), \quad (\text{A4.4})$$

where $f(\sigma_{ij})$ is the yield function and $d\lambda$ is a scalar. If the mathematical form of the yield criterion is known explicitly, the ratios of the strains $d\epsilon_x, d\epsilon_y, \dots, d\epsilon_z$, can be found analytically. For a discussion of the normality principle, see D. C. Drucker [1].

One physical interpretation of Eq. (A4.4) is that the ratios of the strains resulting from yielding can be represented by a vector normal to the yield surface at the stress state, causing the yielding as shown in Fig. A4.2. The strains ϵ_x and ϵ_y are in the same ratio as the components of the vector parallel to the σ_x and σ_y axes. In two dimensions this can be expressed as

$$\frac{d\epsilon_x}{d\epsilon_y} = -\frac{\partial \sigma_y}{\partial \sigma_x}, \quad (\text{A4.5})$$

where $\partial \sigma_y / \partial \sigma_x$ is the slope of the yield locus at the point of loading. This relation is useful in constructing yield loci from either experimental data or theoretical calculations.

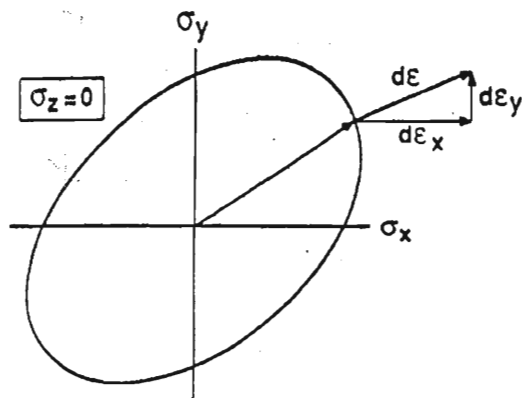


Fig. A4.2. Graphical representation of normality principle. The vector normal to the yield locus gives the ratio of strains during yielding.

A corollary of the principle of normality is that a yield locus cannot be outwardly concave.¹⁴ It may be outwardly convex or have straight sides. If it does have straight sides, all of the stress states on a side will produce the same shape change. There may be corners, and, if so, the stress state at the corner may produce a range of shape changes, as shown in Fig. A4.3.

A4.1.1. Example A4.1

The von Mises criterion for isotropic materials may be written as

$$4f(\sigma_{ij}) = (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 + (\sigma_1 - \sigma_2)^2 = 2Y^2, \quad (\text{A4.6})$$

where σ_1 , σ_2 , and σ_3 are the principal stresses and Y is the yield strength in uniaxial tension. Draw a $\sigma_z = 0$ section of the yield locus and find the stress states corresponding to equal biaxial stretching ($d\epsilon_2 = d\epsilon_1$), plane strain ($d\epsilon_2 = 0$), and plane strain ($d\epsilon_3 = 0$, $d\epsilon_2 = -d\epsilon_1$).

Solution: Substituting $\sigma_3 = 0$ into Eq. (A4.6) leads to

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = Y^2, \quad (\text{A4.7})$$

which plots as an ellipse (Fig. A4.4). Letting α be the stress ratio, σ_2/σ_1 ,

$$\sigma_1 = Y/(1 - \alpha + \alpha^2)^{1/2}. \quad (\text{A4.8})$$

¹⁴ Mathematically a yield function is convex if its Hessian matrix, $H_{ij} = \partial^2 f / \partial \sigma_i \partial \sigma_j$, is positive and

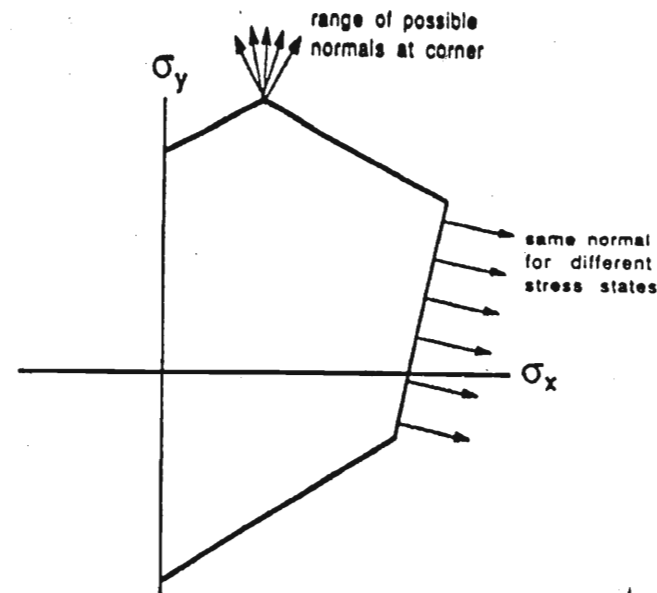


Fig. A4.3. At a corner the range of possible normals represents the possible shape changes. The different stress states along a straight side cause the same strains.

Applying Eq. (A4.4) to Eq. (A4.6) results in the flow rules:

$$\begin{aligned} d\epsilon_1 &= d\lambda \left(\sigma_1 - \frac{\sigma_2 + \sigma_3}{2} \right) \\ d\epsilon_2 &= d\lambda \left(\sigma_2 - \frac{\sigma_3 + \sigma_1}{2} \right) \\ d\epsilon_3 &= d\lambda \left(\sigma_3 - \frac{\sigma_1 + \sigma_2}{2} \right). \end{aligned} \quad (\text{A4.9})$$

Therefore

$$\frac{d\epsilon_2}{d\epsilon_1} = \frac{\sigma_2 - \sigma_1/2}{\sigma_1 - \sigma_2/2} = \frac{\alpha - \frac{1}{2}}{1 - \alpha/2}. \quad (\text{A4.10})$$

Solving for α ,

$$\alpha = \frac{\sigma_2}{\sigma_1} = \frac{(d\epsilon_2/d\epsilon_1) + \frac{1}{2}}{1 + (d\epsilon_2/d\epsilon_1)/2}. \quad (\text{A4.11})$$

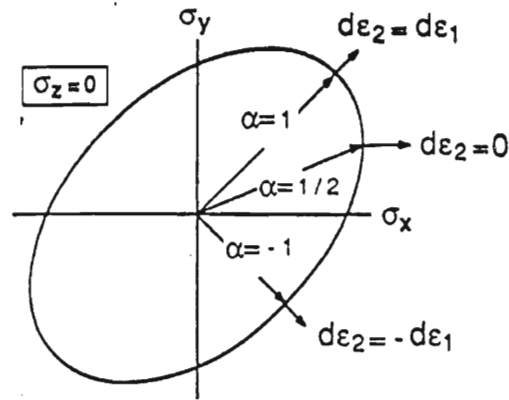


Fig. A4.4. Von Mises yield locus showing loading paths corresponding to several shape changes.

For $d\epsilon_2 = d\epsilon_1$, $\alpha = 1$, and substitution into Eq. (A4.8) gives $\sigma_2 = \sigma_1 = Y$; for $d\epsilon_2 = 0$, $\alpha = 1/2$, $\sigma_1 = 2Y/\sqrt{3}$, $\sigma_2 = Y/\sqrt{3}$; and for $d\epsilon_2 = -d\epsilon_1$, $\sigma_2 = -\sigma_1$, $\sigma_1 = Y/\sqrt{3}$, and $\sigma_2 = -Y/\sqrt{3}$.

Examination of Fig. A4.4 shows these points do correspond to the condition $\partial\sigma_2/\partial\sigma_1 = -d\epsilon_1/d\epsilon_2$. Although this example was for an isotropic material, the principle of normality [e.g., Eqs. (A4.4) and A4.5)] is equally valid for anisotropic materials.

A4.2. Basic assumptions about yield criteria

In continuum plasticity, three simplifying assumptions are commonly made about the yield criteria of materials:

1. Independence of the hydrostatic pressure, or mean stress,

$$\sigma_m = \frac{\sigma_x + \sigma_y + \sigma_z}{3};$$

2. Plastic isotropy;
3. Independence of the sign of the external stresses.

The reasonableness of each of these three assumptions depends on different aspects of structure, and each has a different effect on the shape of the yield locus.

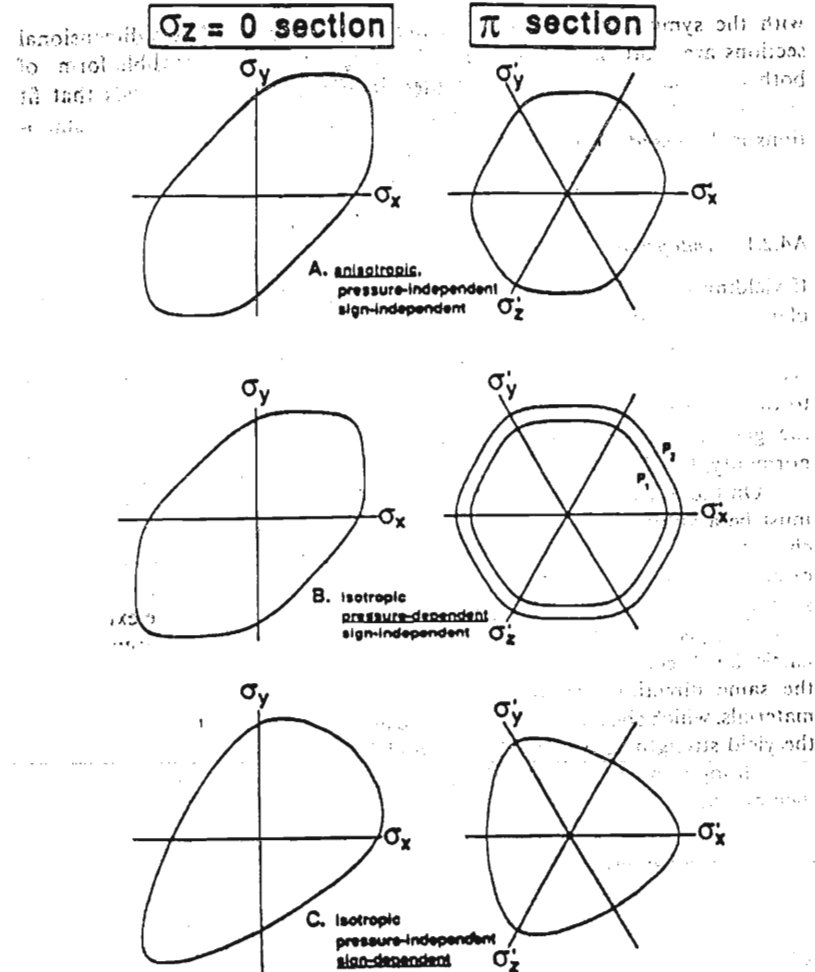


Fig. A4.5. Yield loci corresponding to three different assumptions. The $\sigma_z = 0$ sections are shown at the left, and the π sections at the right. For the π sections, $\sigma'_x = \sigma_x - p$, $\sigma'_y = \sigma_y - p$, and $\sigma'_z = \sigma_z - p$, where $p = (\sigma_x + \sigma_y + \sigma_z)/3$. (a) Represents yield loci that are anisotropic, pressure independent, and sign independent; (b) isotropic, pressure dependent, and sign independent; (c) isotropic, pressure independent, and sign dependent.

The most general yield criteria require six stresses; so a complete plot of the yield locus shape requires six-dimensional stress space. In this treatment, however, yield loci will be represented by two-dimensional sections of three-dimensional plots. The three-dimensional plots are possible if we consider the special cases in which the principal stress axes coincide.

with the symmetry axes of the material. Two types of two-dimensional sections are plotted: the π plane and the $\sigma_z = 0$ section. Possible forms of both of these sections are represented in Fig. A4.5, for materials that fit various combinations of the three assumptions. Each of the three assumptions is discussed below.

A4.2.1. Independence of pressure

If yielding is pressure independent, the shape and size of the locus on all planes parallel to the π plane should be the same, because these planes differ only in the level of mean stress, σ_m . The yield surface in principal stress space is a cylinder whose generator is parallel to the line $\sigma_x = \sigma_y = \sigma_z$. The normal to this surface is parallel to the π plane and has no component parallel to the generator of the cylinder. Therefore, according to the principle of normality, there should be no volume change accompanying yielding.

On the other hand, if yielding is sensitive to hydrostatic pressure, there must be a volume change on yielding, and conversely if there is a volume change on yielding, the yield criterion must be pressure dependent. An example is deformation by martensitic shear, as may occur in some austenitic stainless steels at low temperatures. In this case there is a volume expansion; so yielding is favored by hydrostatic tension. The tensile yield strength in a particular direction should be lower than the compressive yield strength in the same direction. Another example might be the yielding of porous materials, which should shrink in volume as they are deformed. In such cases, the yield strength in tension should be higher.

Although pressure insensitivity is often assumed, there is much experimental evidence that yielding of some materials is pressure sensitive. This has long been recognized in soil mechanics. Compressive yield strengths of polymers are usually significantly higher than their tensile yield strengths.

A4.2.2. Isotropy

A material is isotropic if the position of the principal stress axes relative to the material has no effect on yielding. The π -plane section must have threefold symmetry about the origin. The axes of principal stresses should coincide with the axes of principal strains.

Coincidence of the principal stress and strain axes need not occur in an anisotropic material. They will coincide only if they also coincide with the symmetry axes of the material.

A polycrystalline material will be isotropic only if the crystallographic orientation of its grains is random. Preferred alignment of the grain boundaries, second phases, and residual stresses also can cause anisotropic effects.

A4.2.3. Sign of the stress state

If yielding is independent of the sign of the applied stresses, changing the sign of all of the stress components should have no effect. Such independence is assured if the internal deformation mechanisms are also independent of the sign of the stress. Twinning and martensitic shear are directional mechanisms, and if they occur, yielding will depend on the sign of the stress. Slip in some bcc metals is sign dependent. If the material is both isotropic and sign independent, the π -plane locus must have sixfold rotational symmetry about the origin.

Qualitatively the same strength differential (SD) effect should be predicted for directional slip in bcc metals, since the direction of easy slip is the "forward" twinning direction. Yet the SD effect reported in high-strength steels is of the opposite sign (higher yield strength in compression) and therefore is probably due to a pressure sensitivity of yielding.

A4.3. Yield locus symmetry

The construction of yield loci is greatly simplified by taking advantage of the symmetry. For a material in which yielding is pressure-insensitive and sign-independent, the yield locus has a center of symmetry. For every point on the locus at $\sigma_x = p$ and $\sigma_y = q$, there is a corresponding point at $\sigma_x = -p$ and $\sigma_y = -q$. If, in addition, there is planar symmetry about z , the locus may be divided into four regions (Fig. A4.6). There is mirror symmetry about PP' and QQ' which leads to the following relations:

$$\begin{aligned} \sigma_{xD} &= \sigma_{yA}, & \sigma_{yD} &= \sigma_{xA} \\ \sigma_{xC} &= -\sigma_{xA}, & \sigma_{yC} &= -\sigma_{yA} \\ \sigma_{xB} &= -\sigma_{yA}, & \sigma_{yB} &= -\sigma_{xA} \end{aligned} \quad (A4.12)$$

If the material is also isotropic, region A may be subdivided into three regions according to the size of the Mohr's circles, as shown in Fig. A4.7. The relative sizes of the Mohr's circles are:

in region I, $(\sigma_x - \sigma_z) > (\sigma_x - \sigma_y) > (\sigma_y - \sigma_z)$;
in region II, $(\sigma_x - \sigma_z) > (\sigma_y - \sigma_z) > (\sigma_x - \sigma_y)$; and
in region III, $(\sigma_x - \sigma_y) > (\sigma_x - \sigma_z) > (\sigma_y - \sigma_z)$.

With isotropy, these diameters can be interchanged; so

$$\begin{aligned} \sigma_{xII} &= \sigma_{xI}, & \sigma_{yII} &= \sigma_{xI} - \sigma_{yI}, \\ \sigma_{xIII} &= \sigma_{xI} - \sigma_{yI}, & \sigma_{yIII} &= -\sigma_{yI}. \end{aligned} \quad (A4.13)$$

اینجا دایره‌های موهر را نشان می‌دهد

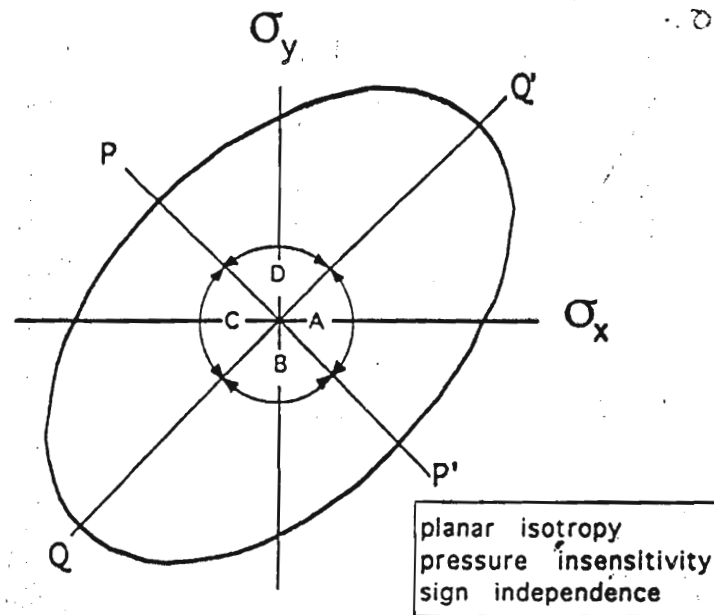


Fig. A4.6. The symmetry of the yield locus for a material in which yielding is anisotropic but insensitive to both pressure and sign of stresses. The locus is symmetric about QQ' and about PP' .

A4.4. Definition of yielding under combined stresses

For a non-work-hardening material, the yield strength is simply defined as the stress required to cause plastic flow. For a work-hardening material, the transition from elastic to plastic flow is gradual. In tension testing, the yield strength is conventionally defined as that level of stress at which the plastic strain reaches some critical level (e.g., $\epsilon = 0.2\%$). Under combined-stress loading, some combination of strain components should be used to define yielding. Such a combination is called the *effective strain*

$$\bar{\epsilon} = f(\epsilon_x, \epsilon_y, \epsilon_z, \dots, \gamma_{xy}), \quad (\text{A4.14})$$

where the effective strain function is related to the yield criterion. This definition is not very useful in situations where the yield criterion and hence the effective strain function is not already known. However, the functional form of Eq. (A4.14) must be consistent with the form of the effective stress, $\bar{\sigma}$, so that the incremental plastic work per volume, dw , can be expressed as

$$dw = \bar{\sigma} d\bar{\epsilon} = \sigma_x d\epsilon_x + \sigma_y d\epsilon_y + \sigma_z d\epsilon_z + \dots + \sigma_{xy} d\gamma_{xy}. \quad (\text{A4.15})$$

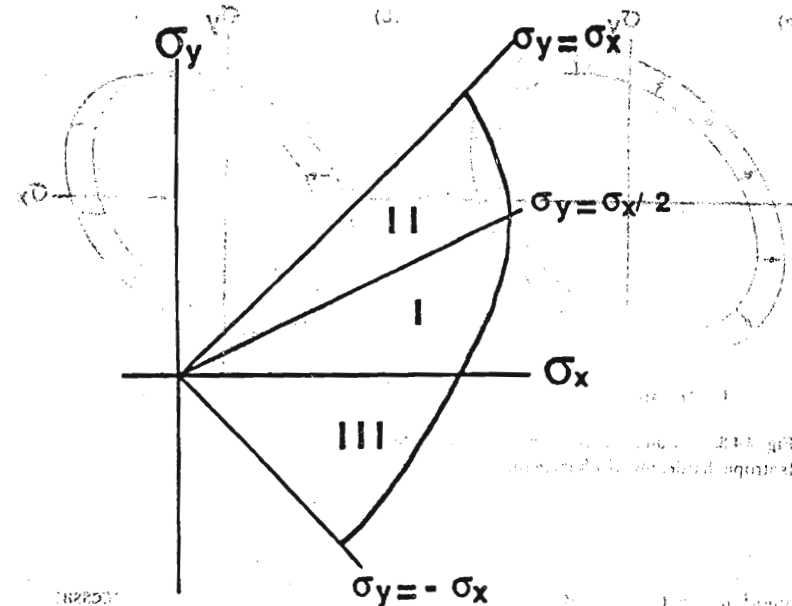


Fig. A4.7. Equivalent regions of isotropic yield locus.

Here the effective stress is defined as the combination of stress components,

$$\bar{\sigma} = f(\sigma_x, \sigma_y, \sigma_z, \dots, \sigma_{xy}). \quad (\text{A4.16})$$

which governs whether yielding occurs. For plane-stress loading with $\sigma_z = \sigma_{xz} = \sigma_{yz} = \sigma_{xy} = 0$,

$$\bar{\sigma} d\bar{\epsilon} = (1 - \alpha\rho)\sigma_x d\epsilon_x, \quad (\text{A4.17})$$

where $\alpha = \sigma_y/\sigma_x$ and $\rho = -d\epsilon_y/d\epsilon_x$. Therefore, with a linear strain path, a fixed level of $\bar{\epsilon}$ corresponds to a value of

$$\epsilon_x = \frac{\bar{\sigma}/\sigma_x}{1 - \alpha\rho} \bar{\epsilon}. \quad (\text{A4.18})$$

A4.5. Work hardening

Two basic assumptions are made about the effect of work hardening on the yield locus. The more common assumption, and the one used here, is that of isotropic hardening, by which is meant that the deformation causes the

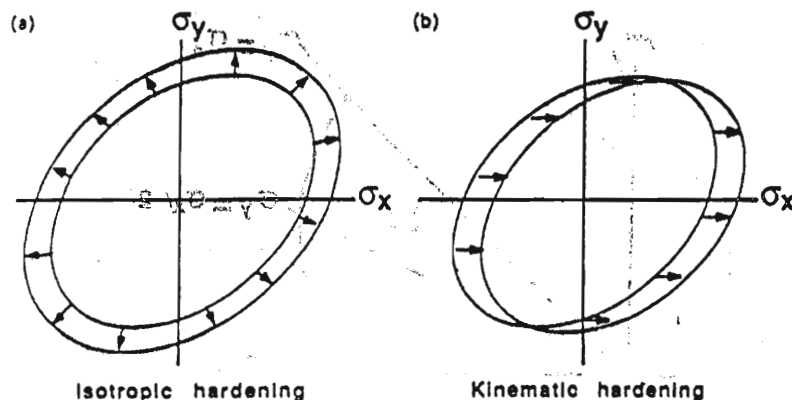


Fig. A4.8. Assumed effects on the yield locus of prestraining in x-direction tension. (a) Isotropic hardening; (b) Kinematic hardening.

yield locus to expand uniformly. The level of effective stress necessary to continue plastic deformation increases with effective strain, regardless of the strain path, so that

$$\bar{\sigma} = f(\bar{\epsilon}). \quad (\text{A4.19})$$

The other extreme assumption is called *kinematic hardening*, by which is meant that strain hardening translates the yield locus in the direction of loading rather than expanding it. The difference between these two assumptions is illustrated in Fig. A4.8. If loading is monotonic (the stress ratio, σ_y/σ_x , and strain ratio, ϵ_y/ϵ_x , held constant), the predictions of both models are the same. For changing loading paths, experimental evidence agrees better with the kinematic model if the new yield stress is measured after a very small amount of strain along the new path; but the isotropic hardening model is better if the new yield stress is measured after a large amount of strain along the new path.

(31.1.1)

A4.6. Plotting yield loci on the π plane

Most of the yield loci in this book are $\sigma_z = 0$ sections of the yield surface. A method of converting $\sigma_z = 0$ section data so that it can be plotted on the π -plane is described below.

Let the stresses on the $\sigma_z = 0$ plane be σ_x and σ_y . In general the reduced

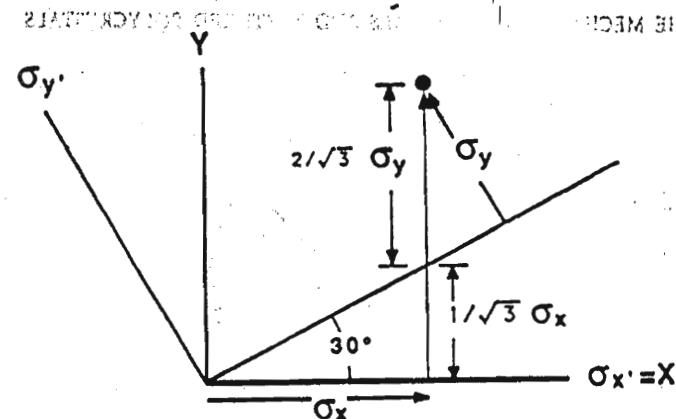


Fig. A4.9. Cartesian coordinates for π -plane plotting.

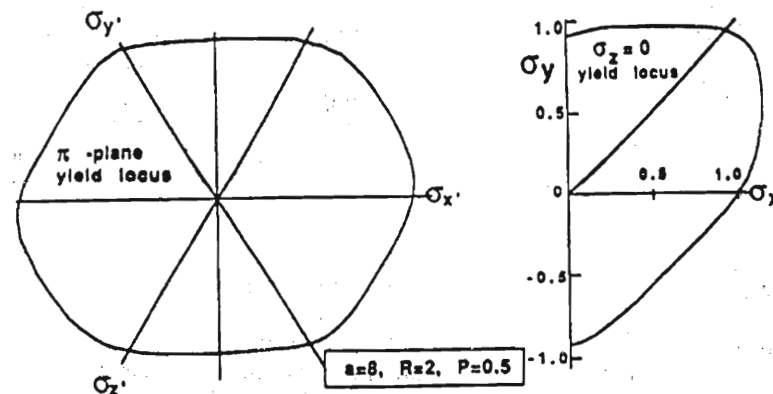


Fig. A4.10. π -plane (left) and $\sigma_z = 0$ plane (right) plots of the yield locus for $P\sigma_x' + R\sigma_y' + RP(\sigma_x - \sigma_y)^2 = P(R+1)X^2$, with $a = 8$, $R = 2$, $P = 0.5$.

stresses are $\sigma_{ii} = \sigma_{ii} - \sigma_H$, where the hydrostatic stress σ_H is

(31.1.2)

$$\sigma_H = \frac{\sigma_x + \sigma_y + \sigma_z}{3}, \quad \text{so} \quad \begin{aligned} \sigma_x &= \frac{2}{3}\sigma_x - \frac{1}{3}\sigma_y, \\ \sigma_y &= \frac{2}{3}\sigma_y - \frac{1}{3}\sigma_x, \\ \sigma_z &= -\frac{1}{3}\sigma_x - \frac{1}{3}\sigma_y. \end{aligned} \quad (\text{A4.20})$$

Figure A4.9 shows that the corresponding coordinates, X and Y on a π plane, are:

$$\begin{aligned} X &= \sigma'_x \\ Y &= \frac{1}{\sqrt{3}} \sigma'_x + \frac{2}{\sqrt{3}} \sigma'_y \end{aligned} \quad (\text{A4.21})$$

Substituting Eq. (A4.20) simplifies Eq. (A4.21) to

$$\left. \begin{aligned} X &= \frac{1}{3} \sigma_x - \frac{1}{3} \sigma_y \\ Y &= \frac{1}{\sqrt{3}} \sigma_y \end{aligned} \right\} \quad (\text{A4.22})$$

Figure A4.10(a) and (b) show the yield locus for the same material plotted on the $\sigma_x = 0$ and π planes. The criterion for these is Eq. (7.24).

Reference

1. D. C. Drucker, *Proc. 1st U.S. Nat. Cong. of Appl. Mech.*, 1951, p. 487.

APPENDIX 5

PRINCIPLE OF MAXIMUM VIRTUAL WORK

The principle of maximum virtual work states that among all possible stress states that can cause a body to yield (i.e., all points on the yield surface) which stress state is appropriate for a specific shape change. It says that the appropriate stress state is the one for which the calculated plastic work per volume, dw , is the largest, where $dw = \sigma_x d\epsilon_x + \sigma_y d\epsilon_y + \dots + \sigma_{xy} d\gamma_{xy}$. Two simple examples will illustrate this principle.

First consider a material that obeys the Tresca yield criterion (Fig. A5.1). We wish to decide which stress state with $\sigma_z = 0$ will produce axially symmetric flow about x , $d\epsilon_x = d\epsilon_y = -\frac{1}{2} d\epsilon_z$, and $d\epsilon_z > 0$, so the virtual work is given by: $dw = \sigma_x d\epsilon_x + \sigma_y (-d\epsilon_z/2)$; so $dw/d\epsilon_x = \sigma_x - \sigma_y/2$. The stress states that can cause yielding are:

1. $\sigma_x = +Y, 0 \leq \sigma_y \leq +Y$
2. $\sigma_y = +Y, 0 \leq \sigma_x \leq Y$
3. $\sigma_x - \sigma_y = +Y$

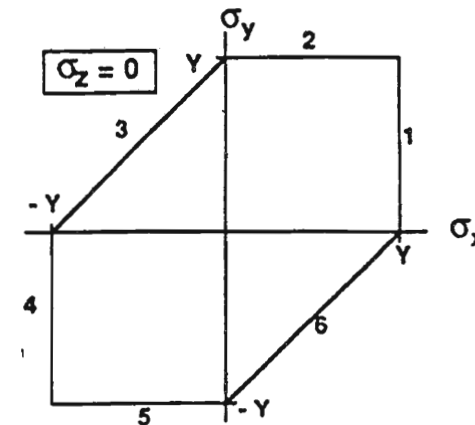


Fig. A5.1. Yield locus for the Tresca criterion.

$$4. \sigma_x = -Y, 0 \geq \sigma_y \geq -Y$$

$$5. \sigma_y = -Y, 0 \geq \sigma_x \geq 0$$

$$6. \sigma_x - \sigma_y = -Y, \sigma_x \geq 0, \sigma_y \leq 0$$

For stress states II and III with $\sigma_y = 0$, $d\epsilon_y/d\epsilon_x = Y$. The calculated value of $d\epsilon_y/d\epsilon_x$ is smaller for all other stress states. Thus we conclude that the appropriate stress state is $\sigma_x = Y, \sigma_y = 0$.

The second example is a frictional system where the normal force, N , and the coefficient of friction, μ , are fixed. Sliding can be caused by a shear force of μN in any direction parallel to the frictional surface. However, for a given displacement d (i.e., shape change), only a shear force acting in the direction of d is appropriate. Otherwise the calculated work, $F \cdot d$, would not be a maximum.

The principle of maximum virtual work is a corollary to the normality principle; when a material yields under a given stress state, the resulting strains are in the ratio that maximizes the dissipated energy.

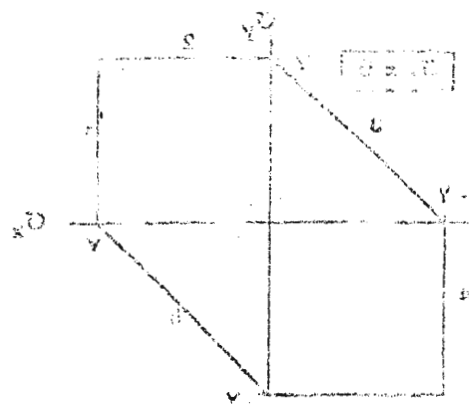


Figure 2.4A shows the corresponding coordinates X and Y on a

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